Monetary Operating Procedures in the Fed Funds Market: Theory and Policy Analysis*

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Abstract

The federal funds market changed drastically during the last decade, as new policy tools were implemented while large-scale asset purchases programs increased reserve balances to unprecedented levels. We develop a model of the federal funds market that incorporates most of its salient features and recent changes. The model includes heterogeneous types of banks and we estimate the parameters governing this heterogeneity using bank-level transaction data. Consistent with the data, a small set of very active banks in the model participate in the majority of loan transactions. We use the model to analyze the consequences of a balance sheet normalization. We argue that the increase in interest rates will be larger if reserves decrease disproportionately more for the set of active banks.

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1 Introduction

The federal funds market is an interbank over-the-counter market where financial institutions lend and borrow reserve balances at the Federal Reserve Banks in order to meet requirements and earn interest. The Federal Reserve implements its monetary policy targets by affecting lending conditions in this market. However, the federal funds market experienced substantial changes during the last decade. On one hand, the Federal Reserve introduced new policy instruments, such as interest payments on reserves. On the other hand, the amount of reserve balances held by banks increased substantially as a result of the large-scale asset purchase program. However, going forward, the Federal Reserve is considering to shrink or "normalize" its balance sheet from the current exceptional level. In light of all these changes, a natural question arises: what will happen with interbank lending in the federal funds market as the Federal Reserve normalizes its balance sheet, and how will this affect the implementation of monetary policy targets?

This is the question we tackle in this paper. We develop a model of the federal funds market that incorporates most of its salient features and recent changes. Two particularly important features are present in the model. First, we incorporate all of the currently available policy instruments for the Federal Reserve (Fed), including the interest rates on reserves (IOR), the discount window, and the overnight reverse repurchase (ONRRP) facility. Second, we include heterogeneous type of banks in the model and estimate the parameters governing this heterogeneity to match cross-sectional moments. Adding heterogeneous banks helps us include special type of institutions, such as Government Sponsored Enterprises (GSEs), but it also allows to consider the role of a small set of banks that play a crucial role in the federal funds market.

We then use the model to analyze the effects the Fed's balance sheet normalization. We argue that the normalization will increase interest rates in the federal funds market, but its effect will crucially depend on the how the distribution of reserves across banks changes. In particular, the largest increase in rates will occur if reserves decrease more for most active banks in the federal funds market, even if these are very few banks. We argue that is likely to be the more plausible case.

We model the over-the-counter nature of the federal funds market using a random-search model. We assume that banks are heterogeneous on the probability in which they are matched with other banks. We use bank-level transaction data (*Fedwire*) to estimate the heterogeneity of matching probabilities. The estimation results in a small set of banks that account for the vast majority of lending transactions. As in the data, the top 1% of most actives banks in the model are counter-parties for 97% of total lending in the federal funds market. We show that this set of very active banks constitutes a core in a network of lending, and most reallocation of reserves across banks goes through these active banks. In turn, the trading behavior of the most active banks has large effects on the interest rates in the market.

Consequently, if the balance sheet normalization decreases reserves mostly for active banks, they hoard on to reserves and reduce their reallocation behavior. The interbank market quickly "dries up", and interest rate sharply increase. If the normalization decreases reserves for less active banks, rates increase by less.

The rest of the paper is organized as follows. Section 2 lays out the model and discusses how it incorporates key features of the federal funds market. Section 3 characterizes the equilibrium. Sections 4 and 5 analyze versions of the model in which analytical solutions are available. Section 6 derives positive implications of the model, and Section 7 uses this implications to estimate the model. Section 8 contains the quantitative analysis of the model, and Section 9 performs the policy analysis. Section 10 concludes.

2 Model

There is a unit measure of *banks* and a unit measure of identical *brokers*. Banks are ex ante heterogeneous along several dimensions. To represent this heterogeneity, we define a set $\mathbb{N} \equiv \{1, 2 \ldots, N\}$ of bank types and let $n_i \in [0, 1]$ represent the proportion of banks of type $i \in \mathbb{N}$, with $\sum_{i \in \mathbb{N}} n_i = 1$. Banks can hold an asset we interpret as (claims to) *reserve balances* that can be traded with other banks, directly or through brokers, during a trading session set in continuous time. The trading session starts at time 0 and ends at time T, so trade takes place during the time interval $\mathbb{T} = [0, T]$. The aggregate supply of reserve balances, A, is given at $t = 0$. Brokers do not hold reserves. The reserve balance that a bank holds (e.g., at its Federal Reserve account) at a given time is represented by a real number, e.g., $a \in \mathbb{R}$. The cumulative distribution function of reserve balances across all banks at time $t \in \mathbb{T}$ is denoted $F_t(a) = \sum_{i \in \mathbb{N}} n_i F_t^i(a)$, where $F_t^i(a) : \mathbb{R} \times \mathbb{T} \to [0, 1]$ is the cumulative distribution of reserve balances across banks of type i at time t. The initial distribution, $\left\{F_0^i(a)\right\}_{i \in \mathbb{N}}$, is given, and $A \equiv \int a dF_t(a)$ for all $t \in \mathbb{T}$.

Banks can trade reserves in two ways: multilaterally through brokers, or bilaterally with other banks. The brokered segment of the market is modeled as follows. Brokers have continuous access to a competitive market that allows them to intermediate trades among banks.[1] Each bank periodically comes in contact with a broker who can trade in this market on the bank's behalf. A bank of type $i \in \mathbb{N}$ can execute trades through some broker at random times induced by a Poisson process with arrival rate α_i (this process is independent across banks). We model trades as loans of reserve balances intermediated by brokers. Specifically, once a bank and a broker have made contact, they bargain over the broker's intermediation fee and the terms of the loan, i.e., the size of the loan the bank is issuing or receiving at the interest rate quoted by the broker. The bargaining outcome is determined by Nash bargaining. We use $\theta_i \in [0, 1]$ to denote the bargaining strength of a bank of type $i \in \mathbb{N}$. After the terms of the transaction have been agreed upon, the bank and the broker part ways.

The bilateral segment of the fed funds market is modeled as a conventional over-the-counter market where bank of type $i \in \mathbb{N}$ can trade with another bank at random times induced by a Poisson process with arrival rate β_i (this process is independent across banks and independent of the process that drives trading opportunities with brokers). Conditional on meeting, the counterparty is a random (uniform) draw from the population of all banks. Once two banks have made contact, they bargain over the size of the loan and the quantity of reserve balances to be repaid by the borrower. The bargaining outcome is determined by Nash bargaining. We assume that when a bank of type $i \in \mathbb{N}$ negotiates with a bank of type $j \in \mathbb{N}$, the bargaining power of the former is $\theta_{ij} \equiv \mathbb{I}_{\{i \neq j\}}\theta_i/(\theta_i + \theta_j) + \mathbb{I}_{\{i=j\}}(1/2)$. After the terms of the transaction have been agreed upon, the banks part ways. We assume all loans, bilateral and brokered, are settled at time $\bar{T} > T$.

We assume that all reserve loans and broker fees are settled at time $\bar{T} > T$ (i.e., some time in the following, unmodelled, trading day) and that banks value reserve balances linearly at that time. Specifically, if $c \in \mathbb{R}$ is a bank's net credit position (including loans and broker fees) to be settled at $\bar{T}$ that has resulted from a certain history of trades, then $e^{-r(\bar{T}-t)}c$ is the bank's payoff from this credit balance at time $t \in [0, T]$, where $r \in \mathbb{R}_+$ is the discount rate common to all banks and brokers.

Throughout the session, banks receive payment shocks that induce reallocations of reserve

[1] The interbroker competitive is a convenient approximation to a market structure where trade between brokers is decentralized, but brokers can contact each other at very high rates.

balances among pairs of banks. Specifically, with Poisson rate λ_i, a bank of type $i \in \mathbb{N}$ is forced to make an immediate transfer of reserves to a counterparty drawn randomly (uniformly) from the population of banks. This process for payment shocks is independent across banks and independent of the processes that drive bilateral trading opportunities between banks and trading opportunities with brokers. Once a bank of type i and a bank of type j are connected by a payment shock initiated by the bank of type i (i.e., according to the Poisson process with rate λ_i), the amount of reserves that the bank of type i sends the bank of type j is modeled as a random variable with cumulative distribution function $G^{ij} : \mathbb{Z} \to [0,1]$, where $\mathbb{Z} \subseteq \mathbb{R}$. To capture the notion that these payments are *transfers* between pairs of banks, we assume $\int z dG^{ii}(z) = 0$, and $dG^{ij}(z) = dG^{ji}(-z)$ for $i \neq j$.

For each $i \in \mathbb{N}$, define the function $U^i : \mathbb{R} \to \mathbb{R}$, where $U^i(a)$ represents the payoff to a bank of type i from holding reserve balance $a \in \mathbb{R}$ at the end of the trading session. Similarly, for each $i \in \mathbb{N}$, define the function $u^i : \mathbb{R} \to \mathbb{R}$, where $u^i(a)$ represents the flow payoff to a bank of type i from holding reserve balance $a \in \mathbb{R}$ during the trading session. The type of a bank is defined by a vector of primitives, i.e., type $i \in \mathbb{N}$ is defined by $(n_i, \alpha_i, \beta_i, \theta_i, \lambda_i, \{G^{ij}\}_{j \in \mathbb{N}}, u^i, U^i)$, in the sense that each of the n_i banks of type i has trading frequency α_i for trades with brokers, trading frequency β_i for direct, non-brokered bilateral trades with other banks, bargaining power θ_i, payment frequency λ_i, probability distribution $\{G^{ij}\}_{j \in \mathbb{N}}$ of payment shock sizes, intraday payoff function u^i, and end-of-day payoff function U^i.

2.1 Discussion

XXX

3 Equilibrium

Let $J_t^i(a,c) : \mathbb{N} \times \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ be the maximum attainable payoff to a bank of type i that at time $t \in \mathbb{T}$ has reserve balance $a \in \mathbb{R}$ and net credit position $c \in \mathbb{R}$. It can be shown that $J_t^i(a,c) = V_t^i(a) + e^{-r(\bar{T}-t)}c$, where $V_t^i(a) : \mathbb{N} \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is the maximum expected discounted payoff a bank of type $i \in \mathbb{N}$ can obtain when holding $a \in \mathbb{R}$ reserve balances at time $t \in \mathbb{T}$.[2]

Whenever a bank and a broker make contact during the trading session, they bargain over the broker's intermediation fee and the terms of the loan. Specifically, consider a meeting at time

[2]See Lemma 3 in the appendix.

t between a broker and a bank whose reserve balance is a. Let a' denote the bank's post-trade balance, ϕ the intermediation fee earned by the broker, and ρ_t the gross interest rate on loans prevailing at time t in the competitive market where brokers interact with each other on behalf of their clients. We take (a', ϕ) to be the outcome corresponding to the Nash solution to a bargaining problem where the bank has bargaining power $\theta_i \in [0,1]$. The bank's continuation payoff is $V_t^i(a') - e^{-r(\bar{T}-t)}[\rho_t(a'-a)+\phi]$ if an agreement (a', ϕ) is reached, and $V_t^i(a)$ in case of disagreement. Thus, the bank's gain from trade is $V_t^i(a') - V_t^i(a) - e^{-r(\bar{T}-t)}[\rho_t(a'-a)+\phi]$. The dealer's gain from this trade is equal to the present value of the intermediation fee, $e^{-r(\bar{T}-t)}\phi$. The bargaining outcome is

$$\left[a_t^{*i}, \phi_t^{*i}(a)\right] = \arg\max_{(a',\phi)\in\mathbb{R}^2} \left[V_t^i(a') - V_t^i(a) - e^{-r(\bar{T}-t)}\left[\rho_t(a'-a)+\phi\right]\right]^{\theta_i} \left[e^{-r(\bar{T}-t)}\phi\right]^{1-\theta_i}.$$

This solution can be written as

$$a_t^{*i} = \arg\max_{a'\in\mathbb{R}} \left[V_t^i(a') - p_t a'\right] \tag{1}$$

$$e^{-r(\bar{T}-t)}\phi_t^{*i}(a) = (1-\theta_i)\left[V_t^i\left(a_t^{*i}\right) - V_t^i(a) - p_t(a_t^{*i}-a)\right], \tag{2}$$

where $p_t \equiv e^{-r(\bar{T}-t)}\rho_t$.

Whenever two banks meet during the trading session, they bargain over the size of the loan and the size of the repayment. Consider a bank of type $i \in \mathbb{N}$ with reserve balance a that contacts a bank of type $j \in \mathbb{N}$ with reserve balance $\tilde{a}$; the pair $(b_t^{ij}(a,\tilde{a}), R_t^{ji}(\tilde{a},a))$ denotes the bilateral terms of trade negotiated by these banks at time t. That is, $b_t^{ij}(a,\tilde{a})$ is the quantity of reserves that the bank of type i with balance a lends to the bank of type j with balance $\tilde{a}$, and $R_t^{ji}(\tilde{a},a)$ is the quantity of balances that the latter commits to repay at time $\bar{T}$. Then for any $(a,\tilde{a},t) \in \mathbb{R}^2 \times \mathbb{T}$, $[b_t^{ij}(a,\tilde{a}), R_t^{ji}(\tilde{a},a)]$ is the solution to

$$\max_{(b,R)\in\bar{\mathbb{R}}\times\mathbb{R}^2} \left[V_t^i(a-b) + e^{-r(\bar{T}-t)}R - V_t^i(a)\right]^{\theta_{ij}} \left[V_t^j(\tilde{a}+b) - e^{-r(\bar{T}-t)}R - V_t^j(\tilde{a})\right]^{\theta_{ji}},$$

with $\bar{\mathbb{R}} \equiv \left[-\bar{b}, \bar{b}\right]$, where $\bar{b} \in \mathbb{R}_+ \cup \{\infty\}$ represents a limit on bilateral credit exposures (there is no borrowing limit if $\bar{b} = \infty$). The first-order conditions for this problem imply

$$b_t^{ij}(a,\tilde{a}) \in \arg\max_{b\in\bar{\mathbb{R}}} S_t^{ij}(a,\tilde{a},b) \tag{3}$$

$$e^{-r(\bar{T}-t)}R_t^{ji}(\tilde{a},a) = \theta_{ij}\left[V_t^j(\tilde{a}+b_t^{ij}(a,\tilde{a})) - V_t^j(\tilde{a})\right] + \theta_{ji}\left[V_t^i(a) - V_t^i(a-b_t^{ij}(a,\tilde{a}))\right], \tag{4}$$

where

$$S_t^{ij}(a,\tilde{a},b) \equiv V_t^i(a-b) + V_t^j(\tilde{a}+b) - V_t^i(a) - V_t^j(\tilde{a}).$$

The gross interest rate on this loan of size $\left|b_t^{ij}(a,\tilde{a})\right| > 0$ is

$$1 + \rho_t^{ji}(\tilde{a}, a) = \frac{R_t^{ji}(\tilde{a}, a)}{b_t^{ij}(a, \tilde{a})}. \tag{5}$$

The value function $V_t^i(a)$ satisfies

$$\begin{aligned}
V_t^i(a) = \mathbb{E}_t \Bigg\{ & \mathbb{I}_{\{T-t \leq \min[\tau(\alpha_i),\tau(\beta_i),\tau(\lambda_i)]\}} \left[\int_t^T e^{-r(s-t)} u^i(a)\, ds + e^{-r(T-t)} U^i(a) \right] \\
& + \mathbb{I}_{\{\tau(\lambda_i) < \min[\tau(\alpha_i),\tau(\beta_i),T-t]\}} \Bigg[\int_t^{t+\tau(\lambda_i)} e^{-r(s-t)} u^i(a)\, ds \\
& + e^{-r\tau(\lambda_i)} \sum_{j \in \mathbb{N}} \pi_j \int V_{t+\tau(\lambda_i)}^i (a - z)\, dG^{ij}(z) \Bigg] \\
& + \mathbb{I}_{\{\tau(\alpha_i) < \min[\tau(\beta_i),\tau(\lambda_i),T-t]\}} \Bigg[\int_t^{t+\tau(\alpha_i)} e^{-r(s-t)} u^i(a)\, ds \\
& + e^{-r\tau(\alpha_i)} \Bigg\{ V_{t+\tau(\alpha_i)}^i \left(a_{t+\tau(\alpha_i)}^{*i} \right) \\
& - e^{-r\left\{\bar{T}-[t+\tau(\alpha_i)]\right\}} \left[\rho_{t+\tau(\alpha_i)} (a_{t+\tau(\alpha_i)}^{*i} - a) + \phi_{t+\tau(\alpha_i)}^{*i}(a) \right] \Bigg\} \Bigg] \\
& + \mathbb{I}_{\{\tau(\beta_i) < \min[\tau(\alpha_i),\tau(\lambda_i),T-t]\}} \Bigg[\int_t^{t+\tau(\beta_i)} e^{-r(s-t)} u^i(a)\, ds \\
& + e^{-r\tau(\beta_i)} \sum_{j \in \mathbb{N}} \sigma_j \int \left[V_{t+\tau(\beta_i)}^i (a - b_{t+\tau(\beta_i)}^{ij}(a,\tilde{a})) + \bar{R}_{t+\tau(\beta_i)}^{ji}(\tilde{a}, a) \right] dF_{t+\tau(\beta_i)}^j(\tilde{a}) \Bigg] \Bigg\},
\end{aligned} \tag{6}$$

where

$$\pi_j \equiv \frac{\lambda_j n_j}{\sum_{i \in \mathbb{N}} \lambda_i n_i}$$

is the probability the counterparty in a random bilateral payment shock is a bank of type j,

$$\sigma_j \equiv \frac{\beta_j n_j}{\sum_{k \in \mathbb{N}} \beta_k n_k}$$

is the probability the counterparty in a random bilateral trade is a bank of type j, $\bar{R}_{t+\tau(\beta_i)}^{ji}(\tilde{a}, a) \equiv e^{-r\left\{\bar{T}-[t+\tau(\beta_i)]\right\}} R_{t+\tau(\beta_i)}^{ji}(\tilde{a}, a)$, and $\tau(\zeta)$ denotes the exponentially distributed random time until the first passage time of the Poisson process with rate ζ.

Lemma 1 *The Bellman equation (6) can be written as*

$$
\begin{aligned}
V_t^i(a) = {} & \left[1 - e^{-(r+\kappa_i+\beta_i+\lambda_i)(T-t)}\right] \frac{u^i(a)}{r+\kappa_i+\beta_i+\lambda_i} + e^{-(r+\kappa_i+\beta_i+\lambda_i)(T-t)} U^i(a) \\
& + \lambda_i \int_t^T e^{-(r+\kappa_i+\beta_i+\lambda_i)(\tau-t)} \left[\sum_{j\in\mathbb{N}} \pi_j \int V_\tau^i(a-z)\, dG^{ij}(z)\right] d\tau \\
& + \kappa_i \int_t^T e^{-(r+\kappa_i+\beta_i+\lambda_i)(\tau-t)} \max_{a'\in\mathbb{R}} \left[V_\tau^i(a') - p_\tau(a'-a)\right] d\tau \\
& + \beta_i \int_t^T e^{-(r+\kappa_i+\beta_i+\lambda_i)(\tau-t)} \left[V_\tau^i(a) + \sum_{j\in\mathbb{N}} \sigma_j \theta_{ij} \int \max_{b\in\bar{\mathbb{R}}} S_\tau^{ij}(a,\tilde{a},b)\, dF_\tau^j(\tilde{a})\right] d\tau
\end{aligned}
\tag{7}
$$

(with $\kappa_i \equiv \alpha_i\theta_i$), or equivalently, as

$$
\begin{aligned}
rV_t^i(a) = {} & \dot{V}_t^i(a) + u^i(a) \\
& + \kappa_i \max_{a'\in\mathbb{R}} \left[V_t^i(a') - p_t(a'-a) - V_t^i(a)\right] \\
& + \lambda_i \sum_{j\in\mathbb{N}} \pi_j \int \left[V_t^i(a-z) - V_t^i(a)\right] dG^{ij}(z) \\
& + \beta_i \sum_{j\in\mathbb{N}} \sigma_j \theta_{ij} \int \max_{b\in\bar{\mathbb{R}}} S_t^{ij}(a,\tilde{a},b)\, dF_t^j(\tilde{a})
\end{aligned}
\tag{8}
$$

with boundary condition $V_T^i(a) = U^i(a)$.

Lemma 2 *The value function $V_t^i(a) : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ that satisfies (7) takes the form*

$$V_t^i(a) = v_t^i(a) + q_t^i a, \tag{9}$$

where $q_t^i \in \mathbb{R}$ satisfies

$$q_t^i = \int_t^T e^{-(r+\kappa_i+\beta_i+\lambda_i)(\tau-t)} \left[\kappa_i p_\tau + (\beta_i+\lambda_i)\, q_\tau^i\right] d\tau \tag{10}$$

for every t, and $v_t^i(a) : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ solves

$$
\begin{aligned}
v_t^i(a) = {} & \left[1 - e^{-(r+\kappa_i+\beta_i+\lambda_i)(T-t)}\right] \frac{u^i(a)}{r+\kappa_i+\beta_i+\lambda_i} + e^{-(r+\kappa_i+\beta_i+\lambda_i)(T-t)} U^i(a) \\
& + \int_t^T e^{-(r+\kappa_i+\beta_i+\lambda_i)(\tau-t)} \Bigg\{ \beta_i \left[v_\tau^i(a) + \sum_{j\in\mathbb{N}} \sigma_j \theta_{ij} \int \max_{b\in\bar{\mathbb{R}}} s_\tau^{ij}(a,\tilde{a},b)\, dF_\tau^j(\tilde{a})\right] \\
& + \lambda_i \sum_{j\in\mathbb{N}} \pi_j \int v_\tau^i(a-z)\, dG^{ij}(z) \Bigg\} d\tau + K_t^i
\end{aligned}
\tag{11}
$$

with

$$s_{\tau}^{ij}\left(a,\tilde{a},b\right)\equiv v_{\tau}^{i}\left(a-b\right)+v_{\tau}^{j}\left(\tilde{a}+b\right)+(q_{\tau}^{j}-q_{\tau}^{i})b-v_{\tau}^{i}\left(a\right)-v_{\tau}^{j}\left(\tilde{a}\right)$$

and

$$\begin{aligned} K_t^i \equiv \kappa_i &\int_t^T e^{-(r+\kappa_i+\beta_i+\lambda_i)(\tau-t)} \max_{a'\in\mathbb{R}}\left[v_{\tau}^i\left(a'\right)-\left(p_{\tau}-q_{\tau}^i\right)a'\right]d\tau \\ &-\lambda_i\left[\int_t^T e^{-(r+\kappa_i+\beta_i+\lambda_i)(\tau-t)}q_{\tau}^i d\tau\right]\bar{z}_i, \end{aligned} \tag{12}$$

where

$$\bar{z}_i\equiv\sum_{j\in\mathbb{N}}\pi_j\int zdG^{ij}\left(z\right). \tag{13}$$

For every $i\in\mathbb{N}$, let u^i and U^i be continuous and bounded functions, and assume $\bar{b}<\infty$, and $0\leq p_t-q_t^i$ for all $t\in\mathbb{T}$. Then there exists a unique function $V_t^i\left(a\right):\mathbb{N}\times\mathbb{T}\times\mathbb{R}\rightarrow\mathbb{R}$ that satisfies (7). If in addition, u^i and U^i are increasing and concave functions of a, then for every $(i,t)\in\mathbb{N}\times[0,T)$, v_t^i is an increasing and concave function of a. Moreover, v_t^i is strictly increasing if either u^i or U^i is strictly increasing, and strictly concave if either u^i or U^i is strictly concave.

Consider a bank with preference type $i\in\mathbb{N}$ and reserve balance $a\in\mathbb{R}$ who trades with a broker at time $t\in\mathbb{T}$. Given (1) and Lemma 2, this bank's post-trade reserve balance is

$$a_t^{*i}=\arg\max_{a'\in\mathbb{R}}\left[v_t^i(a')-\left(p_t-q_t^i\right)a'\right]. \tag{14}$$

Since each bank of type i contacts a broker at the same rate, α_i, the measure of banks of type i that can trade reserves in any instant is $\alpha_i n_i$. Thus the total quantity of reserves held by banks of type i that is available for trade in the interbroker market in any instant is $\alpha_i A_t^i$, and the total quantity of reserves demanded by those type i banks who contact brokers is $A_t^{*i}\equiv n_i a_t^{*i}$. Hence the clearing condition in the interbroker market is

$$\sum_{i\in\mathbb{N}}\alpha_i\left(A_t^{*i}-A_t^i\right)=0. \tag{15}$$

With the notation in Lemma 2, we can write (3) and (4) as

$$b_t^{ij}\left(a,\tilde{a}\right)\in\arg\max_{b\in\bar{\mathbb{R}}}s_{\tau}^{ij}\left(a,\tilde{a},b\right) \tag{16}$$

$$\begin{aligned} e^{-r(\bar{T}-t)}R_t^{ji}\left(\tilde{a},a\right)=\theta_{ij}&\left[v_t^j(\tilde{a}+b_t^{ij}\left(a,\tilde{a}\right))-v_t^j\left(\tilde{a}\right)\right] \\ &+\theta_{ji}\left[v_t^i\left(a\right)-v_t^i(a-b_t^{ij}\left(a,\tilde{a}\right))\right] \\ &+(\theta_{ij}q_t^j+\theta_{ji}q_t^i)b_t^{ij}\left(a,\tilde{a}\right). \end{aligned} \tag{17}$$

Let $f_t^i(a) = dF_t^i(x)$ denote the probability density function of reserve holdings among banks of type i at time t. This density follows the law of motion

$$\dot{f}_t^i(a) + (\alpha_i + \beta_i + \lambda_i) f_t^i(a) = \alpha_i \mathbb{I}_{\{a_t^{*i}=a\}} + \lambda_i \sum_{j\in\mathbb{N}} \pi_j \int\int \mathbb{I}_{\{x-z=a\}} dF_t^i(x)\, dG^{ij}(z)$$
$$+ \beta_i \sum_{j\in\mathbb{N}} \sigma_j \int\int \mathbb{I}_{\{x-b_t^{ij}(x,\tilde{a})=a\}} dF_t^i(x)\, dF_t^j(\tilde{a}). \tag{18}$$

Let A_t^i denote the consolidated reserve balance of all banks of type i at time t, i.e.,

$$A_t^i \equiv n_i \int a dF_t^i(a). \tag{19}$$

Then,

$$\dot{A}_t^i = \alpha_i \left(A_t^{*i} - A_t^i\right) - \lambda_i n_i \sum_{j\in\mathbb{N}} \pi_j \int z dG^{ij}(z)$$
$$- \beta_i n_i \sum_{j\in\mathbb{N}} \sigma_j \int\int b_t^{ij}(a,\tilde{a})\, dF_t^j(\tilde{a})\, dF_t^i(a). \tag{20}$$

Hereafter, let $\boldsymbol{q}_t = \left\{q_t^i\right\}_{i\in\mathbb{N}}$, $\boldsymbol{v}_t(\cdot) = \left\{v_t^i(\cdot)\right\}_{i\in\mathbb{N}}$, $\boldsymbol{a}_t^* = \left\{a_t^{*i}\right\}_{i\in\mathbb{N}}$, $\boldsymbol{b}_t(\cdot,\cdot) = \{b_t^{ij}(\cdot,\cdot)\}_{i,j\in\mathbb{N}}$, $\boldsymbol{R}_t(\cdot,\cdot) = \{R_t^{ij}(\cdot,\cdot)\}_{i,j\in\mathbb{N}}$, $\boldsymbol{F}_t(\cdot) = \left\{F_t^i(\cdot)\right\}_{i\in\mathbb{N}}$, and $\boldsymbol{A}_t = \left\{A_t^i\right\}_{i\in\mathbb{N}}$.

Definition 1 *An equilibrium is a time-path* $\{\boldsymbol{q}_t, \boldsymbol{v}_t(\cdot), \boldsymbol{a}_t^*, p_t, \boldsymbol{b}_t(\cdot,\cdot), \boldsymbol{R}_t(\cdot,\cdot), \boldsymbol{F}_t(\cdot), \boldsymbol{A}_t\}_{t\in\mathbb{T}}$ *that satisfies (10), (11), (14), (15), (16), (17), (18), and (20), given an initial condition* $\boldsymbol{F}_0$.

4 Quadratic end-of-day payoffs

Suppose $\left\{u^i, U^i\right\}_{i\in\mathbb{N}}$ are given by

$$u^i(a) = c_i a - \frac{1}{2} b_i a^2 \tag{21}$$
$$U^i(a) = C_i a - \frac{1}{2} B_i a^2, \tag{22}$$

with $b_i, c_i, B_i, C_i \in \mathbb{R}_{++}$ for all $i \in \mathbb{N}$. Then function $v_t^i(a)$ that solves (11) is

$$v_t^i(a) = \eta_t^i a - \frac{1}{2}\mu_t^i a^2 + \varsigma_t^i, \tag{23}$$

where ς_t^i satisfies

$$\dot{\varsigma}_t^i - r\varsigma_t^i = -\kappa_i \frac{\left(p_t - q_t^i - \eta_t^i\right)^2}{2\mu_t^i} + \lambda_i \left[\left(\eta_t^i + q_t^i\right) \bar{z}_i + \mu_t^i \frac{1}{2}\left(\sigma_i^2 + \bar{z}_i^2\right)\right]$$
$$- \beta_i \sum_{j\in\mathbb{N}} \frac{\sigma_j \theta_{ij}}{2(\mu_t^i + \mu_t^j)} \left[(\eta_t^j + q_t^j - \eta_t^i - q_t^i)(\eta_t^j + q_t^j - \eta_t^i - q_t^i - 2\mu_t^j \frac{1}{n_j} A_t^j) + (\mu_t^j)^2 \Sigma_t^j\right] \tag{24}$$

with boundary condition $\varsigma_T^i = 0$, where $\bar{z}_i$ is given by (13), A_t^i is given by (19),

$$\sigma_i^2 \equiv \sum_{j\in\mathbb{N}} \pi_j \int z^2 dG^{ij}(z) - \bar{z}_i^2$$
$$\Sigma_t^i \equiv \int a^2 dF_t^i(a),$$

and η_t^i and μ_t^i satisfy

$$\dot{\eta}_t^i - (r + \kappa_i + \beta_i)\,\eta_t^i = -c_i - \lambda_i \bar{z}_i \mu_t^i$$
$$- \beta_i \left[\eta_t^i + \sum_{j\in\mathbb{N}} \sigma_j \theta_{ij} \frac{\mu_t^i}{\mu_t^i + \mu_t^j} \left(\eta_t^j + q_t^j - \eta_t^i - q_t^i - \mu_t^j \frac{1}{n_j} A_t^j\right)\right] \tag{25}$$

$$\dot{\mu}_t^i - (r + \kappa_i + \beta_i)\,\mu_t^i = -b_i - \beta_i \left[\mu_t^i + \sum_{j\in\mathbb{N}} \sigma_j \theta_{ij} \frac{\left(\mu_t^i\right)^2}{2(\mu_t^i + \mu_t^j)}\right] \tag{26}$$

with boundary conditions $\eta_T^i = C_i$ and $\mu_T^i = B_i$.

The maximization on the right side of (14) implies

$$a_t^{i*} = \frac{\eta_t^i - \left(p_t - q_t^i\right)}{\mu_t^i}. \tag{27}$$

Condition (15) then implies

$$p_t = \sum_{i\in\mathbb{N}} \omega_t^i \eta_t^i + \sum_{i\in\mathbb{N}} \omega_t^i q_t^i - \sum_{i\in\mathbb{N}} \hat{\alpha}_t^i A_t^i, \tag{28}$$

where

$$\nu_t^i \equiv \frac{\alpha_i n_i}{\mu_t^i}$$
$$\hat{\alpha}_t^i \equiv \frac{\alpha_i}{\sum_{j\in\mathbb{N}} \nu_t^j}$$
$$\omega_t^i \equiv \frac{\nu_t^i}{\sum_{j\in\mathbb{N}} \nu_t^j}.$$

With (28), condition (10) implies

$$\dot{q}_t^i = (r + \kappa_i)\, q_t^i - \kappa_i \left[\sum_{i \in \mathbb{N}} \omega_t^i \eta_t^i + \sum_{i \in \mathbb{N}} \omega_t^i q_t^i - \sum_{i \in \mathbb{N}} \hat{\alpha}_t^i A_t^i \right] \tag{29}$$

with boundary condition $q_T^i = 0$ for all $i \in \mathbb{N}$.

Consider a bilateral trade at time t between a bank of type i with balance a and a bank of type j with balance $\tilde{a}$; the loan from the former to the latter is

$$b_t^{ij}(a, \tilde{a}) = \frac{\eta_\tau^j - \mu_\tau^j \tilde{a} + q_\tau^j - \left(\eta_\tau^i - \mu_\tau^i a + q_\tau^i\right)}{\mu_\tau^i + \mu_\tau^j}, \tag{30}$$

and their respective post trade balances are

$$\begin{aligned} a - b_t^{ij}(a, \tilde{a}) &= \frac{\eta_t^i + q_t^i - \eta_t^j - q_t^j}{\mu_t^i + \mu_t^j} + \frac{\mu_t^j}{\mu_t^i + \mu_t^j}(a + \tilde{a}) \\ \tilde{a} + b_t^{ij}(a, \tilde{a}) &= \frac{\eta_t^j + q_t^j - \eta_t^i - q_t^i}{\mu_t^i + \mu_t^j} + \frac{\mu_t^i}{\mu_t^i + \mu_t^j}(a + \tilde{a}). \end{aligned}$$

Remark 1 *Need to write down the laws of motion for*

$$\dot{A}_t^i = \tag{31}$$

and

$$\dot{\Sigma}_t^i = \tag{32}$$

with initial conditions A_0^i and $\dot{\Sigma}_0^i$ given for all $i \in \mathbb{N}$.

Let $\boldsymbol{q}_t = \left\{q_t^i\right\}_{i \in \mathbb{N}}$, $\boldsymbol{\eta}_t = \left\{\eta_t^i\right\}_{i \in \mathbb{N}}$, $\boldsymbol{\mu}_t = \left\{\mu_t^i\right\}_{i \in \mathbb{N}}$, $\boldsymbol{\varsigma}_t = \left\{\varsigma_t^i\right\}_{i \in \mathbb{N}}$, $\boldsymbol{a}_t^* = \left\{a_t^{*i}\right\}_{i \in \mathbb{N}}$, $\boldsymbol{b}_t(\cdot,\cdot) = \{b_t^{ij}(\cdot,\cdot)\}_{i,j \in \mathbb{N}}$, $\boldsymbol{A}_t = \left\{A_t^i\right\}_{i \in \mathbb{N}}$, and $\boldsymbol{\Sigma}_t = \left\{\Sigma_t^i\right\}_{i \in \mathbb{N}}$.

Definition 2 *An equilibrium is characterized by a time-path $\{\boldsymbol{\varsigma}_t, \boldsymbol{\eta}_t, \boldsymbol{\mu}_t, \boldsymbol{a}_t^*, p_t, \boldsymbol{q}_t, \boldsymbol{b}_t(\cdot,\cdot), \boldsymbol{A}_t, \boldsymbol{\Sigma}_t\}_{t \in \mathbb{T}}$ that satisfies (24), (25), (26), (27), (28), (29), (30), (31), and (32).*

5 Pure Brokered Market

Assume $\beta_i = 0$ for all $i \in \mathbb{N}$.

Proposition 1 *Assume that for all $i \in \mathbb{N}$, u^i and U^i are nondecreasing (at least one increasing), concave (at least one strictly concave), and smooth functions of a. Let $h_t^i(x) \equiv \arg\max_{a\in\mathbb{R}}\left[v_t^i(a) - xa\right]$, and suppose $\lim_{x\to\infty} h_t^i(x) \leq 0$ for each $t \in \mathbb{T}$ and all $i \in \mathbb{N}$. Then an equilibrium is summarized by a path $\{\boldsymbol{q}_t, \boldsymbol{A}_t\}_{t\in\mathbb{T}}$ that satisfies the following system of $2N$ first-order differential equations*

$$\dot{q}_t^i = (r+\kappa_i)\, q_t^i - \kappa_i p_t^*(\boldsymbol{q}_t, \boldsymbol{A}_t) \tag{33}$$

$$\dot{A}_t^i = \alpha_i\left[n_i h_t^i\left(p_t^*(\boldsymbol{q}_t, \boldsymbol{A}_t) - q_t^i\right) - A_t^i\right] - \lambda_i n_i \bar{z}_i \tag{34}$$

with N initial conditions $\boldsymbol{A}_0$, and N terminal conditions $\boldsymbol{q}_T = \boldsymbol{0}$, where for each $t \in \mathbb{T}$, $p_t^*(\boldsymbol{q}_t, \boldsymbol{A}_t)$ is the unique price of reserves, p_t^*, that satisfies

$$\sum_{i\in\mathbb{N}} \alpha_i\left[n_i h_t^i\left(p_t^* - q_t^i\right) - A_t^i\right] = 0.$$

5.1 Piecewise linear end-of-day payoff

Consider the following special case. Assume no payment shocks, i.e., $\lambda_i = 0$ for all $i \in \mathbb{N}$. Also for all $a \in \mathbb{R}$ and $i \in \mathbb{N}$, assume $u^i(a) = 0$ and

$$U^i(a) = \left[\mathbb{I}_{\{0\leq a-\underline{a}\}}\mu_L + \mathbb{I}_{\{a-\underline{a}<0\}}\mu_H\right](a-\underline{a}), \tag{35}$$

where $\underline{a}, \mu_L, \mu_H \in \mathbb{R}_+$, with $\mu_L < \mu_H$. Define $R^i \equiv n_i\underline{a}$, and let $R \equiv \sum_{i\in\mathbb{N}} R^i$. In this case, the function v_t^i that solves (11) is

$$v_t^i(a) = \mu_t^i(a)(a-\underline{a}) + \varsigma_t^i,$$

where

$$\mu_t^i(a) \equiv e^{-(r+\kappa_i)(T-t)}\left[\mathbb{I}_{\{0\leq a-\underline{a}\}}\mu_L + \mathbb{I}_{\{a-\underline{a}<0\}}\mu_H\right]$$

and

$$\varsigma_t^i = \kappa_i \int_t^T e^{-(r+\kappa_i)(s-t)} \max_{a'}\left[v_s^i(a') - p_s a'\right] ds.$$

The maximizer on the right of (14) can be written as

$$a_t^{*i}\left(P_t^i\right) = \arg\max_{a'\in\mathbb{R}}\left\{\left[\mathbb{I}_{\{0\leq a'-\underline{a}\}}\mu_L + \mathbb{I}_{\{a'-\underline{a}<0\}}\mu_H\right]\left(a'-\underline{a}\right) - P_t^i a'\right\}$$

or equivalently,

$$a_t^{*i}\left(P_t^i\right)=\begin{cases}\infty & \text{if } P_t^i<\mu_L\\ \varkappa_+ & \text{if } \mu_L=P_t^i\\ \underline{a} & \text{if } \mu_L<P_t^i<\mu_H\\ \varkappa_- & \text{if } P_t^i=\mu_H\\ -\infty & \text{if } \mu_H<P_t^i,\end{cases} \tag{36}$$

where $\varkappa_+\in[\underline{a},\infty]$, and $\varkappa_-\in[-\infty,\underline{a}]$, and

$$P_t^i\equiv e^{(r+\kappa_i)(T-t)}\left(p_t-q_t^i\right).$$

Condition (15) then implies

$$\sum_{i\in\mathbb{N}}\alpha_i\left[n_i a_t^{*i}\left(P_t^i\right)-A_t^i\right]=0. \tag{37}$$

Let $\mathbb{N}_t^0=\left\{i\in\mathbb{N}:\mu_L<P_t^i<\mu_H\right\}$, $\mathbb{N}_t^+=\left\{i\in\mathbb{N}:P_t^i=\mu_L\right\}$, and $\mathbb{N}_t^-=\left\{i\in\mathbb{N}:P_t^i=\mu_H\right\}$. Notice (37) cannot hold if there exist some $i\in\mathbb{N}_t\backslash\left(\mathbb{N}_t^+\cup\mathbb{N}_t^0\cup\mathbb{N}_t^-\right)$. With this notation, (36) and (37) imply that, taking $\left\{q_t^i\right\}_{i\in\mathbb{N}}$ as given, the market-clearing price, p_t, satisfies

$$\sum_{i\in\mathbb{N}}\alpha_i\left[n_i\left(\mathbb{I}_{\{i\in\mathbb{N}_t^+\}}\varkappa_++\mathbb{I}_{\{i\in\mathbb{N}_t^0\}}\underline{a}+\mathbb{I}_{\{i\in\mathbb{N}_t^-\}}\varkappa_-\right)-A_t^i\right]=0. \tag{38}$$

Generically (given $\kappa_i\neq\kappa_j$ for all $i,j\in\mathbb{N}$), the equilibrium can take one of three configurations: (i) $\mathbb{N}_t^-\neq\varnothing$ and $\mathbb{N}_t^0=\mathbb{N}\backslash\mathbb{N}_t^-$, (ii) $\mathbb{N}_t^0=\mathbb{N}$, (iii) $\mathbb{N}_t^+\neq\varnothing$ and $\mathbb{N}_t^0=\mathbb{N}\backslash\mathbb{N}_t^+$. In case (i), (38) becomes $\sum_{i\in\mathbb{N}_t^-}\alpha_i n_i\varkappa_-+\sum_{i\in\mathbb{N}\backslash\mathbb{N}_t^-}\alpha_i n_i\underline{a}=\sum_{i\in\mathbb{N}}\alpha_i A_t^i$, with $\varkappa_-\in[-\infty,\underline{a}]$, and $p_t=e^{-(r+\kappa_{s_t})(T-t)}\mu_H+q_t^{s_t}\equiv\bar{p}_t$, where $s_t\in\mathbb{N}_t^-$. In case (iii), (38) becomes $\sum_{i\in\mathbb{N}_t^+}\alpha_i n_i\varkappa_++\sum_{i\in\mathbb{N}\backslash\mathbb{N}_t^+}\alpha_i n_i\underline{a}=\sum_{i\in\mathbb{N}}\alpha_i A_t^i$, with $\varkappa_+\in[\underline{a},\infty]$, and $p_t=e^{-(r+\kappa_{l_t})(T-t)}\mu_L+q_t^{l_t}\equiv\underline{p}_t$, where $l_t\in\mathbb{N}_t^+$. In case (ii), (38) becomes $\sum_{i\in\mathbb{N}}\alpha_i n_i\underline{a}=\sum_{i\in\mathbb{N}}\alpha_i A_t^i$, and $p_t\in[\underline{p}_t,\bar{p}_t]$. Therefore, the market-clearing price at time t is

$$p_t=\begin{cases}\bar{p}_t & \text{if } 0<\mathcal{R}-\mathcal{A}_t\\ \tilde{p}_t & \text{if } \mathcal{R}-\mathcal{A}_t=0\\ \underline{p}_t & \text{if } \mathcal{R}-\mathcal{A}_t<0,\end{cases} \tag{39}$$

where

$$\mathcal{A}_t\equiv\sum_{i\in\mathbb{N}}\alpha_i A_t^i$$
$$\mathcal{R}\equiv\sum_{i\in\mathbb{N}}\alpha_i R^i,$$

$\bar{p}_t \equiv e^{-(r+\kappa_{s_t})(T-t)}\mu_H + q_t^{s_t}$, $\underline{p}_t \equiv e^{-(r+\kappa_{l_t})(T-t)}\mu_L + q_t^{l_t}$, and $\tilde{p}_t \in [\underline{p}_t, \bar{p}_t]$. With (36) and (39), condition (20) becomes $\dot{A}_t^i = \alpha_i \left[n_i a_t^{*i}\left(P_t^i\right) - A_t^i\right]$. If we assume $\mathbb{N}_t^- = \{s_t\}$ and $\mathbb{N}_t^+ = \{l_t\}$, then we can write

$$\dot{A}_t^i = \begin{cases} \alpha_i \left(R^i - A_t^i\right) \mathbb{I}_{\{i \neq s_t\}} - \mathbb{I}_{\{i = s_t\}} \sum_{j \in \mathbb{N}\setminus\{s_t\}} \alpha_j (R^j - A_t^j) & \text{if } 0 < \mathcal{R} - \mathcal{A}_t \\ \alpha_i \left(R^i - A_t^i\right) & \text{if } \mathcal{R} - \mathcal{A}_t = 0 \\ \alpha_i \left(R^i - A_t^i\right) \mathbb{I}_{\{i \neq l_t\}} - \mathbb{I}_{\{i = l_t\}} \sum_{j \in \mathbb{N}\setminus\{l_t\}} \alpha_j (R^j - A_t^j) & \text{if } \mathcal{R} - \mathcal{A}_t < 0. \end{cases} \tag{40}$$

In order to find an equilibrium, we need to find a path $\left\{q_t^i, p_t, A_t^i\right\}_{i \in \mathbb{N}, t \in \mathbb{T}}$ that satisfies (10), (39), and (40).

The following proposition characterizes the equilibrium set for a leading example.

Proposition 2 *Assume $N = 2$, with $\alpha_1 < \alpha_2$, and all other parameters the same across types.*
(i) Suppose $0 \leq R - A$ and $0 < \mathcal{R} - \mathcal{A}_0$. The equilibrium price is $p_t = e^{-r(T-t)}\mu_H$ for all $t \in \mathbb{T}$, and no bank provides liquidity for other banks.
(ii) Suppose $R - A \leq 0$ and $\mathcal{R} - \mathcal{A}_0 < 0$. The equilibrium price is $p_t = e^{-r(T-t)}\mu_L$ for all $t \in \mathbb{T}$, and no bank provides liquidity for other banks.
(iii) Suppose $0 < R - A$ and $\mathcal{R} - \mathcal{A}_0 \leq 0$. (a) If

$$R - A \leq -\frac{e^{-\alpha_1 T}}{\alpha_N \left(1 - e^{-\alpha_1 T}\right)} \left(\mathcal{R} - \mathcal{A}_0\right), \tag{41}$$

the equilibrium is as in part (ii). (b) If

$$-\frac{e^{-\alpha_1 T}}{\alpha_N \left(1 - e^{-\alpha_1 T}\right)} \left(\mathcal{R} - \mathcal{A}_0\right) < R - A, \tag{42}$$

the equilibrium price is

$$p_t = \begin{cases} e^{-r(t^*-t)} \left[e^{-r(T-t^*)}\mu_H - e^{-(r+\kappa_N)(T-t^*)}\left(\mu_H - \mu_L\right)\right] & \text{for } t \in [0, t^*) \\ e^{-r(T-t)}\mu_H & \text{for } t \in [t^*, T] \end{cases} \tag{43}$$

with

$$t^* = \frac{1}{\alpha_1} \ln\left(1 - \frac{1}{\alpha_N}\frac{\mathcal{R} - \mathcal{A}_0}{R - A}\right), \tag{44}$$

and for any $t \in [0, t^*)$, the distribution of balances, $\left\{A_t^i\right\}_{i \in \mathbb{N}}$, is given by

$$A_t^N = A_0^N - \left(1 - e^{-\alpha_1 t}\right)\left(R^1 - A_0^1\right) \tag{45}$$

$$A_t^1 = R^1 - e^{-\alpha_1 t}\left(R^1 - A_0^1\right), \tag{46}$$

so banks of type N provide liquidity to banks of type 1 during $[0, t^*)$.

(iv) Suppose $R - A < 0$ and $0 \leq \mathcal{R} - \mathcal{A}_0$. (a) If (42) holds (with "$\leq$"), equilibrium is as in part (i). (b) If (41) holds with "$<$", the equilibrium price is

$$p_t = \begin{cases} e^{-r(t^*-t)} \left[e^{-r(T-t^*)}\mu_L + e^{-(r+\kappa_N)(T-t^*)} \left(\mu_H - \mu_L\right)\right] & \text{for } t \in [0, t^*) \\ e^{-r(T-t)}\mu_L & \text{for } t \in [t^*, T] \end{cases} \tag{47}$$

with t^* given by (44), the distribution of balances, $\left\{A_t^i\right\}_{i \in \mathbb{N}}$, is given by (45) and (46), so banks of type N provide liquidity to banks of type 1 during $[0, t^*)$.

(v) Suppose $R - A = \mathcal{R} - \mathcal{A}_0 = 0$. The equilibrium price is indeterminate, i.e., any p_t such that $e^{r(T-t)}p_t \in [\mu_L, \mu_H]$ is an equilibrium price, and no bank provides liquidity for other banks.

Figure **??** shows the regions of parameter space where each type of equilibrium described in Proposition 2 exists. The figure has $R^2 - A_0^2$ on the vertical axis (i.e., the time-0 pre-trade "excess demand" by all banks of type 2), and $R^1 - A_0^1$ on the horizontal axis (i.e., the time-0 pre-trade "excess demand" by all banks of type 1). There are three boundaries: $R^N - A_0^N = -\left(R^1 - A_0^1\right)$, $R^N - A_0^N = -\frac{\alpha_1}{\alpha_2}\left(R^1 - A_0^1\right)$, and $R^N - A_0^N = -\frac{\tilde{\alpha}_1}{\alpha_N}\left(R^1 - A_0^1\right)$, where $\tilde{\alpha}_1 \equiv e^{-\alpha_1 T}\alpha_1 + \left(1 - e^{-\alpha_1 T}\right)\alpha_N$. Combinations of $R^1 - A_0^1$ and $R^2 - A_0^2$ that lie above (below) the first boundary correspond to situations in which the market as a whole has insufficient (excessive) balances relative to the aggregate reserve requirement, i.e., $0 < R - A$ ($R - A < 0$). Combinations of $R^1 - A_0^1$ and $R^2 - A_0^2$ that lie above (below) the second boundary correspond to situations in which the consolidated reserves of the banks participating in the interbroker market at $t = 0$ are lower (larger) than their aggregate reserve requirements, i.e., $0 < \mathcal{R} - \mathcal{A}_0$. The proposition shows that faster banks play a role in liquidity provision when their initial net position (reserves minus requirement) has the opposite sign to the marketwide net position and is large enough in magnitude. Consider region *(iii)(b)*, for example, where the marketwide net reserve position is negative, i.e., $A < R$, but the pre-trade reserve position of the banks of type 2 that participate of the interbroker market at $t = 0$ is positive but moderate in magnitude, i.e., $0 < \frac{\alpha_1}{\alpha_N}\left(R^1 - A_0^1\right) \leq A_0^N - R^N < \frac{\tilde{\alpha}_1}{\alpha_N}\left(R^1 - A_0^1\right)$. In this case, type 2 banks accommodate the trading needs of type 1 banks by lending them their excess reserves early in the day, i.e., during $[0, t^*]$. Conversely, consider region $(iv)(b)$, where the marketwide net reserve position is positive, i.e., $R < A$, but the pre-trade position of the banks of type 2 that participate of the interbroker market at $t = 0$ is negative and moderate in magnitude, i.e., $\frac{\tilde{\alpha}_1}{\alpha_N}\left(R^1 - A_0^1\right) < A_0^N - R^N \leq \frac{\alpha_1}{\alpha_N}\left(R^1 - A_0^1\right) < 0$. In this case, type 2 banks accommodate the trading needs of type 1 banks by purchasing their excess reserves early in the day, i.e., during $[0, t^*]$.

The following proposition shows the role that heterogeneity in α plays in generating liquidity provision.

Proposition 3 *Assume $\alpha_i = \alpha$ for all $i \in \mathbb{N}$. The equilibrium price for all $t \in \mathbb{T}$ is*

$$e^{r(T-t)} p_t \begin{cases} = \mu_H & \text{if } 0 < R - A \\ \in [\mu_L, \mu_H] & \text{if } R - A = 0 \\ = \mu_L & \text{if } R - A < 0. \end{cases}$$

The path for the distribution of balances is indeterminate, and there is no liquidity provision in equilibrium.

Notice we still have $\theta_i \neq \theta_j$ for all $i, j \in \mathbb{N}$, and therefore $\kappa_i \neq \kappa_j$ for all $i, j \in \mathbb{N}$, yet there is no liquidity provision by banks.

6 Positive Implications

The performance of the fed funds market is typically assessed with measures of volume traded, reserves reallocation across banks, and the distribution of interest rates resulting from these trades. In this section, we derive theoretical counterparts of these measures. In Section 7 we compute the empirical counterpart of the measures derived in this section, and use them to estimate the key model parameters.

6.1 Trade volume

Daily trade volume in the *brokered market* is $\bar{\upsilon}^{\mathrm{bm}} = \int_0^T \upsilon_t^{\mathrm{bm}} dt$, where

$$\upsilon_t^{\mathrm{bm}} = \frac{1}{2} \sum_{i \in \mathbb{N}} \alpha_i \left| n_i a_t^{*i} - A_t^i \right|$$

is the volume traded at time t. Similarly, the daily volume in *bilateral trades* is given as $\bar{\upsilon}^{\mathrm{bt}} = \int_0^T \upsilon_t^{\mathrm{bt}} dt$, where

$$\upsilon_t^{\mathrm{bt}} = \frac{1}{2} \sum_{i \in \mathbb{N}} n_i \beta_i \int_a \left(\sum_{j \in \mathbb{N}} \sigma_j \int_{\tilde{a}} b_t^{ij}(a, \tilde{a}) dF_t^j(\tilde{a}) \right) dF_t^i(a)$$

is the volume of bilateral trades at time t. The total volume traded in a day is $\bar{\upsilon} = \int_0^T \upsilon_t dt$ where $\upsilon_t = \upsilon_t^{\mathrm{bm}} + \upsilon_t^{\mathrm{bt}}$.

6.2 Effective fed funds rate

The (gross) interest rate on loans extended at time $t \in T$ in the brokered market is ρ_t. So the average volume-weighted daily rate in the brokered market is $\bar{\rho} = \int_0^T \varpi_t \rho_t dt$, where $\varpi_t \equiv v_t^{\mathrm{bm}}/\bar{v}^{\mathrm{bm}}$. This is the rate on intermediated transactions that would be reported by brokers, so it is the model equivalent of the rate used to compute *effective fed funds rate.* Additionaly, brokers charge a fee, which introduces a spread between the borrowing and lending rates. Specifically, the (gross) cum-fee interest rate that a broker offers a bank of type i with pre-trade balance a who trades a loan of size $a' - a \neq 0$ at time t, is

$$\rho_t^i \left(a', a\right) \equiv \rho_t + \frac{\phi_t^{*i}(a)}{a' - a}.$$

The interest rate on bilateral trades is given by $\rho_t^{ij}(a, \tilde{a})$. The volume-weighted interest rate on bilateral transactions is given as

$$\bar{\rho}^{\mathrm{bt}} = \int_0^T \sum_{i,j \in \mathbb{N}^2} \left(\int_a \int_{\tilde{a}} \rho_t^{ij}(a, \tilde{a}) \varpi_t^{ij}(a, \tilde{a}) d\tilde{a} da \right) dt$$

where $\varpi_t^{ij}(a, \tilde{a})$ is the volume transacted bilaterally between bank i with holdings a and bank j with holdings $\tilde{a}$ at time t, relative to the volume of all bilateral transactions

$$\varpi_t^{ij}(a, \tilde{a}) = \frac{1}{2} \frac{n_i \beta_i \int_a \left(\sum_{j \in \mathbb{N}} \sigma_j \int_{\tilde{a}} \left| b_t^{ij}(a, \tilde{a}) \right| dF_t^j(\tilde{a}) \right) dF_t^i(a)}{\bar{v}^{\mathrm{bt}}}$$

The Effective Fed Funds Rate (EFFR) is then given as $\mathrm{EFFR} = \frac{\bar{v}^{\mathrm{bm}}}{\bar{v}} \bar{\rho}^{\mathrm{bt}} + \frac{\bar{v}^{\mathrm{bt}}}{\bar{v}} \bar{\rho}$.

6.3 Intermediation, Reallocation and Network of Trades

6.3.1 Intermediation

A key measure of intermediation is the fraction of total volume of trade transacted by a particular bank type. Let m_i^b be the total dollar-volume of loans either received or extended by banks type i

$$m_i^b = \int_0^T n_i \beta_i \int_a \left(\sum_{j \in \mathbb{N}} \sigma_j \int_{\tilde{a}} \left| b_t^{ij}(a, \tilde{a}) \right| dF_t^j(\tilde{a}) \right) dF_t^i(a) dt$$

The fraction of total volume intermediated by bank type i then is

$$\mathcal{B}_i = \frac{m_i^b}{\frac{1}{2} \sum_{j \in \mathbb{N}} m_j^b} \tag{48}$$

Thus, $\mathcal{B}_i$ is the fraction of total bilateral loans that had banks' type i as a counterpart. For instance, if $\mathcal{B}_i = 1$, banks type i participated in every bilateral transaction. Measures $\mathcal{B}_i$ will be very useful in identifying meeting meeting probabilities $\{\beta_i, \alpha_i\}_{i\in\mathbb{N}}$.

6.3.2 Reallocation

Banks trade partially because they profit from reallocating reserves from those who are long to those who are short. We measure this reallocation by computing the net amount of reserves that banks trade, subtracting purchases from sales. Let $\mathcal{O}_i^s$ be the amount of reserves that banks type i lent to others (sold reserves), and $\mathcal{O}_i^p$ be the amount that it borrowed from others (purchased reserves). Formally

$$\mathcal{O}_i^s = \int_0^T n_i\beta_i \int_a \left(\sum_{j\in\mathbb{N}} \sigma_j \int_{\tilde{a}} b_t^{ij}(a,\tilde{a})\mathbb{I}_{\left\{b_t^{ij}(a,\tilde{a})>0\right\}} dF_t^j(\tilde{a})\right) dt$$

$$\mathcal{O}_i^p = \int_0^T n_i\beta_i \int_a \left(\sum_{j\in\mathbb{N}} \sigma_j \int_{\tilde{a}} -b_t^{ij}(a,\tilde{a})\mathbb{I}_{\left\{b_t^{ij}(a,\tilde{a})<0\right\}} dF_t^j(\tilde{a})\right) dt$$

Notice that $\mathcal{O}_i^s, \mathcal{O}_i^p \geq 0$. We measure reallocation as

$$\mathcal{R}_i = \frac{\mathcal{O}_i^s - \mathcal{O}_i^p}{\mathcal{O}_i^s + \mathcal{O}_i^p}$$

Notice that $\mathcal{R}_i \in [-1, 1]$. If $\mathcal{R}_i = 1$, banks type i only lent to other banks and did not borrow at all. On the contrary, if $\mathcal{R}_i = -1$, banks type i only borrowed. If $\mathcal{R}_i = 0$, then banks type i only intermediated reserves by lending every dollar that they borrowed. We can think of banks with $\mathcal{R}_i = 0$ as being market makers, connecting banks in need of reserves to those in excess.

6.3.3 Network of Trades

Reallocation occurs through a network of trades. We compute the network of flows from on set of banks to another as

$$\bar{\eta}_{ij} = \int_0^T n_i\beta_i \int_a \left(\sum_{j\in\mathbb{N}} \sigma_j \int_{\tilde{a}} b_t^{ij}(a,\tilde{a})\mathbb{I}_{\left\{b_t^{ij}(a,\tilde{a})>0\right\}} dF_t^j(\tilde{a})\right) dF_t^i(a)dt$$

Thus, $\bar{\eta}_{ij}$ measures the amount of reserves that banks i lent to banks j. Notice that $\bar{\eta}_{ij} > 0$. We normalize to $\bar{\eta}_{ij}$ to fractions as

$$\eta_{ij} = \frac{\bar{\eta}_{ij}}{\sum_{\tilde{i}\tilde{j}\in\mathbb{N}^2} \bar{\eta}_{\tilde{i}\tilde{j}}}$$

7 Micro Data and Model Estimation

In this section, we compute the empirical counterpart of the theoretical measures derive in Section 6. We use these measures to estimate the model parameters. We start by describing the data we have, and then move to the model estimation.

7.1 Transaction Level Data

We use two databases for estimating the model. The first database includes all transactions done using *Fedwire Funds* system. This is an electronic system that banks use to settle payments among them, and is also the system used to lend/borrow federal funds loans. We use a version of the *Furfine* algorithm to identify fed funds loans, and label the remaining transactions as payment shpcks. The second database we use is maintained by Monetary Affairs Division at the Federal Reserve Board, and it includes reserves balances as well as required balances for each bank on a daily basis. We use this secon database to compute the distribution of reserves across banks.

A more detailed explanation of each database can be found in Appendix C.

7.2 Model Estimation

The model has N type of banks, and each type is characterized by parameters

$$\Theta_i = \left(n_i, \alpha_i, \beta_i, \theta_i, \lambda_i, F_0^i, \left\{G^{ij}\right\}_{j\in\mathbb{N}}, u^i, U^i\right).$$

We start by discussing how we identify the N types of banks, and then move to estimate Θ_i. We will use hats "$\hat{x}$" to denote the empirical counterpart of of any variable/moment x in the model.

7.2.1 Identifying Banks' Types

We allocate banks into types by measuring their participation on the fed funds market. For each bank $\mathbf{i}$, we compute their intermediation volume as

$$\hat{\mathcal{B}}_{\mathbf{i}} = \frac{\hat{m}^b_{\mathbf{i}}}{\frac{1}{2}\sum_{\mathbf{j}} \hat{m}^b_{\mathbf{j}}} \tag{49}$$

where $\hat{m}^b_{\mathbf{i}}$ is the total dollar volume of loans either received or extended by bank $\mathbf{i}$. The measure $\hat{\mathcal{B}}_{\mathbf{i}}$ is the empirical counterpart to $\mathcal{B}_i$ in equation 48 for an individual bank.

A small set of banks account for a large amount of intermediation. Figure 1 shows this by plotting $\hat{\mathcal{B}}_{\mathbf{i}}$ for each bank in our sample during 2005. In this year, the four banks with the largest $\hat{\mathcal{B}}_{\mathbf{i}}$ were counterparties for about 97% of total trades.[3] We think of these four top banks as very active/fast group of traders, while the other banks are less active and largely homogeneous on their trading intensities.

In light of this, we separate banks into three groups: $N = 3$. The first group is composed by the top four banks, which we label F for fast. As a second group, we put together all Government Sponsored Enterprises (GSE), which we label as G. Having a separate group for GSE is important because of the different legal structure they face, as well as the special role they play in the fed funds market. The third group is the remaining of banks, which we label S for slow. As before, we denote $\mathbb{N} = \{F, S, G\}$ to the group of banks' types.

7.2.2 Banks Parameters Θ_i

We use year 2005 as baseline to estimate most parameters. We take n_i to be the fraction of each bank type. We calibrate $\{\beta_i\}_{i\in\mathbb{N}}$ so that intermediation volume $\mathcal{B}_i$ matches its empirical counterpart $\hat{\mathcal{B}}_i$.[4] Because $\sum_i \mathcal{B}_i = 2$, we need an additional condition to identify $\{\beta_i\}_{i\in\mathbb{N}}$. We chose the average of βs to match the ratio of total bilateral trades to total excess reserves in the economy. We assume that all banks have the same contact probability in the brokered market $\alpha_i = \alpha\ \forall i = F, S, G$, and calibrate α to match the ratio of brokered trades relative to the total. We assume $\theta_i = 1\ \forall i = F, S, G$ so that all banks have the same bargaining power.

We take the end-of-day pay-off $U^i(a)$ to be to be return a bank receive on its reserves at the Fed. Let $\underline{a}_i$ be the required reserves for bank i. Then,

$$U_i(a) = \begin{cases} a + i_i^r \underline{a}_i + i_i^e (a - \underline{a}_i) & \text{if } a \geq \underline{a}_i \\ \\ a + i_i^r \underline{a}_i - i_i^w (\underline{a}_i - a) & \text{if } a < \underline{a}_i \end{cases}$$

where i_i^r is the interest paid on required reserves, i_i^e is the return paid on excess reserves, and i_i^w is the discount window rate. We allow for interest rates to depend on bank's type i to account for the different returns received by GSE. We assume there are no intra-day payoffs $u^i(a) = 0$.

We set reserves requirement to $\bar{a}_i = 0\ \forall i$, and consider a as the excess reserves of each

[3] Very similar can be found for other years, see Appendix C.

[4] We compute $\hat{\mathcal{B}}_i$ as the intermediation made by all banks in that group. This is $\hat{\mathcal{B}}_i = \frac{\sum_{\mathbf{i}\in i} \hat{m}_{\mathbf{i}}}{\sum_{\mathbf{j}} \hat{m}_{\mathbf{j}}}$. See Appendix X for more details.

bank.[5]. We compute compute $F_0^i(a)$ as the distribution of excess reserves at 9:00 AM across banks within each type, which we obtain using *reserves data* as explained in Section 7.1. Appendix C contains more details on the estimation of the distribution of $F_0^i(a)$.

We test the model predictions for several years. The only parameter we change across years is the initial distribution $F_0^i(a)$. Figure 2 shows the excess reserves distribution for each type of bank in years 2005 and 2018. There are two things worth noticing. First, during 2005, excess reserves were positive on average but largely concentrated around zero. The exception is for bank type F, whose reserves distribution is more disperse. Second, during 2018, most banks started their day in excess of reserves and only a few were short in reserves. This is the counterpart of the quantitative easing program implemented by the Fed. Importantly, banks type F are the ones whose reserves increased the most. This accumulation of reserves for F banks will be key in the policy discussions of Section 9.

8 Quantitative Analysis

In this section, we analyze the quantitative properties of the model and compare them with the empirical counterparts. We focus on two dimension. First, we analyze how banks intermediate to reallocate reserves among them. Second, we compute the distribution of interest rates that result from this intermediation. Finally, we put these two part pieces together to analyze the network of trades across banks. We evaluate the model in 2005 and 2018, and compute identical measures in the data.

Banks efficiently reallocate reserves from those who are in excess to those who are in need. Figure 3 shows the distribution of excess reserves for each bank type at different points in the day. The distributions are more disperse at the beginning of the day, and they concentrate around zero as they trading sessions go by. Interestingly, this reallocation pattern is also true for the 2018 calibration, even if most banks are long in reserves. Thus, banks trade among them and efficiently reallocate reserves.

Banks type F are largely responsible for the reallocation just discussed, despite of being a small fraction of total banks. For both 2005 and 2018, the measure of banks type F is less than 1% but they intermediated about 97% of all dollars traded, as Figure 4 shows. Relative to 2005, GSEs doubled their intermediation share in 2018, which the model captures

[5] In Appendix A, we show that normalizing $\bar{a}_i = 0$ is without loss of generality.

accurately. However, the model misses the decline in intermediation for S banks.

Banks may trade because they are short/long in reserves, or because they profit from reallocating reserves. Our reallocation measure $\mathcal{R}_i$ measures the trading motive: if $\mathcal{R}_i \approx 0$, banks are only reallocating reserves by selling roughly every dollar they buy; while $\mathcal{R}_i \approx \pm 1$ implies a banks is a net lender/borrower. Figure 5 shows the reallocation measure $\mathcal{R}_i$ for each bank type and year, in the data and the model

Banks type F are clear market-makers since they participate in most dollars transacted ($\mathcal{B}_F \approx 1$) but largely reallocate reserves across banks ($\mathcal{R}_F \approx 0$). On the contrary, GSEs are largely lenders ($\mathcal{R}_G \approx 1$) and they participate in fewer transactions (small $\mathcal{B}_G$). The S banks are the bulk of traders, their measure is 96% of total banks, but they trade less frequently and mostly to borrow (negative $\mathcal{R}_S$). Our model captures this trade dynamics in the Fed Funds market, and is thus suitable for policy analysis.

There is a rich distribution of interest rates transacted as a result of the heterogeneity of banks in the *fed funds market*. Figure 6 shows the cumulative density function of interest rate for years 2005 and 2018, for both model and data. We normalize the data (and the model) by subtracting the lowest rate paid by the Federal Reserve on that day—which was always zero in 2005, and the ONRRP in 2018.

For year 2018, the model generates about 50% of trades with rates between the IOR and the ONRRP. The model generates this because GSEs are willing to lend at any rate above the ONRRP and, as in the data, their share in the volume of intermediation increases. Trades with interest rates below IOR have been pervasive since the Great Recession, and the focus of recent policy debates. In particular, there is still uncertainty on when the IOR will be a floor on interest rates instead of a ceiling. We use the model to discuss this topic on Section 9.

Figure 7 puts together the pieces discussed so far in this Section by plotting the network of loans implied by the model. As can be seen, banks type F are at the core of the network loans and virtually all reallocation of reserves involves them.

9 Policy Analysis

Reserve balances held by banks at the Fed increased substantially during 2008-2014 as a result of the large-scale asset purchases program that the Fed implemented. Recently, the Fed has started to reduce its assets holdings with the consequent decrease in reserves balances. In this scenario, a natural question arises: what will happen with the Effective Fed Funds Rate (EFFR) during the normalization period? We tackle this question in this section.

We analyze the effect of a \$15 billions decrease in reserves, which was about 10% of the excess reserves by the end of 2018. We consider two possible scenarios for the reduction in reserves. First, an scenario in which all bank loose the same amount of reserves. We refer to this as *Scenario All*. We consider a second scenario in which only F banks experience a decline in reserves. We refer to this as *Scenario F*. Interest rates are kept constant.

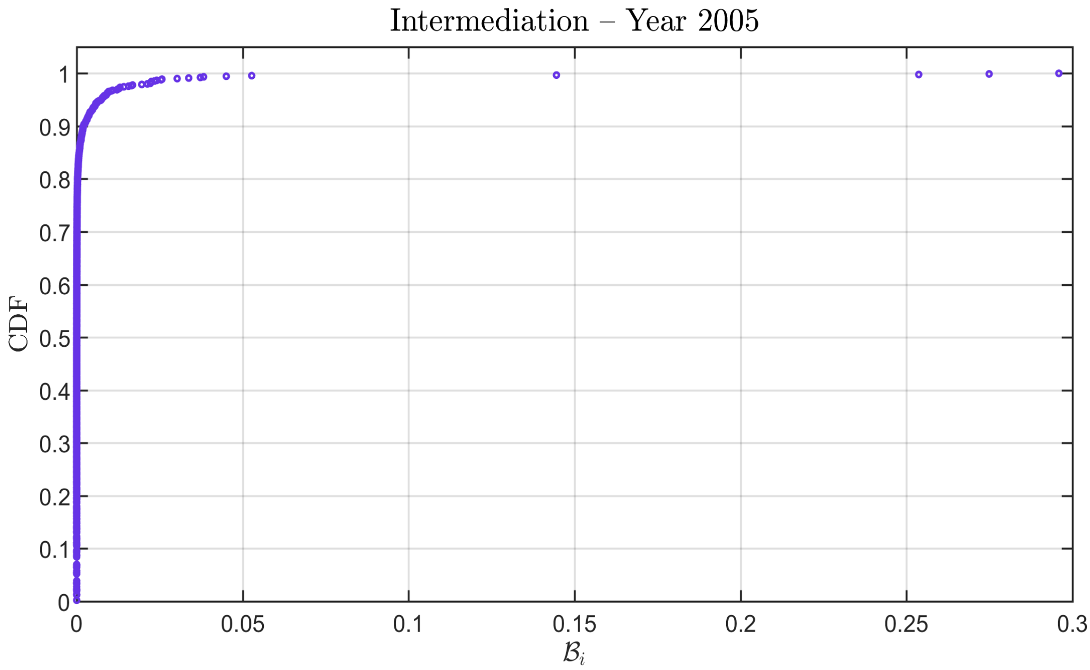
Our model predicts that the dispersion of interest rates will increase substantially more under *Scenario F*, although the EFFR will increases more under *Scenario All*. Figure 8 shows the box plot of interest rate in each scenario—as well as the model prediction during 2018 for comparisons. The intuition for these results are as follows. The model predicts a high intermediation level of banks type S (Figure 4), which is the bank type shorter in reserves (Figure 2). Consequently, *Scenario All* implies a larger decline in the “effective” supply of reserves—in line with the rationale discussed in Section 5—and thus a higher increase in interest rates. In *Scenario F*, reserves decline for banks type F, who are less active. Thus, the “effective” decline in the supply of reserves is smaller in *Scenario F* and the EFFR increases by less.[6] However, banks type F are now more likely to end with negative excess reserves, and thus decide to hoard reserves more and exploit less the potential profits of reallocating reserves. Since banks type F are the market-makers, their decline in trading induces a higher dispersion of interest rates.

10 Conclusions

We developed a model of the federal funds market which contains its salient institutional featues. The model generates patterns of trade across banks that resembles those of the data. We used the model to analyze the effects of a balance sheet normalization. We argue that the resulting increase in interest rates will depend on how the normalization changes the distribution of reserves across banks.

[6] We understand this results may revert if we were to match the data counterpart of Figure 4 for 2018 better.

Figure 1: Intermediation $\hat{\mathcal{B}}_i$ – Empirical CDF



11 Figures

Figure 2: Excess Reserves Distribution

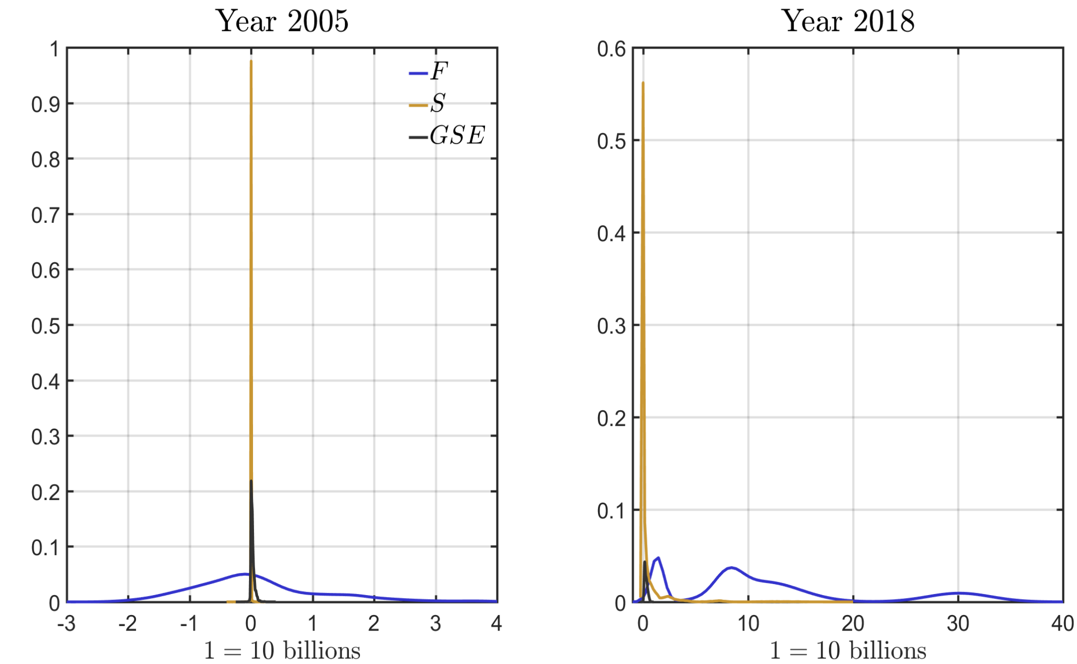


Figure 3: Excess Reserves Distribution during Trading Sessions

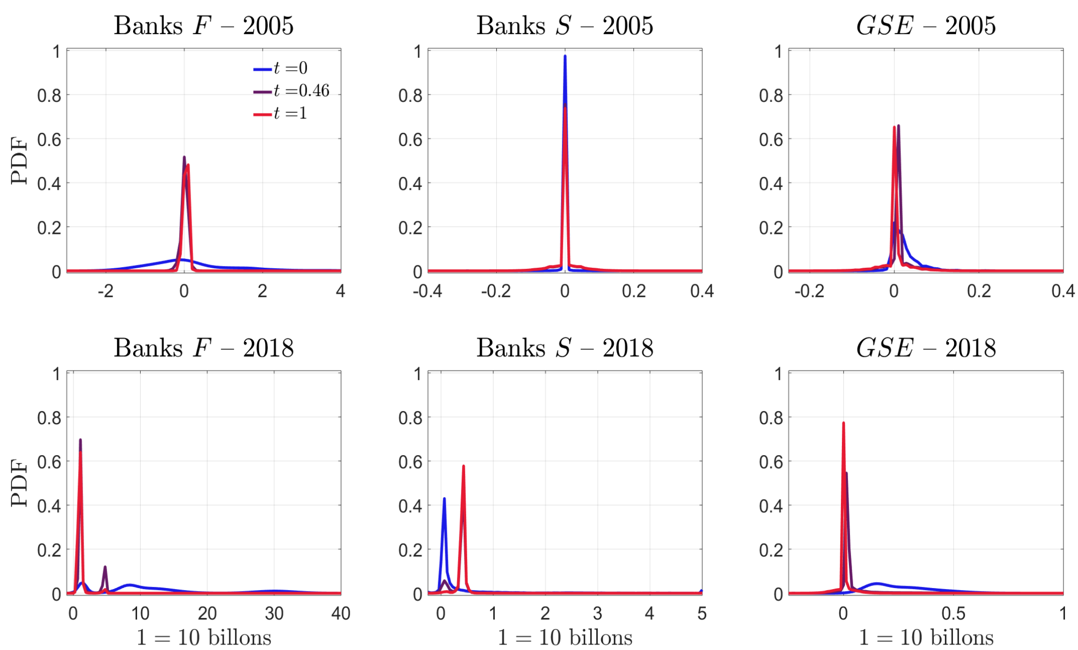


Figure 4: Intermediation $\mathcal{B}_i$ by bank type

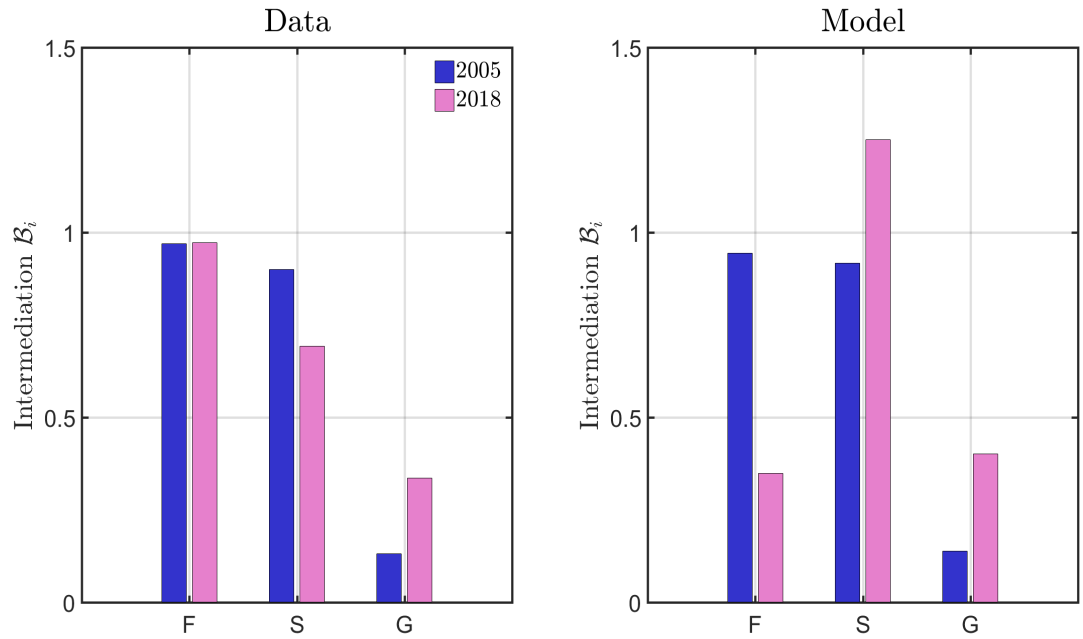


Figure 5: Reallocation $\mathcal{R}_i$ by bank type

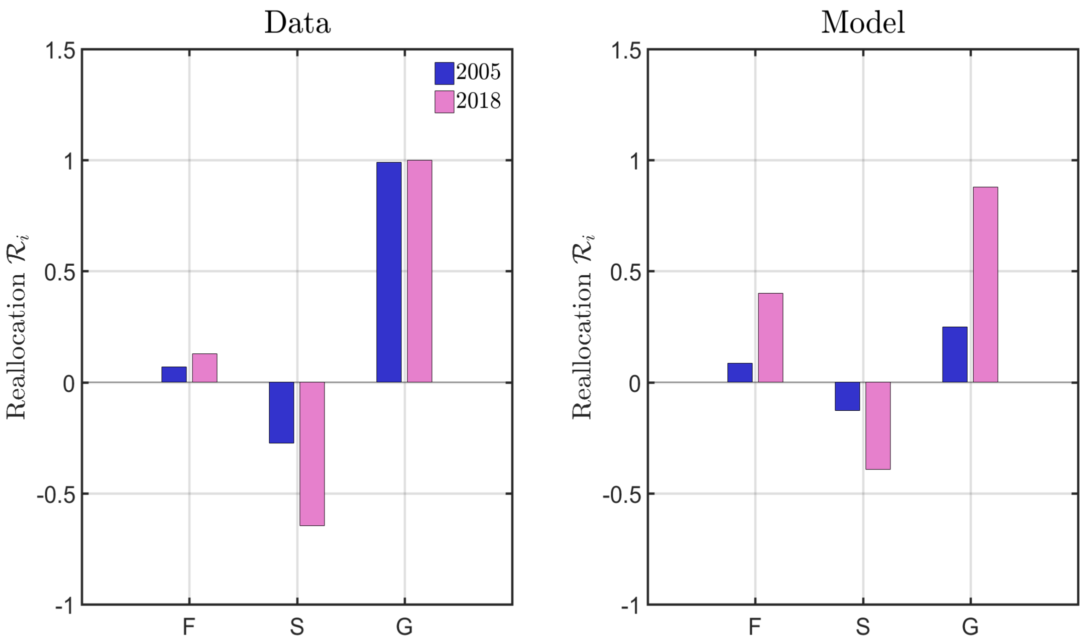


Figure 6: Interest rate distribution – data and model

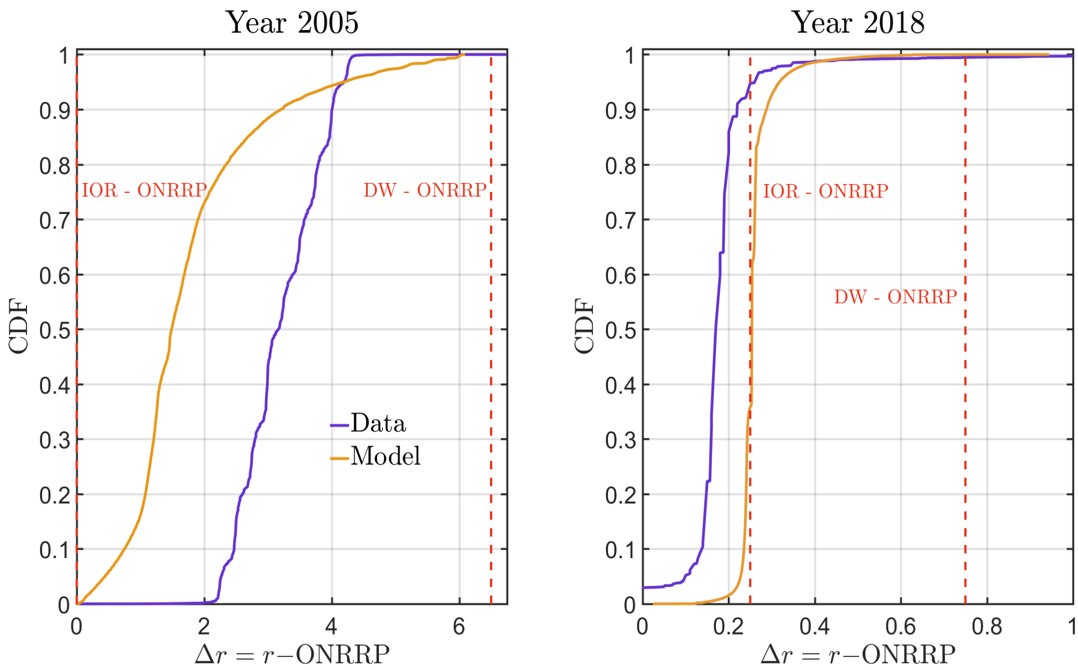
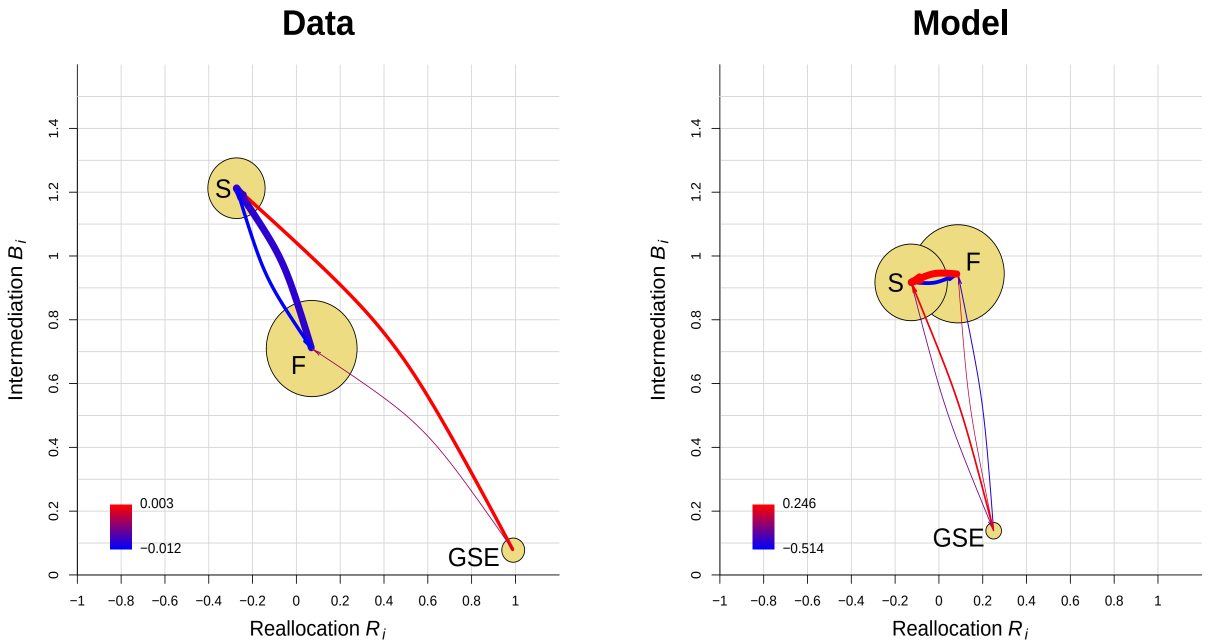


Figure 7: Network of Loans

(a) Year 2005



(b) Year 2018

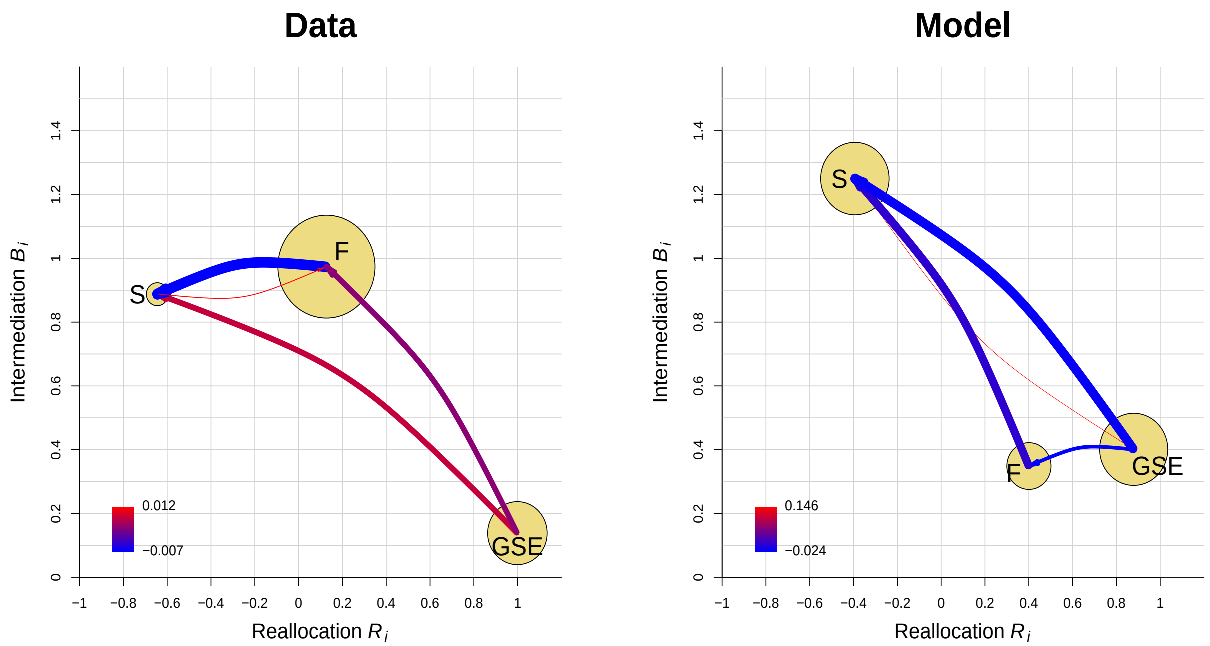
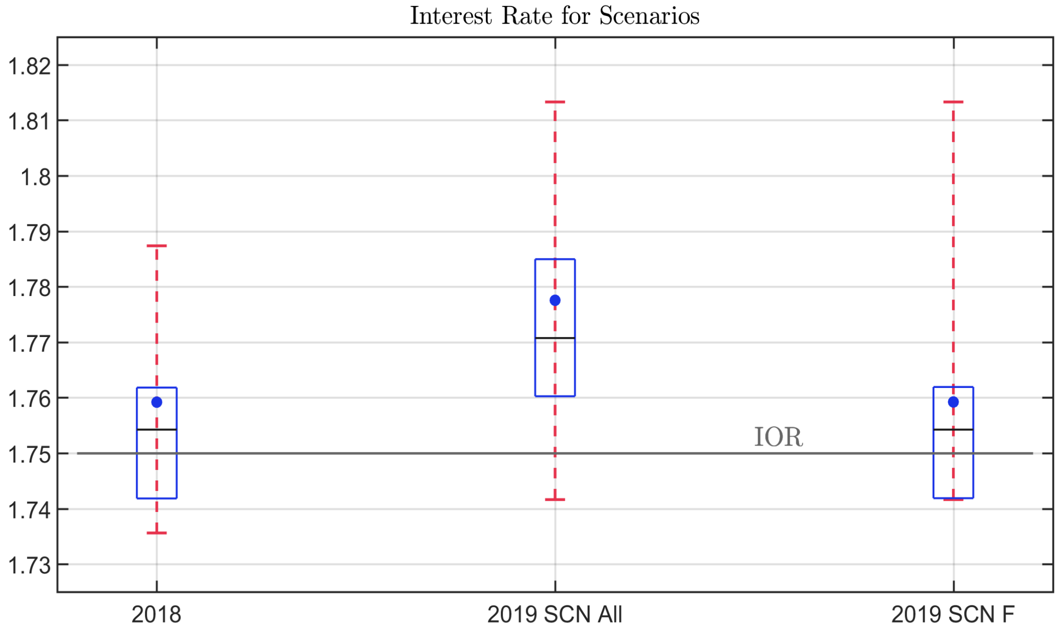


Figure 8: Interest rate for each Scenario



A Proofs

A.1 Value function

Let $J_t^i(a,c) : \mathbb{N} \times \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ denote the maximum attainable payoff to a bank of type i that at time $t \in \mathbb{T}$ has reserve balance $a \in \mathbb{R}$ and net credit position $c \in \mathbb{R}$. Then, $J_t^i(a,c)$ satisfies

$$
\begin{aligned}
&J_t^i(a,c) \\
&= \mathbb{E}_t \Bigg\{ \mathbb{I}_{\{T-t \le \min[\tau(\alpha_i),\tau(\beta_i),\tau(\lambda_i)]\}} \Bigg[\int_t^T e^{-r(s-t)} u^i(a)\, ds \\
&+ e^{-r(T-t)} \left[U^i(a) + e^{-r(\bar{T}-T)} c \right] \Bigg] \\
&+ \mathbb{I}_{\{\tau(\lambda_i) < \min[\tau(\alpha_i),\tau(\beta_i),T-t]\}} \Bigg[\int_t^{t+\tau(\lambda_i)} e^{-r(s-t)} u^i(a)\, ds \\
&+ e^{-r\tau(\lambda_i)} \sum_{j\in\mathbb{N}} \pi_j \int J_{t+\tau(\lambda_i)}^i (a-z,c)\, dG^{ij}(z) \Bigg] \\
&+ \mathbb{I}_{\{\tau(\alpha_i) < \min[\tau(\beta_i),\tau(\lambda_i),T-t]\}} \Bigg[\int_t^{t+\tau(\alpha_i)} e^{-r(s-t)} u^i(a)\, ds \\
&+ e^{-r\tau(\alpha_i)} J_{t+\tau(\alpha_i)}^i \left[a_{t+\tau(\alpha_i)}^{*i}, c - \rho_{t+\tau(\alpha_i)} (a_{t+\tau(\alpha_i)}^{*i} - a) - \phi_{t+\tau(\alpha_i)}^{*i} \right] \Bigg] \\
&+ \mathbb{I}_{\{\tau(\beta_i) < \min[\tau(\alpha_i),\tau(\lambda_i),T-t]\}} \Bigg[\int_t^{t+\tau(\beta_i)} e^{-r(s-t)} u^i(a)\, ds \\
&+ e^{-r\tau(\beta_i)} \sum_{j\in\mathbb{N}} \sigma_j \int J_{t+\tau(\beta_i)}^i \left[a - b_{t+\tau(\beta_i)}^{ij}(a,\tilde{a}), c + R_{t+\tau(\beta_i)}^{ji}(\tilde{a},a) \right] dF_{t+\tau(\beta_i)}^j(\tilde{a}) \Bigg] \Bigg\}, \qquad (50)
\end{aligned}
$$

where

$$\left(a_t^{*i},\phi_t^{*i}\right)=\arg\max_{(a',\phi)\in\mathbb{R}^2}\left[J_t^i\left(a',c-\rho_t(a'-a)-\phi\right)-J_t^i\left(a,c\right)\right]^{\theta_i}[e^{-r(\bar{T}-t)}\phi]^{1-\theta_i} \tag{51}$$

and

$$\begin{aligned}\left(b_t^{ij}\left(a,\tilde{a}\right),R_t^{ji}\left(\tilde{a},a\right)\right)=\arg\max_{(b,R)\in\mathbb{R}^2}\Big\{&\left[J_t^i\left(a-b,c+R\right)-J_t^i\left(a,c\right)\right]^{\theta_{ij}}\\&[J_t^j\left(\tilde{a}+b,c-R\right)-J_t^j\left(\tilde{a},c\right)]^{\theta_{ji}}\Big\}.\end{aligned} \tag{52}$$

Lemma 3 *The function*

$$J_t^i\left(a,c\right)=V_t^i\left(a\right)+e^{-r(\bar{T}-t)}c \tag{53}$$

satisfies (50) if and only if $V_t^i\left(a\right)$ satisfies (6), with (1), (2), (3), and (4).

Proof. With (53), (51) becomes equivalent to (1) and (2), and (52) becomes equivalent to (3) and (4). Substitute (53) into (50) to get

$$\begin{aligned}&V_t^i\left(a\right)+e^{-r(\bar{T}-t)}c\\
&=\mathbb{E}_t\Bigg\{\mathbb{I}_{\{T-t\leq\min[\tau(\alpha_i),\tau(\beta_i),\tau(\lambda_i)]\}}\Bigg[\int_t^T e^{-r(s-t)}u^i\left(a\right)ds\\
&+e^{-r(T-t)}\left[U^i\left(a\right)+e^{-r(\bar{T}-T)}c\right]\Bigg]\\
&+\mathbb{I}_{\{\tau(\lambda_i)<\min[\tau(\alpha_i),\tau(\beta_i),T-t]\}}\Bigg[\int_t^{t+\tau(\lambda_i)} e^{-r(s-t)}u^i\left(a\right)ds\\
&+e^{-r\tau(\lambda_i)}\sum_{j\in\mathbb{N}}\pi_j\int\left[V_{t+\tau(\lambda_i)}^i\left(a-z\right)+e^{-r\{\bar{T}-[t+\tau(\lambda_i)]\}}c\right]dG^{ij}\left(z\right)\Bigg]\\
&+\mathbb{I}_{\{\tau(\alpha_i)<\min[\tau(\beta_i),\tau(\lambda_i),T-t]\}}\Bigg[\int_t^{t+\tau(\alpha_i)} e^{-r(s-t)}u^i\left(a\right)ds\\
&+e^{-r\tau(\alpha_i)}\Bigg\{V_{t+\tau(\alpha_i)}^i(a_{t+\tau(\alpha_i)}^{*i})\\
&+e^{-r\left\{\bar{T}-[t+\tau(\alpha_i)]\right\}}\left[c-\rho_{t+\tau(\alpha_i)}(a_{t+\tau(\alpha_i)}^{*i}-a)-\phi_{t+\tau(\alpha_i)}^{*i}\right]\Bigg\}\Bigg]\\
&+\mathbb{I}_{\{\tau(\beta_i)<\min[\tau(\alpha_i),\tau(\lambda_i),T-t]\}}\Bigg[\int_t^{t+\tau(\beta_i)} e^{-r(s-t)}u^i\left(a\right)ds\\
&+e^{-r\tau(\beta_i)}\sum_{j\in\mathbb{N}}\sigma_j\int\left[V_{t+\tau(\beta_i)}^i(a-b_{t+\tau(\beta_i)}^{ij}\left(a,\tilde{a}\right))+e^{-r\{\bar{T}-[t+\tau(\beta_i)]\}}[c+R_{t+\tau(\beta_i)}^{ji}\left(\tilde{a},a\right)]\right]dF_{t+\tau(\beta_i)}^j\left(\tilde{a}\right)\Bigg]\Bigg\},\end{aligned}$$

which after cancelling the terms in c, becomes identical to (6). ■

Proof of Lemma 1. With the bargaining outcomes (1) and (2), and (3) and (4), (6) can be rewritten as

$$\begin{aligned}
V_t^i(a) = \mathbb{E}_t\Bigg\{ & \mathbb{I}_{\{T-t\le\min[\tau(\alpha_i),\tau(\beta_i),\tau(\lambda_i)]\}}\left[\int_t^T e^{-r(s-t)}u^i(a)\,ds + e^{-r(T-t)}U^i(a)\right] \\
& + \mathbb{I}_{\{\tau(\lambda_i)<\min[\tau(\alpha_i),\tau(\beta_i),T-t]\}}\Bigg[\int_t^{t+\tau(\lambda_i)} e^{-r(s-t)}u^i(a)\,ds \\
& + e^{-r\tau(\lambda_i)}\sum_{j\in\mathbb{N}}\pi_j\int V^i_{t+\tau(\lambda_i)}(a-z)\,dG^{ij}(z)\Bigg] \\
& + \mathbb{I}_{\{\tau(\alpha_i)<\min[\tau(\beta_i),\tau(\lambda_i),T-t]\}}\Bigg[\int_t^{t+\tau(\alpha_i)} e^{-r(s-t)}u^i(a)\,ds \\
& + e^{-r\tau(\alpha_i)}\left\{\theta_i\left[V^i_{t+\tau(\alpha_i)}(a^{*i}_{t+\tau(\alpha_i)}) - p_{t+\tau(\alpha_i)}(a^{*i}_{t+\tau(\alpha_i)}-a)\right] + (1-\theta_i)\,V^i_{t+\tau(\alpha_i)}(a)\right\}\Bigg] \\
& + \mathbb{I}_{\{\tau(\beta_i)<\min[\tau(\alpha_i),\tau(\lambda_i),T-t]\}}\Bigg[\int_t^{t+\tau(\beta_i)} e^{-r(s-t)}u^i(a)\,ds \\
& + e^{-r\tau(\beta_i)}\sum_{j\in\mathbb{N}}\sigma_j\int\left[V^i_{t+\tau(\beta_i)}(a) + \theta_{ij}\max_{b\in\bar{\mathbb{R}}} S^{ij}_{t+\tau(\beta_i)}(a,\tilde{a},b)\right]dF^j_{t+\tau(\beta_i)}(\tilde{a})\Bigg]\Bigg\},
\end{aligned}$$

where

$$S_t^{ij}(a,\tilde{a},b) \equiv V_t^i(a-b) + V_t^j(\tilde{a}+b) - V_t^i(a) - V_t^j(\tilde{a}).$$

The first term on the right side of $V_t^i(a)$ can be written as

$$\begin{aligned}
& \mathbb{E}_t\left\{\mathbb{I}_{\{T-t\le\min[\tau(\alpha_i),\tau(\beta_i),\tau(\lambda_i)]\}}\left[\int_t^T e^{-r(s-t)}u^i(a)\,ds + e^{-r(T-t)}U^i(a)\right]\right\} \\
& = e^{-(\alpha_i+\beta_i+\lambda_i)(T-t)}\left\{\left[1-e^{-r(T-t)}\right]\frac{u^i(a)}{r} + e^{-r(T-t)}U^i(a)\right\}.
\end{aligned}$$

The second term on the right side of $V_t^i(a)$ can be written as

$$\begin{aligned}
& \mathbb{E}_t\left\{\mathbb{I}_{\{\tau(\lambda_i)<\min[\tau(\alpha_i),\tau(\beta_i),T-t]\}}\left[\int_t^{t+\tau(\lambda_i)} e^{-r(s-t)}u^i(a)\,ds + e^{-r\tau(\lambda_i)}\sum_{j\in\mathbb{N}}\pi_j\int V^i_{t+\tau(\lambda_i)}(a-z)\,dG^{ij}(z)\right]\right\} \\
& = \frac{\lambda_i}{\alpha_i+\beta_i+\lambda_i}\,\frac{r\left[1-e^{-(\alpha_i+\beta_i+\lambda_i)(T-t)}\right] - (\alpha_i+\beta_i+\lambda_i)\,e^{-(\alpha_i+\beta_i+\lambda_i)(T-t)}\left[1-e^{-r(T-t)}\right]}{r+\alpha_i+\beta_i+\lambda_i}\,\frac{u^i(a)}{r} \\
& + \int_0^{T-t}\lambda_i e^{-(r+\alpha_i+\beta_i+\lambda_i)y}\left[\sum_{j\in\mathbb{N}}\pi_j\int V^i_{t+y}(a-z)\,dG^{ij}(z)\right]dy.
\end{aligned}$$

The third term on the right side of $V_t^i(a)$ can be written as

$$\mathbb{E}_t\left\{\mathbb{I}_{\{\tau(\alpha_i)<\min[\tau(\beta_i),\tau(\lambda_i),T-t]\}}\left[\int_t^{t+\tau(\alpha_i)} e^{-r(s-t)}u^i(a)\,ds\right.\right.$$
$$\left.\left.+\,e^{-r\tau(\alpha_i)}\left\{\theta_i\left[V^i_{t+\tau(\alpha_i)}(a^{*i}_{t+\tau(\alpha_i)})-p_{t+\tau(\alpha_i)}(a^{*i}_{t+\tau(\alpha_i)}-a)\right]+(1-\theta_i)\,V^i_{t+\tau(\alpha_i)}(a)\right\}\right]\right\}$$
$$=\frac{\alpha_i}{\alpha_i+\beta_i+\lambda_i}\frac{r\left[1-e^{-(\alpha_i+\beta_i+\lambda_i)(T-t)}\right]-(\alpha_i+\beta_i+\lambda_i)\,e^{-(\alpha_i+\beta_i+\lambda_i)(T-t)}\left[1-e^{-r(T-t)}\right]}{r+\alpha_i+\beta_i+\lambda_i}\frac{u^i(a)}{r}$$
$$+\int_0^{T-t}\alpha_i e^{-(r+\alpha_i+\beta_i+\lambda_i)x}\left\{\theta_i\left[V^i_{t+x}(a^{*i}_{t+x})-p_{t+x}(a^{*i}_{t+x}-a)\right]+(1-\theta_i)\,V^i_{t+x}(a)\right\}dx.$$

The fourth term on the right side of $V_t^i(a)$ can be written as

$$V_t^i(a)=\mathbb{E}_t\left\{\mathbb{I}_{\{\tau(\beta_i)<\tau(\alpha_i)\}}\mathbb{I}_{\{\tau(\beta_i)<\tau(\lambda_i)\}}\mathbb{I}_{\{\tau(\beta_i)<T-t\}}\left[\int_t^{t+\tau(\beta_i)} e^{-r(s-t)}u^i(a)\,ds\right.\right.$$
$$\left.\left.+\,e^{-r\tau(\beta_i)}\sum_{j\in\mathbb{N}}\sigma_j\int\left[V^i_{t+\tau(\beta_i)}(a)+\theta_{ij}\max_{b\in\bar{\mathbb{R}}}S^{ij}_{t+\tau(\beta_i)}(a,\tilde{a},b)\right]dF^j_t(\tilde{a})\right]\right\}$$
$$=\frac{\beta_i}{\alpha_i+\beta_i+\lambda_i}\frac{r\left[1-e^{-(\alpha_i+\beta_i+\lambda_i)(T-t)}\right]-(\alpha_i+\beta_i+\lambda_i)\,e^{-(\alpha_i+\beta_i+\lambda_i)(T-t)}\left[1-e^{-r(T-t)}\right]}{r+\alpha_i+\beta_i+\lambda_i}\frac{u^i(a)}{r}$$
$$+\int_0^{T-t}\beta_i e^{-(r+\alpha_i+\beta_i+\lambda_i)z}\left[\sum_{j\in\mathbb{N}}\sigma_j\int\left[V^i_{t+z}(a)+\theta_{ij}\max_{b\in\bar{\mathbb{R}}}S^{ij}_{t+z}(a,\tilde{a},b)\right]dF^j_{t+z}(\tilde{a})\right]dz.$$

Thus we can write

$$V_t^i(a)=\left[1-e^{-(r+\alpha_i+\beta_i+\lambda_i)(T-t)}\right]\frac{u^i(a)}{r+\alpha_i+\beta_i+\lambda_i}+e^{-(r+\alpha_i+\beta_i+\lambda_i)(T-t)}U^i(a)$$
$$+\lambda_i\int_0^{T-t}e^{-(r+\alpha_i+\beta_i+\lambda_i)y}\left[\sum_{j\in\mathbb{N}}\pi_j\int V^i_{t+y}(a-s)\,dG^{ij}(s)\right]dy$$
$$+\alpha_i\int_0^{T-t}e^{-(r+\alpha_i+\beta_i+\lambda_i)x}\left\{\theta_i\max_{a'\in\mathbb{R}}\left[V^i_{t+x}(a')-p_{t+x}(a'-a)\right]+(1-\theta_i)\,V^i_{t+x}(a)\right\}dx$$
$$+\beta_i\int_0^{T-t}e^{-(r+\alpha_i+\beta_i+\lambda_i)z}\left[\sum_{j\in\mathbb{N}}\sigma_j\int\left[V^i_{t+z}(a)+\theta_{ij}\max_{b\in\bar{\mathbb{R}}}S^{ij}_{t+z}(a,\tilde{a},b)\right]dF^j_{t+z}(\tilde{a})\right]dz.$$

With a change of variables in the integrals with respect to time,

$$\begin{aligned}
V_t^i(a) &= \left[1 - e^{-(r+\alpha_i+\beta_i+\lambda_i)(T-t)}\right] \frac{u^i(a)}{r+\alpha_i+\beta_i+\lambda_i} + e^{-(r+\alpha_i+\beta_i+\lambda_i)(T-t)} U^i(a) \\
&+ \lambda_i \int_t^T e^{-(r+\alpha_i+\beta_i+\lambda_i)(\tau-t)} \left[\sum_{j\in\mathbb{N}} \pi_j \int V_\tau^i(a-z)\, dG^{ij}(z)\right] d\tau \\
&+ \alpha_i \int_t^T e^{-(r+\alpha_i+\beta_i+\lambda_i)(\tau-t)} \left[\theta_i \max_{a'\in\mathbb{R}} \left[V_\tau^i(a') - p_\tau(a'-a)\right] + (1-\theta_i) V_\tau^i(a)\right] d\tau \\
&+ \beta_i \int_t^T e^{-(r+\alpha_i+\beta_i+\lambda_i)(\tau-t)} \left[V_\tau^i(a) + \sum_{j\in\mathbb{N}} \sigma_j \theta_{ij} \int \max_{b\in\bar{\mathbb{R}}} S_\tau^{ij}(a,\tilde{a},b)\, dF_\tau^j(\tilde{a})\right] d\tau. \qquad (54)
\end{aligned}$$

As in Lagos and Rocheteau (2009), the value function that satisfies (54) is the same value function that would correspond to the expected discounted payoff in an alternative environment where a bank of type i meets brokers according to a Poisson process with arrival rate α_i, but instead of bargaining, he readjusts his asset position and extracts the whole surplus with probability θ_i, whereas with probability $1-\theta_i$, the bank cannot change its reserve balance and therefore enjoys no gain from trade. Thus, from the standpoint of the individual bank, keeping the paths of the aggregate variables unchanged, the environment in which the bank has contact rate α_i and Nash bargaining parameter θ_i is payoff-equivalent to an alternative environment in which the bank meets brokers according to a Poisson process with arrival rate $\alpha_i\theta_i$ and has all the bargaining power in bilateral negotiations with brokers. This observation leads to (7). To obtain (8), simply differentiate (7) with respect to t. ■

Proof Lemma 2. Substitute (9) into (7) to obtain

$$\begin{aligned}
v_t^i(a) + q_t^i a &= \left[1 - e^{-(r+\kappa_i+\beta_i+\lambda_i)(T-t)}\right] \frac{u^i(a)}{r+\kappa_i+\beta_i+\lambda_i} + e^{-(r+\kappa_i+\beta_i+\lambda_i)(T-t)} U^i(a) \\
&+ \int_t^T e^{-(r+\kappa_i+\beta_i+\lambda_i)(\tau-t)} \Bigg\{ \beta_i \left[v_\tau^i(a) + \sum_{j\in\mathbb{N}} \sigma_j \theta_{ij} \int \max_{b\in\bar{\mathbb{R}}} s_\tau^{ij}(a,\tilde{a},b)\, dF_\tau^j(\tilde{a})\right] \\
&+ \lambda_i \sum_{j\in\mathbb{N}} \pi_j \int v_\tau^i(a-z)\, dG^{ij}(z) \Bigg\} d\tau \\
&+ \left[\int_t^T e^{-(r+\kappa_i+\beta_i+\lambda_i)(\tau-t)} \left[\kappa_i p_\tau + (\beta_i+\lambda_i) q_\tau^i\right] d\tau\right] a + K_t^i \qquad (55)
\end{aligned}$$

where with K_t^i defined as in (12). Thus, (55) implies (9) satisfies (7) if and only if q_t^i satisfies (10), and $v_t^i(a)$ satisfies (11).

Let $\mathfrak{B}$ denote the space of continuous and bounded real-valued functions defined on $\mathbb{N} \times \mathbb{T} \times \mathbb{R}$. For any $w, w' \in \mathfrak{B}$, define the metric $D : \mathfrak{B} \times \mathfrak{B} \to \mathbb{R}$ by

$$D\left(w, w'\right) = \sup_{(i,t,a) \in \mathbb{N} \times \mathbb{T} \times \mathbb{R}} e^{\delta t} \left| w\left(i, t, a\right) - w'\left(i, t, a\right) \right|$$

with

$$\delta = \max\left(0, 4\bar{\beta} - r\right), \tag{56}$$

where $\bar{\beta} \equiv \max_{i \in \mathbb{N}} \beta_i$. For the case with $\delta = 0$, D reduces to the standard sup metric, d_∞. The metric space $(\mathfrak{B}, d_\infty)$ is complete, and since $(\mathfrak{B}, d_\infty)$ and $(\mathfrak{B}, D)$ are strongly equivalent, it follows that $(\mathfrak{B}, D)$ is also a complete metric space (see Ok (2007, pp. 136 and 167). The right side of (11) defines a mapping $\mathcal{M}$ on $\mathfrak{B}$. That is, for any $w \in \mathfrak{B}$, and all $(i, t, a) \in \mathbb{N} \times \mathbb{T} \times \mathbb{R}$,

$$\begin{aligned}
(\mathcal{M}w)\left(i, t, a\right) = {} & \left[1 - e^{-(r+\kappa_i+\beta_i+\lambda_i)(T-t)}\right] \frac{u^i\left(a\right)}{r + \kappa_i + \beta_i + \lambda_i} + e^{-(r+\kappa_i+\beta_i+\lambda_i)(T-t)} U^i\left(a\right) \\
& + \lambda_i \int_t^T e^{-(r+\kappa_i+\beta_i+\lambda_i)(\tau-t)} \left[\sum_{j \in \mathbb{N}} \pi_j \int w\left(i, \tau, a - z\right) dG^{ij}\left(z\right)\right] d\tau \\
& - \lambda_i \left[\int_t^T e^{-(r+\kappa_i+\beta_i+\lambda_i)(\tau-t)} q_\tau^i d\tau\right] \bar{z}_i \\
& + \kappa_i \int_t^T e^{-(r+\kappa_i+\beta_i+\lambda_i)(\tau-t)} \max_{a' \in \mathbb{R}} \left[w\left(i, \tau, a'\right) - \left(p_\tau - q_\tau^i\right) a'\right] d\tau \\
& + \beta_i \int_t^T e^{-(r+\kappa_i+\beta_i+\lambda_i)(\tau-t)} w\left(i, \tau, a\right) d\tau \\
& + \beta_i \int_t^T e^{-(r+\kappa_i+\beta_i+\lambda_i)(\tau-t)} \left[\sum_{j \in \mathbb{N}} \sigma_j \theta_{ij} \int \max_{b \in \bar{\mathbb{R}}} \mathcal{S}_\tau^{ij}\left(a, \tilde{a}, b; w\right) dF_\tau^j\left(\tilde{a}\right)\right] d\tau
\end{aligned}$$

where

$$\mathcal{S}_\tau^{ij}\left(a, \tilde{a}, b; w\right) \equiv w\left(i, \tau, a - b\right) + w\left(j, \tau, \tilde{a} + b\right) + (q_\tau^j - q_\tau^i) b - w\left(i, \tau, a\right) - w\left(j, \tau, \tilde{a}\right).$$

Given $u^i, U^i \in \mathfrak{B}$, $w \in \mathfrak{B}$ and $0 \leq p_t - q_t^i$ for all $t \in \mathbb{T}$ (a condition that will hold in any equilibrium) imply $\max_{a' \in \mathbb{R}} \left[w\left(i, \tau, a'\right) - \left(p_\tau - q_\tau^i\right) a'\right] < \infty$ for all $(i, \tau) \in \mathbb{N} \times \mathbb{T}$. Also, since $w \in \mathfrak{B}$,

$$\sum_{j \in \mathbb{N}} \sigma_j \theta_{ij} \int \max_{b \in \bar{\mathbb{R}}} \mathcal{S}_\tau^{ij}\left(a, \tilde{a}, b; w\right) dF_\tau^j\left(\tilde{a}\right)$$

is a continuous and bounded function of a. Hence, $\mathcal{M} : \mathfrak{B} \to \mathfrak{B}$.

For any $w, w' \in \mathfrak{B}$,

$$
\begin{aligned}
&\left|\left(\mathcal{M}w\right)(i,t,a)-\left(\mathcal{M}w'\right)(i,t,a)\right| \\
&= \Bigg|\lambda_i \int_t^T e^{-(r+\kappa_i+\beta_i+\lambda_i)(\tau-t)} \left[\sum_{j\in\mathbb{N}} \pi_j \int w(i,\tau,a-z)\, dG^{ij}(z)\right] d\tau \\
&- \lambda_i \int_t^T e^{-(r+\kappa_i+\beta_i+\lambda_i)(\tau-t)} \left[\sum_{j\in\mathbb{N}} \pi_j \int w'(i,\tau,a-z)\, dG^{ij}(z)\right] d\tau \\
&+ \kappa_i \int_t^T e^{-(r+\kappa_i+\beta_i+\lambda_i)(\tau-t)} \max_{x\in\mathbb{R}} \left[w(i,\tau,x)-\left(p_\tau - q_\tau^i\right)x\right] d\tau \\
&- \kappa_i \int_t^T e^{-(r+\kappa_i+\beta_i+\lambda_i)(\tau-t)} \max_{x\in\mathbb{R}} \left[w'(i,\tau,x)-\left(p_\tau - q_\tau^i\right)x\right] d\tau \\
&+ \beta_i \int_t^T e^{-(r+\kappa_i+\beta_i+\lambda_i)(\tau-t)} w(i,\tau,a)\, d\tau \\
&- \beta_i \int_t^T e^{-(r+\kappa_i+\beta_i+\lambda_i)(\tau-t)} w'(i,\tau,a)\, d\tau \\
&+ \beta_i \int_t^T e^{-(r+\kappa_i+\beta_i+\lambda_i)(\tau-t)} \left[\sum_{j\in\mathbb{N}} \sigma_j \theta_{ij} \int \max_{b\in\bar{\mathbb{R}}} \mathcal{S}_\tau^{ij}(a,\tilde{a},b;w)\, dF_\tau^j(\tilde{a})\right] d\tau \\
&-\beta_i \int_t^T e^{-(r+\kappa_i+\beta_i+\lambda_i)(\tau-t)} \left[\sum_{j\in\mathbb{N}} \sigma_j \theta_{ij} \int \max_{b\in\bar{\mathbb{R}}} \mathcal{S}_\tau^{ij}\left(a,\tilde{a},b;w'\right) dF_\tau^j(\tilde{a})\right]\Bigg|.
\end{aligned}
$$

Let

$$
a_\tau^i = \max_{x\in\mathbb{R}} \left[w(i,\tau,x)-\left(p_\tau - q_\tau^i\right)x\right]
$$

and

$$
\begin{aligned}
y_\tau^{ij}(a,\tilde{a}) &= a - b_\tau^{ij}(a,\tilde{a}) \\
y_\tau^{ji}(\tilde{a},a) &= \tilde{a} + b_\tau^{ij}(a,\tilde{a}),
\end{aligned}
$$

where

$$
b_\tau^{ij}(a,\tilde{a}) = \arg\max_{b\in\bar{\mathbb{R}}} \mathcal{S}_\tau^{ij}(a,\tilde{a},b;w).
$$

Then,

$$\begin{aligned}
&e^{\delta t}\left|(\mathcal{M}w)(i,t,a)-\left(\mathcal{M}w'\right)(i,t,a)\right| \\
&\leq \lambda_i\int_t^T\sum_{j\in\mathbb{N}}\pi_j\int e^{\delta\tau}\left|w(i,\tau,a-z)-w'(i,\tau,a-z)\right|dG^{ij}(z)\,e^{-(r+\kappa_i+\beta_i+\lambda_i+\delta)(\tau-t)}d\tau \\
&+\kappa_i\int_t^T e^{\delta\tau}\left|w\left(i,\tau,a_\tau^i\right)-w'\left(i,\tau,a_\tau^i\right)\right|e^{-(r+\kappa_i+\beta_i+\lambda_i+\delta)(\tau-t)}d\tau \\
&+\beta_i\int_t^T e^{\delta\tau}\left|w(i,\tau,a)-w'(i,\tau,a)\right|e^{-(r+\kappa_i+\beta_i+\lambda_i+\delta)(\tau-t)}d\tau \\
&+\beta_i\int_t^T\Bigg\{\int\sum_{j\in\mathbb{N}}\sigma_j\theta_{ij}\Bigg[e^{\delta\tau}\left|w(i,\tau,y_\tau^{ij}(a,\tilde{a}))-w'(i,\tau,y_\tau^{ij}(a,\tilde{a}))\right| \\
&+e^{\delta\tau}\left|w(j,\tau,\tilde{a}+y_\tau^{ji}(\tilde{a},a))-w'(j,\tau,\tilde{a}+y_\tau^{ji}(\tilde{a},a))\right| \\
&+e^{\delta\tau}\left|w'(i,\tau,a)-w(i,\tau,a)\right| \\
&+e^{\delta\tau}\left|w'(j,\tau,\tilde{a})-w(j,\tau,\tilde{a})\right|\Bigg]dF_\tau^j(\tilde{a})\Bigg\}e^{-(r+\kappa_i+\beta_i+\lambda_i+\delta)(\tau-t)}d\tau \\
&\leq\frac{\kappa_i+\lambda_i+5\beta_i}{r+\kappa_i+\beta_i+\lambda_i+\delta}D\left(w,w'\right),
\end{aligned}$$

which implies

$$e^{\delta t}\left|(\mathcal{M}w)(i,t,a)-\left(\mathcal{M}w'\right)(i,t,a)\right|\leq mD\left(w,w'\right) \tag{57}$$

with

$$m\equiv\max_{i\in\mathbb{N}}\frac{\kappa_i+\lambda_i+5\beta_i}{r+\kappa_i+\beta_i+\lambda_i+\delta}.$$

Since (57) holds for all $(i,a,t)\in\mathbb{N}\times\mathbb{T}\times\mathbb{R}$, and w and w' are arbitrary, we have

$$D\left(\mathcal{M}w,\mathcal{M}w'\right)\leq mD\left(w,w'\right)\text{ for all }w,w'\in\mathfrak{B}.$$

Finally, notice (56) implies $\frac{\kappa_i+\lambda_i+5\beta_i}{r+\kappa_i+\beta_i+\lambda_i+\delta}<1$ for all $i\in\mathbb{N}$, and therefore $m\in(0,1)$, so $\mathcal{M}$ is a contraction mapping on the complete metric space $(\mathfrak{B},D)$. By the Contraction Mapping Theorem (Theorem 3.2 in Stokey and Lucas (1989)), for any given path $\{\boldsymbol{F}_t\}_{t\in\mathbb{T}}$, there exists a unique $w^*\in\mathfrak{B}$ that satisfies $\mathcal{M}w^*=w^*$. In the body of the paper, this fixed point is denoted $\{\boldsymbol{v}_t(a)\}_{t\in\mathbb{T},a\in\mathbb{R}}$. The remaining properties listed in the statement of the lemma follow from an application of Corollary 1 in Stokey and Lucas (1989, p. 52). ■

Lemma 4 *Condition (10) is equivalent to*

$$\dot{q}_t^i = (r + \kappa_i)\, q_t^i - \kappa_i p_t \tag{58}$$

with boundary condition $q_T^i = 0$*, and also to*

$$q_t^i = \kappa_i \int_t^T p_s e^{-(r+\kappa_i)(s-t)} ds \tag{59}$$

Proof. Differentiate (10) with respect to t to obtain (58). Then integrate (58) and use the boundary condition $q_T^i = 0$ to obtain (59). ■

A.2 Pure brokered market

Proof of Proposition 1. Under the conditions in the statement of the proposition, by Lemma 2 there exists a unique $v_t^i(a)$ that satisfies (11), and it is continuous, bounded, strictly increasing, and strictly concave. Hence the maximization on the right of (14) has a unique solution, namely

$$a_t^{*i} = h_t^i\left(p_t - q_t^i\right), \tag{60}$$

where for every $(i,t) \in \mathbb{N} \times \mathbb{T}$, the decision rule $h_t^i : \mathbb{R} \to \mathbb{R}$ is strictly decreasing and smooth. Condition (15) then becomes

$$\sum_{i \in \mathbb{N}} \alpha_i \left[n_i h_t^i\left(p_t - q_t^i\right) - A_t^i\right] = 0. \tag{61}$$

Taking $\boldsymbol{q}_t = \left(q_t^1, \ldots, q_t^N\right)$ and $\boldsymbol{A}_t = \left(A_t^1, \ldots, A_t^N\right)$ for each $t \in \mathbb{T}$ as given, we have

$$\lim_{p_t \to \infty} \sum_{i \in \mathbb{N}} \alpha_i \left[n_i h_t^i\left(p_t - q_t^i\right) - A_t^i\right] < 0 < \lim_{p_t \to 0} \sum_{i \in \mathbb{N}} \alpha_i \left[n_i h_t^i\left(p_t - q_t^i\right) - A_t^i\right],$$

and since h_t^i is strictly decreasing and continuous for all $i \in \mathbb{N}$, (61) implies there exists a unique

$$p_t = p_t^*\left(\boldsymbol{q}_t, \boldsymbol{A}_t\right) \tag{62}$$

that satisfies (15) for each $t \in \mathbb{T}$. Condition (10) is equivalent to $\dot{q}_t^i = (r + \kappa_i)\, q_t^i - \kappa_i p_t$ with $q_T^i = 0$ for all $i \in \mathbb{N}$ (see Lemma 4), so using (62) we obtain (33), with $\boldsymbol{q}_T = \mathbf{0}$. Given $p_t^*\left(\boldsymbol{q}_t, \boldsymbol{A}_t\right)$ and $A_t^{*i} = n_i a_t^{*i} = n_i h_t^i\left(p_t^*\left(\boldsymbol{q}_t, \boldsymbol{A}_t\right) - q_t^i\right)$, the law of motion (20) can be written as (34), with $\boldsymbol{A}_0$ given. ■

Lemma 5 *Consider a time interval* $[t_0, t_1] \subseteq \mathbb{T}$, *with* $\left\{A^i_{t_0}, q^i_{t_1}\right\}_{i\in\mathbb{N}}$ *given.*

(i) Suppose $0 < \mathcal{R} - \mathcal{A}_t$ *and* $\mathbb{N}^-_t = \{N\}$ *for all* $t \in [t_0, t_1]$. *Then, for all* $t \in [t_0, t_1]$,

$$p_t = e^{-r(t_1-t)} \left[q^N_{t_1} + e^{-(r+\kappa_N)(T-t_1)}\mu_H\right]$$

and

$$q^i_t = e^{-r(t_1-t)} \left\{q^N_{t_1} + e^{-(r+\kappa_N)(T-t_1)} \left[1 - e^{-\kappa_i(t_1-t)}\right] \mu_H + e^{-\kappa_i(t_1-t)} \left(q^i_{t_1} - q^N_{t_1}\right)\right\}$$

for all $i \in \mathbb{N}$, *and the distribution of balances is*

$$A^i_t = \begin{cases} A^N_{t_0} - \sum_{i\in\mathbb{N}\setminus\{N\}} \left[1 - e^{-\alpha_i(t-t_0)}\right] \left(R^i - A^i_{t_0}\right) & \text{if } i = N \\ R^i - e^{-\alpha_i(t-t_0)} \left(R^i - A^i_{t_0}\right) & \text{if } i \in \mathbb{N}^0_t \text{ for all } t \in [t_0, t_1]\,. \end{cases} \tag{63}$$

(ii) Suppose $\mathcal{R} - \mathcal{A}_t < 0$ *and* $\mathbb{N}^+_t = \{N\}$ *for all* $t \in [t_0, t_1]$. *Then, for all* $t \in [t_0, t_1]$,

$$p_t = e^{-r(t_1-t)} \left[q^N_{t_1} + e^{-(r+\kappa_N)(T-t_1)}\mu_L\right].$$

and

$$q^i_t = e^{-r(t_1-t)} \left\{q^N_{t_1} + e^{-(r+\kappa_N)(T-t_1)} \left[1 - e^{-\kappa_i(t_1-t)}\right] \mu_L + e^{-\kappa_i(t_1-t)} \left(q^i_{t_1} - q^N_{t_1}\right)\right\}$$

for all $i \in \mathbb{N}$, *and the distribution of balances is given by (63).*

(iii) Suppose $\mathcal{R} - \mathcal{A}_t = 0$. *Then, for all* $t \in [t_0, t_1]$, p_t *and* q_t *are indeterminate, and the distribution of balances is given by*

$$A^i_t = R^i - e^{-\alpha_i(t-t_0)} \left(R^i - A^i_{t_0}\right) \tag{64}$$

for all $i \in \mathbb{N}$.

Proof. (i) Condition (39) implies $p_t = \bar{p}_t$ for all $t \in [t_0, t_1]$, or equivalently, $p_t = e^{-(r+\kappa_N)(T-t)}\mu_H + q^N_t$, where $N \in \mathbb{N}^-_t$. Then (10) implies

$$\begin{aligned} q^N_t &= e^{-r(t_1-t)} \left\{q^N_{t_1} + e^{-r(T-t_1)} \left[e^{-\kappa_N(T-t_1)} - e^{-\kappa_N(T-t)}\right] \mu_H\right\} \\ &= p_t - e^{-(r+\kappa_N)(T-t)}\mu_H \end{aligned}$$

for type N. Given q^N_t, the equilibrium price for $t \in [t_0, t_1]$ is

$$p_t = e^{-r(t_1-t)} \left[q^N_{t_1} + e^{-(r+\kappa_N)(T-t_1)}\mu_H\right].$$

Then for all types $i \notin \mathbb{N}_t^-$, (10) implies

$$\begin{aligned} q_t^i &= e^{-(r+\kappa_i)(t_1-t)} q_{t_1}^i + \left[1 - e^{-\kappa_i(t_1-t)}\right] e^{-r(t_1-t)} \left[q_{t_1}^N + e^{-(r+\kappa_N)(T-t_1)} \mu_H\right] \\ &= e^{-(r+\kappa_i)(t_1-t)} q_{t_1}^i + \left[1 - e^{-\kappa_i(t_1-t)}\right] p_t. \end{aligned}$$

For any i such that $i \in \mathbb{N}_t^0$ for all $t \in [t_0, t_1]$, (40) implies

$$A_t^i = e^{-\alpha_i(t-t_0)} A_{t_0}^i + \left[1 - e^{-\alpha_i(t-t_0)}\right] R^i,$$

and for $i = N$, (40) implies

$$A_t^N = A_{t_0}^N - \sum_{i \in \mathbb{N} \setminus \{N\}} \left[1 - e^{-\alpha_i(t-t_0)}\right] \left(R^i - A_{t_0}^i\right).$$

(ii) Condition (39) implies $p_t = \underline{p}_t$ for all $t \in [t_0, t_1]$, or equivalently, $p_t \equiv e^{-(r+\kappa_N)(T-t)} \mu_L + q_t^N$, where $N \in \mathbb{N}_t^+$. Then (10) implies

$$\begin{aligned} q_t^N &= e^{-r(t_1-t)} \left\{ q_{t_1}^N + e^{-r(T-t_1)} \left[e^{-\kappa_N(T-t_1)} - e^{-\kappa_N(T-t)}\right] \mu_L \right\} \\ &= p_t - e^{-(r+\kappa_N)(T-t)} \mu_L \end{aligned}$$

for type N. Given q_t^N, the equilibrium price for $t \in [t_0, t_1]$ is

$$p_t = e^{-r(t_1-t)} \left[q_{t_1}^N + e^{-(r+\kappa_N)(T-t_1)} \mu_L\right].$$

Then for all types $i \notin \mathbb{N}_t^+$, (10) implies

$$\begin{aligned} q_t^i &= e^{-(r+\kappa_i)(t_1-t)} q_{t_1}^i + \left[1 - e^{-\kappa_i(t_1-t)}\right] e^{-r(t_1-t)} \left[q_{t_1}^N + e^{-(r+\kappa_N)(T-t_1)} \mu_L\right] \\ &= e^{-(r+\kappa_i)(t_1-t)} q_{t_1}^i + \left[1 - e^{-\kappa_i(t_1-t)}\right] p_t. \end{aligned}$$

Condition (40) implies (63) just as in part *(i)*.

(iii) The fact that p_t is indeterminate is immediate from (39). Given p_t is indeterminate on $[t_0, t_1]$, (10) implies that so is q_t. The path for the distribution of balances is immediate from (40). ■

Corollary 1 *Consider a time interval $[t_0, t_1] \subseteq \mathbb{T}$, with $\left\{A_{t_0}^i, q_{t_1}^i\right\}_{i \in \mathbb{N}}$ given. Suppose that either $0 < \sum_{i \in \mathbb{N}} \alpha_i \left(R^i - A_t^i\right)$ and $\mathbb{N}_t^- = \{N\}$ for all $t \in [t_0, t_1]$, or $\sum_{i \in \mathbb{N}} \alpha_i \left(R^i - A_t^i\right) < 0$ and*

$\mathbb{N}_t^+ = \{N\}$ *for all* $t \in [t_0, t_1]$. *Then, for all* $t \in [t_0, t_1]$,

$$\mathcal{A}_t = \sum_{i \in \mathbb{N}} \alpha_i A_{t_0}^i - \sum_{i \in \mathbb{N}} (\alpha_N - \alpha_i) \left[1 - e^{-\alpha_i (t - t_0)}\right] \left(R^i - A_{t_0}^i\right)$$
$$\mathcal{R} - \mathcal{A}_t = \sum_{i \in \mathbb{N}} \left\{e^{-\alpha_i (t - t_0)} \alpha_i + \left[1 - e^{-\alpha_i (t - t_0)}\right] \alpha_N\right\} \left(R^i - A_{t_0}^i\right),$$

and

$$\dot{\mathcal{A}}_t = -\sum_{i \in \mathbb{N}} \alpha_i (\alpha_N - \alpha_i) e^{-\alpha_i (t - t_0)} \left(R^i - A_{t_0}^i\right).$$

Proof. Immediate from (63). ■

Proof of Proposition 2. In this case,

$$\begin{aligned} \mathcal{R} - \mathcal{A}_t &= \alpha_1 \left(R^1 - A_t^1\right) + \alpha_N \left(R^N - A_t^N\right) \\ &= \alpha_N (R - A) - (\alpha_N - \alpha_1) \left(R^1 - A_t^1\right) \end{aligned} \tag{65}$$

so $0 < \mathcal{R} - \mathcal{A}_0$ if and only if $-\alpha_1 \left(R^1 - A_0^1\right) < \alpha_N \left(R^N - A_0^N\right)$. We divide the analysis into the following cases. (A1) $0 < R - A$ and $0 < \mathcal{R} - \mathcal{A}_0$, (A2) $0 < R - A$ and $\mathcal{R} - \mathcal{A}_0 \leq 0$, (B1) $R - A < 0$ and $\mathcal{R} - \mathcal{A}_0 < 0$, (B2) $R - A < 0$ and $0 \leq \mathcal{R} - \mathcal{A}_0$, (C1) $R - A = 0$ and $\mathcal{R} - \mathcal{A}_0 < 0$, (C2) $R - A = 0$ and $0 \leq \mathcal{R} - \mathcal{A}_0$.

Case (A1). Conditions $0 < R - A$ and $0 < \mathcal{R} - \mathcal{A}_0$ imply

$$\max\left\{0, (\alpha_N - \alpha_1) \left(R^1 - A_0^1\right)\right\} < \alpha_N (R - A). \tag{66}$$

Guess that, for all $t \in \mathbb{T}$, the equilibrium has $p_t = \bar{p}_t$ (or equivalently, $0 < \mathcal{R} - \mathcal{A}_t$), and $2 \in \mathbb{N}_t^-$. From Corollary 1,

$$\mathcal{R} - \mathcal{A}_t = \alpha_N (R - A) - e^{-\alpha_1 t} (\alpha_N - \alpha_1) \left(R^1 - A_0^1\right) \tag{67}$$

for all $t \in \mathbb{T}$. Clearly, (66) and (67) imply $0 < \mathcal{R} - \mathcal{A}_t$ for all $t \in \mathbb{T}$, which confirms the conjecture that $p_t = \bar{p}_t$ for all $t \in \mathbb{T}$. Part (i) of Lemma 5 then implies $\left\{p_t, q_t^i\right\}_{i \in \mathbb{N}, t \in \mathbb{T}}$ are given by

$$p_t = e^{-r(T-t)} \mu_H \tag{68}$$
$$q_t^i = \left[1 - e^{-\kappa_i (T-t)}\right] e^{-r(T-t)} \mu_H. \tag{69}$$

Notice that (68) and (69) imply $P_t^1 = P_t^N = \mu_H$ for all $t \in \mathbb{T}$, which confirms the conjecture that $2 \in \mathbb{N}_t^-$ for all $t \in \mathbb{T}$. However, in this case it turns out we also have $1 \in \mathbb{N}_t^-$, i.e., $\mathbb{N}_t^- = \mathbb{N}$. This means the path for the distribution of balances $\{A_t^i\}_{i\in\mathbb{N},t\in\mathbb{T}}$ is indeterminate, i.e., for any $\epsilon \in [0,1]$, the path

$$A_t^N = \epsilon \tilde{A}_t^N + (1-\epsilon)\hat{A}_t^N \tag{70}$$

$$A_t^1 = \epsilon \tilde{A}_t^1 + (1-\epsilon)\hat{A}_t^1 \tag{71}$$

with

$$\begin{aligned}
\tilde{A}_t^N &\equiv A_0^N - \left(1-e^{-\alpha_1 t}\right)\left(R^1 - A_0^1\right)\\
\tilde{A}_t^1 &\equiv R^1 - e^{-\alpha_1 t}\left(R^1 - A_0^1\right)\\
\hat{A}_t^N &\equiv R^N - e^{-\alpha_N t}\left(R^N - A_0^N\right)\\
\hat{A}_t^1 &\equiv A_0^1 - \left(1-e^{-\alpha_N t}\right)\left(R^N - A_0^N\right),
\end{aligned}$$

is consistent with equilibrium. Hence, (68)-(71) constitute an equilibrium.

Case (A2). Conditions $0 < R - A$ and $\mathcal{R} - \mathcal{A}_0 \le 0$ imply

$$0 < \alpha_N (R - A) \le (\alpha_N - \alpha_1)\left(R^1 - A_0^1\right). \tag{72}$$

Guess that the equilibrium is as follows. There exists $t_1 \in [0,T]$, such that (i) for all $t \in [0,t_1)$, $p_t = \underline{p}_t$ (or equivalently, $\mathcal{R} - \mathcal{A}_t < 0$), and $\mathbb{N}_t^+ = \{2\}$; and (ii) for all $t \in (t_1,T]$, $p_t = \bar{p}_t$ (or equivalently, $0 < \mathcal{R} - \mathcal{A}_t$), and $\mathbb{N}_t^- = \mathbb{N}$. We solve for the equilibrium backwards. From time t_1 onwards, we know from part (i) of Lemma 5 that the equilibrium prices, $\{p_t, q_t^i\}_{i\in\mathbb{N},t\in(t_1,T]}$, are given by (68) and (69), and the distribution of balances, $\{A_t^i\}_{i\in\mathbb{N},t\in[t_1,T]}$, is given by (70) and (71) but with

$$\tilde{A}_t^N \equiv A_{t_1}^N - \left(1-e^{-\alpha_1 t}\right)\left(R^1 - A_{t_1}^1\right) \tag{73}$$

$$\tilde{A}_t^1 \equiv R^1 - e^{-\alpha_1 t}\left(R^1 - A_{t_1}^1\right) \tag{74}$$

$$\hat{A}_t^N \equiv R^N - e^{-\alpha_N t}\left(R^N - A_{t_1}^N\right) \tag{75}$$

$$\hat{A}_t^1 \equiv A_{t_1}^1 - \left(1-e^{-\alpha_N t}\right)\left(R^N - A_{t_1}^N\right). \tag{76}$$

For the time interval $[0,t_1)$, $\{p_t, q_t^i\}_{i\in\mathbb{N}}$ are as in part (ii) of Lemma 5, but with $\{q_{t_1}^i\}_{i\in\mathbb{N}}$ given by (69), i.e.,

$$\begin{aligned}
p_t &= e^{-r(t_1-t)}\left[e^{-r(T-t_1)}\mu_H - e^{-(r+\kappa_N)(T-t_1)}\left(\mu_H - \mu_L\right)\right]\\
q_t^i &= e^{-r(T-t)}\left\{\left[1-e^{-\kappa_i(T-t)}\right]\mu_H - e^{-\kappa_N(T-t_1)}\left[1-e^{-\kappa_i(t_1-t)}\right]\left(\mu_H \mathbb{I}_{\{i\neq N\}} - \mu_L\right)\right\}.
\end{aligned}$$

For any $t \in [0, t_1)$, the distribution of balances, $\left\{A_t^i\right\}_{i\in\mathbb{N}}$, is given by (63), i.e.,

$$A_t^N = A_0^N - \left(1 - e^{-\alpha_1 t}\right)\left(R^1 - A_0^1\right) \tag{77}$$

$$A_t^1 = R^1 - e^{-\alpha_1 t}\left(R^1 - A_0^1\right). \tag{78}$$

For any $t \in [t_1, T]$, the distribution of balances, $\left\{A_t^i\right\}_{i\in\mathbb{N}}$, is given by (70) and (71); and to construct this equilibrium, we select $\epsilon = 1$ (this ensures a continuous path for $\mathcal{R} - \mathcal{A}_t$). Then (65) implies that for all $t \in \mathbb{T}$,

$$\mathcal{R} - \mathcal{A}_t = \alpha_N (R - A) - e^{-\alpha_1 t}(\alpha_N - \alpha_1)\left(R^1 - A_0^1\right). \tag{79}$$

The last step in the construction of this equilibrium is to verify that, as conjectured, $\mathcal{R} - \mathcal{A}_t < 0$ for $t \in [0, t_1)$, $\mathcal{R} - \mathcal{A}_{t_1} = 0$, and $0 < \mathcal{R} - \mathcal{A}_t$ for $t \in [t_1, T]$. First, notice that given (72), $\mathcal{R} - \mathcal{A}_t$ is strictly increasing in t. Hence if there exists a $t_1 \in [0, T]$ such that $\mathcal{R} - \mathcal{A}_{t_1} = 0$, then it is unique. Second, notice that, by (72),

$$\mathcal{R} - \mathcal{A}_0 = \alpha_N (R - A) - (\alpha_N - \alpha_1)\left(R^1 - A_0^1\right) \leq 0$$

(with "<" unless the weak inequality in (72) holds with equality). Third, we have

$$\mathcal{R} - \mathcal{A}_T = \alpha_N (R - A) - e^{-\alpha_1 T}(\alpha_N - \alpha_1)\left(R^1 - A_0^1\right), \tag{80}$$

so $0 < \mathcal{R} - \mathcal{A}_T$ if and only if

$$e^{-\alpha_1 T}(\alpha_N - \alpha_1)\left(R^1 - A_0^1\right) < \alpha_N (R - A). \tag{81}$$

Hence this equilibrium exists provided conditions (72) and (81) hold (with strict inequality in (72)). If condition (81) does not hold, the equilibrium looks like in Case B1.

Case (B1). Conditions $R - A < 0$ and $\mathcal{R} - \mathcal{A}_0 < 0$ imply

$$\alpha_N (R - A) < \min\left\{0, (\alpha_N - \alpha_1)\left(R^1 - A_0^1\right)\right\}. \tag{82}$$

Guess that, for all $t \in \mathbb{T}$, the equilibrium has $p_t = \underline{p}_t$ (or equivalently, $\mathcal{R} - \mathcal{A}_t < 0$), and $2 \in \mathbb{N}_t^+$. From Corollary 1 we get that (67) holds for all $t \in \mathbb{T}$. Clearly, (67) and (82) imply $\mathcal{R} - \mathcal{A}_t < 0$ for all $t \in \mathbb{T}$, which confirms the conjecture that $p_t = \underline{p}_t$ for all $t \in \mathbb{T}$. Part (*ii*) of Lemma 5 then implies $\left\{p_t, q_t^i\right\}_{i\in\mathbb{N}, t\in\mathbb{T}}$ are given by

$$p_t = e^{-r(T-t)}\mu_L \tag{83}$$

$$q_t^i = \left[1 - e^{-\kappa_i (T-t)}\right] e^{-r(T-t)}\mu_L. \tag{84}$$

Notice that (83) and (84) imply $P_t^1 = P_t^N = \mu_L$ for all $t \in \mathbb{T}$, which confirms the conjecture that $2 \in \mathbb{N}_t^+$ for all $t \in \mathbb{T}$. However, in this case it turns out we also have $1 \in \mathbb{N}_t^+$, i.e., $\mathbb{N}_t^+ = \mathbb{N}$. This means the path for the distribution of balances $\{A_t^i\}_{i\in\mathbb{N},t\in\mathbb{T}}$ is indeterminate, i.e., it is given by (70) and (71) for any $\epsilon \in [0,1]$. In summary, (83) and (84) together with (70) and (71) constitute an equilibrium.

Case (B2). Conditions $R - A < 0$ and $0 \leq \mathcal{R} - \mathcal{A}_0$ imply

$$(\alpha_N - \alpha_1)\left(R^1 - A_0^1\right) \leq \alpha_N (R - A) < 0. \tag{85}$$

Guess that the equilibrium is as follows. There exists $t_1 \in [0,T]$, such that (i) for all $t \in [0,t_1)$, $p_t = \bar{p}_t$ (or equivalently, $0 < \mathcal{R} - \mathcal{A}_t$), and $\mathbb{N}_t^- = \{2\}$; and (ii) for all $t \in (t_1,T]$, $p_t = \underline{p}_t$ (or equivalently, $\mathcal{R} - \mathcal{A}_t < 0$), and $\mathbb{N}_t^+ = \mathbb{N}$. We solve for the equilibrium backwards. From time t_1 onwards, we know from part (ii) of Lemma 5 that the equilibrium prices, $\{p_t, q_t^i\}_{i\in\mathbb{N},t\in\mathbb{T}}$, are given by (83) and (84), and the distribution of balances, $\{A_t^i\}_{i\in\mathbb{N},t\in[t_1,T]}$, is given by (70) and (71), with $\left\{\tilde{A}_t^i, \hat{A}_t^i\right\}_{i\in\mathbb{N},t\in[t_1,T]}$ given by (73)-(76). For the time interval $[0,t_1)$, $\{p_t, q_t^i\}_{i\in\mathbb{N}}$ are as in part (i) of Lemma 5, but with $\{q_{t_1}^i\}_{i\in\mathbb{N}}$ given by (84), i.e.,

$$\begin{aligned}
p_t &= e^{-r(t_1-t)}\left[e^{-r(T-t_1)}\mu_L + e^{-(r+\kappa_N)(T-t_1)}\left(\mu_H - \mu_L\right)\right] \\
q_t^i &= e^{-r(T-t)}\left\{\left[1 - e^{-\kappa_i(T-t)}\right]\mu_L + e^{-\kappa_N(T-t_1)}\left[1 - e^{-\kappa_i(t_1-t)}\right]\left(\mu_H - \mathbb{I}_{\{i\neq N\}}\mu_L\right)\right\}.
\end{aligned}$$

For any $t \in [0,t_1)$, the distribution of balances, $\{A_t^i\}_{i\in\mathbb{N}}$, is given by (63), i.e., by (77) and (78). For any $t \in [0,t_1)$, the distribution of balances, $\{A_t^i\}_{i\in\mathbb{N}}$, is given by (63), i.e., by (77) and (78). For any $t \in [t_1,T]$, the distribution of balances, $\{A_t^i\}_{i\in\mathbb{N}}$, is given by (70) and (71); and to construct this equilibrium, we select $\epsilon = 1$ (this ensures a continuous path for $\mathcal{R} - \mathcal{A}_t$). Then (65) implies $\mathcal{R} - \mathcal{A}_t$ is given by (79) for all $t \in \mathbb{T}$. The last step in the construction of this equilibrium is to verify that, as conjectured, $0 < \mathcal{R} - \mathcal{A}_t$ for $t \in [0,t_1)$, $\mathcal{R} - \mathcal{A}_{t_1} = 0$, and $\mathcal{R} - \mathcal{A}_t < 0$ for $t \in [t_1,T]$. First, notice that given (85), $\mathcal{R} - \mathcal{A}_t$ is strictly decreasing in t. Hence if there exists a $t_1 \in [0,T]$ such that $\mathcal{R} - \mathcal{A}_{t_1} = 0$, then it is unique. Second, notice that, by (85),

$$0 < \mathcal{R} - \mathcal{A}_0 = \alpha_N (R - A) - (\alpha_N - \alpha_1)\left(R^1 - A_0^1\right)$$

(with "<" unless the weak inequality in (85) holds with equality). Third, $\mathcal{R} - \mathcal{A}_T$ is given by (80), so $\mathcal{R} - \mathcal{A}_T < 0$ if and only if

$$\alpha_N (R - A) < e^{-\alpha_1 T}(\alpha_N - \alpha_1)\left(R^1 - A_0^1\right). \tag{86}$$

Hence this equilibrium exists provided conditions (85) and (86) hold (with strict inequality in (85)).

Case (C1). Conditions $R - A = 0$ and $\mathcal{R} - \mathcal{A}_0 < 0$ imply

$$0 < (\alpha_N - \alpha_1)\left(R^1 - A_0^1\right). \tag{87}$$

Guess for all $t \in \mathbb{T}$, the equilibrium has $p_t = \underline{p}_t$ (or equivalently, $\mathcal{R} - \mathcal{A}_t < 0$), and $2 \in \mathbb{N}_t^+$. From part (*ii*) of Lemma (5), $A_t^1 = R^1 - e^{-\alpha_1 t}\left(R^1 - A_0^1\right)$ for all $t \in \mathbb{T}$, so (65) implies

$$\mathcal{R} - \mathcal{A}_t = -e^{-\alpha_1 t}(\alpha_N - \alpha_1)\left(R^1 - A_0^1\right) \tag{88}$$

for all $t \in \mathbb{T}$. Together, (87) and (88) imply $\mathcal{R} - \mathcal{A}_t < 0$ for all $t \in \mathbb{T}$, which confirms the conjecture that $p_t = \underline{p}_t$ for all $t \in \mathbb{T}$. Part (*ii*) of Lemma 5 then implies $\left\{p_t, q_t^i\right\}_{i\in\mathbb{N}, t\in\mathbb{T}}$ are given by (83) and (84). Notice that (83) and (84) imply $P_t^1 = P_t^N = \mu_L$ for all $t \in \mathbb{T}$, which confirms the conjecture that $2 \in \mathbb{N}_t^+$ for all $t \in \mathbb{T}$. However, in this case it turns out we also have $1 \in \mathbb{N}_t^+$, i.e., $\mathbb{N}_t^+ = \mathbb{N}$. This means the path for the distribution of balances $\left\{A_t^i\right\}_{i\in\mathbb{N}, t\in\mathbb{T}}$ is indeterminate, i.e., it is given by (70) and (71) for any $\epsilon \in [0, 1]$.

Case (C2). Conditions $R - A = 0$ and $0 \leq \mathcal{R} - \mathcal{A}_0$ imply

$$(\alpha_N - \alpha_1)\left(R^1 - A_0^1\right) \leq 0. \tag{89}$$

Guess that, for all $t \in \mathbb{T}$, the equilibrium has $p_t = \bar{p}_t$ (or equivalently, $0 < \mathcal{R} - \mathcal{A}_t$), and $2 \in \mathbb{N}_t^-$. From Corollary 1, $\mathcal{R} - \mathcal{A}_t$ is again given by (88). Clearly, if $R^1 - A_0^1 = 0$, then (88) implies $\mathcal{R} - \mathcal{A}_t = 0$ for all $t \in \mathbb{T}$. In this case, the distribution of balances is given by (64), and $\left\{p_t, q_t^i\right\}_{i\in\mathbb{N}, t\in\mathbb{T}}$ is indeterminate. So suppose (89) holds with strict inequality. In this case, (88) implies $0 < \mathcal{R} - \mathcal{A}_t$ for all $t \in \mathbb{T}$, which confirms the conjecture that $p_t = \bar{p}_t$ for all $t \in \mathbb{T}$. Part (*i*) of Lemma 5 then implies $\left\{p_t, q_t^i\right\}_{i\in\mathbb{N}, t\in\mathbb{T}}$ are given by (68) and (69). Notice that (68) and (69) imply $P_t^1 = P_t^N = \mu_H$ for all $t \in \mathbb{T}$, which confirms the conjecture that $2 \in \mathbb{N}_t^-$ for all $t \in \mathbb{T}$. However, in this case it turns out we also have $1 \in \mathbb{N}_t^-$, i.e., $\mathbb{N}_t^- = \mathbb{N}$. This means the path for the distribution of balances $\left\{A_t^i\right\}_{i\in\mathbb{N}, t\in\mathbb{T}}$ is indeterminate, and given by (70) and (71), for any $\epsilon \in [0, 1]$.

Part (*i*) in the statement of the proposition follows from combining cases (A1) and (C2). Part (*ii*) follows from combining cases (B1) and (C1). Part (*iii*) corresponds to case (A2). Part (*iv*) corresponds to case (B2). ■

Proof of Proposition 3. In this case, (39) and (40) simplify to

$$p_t = \begin{cases} \bar{p}_t & \text{if } 0 < R - A \\ \tilde{p}_t & \text{if } R - A = 0 \\ \underline{p}_t & \text{if } R - A < 0 \end{cases} \tag{90}$$

and

$$\dot{A}_t^i = \begin{cases} \alpha \left\{ \left(R^i - A_t^i\right) \mathbb{I}_{\{i \neq s_t\}} + \mathbb{I}_{\{i = s_t\}} \left[\left(R^{s_t} - A_t^{s_t}\right) - (R - A) \right] \right\} & \text{if } 0 < R - A \\ \alpha \left(R^i - A_t^i\right) & \text{if } R - A = 0 \\ \alpha \{ \left(R^i - A_t^i\right) \mathbb{I}_{\{i \neq l_t\}} + \mathbb{I}_{\{i = l_t\}} [(R^{l_t} - A_t^{l_t}) - (R - A)] \} & \text{if } R - A < 0. \end{cases}$$

In this special case, (10) and (90) can be solved for $\left\{q_t^i, p_t\right\}_{i \in \mathbb{N}, t \in \mathbb{T}}$. We consider three cases in turn, depending on whether parameters are such that (i) $0 < R - A$, (ii) $R - A < 0$, or (iii) $R - A = 0$. (i) Suppose $0 < R - A$. Guess that $N \in \mathbb{N}_t^-$ for all t. Then $p_t = \bar{p}_t \equiv e^{-(r+\kappa_N)(T-t)}\mu_H + q_t^N$ and (10) imply

$$q_t^i = \left[1 - e^{-\kappa_i (T-t)}\right] p_t \text{ for all } i \in \mathbb{N}, \tag{91}$$

with $p_t = e^{-r(T-t)}\mu_H$. Notice that $P_t^i = \mu_H$ for all $i \in \mathbb{N}$, so $\mathbb{N}_t^- = \mathbb{N}$ for all t. This implies the path for the distribution of balances $\left\{A_t^i\right\}_{i \in \mathbb{N}, t \in \mathbb{T}}$ is indeterminate, i.e., for any $\epsilon \in [0, 1]$, the path

$$A_t^N = \epsilon \tilde{A}_t^N + (1 - \epsilon) \hat{A}_t^N$$
$$A_t^1 = \epsilon \tilde{A}_t^1 + (1 - \epsilon) \hat{A}_t^1$$

with $\tilde{A}_t^N \equiv A_0^N - \left(1 - e^{-\alpha_1 t}\right)\left(R^1 - A_0^1\right)$, $\tilde{A}_t^1 \equiv R^1 - e^{-\alpha_1 t}\left(R^1 - A_0^1\right)$, $\hat{A}_t^N \equiv R^N - e^{-\alpha_N t}\left(R^N - A_0^N\right)$, and $\hat{A}_t^1 \equiv A_0^1 - \left(1 - e^{-\alpha_N t}\right)\left(R^N - A_0^N\right)$, is consistent with equilibrium. (ii) Suppose $R - A < 0$. Guess that $N \in \mathbb{N}_t^+$ for all t. Then $p_t = \underline{p}_t \equiv e^{-(r+\kappa_N)(T-t)}\mu_L + q_t^N$ and (10) imply (91), with $p_t = e^{-r(T-t)}\mu_L$. Notice that $P_t^i = \mu_L$ for all $i \in \mathbb{N}$, so $\mathbb{N}_t^+ = \mathbb{N}$ for all t. This implies the equilibrium distribution of balances is indeterminate, as in part (i). (iii) Suppose $R - A = 0$. In this case, the prices $\left\{q_t^i, p_t\right\}_{i \in \mathbb{N}, t \in \mathbb{T}}$ and the distribution of balances are indeterminate. ■

B Supplementary material

Claim 1 $\sum_{i\in\mathbb{N}} \dot{A}_t^i = 0$.

Proof. With (15), $\int z dG^{ii}(z) = 0$, and $dG^{ij}(z) = dG^{ji}(-z)$ for $i \neq j$,

$$\begin{aligned}
\sum_{i\in\mathbb{N}} \dot{A}_t^i &= \sum_{i\in\mathbb{N}} \alpha_i \left(A_t^{*i} - A_t^i\right) - \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} \lambda_i n_i \pi_j \int z dG^{ij}(z) \\
&= -\sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} \lambda_i n_i \pi_j \int z dG^{ij}(z) \\
&= -\frac{1}{\sum_{k\in\mathbb{N}} \lambda_k n_k} \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} \lambda_i n_i \lambda_j n_j \int z dG^{ij}(z) \\
&= -\frac{1}{\sum_{k\in\mathbb{N}} \lambda_k n_k} \left[\sum_{i\in\mathbb{N}} (\lambda_i n_i)^2 \int z dG^{ii}(z) + \sum_{i,j\in\{\mathbb{N}^2 : i\neq j\}} \lambda_i n_i \lambda_j n_j \int z dG^{ij}(z)\right] \\
&= -\frac{1}{\sum_{k\in\mathbb{N}} \lambda_k n_k} \sum_{i,j\in\{\mathbb{N}^2 : i>j\}} \lambda_i n_i \lambda_j n_j \left(\int z dG^{ij}(z) + \int z dG^{ji}(z)\right) \\
&= -\frac{1}{\sum_{k\in\mathbb{N}} \lambda_k n_k} \sum_{i,j\in\{\mathbb{N}^2 : i>j\}} \lambda_i n_i \lambda_j n_j \left(\int z dG^{ij}(z) + \int z dG^{ij}(-z)\right) \\
&= -\frac{1}{\sum_{k\in\mathbb{N}} \lambda_k n_k} \sum_{i,j\in\{\mathbb{N}^2 : i>j\}} \lambda_i n_i \lambda_j n_j \left(\int z dG^{ij}(z) - \int x dG^{ij}(x)\right) = 0.
\end{aligned}$$

■

B.1 Pure brokered market with quadratic end-of-day payoff

Assume $\beta_i = 0$ for all $i \in \mathbb{N}$, and suppose $\{u^i, U^i\}_{i\in\mathbb{N}}$ are given by (21) and (22). Then the function v_t^i that solves (11) with $\beta_i = 0$ takes the form (23), where ς_t^i satisfies

$$\begin{aligned}
\dot{\varsigma}_t^i - r\varsigma_t^i = &-\kappa_i \frac{\left[\eta_t^i - \left(p_t - q_t^i\right)\right]^2}{2\mu_t^i} \\
&+ \lambda_i \left[\left(\eta_t^i + q_t^i\right) \bar{z}_i + \frac{1}{2}\mu_t^i \left(\sigma_i^2 + \bar{z}_i^2\right)\right],
\end{aligned} \tag{92}$$

with boundary condition $\varsigma_T^i = 0$, where $\bar{z}_i$ is given by (13), and $\sigma_i^2 \equiv \sum_{j\in\mathbb{N}} \pi_j \int z^2 dG^{ij}(z) - \bar{z}_i^2$, and

$$\begin{aligned}\eta_t^i &= e^{-(r+\kappa_i)(T-t)}\left[C_i + \lambda_i \bar{z}_i (T-t)\left(B_i - \frac{b_i}{r+\kappa_i}\right)\right] \\ &\quad + \left[1 - e^{-(r+\kappa_i)(T-t)}\right]\frac{1}{r+\kappa_i}\left(c_i + \frac{\lambda_i \bar{z}_i b_i}{r+\kappa_i}\right) && (93) \\ \mu_t^i &= e^{-(r+\kappa_i)(T-t)} B_i + \left[1 - e^{-(r+\kappa_i)(T-t)}\right]\frac{b_i}{r+\kappa_i}. && (94)\end{aligned}$$

The maximization on the right side of (14) implies (27), and condition (15) implies (28), and with (28), condition (10) implies (29) with boundary condition $q_T^i = 0$ for all $i \in \mathbb{N}$.

Given (27) and (28), the law of motion (20) implies

$$\begin{aligned}\dot{A}_t^i &= -\alpha_i A_t^i - \lambda_i n_i \sum_{j\in\mathbb{N}} \pi_j \int z dG^{ij}(z) \\ &\quad + \nu_t^i \left[\eta_t^i - \sum_{j\in\mathbb{N}} \omega_t^j \eta_t^j + q_t^i - \sum_{j\in\mathbb{N}} \omega_t^j q_t^j + \sum_{j\in\mathbb{N}} \hat{\alpha}_t^j A_t^j\right] && (95)\end{aligned}$$

with A_0^i given for all $i \in \mathbb{N}$. Notice that (29)-(95) is a system of $2N$ linear first-order differential equations with variable coefficients.

B.1.1 A special case

Consider the following special case. Assume no payment shocks, i.e., $\lambda_i = 0$ for all $i \in \mathbb{N}$, $U^i(a) = a - \frac{\mu_i}{2} a^2$, and $u^i(a) = (r+\kappa_i) U^i(a)$, with $\mu_i \in \mathbb{R}_{++}$ for all $i \in \mathbb{N}$. In this case, the function v_t^i that solves (11) is

$$v_t^i(a) = a - \frac{\mu_i}{2} a^2 + \varsigma_t^i,$$

where ς_t^i satisfies

$$\varsigma_t^i = \kappa_i \int_t^T e^{-(r+\kappa_i)(s-t)} \left[\frac{1}{2}\frac{(1-p_s)^2}{\mu_i} + \varsigma_s^i\right] ds,$$

or equivalently,

$$\dot{\varsigma}_t^i - r\varsigma_t^i = -\kappa_i \frac{\left[1 - \left(p_t - q_t^i\right)\right]^2}{2\mu_i}$$

with terminal condition $\varsigma_T^i = 0$. Hence the maximization on the right of (14) implies

$$a_t^{*i} = \frac{1 - \left(p_t - q_t^i\right)}{\mu_i}. \tag{96}$$

Condition (15) then implies

$$p_t = 1 + \sum_{i \in \mathbb{N}} \omega^i q_t^i - \sum_{i \in \mathbb{N}} \hat{\alpha}^i A_t^i, \tag{97}$$

where

$$\begin{aligned} \nu^i &\equiv \frac{\alpha_i n_i}{\mu_i} \\ \hat{\alpha}^i &\equiv \frac{\alpha_i}{\sum_{j \in \mathbb{N}} \nu^j} \\ \omega^i &\equiv \frac{\nu^i}{\sum_{j \in \mathbb{N}} \nu^j}. \end{aligned}$$

With (97), condition (10) implies

$$\dot{q}_t^i = (r + \kappa_i)\, q_t^i - \kappa_i \left[1 + \sum_{j \in \mathbb{N}} \omega^j q_t^j - \sum_{j \in \mathbb{N}} \hat{\alpha}^j A_t^j \right] \tag{98}$$

with $q_T^i = 0$ for all $i \in \mathbb{N}$. Given (96) and (97), the law of motion (20) implies

$$\dot{A}_t^i = -\alpha_i A_t^i + \nu^i \left[q_t^i - \sum_{j \in \mathbb{N}} \omega^j q_t^j + \sum_{j \in \mathbb{N}} \hat{\alpha}^j A_t^j \right] \tag{99}$$

with A_0^i given for all $i \in \mathbb{N}$. Notice that (98)-(99) is a system of $2N$ linear first-order differential equations with constant coefficients. We can write the system as

$$\begin{aligned} \dot{q}_t^i &= \sum_{j \in \mathbb{N}} \left[(r + \kappa_i)\, \mathbb{I}_{\{j=i\}} - \kappa_i \omega^j \right] q_t^j + \sum_{j \in \mathbb{N}} \kappa_i \hat{\alpha}^j A_t^j - \kappa_i \\ \dot{A}_t^i &= \nu^i \sum_{j \in \mathbb{N}} \left(\mathbb{I}_{\{j=i\}} - \omega^j \right) q_t^j + \sum_{j \in \mathbb{N}} \left(-\alpha_i \mathbb{I}_{\{j=i\}} + \omega^i \alpha_j \right) A_t^j \end{aligned}$$

or equivalently,

$$\dot{\boldsymbol{y}}_t = M \boldsymbol{y}_t + \boldsymbol{b} \tag{100}$$

where $\dot{\boldsymbol{y}}$, $\boldsymbol{y}$, and $\boldsymbol{b}$ are $N \times 1$ vectors,

$$\dot{\boldsymbol{y}}_t = \left[\begin{array}{c} \dot{q}_t^1 \\ \vdots \\ \dot{q}_t^N \\ \hline \dot{A}_t^1 \\ \vdots \\ \dot{A}_t^N \end{array} \right], \ \boldsymbol{y}_t = \left[\begin{array}{c} q_t^1 \\ \vdots \\ q_t^N \\ \hline A_t^1 \\ \vdots \\ A_t^N \end{array} \right], \ \boldsymbol{b} = \left[\begin{array}{c} -\kappa_1 \\ \vdots \\ -\kappa_N \\ \hline 0 \\ \vdots \\ 0 \end{array} \right]$$

and M is a $2N \times 2N$ matrix,

$$M = \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right]$$

where M_{11}, M_{12}, M_{21}, and M_{22} are $N \times N$ matrices given by

$$M_{11} = \begin{bmatrix} r + \kappa_1 \left(1 - \omega^1\right) & -\kappa_1 \omega^2 & \cdots & -\kappa_1 \omega^N \\ -\kappa_2 \omega^1 & r + \kappa_2 \left(1 - \omega^2\right) & \cdots & -\kappa_2 \omega^N \\ \vdots & \vdots & \ddots & \vdots \\ -\kappa_N \omega^1 & -\kappa_N \omega^2 & \cdots & r + \kappa_N \left(1 - \omega^N\right) \end{bmatrix}$$

$$M_{12} = \begin{bmatrix} \kappa_1 \hat{\alpha}^1 & \kappa_1 \hat{\alpha}^2 & \cdots & \kappa_1 \hat{\alpha}^N \\ \kappa_2 \hat{\alpha}^1 & \kappa_2 \hat{\alpha}^2 & \cdots & \kappa_2 \hat{\alpha}^N \\ \vdots & \vdots & \ddots & \vdots \\ \kappa_N \hat{\alpha}^1 & \kappa_N \hat{\alpha}^2 & \cdots & \kappa_N \hat{\alpha}^N \end{bmatrix}$$

$$M_{21} = \begin{bmatrix} \nu^1 \left(1 - \omega^1\right) & -\nu^1 \omega^2 & \cdots & -\nu^1 \omega^N \\ -\nu^2 \omega^1 & \nu^2 \left(1 - \omega^2\right) & \cdots & -\nu^2 \omega^N \\ \vdots & \vdots & \ddots & \vdots \\ -\nu^N \omega^1 & -\nu^N \omega^2 & \cdots & \nu^N \left(1 - \omega^N\right) \end{bmatrix}$$

$$M_{22} = \begin{bmatrix} -\alpha_1 \left(1 - \omega^1\right) & \omega_1 \alpha_2 & \cdots & \omega_1 \alpha_N \\ \omega_2 \alpha_1 & -\alpha_2 \left(1 - \omega^2\right) & \cdots & \omega_2 \alpha_N \\ \vdots & \vdots & \ddots & \vdots \\ \omega_N \alpha_1 & \omega_N \alpha_2 & \cdots & -\alpha_N \left(1 - \omega^N\right) \end{bmatrix}.$$

Remark 2 *Notice that (100) is an inhomogeneous $2N \times 2N$ system of linear first-order ODEs with constant coefficients. The $2N$ boundary conditions are $q_T^i = 0$ for all i, and A_0^i as given for all i. To solve it, proceed as follows.*

Let $\boldsymbol{\lambda} \equiv \left(\lambda^i\right)_{i=1}^{2N}$ denote the $2N \times 1$ vector of eigenvalues of M, and let $\boldsymbol{x}^1$ denote the $2N \times 1$ eigenvector associated with eigenvalue λ_i. Let S be the $2N \times 2N$ matrix with the property that the i^{th} column of S is $\boldsymbol{x}^i$ (i.e., the eigenvector associated to the eigenvalue λ_i), and let S^{-1} denote the inverse of S. Let Λ be the $2N \times 2N$ diagonal matrix with the eigenvalues of M on the diagonal (i.e., the entry in row i and column i of Λ is λ_i). Then, we know that $M = S \Lambda S^{-1}$, so we can write (100) as

$$\dot{\boldsymbol{y}}_t = \left(S \Lambda S^{-1}\right) \boldsymbol{y}_t + \boldsymbol{b}.$$

Left-multiplying this equation by S^{-1} wet get

$$\dot{\boldsymbol{u}}_t = \Lambda \boldsymbol{u}_t + \boldsymbol{c}. \tag{101}$$

where

$$\dot{\boldsymbol{u}}_t \equiv S^{-1}\dot{\boldsymbol{y}}_t$$
$$\boldsymbol{u}_t \equiv S^{-1}\boldsymbol{y}_t$$
$$\boldsymbol{c} \equiv S^{-1}\boldsymbol{b}.$$

Notice (101) can be written as

$$\dot{u}_t^i = \lambda_i u_t^i + c^i \textit{ for } i = 1, ..., 2N$$

where $\dot{u}_t^i$ *is the* i^{th} *element of* $\dot{\boldsymbol{u}}_t$, u_t^i *is the* i^{th} *element of* $\boldsymbol{u}_t$, *and* c^i *is the* i^{th} *element of* $\boldsymbol{c}$. *Hence*

$$u_t^i = -\frac{c^i}{\lambda_i} + e^{\lambda_i t} k_i \textit{ for } i = 1, ..., 2N \tag{102}$$

where $\boldsymbol{k} = (k_i)_{i=1}^{2N}$ *is a vector of unknown constants. The solution (102) can be written as*

$$\boldsymbol{u}_t = \Delta\tilde{\boldsymbol{\lambda}} + \boldsymbol{z}_t \tag{103}$$

where $\tilde{\boldsymbol{\lambda}} \equiv \left(1/\lambda^i\right)_{i=1}^{2N}$ *is the* $2N \times 1$ *vector of the reciprocal of the eigenvalues of* M, *and* Δ *is the* $2N \times 2N$ *diagonal matrix with* $-c^i$ *as the entry in row* i *and column* i, *and* $\boldsymbol{z}_t$ *is the* $2N \times 1$ *vector whose* i^{th} *element is* $e^{\lambda_i t} k_i$, *i.e.,*

$$\Delta = \begin{bmatrix} -c^1 & 0 & \cdots & 0 \\ 0 & -c^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -c^{2N} \end{bmatrix}$$
$$\boldsymbol{z}_t = \begin{bmatrix} e^{\lambda_1 t} k_1 \\ \vdots \\ e^{\lambda_{2N} t} k_{2N} \end{bmatrix} = e^{\Lambda t}\boldsymbol{k}.$$

(Notice that the $2N$ *equations stacked in (103) are independent of each other.) If we pre-multiply (103) by* S, *we get*

$$\boldsymbol{y}_t = S\Delta\tilde{\boldsymbol{\lambda}} + S\boldsymbol{z}_t. \tag{104}$$

Notice that (104) gives the solution to (100), but it contains $2N$ *constants still to be determined.*

To determine these constants (i.e., the vector $\boldsymbol{k}$*), we use the boundary conditions (terminal conditions for the* q^i*'s and initial conditions for the* A^i*'s) as follows. Evaluate (104) at* $t = 0$

and at $t = T$, to get the following system of $4N$ linear equations

$$\boldsymbol{y}_0 = S\Delta\tilde{\boldsymbol{\lambda}} + S\boldsymbol{z}_0$$
$$\boldsymbol{y}_T = S\Delta\tilde{\boldsymbol{\lambda}} + S\boldsymbol{z}_T$$

that can be solved for the $4N$ unknowns, $\left(q_0^1, \ldots, q_0^N, A_T^1, \ldots A_T^N, k_1, \ldots k_{2N}\right)$. We can write this system as follows

$$\left[\frac{\boldsymbol{y}_0 - S\boldsymbol{z}_0}{\boldsymbol{y}_T - S\boldsymbol{z}_T}\right] = \left[\frac{S\Delta\tilde{\boldsymbol{\lambda}}}{S\Delta\tilde{\boldsymbol{\lambda}}}\right]$$

or equivalently,

$$\left[\frac{\boldsymbol{y}_0 - S\boldsymbol{k}}{\boldsymbol{y}_T - Se^{\Lambda T}\boldsymbol{k}}\right] = \left[\frac{S\Delta\tilde{\boldsymbol{\lambda}}}{S\Delta\tilde{\boldsymbol{\lambda}}}\right] \tag{105}$$

where

$$\boldsymbol{y}_0 = \left[\begin{array}{c} q_0^1 \\ \vdots \\ q_0^N \\ \hline A_0^1 \\ \vdots \\ A_0^N \end{array}\right], \boldsymbol{y}_T = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline A_T^1 \\ \vdots \\ A_T^N \end{array}\right] \tag{106}$$

and $e^{\Lambda T}$ is the $2N \times 2N$ matrix given by

$$e^{\Lambda T} = \begin{bmatrix} e^{\lambda_1 T} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 T} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_{2N} T} \end{bmatrix}.$$

Once we have found $\boldsymbol{k}$ by solving (105), we can plug them into (104) to find the final solution to the system of ODEs, namely

$$\boldsymbol{y}_t = S\Delta\tilde{\boldsymbol{\lambda}} + Se^{\Lambda t}\boldsymbol{k}. \tag{107}$$

C Data Appendix

TO BE ADDED.

D Quantitative analysis

D.1 Calibration

The parameters are

$$r, n_i, \alpha_i, \beta_i, \theta_i, \lambda_i, \left\{G^{ij}\right\}_{j\in\mathbb{N}}, u^i, U^i, F_0^i\left(a\right)$$

Moments:

- Set $r = 0$
- Distribution of trading activity β_i
- Distribution of payments λ_i
- Assume symmetry for α_i and θ_i, i.e., set $\alpha_i = \alpha$ and $\theta_i = 1/2$ for all i.
- Choose number of bank types and assign n_i to the empirical share of teach type

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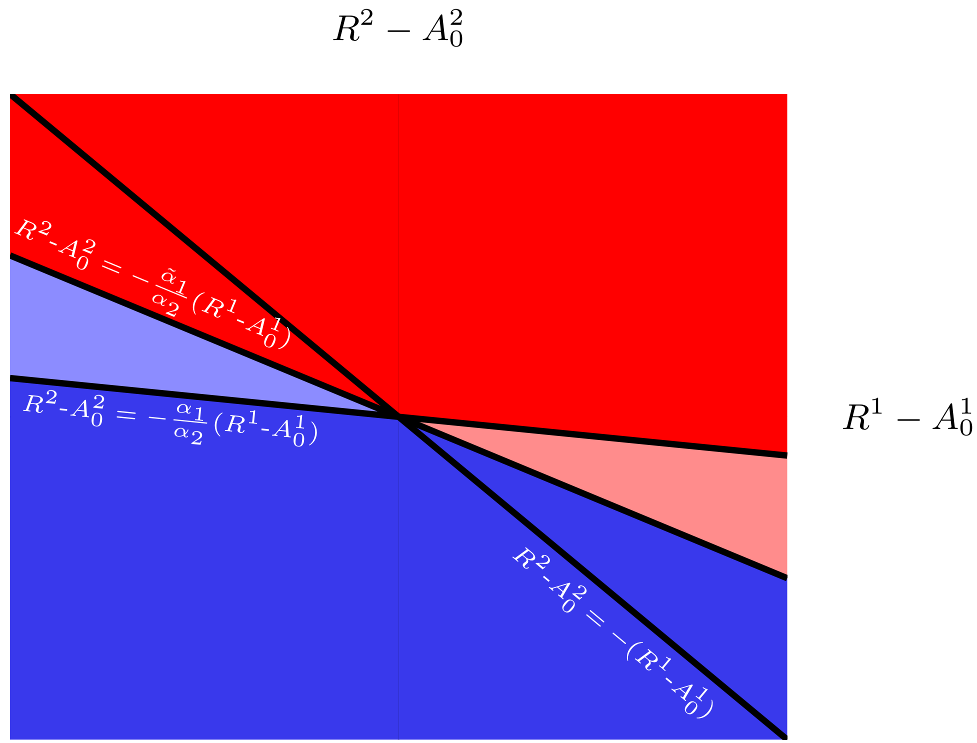

Figure 9: Existence regions for the model with piecewise linear end-of-day payoffs and $N = 2$. Each shaded region corresponds to one of the cases in Proposition 2. The dark (red) shaded region above the upper envelope of the steepest and the flattest boundaries corresponds to case (i). The dark (blue) shaded region below the lower envelope of the steepest and the flattest boundaries corresponds to case (ii). The light (red) shaded region inside the second quadrant corresponds to case $(iii)(b)$. The dark (blue) shaded triangle below it corresponds to case $(iii)(a)$. The light (blue) shaded region inside the fourth quadrant corresponds to case $(iv)(b)$. The dark (red) shaded triangle above it corresponds to case $(iv)(a)$. The origin is case (v). The equilibrium price is: $e^{-r(T-t)}\mu_H$ in the dark (red) region on the top of the graph, $e^{-r(T-t)}\mu_L$ in the dark (blue) region on the bottom of the graph, given by (43) in the light (red) region in the second quadrant, and given by (47) in the light (blue) region in the fourth quadrant.