

Aggregate Dynamics in Lumpy Economies*

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February 15, 2019

Abstract

We develop new tools to analyze the aggregate implications of lumpiness in microeconomic adjustment. We derive a set of structural relationships between the steady state moments and the business cycle dynamics, and we show how to discipline these relationships using micro panel data. As an application, we implement our machinery in a standard framework of firm investment with fixed adjustment costs. We demonstrate analytically that, in order to explain aggregate capital dynamics, that model must match two steady state moments related to capital misallocation and the time since the last investment. Using plant-level data from Chile, we compute these two moments, and discover that there does not exist a calibration of the lumpy investment model that is consistent with the data.

JEL: D30, D80, E20, E30

Keywords: inaction, lumpiness, transitional dynamics, fixed adjustment costs, aggregate shocks, sufficient statistics, firm investment, Ss models.

*We thank Adrien Auclert, Rudi Bachmann, Fernando Broner, Andrea Caggese, Javier Cravino, John Leahy, Kurt Mitman, Matthias Meier, Pablo Ottonello, Jaume Ventura, Edouard Schaal, and seminar participants at CREi, UPF, Michigan, EM³C Conference, Banque de France, Bocconi, Cleveland FED, and PSE for extremely useful feedback and suggestions. Lauri Esala provided outstanding research assistance. Isaac Baley gratefully acknowledges the support of the Spanish Ministry of the Economy and Competitiveness, through the Seed Grant (Aggregate Dynamics in Lumpy Economies) of the Barcelona Graduate School of Economics Severo Ochoa Programme for Centres of Excellence in R&D (SEV-2015-0563).

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1 Introduction

Lumpiness in microeconomic adjustment is pervasive in many economic environments. For instance, capital investment, inventory management, consumption of durable goods, price setting, portfolio adjustment, and many other economic decisions faced by firms and households are characterized by periods of inaction followed by bursts of activity. Recurrent questions that arise in these environments relate to the macroeconomics consequences of this micro lumpiness. How does lumpiness in microeconomic adjustment affect aggregate dynamics? After a policy change or an aggregate shock, how long do transitions last until the lumpy economy reaches its new long-run equilibrium? Understanding these issues is key for the design and implementation of policies aiming at stabilizing the business cycle or generating long-run growth.

Our work contributes by developing new tools to study transitional dynamics in lumpy economies. We consider environments with heterogeneous agents that make decisions subject to adjustment frictions and idiosyncratic shocks. These frictions can take the form of fixed adjustment costs, random opportunities of adjustment, fixed dates of adjustments, among many other. Starting from any arbitrary initial cross-sectional distribution—for instance, one that arises after an aggregate unanticipated shock—we characterize its convergence towards the invariant distribution by focusing on the transition of the moments of the distribution. Following [Alvarez, Le Bihan and Lippi \(2016\)](#), we measure transitional dynamics through the cumulative impulse-response function (CIR), which equals the cumulative deviations in the moment of interest with respect to its steady state value along the transition path.

We contribute by formally characterizing three properties of the CIR that provide a tight link between business cycle dynamics, steady state moments, and observable actions in the micro panel data. The first property, *aggregation*, expresses transitional dynamics as the solution to a representative agent’s recursive problem. The second property, *representation*, expresses the representative agent’s problem as a function of steady state moments. The third property, *observation*, recovers the steady state moments and parameters using information about observable actions, such as the frequency and size of adjustments.

With our theoretical results at hand, we apply our framework to study investment dynamics in an environment with idiosyncratic productivity shocks, a drift due to depreciation and productivity growth, asymmetric policies, and random capital adjustments costs. For this purpose, we set-up a canonical model of lumpy investment à la [Khan and Thomas \(2008\)](#), [Bachmann, Caballero and Engel \(2013\)](#) and [Winberry \(2016\)](#). We demonstrate analytically that, in order to explain aggregate capital dynamics, such a model must match two steady state moments related to capital misallocation and capital age (the time since its last adjustment). In particular, defining the capital gap as the log difference between the current capital and its static optimum, the CIR of average capital in this model depends on the variance of capital gaps and the covariance between capital gaps and capital age. Using annual plant-level data from the Chilean Annual National Manufacturing Survey (ENIA) between 1979 and 2011, we compute these two unobservable moments using data on investment rates and expected capital duration. These moments have not been computed before. We discover that there does not exist a calibration of the lumpy investment model that is consistent with the data. In this spirit, our tools can aid researchers in improving their models to be in line with the empirical evidence on inaction.

Let us now discuss in more detail the three properties of the CIR.

1. Aggregation. Transitional dynamics towards steady state are characterized as the solution to the recursive problem of a representative agent. By requiring certain degree of history independence in the processes and policies, we are able to characterize all ex-post heterogeneity due to idiosyncratic shocks through the problem of one agent. Intuitively, consider an environment in which upon adjustment, all agents adjust to bring their state to the same value. Then, agents become ex-ante identical at the moment of adjustment, and they will only differ ex-post due to the idiosyncratic shocks. Any ex-post heterogeneity due to different initial conditions or conditional dynamics can be summarized by one of these ex-ante agents. Importantly, our aggregation result *does not imply* that heterogeneity is irrelevant for aggregate dynamics; it says that all heterogeneity can be summarized in a compact way.

2. Representation. The solution to the representative agent’s problem can be represented through a combination of steady state cross-sectional moments. The idea behind this result lies in that, in an approximation around the steady state, the ergodic moments encode information about agents’ responsiveness to idiosyncratic shocks, and thus these moments inform us about the representative agent’s policy. For instance, an economy that features substantial ex-post heterogeneity relative to the fundamental volatility of shocks must necessarily have important adjustment frictions, which in turn generate unresponsive policies and persistent transitional dynamics.

3. Observation. Our previous results express transitional dynamics as a function of steady state moments. However, such moments, in many cases, may be hard to observe in the data. Firms’ markups are unobservable, since we only observe average cost and not marginal costs, and it is even harder to think about capital gaps, the distance between firms’ capital and their optimal level. Then, how we can discipline steady state moments of these variables? Our third and most applicable result shows how to recover the steady state moments with observable actions, namely, adjustments and stopping times, both of which are very likely to be observable in micro datasets. The logic behind this mapping is that, by assuming a structure for the state’s evolution during inaction and merging it with information revealed through agents’ adjustments, we can back out ergodic moments and the process of shocks affecting them.

Related literature. Aggregate dynamics in inaction models has been widely studied. The ground-breaking work of [Caplin and Spulber \(1987\)](#), [Caballero and Engel \(1991\)](#) and [Caplin and Leahy \(1991\)](#) provided theoretical guidelines in stylized models to understand the role of micro lumpiness in shaping aggregate dynamics. With the surge of micro data, more realistic models that incorporated idiosyncratic shocks were developed, such as [Cooper and Haltiwanger \(2006\)](#), [Golosov and Lucas \(2007\)](#), [Midrigan \(2011\)](#), [Berger and Vavra \(2015\)](#), [Carvalho and Schwartzman \(2015\)](#) and [Álvarez, Lippi and Paciello \(2016\)](#), with the objective of understanding how the interaction of heterogeneity and lumpiness mattered for aggregate dynamics. We contribute by providing novel theoretical insights and an empirical strategy that exploits the micro data to its maximum while imposing a minimum structure to the inaction model.

Our paper is inspired by [Álvarez, Le Bihan and Lippi \(2016\)](#), who consider a multi-product menu cost model with random opportunities to freely adjust and Brownian innovations to markup gaps. In that setup, they study the real effects of monetary shocks. In our view, one of the key results in their paper is that the cumulative impulse-response (CIR) for average markup gaps—a measure of the real effects of

a money shock—equals the kurtosis of price changes times the average duration of prices divided by 6. They show this result analytically for the case of one product ($n = 1$) and infinite products ($n = \infty$), and more generally, they construct power series of each of the terms in the equality and confirm numerically that the relationship holds. Our contribution lies in proving an intermediate link between the CIR and observable actions, given by the steady state moments. For example, in their menu cost model, the intermediate link between the CIR and the kurtosis of price changes is given by the variance of markups. This strategy allows us to study richer environments and provide new economic insights on the aggregate implications of lumpiness.

Our paper also relates to the pioneer work in Hamermesh (1989), where it is shown how firms' labor decision can be rationalized in a fixed adjustment cost model in which the adjustment trigger depends on the labor gap, i.e. difference between current and static optimal labor. This strategy has been applied in Caballero and Engel (1993) and Caballero, Engel and Haltiwanger (1997) for analyzing the consequences of lumpiness for macroeconomic fluctuations. More recently, a similar approach has been applied in the analysis of cross-country productivity differences due to capital misallocation across firms (see Restuccia and Rogerson (2013) for a survey). While in theory this methodology appears adequate, the challenge lies in finding the optimal target to construct the gap. The standard approach, as in Hsieh and Klenow (2009), consists in specifying a production function at the micro level that allows to recover the optimal input demand, therefore the gap. However, as argued by Cooper and Willis (2004) and Oberfield (2013), such an approach suffers from a specification error, as it is hard to test the validity of the technological assumptions. We propose an alternative way—embedded in the observation property—that consists in directly assuming a stochastic process for the unobserved marginal product of an input, thus its optimal static demand, and then adding discipline to the parameters of the stochastic process using observable micro data on investment that holds for *all* lumpy adjustment models. While this approach clearly depends on the assumptions on the stochastic process (e.g. mean-reversion vs. drift), the theory imposes cross-equation restrictions that allow us to validate such assumptions.

Structure of the paper. Section 2 presents a standard model of lumpy investment that allows us to introduce the objects of interest. Section 3 develops the theory and explains the logic behind the aggregation, representation and observation properties of the CIR. Section 4 applies the theory using micro-level data from Chile. Section 5 generalizes and extends the results.

2 Baseline Model: Lumpy Investment

This section describes the economic environment in which we apply the theory we develop. We build a partial equilibrium lumpy investment model in the spirit of Khan and Thomas (2008), Bachmann, Caballero and Engel (2013) and Winberry (2016), with a few simplifications that are discussed below.

2.1 Environment

Time is continuous and infinite. There is a representative household and a continuum of ex-ante identical firms. There is no aggregate uncertainty and firms face two types of idiosyncratic shocks: productivity shocks and a random fixed cost to be paid with each capital adjustment. We denote with $\omega \in \Omega$ the full

history of these two shocks and consider $(\Omega, P, \mathcal{F})$ to be a probability space equipped with the filtration $\mathcal{F} = \{\mathcal{F}_t : t \geq 0\}$. We use the notation $g_{\omega,t} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ to denote an adapted process (a function $\mathcal{F}_t$ -measurable for any $t \geq 0$) and $\mathbb{E}[g_{\omega,t}]$ to denote its expectation under P .

Firms. Firms operate in competitive markets. They produce output Y using capital K as the only input through a decreasing returns technology:

$$Y_{\omega,t} = E_{\omega,t}^{1-\alpha} K_{\omega,t}^\alpha, \quad (1)$$

where the log of idiosyncratic productivity E evolves according to a Brownian motion with drift μ and volatility σ , as follows:

$$d \log(E_{\omega,t}) = \mu dt + \sigma dW_{\omega,t}, \quad W_{\omega,t} \sim Wiener. \quad (2)$$

A firm chooses capital to maximize its expected stream of profits. For every capital adjustment, a firm pays a random fixed adjustment cost proportional to its productivity $\kappa_t E_{\omega,t}$, where κ_t is described by a compound Poisson process. In a period of length dt , the adjustment cost equals a constant $\kappa > 0$ with probability $1 - \lambda dt$, or it is given by random variable $\xi_{\omega,t}$ with probability λdt , where $\xi_{\omega,t}$ follows the distribution $H(\xi)$ with support $[0, \kappa]$. Letting $\tilde{N}_{\omega,t}$ describe a Poisson process with arrival rate λ , we write the adjustment cost $\kappa_{\omega,t}$ as

$$\kappa_{\omega,t} = \begin{cases} \kappa & \text{if } d\tilde{N}_{\omega,t} = 0 \\ \xi_{\omega,t} & \text{if } d\tilde{N}_{\omega,t} = 1. \end{cases} \quad (3)$$

Profits are discounted at the Arrow–Debreu time-zero price Q_t . Capital between adjustments depreciates at a constant rate ψ . With all the elements above, a firm's problem entering at time t consists in choosing a sequence of adjustment dates $(\tau_{\omega,i})$, and investment rates $(\Delta K_{\tau_{\omega,i}} = K_{\tau_{\omega,i}} - K_{\tau_{\omega,i}}^-)$ that solve the following stopping-time problem:

$$\max_{\{\tau_{\omega,i}, \Delta K_{\tau_{\omega,i}}\}_{i=1}^\infty} \mathbb{E} \left[\int_0^\infty Q_s Y_{\omega,s} ds - \sum_{i=1}^\infty Q_{\tau_{\omega,i}} (\kappa_{\omega,\tau_{\omega,i}} E_{\omega,\tau_{\omega,i}} + \Delta K_{\tau_{\omega,i}}) \right], \quad (4)$$

where output, productivity, and fixed costs follow (1), (2), and (3), respectively, and capital follows

$$\log(K_{\omega,s}) = \log(K_{\omega,0}) - \psi(s-t) + \sum_{\tau_{\omega,i} \leq s} \Delta K_{\tau_{\omega,i}}. \quad (5)$$

Household. The household chooses the stochastic process for consumption to maximize its expected utility subject to a budget constraint. The household problem is given by

$$\int_0^\infty e^{-\rho t} C_t dt, \quad \text{subject to } \int_0^\infty Q_t (C_t - \Pi_t) dt = 0. \quad (6)$$

where $\Pi_t \equiv \mathbb{E}[\pi_{\omega,t}]$ denotes aggregate firm's profits and C_t denotes household's consumption.

Aggregate feasibility. Aggregate output Y_t is used for household's consumption C_t and firms investments I_t , which includes capital adjustments adjustment costs¹:

$$\underbrace{\mathbb{E} [E_{\omega,t}^{1-\alpha} K_{\omega,t}^\alpha]}_{Y_t} = C_t + \underbrace{\mathbb{E} [\mathbb{1}_{\{\tau,t\}} [\kappa_{\omega,t} E_{\omega,t} + \Delta K_{\omega,t}]]}_{I_t}. \quad (7)$$

Equilibrium. Given an initial distribution of $\{K_{\omega,0}, E_{\omega,0}\}$, an equilibrium is a set of stochastic processes for prices $\{Q_t\}$, household's policy $\{C_t\}$, and firms' policies $\{\tau_{\omega,i}, \Delta K_{\omega,i}\}$ such that:

- (i) Given $\{Q_t\}, \{C_t\}$ solve the household's problem (6).
- (ii) Given $\{Q_t\}, \{\tau_{\omega,i}, \Delta K_{\omega,i}\}$ solve the firm's investment problem (4).
- (iii) Goods market clears (7).

Price system dynamics. In principle, the price system may depend on two types of states variables: endogenous state variables, such as the distribution of capital holdings across firms, or exogenous state variables. The challenge under the first scenario, is that firms' values and policies would depend on the aggregate state (which in turn depends on their distribution) and thus steady state policies would not be adequate to characterize the transitional dynamics. To circumvent this issue in order to use steady state policies, we have specified a general equilibrium structure that generates a price system which is independent from the firm distribution. In this case, due to linearity of preferences, the only price in the economy is that of the Arrow security Q_t , which satisfies $Q_t = Q_0 e^{-\rho t}$ and is *independent* of the firm distribution. This assumption does not imply that the aggregate state is independent of the firm distribution, since for example, aggregate output does depend on the joint distribution of capital and productivity; but prices are.

Discussion of simplifying assumptions. Let us compare our environment with one of the benchmark models in this literature in Khan and Thomas (2008). First, in contrast to that paper, we do not consider labor as a factor of production. Given that we consider a partial equilibrium setting, and the labor decision is static in their model, this assumption is innocuous since adding labor would only affect the value of the output-capital elasticity. Second, all the investments in our model, regardless if these fall within a small range, require the payment of the fixed adjustment cost (in the language of these authors, we do not consider *unconstrained* investments). This assumption is quantitatively irrelevant for transitional dynamics, as in the calibration, most investments are constrained anyways due to the large size of idiosyncratic shocks relative to aggregate shocks.

Lastly, we consider a random-walk process for idiosyncratic productivity instead of mean reversion. This assumption is considered to simplify the exposition at this stage and it is relaxed in Section 5. Moreover, this assumption is also motivated by empirical observation: considering mean-reverting shocks generates a negative autocorrelation in investment rates at the firm level that is not observed in the data. A random-walk process generates *iid* investment rates which are more aligned with the data.

¹Here $\mathbb{1}_{\{\tau,t\}} = \{\omega : \exists i \text{ s.t. } \tau_{\omega,i} = t\}$ indicates the set of adjusters.

General equilibrium effects. Our analysis takes as a premise that the steady state policies hold along the transition path. This assumption is valid as long as the general equilibrium feedback from the aggregate distribution to the individual policies through prices is quantitatively insignificant. This is guaranteed by the preferences we have assumed. There are several general equilibrium frameworks in which this is the case.² When general equilibrium effects are quantitatively relevant, the tools developed in this paper do not fully characterize aggregate dynamics. Nevertheless, there exist lumpy environments in which the partial equilibrium mechanisms continue to hold in general equilibrium.³

2.2 Characterization of investment policy

Firms' investment policy. The firm's policy is described by three objects. An inaction region, denoted with $\mathcal{C} = (\underline{k}, \bar{k})$, such that the firm adjusts with probability 1 if $k_t \notin \mathcal{C}$. An adjustment hazard, denoted with $\Lambda(k)$, that describes investments within the inaction region $\mathcal{C}$ that happen with a sufficiently low realization of the fixed cost. A reset capital gap, denoted with $\hat{k}$, that describes the new capital gap after investment. Proposition 1 characterizes these three objects. For convenience, let us define the total drift $\nu \equiv -(\psi + \mu)$ and the adjusted discount $\tilde{\rho} \equiv \rho + \lambda - \mu - \frac{\sigma^2}{2}$. Recall that $H(\xi)$ denotes the *cdf* of the adjustment costs.

Proposition 1. *Let $V(K, E)$ be the value of a firm with initial capital K and initial productivity E :*

$$V(K, E) = \mathbb{E} \left[\int_0^\tau e^{-\rho t} E_t^{1-\alpha} K_t^\alpha dt + e^{-\rho \tau} \left(\kappa_\tau E_\tau + \max_{K^*} V(K^*, E_\tau) - (K^* - K_\tau) \right) \right]. \quad (8)$$

Then we can reexpress the value as $V(K, E) = Ev(\log(\frac{K}{E}))$, where $v(k)$, the continuation region $\mathcal{C}$ and the reset capital $\hat{k}$ satisfy the Hamilton-Jacobi-Bellman equation

$$\tilde{\rho}v(k) = e^{\alpha k} + \nu v'(k) + \frac{\sigma^2}{2}v''(k) + \lambda \int \max \left\{ v(\hat{k}) - \xi - (\hat{k} - k), v(k) \right\} dH(\xi) \quad \forall k \in \mathcal{C}, \quad (9)$$

together with the value matching conditions

$$v(\underline{k}) - e^{\underline{k}} = v(\bar{k}) - e^{\bar{k}} = v(\hat{k}) - e^{\hat{k}} - \kappa, \quad (10)$$

and the optimality for the reset capital and smooth pasting conditions

$$v'(z) = e^z \quad \text{for } z \in \{\underline{k}, \bar{k}, \hat{k}\}. \quad (11)$$

²For the effect of monetary shocks, see Woodford (2009), Golosov and Lucas (2007), and the vast literature that builds on them. For real exchange dynamics, see Carvalho and Nechio (2011) and Kehoe and Midrigan (2008). Regarding investment models, Bachmann, Caballero and Engel (2013) and Winberry (2016), building on Khan and Thomas (2008), show that partial equilibrium dynamics are not undone by general equilibrium effects whenever the model is calibrated to match the cyclical properties of aggregate investment or interest rates. Web Appendix C describes some of these frameworks.

³A concrete example of this logic is found in the context of pricing literature with Calvo-type adjustments. In a model with negligible first order general equilibrium effects, Alvarez, Le Bihan and Lippi (2016) show analytically that the effectiveness of monetary policy is a function of the average duration of pricing spells, independent of any type of heterogeneity. Following this result, Blanco and Cravino (2018) reach a similar conclusion in a model with large general equilibrium effects (arising from real rigidities) in the context of real-exchange dynamics. Therefore, the role of heterogeneity and inaction in shaping aggregate dynamics is not altered by general equilibrium forces.

The adjustment dates $\tau_{\omega,i}$ (when capital is reset to $k_{\tau_{\omega,i}} = \hat{k}$) are given by

$$\tau_{\omega,i+1} = \inf \left\{ t \geq \tau_{\omega,i} : k_{\omega,t} \notin \mathcal{C} \text{ or } N_{\omega,t}^k - N_{\omega,\tau_{\omega,i}}^k \geq 1 \right\}, \quad \tau_{\omega,0} = 0. \quad (12)$$

where $N_{\omega,t}^k$ is a Poisson process with arrival rate $\Lambda(k) = \lambda H(v(\hat{k}) - v(k) - (\hat{k} - k))$.

Capital gaps and aggregate variables. Given the firms investment policy, we are interested in characterizing the log deviations of aggregate capital from its steady state. To this end, we define three variables. First, we define the capital gap $k_{\omega,t} \equiv \log(K_{\omega,t}/E_{\omega,t})$ as the log of the ratio of a firm's capital to its productivity. Second, we define $k_{ss} \equiv \mathbb{E}[\log(K_{\omega}/E_{\omega})]$ to be the average of capital gaps in the steady state.⁴ Lastly, we define the *normalized capital gap* $x_{\omega,t} \equiv k_{\omega,t} - k_{ss}$ as a firm's capital gap minus the steady state average. Redefining the state in this way is convenient to characterize the investment policy, and moreover, it has useful properties to think about aggregate objects.⁵

With these definitions, we compute the aggregate capital log deviation from steady state, denoted with $\hat{K}_t$, which up to a first order approximation, it is equal to the average normalized capital gap $\mathbb{E}[x_{\omega,t}]$:

$$\hat{K}_t \equiv \mathbb{E}[\log(K_{\omega,t})] - \mathbb{E}[\log(K_{\omega})] = \mathbb{E}[k_{\omega,t}] - k_{ss} = \mathbb{E}[x_{\omega,t}], \quad (13)$$

where we use the assumption that productivity distribution is in the steady state. Notice that, in this normalization, we first centralize the capital-gap distribution around its steady state average and then we aggregate across firms. By the previous analysis, we may shift the focus from aggregate capital to moments of the normalized capital gaps. Finally, note that the dynamics of other aggregate variables, such as output deviations from steady state $\hat{Y}_t$, can also be expressed in terms of moments of normalized capital gaps:

$$\hat{Y}_t \equiv \mathbb{E}[\log(Y_{\omega,t})] - \mathbb{E}[\log(Y_{\omega})] = \alpha \hat{K}_t = \alpha \mathbb{E}[x_{\omega,t}] \quad (14)$$

Law of motion of capital gaps. To derive the law of motion of capital gaps, we use the firm policy and its adjustment hazard from Proposition 1. Given the investment policy, we normalize the state to consider deviations from steady state: $(\underline{x}, \hat{x}, \bar{x}) \equiv (\underline{k} - k_{ss}, \hat{k} - k_{ss}, \bar{k} - k_{ss})$. The *uncontrolled* capital gaps—not considering any investments—follow the process

$$d\tilde{x}_{\omega,t} = \nu dt + \sigma dW_{\omega,t}, \quad (15)$$

where we use tildes to show explicitly that these variables evolve exogenously. The initial conditions $\tilde{x}_{\omega,0}$ are exogenously given. By the discussion above, the initial condition of the uncontrolled capital gap is by $\tilde{x}_{\omega,0} = k_{\omega,0} - k_{ss}$.

In contrast, the *controlled* capital gaps—taking into account investments—evolves as

$$x_{\omega,t} = \tilde{x}_{\omega,t} + \sum_{\tau_{\omega,i} \leq t} \Delta x_{\tau_{\omega,i}}, \quad (16)$$

⁴The notation without time index t refers to moments computed with the steady state distribution.

⁵Note that we are able to transform the state from capital and productivity (K, E) to capital gaps x due to the homothetic production function and the shape of the fixed cost. For all the proofs in this example see Web Appendix D.

where the adjustment dates are defined in (12) and the investment rates $\Delta x_{\tau_{\omega,i}}$ are defined implicitly by the difference in the capital stock between date $\tau_{\omega,i}$ and immediately before adjustment $\tau_{\omega,i}^-$:

$$\Delta x_{\tau_{\omega,i}} = \hat{x} - x_{\tau_{\omega,i}}^-. \quad (17)$$

2.3 Steady state and transitional dynamics

Steady state moments. Consider the steady state distribution of the controlled state $F(x)$. Define $G(a|x)$ the distribution of the time since x 's last adjustment, which we refer to it as ‘‘age’’. For any numbers $k, l \in \mathbb{N}$, we define the ergodic cross-sectional moment of capital gaps and age as

$$\mathbb{E}[x^k a^l] \equiv \int_x \int_a x^k a^l dG(a|x) dF(x) \quad \forall k, l \in \mathbb{N}, \quad \text{with} \quad \mathbb{E}[x] = 0. \quad (18)$$

Transitional dynamics. Fix an initial distribution of the state $F_0(x) = \mathbb{E}[\mathbb{1}_{\{x_{\omega,0} \leq x\}}]$. We define the impulse-response function for the m -th moment of the capital gap distribution under the initial distribution F_0 , denoted by $IRF_{m,t}(F_0)$, as the difference between its time t value and its ergodic value:

$$IRF_{m,t}(F_0, t) \equiv \underbrace{\mathbb{E}[x_{\omega,t}^m]}_{\text{transition}} - \underbrace{\mathbb{E}[x^m]}_{\text{steady state}}. \quad (19)$$

Following Alvarez, Le Bihan and Lippi (2016), we define the cumulative impulse-response (CIR), denoted by $CIR_m(F_0)$, as the area under the $IRF_{m,t}(F_0)$ curve across all dates $t \in (0, \infty)$:

$$CIR_m(F_0) \equiv \int_0^\infty IRF_{m,t}(F_0, t) dt. \quad (20)$$

Figure I illustrates these two objects. In the left panel, we plot the initial distribution F_0 and the steady state distribution F , and also highlight the m -th moment of capital gaps $\mathbb{E}[x_0^m]$ which will be tracked in its way towards steady state. In the right panel, the solid line represents the impulse-response of $\mathbb{E}[x_t^m]$, which is a function of time, and the area underneath it is the cumulative impulse-response or CIR.

The CIR is our key measure of the convergence speed towards the steady state. The smaller is the CIR, the faster the convergence. The following Lemma expresses the CIR in a recursive way, and it is a generalization of the result in Alvarez, Le Bihan and Lippi (2016). Although at this stage we are working in a particular example, the result holds true for any moment of the distribution $m > 1$, for an arbitrary Markovian stopping policy, and for any Markovian law of motion of the uncontrolled state.

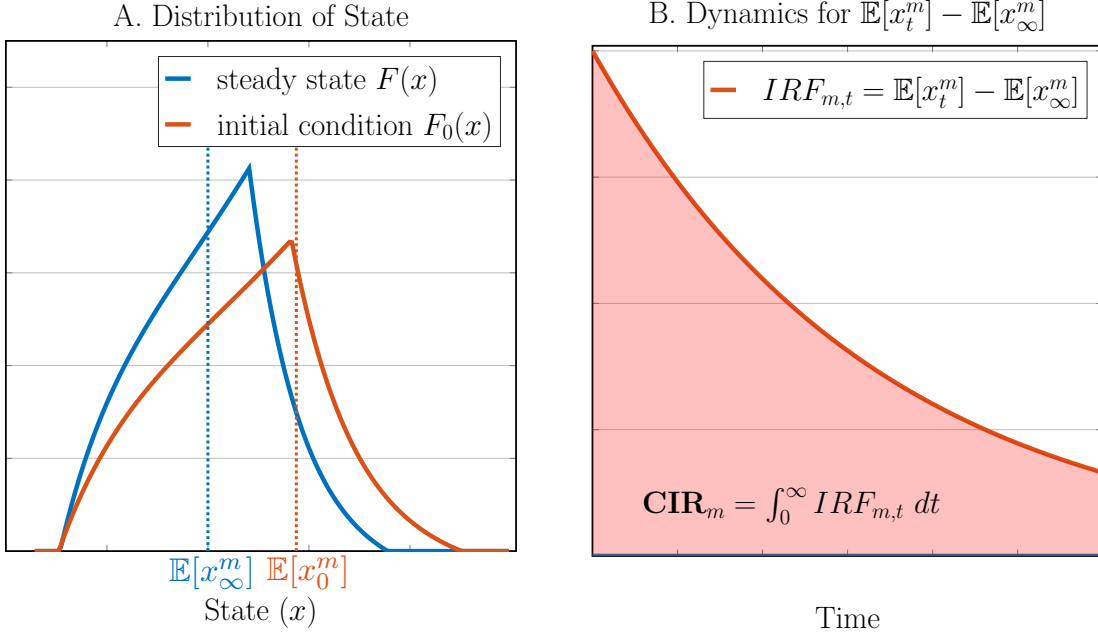
Lemma 1. *The CIR can be written recursively as:*

$$CIR_m(F_0) = \int v_m(x) dF_0(x). \quad (21)$$

where the value function for an agent with initial state x is given by:

$$v_m(x) \equiv \mathbb{E}^x \left[\int_0^\tau (x_t^m - \mathbb{E}[x^m]) dt \right] \quad (22)$$

Figure I – Cumulative Impulse-Response (CIR)



The idea behind Lemma 1 is to exchange the integral across agents (the cross-section) with the infinite time integral (the time-series).⁶ Then, it is key to recognize that the first time a firm adjusts its capital it incorporates the deviations into its policy, and thus we only need to keep track until its *first* adjustment; any additional adjustment is driven by idiosyncratic conditions. The average of these additional adjustments equals the ergodic moment $\mathcal{M}_m[x]$, implying that the value function $v_m(x)$ is equal to zero after the first adjustment. For that reason, the infinite time integral gets substituted for an integral between $t = 0$ and the stopping-time $t = \tau$.

3 The Three Properties in a Baseline Environment

This section derives the three properties of the CIR function. The first result—aggregation—approximates the transitional dynamics, measured via the CIR, as the stopping time problem of a representative firm that captures adjustments through the intensive and the extensive margins. The second result—representation—expresses the intensive and extensive margins in terms of moments of the state’s ergodic distribution. Finally, the third result—observation—connects the state’s ergodic moments with the distribution of policies Δx and τ , which are observable statistics in most micro-data sets.

Our baseline environment focuses on the inaction model of investment presented in Section 2. Here, we study transitional dynamics for average capital gaps, i.e. $m = 1$, when the initial condition consists of a mean translation of the steady state distribution. In Section 5, we extend the results for an arbitrary state space, arbitrary policies, and other additional features.

⁶This can be done due to the ergodic properties of the problem and the fact that moments are finite.

Initial conditions as δ -perturbations around steady state. For ease of exposition, we interpret the initial condition as a perturbation of the steady state distribution, that can be described in terms of one parameter δ . In particular, in the baseline case analyzed here, we consider a perturbation that translates horizontally the distribution of capital gaps. If $f(x - \delta)$ denotes the new density of capital gaps, and we approximate it as $f(x - \delta) \approx f(x) - \delta f'(x)$, we observe that it is equal to a right shift of the steady state density by $\delta f'(x)$. Afterwards, the distribution will evolve according to the agents' policies and will converge back to its steady state. Under this interpretation, we denote $\text{CIR}_m(F_0)$ with $\text{CIR}_m(\delta)$.

3.1 Aggregation

Starting from its recursive representation, Proposition 2 computes the cumulative CIR as the sum of intensive margin Γ and extensive margin Θ components, defined below.

Proposition 2. *To a first order, the transitional dynamics towards the steady state, measured through the CIR, are given by*

$$\text{CIR}_1(\delta) = \delta \times \left(\Gamma_1 + \Theta_1 - \frac{\sigma^2}{2\nu} \Theta_0 \right) + o(\delta^2) \quad (23)$$

where the intensive margin component Γ_1 and the extensive margin components Θ_1 and Θ_0 are:

$$\Gamma_1 \equiv \frac{\mathbb{E}^{\hat{x}} \left[\int_0^\tau \varphi_1^\Gamma(x_t) dt \right]}{\mathbb{E}^{\hat{x}}[\tau]} \quad \text{with} \quad \varphi_1^\Gamma(x) \equiv \frac{1}{\nu} (\mathbb{E}^x[x_\tau] - x) \quad (24)$$

$$\Theta_m \equiv \frac{\mathbb{E}^{\hat{x}} \left[\int_0^\tau \varphi_m^\Theta(x_t) dt \right]}{\mathbb{E}^{\hat{x}}[\tau]} \quad \text{with} \quad \varphi_m^\Theta(x) \equiv \frac{1}{\nu} \left(\frac{\partial \mathbb{E}^x[x_\tau^{m+1}/(m+1)]}{\partial x} - \mathbb{E}^x[x_\tau^m] \right) \quad (25)$$

Equation (23) measures the total effect of the δ -perturbation as an area, with a height of δ , and a base given by two components: Γ_1 , which measure of adjustments through the intensive margin, and Θ_1, Θ_0 , that measure adjustments through the extensive margin. In turn, each component is expressed through two nested HJB equations of a representative firm in (24) and (25), what is sometimes known as recursively squared. The inner HJBs denoted by $\varphi_1^\Gamma(x)$, and $\varphi_m^\Theta(x)$ track conditional dynamics for any initial condition x . They captures the evolution of the state from the initial condition towards the new steady state. The outer HJBs measures the mass of firms at each initial condition by computing the occupancy measure or local time spent at such state between any two adjustments. Together, the nested recursive problems capture the total deviations between the initial condition and the final destination, i.e., the steady state.⁷

The result uses four steps. The first step consists in a first order Taylor approximation of $\text{CIR}_1(\delta)$ together with integration by parts that delivers

$$\text{CIR}_1(\delta) = \int_{\underline{x}}^{\bar{x}} v(x) f(x - \delta) dx \approx \delta \times \int_{\underline{x}}^{\bar{x}} v'(x) f(x) dx. \quad (26)$$

where we have used that there is no mass at the boundary of the inaction region (or the regularity of the boundary of the continuation region).

⁷Convergence to the steady state is not needed for the aggregation result. For example, the property goes through in an Ss model without idiosyncratic shocks that features cycles, as in [Caplin and Spulber \(1987\)](#).

The second step consists in decomposing the base into an intensive and an extensive margin. Since $\mathbb{E}[x] = 0$ by the normalization, we have that $v(x) = \mathbb{E}^x \left[\int_0^\tau x_t dt \right]$ and its derivative is $v'(x) = d\mathbb{E}^x \left[\int_0^\tau x_t dt \right] / dx$. Substitute this derivative into (26); then add and subtract the expectation of the derivative of the state with respect to the initial conditions, $\mathbb{E}^x \left[\int_0^\tau \frac{dx_t}{dx} dt \right]$, which equals $\mathbb{E}^x[\tau]$ to obtain⁸:

$$\text{CIR}_1(\delta) \approx \underbrace{\delta \int_{\underline{x}}^{\bar{x}} \mathbb{E}^x[\tau] f(x) dx}_{\Gamma_1 = \text{intensive margin}} + \underbrace{\delta \int_{\underline{x}}^{\bar{x}} \left(\frac{d\mathbb{E}^x \left[\int_0^\tau x_t dt \right]}{dx} - \mathbb{E}^x[\tau] \right) f(x) dx}_{\Theta_1 - \frac{\sigma^2}{2\nu} \Theta_0 = \text{extensive margin}}. \quad (27)$$

The first component measures aggregate adjustments through the intensive margin: changes in the path x_t due to the new initial condition x *keeping the duration fixed*. The second component measures the aggregate adjustments through the extensive margin: changes in duration due to the new initial condition *keeping the state's path fixed*. To see this clearly, suppose the state evolves deterministically, then the second term becomes $x_t \cdot d\tau/dx$.

The third step consists in finding an equivalent recursive representation for the conditional dynamics inside the integrals in (27). Let us show the steps for the intensive margin. Using the law of motion of x_t , we have that $dx_t = \nu dt + \sigma dW_t$. Then, integrating from 0 to τ and taking expectations with initial condition x , we have that $\mathbb{E}^x[x_\tau] = x + \nu \mathbb{E}^x[\tau] + \sigma \mathbb{E}^x \left[\int_0^\tau dW_t \right]$. By the Optional Sampling Theorem, the last term is equal to zero because it is a martingale with zero initial condition. Therefore, we can express $\mathbb{E}^x[\tau] = \frac{\mathbb{E}^x[x_\tau] - x}{\nu}$. With similar steps, we find the expression for the extensive margin.

In the final step, we focus on the unconditional dynamics, that is, the measure used to integrate the conditional dynamics outlined above. For this purpose, we use an alternative representation of the steady state distribution as the occupancy measure.⁹ Intuitively, the mass of agent in x —given by $f(x)dx$ —is equal to the amount of time spent at x . Formally, we have the following equivalence: $F(x) \equiv \Pr[x_t \leq x] = \mathbb{E}^{\hat{x}} \left[\int_0^\tau \mathbb{1}_{\{x_t \leq x\}} dt \right] / \mathbb{E}^{\hat{x}}[\tau]$. Thus, we rewrite $\frac{\text{CIR}_1(\delta)}{\delta}$ in (27) as

$$\underbrace{\frac{\mathbb{E}^{\hat{x}} \left[\int_0^\tau \overbrace{\left(\frac{\mathbb{E}^x[x_\tau] - x}{\nu} \right)}^{=\varphi_1^\Gamma(x_t)} dt \right]}{\mathbb{E}^{\hat{x}}[\tau]}}_{\Gamma_1} + \underbrace{\frac{\mathbb{E}^{\hat{x}} \left[\int_0^\tau \overbrace{\frac{\partial \mathbb{E}^x[x_\tau^2/2]}{\partial x} / \partial x - \mathbb{E}^x[x_\tau]}^{=\varphi_1^\Theta(x_t)} dt \right]}{\mathbb{E}^{\hat{x}}[\tau]}}_{\Theta_1} - \frac{\sigma^2}{2\nu} \underbrace{\frac{\mathbb{E}^{\hat{x}} \left[\int_0^\tau \overbrace{\frac{\partial \mathbb{E}^x[x_\tau]}{\partial x} - 1}_{=\varphi_0^\Theta(x_t)} dt \right]}{\mathbb{E}^{\hat{x}}[\tau]}}_{\Theta_0} \quad (28)$$

Usefulness of aggregation. We have shown that transitional dynamics (measured through the CIR) can be written as the solution of a recursive problem for a single agent. The intuition lies in the fact that after adjustment, firms become identical, and then any differences in their dynamics arise due to differences in their initial conditions at the moment of the perturbation. All the information regarding these ex-post differences is neatly summarized in the representative agent problem.

The usefulness of the aggregation result in Proposition 2 is twofold. Within the scope of this paper, it allows us to derive a connection between the intensive and extensive margins and the steady state

⁸ $\mathbb{E}^x \left[\int_0^\tau \frac{dx_t}{dx} dt \right] = \mathbb{E}^x \left[\int_0^\tau \frac{d(x + \nu t + \sigma W_t)}{dx} dt \right] = \mathbb{E}^x \left[\int_0^\tau 1 dt \right] = \mathbb{E}^x[\tau]$.

⁹ See [Stokey \(2009\)](#) for details.

moments of the distribution as we show next. More generally, its usefulness lies in aiding researchers with an aggregation result in models with inaction in which the cross-sectional distribution is part of the state.

3.2 Representation

With a recursive expression for the transitional dynamics at hand, the second step consists in expressing the three components of the CIR, Γ_1 and Θ_1 and Θ_0 , as a function of the steady state cross-sectional moments. We derive a mapping that does not depend on a particular inaction model, therefore different models will produce different aggregate effects if and only if they change the ergodic moments.

Characterization of intensive margin Γ_1 . The first component of the CIR, Γ_1 , measures the aggregate effects keeping any changes in aggregate duration fixed. Proposition 3 shows that the intensive margin equals the capital's average age across firms.

Proposition 3. *The intensive margin component is given by:*

$$\Gamma_1 = \mathbb{E}[a] \quad (29)$$

Technically, the proof consists on finding an equivalence between the average age $\mathbb{E}[a]$ and Γ_1 by using the occupancy measure, Itô's lemma, and the law of iterated expectations. First, by the equivalence between the ergodic distribution and the occupancy measure, we have that

$$\Gamma_1 = \frac{\mathbb{E}^{\hat{x}} \left[\int_0^\tau (x_\tau - x_t) dt \right]}{\nu \mathbb{E}^{\hat{x}} [\tau]} = \frac{\mathbb{E}^{\hat{x}} [x_\tau \tau]}{\nu \mathbb{E}^{\hat{x}} [\tau]} - \frac{\mathbb{E}^{\hat{x}} \left[\int_0^\tau x_t \right]}{\nu \mathbb{E}^{\hat{x}} [\tau]} = \frac{\mathbb{E}^{\hat{x}} [x_\tau \tau]}{\nu \mathbb{E}^{\hat{x}} [\tau]}, \quad (30)$$

where we have used that $\frac{\mathbb{E}^{\hat{x}} \left[\int_0^\tau x_t \right]}{\nu \mathbb{E}^{\hat{x}} [\tau]} = \frac{\mathbb{E}[x]}{\nu} = 0$ by the normalization. Lastly, following similar steps as before (using Itô's Lemma and the Optional Sampling Theorem), it is easy to show that $\frac{\mathbb{E}^{\hat{x}} [x_\tau \tau]}{\nu \mathbb{E}^{\hat{x}} [\tau]} = \frac{\mathbb{E}[x]}{\nu} + \mathbb{E}[a] = 0 + \mathbb{E}[a]$. Substituting this equivalence above, we have the result.

How do we understand the connection between the intensive margin and the capital's average age? Average age provides information about the speed at which the average firm adjusts to the perturbation from the steady state. The older is average capital in the economy, the longer the transition. Consider a frictionless limit in which all firms continuously invest to bring capital gaps to zero. Since capital in all firms would have age equal to zero, the economy reaches its steady state immediately. The reason is that any deviation from steady state is immediately absorbed into the representative firm's policy and there are no persistent deviations from steady state.

Characterization of extensive margin Θ_1 and Θ_0 . Now, it is the turn to characterize the evolution of the extensive margin Θ_m in terms of ergodic cross-sectional moments. There are two main challenges. First, the extensive margin does not only depend on the *immediate* response of the aggregate adjustment frequency, but it also reflects all current and *future changes* in frequency. Second, even if we had the whole sequence of adjustment frequency that follows a perturbation, the extensive margin also depends on the

capital-gaps of the particular set of firms selected to invest. This is clearly seen again in the deterministic case, where the extensive margin becomes $x_t \cdot d\tau/dx_0$, thus the change in frequency is scaled by the state, diffculting its characterization. Next, we develop a theory to discipline these two objects.

Proposition 4 presents a characterization of the extensive margin in terms of two objects: how investment responds to idiosyncratic initial conditions (a micro-elasticity) and the aggregate moments of the distribution. Together, these objects imply a macro-elasticity of the extensive margin with respect to the perturbation. Additionally, as known in the literature, we show that in time-dependent models, the extensive margin does not play a role.

Proposition 4. *Let $g_m(x)$ be a smooth function such that for all m*

$$g_m(x) = \mathbb{E}^{\hat{x}+x}[(\hat{x} - \Delta x)^m] - \mathbb{E}^{\hat{x}}[(\hat{x} - \Delta x + x)^m], \quad (31)$$

and define the micro-elasticities as $\theta_{m,j} \equiv \frac{1}{\nu} \sum_{k \geq j}^{\infty} \frac{\hat{x}^{k-j}}{k!j!} \left[\frac{d^{k+1}g_{m+1}(0)/m+1}{dx^{k+1}} - \frac{d^k g_m(0)}{dx^k} \right]$. Then, the extensive margin is given by

$$\Theta_m = \sum_{j=0}^{\infty} \theta_{m,j} \mathbb{E}[x^j]. \quad (32)$$

Moreover, if τ is independent of x , then $g_m(x) = \theta_{m,j} = 0$ for all m and $\Theta_m = 0$ as well.

First, let us describe each object in equation (31). Recall that the expected capital gap at the moment of adjustment is equal to $x_\tau = \hat{x} - \Delta x$. Now, the first term, given by $\mathbb{E}^{\hat{x}+x}[(\hat{x} - \Delta x)^m]$, equals the expected capital gap at the moment of adjustment when the initial condition is $\hat{x} + x$; while the second term, given by $\mathbb{E}^{\hat{x}}[(\hat{x} - \Delta x + x)^m]$, equals the expected capital gap at the moment of adjustment plus a deterministic increase of size x when the initial condition is $\hat{x}$. The difference between these two functions of x provides information of how the *stopping time* depends on the initial condition and how it correlates with the state. To see this more clearly, notice that we can re-express $g_m(x)$ in the following way

$$g_m(x) = \mathbb{E} \left[(\hat{x} + x - \nu\tau^{\hat{x}+x} - \sigma W_{\tau^{\hat{x}+x}})^m \right] - \mathbb{E} \left[(\hat{x} + x - \nu\tau^{\hat{x}} - \sigma W_{\tau^{\hat{x}}})^m \right], \quad (33)$$

where τ^z is the stopping time with initial condition z . In equation (33) we observe that if τ is independent of the initial condition, i.e. $\tau^{\hat{x}+x} = \tau^{\hat{x}}$, as in time-dependent models, then $g_m(x) = 0$ for all x . Thus, $g_m(x)$ provides a micro-elasticity of firms idiosyncratic response to the new initial conditions though changes in its stopping-time τ . To construct the aggregate elasticity, we aggregate the micro-elasticities with weights equal to the capital-gap's ergodic moments, which encode information about the state's distribution.

Constructing $g_m(x)$ from the data. To construct the micro-elasticities, we need to ask: what is observable in the data and what is not? The object $\mathbb{E}^{\hat{x}}[(\hat{x} - \Delta x + x)^m]$ is an observable statistic as it depends on the steady state investment rates.¹⁰ The objects $\mathbb{E}[x^j]$ and $\hat{x}$ can also be recovered from the

¹⁰If idiosyncratic volatility is large enough with respect to aggregate volatility, then steady state micro-statistics can be recovered with average statistics in a model with business cycles. See [Blanco \(2015\)](#) for a verification of this statement in

data as we show in the next section. Therefore, the only object that might not be directly observable is $\mathbb{E}^{\hat{x}+x}[(\hat{x} - \Delta x)^m]$, which measures the elasticity of investment with respect to changes in initial conditions. Guided by the theory, we suggest that this elasticity is the key object that future research should focus on computing, both in the data and in the models.

There are two papers that use an adequate methodology and data to construct the micro-elasticities. First, in the pricing literature, [Karadi and Reiff \(2014\)](#) study the immediate monthly price response to a change in the VAT with Hungarian CPI. Since a change in the VAT proxies a cost-push shock, the experiment is equivalent to an increase in firm's markup in the same proportion; this design would allow to compute the micro-elasticity of the expected price change to initial conditions. Second, in the investment literature, [Zwick and Mahon \(2017\)](#) exploit shifts in accelerated depreciation to estimate the effect of temporary tax incentives on equipment investment; such a design would allow to compute the micro-elasticity of expected investment to initial conditions.

The role of micro-elasticities: two examples. Are the micro-elasticities necessary to discipline the extensive margin? Moreover, can we provide an analytic value for the infinite sum in Θ_m , that can be related to ergodic moments? The answer to these questions is model-dependent as we illustrate with one example and one counterexample. First, we show that in the random fixed cost model presented in Section 2 (known in the pricing literature as the CalvoPlus model, see [Nakamura and Steinsson \(2008\)](#)), the micro-elasticities are not needed to compute the CIR, since there exists a one-to-one mapping from ergodic moments to the CIR. Second, we provide a counterexample where two different models that generate the same ergodic moments have different CIR; thus, in this case, micro-elasticities play a key role in determining the CIR through the extensive margin.

Example 1. *In the random fixed cost model, where $H(\xi) = 1$ for all ξ , the CIR is equal to (up to first order):*

$$CIR_1(\delta)/\delta \approx \frac{\text{Var}[x] - \nu \text{Cov}[a, x]}{\sigma^2}. \quad (34)$$

Equation (34) shows that there exists a one-to-one mapping from the ergodic cross-sectional moments to the CIR, which is determined by the steady state variance of the capital-gaps, normalized by the shock volatility, minus the covariance between capital age (or vintage) and the capital gap. Consequently, under this particular type of adjustment cost, micro-elasticities are not needed to characterize transitional dynamics.

To build the intuition for this result, consider the case $\nu = 0$ so that the CIR is given exclusively by the (ex-post) dispersion of capital gaps $\text{Var}[x]$ normalized by the (ex-ante) volatility of shocks. This dispersion encodes information about agents' *responsiveness* to idiosyncratic shocks (the higher the ratio the less responsive), and in turn, the responsiveness determines the speed of convergence to the steady state. For instance, high levels of capital misallocation or large price dispersion (normalized by the volatility of idiosyncratic shocks) signal little responsiveness to idiosyncratic shocks, from where we infer that there is also slow adjustment to aggregate shocks. In the case $\nu \neq 0$, the covariance between capital vintage and the investment rate (which is negative in this model) appears as a way to correct for the effects of the drift.

the context of firm pricing decisions.

From Example 1 a natural question arises: Is this a general result? Are the micro-elasticities presented in Proposition 4 actually not relevant? The answer to both questions is no, as we show with the following counterexample.¹¹

Example 2. Let $T \equiv \mathbb{E}^{\hat{x}}[\tau]$ denote average duration. Consider an inaction model with adjustments at fixed dates (Taylor-type) and a standard Ss model; assume away idiosyncratic shocks ($\sigma = 0$) and allow for a non-zero drift ($\nu \neq 0$). In these two models there exists a steady state with a uniform distribution of capital-gaps and an investment distribution with an atom at $-\nu T$; thus they produce the same ergodic moments.¹² Now, let us study transitional dynamics for $\delta < 0$. As stated by the theory, in both models the intensive margin is equal to the average age: $\Gamma_1 = T/2$. Since the Taylor model is time-dependent, $\Theta_1 = 0$ and its CIR equals: $\text{CIR}^{\text{Taylor}}(\delta)/\delta \approx T/2$. In the Ss model, the extensive margin is equal to $\Theta_1 = \theta_{1,0}\mathbb{E}[x^0] = -T/2$ (as $\theta_{1,j} = 0$ for $j > 0$), and its CIR equals $\text{CIR}^{\text{Ss}}(\delta)/\delta \approx T/2 - T/2 = 0$. This result mirrors the classic money neutrality outcome in [Caplin and Spulber \(1987\)](#) in lumpy adjustment models.

The previous counterexample illustrates that two models may produce the same steady state statistics, but nevertheless, they can exhibit completely different transitional dynamics. Our explanation lies in the differences in micro-elasticities, zero in the Taylor-type model and $-\Gamma_1$ in the Ss model. Therefore, there exist cases for which the micro-elasticities are relevant objects for characterizing the extensive margin, and our theory can guide researchers in finding experiments or exogenous variation to compute them.

3.3 Observation

In the third set of results, we express the ergodic cross-sectional moments of the state distribution and the structural parameters in terms of the investment distribution Δx and adjustment dates τ . The relevance of this result lies in that in many applications, the state x is likely to be unobservable, but the adjustments Δx and τ are. This is the case in our example, as capital gaps are hard to observe but investment rates are readily available in the data.¹³ As a consequence of the results in the following Proposition 5, we can track unobservable states using observable statistics.

Proposition 5. Let $(\Delta x, \tau)$ be a panel of observations of adjustment size and inaction duration. Denote with $\mathbb{E}^{\hat{x}}[\cdot]$, $\text{Cov}^{\hat{x}}[\cdot]$ and $\mathbb{CV}^{\hat{x}}[\cdot]^2 = \mathbb{V}^{\hat{x}}[\cdot]/\mathbb{E}^{\hat{x}}[\cdot]^2$, respectively, the cross-sectional average, covariance and coefficient of variation squared conditional on adjustment.

1. The reset capital gap is given by:

$$\hat{x} = \frac{\mathbb{E}^{\hat{x}}[\Delta x]}{2} \left(1 - \mathbb{CV}^{\hat{x}}[\tau]^2\right) + \frac{\text{Cov}^{\hat{x}}[\tau, \Delta x]}{\mathbb{E}^{\hat{x}}[\tau]}. \quad (35)$$

¹¹See Web Appendix E.4 for the proof.

¹²These two models have several ergodic time-varying distributions that depend on the initial condition. To generate an unique ergodic distribution for any initial condition, we add a small and random probability of free adjustment. Besides generating a unique ergodic distribution, it gives differentiability to the CIR at $\delta = 0$ in the Ss model.

¹³Our formulas require us to compute the change in log capital gaps Δx in the data. Due to the continuity assumption for the idiosyncratic productivity, the changes in the capital-gap equal the observed investment rates: $\Delta x_{\omega, \tau} = \lim_{t \uparrow \tau_i} \log(K_{\omega, \tau_i}/K_{\omega, t}) - \lim_{t \uparrow \tau_i} \log(E_{\omega, \tau_i}/E_{\omega, t}) = \lim_{t \uparrow \tau_i} \log(K_{\omega, \tau_i}/K_{\omega, t})$. Therefore, we can compute the changes in the capital gap using changes in the capital stock.

2. The drift and volatility of the capital gap process are recovered as:

$$\nu = -\frac{\mathbb{E}^{\hat{x}}[\Delta x]}{\mathbb{E}^{\hat{x}}[\tau]} \quad ; \quad \sigma^2 = \frac{\mathbb{E}^{\hat{x}}[\Delta x^2]}{\mathbb{E}^{\hat{x}}[\tau]} + 2\nu\hat{x}. \quad (36)$$

3. For any $m \leq 1$, the steady state moments are given by:

$$\mathbb{E}[x^m] = \frac{1}{m+1} \left\{ \frac{\hat{x}^{m+1} - \mathbb{E}^{\hat{x}}[(\hat{x} - \Delta x)^{m+1}]}{\mathbb{E}^{\hat{x}}[\Delta x]} - \frac{\sigma^2 m(m+1)}{2\nu} \mathbb{E}[x^{m-1}] \right\}, \quad (37)$$

$$\mathbb{E}[a] = \frac{1}{2} \mathbb{E}^{\hat{x}}[\tau] \left(1 + \mathbb{C}V^{\hat{x}}[\tau]^2 \right), \quad (38)$$

$$\mathbb{E}[x^m a] = \frac{1}{\nu(m+1)} \left\{ \frac{\mathbb{E}^{\hat{x}}[\tau (\hat{x} - \Delta x)^{m+1}]}{\mathbb{E}^{\hat{x}}[\tau]} - \mathbb{E}[x^{m+1}] - \frac{\sigma^2 m(m+1)}{2} \mathbb{E}[x^{m-1} a] \right\}, \quad (39)$$

Equation (35) shows the observability property for the reset state $\hat{x}$, which is derived from the cross-equation restriction imposed by the normalization of the ergodic mean $\mathbb{E}[x] = 0$. The expression has two components: the first one mainly reflects the effect of the drift in the reset state, while the second one mainly reflects the asymmetry in the policies.

To explain the drift component consider an Ss model without idiosyncratic shocks and a negative drift. In such a model, τ and Δx have a degenerate distribution and the capital-gap distribution is uniform in the domain $[\hat{x} - \Delta x, \hat{x}]$. Our formula implies a reset state of $\hat{x} = \frac{\mathbb{E}^{\hat{x}}[\Delta x]}{2} > 0$, which centers the distribution around zero. By compensating the negative drift, the positive reset state ensures that the ergodic mean is zero. To explain the second component, set the drift to zero and consider an asymmetric inaction region $|\underline{x} - \hat{x}| > |\bar{x} - \hat{x}|$ such that the upper trigger is closer to $\hat{x}$ than the lower trigger. In this case, the capital-gap distribution is skewed towards values lower than $\hat{x}$. Our formula implies a reset state given by $\hat{x} = \text{Cov}^{\hat{x}}[\tau, \Delta x] / \mathbb{E}^{\hat{x}}[\tau] > 0$.¹⁴ By reflecting the bias in the policy, the positive reset state shifts the distribution to the right to ensure that the ergodic mean is zero.

Expressions in (36), which extend those in Alvarez, Le Bihan and Lippi (2016) for the case with drift, provide a guide to infer the parameters of the stochastic process. The first expression shows how to infer the drift from the average investment rate in the data, scaled by the adjustment frequency; and the second expression shows how to infer the volatility from the dispersion in investment rates, scaled by the frequency and corrected by the drift (which also generates dispersion, but not due to fundamental volatility).

Equation (38) relates average age to the average and the dispersion in duration, measured through the coefficient of variation. The relationship with the average duration is straightforward. To understand why the dispersion of duration affects average age, it is important to recall a basic property in renewal theory: the probability that a random firm has an expected time between capital changes of τ is increasing in τ , i.e. larger stopping times are more representative in the capital-gap distribution.¹⁵ Therefore, dispersion

¹⁴The covariance is positive since the longer the duration, the higher is the probability to hit the lower trigger and to do an upward adjustment.

¹⁵This property has been widely studied in labor economics when thinking about long-term unemployment. For example, Mankiw (2014)'s textbook Principles of Macroeconomics mentions that: "[...] many spells of unemployment are short, but most weeks of unemployment are attributable to long-term unemployment".

in duration means there are firms that take a long time to adjust, and on top of that, those firms are more representative in the economy; this rises the average age.

Lastly, equation (37) provides a recursively formula to compute the centralized moments using observed investment rates. By assuming a stochastic process for the uncontrolled capital gaps ($dx_t = \nu dt + \sigma dW_t$), together with Itô's lemma and the Optional Sampling Theorem, we can connect the average *slope* of moment $m + 1$ to the *level* of the previous moments m and $m - 1$ as follows:

$$\underbrace{\mathbb{E}_t [dx_t^{m+1}]}_{\text{slope}} = \nu(m+1) \underbrace{x_t^m}_{\text{level}} dt + \frac{\sigma^2}{2}(m+1)m \underbrace{x_t^{m-1}}_{\text{level}} dt + \sigma(m+1) \underbrace{\mathbb{E}_t [x_t^m dW_t]}_{=0} \quad (40)$$

Therefore, the average investment to the power $m + 1$, gives information about the centralized moment m of capital gaps. To see this clearly, set $\hat{x} = 0$ and $m = 2$, then equation (37) reads $\mathbb{E}[x^2] = \frac{\mathbb{E}^{\hat{x}}[(\Delta x)^3]}{3\mathbb{E}^{\hat{x}}[\Delta x]}$, relating the dispersion of capital gaps in the LHS to the skewness of investment rates in the RHS. A similar argument holds for (39).

4 Application: Investment Dynamics

In this section, we revisit the investment model from Section 2 and apply our tools using establishment-level data to gauge the magnitude of capital misallocation in steady state as well as the transitional dynamics of capital gaps.

4.1 Data description

Sources. We use yearly micro data on the cross-section of manufacturing plants in Chile. Information on depreciation rates and price deflators, used to construct the capital series, comes from National Accounts and Penn World Tables. We report statistics for the total capital stock as well as for structures, a capital category that represents approximately 30% of all investment in the manufacturing sector and features the strongest lumpy behavior.¹⁶ We consider all plants that appear in the sample for at least 10 years (more than 60% of the sample).

Capital stock and investment rates. We construct the capital stock series through the perpetual inventory method (PIM).¹⁷ Let firm's ω stock of capital of type j on year t be given by:

$$K_{\omega,j,t} = (1 - \delta_j)K_{\omega,j,t-1} + I_{\omega,j,t}/D_{j,t} \quad \text{for } K_{\omega,j,t_0} \text{ given,} \quad (41)$$

where the depreciation rate δ_j is a type-specific time-invariant depreciation rate; price deflators $D_{j,t}$ are gross fixed capital formation deflators by capital type; and initial capitals K_{i,j,t_0} are given by the firms' self-reported nominal stock of capital of type j at current prices on the first year in which they report a non-negative capital stock. Gross nominal investment $I_{\omega,j,t}$ is based on the information on purchases,

¹⁶The Online Data Appendix presents all the details on the data, construction of variables, and analysis. We repeat the results for each capital category separately: structures, machinery, equipment and vehicles.

¹⁷See Section A.2 for details on the PIM method and several checks on the data.

reforms, improvements, and sales of fixed assets reported by each plant in the survey:

$$I_{\omega,j,t} = purchases_{\omega,j,t} + reforms_{\omega,j,t} + improvements_{\omega,j,t} - sales_{\omega,j,t} \quad (42)$$

Once we construct the investment and capital stock series, we define the investment rate $i_{\omega,j,t}$ as the ratio of real gross investment to the capital stock:

$$i_{\omega,j,t} \equiv \frac{I_{\omega,j,t}/D_{j,t}}{K_{\omega,j,t}}. \quad (43)$$

Table I presents descriptive statistics on investment rates. Inaction is defined as investment below 1% in absolute value; positive spikes are investments above 20% and negative spikes below -20% . Besides the statistics for Chile, we include the numbers reported by [Cooper and Haltiwanger \(2006\)](#) for 7,000 US manufacturing plants between 1972 and 1988.¹⁸ Structures presents an inaction rate of 76% and a large fraction of positive spikes of 31%. Investment rates are very asymmetric (the frequency of positive investments is larger than the frequency of negative investments) and serially uncorrelated, as in the US data. The zero correlation of investment rates in the data is consistent with the model we developed in the previous section.

Table I – Investment Rates Statistics

	Structures	Total	US
Average Investment	10.3	18.3	12.2
Positive Fraction ($i > 1\%$)	22.8	56.7	81.5
Negative Fraction ($i < -1\%$)	1.4	4.0	10.4
Inaction rate ($ i \leq 1\%$)	75.8	39.3	8.1
Spike rate ($ i > 20\%$)	10.2	24.3	20.4
Positive spikes ($i > 20\%$)	9.7	23.2	18.6
Negative spikes ($i < -20\%$)	0.5	1.0	1.8
Serial correlation	0.0	0.0	0.1

Sources: Own calculations using plant-level data for Chile. The time period is 1979-2011 for plants that appear in the sample for at least 10 years. US refers to data in [Cooper and Haltiwanger \(2006\)](#) which covers manufacturing plants in the US from 1972 to 1988. Investment rates reported in this table are computed as investment divided by initial capital. We use perpetual inventories to compute capital stock. We eliminate investment rates below the 1st percentile and above the 99th percentile of the investment rate distribution.

4.2 Construction of capital gaps and duration

To apply our formulas, for each firm ω and each inaction spell k , we record the capital gap change upon action $\Delta x_{\omega,k}$ and the spell's duration $\tau_{\omega,k}$. Recall that the capital gap change is given by the log difference

¹⁸The Data Online Appendix shows numbers reported in [Zwick and Mahon \(2017\)](#) from tax records for US firms. In particular, the weighted inaction rate across firms is 30% in their unbalanced panel.

in the capital stock between an adjustment date $t_{\omega,k}$ and immediately before adjustment $t_{\omega,k}^-$:

$$\Delta x_{\omega,k} = \hat{x} - x_{t_{\omega,k}}^- = \log \left(K_{t_{\omega,k}} / K_{t_{\omega,k}}^- \right) = \log (1 + i_{\omega,t}) \quad (44)$$

Using the information on investment rates, we construct capital gap changes as:

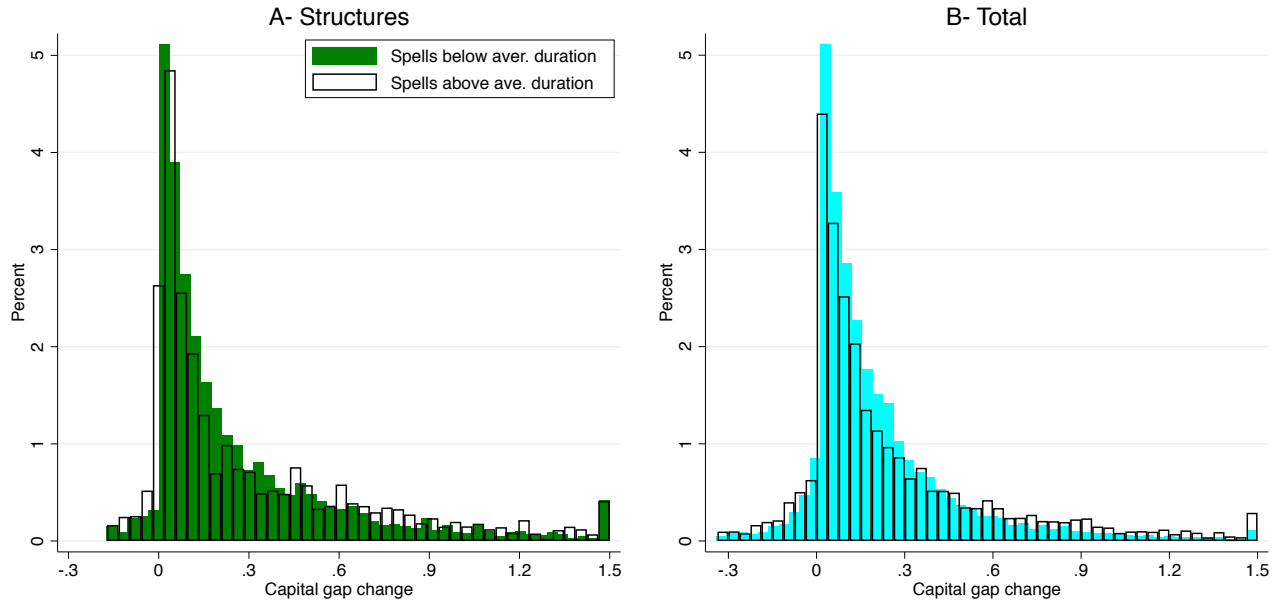
$$\Delta x_{\omega,k} = \begin{cases} \log (1 + i_{\omega,k}) & \text{if } |i_{\omega,k}| > \underline{i} \\ 0 & \text{if } |i_{\omega,k}| < \underline{i}, \end{cases} \quad (45)$$

where $\underline{i} > 0$ is a parameter that captures the idea that small maintenance investments do not incur the fixed cost. Following [Cooper and Haltiwanger \(2006\)](#), we set $\underline{i} = 0.01$, such that all investments smaller than 1% in absolute value are excluded and considered as inaction. Finally, we compute a spell's duration as the difference between two adjacent adjustment dates:¹⁹

$$\tau_{\omega,k} = t_{\omega,k} - t_{\omega,k-1}. \quad (46)$$

Figure II plots the cross-sectional distribution of capital gap changes for structures and the total capital stock. In each figure, we show the distribution for two subsamples: observations with spell duration above the average (dark bars) and spell duration below the average (white bars). Interestingly, the two populations present the same behavior in terms of adjustment size: there is a large mass concentrated at low levels of capital gap changes and there are a few firms that have very large adjustments, and the distribution is asymmetric. Moreover, given that the two distributions lie on top of one another, there is no apparent correlation between adjustment size and duration.

Figure II – Distribution of capital gap changes by duration



¹⁹See the Online Data Appendix for corrections due to duration dependence and unobserved heterogeneity.

As the next step, we put the theory to work by computing the cross-sectional statistics of the capital gap changes and duration in order to back out the parameters of the stochastic process as well as the ergodic moments. With these objects at hand, we then study the transitional dynamics via the CIR implied by different models of inaction.

4.3 Putting the theory to work

The relationships derived in Proposition 5 tell us how to use cross-sectional data on capital gaps and duration to pin down the parameters of the productivity process, the reset point, as well as the ergodic moments, which in turn are used to construct the CIR. Table II summarizes the statistics calculated from the micro data which serve as inputs into the formulas, as well as the theory's output.

Table II shows results by capital type and is divided in two parts. In the upper part, we show the inputs from the data for our observation formulas: cross-sectional statistics for frequency, capital gaps, and covariances between them. The lower part of the table shows the outputs from our theory: estimated parameters $\nu, \sigma^2, \hat{x}$ and ergodic moments $\mathbb{V}ar[x], \mathbb{C}ov[x, a]$, as well as the implied $\text{CIR}_1(\delta)$.

Table II – Inputs from Micro Data and Outputs from the Theory

Inputs from Data			Outputs from Theory		
	Structures	Total		Structures	Total
Frequency			Parameters		
$\mathbb{E}^{\hat{x}}[\tau]$	2.441	1.714	ν	-0.111	-0.129
$\mathbb{C}\mathbb{V}^2[\tau]$	1.093	0.855	σ^2	0.076	0.067
			$\hat{x}$	0.013	0.039
Capital Gaps			Steady State Moments		
$\mathbb{E}^{\hat{x}}[\Delta x]$	0.271	0.220	$\mathbb{V}ar[x]$	0.228	0.149
$\mathbb{E}^{\hat{x}}[\Delta x^2]$	0.192	0.132	$\mathbb{E}[a]$	2.554	1.590
$\mathbb{E}^{\hat{x}}[(\hat{x} - \Delta x)^3]$	-0.186	-0.099	$\mathbb{C}ov[a, x]$	0.914	0.417
Covariances			Transitional Dynamics		
$\mathbb{C}ov^{\hat{x}}[\tau, \Delta x]$	0.063	0.040	$\mathbb{V}ar[x]/\sigma^2$	3.014	2.240
$\mathbb{E}^{\hat{x}}[\tau(\hat{x} - \Delta x)^2]$	0.534	0.254	$-\nu\mathbb{C}ov[a, x]/\sigma^2$	1.340	0.803
			$\text{CIR}_1(\delta)$	4.354	3.043

Sources: Authors' calculations using establishment-level survey data for Chile.

Inputs from Micro Data. Consider first the distribution of expected times τ . We obtain an average expected time to adjustment of $\mathbb{E}^{\hat{x}}[\tau] = 2.4$ years with a large dispersion, suggesting substantial heterogeneity in adjustment times across establishments. Now consider the distribution of capital gaps; it has an average of $\mathbb{E}^{\hat{x}}[\Delta x] = 0.27$ and a second moment of $\mathbb{E}^{\hat{x}}[\Delta x^2] = 0.19$, and it is right-skewed. The covariance between adjustment size and expected time is almost zero $\text{Cov}^{\hat{x}}[\tau, \Delta x] = 0.06$. This zero covariance is surprising, as one would expect a positive covariance: the longer the inaction period, the stronger the effect of the drift; consequently, upon taking action, the investment rate should be larger. We discuss this finding at the end of this section.

Outputs from theory: parameters. Let us now explain the parameter values implied by our formulas. From (36), the implied drift, which captures the depreciation rate, productivity growth, and changes in relative prices, equals

$$\nu = -\frac{\mathbb{E}^{\hat{x}}[\Delta x]}{\mathbb{E}^{\hat{x}}[\tau]} = -\frac{0.271}{2.441} = -0.111, \quad (47)$$

and the volatility of idiosyncratic shocks equals

$$\sigma = \sqrt{\underbrace{\frac{\mathbb{E}^{\hat{x}}[\Delta x^2]}{\mathbb{E}^{\hat{x}}[\tau]}}_{0.08} + \underbrace{2\nu\hat{x}}_{0.002}} = 0.27. \quad (48)$$

Note that the main component that drives the volatility estimate is the second moment of capital gap changes, normalized by expected duration, whereas the drift term is negligible. The calibration for the volatility of innovations in the literature falls within a very wide range, from 0.052 in [Khan and Thomas \(2008\)](#) to 0.117 in [Winberry \(2016\)](#) to 0.202 in [Bachmann, Caballero and Engel \(2013\)](#).²⁰ It is worth noting that these calibrations are done jointly with the fixed adjustment cost within a particular inaction model. In contrast, our volatility estimate is pinned down directly through our model-independent mapping between data and parameters. Lastly, the observation formula for the reset capital gap in (35) implies that, upon adjustment, capital gaps are reset on average 1% above the average capital gap:

$$\hat{x} = \underbrace{\frac{\mathbb{E}^{\hat{x}}[\Delta x]}{2}}_{0.14} \underbrace{\left(1 - \mathbb{CV}^{\hat{x}}[\tau]^2\right)}_{-0.23} + \underbrace{\frac{\text{Cov}^{\hat{x}}[\tau, \Delta x]}{\mathbb{E}^{\hat{x}}[\tau]}}_{0.02} = 0.013 \quad (49)$$

Recall the drift of minus 11%. As capital gaps are reset 1.3% above the average capital gap, the part of the drift that is not compensated by the reset state (9.7%) must be accommodated through an asymmetric policy.

²⁰The original numbers used in those papers are 0.022 and 0.049, respectively. Since we abstract from labor and our productivity is rescaled, we must adjust their volatilities by a factor $1/(1 - \alpha)$ in order to make their numbers comparable to ours. We assume a labor share of $\alpha = 0.58$ and obtain the numbers above. Additionally, for [Bachmann, Caballero and Engel \(2013\)](#) and [Winberry \(2016\)](#), we convert their quarterly volatilities $\sigma^q = 0.021, 0.047$ to yearly taking into account the persistence as follows: $\sigma^a = \sigma^q \sqrt{1 + \rho + \rho^2 + \rho^3}$, with $\rho = 0.94, 0.86$. Lastly, for [Bachmann, Caballero and Engel \(2013\)](#), we only consider the idiosyncratic shocks (excluding the sectorial shocks). Recall that we only consider here structures.

Output from theory: ergodic moments. According to the observation formula (37), the steady state dispersion of capital gaps $\text{Var}[x]$ —a notion of misallocation—can be expressed in terms of capital gap changes and the reset point as follows:

$$\text{Var}[x] = \frac{\hat{x}^3 - \mathbb{E}^{\hat{x}}[(\hat{x} - \Delta x)^3]}{3\mathbb{E}^{\hat{x}}[\Delta x]} = 0.23, \quad (50)$$

where the cubic powers capture asymmetries in the distribution. Note that the ratio $\text{Var}[x]/\sigma^2 = 2.8$ is quantitatively close to the expected duration $\mathbb{E}^{\hat{x}}[\tau] = 2.4$. This suggests that the dispersion—a measure of ex-post heterogeneity—is almost equal to the fundamental volatility σ^2 times expected duration, even if there is a large negative drift of -11% .

The average age $\mathbb{E}[a]$ is recovered using information about the average and the dispersion of adjustment times from (38). Following our earlier discussion on renewal theory—larger stopping times are more representative in the sample—the heterogeneity in expected times increases the average age:

$$\mathbb{E}[a] = \underbrace{\mathbb{E}^{\hat{x}}[\tau]/2}_{1.2} \underbrace{\left(1 + \mathbb{CV}^{\hat{x}}[\tau]^2\right)}_{2.1} = 2.52. \quad (51)$$

Lastly, equation (39) implies that the covariance between age and adjustment size suggested by the data is positive:

$$\text{Cov}[x, a] = \frac{1}{2\nu} \left(\underbrace{\frac{\mathbb{E}^{\hat{x}}[\tau(\hat{x} - \Delta x)^2]}{\mathbb{E}^{\hat{x}}[\tau]}}_{0.22} - \underbrace{\text{Var}[x]}_{0.23} - \underbrace{\sigma^2 \mathbb{E}[a]}_{0.20} \right) = 0.914. \quad (52)$$

This positive covariance between capital gaps and capital age means that the capital holdings of plants that have not adjusted in a long time (their capital is old) are *above* the gap of firms that have recently adjusted.

Discussion. The data throws two surprising results: a tiny covariance between adjustment sizes and duration, $\text{Cov}[\tau, \Delta x] \approx 0$, and a positive covariance between capital gaps and capital age $\text{Cov}[x, a] \approx 1$. While the value of these covariances appears counterintuitive, we want to stress that they are coming directly from the data. Let us recall the only two assumptions we make to establish the link between data and ergodic moments: (i) capital gaps follow a Brownian motion, and (ii) the reset state $\hat{x}$ is *iid*. To be clear, we have not assumed any particular inaction model or parametric restriction of the plants' state besides those imposed to the capital gaps x . Clearly, plants may have other drivers of their investments besides capital gaps, and we do not impose any structure on those.

As a consequence, if a particular model generates a covariance that does not square with the data, one of the two assumptions above must be changed. One option would be to abandon the Brownian motion assumption for capital gaps x , for instance, by introducing mean reversion into productivity. However, this alternative assumption is problematic given the zero autocorrelation of investment rates, as shown at the bottom of Table I. Another option would be to abandon the *iid* reset state, for instance, through the introduction of convex adjustment costs or information frictions that avoid full adjustment. In this case, the substantial lumpy behavior that we document sets a limit to the role of these alternative frictions in

explaining the investment behavior.

4.4 Business Cycle Dynamics.

Now we study the transitional dynamics of the first moment of capital gaps, i.e. CIR_1 . For this purpose, we consider an unanticipated permanent aggregate productivity shock that shifts horizontally the distribution of idiosyncratic productivity of all firms. The transitional dynamics are model dependent, so we provide results for two specifications. If the random fixed cost model is true, where both intensive and extensive margins are active, we obtain that

$$\frac{CIR_1(\delta)}{\delta} \approx \underbrace{\frac{\text{Var}[x]}{\sigma^2}}_{3.014} - \underbrace{\frac{\nu \text{Cov}[a, x]}{\sigma^2}}_{1.340} = 4.354. \quad (53)$$

4.5 Is the random fixed cost model consistent with the data?

Next, we study the quantitative implications of the random fixed cost model. We show that there does not exist a calibration that can simultaneously match the cross-sectional moments implied by our formulas and the data.

Calibration strategy. The discount rate $\rho = 0.04$ and the capital share $\alpha = 0.58$ are set externally. The parameters for the stochastic process of the capital gaps $(\nu, \sigma^2) = (-0.111, 0.076)$ and the reset state $\hat{x} = 0.013$ are taken from the data through our formulas, see Table II. We are left with four parameters to calibrate: the upper and lower border of the inaction region $(\underline{x}, \bar{x})$, the size of the adjustment cost κ , and the arrival rate of free adjustment opportunities λ . To set the remaining parameters, we follow the following strategy. First, we find the arrival rate of free adjustments λ to match the average expected time to adjustment $\mathbb{E}^{\hat{x}}[\tau] = 2.44$. The implied λ is plotted in Figure III. Then, we vary the normalized borders of inaction $(\underline{x} - \hat{x}, \bar{x} - \hat{x})$ and compute the implied capital gap variance $\text{Var}[x]$ and covariance with age $\text{Cov}[x, a]$ for those parameters.

Figure III – Calibration of arrival rate of free adjustment opportunities

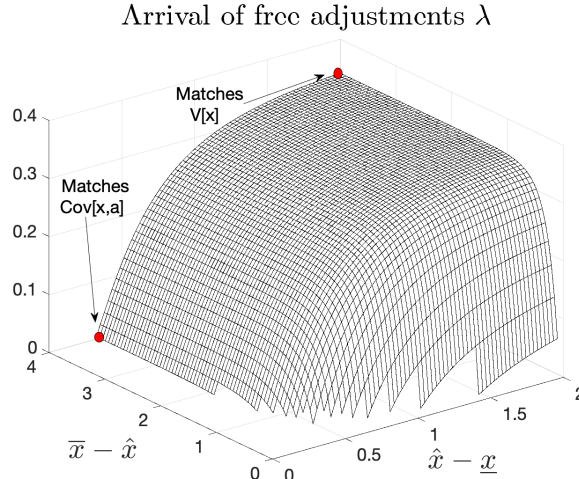
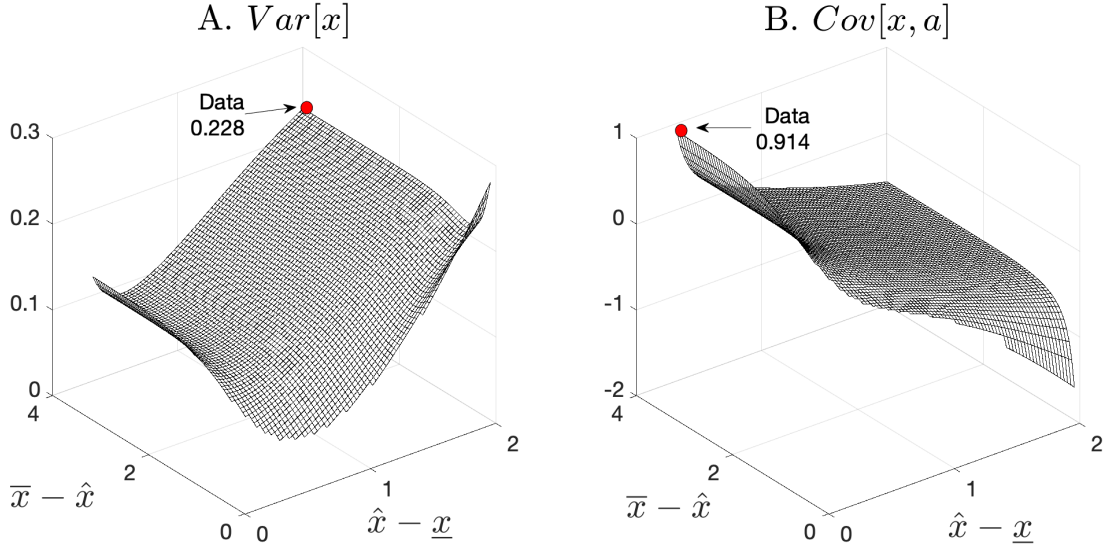


Figure IV plots these key moments for different values of the normalized inaction regions (and the implied λ that matches average duration). The variance is increasing in both sides of the inaction region, while the covariance is decreasing in the lower band and increasing in the first. Given these shapes, we find that, in order to match the variance of capital gaps $\text{Var}[x]$, we need a wide upper band $\bar{x} - \hat{x} = 4$ and a wide lower band $\hat{x} - \underline{x} = 2$ (and a large $\lambda = 0.34$); in other words, the variance calls for wide inaction region with frequent free adjustment opportunities. In this case, the adjustments would be *primarily* driven by the Poisson arrival rate, as in a Calvo model. In contrast, to match the covariance of capital gap and its age, we still need a wide upper band $\bar{x} - \hat{x} = 4$ but a very narrow lower band $\hat{x} - \underline{x} = 0.3$ (as well as a low $\lambda = 0.005$); thus, the covariance calls for a one-sided inaction region and infrequent opportunities to adjust. This is close to a standard menu cost model with very asymmetric inaction regions.

Figure IV – Key moments generated by the random fixed cost model



In summary, the random fixed cost model cannot match simultaneously the two key moments that it must satisfy for the CIR of average capital.

4.6 Heterogeneity concerns

One obvious concern that arises regarding the empirical analysis is that different layers of heterogeneity may affect the computation and the interpretation of cross-sectional statistics. To mitigate this concern, we repeat the analysis considering eight 2-digit subsectors within manufacturing and also different plant sizes in terms of their number of workers.²¹

²¹See Data Online Appendix for details. Table XII reports cross-sectional statistics by subsectors: (1) Food and beverages; (2) Textile, clothing and leather; (3) Wood and furniture; (4) Paper and printing; (5) Chemistry, petroleum, rubber and plastic; (6) Manufacture of non-metallic mineral products; (7) Basic metal; (8) Metal products, machinery and equipment. Table XIII reports cross-sectional statistics by quartiles of the average number of workers during the sample period: small plants (0-25%, S), medium plants (25-50%, M), large plants (50-75%, L), and very large plants (75-100%, XL).

Regarding the sectoral composition, we find that besides textiles and chemicals, all other sectors present very similar investment patters. Table III recomputes our formulas excluding textiles and chemicals and we find that neither the inputs from the data nor the outputs from the theory differ from the numbers that include these sectors. We conclude that heterogeneity across sectors should not be a concern once these two subsectors are excluded. Regarding plant size, things are more interesting. Across all capital categories, average investment, the frequency of non-zero investments, and the fraction of spikes are increasing in plant size; in contrast, the inaction rate decreases with size. Table III recompute our formulas splitting the sample in small and large plants. Not surprisingly, the inputs from the data are different for small and large firms. What is really striking is that the parameters of the stochastic process that we recovered are identical same across sizes: firms are hit with the same type of shocks. In terms of ergodic moments, larger plants have more dispersed capital gaps, capital is younger on average, and have a larger covariance; all of these forces increase the CIR.

Table III – Inputs from Micro Data and Outputs from the Theory by Size and Sector

	Inputs from Data				Outputs from Theory		
	Plants size		Excluding		Plants size		Excluding
	Small	Large	Textile/Chemical		Small	Large	Textile/Chemical
Frequency				Parameters			
$\mathbb{E}^x[\tau]$	2.862	2.300	2.445	ν	-0.111	-0.111	-0.110
$\mathbb{CV}^2[\tau]$	1.167	1.012	1.096	σ^2	0.077	0.078	0.075
				$\hat{x}$	0.004	0.013	0.011
Capital Gaps				Steady State Moments			
$\mathbb{E}^x[\Delta x]$	0.317	0.256	0.270	$\text{Var}[x]$	0.220	0.244	0.228
$\mathbb{E}^x[\Delta x^2]$	0.222	0.185	0.190	$\mathbb{E}[a]$	3.101	2.314	2.563
$\mathbb{E}^x[(\hat{x} - \Delta x)^3]$	-0.209	-0.187	-0.184	$\text{Cov}[a, x]$	0.849	1.016	0.922
Covariances				Transitional Dynamics			
$\text{Cov}^x[\tau, \Delta x]$	0.088	0.033	0.059	$\text{Var}[x]/\sigma^2$	2.870	3.145	3.021
$\mathbb{E}^x[\tau(\hat{x} - \Delta x)^2]$	0.769	0.455	0.531	$-\nu \text{Cov}[a, x]/\sigma^2$	1.227	1.454	1.350
				$\text{CIR}_1(\delta)$	4.097	4.599	4.371

Sources: Authors' calculations using establishment-level survey data for Chile.

Overall, heterogeneity across sectors and across plant size does not change the conclusion.

5 Extensions and Generalization

In the previous sections we specified parametric restrictions to the inaction model and to the firms' state space. Such assumptions exclude from our analysis models with fixed adjustment dates as in Taylor (1980), models with observation costs as in Álvarez, Lippi and Paciello (2011), and several others. Nevertheless, it is possible to extend our theory to accommodate richer models. In this section, we generalize our results to consider any stopping-time model or state space, explaining the assumptions on policies and processes that are key to apply our tools.

Second, we extend the analysis in three directions, to consider: (i) transitions of higher moments ($m > 1$) of the distribution; (ii) transitions starting from any general initial condition F_0 ; and (iii) transitions for a mean-reverting process. In each case, we focus on the one property that delivers the most interesting mechanism.²² We denote conditional distributions as $Z|Y$, conditional expectations with initial condition z as $\mathbb{E}^z[Z]$, and the minimum between two stopping times as $t \wedge s \equiv \min\{t, s\}$.

5.1 Generalization

Let $(\Omega, P, \mathcal{F})$ be a probability space equipped with a filtration $\mathcal{F} = (\mathcal{F}_t; t \geq 0)$. We consider an economy populated by a continuum of agents indexed with $\omega \in \Omega$, where agent ω 's information set at time t is the filtration $\mathcal{F}_t$. Each agent's uncontrolled state is given by $\tilde{S}_t(\omega) = [\tilde{x}_t(\omega), S_t^{-x}(\omega)] \in \mathbb{R}^{1+K-x}$. The state is split between a *main state* $\tilde{x}$ and a set of *complementary states* $\tilde{S}_t^{-x}$. The main state follows a Brownian motion $d\tilde{x}_t(\omega) = \sigma dW_t(\omega)$. Agent's policies consist of a sequence of adjustment dates $\{\tau_k\}_{k=1}^\infty$ and adjustments sizes $\{\Delta S_{\tau_k}\}_{k=1}^\infty$, measurable with respect to $\mathcal{F}_t$. Given these policies $\{\tau_k(\omega), \Delta S_{\tau_k}(\omega)\}_{k=1}^\infty$, the controlled state $S_t(\omega)$ evolves as the sum of the uncontrolled state plus the adjustments: $S_t(\omega) = \tilde{S}_t(\omega) + \sum_{\tau_k(\omega) \leq t} \Delta S_{\tau_k}(\omega)$.

The first premise for our theory is a recursive representation of the conditional CIR, both between and within stopping dates. This demands $S_t(\omega)$ to be a sufficient statistic for the conditional CIR, which in turn requires that the policy is history independent. Formally, this mean that

$$\mathbb{E} \left[\int_{\tau_i \wedge t(\omega)}^{\tau_{i+1}} f(x_t) dt | \mathcal{F}_{\tau_i \wedge t(\omega)} \right] = \mathbb{E} \left[\int_0^\tau f(x_t) dt | S_{\tau_i \wedge t(\omega)} \right] = v^f(S_{\tau_i \wedge t(\omega)}), \quad \text{for all } t(\omega) \leq \tau_{i+1}.$$

Since the main state follows a Brownian motion, the burden of this requirement falls completely on the complementary state and the policy. Assumption 1 and 2 formalize these requirements.

Assumption 1 (Markovian complementary state). *The complementary state $\tilde{S}_t^{-x}$ follows a Strong Markov process:*

$$\tilde{S}_{(t \wedge \tau_k) + h}^{-x}(\omega) | \mathcal{F}_{t \wedge \tau_k} = \tilde{S}_h^{-x}(\omega) | \tilde{S}_{(t \wedge \tau_k)}(\omega), \quad \forall k. \quad (54)$$

To understand this assumption, consider a history ω such that $t < \tau_k(\omega)$. In this case, the complementary state's law of motion depends only on its current value; thus it is independent of its own history. Additionally, the complementary state is an homogenous process, since its law of motion at date t is

²²The Web Appendix presents the full characterization and analysis of the three properties.

equivalent to its law of motion at zero, given an initial condition. In the complementary case $t \geq \tau_k(\omega)$, these properties continue to hold, thus the stopping policy does not reveal new information about the complementary state's law of motion.

Assumption 2 (Markovian policies). *Policies satisfy the following conditions:*

$$\tau_{k+1}|\mathcal{F}_{\tau_k+h} = \tau_1|S_{\tau_k+h} \quad \text{for all } h \in [0, \tau_{k+1} - \tau_k]. \quad (55)$$

A second premise in our theory is that we can characterize the CIR with the *first* stopping time of every agent. This means that, upon taking action, agents fully adjust to include any deviations from their steady state behavior and come back to the steady state process. This would imply that S_{τ_k} is *iid* across time and independent of the history previous to the adjustment. The challenge with stochastic *iid* resets is that it makes it more difficult to identify the parameters of the stochastic process, e.g. differentiating the fundamental volatility σ from the volatility arising from a random reset state. Therefore, in order for the reset state to be sufficiently informative, we ask that it is a constant $x_{\tau_i} = \hat{x}$.²³

Assumption 3 (Constant reset state). *The reset state is constant: $x_{\tau_k} = \hat{x}$ for all k .*

It is straightforward to check that the previous assumptions hold in the investment example developed in Section 2. For Assumption 1, the complementary state is given by the arrival of free adjustment opportunities N_t , which is assumed to be a Poisson counter process and thus a Strong Markov process. The requirements in Assumption 2 and 3 are also satisfied. We showed that the reset capital gap is constant; and since the stopping policy is an inaction set with respect to the controlled state, the stopping policy is history independent within and between adjustments.

Finally, in order to apply the Optional Sampling Theorem, we require several stopping processes to be well-defined (finite moments at the stopping-time).²⁴

Assumption 4 (Well-defined stopping processes). *The processes $\left(\left\{\int_0^t s^j x_s^m dB_s\right\}_t, \tau\right)$ for all m and $j = 0, 1$, are well-defined stopping processes.*

The previous Markovian requirements are enough in order to characterize the aggregation, representation of the intensive margin, and observation properties; however, in order to apply the representation property to the extensive margin, we must require one additional assumption. There must exist an equivalent representation of the extensive margin as a function exclusively of the main state x . For this, we require that there exists a stopping policy τ^* that only depends on the main state x and can fully describe the extensive margin by itself. For instance, a stopping policy given by a Poisson counter with hazard $\Lambda(x)dt$ satisfies this requirement.

Assumption 5 (Hazard). *Assume that there exist a stopping policy τ^* s.t.*

$$\mathbb{E}^{\hat{x}} \left[\int_0^\tau \left(\frac{\partial \mathbb{E}^S [x_\tau^{m+2}/m+2]}{\partial x} - \mathbb{E}^S [x_\tau^{m+1}] \right) dt \right] = \mathbb{E}^{\hat{x}} \left[\int_0^\tau \left(\frac{\partial \mathbb{E}^x [x_{\tau^*}^{m+2}/m+2]}{\partial x} - \mathbb{E}^x [x_{\tau^*}^{m+1}] \right) dt \right], \quad (56)$$

²³In this paper, we ignore ex-ante heterogeneity across agents (that could be reflected in different reset states and policies), but this can be relaxed. Nevertheless, it remains crucial that history is erased at the moment of resetting the state.

²⁴See Web Appendix A for a formal definition of a well-defined stopping process.

and there exist a smooth function $g_m(x)$ such that

$$g_m(x) = \mathbb{E}^{\hat{x}+x}[(\hat{x} - \Delta x)^m] - \mathbb{E}^{\hat{x}}[(\hat{x} - \Delta x + x)^m], \quad \forall m. \quad (57)$$

where Δx is under the policy τ^* .

5.2 Extensions

Now that we have stated the formal requirements needed to apply our theory, we proceed to develop the three extensions. To highlight the new mechanisms, in all the extensions we focus on the driftless case $\nu = 0$, but the proofs are straightforward to extend to consider a non-zero drift.

Extension I: Transitional dynamics for higher moments We first consider the transitional dynamics for higher moments of the distribution ($m \geq 1$). The initial condition remains to be a mean translation of the steady state distribution. In this case, we focus the discussion on the representation of the intensive margin.

Proposition 6. *Assume $d\tilde{x}_t = \sigma dW_t$. To a first order, the transitional dynamics of the m -th moment are given by*

$$CIR_m(\delta) = \delta \times (\Gamma_m + \Theta_m - \mathbb{E}[x^m]\Theta_0) + o(\delta^2) \quad (58)$$

where the intensive margin relates to ergodic moments as follows:

$$\Gamma_m = m\mathbb{E}[x^{m-1}, a], \quad (59)$$

$$\mathbb{E}[x^{m-1}, a] = \frac{2}{m(m+1)} \left[\frac{\mathbb{E}^{\hat{x}}[\tau(\hat{x} - \Delta x)^{m+1}]}{\mathbb{E}^{\hat{x}}[\Delta x^2]} - \frac{\mathbb{E}[x^{m+1}]}{\sigma^2} \right]. \quad (60)$$

To focus on the intensive margin, assume $\Theta_m = 0$ for all m and consider the transitional dynamics for the state's first three moments by setting $m = 1, 2, 3$. We have that

$$CIR_1(\delta)/\delta = \Gamma_1 = \mathbb{E}[a] \quad (61)$$

$$CIR_2(\delta)/\delta = \Gamma_2 = 2\mathbb{E}[xa] \quad (62)$$

$$CIR_3(\delta)/\delta = \Gamma_3 = 3\mathbb{E}[x^2a] \quad (63)$$

As discussed earlier, the dynamics of the first moment ($m = 1$)—average capital gaps—are fully driven by the state's average age. The dynamics of the second moment ($m = 2$)—dispersion of capital gaps or misallocation—are driven by the covariance between the age and the size of capital gaps. If this covariance is zero, then the distribution's second moment remains constant along the transition path. Asymmetry in the agents' investment policy, which generates a skewed ergodic distribution, is one way to generate a non-zero covariance. This interaction between the business cycle dynamics of capital misallocation and the asymmetry of the ergodic capital distribution is studied by [Ehouarne, Kuehn and Schreindorfer \(2016\)](#) and [Jo and Senga \(2014\)](#). Finally, the dynamics of the third moment ($m = 3$)—skewness of capital gaps—are driven by the covariance between age and the square of capital gaps. Note

that if the ergodic distribution features excess kurtosis, then the skewness of the distribution will change along the transition.

Proposition 6 provides formulas for the CIR of the m -th moment. Additionally, these formulas have two useful applications. They can be used to (i) derive bounds for the dynamics of functions of the m -th moments, and (ii) study transitions of any arbitrary function of the state. To illustrate the first application, let us consider the transitional dynamics for the variance. Using Jensen's inequality, we derive an upper bound on the variance's CIR:²⁵

$$\text{CIR}(\text{Var}[x]) \equiv \int_0^\infty (\mathbb{V}_t[x] - \mathbb{V}[x]) dt \leq \text{CIR}_2(\delta) - \text{CIR}_1^2(\delta). \quad (64)$$

To illustrate the second application, consider a smooth function of the state $f(x)$. For example, in many models the aggregate welfare criteria can be written in this form. Using a Taylor approximation around zero, we write the CIR of the $f(x)$ function in terms of the state's CIR, weighted by the Taylor factors.

$$\text{CIR}(f(x)) = \int_0^\infty \mathbb{E}_t[f(x)] - \mathbb{E}[f(x)] dt = \sum_{j=1}^\infty \frac{df^j(0)}{dx^j} \frac{\text{CIR}_j(\delta)}{j!}. \quad (65)$$

Extension II: General initial conditions This extension considers transitional dynamics for general initial conditions. For instance, since the work on uncertainty shocks by Bloom (2009), there has been a large literature interested in the macroeconomic consequences of uncertainty in the business cycle. Within our framework, these aggregate uncertainty shocks can be studied by setting the initial distribution as a mean-preserving spread of the steady state distribution. Moreover, the interaction between first and second moment shocks, as studied by Aastveit, Natvik and Sola (2013), Vavra (2014), Caggiano, Castelnuovo and Nodari (2014), Castelnuovo and Pellegrino (2018), and Baley and Blanco (2019), can be accommodated as well.

For simplicity, we consider perturbations that can be expressed via a single parameter δ . The initial distribution is described through a function $\mathcal{G}(x, \delta)$, such that $F_0(x) = F(\mathcal{G}^{-1}(x, \delta))$. To make progress, we impose certain smoothness and differentiability properties to the function $\mathcal{G}$.²⁶ Additionally, we focus on perturbations to the first and second moments. Since this extension does not affect steady state moments, we omit the characterization of the observation property as it remains as before.

Proposition 7. *Assume $d\tilde{x}_t = \sigma dW_t$ and let $\mathcal{G}(x, \delta)$ be a function that satisfies the following properties:*

1. $\mathcal{G}(x, 0) = x$.
2. $\exists z > 0$ such that $\forall \epsilon \in (-z, z)$, $\mathcal{G}(\cdot, \epsilon)$ is bijective.
3. $\frac{\partial \mathcal{G}(\mathcal{G}^{-1}(y, 0), 0)}{\partial \delta} = -(\mathcal{G}_0 + \mathcal{G}_1 y)$ with $\mathcal{G}_0^2 + \mathcal{G}_1^2 = 1$.

²⁵ $\text{CIR}(\text{Var}[x]) \equiv \int_0^\infty (\mathbb{V}_t[x] - \mathbb{V}[x]) dt = \int_0^\infty (\mathbb{E}_t[x^2] - \mathbb{E}[x^2]) dt - \int_0^\infty \mathbb{E}_t[x]^2 dt \leq \text{CIR}_2(\delta) - \text{CIR}_1^2(\delta)$.

²⁶See proof in Web Appendix B.3 for details.

To a first order, the CIR is given by:

$$CIR_1(\mathcal{G}) = \delta \times \left(\underbrace{\mathcal{G}_0 (\Gamma_{1,0} + \Theta_{1,0})}_{1st\ moment\ shock} + \underbrace{\mathcal{G}_1 (\Gamma_{1,1} + \Theta_{1,1})}_{2nd\ moment\ shock} \right) + o(\delta^2) \quad (66)$$

$$\Gamma_{1,i} = (i+1)\mathbb{E}[x^i a] \quad (67)$$

$$\Theta_{1,i} = \sum_{j=0}^{\infty} \theta_{1,j} \mathbb{E}[x^{j+i}] \quad (68)$$

with $\theta_{1,j}$ are the micro-elasticities.

Proposition 7 points towards the moments that are crucial to characterize the dynamics for a particular type of initial condition. As long as there exists enough differentiability in the perturbation of the initial condition, we can find ergodic moments that perfectly describe the dynamics of the model. Interestingly, the micro-elasticities needed to compute the extensive margin are independent of the number of moments that are shocked.

As an example, consider $\mathcal{G}$ to be a mean preserving spread of the steady state distribution F_0 . This means that $\mathcal{G}(x, \delta) = x(1 + \delta)$ and therefore $\mathcal{G}_0 = 0$ and $\mathcal{G}_1 = 1$. Again, let us focus only in the intensive margin by setting $\Theta_{1,i} = 0$ for all i . Then the CIR is approximated as:

$$\frac{CIR_1(\delta)}{\delta} \approx \mathcal{G}_1 \Gamma_{1,1} = Cov[x, a].$$

Thus mean-preserving perturbations have first order effects if and only if the covariance between age and the state is different from zero. A non-zero covariance is consistent with the data presented in Section 4. Therefore, suggesting that uncertainty shocks (in the form of mean-preserving spreads of the capital gap distribution) would have effects on average investment.

Extension III: Mean-reversion This extension considers a mean-reverting process for the uncontrolled state. This type of process is widely used due to its empirical relevance and because it ensures the existence of an ergodic distribution. For this application, we focus on the observation properties.

Proposition 8. Assume the uncontrolled state follows a Ornstein–Uhlenbeck process $d\tilde{x}_t = \rho\tilde{x}_t dt + \sigma dW_t$. Then, the reset state and structural parameters are recovered through a system of equations:

$$\hat{x} = \frac{\mathbb{E}^{\hat{x}}[e^{-\rho\tau} \Delta x]}{\mathbb{E}^{\hat{x}}[e^{-\rho\tau}] - 1} \quad (69)$$

$$\frac{\sigma^2}{\rho} = 2 \frac{\hat{x}^2 - \mathbb{E}^{\hat{x}}[e^{-2\rho\tau} (\hat{x} - \Delta x)^2]}{\mathbb{E}^{\hat{x}}[e^{-2\rho\tau}] - 1} \quad (70)$$

$$erf\left(\frac{\hat{x}}{\sqrt{\sigma^2/\rho}}\right) = \mathbb{E}^{\hat{x}}\left[erf\left(\frac{\hat{x} - \Delta x}{\sqrt{\sigma^2/\rho}}\right)\right] \quad (71)$$

where $erf(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ is the Gauss error function.

To gain some intuitions about the observation formulas above, let us consider the limiting case $\rho \rightarrow 0$. Using the approximation $e^{-\rho\tau} \approx 1 - \rho\tau$, it is easy to show that equations (69) and (70) converge to our baseline observations expressions in (35) and (36) with $\nu = 0$ (no mean-reversion):

$$\hat{x} \xrightarrow{\rho \rightarrow 0} \frac{\mathbb{E}^{\hat{x}}[\tau \Delta x]}{\mathbb{E}^{\hat{x}}[\tau]}, \quad \sigma^2 \xrightarrow{\rho \rightarrow 0} \frac{\mathbb{E}^{\hat{x}}[\Delta x^2]}{\mathbb{E}^{\hat{x}}[\tau]}. \quad (72)$$

Therefore, as long mean reversion is “sufficiently small”, the mappings between the data and the reset state, and between the data and idiosyncratic volatility do not change.

Let us make a deeper comparison of how $\hat{x}$ is determined with and without mean-reversion. With *iid* shocks, we can write (35) as a weighted sum of investment rates across firms:

$$\hat{x}^{iid} = \mathbb{E}^{\hat{x}}[\eta(\tau)\Delta x], \quad \text{with} \quad \eta(\tau) \equiv \frac{\tau}{\mathbb{E}^{\hat{x}}[\tau]} > 0, \quad \mathbb{E}^{\hat{x}}[\eta(\tau)] = 1,$$

where the weights $\eta(\tau)$ are increasing in τ , i.e. more weight is given to the investment rate of firms with large periods of inaction (with “old” capital). In order to understand this result, note that conditional of surviving, the distribution of the state is more centered around the reset state for “young” capital vintages, which cannot reflect policy asymmetries. The opposite happens for firms with “old” vintages, as the distribution of the state is more centered around *the domain’s middle point*, reflecting the policy asymmetries. Thus investment rates associated with large stopping times are more informative about these asymmetries.

The opposite happens when we consider a mean-reverting process. An analogous decomposition yields

$$\hat{x}^{mr} = \mathcal{R} \mathbb{E}^{\hat{x}}[\eta'(\tau)\Delta x], \quad \text{with} \quad \eta'(\tau) \equiv \frac{e^{-\rho\tau}}{\mathbb{E}^{\hat{x}}[e^{-\rho\tau}]} > 0, \quad \mathbb{E}^{\hat{x}}[\eta'(\tau)] = 1, \quad \mathcal{R} \equiv \frac{\mathbb{E}^{\hat{x}}[e^{-\rho\tau}]}{\mathbb{E}^{\hat{x}}[e^{-\rho\tau}] - 1} < 0,$$

where now the weights are decreasing in duration and it is preceded by a negative number. As the inaction period of increases, the mean-reverting productivity process goes back to its zero long-run mean, and the distribution gets centered *around zero* on its own, so there is no need to correct for policy asymmetries with the initial condition.

6 Conclusion

This paper provides a structural relation in model of inaction between the CIR (a measure of persistence for aggregate dynamics) and micro-data. This relation holds for any moment of the distribution, any inaction model, and any initial condition. In the same way we apply our tools to a model of lumpy investment, we foresee applications in models with labor adjustment costs, inventory models, portfolio management, government debt management, among others.

For developing our theory, we assume that upon taking action, agents fully adjust to include any deviations from their steady state behavior. Thus our results do not accommodate partial adjustments which are due, for instance, to imperfect information or convex adjustment costs. One example of these frameworks is the menu cost model with information frictions in [Baley and Blanco \(2019\)](#). We leave for future research the application of the tools developed here to that type of frameworks.

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A Auxiliary Theorems

The following three theorems are repeatedly used in the proofs.

Theorem 1. [Optional Sampling Theorem (OST)] Let Z be a (sub) martingale on the filtered space $(\Omega, \mathcal{P}, \mathcal{F})$ and τ an stopping time. If $(\{Z_t\}_t, \tau)$ is a well-define stopping process, then

$$\mathbb{E}[Z_\tau](\geq) = \mathbb{E}[Z_0] \quad (\text{A.1})$$

Proof. See Theorem 4.4 in [Stokey \(2009\)](#). $\square$

Theorem 2. [Ergodic distribution and occupancy measure] Let S be a strong Markov process and $g : S \rightarrow \mathbb{R}$ a function of S . Denote with F the ergodic distribution of S and with R the renewal distribution, i.e. the distribution conditional on adjustment. Assume $\int g(S)dF(S) = \lim_{T \rightarrow \infty} \frac{\int_0^T g(S_t)dt}{T}$ for all initial conditions S_0 . Then the following relationship holds:

$$\int g(S)dF(S) = \frac{\int \mathbb{E}^S \left[\int_0^\tau g(S_t)dt \right] dR(S)}{\int \mathbb{E}^S [\tau] dR(S)} \quad (\text{A.2})$$

Moreover, with homogenous resets, i.e. $\Pr[S = \hat{S}] = 1$ under the renewal distribution R , then

$$\int g(S)dF(S) = \frac{\mathbb{E}^{\hat{S}} \left[\int_0^\tau g(S_t)dt \right]}{\mathbb{E}^{\hat{S}} [\tau]} \quad (\text{A.3})$$

Proof. The numbers in the proof refer to the equalities below. (1) Start from the ergodicity assumption and (2) write T as the sum of n stopping times. (3) Take conditional expectations with respect to the filtration at time t . Since S is a strong Markov process, there is history independence across stopping times, and thus (4) we can write $\mathbb{E} \left[\int_{\tau_i}^{\tau_{i+1}} g(S_t)dt | \mathcal{F}_{\tau_i} \right] = \mathbb{E}^{S_{\tau_i}} \left[\int_0^{\tau_1} g(S_t)dt \right]$ for each τ_i . (5) Exchange the order between the outer expectation and the limit and divide by n ; then use the definition of the renewal distribution to (6) substitute the infinite sum $\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n \mathbb{E}^{S_{\tau_i}} \left[\int_0^{\tau_1} g(S_t)dt \right]}{n}$ with $\int \mathbb{E}^S \left[\int_0^\tau g(S_t)dt \right] dR(S)$ and we reach the first result.

$$\begin{aligned} \int g(S)dF(S) & \stackrel{(1)}{=} \lim_{T \rightarrow \infty} \frac{\int_0^T g(S_t)dt}{T} \stackrel{(2)}{=} \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n \int_{\tau_i}^{\tau_{i+1}} g(S_t)dt}{\sum_{i=1}^n \int_{\tau_i}^{\tau_{i+1}} 1dt} \stackrel{(3)}{=} \lim_{n \rightarrow \infty} \frac{\mathbb{E} \left[\sum_{i=1}^n \mathbb{E} \left[\int_{\tau_i}^{\tau_{i+1}} g(S_t)dt | \mathcal{F}_{\tau_i} \right] \right]}{\mathbb{E} \left[\sum_{i=1}^n \mathbb{E} \left[\int_{\tau_i}^{\tau_{i+1}} 1dt | \mathcal{F}_{\tau_i} \right] \right]} \\ & \stackrel{(4)}{=} \frac{\lim_{n \rightarrow \infty} \mathbb{E} \left[\sum_{i=1}^n \mathbb{E}^{S_{\tau_i}} \left[\int_0^{\tau_1} g(S_t)dt \right] \right]}{\lim_{n \rightarrow \infty} \mathbb{E} \left[\sum_{i=1}^n \mathbb{E}^{S_{\tau_i}} \left[\int_0^{\tau_1} 1dt \right] \right]} \stackrel{(5)}{=} \frac{\mathbb{E} \left[\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n \mathbb{E}^{S_{\tau_i}} \left[\int_0^{\tau_1} g(S_t)dt \right]}{n} \right]}{\mathbb{E} \left[\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n \mathbb{E}^{S_{\tau_i}} \left[\int_0^{\tau_1} 1dt \right]}{n} \right]} \\ & \stackrel{(6)}{=} \frac{\mathbb{E} \left[\int \mathbb{E}^S \left[\int_0^\tau g(S_t)dt \right] dR(S) \right]}{\mathbb{E} \left[\int \mathbb{E}^S [\tau] dR(S) \right]} \end{aligned}$$

If the renewal distribution has a mass point at $\hat{S}$, then the last expression simplifies to:

$$\frac{\mathbb{E} \left[\int \mathbb{E}^S \left[\int_0^\tau g(S_t)dt \right] dR(S) \right]}{\mathbb{E} \left[\int \mathbb{E}^S [\tau] dR(S) \right]} = \frac{\mathbb{E} \left[\mathbb{E}^{\hat{S}} \left[\int_0^\tau g(S_t)dt \right] \right]}{\mathbb{E} \left[\mathbb{E}^{\hat{S}} [\tau] \right]} = \frac{\mathbb{E}^{\hat{S}} \left[\int_0^\tau g(S_t)dt \right]}{\mathbb{E}^{\hat{S}} [\tau]}$$

$\square$

B Appendix: Proofs

Proposition 1. Assume that:

- The uncontrolled state follows $d\hat{x}_t = \nu dt + \sigma dW_t$, with W_t a Wiener process;
- $\left(\left\{\int_0^t x_s^m s^n dW_s\right\}_t, \tau\right)$ are a well-defined stopping processes for any m and $n = 0, 1$; and
- The moments of adjustment size can be decomposed as follows:

$$g_m(x) = \mathbb{E}^{\hat{x}+x}[(\hat{x} - \Delta x)^m] - \mathbb{E}^{\hat{x}}[(\hat{x} - \Delta x + x)^m],$$

1. **Aggregation:** To a first order, the CIR is given by

$$\mathcal{A}_m(\delta) = \delta \times (\mathcal{Z}_m - \mathcal{M}_m[x]\Theta_0) + o(\delta^2) \quad (\text{B.4})$$

where the intensive and extensive margin are given by

$$\mathcal{Z}_m = \Theta_m + \Gamma_m - \frac{\sigma^2 m}{2\nu} \mathcal{Z}_{m-1} \quad (\text{B.5})$$

$$\Gamma_m \equiv \frac{\mathbb{E}^{\hat{x}}[\int_0^\tau \varphi_m^\Gamma(x_t) dt]}{\mathbb{E}^{\hat{x}}[\tau]} \quad ; \quad \varphi_m^\Gamma(x_t) \equiv \frac{1}{\nu} (\mathbb{E}^x[x_\tau^m] - x_t^m) \quad (\text{B.6})$$

$$\Theta_m \equiv \frac{\mathbb{E}^{\hat{x}}[\int_0^\tau \varphi_m^\Theta(x_t) dt]}{\mathbb{E}^{\hat{x}}[\tau]} \quad ; \quad \varphi_m^\Theta(x_t) \equiv \frac{1}{\nu} \left[\frac{\partial \mathbb{E}^x[x_\tau^{m+1}/(m+1)]}{\partial x} - \mathbb{E}^x[x_\tau^m] \right] \quad (\text{B.7})$$

2. **Representation for the intensive margin:**

$$\Gamma_m = m\mathcal{M}_{m-1,1}[x, a] + \frac{\mathbb{1}_{\{m \geq 2\}} \sigma^2 m(m-1)}{2\nu} \mathcal{M}_{m-2,1}[x, a] \quad (\text{B.8})$$

3. **Representation for the extensive margin:**

$$\Theta_m = \sum_{j=0}^{\infty} \theta_{m,j} \mathcal{M}_j[x] \quad \text{with} \quad \theta_{m,j} \equiv \frac{2}{\sigma^2(m+1)} \sum_{k \geq j}^{\infty} \frac{\hat{x}^{k-j}}{k!j!} \left[\frac{d^{k+1} g_{m+2}(0)}{dx^{k+1}} / m + 2 - \frac{d^k g_{m+1}(0)}{dx^k} \right]. \quad (\text{B.9})$$

- If $\tau|x_t \sim \tau$, $g_m(x) = \theta(m, j) = 0$ for all m, i .

4. **Observation:** The reset state $\hat{x}$ and structural parameters (ν, σ) are recovered as

$$\hat{x} = \mathbb{E}[\Delta x] \left(\frac{1 - \mathbb{CV}^2[\tau]}{2} \right) + \frac{\text{Cov}[\tau, \Delta x]}{\mathbb{E}[\tau]}, \quad \nu = -\frac{\mathbb{E}[\Delta x]}{\mathbb{E}[\tau]}, \quad \sigma^2 = \frac{\mathbb{E}[\Delta x^2]}{\mathbb{E}[\tau]} + 2\nu\hat{x} \quad (\text{B.10})$$

and the ergodic moments are recovered as:

$$\mathcal{M}_m[x] = \frac{\hat{x}^{m+1} - \mathbb{E}[(\hat{x} - \Delta x)^{m+1}]}{\mathbb{E}[\Delta x](m+1)} - \frac{\sigma^2 m}{2\nu} \mathcal{M}_{m-1}[x], \quad (\text{B.11})$$

$$\mathcal{M}_{m,1}[x, a] = \frac{\mathbb{E}[\tau/\mathbb{E}[\tau] (\hat{x} - \Delta x)^{m+1}] - \mathcal{M}_{m+1}[x]}{\nu(m+1)} - \frac{\sigma^2 m}{2\nu} \mathcal{M}_{m-1,1}[x, a] \quad (\text{B.12})$$

with initial conditions $\mathcal{M}_1[x] = 0$ and $\mathcal{M}_{0,1}[x, a] = \frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]}$.

The proof is divided into 4 Lemmas for clarity.

Lemma 1. [Aggregation] *To a first order, the transitional dynamics of the m -th moment are given by*

$$\mathcal{A}_m(\delta) = \delta \times (\mathcal{Z}_m - \mathcal{M}_m[x]\Theta_0) + o(\delta^2) \quad (\text{B.13})$$

where the intensive and extensive margin are given by

$$\mathcal{Z}_m = \Theta_m + \Gamma_m - \frac{\sigma^2 m}{2\nu} \mathcal{Z}_{m-1} \quad (\text{B.14})$$

$$\Gamma_m \equiv \frac{\mathbb{E}^{\hat{x}} \left[\int_0^\tau \varphi_m^\Gamma(x_t) dt \right]}{\mathbb{E}^{\hat{x}}[\tau]} \quad ; \quad \varphi_m^\Gamma(x_t) \equiv \frac{1}{\nu} (\mathbb{E}^x [x_\tau^m] - x_t^m) \quad (\text{B.15})$$

$$\Theta_m \equiv \frac{\mathbb{E}^{\hat{x}} \left[\int_0^\tau \varphi_m^\Theta(x_t) dt \right]}{\mathbb{E}^{\hat{x}}[\tau]} \quad ; \quad \varphi_m^\Theta(x_t) \equiv \frac{1}{\nu} \left[\frac{\partial \mathbb{E}^x [x_\tau^{m+1}/(m+1)]}{\partial x} - \mathbb{E}^x [x_\tau^m] \right] \quad (\text{B.16})$$

Proof. The proof consists of 6 steps.

Characterization of CIR as the recursive problem of a representative agent. Fix an $m \in \mathbb{N}$. Start from the CIR's definition:

$$\mathcal{A}_m(\delta) = \mathbb{E} \left[\int_0^\infty (x_t(\omega|\delta)^m - \mathcal{M}_m[x]) dt \right], \quad (\text{B.17})$$

where the expectation is taken across agents ω . Let $\{\tau_i\}_{i=1}^\infty$ be the sequence of stopping times after the arrival of the perturbation. In (1), we write the CIR as the cumulative deviations between time $t = 0$ and the first stopping time τ_1 plus the sum of deviations between all future stopping times. In (2), we use the Law of Iterated Expectations to condition on the information set $\mathcal{F}_{\tau_i}$. In (3), we use the Strong Markov Property of the Brownian motion, the assumption of homogenous resets and that $\hat{x}$ is independent of δ for $i \geq 1$ to change the conditioning from $x_{\tau_i+h}|\mathcal{F}_{\tau_i}$ to $x_h|\hat{x}$ and write the problem recursively. To get (4), we show that *every element* inside the infinite sum is equal to zero. For this purpose, recall the relationship between ergodic moments and expected duration derived in Auxiliary Theorem 2, $\mathcal{M}_m[x] = \mathbb{E}^{\hat{x}} [\int_0^\tau x_t(\delta|\omega)^m] / \mathbb{E}^{\hat{x}}[\tau]$, and thus we are left with the simple expression in the fourth line (we also relabel τ_1 as τ):

$$\begin{aligned} \mathcal{A}_m(\delta) & \stackrel{(1)}{=} \mathbb{E} \left[\int_0^{\tau_1} (x_t(\delta|\omega)^m - \mathcal{M}_m[x]) dt + \sum_{i=1}^\infty \int_{\tau_i}^{\tau_{i+1}} (x_t(\delta|\omega)^m - \mathcal{M}_m[x]) dt \right] \\ & \stackrel{(2)}{=} \mathbb{E} \left[\int_0^{\tau_1} (x_t(\delta|\omega)^m - \mathcal{M}_m[x]) dt + \sum_{i=1}^\infty \mathbb{E} \left[\int_{\tau_i}^{\tau_{i+1}} (x_t(\delta|\omega)^m - \mathcal{M}_m[x]) dt \middle| \mathcal{F}_{\tau_i} \right] \right] \\ & \stackrel{(3)}{=} \mathbb{E} \left[\int_0^{\tau_1} (x_t(\delta|\omega)^m - \mathcal{M}_m[x]) dt \right] + \mathbb{E} \left[\underbrace{\sum_{i=1}^\infty \mathbb{E} \left[\int_0^\tau x_t(\delta|\omega)^m dt \middle| \hat{x} \right] - \mathcal{M}_m[x] \mathbb{E}[\tau|\hat{x}]}_{=0} \right] \\ & \stackrel{(4)}{=} \mathbb{E} \left[\int_0^\tau (x_t(\delta|\omega)^m) dt \right] - \mathcal{M}_m[x] \mathbb{E}[\tau]. \end{aligned}$$

As a final step, define the following value function conditional on a particular initial condition x :

$$v^m(x) \equiv \mathbb{E}^x \left[\int_0^\tau x_t(\omega)^m dt \right] - \mathcal{M}_m[x] \mathbb{E}^x[\tau], \quad (\text{B.18})$$

and notice that $\mathcal{A}_m(\delta)$ is equal to the average of $v^m(x)$ across all initial conditions after the perturbation, given by the shift in the ergodic distribution ($F_0(x) = F(x - \delta)$):

$$\mathcal{A}_m(\delta) = \int v^m(x) dF(x - \delta). \quad (\text{B.19})$$

2. State's support. Since Brownian motions are continuous in t , and initial conditions are identical across agents (by the assumption of homogeneous resets), the ergodic set is connected. Thus, the support of x is given by an interval $[\underline{x}, \bar{x}]$.

3. Taylor approximation to $\mathcal{A}_m(\delta)$ and decomposition into two terms. We do a first order Taylor approximation of $\mathcal{A}_m(\delta)$ around zero: $\mathcal{A}_m(\delta) = \mathcal{A}_m(0) + \mathcal{A}'_m(0)\delta$. Since $\mathcal{A}_m(0) = 0$ by definition, we have that: $\mathcal{A}_m(\delta) = \delta \mathcal{A}'_m(0)$, which we now characterize. Start from the representation in (B.19), expressed in terms of the marginal density of x :

$$\mathcal{A}_m(\delta) = \int v^m(x) f(x - \delta) dx. \quad (\text{B.20})$$

The derivative with respect to δ , at $\delta = 0$, is given by:

$$\begin{aligned}\mathcal{A}'_m(0) &= \frac{\partial}{\partial \delta} \int v^m(x) f(x - \delta) dx \Big|_{\delta=0} = - \int v^m(x) f'(x) dx = - v^m(x) f(x) \Big|_{\underline{x}}^{\bar{x}} + \int \frac{d}{dx} v^m(x) f(x) dx \\ &= \int \frac{d}{dx} v^m(x) f(x) dx\end{aligned}$$

where in the third equality we do integration by parts, and in the fourth equality we use the result that there is no mass at the endpoints (or $Pr^{x=\bar{x}}[\tau = 0] = Pr^{x=\underline{x}}[\tau = 0] = 1$). The previous expression says that the effect of the perturbation is equivalent to the changes in the stopping time problem of one agent when her initial conditions change (derivative of v^m with respect to x), averaged across all the possible initial conditions (the steady state distribution). In turn, as we show next, changes in the stopping time problem are reflected by alterations in the state paths and by shifts in duration.

From v^m 's definition in (B.18), take its derivative with respect to initial conditions and substitute it back into $\mathcal{A}'_m(0)$

$$\mathcal{A}'_m(0) = \int \frac{\partial}{\partial x} \mathbb{E}^x \left[\int_0^\tau x_t(\omega)^m dt \right] dF(x) - \mathcal{M}_m[x] \int \frac{\partial \mathbb{E}^S[\tau]}{\partial x} dF(x)$$

Lastly, by adding and subtracting the term $\int \mathbb{E}^x \left[\int_0^\tau \frac{\partial x_t^m}{\partial x} dt \right] dF(x)$, we re-express $\mathcal{A}'_m(0)$ as the sum of three terms Γ_m , Θ_m , and Θ_0 defined in the brackets.

$$\mathcal{A}'_m(0) = \underbrace{\int \mathbb{E}^x \left[\int_0^\tau \frac{\partial x_t^m}{\partial x} dt \right] dF(x)}_{\mathcal{B}_m} - \underbrace{\mathcal{M}_m[x] \int \frac{\partial \mathbb{E}^S[\tau]}{\partial x} dF(x)}_{\Theta_0} + \underbrace{\int \left(\frac{\partial}{\partial x} \mathbb{E}^x \left[\int_0^\tau x_t^m dt \right] - \mathbb{E}^x \left[\int_0^\tau \frac{\partial x_t^m}{\partial x} dt \right] \right) dF(x)}_{\mathcal{C}_m}. \quad (\text{B.21})$$

Now we further characterize each of these terms. Note that for various extensions, the proof up to this point is exactly the same. The results change from this point onwards as we make use of the particular stochastic process for the uncontrolled state.

4. Characterize $\mathcal{B}_m$. Since $x_t = x + \nu t + \sigma W_t$, for all $t \leq \tau$ we have that

$$\mathcal{B}_m \equiv \int \mathbb{E}^x \left[\int_0^\tau \frac{\partial x_t^m}{\partial x} dt \right] dF(x) = \int \mathbb{E}^x \left[\int_0^\tau m x_t^{m-1} dt \right] dF(x).$$

Applying Itô's Lemma to x_t^m we have $dx_t^m = \nu m x_t^{m-1} dt + \sigma m x_t^{m-1} dW_t + \frac{\sigma^2}{2} m(m-1) x_t^{m-2} dt$, and integrating both sides from 0 to τ and taking expectations with initial condition x we get

$$\mathbb{E}^x [x_\tau^m] - x^m = \underbrace{m\sigma \mathbb{E}^x \left[\int_0^\tau x_t^{m-1} dW_t \right]}_{=0 \text{ by OST}} + \nu \mathbb{E}^x \left[\int_0^\tau m x_t^{m-1} dt \right] + \frac{\sigma^2 m}{2} \mathbb{E}^x \left[\int_0^\tau (m-1) x_t^{m-2} dt \right].$$

Given that $\int_0^\tau x_s^m dW_s$ is a martingale with zero initial condition and it is well-defined by assumption, we apply the Optional Sampling Theorem (OST) to conclude that $\mathbb{E}^x [\int_0^\tau x_t^{m-1} dW_t] = 0$. Solve for $\mathbb{E}^x [\int_0^\tau m x_t^{m-1} dt]$:

$$\mathbb{E}^x \left[\int_0^\tau m x_t^{m-1} dt \right] = \frac{\mathbb{E}^x [x_\tau^m] - x^m}{\nu} - \frac{\sigma^2 m}{2\nu} \mathbb{E}^x \left[\int_0^\tau (m-1) x_t^{m-2} dt \right].$$

Integrating both sides across all initial conditions, defining $\varphi_m^\Gamma \equiv \frac{1}{\nu} (\mathbb{E}^x [x_\tau^m] - x^m)$ and $\Gamma_m \equiv \frac{\mathbb{E}^{\hat{x}} [\int_0^\tau \varphi_m^\Gamma(x_t) dt]}{\mathbb{E}^{\hat{x}} [\tau]}$, and recognizing $\mathcal{B}_m$ and $\mathcal{B}_{m-1}$ we get:

$$\mathcal{B}_m = \Gamma_m - \frac{\sigma^2 m}{2\nu} \mathcal{B}_{m-1}, \quad \Gamma_0 = 0, \quad (\text{B.22})$$

where we used the Auxiliary Theorem 2, exchanging the ergodic distribution for the local occupancy measure.

5. Characterize $\mathcal{C}_m$. With similar steps as in the previous point, we characterize $\mathcal{C}_m$ as follows.

$$\mathcal{C}_m \equiv \underbrace{\int \frac{\partial}{\partial x} \mathbb{E}^x \left[\int_0^\tau x_t^m dt \right] dF(x)}_A - \underbrace{\mathbb{E}^x \left[\int_0^\tau \frac{\partial x_t^m}{\partial x} dt \right] dF(x)}_B.$$

First we get an expression for the term A. Applying Itô's Lemma to x_t^{m+1} we have $dx_t^{m+1} = (m+1)\nu x_t^m dt + \sigma(m+1)x_t^m dW_t + \frac{\sigma^2}{2} m(m+1)x_t^{m-1} dt$. Integrating both sides from 0 to τ , taking expectations with initial condition x , using the OST, and rearranging we get: $\mathbb{E}^x [\int_0^\tau x_t^m dt] = \frac{1}{\nu(m+1)} (\mathbb{E}^x [x_\tau^{m+1}] - x^{m+1}) - \frac{\sigma^2 m}{2\nu} \mathbb{E}^x [\int_0^\tau x_t^{m-1} dt]$, and its derivative with respect to

initial condition x :

$$A \equiv \frac{\partial}{\partial x} \mathbb{E}^x \left[\int_0^\tau x_t^m dt \right] = \frac{1}{\nu} \left(\frac{\partial \mathbb{E}^x [x_\tau^{m+1}] / (m+1)}{\partial x} - x_t^m \right) - \frac{\sigma^2 m}{2\nu} \frac{\partial \mathbb{E}^x \left[\int_0^\tau x_t^{m-1} dt \right]}{\partial x}$$

Now, for the term B, recall from the characterization of Γ_m that

$$B \equiv \mathbb{E}^x \left[\int_0^\tau \frac{\partial x_t^m}{\partial x} dt \right] = \frac{1}{\nu} (\mathbb{E}^x [x_\tau^m] - x_t^m) - \frac{\sigma^2 m}{2\nu} \mathbb{E}^x \left[\int_0^\tau \frac{\partial x_t^{m-1}}{\partial x} dt \right].$$

Subtract the equations for A and B and simplify to obtain:

$$\frac{\partial \mathbb{E}^x \left[\int_0^\tau x_t^m dt \right]}{\partial x} - \mathbb{E}^x \left[\int_0^\tau \frac{\partial x_t^m}{\partial x} dt \right] = \underbrace{\frac{1}{\nu} \left(\frac{\partial \mathbb{E}^x [x_\tau^{m+1}] / (m+1)}{\partial x} - \mathbb{E}^x [x_\tau^m] \right)}_{\varphi_m^\Theta(x_t)} - \frac{\sigma^2 m}{2\nu} \left\{ \frac{\partial \mathbb{E}^x \left[\int_0^\tau x_t^{m-1} dt \right]}{\partial x} - \mathbb{E}^x \left[\int_0^\tau \frac{\partial x_t^{m-1}}{\partial x} dt \right] \right\}$$

Integrating with the ergodic distribution and using the definition of Θ_m in (B.10) and recognizing $\mathcal{C}_m$ and $\mathcal{C}_{m-1}$ we get:

$$\mathcal{C}_m = \Theta_m - \frac{\sigma^2 m}{2\nu} \mathcal{C}_{m-1}, \quad \mathcal{C}_{-1} = 0. \quad (\text{B.23})$$

Define $\mathcal{Z}_m \equiv \mathcal{B}_m + \mathcal{C}_m$, which implies $\mathcal{Z}_m = \Gamma_m + \Theta_m - \frac{\sigma^2 m}{2\nu} \mathcal{Z}_{m-1}$. Combine the results in (B.16), (B.17) and (B.20) to obtain (B.7): $\mathcal{A}'_m(0) = (\mathcal{Z}_m - \mathcal{M}_m[x]\Theta_0)$.

6. Characterize Θ_0 . We corroborate that the expression $\int \frac{\partial \mathbb{E}^x[\tau]}{\partial x} dF(x)$ is equal to Θ_0 . By the OST, we have $\mathbb{E}^x[x_\tau] - x = \nu \mathbb{E}^x[\tau]$. Thus $\frac{\partial \mathbb{E}^x[\tau]}{\partial x} = \frac{1}{\nu} \left[\frac{\partial \mathbb{E}^x[x_\tau]}{\partial x} - 1 \right]$. Substituting and using Auxiliary Theorem 2 we recover the expression for Θ_0 in the definition of Θ_m :

$$\Theta_0 \equiv \int \frac{\partial \mathbb{E}^x[\tau]}{\partial x} dF(x) = \int \frac{1}{\nu} \left[\frac{\partial \mathbb{E}^x[x_\tau]}{\partial x} - 1 \right] dF(x)$$

□

Lemma 2. [Representation for intensive margin] The intensive margin Γ_m defined as

$$\Gamma_m \equiv \frac{\mathbb{E}^{\hat{x}} \left[\int_0^\tau \varphi^\Gamma(x_t) dt \right]}{\mathbb{E}^{\hat{x}}[\tau]}, \quad \text{with} \quad \varphi_m^\Gamma(x_t) = \frac{1}{\nu} (\mathbb{E}^x [x_\tau^m] - x_t^m).$$

can be represented as a function of steady state moments as:

$$\Gamma_m = m \mathcal{M}_{m-1,1}[x, a] + \frac{\mathbb{1}_{\{m \geq 2\}} \sigma^2 m(m-1)}{2\nu} \mathcal{M}_{m-2,1}[x, a].$$

Proof. Start in (1) from the definition of Γ_m and $\varphi_m^\Gamma(S)$, then (2) exchange the time integral with the expectation conditional on adjustment $\mathbb{E}^{\hat{x}}[\cdot]$, which introduces an indicator $\mathbb{1}_{\{t \leq \tau\}}$. Use the law of iterated expectations in (3) to condition on the set $\{t \leq \tau\}$.

$$\begin{aligned} \nu \mathbb{E}^{\hat{x}}[\tau] \Gamma_m &\stackrel{(1)}{=} \mathbb{E}^{\hat{x}} \left[\int_0^\tau \mathbb{E}^{x_t} [x_\tau^m] - x_t^m dt \right] \stackrel{(2)}{=} \int_0^\infty \mathbb{E}^{\hat{x}} \left[(\mathbb{E}^{x_t} [x_\tau^m] - x_t^m) \mathbb{1}_{\{t \leq \tau\}} \right] dt \\ &\stackrel{(3)}{=} \int_0^\infty \mathbb{E}^{\hat{x}} \left[\mathbb{E} \left\{ (\mathbb{E}^{x_t} [x_\tau^m] - x_t^m) \mathbb{1}_{\{t \leq \tau\}} \mid t \leq \tau \right\} \right] dt \stackrel{(4)}{=} \int_0^\infty \mathbb{E}^{\hat{x}} \left[\mathbb{E} \left\{ \mathbb{E}^{x_t} [x_\tau^m] - x_t^m \mid t \leq \tau \right\} \right] dt \\ &\stackrel{(5)}{=} \int_0^\infty \mathbb{E}^{\hat{x}} \left[\mathbb{E} \left[x_\tau^m - x_t^m \mid t \leq \tau \right] \right] dt \stackrel{(6)}{=} \int_0^\infty \mathbb{E}^{\hat{x}} \left[\mathbb{E} \left\{ (x_\tau^m - x_t^m) \mathbb{1}_{\{t \leq \tau\}} \mid t \leq \tau \right\} \right] dt \\ &\stackrel{(7)}{=} \mathbb{E}^{\hat{x}} \left[\int_0^\tau (x_\tau^m - x_t^m) dt \right] \stackrel{(8)}{=} \mathbb{E}^{\hat{x}} \left[x_\tau^m \int_0^\tau dt \right] - \mathbb{E}^{\hat{x}} \left[\int_0^\tau x_t^m dt \right] \end{aligned} \quad (\text{B.24})$$

We now characterize $\mathbb{E}^{\hat{x}} [x_\tau^{m+1} \int_0^\tau dt]$. Applying Ito's lemma followed by the OST to $Y_t^m \equiv x_t^m \int_0^t ds$

$$dY_t^m = x_t^m dt + \mathbb{1}_{\{m \geq 1\}} \nu m x_t^{m-1} \int_0^t ds dt + \mathbb{1}_{\{m \geq 1\}} \sigma m x_t^{m-1} \int_0^t ds dW_t + \mathbb{1}_{\{m \geq 2\}} \frac{m(m-1)\sigma^2}{2} x_t^{m-2} \int_0^t ds dt \quad (\text{B.25})$$

$$\mathbb{E}^{\hat{x}} [Y_\tau^m] = \mathbb{E}^{\hat{x}} \left[x_\tau^m \int_0^\tau dt \right] = \underbrace{\mathbb{E}^{\hat{x}} \left[\int_0^\tau x_t^m dt \right]}_{\mathcal{M}_m[x] \mathbb{E}^{\hat{x}}[\tau]} + \underbrace{\mathbb{1}_{\{m \geq 1\}} \nu m \frac{\mathbb{E}^{\hat{x}} \left[\int_0^\tau x_t^{m-1} t dt \right]}{\mathbb{E}^{\hat{x}}[\tau]}}_{\mathcal{M}_{m-1,1}[x, a]} + \underbrace{\mathbb{1}_{\{m \geq 2\}} \frac{m(m-1)\sigma^2}{2} \mathbb{E}^{\hat{x}} \left[\int_0^\tau x_t^{m-2} t dt \right]}_{\mathcal{M}_{m-2,1}[x, a]} \quad (\text{B.26})$$

Using equations (B.24) and (B.26), we have that

$$\Gamma_m = \mathbb{1}_{\{m \geq 1\}} m \mathcal{M}_{m-1,1}[x, a] + \frac{\mathbb{1}_{\{m \geq 2\}} \sigma^2 m(m-1)}{2\nu} \mathcal{M}_{m-2,1}[x, a]$$

□

Lemma 3. [Representation for extensive margin] Assume the moments of the adjustment size can be written as:

$$g_m(x) = \mathbb{E}^{\hat{x}+x}[(\hat{x} - \Delta x)^m] - \mathbb{E}^{\hat{x}}[(\hat{x} - \Delta x + x)^m] \quad (\text{B.27})$$

Then the extensive margin given by $\Theta_m \equiv \frac{\mathbb{E}^{\hat{x}}[\int_0^\tau \varphi_m^\Theta(x_t) dt]}{\mathbb{E}^{\hat{x}}[\tau]}$ with $\varphi_m^\Theta(x) \equiv \frac{1}{\nu} \left(\frac{\partial \mathbb{E}^x[x_\tau^{m+1}/(m+1)]}{\partial x} - \mathbb{E}^x[x_\tau^m] \right)$ can be represented as a function of steady state moments as follows:

$$\Theta_m = \sum_{j=0}^{\infty} \theta_{m,j} \mathcal{M}_j[x] \quad \text{with} \quad \theta_{m,j} = \sum_{k \geq j} \frac{1}{\nu} \frac{\left[\frac{d^{k+1} g_{m+1}(x)}{dx^{k+1}} / m + 1 - \frac{d^k g_m(x)}{dx^k} \right] \Big|_{x=0}}{k! j!} \hat{x}^{k-j}. \quad (\text{B.28})$$

- If $\tau|x_t \sim \tau$, $g_m(x) = \theta(m, j) = 0$ for all m, i .

Proof. Using a change of variable in assumption (B.27), we have that :

$$\mathbb{E}^y[x_\tau^m] = g_m(y - \hat{x}) + \mathbb{E}^{\hat{x}}[(y - \hat{x} + x_\tau)^m]. \quad (\text{B.29})$$

Using the previous equation we have that

$$\begin{aligned} \Theta_m &= \frac{\mathbb{E}^{\hat{S}} \left[\int_0^\tau \left(\frac{\partial \mathbb{E}^S[x_\tau^{m+1}/(m+1)]}{\partial x} - \mathbb{E}^S[x_\tau^m] \right) dt \right]}{\nu \mathbb{E}^{\hat{S}}[\tau]} \\ &= \frac{1}{\nu} \frac{\mathbb{E}^{\hat{S}} \left[\int_0^\tau \left[\left[\frac{dg_{m+1}(y-\hat{x})}{dy} / (m+1) + \mathbb{E}^{\hat{x}}[(y-\hat{x}+x_\tau)^{m+1}] \right] - [g_m(y-\hat{x}) + \mathbb{E}^{\hat{x}}[(y-\hat{x}+x_\tau)^m]] \right] dt \right]}{\mathbb{E}^{\hat{S}}[\tau]} \\ &= \frac{1}{\nu} \frac{\mathbb{E}^{\hat{S}} \left[\int_0^\tau \left[\frac{dg_{m+1}(y-\hat{x})}{dy} / (m+1) - g_m(y-\hat{x}) \right] dt \right]}{\mathbb{E}^{\hat{S}}[\tau]} \\ &= \frac{1}{\nu} \frac{\mathbb{E}^{\hat{S}} \left[\int_0^\tau \left[\sum_{j=0}^{\infty} \frac{d^j}{j! dx^j} \left[\frac{dg_{m+1}(x)}{dx} / (m+1) - g_m(x) \right]_{x=0} (y-\hat{x})^j \right] dt \right]}{\mathbb{E}^{\hat{S}}[\tau]} \\ &= \frac{1}{\nu} \sum_{j=0}^{\infty} \sum_{z=0}^j \frac{\left[\frac{d^{j+1} g_{m+1}(0)}{dx^{j+1}} / m + 1 - \frac{d^j g_m(0)}{dx^j} \right] \hat{x}^z}{j!} \binom{j}{z} \frac{\mathbb{E}^{\hat{S}}[\int_0^\tau x_t^{j-z} dt]}{\mathbb{E}^{\hat{S}}[\tau]} \\ &= \frac{1}{\nu} \sum_{j=0}^{\infty} \sum_{z=0}^j \frac{\left[\frac{d^{j+1} g_{m+1}(0)}{dx^{j+1}} / m + 1 - \frac{d^j g_m(0)}{dx^j} \right] \hat{x}^z}{j!} \binom{j}{z} \mathcal{M}_{j-z}[x] \\ &= \sum_{j=0}^{\infty} \sum_{z=0}^j \underbrace{\frac{1}{\nu} \frac{\left[\frac{d^{j+1} g_{m+1}(0)}{dx^{j+1}} / m + 1 - \frac{d^j g_m(0)}{dx^j} \right] \hat{x}^z}{z!(j-z)!}}_{=\mathcal{H}_{m,j,z}} \mathcal{M}_{j-z}[x] \\ &= \sum_{h=0}^{\infty} \theta_{m,h} \mathcal{M}_h[x], \quad \text{with} \quad \theta_{m,h} = \sum_{k \geq h}^{\infty} \mathcal{H}_{m,k,k-h}. \end{aligned} \quad (\text{B.31})$$

If $\tau|x_t \sim \tau$, then we have that

$$\begin{aligned} g_m(x) &= \mathbb{E}^{\hat{x}+x}[(\hat{x} - \Delta x)^m] - \mathbb{E}^{\hat{x}}[(\hat{x} - \Delta x + x)^m] \\ &= \mathbb{E}[(\hat{x} + x + \nu\tau^{x+\hat{x}} + \sigma W_{\tau^{x+\hat{x}}})^m] - \mathbb{E}[(\hat{x} + x + \nu\tau^{\hat{x}} + \sigma W_{\tau^{\hat{x}}})^m] \\ &= \mathbb{E}[(\hat{x} + x + \nu\tau + \sigma W_\tau)^m] - \mathbb{E}[(\hat{x} + x + \nu\tau + \sigma W_\tau)^m] \\ &= 0 \end{aligned} \quad (\text{B.32})$$

□

Lemma 4. *[Observation] The reset state $\hat{x}$ and structural parameters (ν, σ) are recovered as*

$$\hat{x} = \mathbb{E}[\Delta x] \left(\frac{1 - \mathbb{C}\mathbb{V}^2[\tau]}{2} \right) + \frac{\text{Cov}[\tau, \Delta x]}{\mathbb{E}[\tau]}, \quad \nu = -\frac{\mathbb{E}[\Delta x]}{\mathbb{E}[\tau]}, \quad \sigma^2 = \frac{\mathbb{E}[\Delta x^2]}{\mathbb{E}[\tau]} + 2\nu\hat{x}$$

and the ergodic moments are recovered as:

$$\begin{aligned} \mathcal{M}_m[x] &= \frac{\hat{x}^{m+1} - \mathbb{E}[(\hat{x} - \Delta x)^{m+1}]}{\mathbb{E}[\Delta x](m+1)} - \frac{\sigma^2 m}{2\nu} \mathcal{M}_{m-1}[x], \\ \mathcal{M}_{m,1}[x, a] &= \frac{\mathbb{E}[\tau/\mathbb{E}[\tau] (\hat{x} - \Delta x)^{m+1}] - \mathcal{M}_{m+1}[x]}{\nu(m+1)} - \frac{\sigma^2 m}{2\nu} \mathcal{M}_{m-1,1}[x, a] \end{aligned}$$

with initial conditions $\mathcal{M}_1[x] = 0$ and $\mathcal{M}_{0,1}[x, a] = \frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]}$.

Proof. The basis of the proof is the application of Itô's lemma and the OST.

- **Average adjustment size.** We show that $\frac{\mathbb{E}^\hat{x}[\Delta x]}{\mathbb{E}^\hat{x}[\tau]} = -\nu$. From the law of motion $x_t = \hat{x} + \nu t + \sigma W_t$, we find the following equalities: $\sigma W_\tau = -\nu\tau + x_\tau - \hat{x} = -\nu\tau - \Delta x$. Taking expectations on both sides, we have $\sigma\mathbb{E}[W_\tau] = -\nu\mathbb{E}^\hat{x}[\tau] - \mathbb{E}[\Delta x]$. Since W_τ is a martingale, $\mathbb{E}[W_\tau] = W_0 = 0$ by the OST. Therefore, $\nu = -\frac{\mathbb{E}[\Delta x]}{\mathbb{E}[\tau]}$ as well.
- **Observation of fundamental volatility:** For characterizing σ define $Y_t = x_t - \nu t$ with initial condition $Y_0 = \hat{x}$. With similar steps as before we have that

$$\sigma^2 = \frac{\mathbb{E}^\hat{x}[\Delta Y_\tau^2]}{\mathbb{E}^\hat{x}[\tau]} = \frac{\mathbb{E}^\hat{x}[(x_\tau - \nu\tau - x_\tau + \hat{x})^2]}{\mathbb{E}^\hat{x}[\tau]} = \frac{\mathbb{E}^\hat{x}[(\nu\tau + \Delta x)^2]}{\mathbb{E}^\hat{x}[\tau]} \quad (\text{B.33})$$

or equivalently

$$\sigma^2 = \frac{\mathbb{E}^\hat{x}[\Delta x^2]}{\mathbb{E}^\hat{x}[\tau]} + 2\nu \left(\frac{\mathbb{E}^\hat{x}[\Delta x\tau]}{\mathbb{E}^\hat{x}[\tau]} + \nu \frac{\mathbb{E}^\hat{x}[\tau^2]}{\mathbb{E}^\hat{x}[\tau]} \right)$$

Applying the formula (B.35) we have the result.

- **Observation of reset state:** For the reset state $\hat{x}$, we apply Itô's lemma to x_t^2 to obtain $d(x_t^2) = 2x_t dx_t + (dx_t)^2 = (2\nu x_t + \sigma^2) dt + 2\sigma x_t dW_t$. Using the OST $\mathbb{E}^\hat{x}[\int_0^\tau x_s dW_s] = 0$. Moreover, given that $\mathbb{E}^\hat{x}[\int_0^\tau x_s ds] = \mathcal{M}_1[x]\mathbb{E}^\hat{x}[\tau] = 0$, we have that

$$\mathbb{E}^\hat{x}[x_\tau^2] = \hat{x}^2 + \sigma^2 \mathbb{E}^\hat{x}[\tau] \quad (\text{B.34})$$

Completing squares $\mathbb{E}^\hat{x}[x_\tau^2] = \mathbb{E}^\hat{x}[(\hat{x} - (\hat{x} - x_\tau))^2] = \mathbb{E}^\hat{x}[\Delta x^2] - 2\hat{x}\mathbb{E}^\hat{x}[\Delta x] + (\hat{x})^2$, we get

$$\begin{aligned} \hat{x} &= \frac{1}{2\mathbb{E}^\hat{x}[\Delta x]} \left[\mathbb{E}^\hat{x}[\Delta x^2] - \sigma^2 \mathbb{E}^\hat{x}[\tau] \right] \\ &= \frac{1}{2\mathbb{E}^\hat{x}[\Delta x]} \left[\mathbb{E}^\hat{x}[\Delta x^2] - \left(\mathbb{E}^\hat{x}[\Delta x^2] + 2\frac{\mathbb{E}^\hat{x}[\Delta x]\mathbb{E}^\hat{x}[\Delta x\tau]}{\mathbb{E}^\hat{x}[\tau]} + \frac{\mathbb{E}^\hat{x}[\Delta x]^2\mathbb{E}^\hat{x}[\tau^2]}{\mathbb{E}^\hat{x}[\tau]^2} \right) \right] \\ &= \frac{\mathbb{E}^\hat{x}[\Delta x\tau]}{\mathbb{E}^\hat{x}[\tau]} - \frac{\mathbb{E}^\hat{x}[\Delta x]\mathbb{E}^\hat{x}[\tau^2]}{2\mathbb{E}^\hat{x}[\tau]^2}. \end{aligned} \quad (\text{B.35})$$

Applying the formula for the covariance $\mathbb{E}^\hat{x}[\tau\Delta x] + \mathbb{E}^\hat{x}[\tau]\mathbb{E}^\hat{x}[\Delta x] = \text{Cov}[\tau, \Delta x]$ and coefficient of variation square $\mathbb{C}\mathbb{V}^2[\tau] = \frac{\mathbb{V}^\hat{x}[\tau]}{\mathbb{E}^\hat{x}[\tau]^2}$, we have the result.

- **Observation of ergodic moments with respect to the state:** For observability of ergodic moments of x , apply Itô's lemma to x^{m+1} and get $dx_t^{m+1} = (m+1)x_t^m \nu dt + (m+1)x_t^m \sigma dW_t + \frac{\sigma^2}{2}m(m+1)x_t^{m-1}dt$. Integrating from 0 to τ , using the OST to eliminate martingales, and rearranging:

$$\mathbb{E}^\hat{x} \left[\int_0^\tau x_t^m dt \right] = \frac{1}{\nu(m+1)} \left(\mathbb{E}^\hat{x}[x_\tau^{m+1}] - \hat{x}^{m+1} \right) - \frac{\sigma^2}{2\nu} m \mathbb{E}^\hat{x} \left[\int_0^\tau x_t^{m-1} dt \right] \quad (\text{B.36})$$

Substituting the equivalences $\mathcal{M}_m[x] = \mathbb{E}^\hat{x}[\int_0^\tau x_t^m dt] / \mathbb{E}^\hat{x}[\tau]$ and $\mathbb{E}^\hat{x}[\Delta x] = -\nu\mathbb{E}^\hat{x}[\tau]$ yields:

$$\mathcal{M}_m[x] = \frac{\hat{x}^{m+1} - \mathbb{E}^\hat{x}[(\hat{x} - \Delta x)^{m+1}]}{\mathbb{E}^\hat{x}[\Delta x](m+1)} - \frac{\sigma^2 m}{2\nu} \mathcal{M}_{m-1}[x], \quad \mathcal{M}_1[x] = 0 \quad (\text{B.37})$$

- **Observation of ergodic moments with respect to the joint moments of state and age:** For observability of ergodic moments of $x^m a$, where a stand for the duration of the last action, we use Itô's lemma and the OST on $x_t^{m+1}t$:

$$\mathbb{E}^\hat{x}[\tau(\hat{x} - \Delta x)^{m+1}] = \mathbb{E}^\hat{x} \left[\int_0^\tau x_t^{m+1} dt \right] + (m+1)\nu \mathbb{E}^\hat{x} \left[\int_0^\tau x_t^m t dt \right] + \frac{\sigma^2 m(m+1)}{2} \mathbb{E}^\hat{x} \left[\int_0^\tau x_t^{m-1} t dt \right] \quad (\text{B.38})$$

and therefore

$$\mathcal{M}_{m,1}[x, a] = \frac{\mathbb{E}^{\hat{x}}[\tau (\hat{x} - \Delta x)^{m+1}]}{\nu(m+1)\mathbb{E}^{\hat{x}}[\tau]} - \frac{\mathcal{M}_{m+1}[x]}{\nu(m+1)} - \frac{\sigma^2 m}{2\nu} \mathcal{M}_{m-1,1}[x, a] \quad (\text{B.39})$$

with initial condition $\mathcal{M}_{0,1}[x, a] = \frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]}$.

□