

Banks' Liquidity Management and Financial Fragility*

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Abstract

We study the interaction of banks' liquidity management and financial fragility, in an environment where banks hold a portfolio of liquidity and partially-illiquid productive assets, to insure their depositors against fundamental uncertainty, and liquidity and maturity transformation might trigger a depositors' self-fulfilling run, modelled as a "global game". Under some sufficient conditions, there exists a unique recovery rate, associated with early liquidation of the productive assets, below which the banks first deplete liquidity and then liquidate, to finance depositors' withdrawals during a run. In equilibrium, the banks hoard liquidity in anticipation of a run, and narrow banking is not viable.

Keywords: financial intermediation, liquidity, financial fragility, self-fulfilling runs, global games

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1 Introduction

It is a well documented fact that banks hold excess liquidity at times of financial distress. Many explanations have been proposed to this observation, including precautionary savings (Ashcraft et al., 2011; Acharya and Merrouche, 2013) and counterparty risk (Heider et al., 2015), all based on the assumption that banks face fundamental uncertainty, against which they might want to hold safe assets. However, there is also an extensive evidence showing that banks are prone to financial fragility induced by the investors' self-fulfilling expectations of crises. Indeed, the very essence of banking, i.e. liquidity and maturity transformation, creates financial fragility through a mismatch in banks' balance sheets that leads to depositors' self-fulfilling runs. Financial fragility and self-fulfilling runs are not a phenomenon of the past: for example, Argentina in 2001 and Greece in 2015 faced such systemic events. Moreover, many U.S. money market and life insurance funds – that perform liquidity and maturity transformation, and therefore can be likened to traditional banks – experienced self-fulfilling runs after the bankruptcy of Lehman Brothers in 2008 (Foley-Fisher et al., 2015). Moreover, there is a wide consensus that the whole U.S. financial crisis of 2007-2009 can be interpreted as a systemic self-fulfilling run of financial intermediaries on other financial intermediaries (Gorton and Metrick, 2012). Therefore, the need emerges to understand the interaction between banks' liquidity management and financial fragility. This is the aim of the present paper.

The distinctive feature of our argument is that the interaction between banks' liquidity management and financial fragility goes in both directions. In fact, on the one hand, financial fragility has a non-trivial effect on banks' asset portfolios, as banks anticipate that they can finance excessive withdrawals either by rolling over liquidity from previous periods or by liquidating the more productive assets on their balance sheets. On the other hand, liquidity arguably influences not only banks' resilience to fundamental uncertainty, but also the perception of this resilience, thus affecting investors' expectations and as a consequence financial fragility. Accordingly, we propose a positive theory of banking based on the seminal work by Diamond and Dybvig (1983), with three key characteristics. First, banks are exposed to fundamental uncertainty (idiosyncratic liquidity shocks and aggregate productivity shocks) as well as the possibility of strategic self-fulfilling runs by their depositors, due to incomplete contractibility related to the idiosyncratic liquidity shocks and imperfect information about the aggregate pro-

ductivity shocks. Second, banks can invest in a partially illiquid but productive asset and in a liquid asset, to insure against both types of uncertainties. Introducing this realistic feature complicates the analysis in a substantial way, which represents the methodological contribution of the paper. More importantly, it enables us to offer a full analysis of banks' liquidity management from an ex-ante perspective, i.e. in anticipation of fundamental and self-fulfilling uncertainty, as well as from an ex-post perspective, when the latter materializes. Moreover, the concept of excess liquidity is well-defined, by comparison to a benchmark with perfect information, in which the banks hold liquidity only to insure their depositors against fundamental uncertainty. Third, banks' liquidity management and financial fragility, resulting from the strategic behavior of banks and depositors, are jointly and endogenously determined in a global game (Carlsson and van Damme, 1993; Morris and Shin, 1998).

In such an environment, we first characterize an endogenous pecking order that the banks follow when they utilize their assets to cover a self-fulfilling run. The equilibrium pecking order trades off the opportunity cost of liquidating the productive assets, in terms of (i) forgone resources due to their partial illiquidity and (ii) forgone future consumption, with the opportunity costs of depleting liquidity, in terms of lower insurance against fundamental uncertainty. When the agents' relative risk aversion is sufficiently high, the banks first deplete liquidity and then liquidate the productive asset, as long as the latter is sufficiently illiquid. Accordingly, the model predicts the typical chain of events that we observe in the real world during a self-fulfilling run: as the number of depositors withdrawing increases, at first banks are liquid; then, they become illiquid but solvent, when they run out of liquidity and start liquidating the productive assets, but are still able to serve their depositors; finally, they become insolvent, thereby going into bankruptcy.

Our second result shows instead the conditions under which banks hold excess liquidity against possible self-fulfilling runs. Intuitively, the anticipation of self-fulfilling runs imposes a distortion in the intertemporal allocation of resources in banks' balance sheets, that forces them to increase liquidity and lower the consumption that they promise to the depositors against the realization of the idiosyncratic liquidity shocks, with respect to the benchmark with perfect information. In other words, in the presence of financial fragility the banks increase the amount of excess liquidity in portfolio. Finally, we study banks' behavior under the constraint of being

“narrow”, i.e. such that they hold sufficient liquidity to cover all their demand deposits, in line with the famous proposition from the 1930s that was later called “Chicago Plan”. [\[MORE...\]](#)

The approach of connecting banks’ liquidity management to aggregate productivity shocks and self-fulfilling runs is novel in the literature on banking and financial crises. In fact, in the first-generation models of bank runs, Cooper and Ross (1998) and Ennis and Keister (2006) also study banks’ liquidity management, but in an environment without aggregate productivity shocks. In there, self-fulfilling runs arise as a consequence of multiple equilibria, and the depositors select an equilibrium following the realization of an extrinsic event, commonly named “sunspot”, completely uncorrelated to the fundamentals of the economy. In such a framework, banks do hold excess liquidity in equilibrium, but only to be able to serve all depositors in the case of a run, thus making themselves “run-proof”. In other words, contrary to the empirical evidence, these models do not exhibit excess liquidity and self-fulfilling runs simultaneously. Allen and Gale (1998) also study banks’ liquidity management, but in an environment with fundamental uncertainty and no multiple equilibria. Moreover, the banks in their model do not hold excess liquidity, but offer a standard deposit contract coupled with the possibility to default in the bad states of the world, thus allowing optimal risk sharing. In the second-generation models of bank runs (Rochet and Vives, 2004; Goldstein and Pauzner, 2005), instead, the economy does feature aggregate productivity shocks, and the equilibrium selection mechanism, in the presence of multiple equilibria, is endogenously determined via a “global game”. Yet, in these models, there is no liquidity management: in Rochet and Vives (2004), because banks do not provide insurance against fundamental shocks; in Goldstein and Pauzner (2005), because the productive asset in which banks invest always dominates liquidity, as the banks can liquidate it at zero cost. To the best of our knowledge, the work of Anhert and Elamin (2014) is the only example of a second-generation model where the banks have the possibility to invest in a liquid asset, but only ex post (i.e. during a run), to transfer the proceeds from early liquidation of the productive asset to possibly bad states of the world. In other words, there is no liquidity management ex ante, to hedge against fundamental as well as self-fulfilling uncertainty.

The rest of the paper is organized as follows: in section 2, we lay down the basic features of the environment; in section 3, we study the withdrawing decisions of the depositors, and characterize the optimal pecking order with which the banks employ their assets to finance

depositors' withdrawals; in section 4, we solve for the banking equilibrium; finally, section 5 concludes.

2 Environment

The economy lives for three periods, labeled $t = 0, 1, 2$, and is populated by a continuum of ex-ante identical agents, all endowed with 1 unit of a consumption good at date 0, and 0 afterwards. At date 1, all agents are hit by a privately-observed idiosyncratic liquidity shock θ , taking value 0 with probability λ and 1 with probability $1 - \lambda$. The law of large numbers holds, hence the probability distribution of the idiosyncratic liquidity shocks is equivalent to their cross-sectional distribution: at date 1, there is a fixed fraction λ of agents in the whole economy whose realized shock is $\theta = 0$, and a fraction $1 - \lambda$ whose realized shock is $\theta = 1$. The idiosyncratic liquidity shocks affect the point in time when the agents want to consume, according to the welfare function $U(c_1, c_2, \theta) = (1 - \theta)u(c_1) + \theta u(c_2)$. In other words, those agents receiving a shock $\theta = 0$ are only willing to consume at date 1, and those receiving a shock $\theta = 1$ are only willing to consume at date 2. Thus, in line with the literature, we refer to them as early (or impatient) consumers and late (or patient) consumers, respectively. The utility function $u(c)$ is increasing, concave and twice-continuously differentiable, and is such that $u(0) = 0$ and the coefficient of relative risk aversion is strictly larger than 1. Importantly, it also satisfies the Inada conditions: $\lim_{c \rightarrow 0} u'(c) = +\infty$ and $\lim_{c \rightarrow +\infty} u'(c) = 0$.

There are two technologies available in the economy to hedge against the idiosyncratic liquidity shocks. The first one is a storage technology, here called “liquidity”, yielding 1 unit of consumption at date $t+1$ for each unit invested in t . The second one is instead a productive asset that, for each unit invested at date 0, yields a stochastic return A at date 2. This stochastic return takes values $R > 1$ with probability p , and 0 with probability $1 - p$. The probability of success of the productive asset p represents the aggregate state of the economy, and is distributed uniformly over the interval $[0, 1]$, with $\mathbb{E}[p]R > 1$. Moreover, the productive asset can be liquidated at date 1 via a liquidation technology, that allows to recover $r < 1$ units of consumption for each unit liquidated. Intuitively, this means that the economy features a liquid asset, with low but safe yields, and a partially illiquid asset, that yields a low return in the short run, but a possible high return in the long run, which is subject to the realization of an aggregate productivity shock.

The economy is also populated by a large number of banks, operating in a perfectly-competitive market with free entry. The banks collect the endowments of the agents in the form of deposits, and invest them so as to maximize their profits, subject to agents' participation. Perfect competition and free entry ensure that the banks solve the equivalent problem of maximizing the expected welfare of the agents/depositors, subject to their budget constraint. To this end, they offer a standard deposit contract $\{c, c_L(A)\}$, stating the uncontingent amount c that the depositors can withdraw at date 1, and the state-dependent amount $c_L(A)$ that they can withdraw at date 2.¹ As the realizations of the idiosyncratic types are private information, the deposit contract must be incentive compatible, i.e. the depositors must have the incentives to truthfully report their types. This implies that the deposit contract must satisfy the incentive compatibility constraint $c \leq c_L(R)$.

To finance the deposit contract, the banks invest the deposits – which are the only liability on their balance sheets – in an amount L of liquidity and $1 - L$ of the productive asset, respectively. Then, given the deposit contract and asset portfolio chosen at date 0, the banks commit to pay c to whoever depositor comes to withdraw at date 1, until their resources are exhausted. To this end, the banks also choose the “pecking order” with which to use the assets in order to finance the early withdrawals: {Liquidation, Liquidity} or {Liquidity, Liquidation}. When resources are exhausted, and the banks are not able to fulfill their contractual obligations anymore, they instead go into bankruptcy, at which they must liquidate all the productive assets in portfolio, and serve their depositors according to an “equal service constraint”, i.e. such that all depositors get an equal share of the available resources. Finally, at date 2 the depositors who have not withdrawn at date 1 are residual claimants of an equal share of the remaining resources.

We assume that depositors cannot observe the true value of the realization of the fundamental p , but receive at date 1 a “noisy” signal $\sigma = p + e$ about it. The term e is an idiosyncratic noise, indistinguishable from the true value of p , that is uniformly distributed over the interval $[-\epsilon, +\epsilon]$, where ϵ is positive but small. Given the received signal, each late consumer decides whether to withdraw from her bank at date 2, as the realization of her idiosyncratic shock would command, or “run on her bank” and withdraw at date 1, in accordance to the scheme to be

¹In order to rule out uninteresting run equilibria, the amount of early consumption c must be smaller than $\min\{1/\lambda, R\}$. The fact that the banks have to offer a standard deposit contract here is assumed. However, Farhi et al. (2009) show that a standard deposit contract, with an uncontingent amount of early consumption, endogenously emerge as part of the banking equilibrium, in the presence of non-exclusive contracts.

described in the incoming section.

The timing of actions is as follows: at date 0, the banks collect the deposits, and choose the deposit contract $\{c, c_L(A)\}$ and asset portfolio $\{L, 1 - L\}$; at date 1, the banks choose the pecking order with which to finance early withdrawals; then, all agents get to know their private types and private signals, and the early consumers withdraw, while the late consumers, once observed the signals, decide whether to run or not; finally, at date 2, those late consumers who have not withdrawn at date 1 withdraw an equal share of the available resources left. We solve the model by backward induction, and characterize a pure-strategy symmetric Bayesian Nash equilibrium. Hence, we focus our attention on the behavior of a representative bank. The definition of equilibrium is the following:

Definition 1. *Given the distributions of the idiosyncratic liquidity shocks, of the aggregate productivity shock and of the individual signals, a banking equilibrium is a deposit contract $\{c, c_L(A)\}$, an asset portfolio $\{L, 1 - L\}$, a pecking order and depositors' decisions to run such that, for every realization of signals and idiosyncratic types $\{\sigma, \theta\}$:*

- *the depositors' decisions to run maximize their expected welfare;*
- *the pecking order, the deposit contract and the asset portfolio maximize the depositors' expected welfare, subject to budget constraints.*

2.1 Autarkic Equilibrium

As a benchmark to study the viability of the banking equilibrium in the incoming sections, we start our analysis with the characterization of the equilibrium in autarky. To this end, assume that the agents cannot access the banking system at date 0, but can independently invest an amount L of their initial endowment in liquidity and an amount $1 - L$ in the productive asset, in anticipation of the idiosyncratic liquidity shock θ and of the aggregate productivity shock. Then, if an agent turns out to be an early consumer, she will consume the liquidation value of her asset portfolio, namely $c^A = L + r(1 - L)$. If instead she turns out to be a late consumer, she will consume an amount which depends on the realization of the aggregate shock plus the amount of liquidity which is rolled over to date 2, i.e. $c_2^A(R) = R(1 - L) + L$ or $c_2^A(0) = L$.

Then, at date 0, the portfolio problem boils down to:

$$\max_L \lambda u(L + r(1 - L)) + (1 - \lambda) \int_0^1 [pu(R(1 - L) + L) + (1 - p)u(L)] dp, \quad (1)$$

subject to $L \leq 1$. Attach the Lagrange multiplier χ to the latter constraint. The first-order condition of the problem reads:²

$$\lambda(1 - r)u'(L + r(1 - L)) = (1 - \lambda)\mathbb{E}[p] \left[u'(R(1 - L) + L)(R - 1) - u'(L) \right] + \chi. \quad (2)$$

It can be proved that, if the condition:

$$\frac{\lambda(1 - r)}{1 - \lambda} < \mathbb{E}[p](R - 2) \quad (3)$$

holds, L^A cannot be equal to 1. To see that, notice that, if $L^A = 1$, the equilibrium condition would yield a Lagrange multiplier:

$$\chi = \left[\lambda(1 - r) - (1 - \lambda)\mathbb{E}[p](R - 2) \right] u'(1). \quad (4)$$

Under condition (3), this expression is negative, which is impossible by the definition of Lagrange multiplier. Hence, if condition (3) holds, the autarkic allocation is characterized by the equilibrium condition:

$$\lambda(1 - r)u'(L + r(1 - L)) + (1 - \lambda)\mathbb{E}[p]u'(L) = (1 - \lambda)\mathbb{E}[p](R - 1)u'(R(1 - L) + L). \quad (5)$$

Intuitively, this expression states that an agent in autarky chooses an equilibrium asset portfolio such that the expected marginal benefits of holding liquidity, in terms of early consumption and late consumption in the bad state of the world (as $c_L^A(0) = L$), must be equal to the expected marginal costs of holding liquidity, in terms of late consumption $c_L^A(R)$ lost in the good state of the world.

²Notice that in equilibrium L^A is positive, as $L^A = 0$ would not satisfy the first-order condition because of the Inada conditions.

2.2 Equilibrium with Perfect Information

As a further benchmark for the results that follow, here we characterize a banking equilibrium with perfect information, provided by a representative bank who can observe the realization of the idiosyncratic liquidity shocks hitting the depositors, and maximizes their expected welfare subject to budget constraints. More formally, the bank solves:

$$\max_{c, c_L(A), L, D} \lambda u(c) + (1 - \lambda) \mathbb{E}[u(c_L(A))], \quad (6)$$

subject to the resource constraints:

$$L + rD \geq \lambda c, \quad (7)$$

$$(1 - \lambda)c_L(A) + \lambda c = A(1 - L - D) + L + rD, \quad (8)$$

where the last constraint has to hold for any $A \in \{0, R\}$, and to the non-negativity constraint $D \geq 0$.³ At date 0, the bank collects all endowments, and invests them in an amount L of liquidity and $1 - L$ of productive assets. At date 1, the liquidity constraint (7) states that the amount of liquid assets, given by the sum of liquidity plus the extra resources generated by liquidating an amount D of productive assets at rate r , must be sufficient to pay early consumption c to the λ early consumers. Any resources left are rolled over to date 2 and, together with the return from the remaining productive assets, pay late consumption $c_L(A)$ to the $1 - \lambda$ late consumers: if the return on the productive asset is positive, a late consumer gets $c_L(R) = (R(1 - L - D) + L + rD - \lambda c)/(1 - \lambda)$; if instead the return turns out to be zero, she gets $c_L(0) = (L + rD - \lambda c)/(1 - \lambda)$. Plugging the resource constraints in the objective function, the bank's problem reads::

$$\max_{c, L, D} \lambda u(c) + (1 - \lambda) \int_0^1 \left[pu \left(\frac{R(1 - L - D) + L + rD - \lambda c}{1 - \lambda} \right) + (1 - p)u \left(\frac{L + rD - \lambda c}{1 - \lambda} \right) \right] dp, \quad (9)$$

subject to the liquidity constraint $L + rD \geq \lambda c$ and $D \geq 0$. In this framework, we can prove the following:

³The non-negativity constraints on the other choice variables are always satisfied in equilibrium, given the assumption that the Inada conditions hold.

Lemma 1. *The equilibrium with perfect information exhibits no liquidation of the productive asset ($D^{PI} = 0$) and excess liquidity ($L^{PI} > \lambda c^{PI}$). The deposit contract and asset portfolio satisfy the Euler equation:*

$$u'(c^{PI}) = \mathbb{E}[p]Ru' \left(\frac{R(1 - L^{PI}) + L^{PI} - \lambda c^{PI}}{1 - \lambda} \right). \quad (10)$$

Moreover, if λ is sufficiently large, the deposit contract satisfies:

$$0 < c_L^{PI}(0) < 1 < c^{PI} < c_L^{PI}(R). \quad (11)$$

Proof. In Appendix A. ■

The lemma shows that liquidating the productive asset to create liquidity at date 1 is never part of an equilibrium with perfect information, because the recovery rate r is too low. More importantly, if the probability of a depositor being hit by the idiosyncratic liquidity shock is sufficiently large, the bank provides insurance against it by transferring part of the available resources from late consumption to early consumption. Moreover, the bank provides insurance against the aggregate productivity shock by engaging in precautionary savings, i.e. by holding more liquidity than the one needed to cover early consumption and insure against the idiosyncratic liquidity shock. In equilibrium, the bank achieves these objectives by allocating the deposits according to an Euler equation, i.e. so that the marginal rate of substitution between early and late consumption is equal to the expected marginal rate of transformation of the productive asset. Finally, the concavity of the utility function and the assumption that $\mathbb{E}[p]R > 1$ imply that the incentive compatibility constraint is satisfied, hence this allocation is equivalent to a constrained efficient allocation, in which a bank has to induce truth-telling among the depositors.

3 Strategic Complementarities

We now move to the analysis of the competitive banking equilibrium. As stated above, we characterize it by backward induction, hence in this section we start by studying the withdrawing decisions of the late consumers (as the early consumers withdraw for sure at date 1) who choose whether to withdraw (“run”) at date 1 or wait until date 2. Then, in the following section, we

characterize the equilibrium deposit contract and portfolio allocation.

A late consumer receives a signal σ at date 1, and takes as given the deposit contract and asset portfolio, fixed at date 0, and the pecking order, fixed at date 1 before the signal is realized. Based on these, she creates her posterior beliefs about how many depositors are withdrawing at date 1 (call this number n), and the probability of the realization of the aggregate state A , and decides whether to withdraw or not. We assume the existence of two regions of extremely high and extremely low signals, where the decision of a late consumer is independent of her posterior beliefs. In the “upper dominance region”, the signal is so high that a late consumer always prefers to wait until date 2 to withdraw. Following Goldstein and Pauzner (2005), we assume that this happens above a threshold $\bar{\sigma}$, where the investment is safe, i.e. $p = 1$, and gives the same return R at date 1 and 2. In this way, a late consumer is sure to get $\frac{R(1-L)+L-\lambda c}{1-\lambda}$ at date 2, irrespective of the behavior of all the other late consumers, and prefers to wait for any possible realization of the aggregate state. In the “lower dominance region”, instead, the signal is so low that a late consumer always runs, irrespective of the behavior of the other depositors, thus triggering a “fundamental run”. This happens below the threshold signal $\underline{\sigma}_j$, that makes her indifferent between withdrawing or not, and depends on the pecking order j chosen by the bank (we characterize the thresholds in the incoming sections).

The existence of the lower and upper dominance regions, regardless of their size, ensures the existence of an equilibrium in the intermediate region $[\underline{\sigma}_j, \bar{\sigma}]$, where the late consumers decide whether to run or not based on a threshold strategy: they run if the signal is lower than a threshold signal σ_j^* .⁴ Let $Prob(\sigma \leq \sigma_j^*)$ be the probability that $\sigma \leq \sigma_j^*$ under pecking order j . Then, given $\sigma = p + e$, we have:

$$Prob(\sigma \leq \sigma_j^*) = \int_{-\epsilon}^{\sigma_j^* - p} \frac{1}{2\epsilon} de = \max\left(\frac{\sigma_j^* - p + \epsilon}{2\epsilon}, 0\right). \quad (12)$$

Define as $c_L(A, n)$ the amount of late consumption that a late consumer would get if the realized state is A and n depositors are withdrawing at date 1. Arguably, it should be the case that the higher the number of depositors who run is, the lower late consumption is, or $\partial c_L(A, n)/\partial n$. Moreover, define n_j^{**} as the maximum number of depositors that a bank can

⁴In the present environment, Goldstein and Pauzner (2005) prove that the equilibrium strategy is always a threshold strategy.

serve under pecking order j without breaking the deposit contract, i.e. while still being able to pay c to all those depositors who withdraw at date 1. After n_j^{**} , the bank goes into bankruptcy: there are no more resources for late consumption, and the bank pays $c^B(n)$ according to an equal service constraint, i.e. it equally splits the available resources among the n depositors who withdraw.

Define the expected utility from waiting $\mathbb{E}[u(c_L(A, n))]$, given the signal σ and the number n of depositors who withdraw, as:

$$\mathbb{E}[u(c_L(A, n))] = \int_{-\epsilon}^{\epsilon} (\sigma - e)u(c_L(R, n))\frac{1}{2\epsilon}de + \int_{-\epsilon}^{\epsilon} (1 - \sigma + e)u(c_L(0, n))\frac{1}{2\epsilon}de. \quad (13)$$

It is immediate to verify that this reduces to:

$$\mathbb{E}[u(c_L(A, n))] = \sigma u(c_L(R, n)) + (1 - \sigma)u(c_L(0, n)). \quad (14)$$

Then, the utility advantage of waiting versus running, for a given number n of depositors who withdraw and pecking order j , is:

$$v_j(A, n) = \begin{cases} \sigma u(c_L(R, n)) + (1 - \sigma)u(c_L(0, n)) - u(c) & \text{if } \lambda \leq n < n_j^{**}, \\ -u(c(n)) & \text{if } n_j^{**} \leq n < 1. \end{cases} \quad (15)$$

The number of depositors who withdraw at date 1 is given by the sum of the λ early consumers and those among the $1 - \lambda$ late consumers who receive a signal lower than the threshold signal σ^* :

$$n = \lambda + (1 - \lambda)Prob(\sigma \leq \sigma_j^*) = \lambda + (1 - \lambda) \max\left(\frac{\sigma_j^* - p + \epsilon}{2\epsilon}, 0\right). \quad (16)$$

Thus, n is a random variable that depends on the state of the economy. Importantly, as σ is a random variable, the Laplacian Property (Morris and Shin, 1998) tells us that its cumulative distribution function $Prob(\sigma \leq \sigma_j^*)$ is uniformly distributed over the interval $[0, 1]$, thus the number of depositors who withdraw n must be uniformly distributed over the interval $[\lambda, 1]$.

This allows us to calculate the expected value of waiting versus running as:

$$\mathbb{E}[v_j(A, n)|\sigma] = \int_{\lambda}^1 \frac{v_j(A, n)}{1 - \lambda} dn, \quad (17)$$

and to characterize the threshold signal σ_j^* as the one such that $\mathbb{E}[v_j(A, n)|\sigma_j^*] = 0$.

From what said so far, it is clear that the decision of the late consumers about whether to join a run depends, in turns, on the decision of the bank about how to finance early withdrawals, i.e. on the pecking order with which it employs liquidation of the productive asset and liquidity. In what follows, we characterize and compare the withdrawing behavior of the depositors under each pecking order, in particular by studying its effects on the lower dominance region and the threshold strategies.

3.1 Pecking order 1: {Liquidation; Liquidity}

In this first case, the bank serves the depositors who withdraw at date 1 first by liquidating the productive asset, and then by employing the liquidity in portfolio. Under this pecking order, the threshold signal $\underline{\sigma}_1$ characterizing the lower dominance region is the one that equalizes:

$$u(c) = \underline{\sigma}_1 u\left(\frac{R(1-L-\frac{\lambda c}{r})+L}{1-\lambda}\right) + (1-\underline{\sigma}_1)u\left(\frac{L}{1-\lambda}\right). \quad (18)$$

This expression states that a late consumer receiving a signal $\underline{\sigma}_1$ must be indifferent between withdrawing at date 1 and getting c and waiting until date 2 and getting $\frac{R(1-L-\frac{\lambda c}{r})+L}{1-\lambda}$ with probability $\underline{\sigma}_1$ or $\frac{L}{1-\lambda}$ with probability $1-\underline{\sigma}_1$. These values come from the fact that, by liquidating the productive asset first, the bank withholds liquidity, that finances late consumption irrespective of the realization of the aggregate state. Moreover, the bank has to pay an amount of early consumption c to λ early consumers, by liquidating an amount D of productive assets at rate r , hence $D = \lambda c/r$. Rearranging the equality above, we obtain the threshold:

$$\underline{\sigma}_1 = \frac{u(c) - u\left(\frac{L}{1-\lambda}\right)}{u\left(\frac{R(1-L-\frac{\lambda c}{r})+L}{1-\lambda}\right) - u\left(\frac{L}{1-\lambda}\right)}, \quad (19)$$

which is clearly increasing in the amount of early consumption c set in the deposit contract.

The threshold strategy in the intermediate region $[\underline{\sigma}_1, \bar{\sigma}]$, instead, depends on the late con-

sumers' advantage of waiting versus running:

$$v_1(A, n) = \begin{cases} \sigma u \left(\frac{R(1-L-\frac{nc}{r})+L}{1-n} \right) + (1-\sigma)u \left(\frac{L}{1-n} \right) - u(c) & \text{if } \lambda \leq n < n_1^* \\ \sigma u \left(\frac{r(1-L)+L-nc}{1-n} \right) + (1-\sigma)u \left(\frac{r(1-L)+L-nc}{1-n} \right) - u(c) & \text{if } n_1^* \leq n < n_1^{**} \\ -u \left(\frac{r(1-L)+L}{n} \right) & \text{if } n_1^{**} \leq n < 1. \end{cases} \quad (20)$$

In this expression, $n_1^* = \frac{r(1-L)}{c}$ and $n_1^{**} = \frac{r(1-L)+L}{c}$ are the maximum number of depositors that a bank can serve at date 1 without breaking the deposit contract, and either liquidating the whole amount of productive assets in portfolio (up to n_1^*) or using also liquidity (up to n_1^{**}). When the number of depositors who withdraw at date 1 lies in the interval $[\lambda, n_1^*]$, the bank fulfills its contractual obligation by retaining liquidity, and liquidating the productive asset: it needs to pay an amount of early consumption c to n depositors via rD resources from liquidation, hence $D = \frac{nc}{r}$. Then, if n depositors withdraw, the consumption of a late consumer who waits is either $c_L(R, n) = \frac{R(1-L-\frac{nc}{r})+L}{1-n}$ or $c_L(0, n) = \frac{L}{1-n}$, depending on the realization of the aggregate state. When the number of depositors who withdraw lies in the interval $[n_1^*, n_1^{**}]$, the bank, instead, fulfills its contractual obligation by liquidating all productive assets in portfolio (thus generating resources equal to $r(1-L)$) and by employing the liquidity. Thus, if n depositors withdraw, the consumption of a late consumer who waits is independent of the realization of the aggregate state (as the productive assets have all been liquidated) and equal to $c_L^L(n) = \frac{r(1-L)+L-nc}{1-n}$. Finally, when the number of depositors who withdraw lies in the interval $[n_1^{**}, 1]$, the bank goes bankrupt, as it does not hold sufficient resources to pay an amount of early consumption c to all depositors. In this case, the bank is forced to liquidate all productive assets and close down, so a late consumer who waits gets zero. Moreover, the available resources (equal to $r(1-L) + L$) are equally split among all the n depositors who withdraw, and each one gets $c^B(n) = \frac{r(1-L)+L}{n}$.

The sign of the strategic complementarity affecting the decision of a late consumer to run depends on how the advantage of waiting versus running depends on the number of depositors

withdrawing. More formally:

$$\frac{\partial v_1}{\partial n} = \begin{cases} \sigma u'(c_L(R, n)) \frac{-\frac{R}{r}c(1-n) + [R(1-L-\frac{nc}{r})+L]}{(1-n)^2} + \frac{(1-\sigma)u'(c_L(0, n))L}{(1-n)^2}, & \text{if } \lambda \leq n < n_1^* \\ u'(c_L^L(n)) \frac{r(1-L)+L-c}{(1-n)^2}, & \text{if } n_1^* \leq n < n_1^{**} \\ u'(c^B(n)) \frac{c^B(n)}{n} & \text{if } n_1^{**} \leq n < 1. \end{cases} \quad (21)$$

On the one side, in the interval $[n_1^{**}, 1]$ the derivative is positive as, after bankruptcy, equal service prescribes total resources to be shared pro-rata to all depositors; on the other side, in the interval $[n_1^*, n_1^{**}]$ the derivative is negative by definition of n_1^{**} , highlighting the presence of one-sided strategic complementarities. We characterize the direction of the strategic complementarity in the interval $[\lambda, n_1^*]$ in the following lemma:

Lemma 2. *In the interval $[\lambda, n_1^*]$, $v_1(A, n)$ is decreasing in n .*

Proof. In Appendix A. ■

Figure 1 shows that, despite the different environment, the economy exhibits one sided strategic complementarities as in Goldstein and Pauzner (2005): the advantage of waiting versus running is decreasing in the number of depositors running before bankruptcy, and increasing after bankruptcy. However, despite not knowing the sign of $v_1(A, n_1^*)$, the function $v_1(A, n)$ crosses zero only once, because is decreasing in n in both intervals $[\lambda, n_1^*]$ and $[n_1^*, n_1^{**}]$, and this guarantees the uniqueness of the equilibrium in the intermediate region $[\underline{\sigma}_1, \bar{\sigma}]$.

Lemma 3. *Under the pecking order $\{\text{Liquidation}; \text{Liquidity}\}$, in the intermediate region $[\underline{\sigma}_1, \bar{\sigma}]$ a late consumer runs if her signal is lower than the threshold signal:*

$$\sigma_1^* = \frac{\int_{\lambda}^{n_1^{**}} u(c) dn + \int_{n_1^*}^1 u\left(\frac{L+r(1-L)}{n}\right) dn - \int_{\lambda}^{n_1^*} u\left(\frac{L}{1-n}\right) dn - \int_{n_1^*}^{n_1^{**}} u\left(\frac{L+r(1-L)-nc}{1-n}\right) dn}{\int_{\lambda}^{n_1^*} \left[u\left(\frac{R(1-L-\frac{nc}{r})+L}{1-n}\right) - u\left(\frac{L}{1-n}\right) \right] dn}. \quad (22)$$

The threshold signal σ_1^ is increasing in c and decreasing in L .*

Proof. In Appendix A. ■

The lemma characterizes the endogenous threshold signal in the case of pecking order $\{\text{Liquidation}; \text{Liquidity}\}$, and shows the effect that the bank's deposit contract and asset portfolio

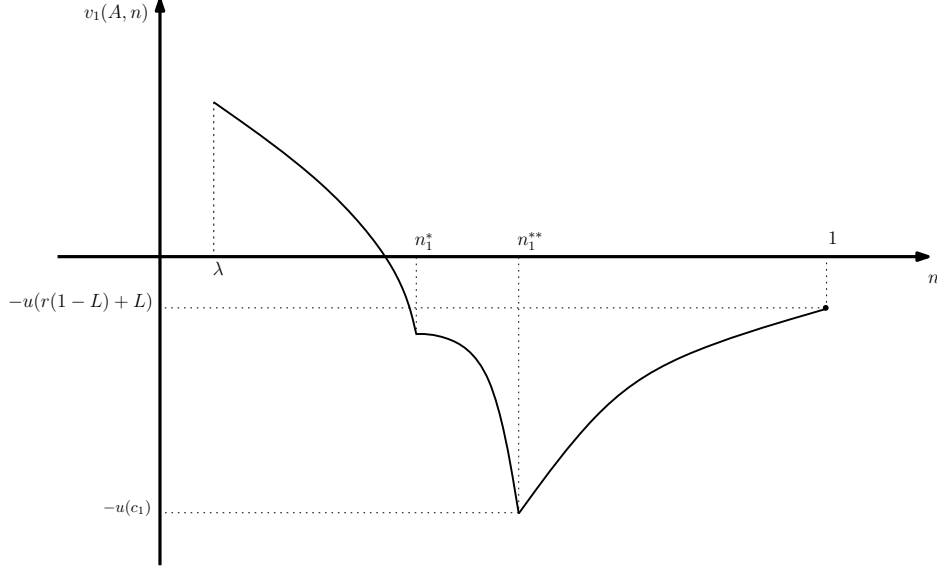


Figure 1: The advantage of waiting versus running, as a function of the number of depositors running, when the bank chooses the pecking order {Liquidation; Liquidity}.

lio have on it. In particular, increasing early consumption c has a threefold positive effect on the threshold signal σ_1^* : it directly increases the advantages for a late consumer to run, both before and after bankruptcy; it lowers the maximum fraction of depositors that a bank can serve before bankruptcy; it decreases the advantages of waiting until date 2. The effect that increasing the total amount of liquidity in the bank's portfolio has on the threshold signal σ_1^* instead looks ambiguous: on the one side, more liquidity increases consumption after bankruptcy and lowers late consumption, thus also increasing the threshold and the incentives to run; on the other side, more liquidity also increases the amount of insurance against the aggregate productivity shock that a bank can provide, thus lowering the threshold and the incentives to run. However, the effect that one more unit of liquidity has on the marginal utility of those depositors running just before bankruptcy (i.e. at n_1^{**}) is large because of the Inada Conditions. Thus, the second effect dominates the first, and the threshold signal σ_1^* turns out to be decreasing in L .

3.2 Pecking order 2: {Liquidity; Liquidation}

In this second case, we assume that the bank serves the withdrawers at date 1 first by employing the liquidity, and then by liquidating the productive asset. Under this pecking order, the threshold signal $\underline{\sigma}_2$ characterizing the lower dominance region is the one that equalizes:

$$u(c) = \underline{\sigma}_2 u\left(\frac{R(1-L) + L - \lambda c}{1 - \lambda}\right) + (1 - \underline{\sigma}_2) u\left(\frac{L - \lambda c}{1 - \lambda}\right). \quad (23)$$

This expression states that a late consumer receiving a signal $\underline{\sigma}_2$ must be indifferent between withdrawing at date 1 and getting c and waiting until date 2 and getting $\frac{R(1-L)+L-\lambda c}{1-\lambda}$ with probability $\underline{\sigma}_2$ or $\frac{L-\lambda c}{1-\lambda}$ with probability $1 - \underline{\sigma}_2$. These values come from the fact that, by employing liquidity first, the bank withholds the productive asset. Hence, having to pay an amount of early consumption c to λ early consumers, it rolls over an amount $L - \lambda c$ of excess liquidity from date 1 to date 2. Rearranging the equality above, we obtain the threshold:

$$\underline{\sigma}_2 = \frac{u(c) - u\left(\frac{L-\lambda c}{1-\lambda}\right)}{u\left(\frac{R(1-L)+L-\lambda c}{1-\lambda}\right) - u\left(\frac{L-\lambda c}{1-\lambda}\right)}. \quad (24)$$

As for the previous pecking order, this value is increasing in the amount of early consumption c set in the deposit contract. To see that, it suffices to calculate:

$$\frac{\partial \underline{\sigma}_2}{\partial c} = \frac{u'(c) + \frac{\lambda}{1-\lambda} u'\left(\frac{L-\lambda c}{1-\lambda}\right) + \underline{\sigma}_2 \frac{\lambda}{1-\lambda} \left[u'\left(\frac{R(1-L)+L-\lambda c}{1-\lambda}\right) - u'\left(\frac{L-\lambda c}{1-\lambda}\right) \right]}{u\left(\frac{R(1-L)+L-\lambda c}{1-\lambda}\right) - u\left(\frac{L-\lambda c}{1-\lambda}\right)}. \quad (25)$$

This expression is always positive, as $\underline{\sigma}_2$ is lower than 1.

The threshold strategy in the intermediate region $[\underline{\sigma}_1, \bar{\sigma}]$, instead, depends on the late consumers' advantage of waiting versus running:

$$v_2(A, n) = \begin{cases} \sigma u\left(\frac{R(1-L)+L-nc}{1-n}\right) + (1-\sigma)u\left(\frac{L-nc}{1-n}\right) - u(c) & \text{if } \lambda \leq n < n_2^* \\ \sigma u\left(\frac{R(1-L-D)}{1-n}\right) - u(c) = \sigma u\left(\frac{R(1-L-\frac{nc-L}{r})}{1-n}\right) - u(c) & \text{if } n_2^* \leq n < n_2^{**} \\ -u\left(\frac{L+r(1-L)}{n}\right) & \text{if } n_2^{**} \leq n < 1 \end{cases} \quad (26)$$

where, similarly to the previous case, $n_2^* = \frac{L}{c}$ and $n_2^{**} = \frac{r(1-L)+L}{c}$ are the maximum number of depositors that a bank can serve at date 1 without breaking the deposit contract and using liquidity (up to n_2^*), and also liquidating the whole amount of productive assets in portfolio (up to n_2^{**}). When the number of depositors who withdraw lies in the interval $[\lambda, n_2^*]$, the bank fulfills its contractual obligation by keeping the productive asset and using liquidity. Hence, if n depositors are withdrawing, the consumption of a late consumer who waits is either $c_L(R, n) = \frac{R(1-L)+L-nc}{1-n}$ or $c_L(0, n) = \frac{L-nc}{1-n}$, depending on the realization of the aggregate productivity shock. When the number of depositors who withdraw lies instead in the interval $[n_2^*, n_2^{**}]$, the

bank is forced to fulfill its contractual obligation also by liquidating the productive assets in portfolio (thus generating resources equal to $r(1 - L)$). Hence, the total available resources to provide early consumption c to n depositors who withdraw are $L + rD$, meaning that the amount that the bank liquidates is equal to $D = \frac{nc - L}{r}$. Moreover, as the liquidity has been exhausted, the consumption of a late consumer who decides to wait and finds herself in the aggregate state where $A = 0$ is zero, while in the aggregate state when A is positive is $c_L^D(R, n) = \frac{R(1 - L - \frac{nc - L}{r})}{1 - n}$. Finally, when the number of depositors who withdraw lies in the interval $[n_2^{**}, 1]$, the bank is bankrupt. Thus, by the equal service constraint, all the n depositors who withdraw get $\frac{r(1 - L) + L}{n}$, and those $1 - n$ who do not withdraw get zero. Notice that the total number of depositors that can be served before bankruptcy is the same under the two pecking orders. Hence, to economize on notation, we write $n_1^{**} = n_2^{**} = n^{**}$.

We again study the sign of the strategic complementarities by taking the derivative of $v_2(A, n)$ with respect to n :

$$\frac{\partial v_2}{\partial n} = \begin{cases} \sigma u'(c_L(R, n)) \frac{c_L(R, n) - c}{1 - n} - (1 - \sigma) u'(c_L(0, n)) \frac{c - c_L(0, n)}{1 - n} & \text{if } \lambda \leq n < n_2^* \\ \sigma u'(c_L^D(R, n)) \frac{c_L^D(R, n) - \frac{Rc}{1 - n}}{1 - n} & \text{if } n_2^* \leq n < n^{**} \\ u'(c^B(n)) \frac{c^B(n)}{n} & \text{if } n^{**} \leq n < 1 \end{cases} \quad (27)$$

As before, in the interval $[n^{**}, 1]$ the derivative is positive, while in the interval $[n_2^*, n^{**}]$ is negative by definition of n^{**} . We characterize the sign of the strategic complementarity in the interval $[\lambda, n_2^*]$ in the following lemma:

Lemma 4. *In the interval $[\lambda, n_2^*]$, $v_2(A, n)$ is decreasing in n whenever is non-positive.*

Proof. In Appendix A. ■

In order to guarantee the uniqueness of the equilibrium, we need to show that $v(A, n_2^*) < 0$. To this end, notice that:

$$v_2(A, n_2^*) = \sigma u\left(\frac{R(1 - L)}{c - L}c\right) - u(c). \quad (28)$$

This expression is negative if:

$$\sigma < \frac{u(c)}{u\left(\frac{R(1 - L)}{c - L}c\right)} \equiv \tilde{\sigma}, \quad (29)$$

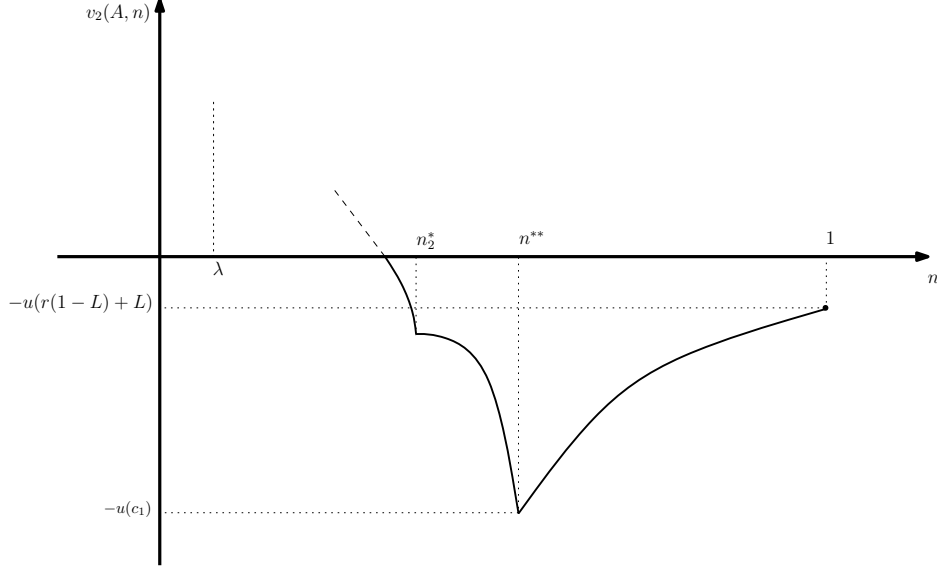


Figure 2: The advantage of waiting versus running, as a function of the number of depositors running, when the bank chooses the pecking order {Liquidity; Liquidation}.

where $\tilde{\sigma} > 1$ whenever $R < \frac{c-L}{1-L}$. In the proof of lemma 6, we show that this condition holds in the banking equilibrium under the pecking order {Liquidity, Liquidation}. Hence, $v_2(A, n_2^*) < 0$, because σ is always lower than 1 by definition. In this way, $v_2(A, n)$ crosses zero only once in the interval $[\lambda, n_2^*]$, and that is sufficient for the solution to exist and be unique.

With this result in hand, we can characterize the threshold signal that makes a late consumer indifferent between waiting or running:

Lemma 5. *Under the pecking order {Liquidity, Liquidation}, in the intermediate region $[\underline{\sigma}_2, \bar{\sigma}]$ a late consumer runs if her signal is lower than the threshold signal:*

$$\sigma_2^* = \frac{\int_{\lambda}^{n^{**}} u(c) dn + \int_{n^{**}}^1 u\left(\frac{L+r(1-L)}{n}\right) dn - \int_{\lambda}^{n_2^*} u\left(\frac{L-nc}{1-n}\right) dn}{\int_{\lambda}^{n_2^*} \left[u\left(\frac{R(1-L)+L-nc}{1-n}\right) - u\left(\frac{L-nc}{1-n}\right) \right] dn + \int_{n_2^*}^{n^{**}} u\left(\frac{R(1-L-\frac{nc-L}{r})}{1-n}\right) dn}. \quad (30)$$

The threshold signal σ_2^* is increasing in c , and decreasing in L .

Proof. In Appendix A. ■

Intuitively, the lemma shows that increasing early consumption c has a positive effect on the threshold signal σ_2^* for many concurrent reasons. First, as in the pecking order {Liquidation, Liquidity}, it directly increases the advantages of running before bankruptcy. Moreover, it de-

creases the advantages of waiting until date 2, either by decreasing the amount of excess liquidity rolled over to date 2 or by forcing the bank to liquidate more productive assets, whenever the liquidity has been completely exhausted. Finally, increasing c has a negative effect on the amount of insurance against the aggregate productivity shock that a bank can provide, and that in turns increases the threshold signal and the incentives to run. In contrast, increasing the amount of liquidity has an ambiguous effect on the threshold signal: on the one side, more liquidity increases the available resources after bankruptcy, and lower the investment in the productive asset, thus increasing the threshold and the incentives for a late consumer to run; on the other side, it allows the bank to provide more insurance against the aggregate productivity shock, and to lower the amount of productive assets to liquidate, thus lowering the incentives to run. However, the effect that one more unit of liquidity has on the marginal utility of those depositors running just before the bank runs out of liquidity (i.e. at n_1^*) and bankruptcy (i.e. at n^{**}) is again large by the Inada Conditions. Thus, this second effect dominates the first, and the threshold probability σ_1^* is decreasing in L .

3.3 Endogenous Pecking Order

At date $t = 1$, given the deposit contract and the asset portfolio, the bank decides the optimal pecking order with which to employ the assets in its portfolio, as a best response to the withdrawing decisions of the depositors. More formally, define:

$$V_j(c, L) \equiv \int_0^{\sigma_j^*} u(L + r(1 - L))dp + \int_{\sigma_j^*}^1 \left[\lambda u(c) + (1 - \lambda) \left[pu(c_L(R)) + (1 - p)u(c_L(0)) \right] \right] dp \quad (31)$$

as the expected utility of a depositor, when her bank offers an amount c of early consumption, holds an amount L of liquidity, and chooses the pecking order j . If $c \geq L + r(1 - L)$ and $L < 1$, the above expression is decreasing in σ_j^* . Hence, maximizing the expected utility of a depositor is equivalent to choosing the pecking order with the lowest threshold signal σ_j^* . That will crucially depend on the recovery rate from liquidating the productive asset, as the following proposition shows:

Proposition 1. *Assume that the utility function belongs to the CRRA class, and that the coefficient of relative risk aversion is sufficiently high. Then, there exists a threshold $\tilde{r} \in [0, 1]$ such that, for any $r \leq \tilde{r}$, the optimal pecking order is $\{Liquidity, Liquidation\}$, and for any*

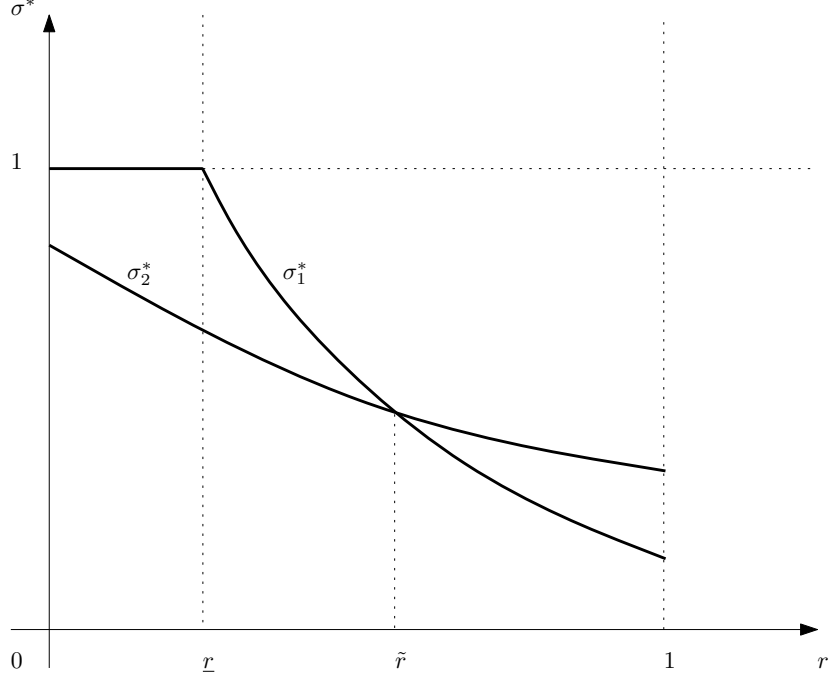


Figure 3: The threshold signals in the two pecking orders (on the y-axis), for different values of the recovery rate of the productive asset (on the x-axis).

$r > \tilde{r}$, the optimal pecking order is $\{Liquidation, Liquidity\}$.

Proof. In Appendix A. ■

The proof of this result is based on showing that the threshold signals under the two pecking orders adjust to changes in the recovery rate of the productive asset as Figure 3 shows. First, both threshold signals σ_1^* and σ_2^* are decreasing and convex functions of the recovery rate. This happens because, when the number of depositors who are running is n^{**} (i.e. the value that triggers bankruptcy under both pecking orders) a late consumer who does not join a run gets zero. Hence, increasing the recovery rate by one marginal unit makes her consumption go from zero to a positive value. This, by the Inada conditions, has a large positive effect on the utility of waiting (although decreasing, because of the concavity of $u(c)$) and lowers both threshold signals.

Second, the comparison between the two pecking orders essentially boils down to comparing the costs associated with using either liquidation or liquidity to finance early withdrawals. On the one hand, liquidation of the productive asset at date $t = 1$ is costly in terms of (i) forgone resources at date $t = 1$ due to a low recovery rate (i.e. $r < 1$) and (ii) forgone late consumption in the good state of the world. On the other hand, using liquidity is costly in terms of forgone

late consumption in the bad state of the world, i.e. in terms of diminished insurance against the aggregate productivity shock. If the depositors are sufficiently risk averse and the recovery rate r is close to 1, both costs associated with liquidation become less relevant, because the depositors care relatively less about high late consumption in the good state of the word and the bank waists less resources when liquidating the productive asset. The opposite is true with respect to the cost associated with using liquidity because, being very risk averse, the depositors care a lot about late consumption in the bad state of the world. Therefore, with sufficiently high relative risk aversion and a recovery rate r close to 1, $\{\text{Liquidation}; \text{Liquidity}\}$ is the optimal pecking order.

If instead the recovery rate is close to zero, liquidation becomes very costly, and this is enough to ensure that $\{\text{Liquidity}; \text{Liquidation}\}$ is the optimal pecking order. This happens because a late consumer who does not join a run is worse off under the pecking order $\{\text{Liquidation}; \text{Liquidity}\}$ than under $\{\text{Liquidity}; \text{Liquidation}\}$: on the one side, the threshold signal σ_1^* under the pecking order $\{\text{Liquidation}; \text{Liquidity}\}$ is constant and equal to one, i.e. there exists a lower bound $\underline{r}$ for the recovery rate, below which all late consumers would rather withdraw early than wait, irrespective of the number of depositors running, hence any signal would lead to a run; on the other side, the threshold signal σ_2^* under the pecking order $\{\text{Liquidity}; \text{Liquidation}\}$ is always lower than 1 when the recovery rate is equal to zero.

To sum up, under the assumptions of Proposition 1, the graphs of the two threshold signals can meet at most once for any recovery rate in the interval $[0, 1]$. This means that the bank prefers the pecking order $\{\text{Liquidation}; \text{Liquidity}\}$ only if the recovery rate of the productive asset is sufficiently high, so that it can liquidate at lower costs and roll over liquidity to the final period to ensure the depositors against the aggregate productivity shock. If instead the recovery rate of the productive asset is low, the bank prefers the pecking order $\{\text{Liquidity}; \text{Liquidation}\}$. In other words, Proposition 1 rationalizes the typical sequence of events emerging when a bank faces a self-fulfilling run, and makes it explicitly contingent on the recovery rate: if this last one is sufficiently low, a bank facing a run is first liquid, then illiquid but solvent, and finally insolvent.

4 Banking Equilibrium

With the results of the previous section in hand, that characterize the behavior of the depositors and the optimal pecking order, we can solve for the banking equilibrium. At date 0, the bank chooses the deposit contract and asset portfolio so as to maximize the expected welfare of the depositors. More formally, under any pecking order j , it solves:

$$\max_{c,L} \int_0^{\sigma_j^*} u(L + r(1-L))dp + \int_{\sigma_j^*}^1 \left[\lambda u(c) + (1-\lambda) \left[pu \left(\frac{R(1-L) + L - \lambda c}{1-\lambda} \right) + (1-p)u \left(\frac{L - \lambda c}{1-\lambda} \right) \right] \right] dp, \quad (32)$$

subject to the liquidity constraint $L \geq \lambda c$, and to the incentive compatibility constraint $c \leq \frac{R(1-L) + L - \lambda c}{1-\lambda}$. When the signal is below the threshold signal σ_j^* , a run happens, either fundamental or self-fulfilling: all depositors get an equal share of the liquidation value of the whole asset portfolio. When instead the signal is above the threshold signal σ_j^* , a run does not happen: a fraction λ of depositors are early consumers, and consume c , while a fraction $1 - \lambda$ of them are late consumers, and consume either $c_L(R) = \frac{R(1-L) + L - \lambda c}{1-\lambda}$, if the productive assets yields a positive return, or $c_L(0) = \frac{L - \lambda c}{1-\lambda}$, if it yields zero. Under both pecking orders j , define the difference between the utility in the case of no-run and the utility in the case of run as:

$$\Delta U(c, L) = \lambda u(c) + (1-\lambda) \left[\sigma_j^* u \left(\frac{R(1-L) + L - \lambda c}{1-\lambda} \right) + (1 - \sigma_j^*) u \left(\frac{L - \lambda c}{1-\lambda} \right) \right] + u(L + r(1-L)), \quad (33)$$

Then, from the first-order conditions of the program, we prove the following:

Lemma 6. *Under any pecking order j , the banking equilibrium features excess liquidity ($L > \lambda c$). The deposit contract and asset portfolio satisfy the distorted Euler equation:*

$$\int_{\sigma_j^*}^1 \left[u'(c) - p R u'(c_L(R)) \right] dp + \sigma_j^* (1-r) u'(L + r(1-L)) = \left[\frac{\partial \sigma_j^*}{\partial L} + \frac{1}{\lambda} \frac{\partial \sigma_j^*}{\partial c} \right] \Delta U(c, L), \quad (34)$$

for any pecking order j .

Proof. In Appendix A. ■

The proof of the Lemma is in part similar to the one of the equilibrium with perfect information: by the Inada conditions, the bank finds optimal to provide insurance against the aggregate productivity shock by engaging in precautionary savings, i.e. by holding more liquidity than the one needed to insure the depositors against the idiosyncratic liquidity shocks. However, differently from the equilibrium with perfect information, the bank imposes a wedge between the marginal rate of substitution between early and late consumption and the marginal rate of transformation of the productive asset (as represented by the first term on the left-hand side of (34)). This happens through two different channels: first, the bank takes into account that it needs higher liquidity to finance consumption in the case of a run (the second term on the left-hand side of (34)); second, it also takes into account that the equilibrium deposit contract and asset portfolio affect the endogenous threshold signal σ_j^* and, therefore, the probability that a run is realized (the right-hand side of (34)). The direction of the distortion with respect to the equilibrium with perfect information depends on the sign of the wedge:

Proposition 2. *Under any pecking order j , if λ is sufficiently large, the banking equilibrium features more excess liquidity than the equilibrium with perfect information, i.e. $c^{BE} < c^{PI}$ and $L^{BE} > L^{PI}$.*

Proof. In Appendix A. ■

Intuitively, the anticipation of self-fulfilling runs imposes a positive wedge between the marginal rate of substitution between early and late consumption and the expected marginal rate of transformation of the productive asset, and this wedge forces the bank to lower early consumption and increase liquidity with respect to the equilibrium with perfect information. In other words, the bank reacts to the anticipation of self-fulfilling runs by increasing the amount of excess liquidity ($L^{BE} - \lambda c^{BE}$) in portfolio. This happens whenever the marginal effect of the bank portfolio choice on the threshold signals σ_j^* (the right-hand side of the distorted Euler equation (34)) is larger than the expected marginal utility of consumption at a run (the second term on the left-hand side of (34)). A sufficient condition for this to happen is that the utility differential $\Delta U(c, L)$ between no-run and run is sufficiently large, which in turns is guaranteed if the probability of the idiosyncratic liquidity shock λ is sufficiently large, too. Moreover, the marginal effect of one additional unit of liquidity on the threshold signals σ_j^* is always negative, and is stronger under the pecking order {Liquidation; Liquidity} than under {Liquidity;

Liquidation} because, under the former, liquidity is not depleted in the initial stages of a run and is consumed in the last period in the bad state of the world, and therefore has a larger diminishing effect on the depositors' incentives to run. Thus, in order to induce a positive wedge and excess liquidity, the requirement on the utility differential $\Delta U(c, L)$ (and consequently on λ) is stronger under the pecking order {Liquidation; Liquidity} than under {Liquidity; Liquidation}.

The last question to be addressed regards the viability of the banking equilibrium. In other words, would the agents prefer the banking equilibrium, where the bank provides insurance against the idiosyncratic liquidity shocks at the cost of possible self-fulfilling runs, or the autarkic equilibrium of section 2.1, where banks are not available and the agents have to independently invest their endowments, but are run free? To answer this question, we compare the equilibrium conditions of the two equilibria, in equation (5) and (34). By doing this, the following can be proved:

Corollary 1. *[**WRONG!!!**] Under any pecking order j , if:*

$$1 - \frac{1}{R} < \frac{\lambda(1-r)}{1-\lambda} \quad (35)$$

holds, then $c^{BE} > c^A$.

Proof. In Appendix A. ■

The intuition for this result highlights the key role played by idiosyncratic risk in making the banking system desirable: if the probability of being hit by the idiosyncratic liquidity shock θ is sufficiently high, the advantage of making a deposit into the banking system, and enjoy risk pooling, is of first order with respect to the risk of facing financial fragility via a possible self-fulfilling run. Moreover, if the recovery rate r were equal to 1, the bank would be able to fully recover the value of the initial deposits in the case of a run, and therefore no run would happen in equilibrium, if the bank also offered a deposit contract equivalent to autarky (i.e. without insurance against the idiosyncratic shocks). However, as the recovery rate r is indeed smaller than 1, the bank is only able to serve a fraction smaller than 1 of depositors in the case of a run: in other words, offering a deposit contract equivalent to the autarkic allocation would not make the bank run proof.

4.1 Narrow Banking

The concluding argument of the previous section raises the question of what can actually make the banking system run proof, and whether a run-proof banking system can be viable at all. Specifically, the subject of the present section pertains the effectiveness of a business model that is generally labelled “narrow banking”. According to Pennacchi (2012), “A narrow bank is a financial institution that issues demandable liabilities and invests in assets that have little or no nominal interest rate and credit risk”. An example of narrow banking, proposed in the 1930s by a group of University of Chicago economists and since then called the “Chicago Plan”, has regained momentum after the recent U.S. financial crisis (Benes and Kumhof, 2012): its intended aim is to gain a better control of the credit cycle by reducing harmful liquidations, and eliminate bank runs by forcing the banks to hold an amount of cash reserves equal to their demand deposits (Fisher, 1936).

In what follows, we analyze the effect of forcing the banks to be narrow, by imposing on the banking problem the narrow-banking constraint $L \geq c$, i.e. such that the bank holds sufficient liquidity to pay early consumption to all depositors, even in the case of a run. Remember that, under both pecking orders, the total fraction of depositors that can be served before bankruptcy is $n^{**} = (L + r(1 - L))/c$. Hence, imposing the narrow-banking constraint $L \geq c$ makes n^{**} larger than or equal to 1: in other words, narrow banking rules out self-fulfilling runs, as all depositors anticipate that the liquidation value of the bank asset portfolio is sufficient to pay early consumption to all of them. This is true under both pecking orders, with the caveat that under {Liquidity; Liquidation} the bank can also avoid liquidation as $n_2^* = L/c$, while this might not be true under {Liquidation; Liquidity}. However, the effect of narrow banking is wider. In fact, remember the thresholds $\underline{\sigma}_1$ and $\underline{\sigma}_2$ for the lower dominance region under the two pecking orders in (19) and (24), respectively. If $L \geq c$, it is easy to see that both thresholds become smaller than or equal to 1. Thus, imposing the narrow-banking constraint $L \geq c$ makes the bank immune to self-fulfilling as well as fundamental runs.

To sum up, narrow banking is a business model that has the advantage of being completely run-proof. To characterize its equilibrium deposit contract and asset portfolio, we solve the

following problem:

$$\max_{L,c} \int_0^1 \left[\lambda u(c) + (1-\lambda) \left[pu \left(\frac{R(1-L) + L - \lambda c}{1-\lambda} \right) + (1-p)u \left(\frac{L - \lambda c}{1-\lambda} \right) \right] \right] dp, \quad (36)$$

subject to the incentive compatibility constraint $c \leq c_L(R)$ and to the narrow-banking constraint $L \geq c$. As the latter implies the former, we get rid of the incentive compatibility constraint, and start characterizing the equilibrium in the following Lemma:

Lemma 7. *In the narrow-banking equilibrium, $L^{NB} = c^{NB}$.*

The proof of the Lemma is straightforward, and is based on showing that, if the narrow-banking constraint were slack, the narrow-banking equilibrium would be equivalent to the equilibrium with perfect information. However, that would mean $L^{NB} > c^{NB} = c^{PI} > 1$, which would clearly contradict the bank budget constraint. Then, imposing a binding narrow-banking constraint $L^{NB} = c^{NB}$, the equilibrium allocation becomes $c_L^{NB}(0) = c^{NB} < 1$ and $c_L^{NB}(R) = c^{NB} + \frac{R(1-c^{NB})}{1-\lambda} > 1$, while the equilibrium asset portfolio satisfies the condition:

$$[\lambda + (1-\lambda)\mathbb{E}[p]]u'(c^{NB}) = \mathbb{E}[p](R-1+\lambda)u'(c_L^{NB}(R)). \quad (37)$$

Intuitively, this expression states that a bank subject to narrow banking chooses an asset portfolio such that the expected marginal benefits of holding liquidity, in terms of early consumption and late consumption in the bad state of the world (as $c^{NB} = c_L^{NB}(0)$), must be equal to the expected marginal costs of holding liquidity, in terms of late consumption $c_L^{NB}(R)$ lost in the good state of the world. Yet, the fact that in equilibrium $L^{NB} = c^{NB} < 1$ means that narrow banking may not be viable. To check on that, the following Proposition compare the narrow-banking equilibrium to autarky:

Proposition 3. *If:*

$$\lambda + (1-\lambda)\mathbb{E}[p] \leq \frac{\lambda(1-r)}{1-\lambda} \quad (38)$$

holds, then $c^{NB} < c^A$.

Proof. In Appendix A. ■

Once again, the intuition for this result highlights the crucial role played by idiosyncratic risk. In fact, at the autarkic allocation, the marginal costs of one more unit of liquidity, in

terms of loss of late consumption in the good state of the world, is of first-order with respect to its marginal benefits, in terms of early consumption and late consumption in the bad state of the world. This happens because, in contrast to the narrow-banking equilibrium, in autarky the agents can also liquidate the productive asset, and thus enjoy higher early consumption. Therefore, if the probability of the idiosyncratic liquidity shock is sufficiently high, a narrow bank has to offer lower early consumption than what the agents would get in autarky. In other words, the narrow banking equilibrium is not viable.

5 Concluding Remarks **[REWRITE. MORE INTUITION. MORE POLICY]**

With the present paper, we study a novel mechanism through which systemic risk, in the form of self-fulfilling runs, triggers excessive liquidity holdings in the banking system. To this end, we develop a positive theory of banking with three main features: a liquid asset is available to the banks, to store resources and roll them over time; the concept of excess liquidity is well-defined, by comparison to a benchmark economy with perfect information; banks' asset portfolios and depositors' expectations are jointly and endogenously determined. In such an environment, we show that an endogenous pecking order emerges, with which the banks employ their assets in order to finance withdrawals during a run: if the recovery rate from liquidating the productive asset is sufficiently low, the banks first employs liquidity, then liquidate the productive asset. In this way, our model is the first, to the best of our knowledge, to provide a rationale for the typical chain of events during a bank run: at first, banks are liquid, when they hold sufficient liquidity to honor the deposit contract with the depositors who withdraw; then, they become illiquid but solvent, when they run out of liquidity and start liquidating the productive assets held in portfolio; finally, they are insolvent, when so many depositors are withdrawing early that they do not have sufficient resources to cover their withdrawals. Moreover, we find that the deposit contract and the amount of liquidity in portfolio do have opposite effects on the probability of a run. On the one side, by increasing the amount of risk sharing against the idiosyncratic liquidity shocks (i.e., increasing early consumption), the banks open themselves to the possibility of not being able to repay all depositors in the case of a run: in other words, high early consumption induces a high probability of a run. On the other side, by increasing

the amount of risk sharing against the aggregate productivity shock (i.e., increasing liquidity), the banks also lower the incentive of the depositors to join a run: in other words, high liquidity induces a low probability of a run. In turns, the banks take into account the effects that their portfolio decisions have on the expectations of the depositors, and, as a consequence, on the probability of a run, and distort the intertemporal allocation of the resources in the economy, by optimally choosing to lower the amount of early consumption and increase their holdings of liquidity, with respect to an economy with perfect information. In other words, the possibility of self-fulfilling bank runs, triggered by the contemporaneous presence of banks' risky behaviors and depositors' self-fulfilling expectations, forces the banks to hoard liquidity.

Despite our focus being mainly positive, we can easily extend the present framework, in which banks explicitly hold a portfolio of liquid and illiquid assets, to analyze policy issues. In particular, we could study the effect that ex-ante liquidity requirements and ex-post liquidity guarantees have on the probability of runs and, in turns, on banks' asset portfolios and liquidity holdings. In principle, we expect such government interventions to be considerably effective at reducing the probability of runs. However, the effect on banks' asset portfolios and liquidity holdings are not trivial, as regulation might strengthen or weaken the marginal effects of liquidity and early consumption on the probability of runs. We keep a formal analysis of these issues open to future research.

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Appendices

A Proofs

Proof of Lemma 1. Attach the Lagrange multipliers μ to the liquidity constraint (7) and ξ to the non-negativity constraint of D . The first-order conditions of the program are:

$$c : \quad u'(c) - \int_0^1 \left[pu'(c_L(R)) + (1-p)u'(c_L(0)) \right] dp - \mu = 0, \quad (39)$$

$$L : \quad \int_0^1 \left[pu'(c_L(R))(1-R) + (1-p)u'(c_L(0)) \right] dp + \mu = 0, \quad (40)$$

$$D : \quad \int_0^1 \left[pu'(c_L(R))(r-R) + (1-p)u'(c_L(0))r \right] dp + r\mu + \xi = 0, \quad (41)$$

where $c_L(R)$ and $c_L(0)$ are the state-dependent amounts of late consumption in the cases when the productive assets yields a positive return or zero return, respectively. For the first part of the lemma, plug (40) into (41) and solve for the Lagrange multiplier:

$$\xi = (1-r) \int_0^1 \left[pu'(c_L(R))R + (1-p)u'(c_L(0)) \right] dp, \quad (42)$$

which is clearly positive, implying that $D = 0$ by complementary slackness. For the second part of the lemma, assume that $\mu > 0$, so that $L - \lambda c = 0$ by complementary slackness. Then, $c_L(0) = 0$ and $u'(c_L(0)) \rightarrow \infty$ by the Inada conditions. Then, for the first-order condition with respect to L to hold, it has to be the case that $u'(c_L(R)) \rightarrow \infty$, hence $c_L(R) = 0$ as well. As a consequence, for the first-order condition with respect to c to hold, also $u'(c) \rightarrow \infty$, hence $c = 0$, implying $L = D = 0$. However, $c_L(R) = 0$ also implies that $L + D = 1$, which leads to a contradiction. Finally, use (39) and (40) to derive (10). This ends the proof of the first part of the lemma.

As far as the second part is concerned, rewrite the bank budget constraints as:

$$L + Y = 1, \quad (43)$$

$$L = \lambda c + Z, \quad (44)$$

$$c_L(R)(1 - \lambda) = RY + Z, \quad (45)$$

$$c_L(0)(1 - \lambda) = Z, \quad (46)$$

Where Y is the amount invested at date 0 in the productive asset, and Z is excess liquidity.

Aggregate the budget constraints to derive the intertemporal budget constraint:

$$\lambda c + (1 - \lambda) \left[\frac{c_L(R)}{R} + \left(1 - \frac{1}{R} \right) c_L(0) \right] = 1. \quad (47)$$

Assume that $c \leq 1$. By the Euler equation (10):

$$\mathbb{E}[p]R = \frac{u'(c)}{u'(c_L(R))} > \frac{c_L(R)}{c} \geq c_L(R) > c_L(0), \quad (48)$$

where the first inequality is a consequence of the assumption of relative risk aversion being larger than one,⁵ and the third one holds by construction. Thus, the term in the square brackets of the intertemporal budget constraint (47), as a linear combination of two terms smaller than $\mathbb{E}[p]R$, must be also smaller than $\mathbb{E}[p]R$. Hence:

$$\lambda c + (1 - \lambda)\mathbb{E}[p]R > 1. \quad (49)$$

By continuity, there exists a λ sufficiently large such that this inequality does not hold. Hence, under this condition, we get a contradiction, implying that $c > 1$. By incentive compatibility, this also implies that $c_L(R) > 1$. Finally, for this to be consistent with the intertemporal budget constraint (47), it must be the case that $c_L(0) < 1$. This ends the proof. ■

⁵To see that, rewrite $-\frac{u''(c)c}{u'(c)} > 1$ as $-\frac{u''(c)}{u'(c)} > \frac{1}{c}$. This, in turn, means that $-(\log[u'(c)])' > (\log[c])'$. Integrate between z_1 and $z_2 > z_1$ so as to obtain $\log[u'(z_1)] - \log[u'(z_2)] > \log[z_2] - \log[z_1]$. Once taken the exponent, the last expression gives $\frac{u'(z_1)}{u'(z_2)} > \frac{z_2}{z_1}$. If $z_1 > z_2$, the inequality is reversed.

Proof of Lemma 2. Rewrite the derivative as:

$$\frac{\partial v_1}{\partial n} = \sigma u'(c_L(R, n)) \frac{R(1 - L - \frac{nc}{r}) + L - \frac{R}{r}c(1 - n)}{(1 - n)^2} + (1 - \sigma)u'(c_L(0, n)) \frac{L}{(1 - n)^2}. \quad (50)$$

This expression is negative whenever:

$$\begin{aligned} \sigma u'(c_L(R, n))R \left(\frac{c}{r} - 1 \right) &> L \left[\sigma u'(c_L(R, n))(1 - R) + (1 - \sigma)u'(c_L(0, n)) \right] > \\ &> Lu'(c_L(R, n))(1 - R), \end{aligned} \quad (51)$$

where the last inequality follows from the term in the square bracket being a linear combination of two terms, with $u'(c_L(0, n)) > u'(c_L(R, n))(1 - R)$. Hence, the derivative is negative, provided that:

$$\sigma R \left(\frac{c}{r} - 1 \right) > L(1 - R). \quad (52)$$

As $R > 1$ by assumption, this last expression is always true if $c > r$ (as it turns out in the banking equilibrium). ■

Proof of Lemma 3. The threshold signal σ_1^* is the value of σ that makes a late consumer indifferent between waiting or running, given her posterior believes:

$$\begin{aligned} \int_{\lambda}^{n_1^*} \left[\sigma_1^* u(c_L(R, n)) + (1 - \sigma_1^*)u(c_L(0, n)) \right] dn + \int_{n_1^*}^{n_1^{**}} u(c_L^L(n)) dn = \\ = \int_{\lambda}^{n_1^{**}} u(c) dn + \int_{n_1^*}^1 u(c^B(n)) dn. \end{aligned} \quad (53)$$

Rearranging this expression, we get the threshold signal in (22), below which a late consumer runs. The derivative of the threshold signal σ_1^* with respect to c reads:

$$\begin{aligned} \frac{\partial \sigma_1^*}{\partial c} &= \frac{1}{\int_{\lambda}^{n_1^*} \left[u(c_L(R, n)) - u(c_L(0, n)) \right] dn} \times \\ &\times \left[(n_1^{**} - \lambda)u'(c) + \int_{n_1^*}^{n_1^{**}} u'(c_L^L(n)) \frac{n}{1 - n} dn + \sigma_1^* \int_{\lambda}^{n_1^*} u'(c_L(R, n)) \frac{Rn}{r(1 - n)} dn \right], \end{aligned} \quad (54)$$

which is always positive as the utility function is increasing. In a similar way, the derivative of

the threshold signal σ_1^* with respect to L reads:

$$\begin{aligned} \frac{\partial \sigma_1^*}{\partial L} = & \frac{1}{\int_{\lambda}^{n_1^*} [u(c_L(R, n)) - u(c_L(0, n))] dn} \times \\ & \times \left[\int_{n_1^{**}}^1 u'(c^B(n)) \frac{1-r}{n} dn - \int_{\lambda}^{n_1^*} u'(c_L(0, n)) \frac{1}{1-n} dn - \int_{n_1^*}^{n_1^{**}} u'(c_L^L(n)) \frac{1-r}{1-n} dn + \right. \\ & \left. - \sigma_1^* \int_{\lambda}^{n_1^*} \left[u'(c_L(R, n)) \frac{1-R}{1-n} - u'(c_L(0, n)) \frac{1}{1-n} \right] dn \right]. \end{aligned} \quad (55)$$

Notice that $\lim_{n \rightarrow n_1^{**}} u'(c_L^L(n)) = \lim_{c \rightarrow 0} u'(c)$, which is equal to $+\infty$ by the Inada conditions.

Hence, this expression is negative. This ends the proof. $\blacksquare$

Proof of Lemma 4. By definition, $u(c)$ is strictly concave on an open interval X if and only if:

$$u(x) - u(y) < u'(y)(x - y) \quad (56)$$

for all x and y in X . Hence, when $\lambda \leq n < n_2^*$, it must be the case that:

$$\begin{aligned} \frac{\partial v_2}{\partial n} & < \sigma \frac{u(c_L(R, n)) - u(c)}{1-n} - (1-\sigma) \frac{u(c) - u(c_L(0, n))}{1-n} \\ & < \frac{\sigma u(c_L(R, n)) + (1-\sigma)u(c_L(0, n)) - u(c)}{1-n} = \frac{v_2(A, n)}{1-n} \end{aligned} \quad (57)$$

Thus, whenever $v_2(A, n) \leq 0$, the derivative is negative. This ends the proof. $\blacksquare$

Proof of Lemma 5. The threshold signal σ_2^* is the value of σ that equalizes:

$$\begin{aligned} & \int_{\lambda}^{n_2^*} \left[\sigma_2^* u \left(\frac{R(1-L) + L - nc}{1-n} \right) + (1-\sigma_2^*) u \left(\frac{L - nc}{1-n} \right) \right] dn + \\ & + \int_{n_2^*}^{n^{**}} \sigma_2^* u \left(\frac{R(1-L - \frac{nc-L}{r})}{1-n} \right) dn = \int_{\lambda}^{n^{**}} u(c) dn + \int_{n^{**}}^1 u \left(\frac{L + r(1-L)}{n} \right) dn \end{aligned} \quad (58)$$

Rearranging this expression, we get the threshold signal in (30), below which a late consumer runs. The derivative of the threshold signal σ_2^* with respect to c reads:

$$\frac{\partial \sigma_2^*}{\partial c} = \frac{1}{\int_{\lambda}^{n_2^*} \left[u \left(\frac{R(1-L) + L - nc}{1-n} \right) - u \left(\frac{L - nc}{1-n} \right) \right] dn + \int_{n_2^*}^{n^{**}} u \left(\frac{R(1-L - \frac{nc-L}{r})}{1-n} \right) dn} \times$$

$$\begin{aligned}
& \times \left[(n^{**} - \lambda)u'(c) + \sigma_2^* \left[\int_{\lambda}^{n_2^*} u'(c_L(R, n)) \frac{n}{1-n} dn + \int_{n_2^*}^{n^{**}} u'(c_L^D(R, n)) \frac{Rn}{r(1-n)} dn \right] + \right. \\
& \left. + (1 - \sigma_2^*) \int_{\lambda}^{n_2^*} u'(c_L(0, n)) \frac{n}{1-n} dn \right]. \tag{59}
\end{aligned}$$

This derivative is positive, because the utility function is increasing and $\sigma_2^* \leq 1$. The derivative of the threshold signal σ_2^* with respect to L instead reads:

$$\begin{aligned}
\frac{\partial \sigma_2^*}{\partial L} &= \frac{1}{\int_{\lambda}^{n_2^*} \left[u\left(\frac{R(1-L)+L-nc}{1-n}\right) - u\left(\frac{L-nc}{1-n}\right) \right] dn + \int_{n_2^*}^{n^{**}} u\left(\frac{R(1-L-\frac{nc-L}{r})}{1-n}\right) dn} \times \\
& \times \left[\int_{n^{**}}^1 u'(c^B(n)) \frac{1-r}{n} dn - \int_{\lambda}^{n_2^*} u'(c_L(0, n)) \frac{1}{1-n} dn + \right. \\
& - \sigma_2^* \left[\int_{\lambda}^{n_2^*} u'(c_L(R, n)) \frac{1-R}{1-n} dn - \int_{\lambda}^{n_2^*} u'(c_L(0, n)) \frac{1}{1-n} dn + \right. \\
& \left. \left. + \int_{n_2^*}^{n^{**}} u'(c_L^D(R, n)) R \left(\frac{1}{r} - 1\right) \frac{1}{1-n} dn \right] \right]. \tag{60}
\end{aligned}$$

As $\sigma_2^* < 1$ and:

$$\lim_{n \rightarrow n_2^*} u'(c_L(0, n)) = \lim_{n \rightarrow n^{**}} u'(c_L^D(R, n)) = \lim_{c \rightarrow 0} u'(c) = +\infty \tag{61}$$

by the Inada conditions, this expression is negative.⁶ This ends the proof. ■

Proof of Proposition 1. We study σ_1^* and σ_2^* as functions of the recovery rate r . As $r \rightarrow \underline{r} = \frac{\lambda c}{1-L}$, we have that $n_1^* \rightarrow \lambda$ and the first interval of $v_1(A, n)$ reduces to zero. Thus, the expected value of waiting versus running under the pecking order {Liquidation, Liquidity} becomes:

$$\begin{aligned}
\mathbb{E}[v_1(A, n)] &= \int_{\lambda}^{n^{**}} \left[u\left(\frac{L + r(1-L) - nc}{1-n}\right) - u(c) \right] dn + \\
& - \int_{n^{**}}^1 u\left(\frac{L + r(1-L)}{n}\right) dn. \tag{62}
\end{aligned}$$

This expression is always negative, as the numerator of σ_1^* must be positive. Hence, σ_1^* is constant and equal to 1 in the interval $[0, \underline{r}]$. In the interval $[\underline{r}, 1]$, instead, the threshold signal σ_1^* is a

⁶The derivative is also finite, as $u(c)$ is not differentiable at zero.

decreasing and convex function of the recovery rate r . To see that, calculate:

$$\begin{aligned}
\frac{\partial \sigma_1^*}{\partial r} = & \frac{1}{\left[\int_{\lambda}^{n_1^*} [u(c_L(R, n)) - u(c_L(0, n))] dn \right]^2} \times \\
& \times \left[\int_{n_1^{**}}^1 u'(c^B(n)) \frac{1-L}{n} dn - \int_{n_1^*}^{n_1^{**}} u'(c_L^L(n)) \frac{1-L}{1-n} dn \right] \times \\
& \times \left[\int_{\lambda}^{n_1^*} [u(c_L(R, n)) - u(c_L(0, n))] dn \right] + \\
& - \left[- \int_{\lambda}^{n_1^*} u'(c_L(R, n)) \frac{Rnc}{r^2(1-n)} dn \right] \times \\
& \times \left[\int_{\lambda}^{n_1^{**}} u(c) dn + \int_{n_1^{**}}^1 u(c^B(n)) dn - \int_{\lambda}^{n_1^*} u(c_L(0, n)) dn - \int_{n_1^*}^{n_1^{**}} u(c_L^L(n)) dn \right]. \quad (63)
\end{aligned}$$

By the Inada conditions, we know that $\lim_{n \rightarrow n_1^{**}} u'(c_L^L(n)) = \lim_{c \rightarrow 0} u'(c) = +\infty$. Hence, the derivative must be negative.⁷

To show that the threshold signal σ_1^* is instead a convex function of r , calculate:

$$\begin{aligned}
\frac{\partial^2 \sigma_1^*}{\partial r^2} = & \frac{1}{\left[\int_{\lambda}^{n_1^*} [u(c_L(R, n)) - u(c_L(0, n))] dn \right]^4} \times \\
& \times \left[\left[\left[-\frac{1-L}{c} u'(c) \frac{1-L}{n_1^{**}} + \int_{n_1^{**}}^1 u''(c^B(n)) \left(\frac{1-L}{c} \right)^2 dn + \right. \right. \right. \\
& - \int_{n_1^*}^{n_1^{**}} u''(c_L^L(n)) \left(\frac{1-L}{1-n} \right)^2 dn + \frac{1-L}{c} u' \left(\frac{L}{1-n} \right) \frac{1-L}{1-n_1^*} \Big] \times \\
& \times \left[\int_{\lambda}^{n_1^*} [u(c_L(R, n)) - u(c_L(0, n))] dn \right] + \\
& + \left[\int_{\lambda}^{n_1^*} u'(c_L(R, n)) \frac{Rnc}{r^2(1-n)} dn \right] \times \\
& \times \left[\int_{n_1^{**}}^1 u'(c^B(n)) \frac{1-L}{n} dn - \int_{n_1^*}^{n_1^{**}} u'(c_L^L(n)) \frac{1-L}{1-n} dn \right] + \\
& - \left[\frac{1-L}{c} u' \left(\frac{L}{1-n_1^*} \right) \frac{R(1-L)}{r(1-n_1^*)} dn + \right. \\
& \left. + \int_{\lambda}^{n_1^*} \left[u''(c_L(R, n)) \left(\frac{Rnc}{r^2(1-n)} \right)^2 - 2u'(c_L(R, n)) \frac{Rnc}{r^3(1-n)} \right] dn \right] \times
\end{aligned}$$

⁷Notice that, for any pecking order j , $v_j(A, n)$ is continuous everywhere, but has kinks at n_j^* and n_1^{**} , so it is not differentiable at those points.

$$\begin{aligned}
& \times \left[\int_{\lambda}^{n_1^{**}} u(c) dn + \int_{n_1^{**}}^1 u(c^B(n)) dn - \int_{\lambda}^{n_1^*} u(c_L(0, n)) dn - \int_{n_1^*}^{n_1^{**}} u(c_L^L(n)) dn \right] + \\
& - \left[\int_{\lambda}^{n_1^*} u'(c_L(R, n)) \frac{Rnc}{r^2(1-n)} dn \right] \times \\
& \times \left[\int_{n^{**}}^1 u'(c^B(n)) \frac{1-L}{n} dn - \int_{n_1^*}^{n^{**}} u'(c_L^L(n)) \frac{1-L}{1-n} dn \right] \times \\
& \times \left[\int_{\lambda}^{n_1^*} \left[u(c_L(R, n)) - u(c_L(0, n)) \right] dn \right]^2 + \\
& - \left[\int_{n^{**}}^1 u'(c^B(n)) \frac{1-L}{n} dn - \int_{n_1^*}^{n^{**}} u'(c_L^L(n)) \frac{1-L}{1-n} dn \right] \times \\
& \times \left[\int_{\lambda}^{n_1^*} \left[u(c_L(R, n)) - u(c_L(0, n)) \right] dn \right] + \\
& - \left[\int_{\lambda}^{n_1^{**}} u(c) dn + \int_{n_1^{**}}^1 u(c^B(n)) dn - \int_{\lambda}^{n_1^*} u(c_L(0, n)) dn - \int_{n_1^*}^{n_1^{**}} u(c_L^L(n)) dn \right] \times \\
& \times \left[\int_{\lambda}^{n_1^*} u'(c_L(R, n)) \frac{Rnc}{r^2(1-n)} dn \right] \times \\
& \times 2 \left[\int_{\lambda}^{n_1^*} \left[u(c_L(R, n)) - u(c_L(0, n)) \right] dn \right] \int_{\lambda}^{n_1^*} u'(c_L(R, n)) \frac{Rnc}{r^2(1-n)} dn \Big]. \tag{64}
\end{aligned}$$

By definition of CRRA utility:

$$\frac{u''(c)}{u'(c)} = -\frac{\gamma}{c}, \tag{65}$$

where γ is the constant coefficient of relative risk aversion. This implies that:

$$\lim_{n \rightarrow n^{**}} \frac{u''(c_L^L(n))}{u'(c_L^L(n))} = \lim_{c \rightarrow 0} \frac{u''(c)}{u'(c)} = -\lim_{c \rightarrow 0} \frac{\gamma}{c} = -\infty. \tag{66}$$

Hence, $u''(c)$ goes to $-\infty$ at a speed faster than the one at which $u'(c)$ goes to $+\infty$, when $c \rightarrow 0$.

This, together with the Inada conditions, ensures that the second derivative is positive, meaning that σ_1^* is a convex function of r .

In contrast, σ_2^* at $r = 0$ is always lower than 1 if R is sufficiently large. To see that, notice that:

$$\sigma_2^*(r = 0) = \frac{\int_{\lambda}^{\frac{L}{c}} u(c) dn + \int_{\frac{L}{c}}^1 u\left(\frac{L}{n}\right) dn - \int_{\lambda}^{\frac{L}{c}} u\left(\frac{L-nc}{1-n}\right) dn}{\int_{\lambda}^{\frac{L}{c}} \left[u\left(\frac{R(1-L)+L-nc}{1-n}\right) - u\left(\frac{L-nc}{1-n}\right) \right] dn}. \tag{67}$$

This expression is lower than 1 if:

$$\int_{\lambda}^{\frac{L}{c}} u(c) dn + \int_{\frac{L}{c}}^1 u\left(\frac{L}{n}\right) dn < \int_{\lambda}^{\frac{L}{c}} u\left(\frac{R(1-L) + L - nc}{1-n}\right) dn. \quad (68)$$

This condition is true if R is sufficiently high, given that $R > c$ must hold. We show that in an example with CRRA utility $u(c) = \frac{(c+\psi)^{1-\gamma} - \psi^{1-\gamma}}{1-\gamma}$, where ψ is a positive but negligible constant (we assume $\psi \simeq 0$) that ensures that $u(0) = 0$, and $\gamma > 1$ is the coefficient of relative risk aversion.⁸ Rewrite (68) as:

$$\int_{\lambda}^{\frac{L}{c}} \left[\frac{\left(\frac{R(1-L) + L - nc}{1-n}\right)^{1-\gamma}}{\gamma-1} - \frac{c^{1-\gamma}}{\gamma-1} \right] dn - \int_{\frac{L}{c}}^1 \frac{\left(\frac{L}{n}\right)^{1-\gamma}}{\gamma-1} dn < 0. \quad (69)$$

This is equivalent to:

$$\int_{\lambda}^{\frac{L}{c}} \left[\left(\frac{(1-L) + \frac{L-nc}{R}}{1-n} \right)^{1-\gamma} R^{1-\gamma} - c^{1-\gamma} \right] dn - \int_{\frac{L}{c}}^1 \left(\frac{L}{n} \right)^{1-\gamma} dn < 0. \quad (70)$$

Multiply the previous expression by $R^{\gamma-1}$, and rewrite it as:

$$\int_{\lambda}^{\frac{L}{c}} \left[\left(\frac{(1-L) + \frac{L-nc}{R}}{1-n} \right)^{1-\gamma} - \left(\frac{c}{R} \right)^{1-\gamma} \right] dn - \int_{\frac{L}{c}}^1 \left(\frac{L}{n} \right)^{1-\gamma} R^{\gamma-1} dn < 0. \quad (71)$$

As $R > c$, $\frac{c}{R}$ is bounded. Therefore, this condition is always satisfied for $R \rightarrow \infty$, as the last integral goes to $-\infty$. By continuity, there must be a sufficiently large and finite value of R such that this is also true.

Having proved that the threshold signal $\sigma_2^* < 1$ at $r = 0$, we want to show that it is also a decreasing and convex function of the recovery rate r . To this end, we first calculate:

$$\begin{aligned} \frac{\partial \sigma_2^*}{\partial r} &= \left[\int_{\lambda}^{n_2^*} \left[u\left(\frac{R(1-L) + L - nc}{1-n}\right) - u\left(\frac{L - nc}{1-n}\right) \right] dn + \int_{n_2^*}^{n^{**}} u\left(\frac{R(1-L - \frac{nc-L}{r})}{1-n}\right) dn \right]^{-2} \times \\ &\times \left[\int_{n^{**}}^1 u'(c^B(n)) \frac{1-L}{n} dn \right] \times \end{aligned}$$

⁸The parameter ψ can be interpreted as a minimum level of consumption that the depositors can enjoy, for example from an extra endowment that they can consume at date 1 or 2, but not deposit in the bank at date 0. The proof would hold even for positive but non-negligible values of the ψ , as long as they are lower than R .

$$\begin{aligned}
& \times \left[\int_{\lambda}^{n_2^*} \left[u \left(\frac{R(1-L) + L - nc}{1-n} \right) - u \left(\frac{L - nc}{1-n} \right) \right] dn + \int_{n_2^*}^{n^{**}} u \left(\frac{R \left(1 - L - \frac{nc-L}{r} \right)}{1-n} \right) dn \right] + \\
& - \left[\int_{n_2^*}^{n^{**}} u' \left(\frac{R \left(1 - L - \frac{nc-L}{r} \right)}{1-n} \right) \frac{Rnc}{r^2(1-n)} dn \right] \times \\
& \times \left[\int_{\lambda}^{n^{**}} u(c) dn + \int_{n^{**}}^1 u(c^B(n)) dn - \int_{\lambda}^{n_2^*} u \left(\frac{L - nc}{1-n} \right) dn \right]. \tag{72}
\end{aligned}$$

By the same considerations as before regarding the Inada conditions, notice that:

$$\lim_{n \rightarrow n^{**}} u' \left(\frac{R \left(1 - L - \frac{nc-L}{r} \right)}{1-n} \right) = +\infty. \tag{73}$$

Hence, the derivative must be negative.

To show that the threshold signal σ_2^* is instead a convex function of r , calculate:

$$\begin{aligned}
\frac{\partial^2 \sigma_2^*}{\partial r^2} &= \left[\int_{\lambda}^{n_2^*} \left[u \left(\frac{R(1-L) + L - nc}{1-n} \right) - u \left(\frac{L - nc}{1-n} \right) \right] dn + \int_{n_2^*}^{n^{**}} u \left(\frac{R \left(1 - L - \frac{nc-L}{r} \right)}{1-n} \right) dn \right]^{-4} \times \\
& \times \left[\left[\left[-\frac{1-L}{c} u'(c) \frac{1-L}{n^{**}} + \int_{n^{**}}^1 u''(c^B(n)) \left(\frac{1-L}{n} \right)^2 dn \right] \times \right. \right. \\
& \times \left[\int_{\lambda}^{n_2^*} \left[u \left(\frac{R(1-L) + L - nc}{1-n} \right) - u \left(\frac{L - nc}{1-n} \right) \right] dn + \int_{n_2^*}^{n^{**}} u \left(\frac{R \left(1 - L - \frac{nc-L}{r} \right)}{1-n} \right) dn \right] + \\
& + \left[\int_{n^{**}}^1 u'(c^B(n)) \frac{1-L}{n} dn \right] \left[\int_{n_2^*}^{n^{**}} u' \left(\frac{R \left(1 - L - \frac{nc-L}{r} \right)}{1-n} \right) \frac{Rnc}{r^2(1-n)} dn \right] + \\
& - \left[\int_{\lambda}^{n^{**}} u(c) dn + \int_{n^{**}}^1 u(c^B(n)) dn - \int_{\lambda}^{n_2^*} u \left(\frac{L - nc}{1-n} \right) dn \right] \times \\
& \times \left[\int_{n_2^*}^{n^{**}} \left[u'' \left(\frac{R \left(1 - L - \frac{nc-L}{r} \right)}{1-n} \right) \left(\frac{Rnc}{r^2(1-n)} \right)^2 - 2u' \left(\frac{R \left(1 - L - \frac{nc-L}{r} \right)}{1-n} \right) \frac{Rnc}{r^3(1-n)} \right] dn \right] + \\
& - \left[\int_{n_2^*}^{n^{**}} u' \left(\frac{R \left(1 - L - \frac{nc-L}{r} \right)}{1-n} \right) \frac{Rnc}{r^2(1-n)} dn \right] \times \\
& \times \left[\int_{n^{**}}^1 u'(c^B(n)) \frac{1-L}{n} dn \right] \times \\
& \times \left[\int_{\lambda}^{n_2^*} \left[u \left(\frac{R(1-L) + L - nc}{1-n} \right) - u \left(\frac{L - nc}{1-n} \right) \right] dn + \int_{n_2^*}^{n^{**}} u \left(\frac{R \left(1 - L - \frac{nc-L}{r} \right)}{1-n} \right) dn \right]^2 + \\
& - 2 \left[\int_{\lambda}^{n_2^*} \left[u \left(\frac{R(1-L) + L - nc}{1-n} \right) - u \left(\frac{L - nc}{1-n} \right) \right] dn + \int_{n_2^*}^{n^{**}} u \left(\frac{R \left(1 - L - \frac{nc-L}{r} \right)}{1-n} \right) dn \right] \times \\
& \times \left[\int_{n_2^*}^{n^{**}} u' \left(\frac{R \left(1 - L - \frac{nc-L}{r} \right)}{1-n} \right) \frac{Rnc}{r^2(1-n)} dn \right] \times
\end{aligned}$$

$$\begin{aligned}
& \times \left[\left[\int_{n^{**}}^1 u'(c^B(n)) \frac{1-L}{n} dn \right] \times \right. \\
& \times \left[\int_{\lambda}^{n_2^*} \left[u \left(\frac{R(1-L) + L - nc}{1-n} \right) - u \left(\frac{L - nc}{1-n} \right) \right] dn + \int_{n_2^*}^{n^{**}} u \left(\frac{R(1-L - \frac{nc-L}{r})}{1-n} \right) dn \right] + \\
& - \left[\int_{n_2^*}^{n^{**}} u' \left(\frac{R(1-L - \frac{nc-L}{r})}{1-n} \right) \frac{Rnc}{r^2(1-n)} dn \right] \times \\
& \times \left[\int_{\lambda}^{n^{**}} u(c) dn + \int_{n^{**}}^1 u(c^B(n)) dn - \int_{\lambda}^{n_2^*} u \left(\frac{L - nc}{1-n} \right) dn \right] \Bigg]. \tag{74}
\end{aligned}$$

By the same consideration regarding the Inada conditions, we get that this derivative is positive, meaning that σ_2^* is a convex function of r .

Being the two threshold signals σ_1^* and σ_2^* both decreasing and convex functions of the recovery rate r , to prove that they cross only once in the interval $[0, 1]$ it suffices to prove that $\sigma_2^* > \sigma_1^*$ at $r = 1$:

$$\sigma_1^*(r=1) = \frac{\int_{\lambda}^{\frac{1}{c}} u(c) dn + \int_{\frac{1}{c}}^1 u\left(\frac{1}{n}\right) dn - \int_{\lambda}^{\frac{1-L}{c}} u\left(\frac{L}{1-n}\right) dn - \int_{\frac{1-L}{c}}^{\frac{1}{c}} u\left(\frac{1-nc}{1-n}\right) dn}{\int_{\lambda}^{\frac{1-L}{c}} \left[u\left(\frac{R(1-L-nc)+L}{1-n}\right) - u\left(\frac{L}{1-n}\right) \right] dn}, \tag{75}$$

$$\sigma_2^*(r=1) = \frac{\int_{\lambda}^{\frac{1}{c}} u(c) dn + \int_{\frac{1}{c}}^1 u\left(\frac{1}{n}\right) dn - \int_{\lambda}^{\frac{L}{c}} u\left(\frac{L-nc}{1-n}\right) dn}{\int_{\lambda}^{\frac{L}{c}} \left[u\left(\frac{R(1-L)+L-nc}{1-n}\right) - u\left(\frac{L-nc}{1-n}\right) \right] dn + \int_{\frac{L}{c}}^{\frac{1}{c}} u\left(\frac{R(1-nc)}{1-n}\right) dn}. \tag{76}$$

Define as NUM_j and DEN_j the numerator and denominator of σ_j^* , respectively, for any pecking order $j = \{1, 2\}$. The following relationship holds:

$$\begin{aligned}
NUM_1 = & NUM_2 + \int_{\lambda}^{n_2^*} u\left(\frac{L-nc}{1-n}\right) dn - \int_{\lambda}^{n_1^*} u\left(\frac{L}{1-n}\right) dn + \\
& - \int_{n_1^*}^{n^{**}} u\left(\frac{r(1-L) + L - nc}{1-n}\right) dn. \tag{77}
\end{aligned}$$

As a preliminary, step, we want to show that:

$$H \equiv \int_{\lambda}^{n_2^*} u\left(\frac{L-nc}{1-n}\right) dn - \int_{\lambda}^{n_1^*} u\left(\frac{L}{1-n}\right) dn - \int_{n_1^*}^{n^{**}} u\left(\frac{r(1-L) + L - nc}{1-n}\right) dn \tag{78}$$

is negative. If $n_1^* \leq n_2^*$, the previous expression can be re-written as:

$$H = \int_{\lambda}^{n_1^*} u \left(\frac{L - nc}{1 - n} \right) dn + \int_{n_1^*}^{n_2^*} u \left(\frac{L - nc}{1 - n} \right) dn - \int_{\lambda}^{n_1^*} u \left(\frac{L}{1 - n} \right) dn + \\ - \int_{n_1^*}^{n_2^*} u \left(\frac{r(1 - L) + L - nc}{1 - n} \right) dn - \int_{n_2^*}^{n^{**}} u \left(\frac{r(1 - L) + L - nc}{1 - n} \right) dn, \quad (79)$$

which is clearly negative. In a similar way, if $n_1^* > n_2^*$, we can re-write:

$$H = \int_{\lambda}^{n_2^*} u \left(\frac{L - nc}{1 - n} \right) dn - \int_{\lambda}^{n_2^*} u \left(\frac{L}{1 - n} \right) dn - \int_{n_2^*}^{n_1^*} u \left(\frac{L}{1 - n} \right) dn + \\ - \int_{n_1^*}^{n^{**}} u \left(\frac{r(1 - L) + L - nc}{1 - n} \right) dn, \quad (80)$$

which, again, is always negative. Thus, $NUM_1 < NUM_2$. Given this result, a sufficient condition for $\sigma_2^* \geq \sigma_1^*$ is $DEN_2 \leq DEN_1$, or:

$$f(c, L) = \int_{\lambda}^{\frac{L}{c}} \left[u \left(\frac{R(1 - L) + L - nc}{1 - n} \right) - u \left(\frac{L - nc}{1 - n} \right) \right] dn + \\ + \int_{\frac{L}{c}}^{\frac{1}{c}} u \left(\frac{R(1 - nc)}{1 - n} \right) dn + \\ - \int_{\lambda}^{\frac{1-L}{c}} \left[u \left(\frac{R(1 - L - nc) + L}{1 - n} \right) - u \left(\frac{L}{1 - n} \right) \right] dn \leq 0. \quad (81)$$

We study how $f(c, L)$ changes with c and L . On the one side:

$$\frac{\partial f(c, L)}{\partial c} = - \int_{\lambda}^{\frac{L}{c}} \left[u' \left(\frac{R(1 - L) + L - nc}{1 - n} \right) - u' \left(\frac{L - nc}{1 - n} \right) \right] \frac{n}{1 - n} dn + \\ - \int_{\frac{L}{c}}^{\frac{1}{c}} u' \left(\frac{R(1 - nc)}{1 - n} \right) \frac{Rn}{1 - n} dn + \\ + \int_{\lambda}^{\frac{1-L}{c}} u' \left(\frac{R(1 - L - nc) + L}{1 - n} \right) \frac{Rn}{1 - n} dn \quad (82)$$

The sign of this derivative is positive. To see that, notice that $\frac{R(1 - nc)}{1 - n} > \frac{L - nc}{1 - n}$. Hence, by the fact that the coefficient of relative risk aversion is larger than 1:⁹

$$\frac{u' \left(\frac{R(1 - nc)}{1 - n} \right)}{u' \left(\frac{L - nc}{1 - n} \right)} < \frac{L - nc}{R(1 - nc)}, \quad (83)$$

⁹See footnote 5.

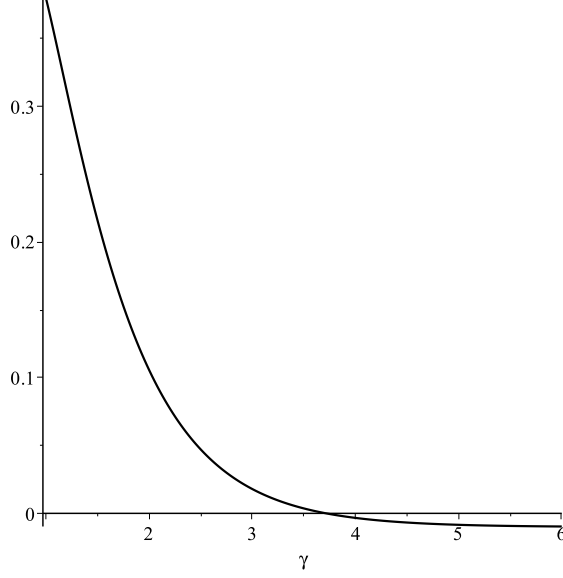


Figure 4: The condition (86) as a function of the coefficient of relative risk aversion.

and this implies that:

$$u' \left(\frac{R(1 - nc)}{1 - n} \right) \frac{Rn}{1 - n} < u' \left(\frac{L - nc}{1 - n} \right) \frac{n}{1 - n} \frac{L - nc}{1 - nc} < u' \left(\frac{L - nc}{1 - n} \right) \frac{n}{1 - n}. \quad (84)$$

On the other side:

$$\begin{aligned} \frac{\partial f(c, L)}{\partial L} = & - \int_{\lambda}^{\frac{L}{c}} \left[u' \left(\frac{R(1 - L) + L - nc}{1 - n} \right) (R - 1) + u' \left(\frac{L - nc}{1 - n} \right) \right] \frac{1}{1 - n} dn + \\ & + \int_{\lambda}^{\frac{1-L}{c}} \left[u' \left(\frac{R(1 - L - nc) + L}{1 - n} \right) (R - 1) + u' \left(\frac{L}{1 - n} \right) \right] \frac{1}{1 - n} dn, \end{aligned} \quad (85)$$

which is negative, because of the Inada Conditions, that make the second integral in the first line become large and negative. Since $f(c, L)$ is increasing in c and decreasing in L , a sufficient condition for it to be less than or equal to zero everywhere is that it is less than or equal to zero at $L^{\min} = \lambda$ and $c^{\max}$ when $L = \lambda$, which is $c^{\max} = 1$. At those points, the condition $f(c, L) \leq 0$ reads:

$$u(R)(1 - \lambda) - \int_{\lambda}^{1 - \lambda} \left[u \left(\frac{R(1 - \lambda - n) + \lambda}{1 - n} \right) - u \left(\frac{\lambda}{1 - n} \right) \right] dn \leq 0. \quad (86)$$

We show how this expression relates to the coefficient of relative risk aversion in a numerical example. We assume that $u(c) = \frac{(c + \psi)^{1 - \gamma} + \psi^{1 - \gamma}}{1 - \gamma}$, with $\psi = 2$ and $\gamma > 1$. Moreover, we pick

$R = 2.01$ and $\lambda = .01$.¹⁰ Figure 4 shows that condition (86) holds for high values of the coefficient of relative risk aversion. This ends the proof. ■

Proof of Lemma 6. Attach the Lagrange multipliers μ and ξ to the liquidity constraint and the incentive compatibility constraint, respectively. The first-order conditions of the program reads:

$$\begin{aligned} c : \quad & -\frac{\partial \sigma_j^*}{\partial c} \Delta U(c, L) + \lambda \int_{\sigma_j^*}^1 [u'(c) - [pu'(c_L(R)) + (1-p)u'(c_L(0))]] dp + \\ & - \lambda \mu - \xi = 0, \end{aligned} \tag{87}$$

$$\begin{aligned} L : \quad & -\frac{\partial \sigma_j^*}{\partial L} \Delta U(c, L) + \sigma_j^* (1-r)u'(L + r(1-L)) + \\ & + \int_{\sigma_j^*}^1 [pu'(c_L(R))(1-R) + (1-p)u'(c_L(0))] dp + \mu + \xi(1-R) = 0. \end{aligned} \tag{88}$$

Clearly, as in the proof for the equilibrium with perfect information, a binding liquidity constraint (with the Lagrange multiplier μ strictly positive) cannot satisfy the equilibrium conditions: in fact, if that was not the case, by complementary slackness we would have $L = \lambda c$ and $c_L(0) = 0$. Plugging (87) into (88) gives:

$$\begin{aligned} \int_{\sigma_j^*}^1 [u'(c) - pRu'(c_L(R))] dp + \sigma_j^* (1-r)u'(L + r(1-L)) = \\ = \left[\frac{\partial \sigma_j^*}{\partial L} + \frac{1}{\lambda} \frac{\partial \sigma_j^*}{\partial c} \right] \Delta U(c, L) + \xi \left(\frac{1}{\lambda} + R - 1 \right). \end{aligned} \tag{89}$$

In equilibrium, either the incentive compatibility constraint is binding or it is not. However, a binding constraint would mean that $c = c_L(R)$, and this in turns would imply that $\underline{\sigma}_2 = 1$, which cannot be an equilibrium. Hence, by complementary slackness, ξ must be equal to zero, and (89) boils down to the distorted Euler equation (34). A point to prove for the sign of the strategic complementarity in the interval $[\lambda, n_2^*]$ was that $\frac{R(1-L)}{c-L} > 1$. Using the previous inequality and the concavity of $u(c)$, this is satisfied by the incentive compatibility constraint. Moreover, this also implies that $c < R$, thus confirming the condition for the existence of the upper dominance region. This ends the proof. ■

Proof of Proposition 2. In order to characterize the sign of the distortion in the Euler equa-

¹⁰The results are robust to different choices of the parameters.

tion, we start by deriving the sign of the sum of the marginal effects, for the pecking order {Liquidity; Liquidation}:

$$\begin{aligned} \left[\frac{\partial \sigma_2^*}{\partial L} + \frac{1}{\lambda} \frac{\partial \sigma_2^*}{\partial c} \right] &= \frac{1}{\int_{\lambda}^{n_2^*} \left[u \left(\frac{R(1-L)+L-nc}{1-n} \right) - u \left(\frac{L-nc}{1-n} \right) \right] dn + \int_{n_2^*}^{n^{**}} u \left(\frac{R(1-L-\frac{nc-L}{r})}{1-n} \right) dn} \times \\ &\quad \times \left[\int_{n^{**}}^1 u'(c^B(n)) \frac{1-r}{n} dn + \frac{n^{**}-\lambda}{\lambda} u'(c) + \right. \\ &\quad \left. + \sigma_2^* \left[\int_{\lambda}^{n_2^*} u'(c_L(R, n)) \frac{R-1+\frac{n}{\lambda}}{1-n} dn + \int_{n_2^*}^{n^{**}} u'(c_L^D(R, n)) \frac{R(\frac{n}{\lambda}-1+r)}{r(1-n)} dn \right] + \right. \\ &\quad \left. + (1-\sigma_2^*) \int_{\lambda}^{n_2^*} u'(c_L(0, n)) \frac{\frac{n}{\lambda}-1}{1-n} dn \right]. \end{aligned} \quad (90)$$

This expression is positive because $n \geq \lambda$ and $\sigma_2^* \leq 1$. We rearrange the distorted Euler equation (34) and rewrite:

$$\begin{aligned} \left[\frac{\partial \sigma_2^*}{\partial L} + \frac{1}{\lambda} \frac{\partial \sigma_2^*}{\partial c} \right] \Delta U(c, L) - \sigma_2^*(1-r)u'(L+r(1-L)) &= \\ = \frac{\Delta U(c, L)}{DEN_2} \int_{n^{**}}^1 u' \left(\frac{L+r(1-L)}{n} \right) \frac{1-r}{n} dn - \sigma_2^*(1-r)u'(L+r(1-L)) + \dots = \\ = \lim_{\epsilon \rightarrow 0} \frac{\Delta U(c, L)}{DEN_2} \left[\int_{n^{**}}^{1-\epsilon} u' \left(\frac{L+r(1-L)}{n} \right) \frac{1-r}{n} dn + \int_{\epsilon}^1 u' \left(\frac{L+r(1-L)}{n} \right) \frac{1-r}{n} dn \right] + \\ - \sigma_2^*(1-r)u'(L+r(1-L)) + \dots = \\ = \lim_{\epsilon \rightarrow 0} \frac{\Delta U(c, L)}{DEN_2} \int_{n^{**}}^{1-\epsilon} u' \left(\frac{L+r(1-L)}{n} \right) \frac{1-r}{n} dn + \\ + (1-r)u'(L+r(1-L)) \left[\frac{\Delta U(c, L)}{DEN_2} - \sigma_2^* \right] + \dots, \end{aligned} \quad (91)$$

where the remaining terms are positive, as proved in (90). Hence, (91) is positive if $\Delta U(c, L) - NUM_2 \geq 0$. The grey area of Figure 5 represents NUM_2 , and clearly shows that $(1-\lambda)u(c) > NUM_2$. Hence, to prove that $\Delta U(c, L) \geq NUM_2$, it is sufficient to prove that $\Delta U(c, L) \geq (1-\lambda)u(c)$. As $u(c) < \sigma_2^*u(c_L(R)) + (1-\sigma_2^*)u(c_L(0))$ by definition of $\underline{\sigma}_2$, a sufficient condition for $\Delta U(c, L) \geq (1-\lambda)u(c)$ is that:

$$\lambda u(c) > u(L+r(1-L)). \quad (92)$$

As $c > L+r(1-L)$, this condition is always satisfied if λ is sufficiently large.

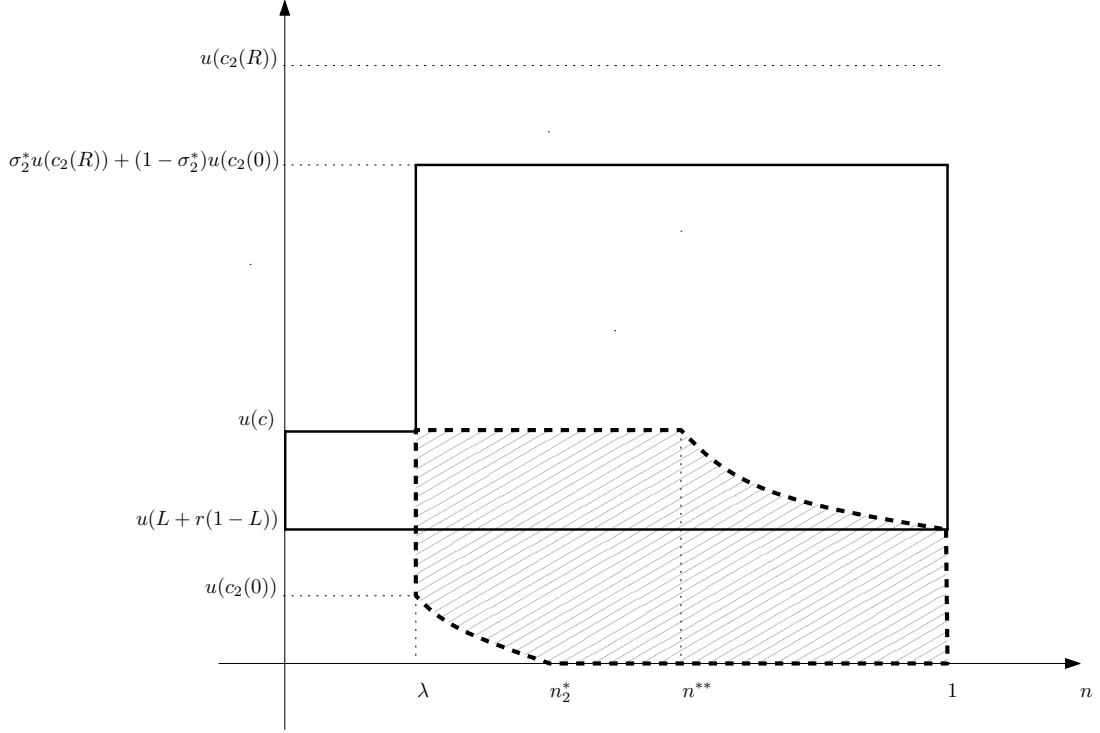


Figure 5: The condition under which $(1 - \lambda)u(c) > NUM_2$.

We follow a similar procedure for the pecking order {Liquidation; Liquidity}.

$$\begin{aligned}
\left[\frac{\partial \sigma_1^*}{\partial L} + \frac{1}{\lambda} \frac{\partial \sigma_1^*}{\partial c} \right] &= \frac{1}{\int_{\lambda}^{n_1^*} \left[u \left(\frac{R(1-L-\frac{nc}{r})+L}{1-n} \right) - u \left(\frac{L}{1-n} \right) \right] dn} \times \\
&\times \left[\int_{n^{**}}^1 u'(c^B(n)) \frac{1-r}{n} dn - \int_{\lambda}^{n_1^*} u'(c_L(0, n)) \frac{1}{1-n} dn + \right. \\
&+ \int_{n_1^*}^{n^{**}} u'(c_L^L(n)) \frac{\frac{n}{\lambda} - 1 + r}{1-n} dn + \left(\frac{n^{**}}{\lambda} - 1 \right) u'(c) + \\
&\left. + \sigma_1^* \int_{\lambda}^{n_1^*} \left[u'(c_L(R, n)) \frac{R \left(\frac{n}{r\lambda} + 1 \right) - 1}{1-n} + u'(c_L(0, n)) \frac{1}{1-n} \right] dn \right]. \quad (93)
\end{aligned}$$

This expression is positive because $n \geq \lambda$ and $n^{**} \geq \lambda$. Hence, we rearrange the distorted Euler equation and write:

$$\begin{aligned}
&\left[\frac{\partial \sigma_1^*}{\partial L} + \frac{1}{\lambda} \frac{\partial \sigma_1^*}{\partial c} \right] \Delta U(c, L) - \sigma_1^* (1-r) u'(L + r(1-L)) = \\
&= (1-r) u'(L + r(1-L)) \left[\frac{\Delta U(c, L)}{DEN_1} - \sigma_1^* \right] - \int_{\lambda}^{n_1^*} u'(c_L(0, n)) \frac{1}{1-n} dn + \dots, \quad (94)
\end{aligned}$$

where the remaining terms are all positive. The previous expression is positive if:

$$\Delta U(c, L) \geq NUM_1 + DEN_1 \int_{\lambda}^{n_1^*} \frac{u'(c_L(0, n))}{u'(L + r(1 - L))} \frac{1}{1 - r} \frac{1}{1 - n} dn. \quad (95)$$

Similarly to the previous case, it can be proved that $NUM_1 < (1 - \lambda)u(c)$, and $DEN_1 < (n_1^* - \lambda) u(c_L(R, \lambda))$. Finally, by the coefficient of relative risk aversion being larger than 1 and the definition of $c_L(0, n)$, we can prove that:

$$\int_{\lambda}^{n_1^*} \frac{u'(c_L(0, n))}{u'(L + r(1 - L))} \frac{1}{1 - r} \frac{1}{1 - n} dn < (n_1^* - \lambda) \frac{L + r(1 - L)}{L(1 - r)}. \quad (96)$$

Hence, a sufficient condition for (95) to hold is:

$$\Delta U(c, L) \geq (1 - \lambda)u(c) + (n_1^* - \lambda)^2 u(c_L(R, \lambda)) \frac{L + r(1 - L)}{L(1 - r)}. \quad (97)$$

By the definition of $\underline{\sigma}_1$ in (19), and the fact that $\underline{\sigma}_1 < \sigma_1^*$, we have:

$$u(c) < \sigma_1^* u\left(\frac{R(1 - L - \frac{\lambda c}{r}) + L}{1 - \lambda}\right) + (1 - \sigma_1^*) u\left(\frac{L}{1 - \lambda}\right) < \sigma_1^* u(c_L(R)) + (1 - \sigma_1^*) u\left(\frac{L}{1 - \lambda}\right), \quad (98)$$

where the last inequality comes from the definition of $c_L(R)$. Using this expression in the definition of $\Delta U(c, L)$, we can express a sufficient condition for (97) to hold as:

$$\lambda u(c) - u(L + r(1 - L)) \geq (n_1^* - \lambda)^2 u(c_L(R, \lambda)) \frac{L + r(1 - L)}{L(1 - r)} + (1 - \sigma_1^*) \left[u\left(\frac{L}{1 - \lambda}\right) - u\left(\frac{L - \lambda}{1 - \lambda}\right) \right]. \quad (99)$$

The right-hand side of (99) is positive, and tends to zero as λ tends to 1, as also n_1^* and σ_1^* tend to 1 when λ tends to 1. Thus, as $c > L + r(1 - L)$, (99) holds only if λ is sufficiently large.

To sum up, the previous results show under which conditions:

$$\left[\frac{\partial \sigma_j^*}{\partial L} + \frac{1}{\lambda} \frac{\partial \sigma_j^*}{\partial c} \right] \Delta U(c, L) - \sigma_j^* (1 - r) u'(L + r(1 - L)) > 0, \quad (100)$$

for any pecking order j . Then, for this to be consistent with the distorted Euler equation (34),

it must also be the case that:

$$\int_{\sigma_j^*}^1 \left[u'(c) - pRu'(c_L(R)) \right] dp > 0, \quad (101)$$

which can be rewritten as:

$$(1 - \sigma_j^*)u'(c) - \frac{1 - \sigma_j^{*2}}{2}Ru'(c_L(R)) > 0. \quad (102)$$

Notice that, by the fact that $R(1 + \sigma_j^*)/2 > 1$ and the concavity of the utility function, $c < c_L(R)$.

Moreover, rearrange the previous expression as:

$$\frac{u'(c)}{u'(c_L(R))} > R \frac{1 + \sigma_j^*}{2} \geq \mathbb{E}[p]R = \frac{u'(c^{PI})}{u'(c_L^{PI}(R))} > 1, \quad (103)$$

where the second inequality holds as $\mathbb{E}[p] = 1/2$ and $\sigma_j^* \geq 0$, and $\{c^{PI}, c_L^{PI}(R)\}$ is the deposit contract in the equilibrium with perfect information. By concavity of the utility function, for the ratio $\frac{u'(c)}{u'(c_L(R))}$ to be higher in the banking equilibrium than in the equilibrium with perfect information, it must be the case that $\frac{c}{c_L(R)} < \frac{c^{PI}}{c_L^{PI}(R)}$. To this end, we calculate:

$$\frac{\partial}{\partial L} \left[\frac{c}{c_L(R)} \right] = -\frac{(1 - \lambda)c}{[R(1 - L) + L - \lambda c]^2} (1 - R) > 0, \quad (104)$$

$$\frac{\partial}{\partial c} \left[\frac{c}{c_L(R)} \right] = \frac{(1 - \lambda)[R(1 - L) + L - \lambda c] + \lambda(1 - \lambda)c}{[R(1 - L) + L - \lambda c]^2} > 0. \quad (105)$$

We take the total differential of the ratio $\frac{c}{c_L(R)}$, evaluated at the equilibrium with perfect information, and look for the condition that makes it negative:

$$\frac{\partial}{\partial L} \left[\frac{c}{c_L(R)} \right] dL + \frac{\partial}{\partial c} \left[\frac{c}{c_L(R)} \right] dc < 0. \quad (106)$$

This implies that:

$$\frac{dL}{dc} < -\frac{\frac{\partial}{\partial c} \left[\frac{c}{c_L(R)} \right]}{\frac{\partial}{\partial L} \left[\frac{c}{c_L(R)} \right]}. \quad (107)$$

As the right-hand side is negative, it must be the case that $dL/dc < 0$. Finally, evaluate the first-order condition with respect to c in (87) at the equilibrium with perfect information. As the term in the integral has to go up when moving from c^{PI} to c^{BE} , then it must be the case

that $c^{BE} < c^{PI}$, hence $dc < 0$. This, together with $\frac{dL}{dc} < 0$, implies that $dL > 0$, or $L^{BE} > L^{PI}$. This ends the proof. ■

Proof of Corollary 1. **[WRONG!!!]** To prove the Corollary, plug the allocation of the banking equilibrium into the equilibrium condition of the autarkic equilibrium in (5):

$$g(c^{BE}, L^{BE}) = \lambda(1-r)u'(c^{BE}) - (1-\lambda)\mathbb{E}[p]\left[u'(c_L^{BE}(R))(R-1) - u'(c_L^{BE}(0))\right]. \quad (108)$$

Remembering that, in the banking equilibrium:

$$u'(c^{BE}) - \frac{1+\sigma_j^*}{2}Ru'(c_L^{BE}(R)) > 0, \quad (109)$$

and that $\mathbb{E}[p] = 1/2$, this expression becomes:

$$g(c^{BE}, L^{BE}) > \mathbb{E}[p]u'(c_L^{BE}(R))\left[\lambda(1-r)(1+\sigma_j^*)R - (1-\lambda)(R-1)\right] + (1-\lambda)u'(c_L^{BE}(0)). \quad (110)$$

A sufficient condition for this to be always positive is that the term in the square brackets is positive, meaning:

$$\sigma_j^* > \frac{1-\lambda}{\lambda(1-r)}\left(1 - \frac{1}{R}\right) - 1. \quad (111)$$

As σ_j^* is non-negative, the expression in (111) is always satisfied if its right-hand side is negative, i.e. if:

$$1 - \frac{1}{R} < \frac{\lambda(1-r)}{1-\lambda}. \quad (112)$$

Thus, under this condition, $g(c^{BE}, L^{BE}) > 0$, so it must be the case that $c^{BE} > c^A$ **[WRONG!!!]**, because of the concavity of the utility function $u(x)$. This ends the proof. ■

Proof of Proposition 3. Plug the autarkic allocation of section 2.1 into the equilibrium condition of the narrow-banking problem in (37):

$$h(L^A) = [\lambda + (1-\lambda)\mathbb{E}[p]]u'(L^A + r(1-L^A)) - \mathbb{E}[p](R-1+\lambda)u'(R(1-L^A) + L^A). \quad (113)$$

Using the equilibrium condition of autarky in (5), we rewrite this expression as:

$$h(L^A) = \left[\lambda + (1 - \lambda)\mathbb{E}[p] - \frac{\lambda(1 - r)}{1 - \lambda} \right] u'(L^A + r(1 - L^A)) - \mathbb{E}[p] \left[u'(L^A) + \lambda u'(R(1 - L^A) + L^A) \right]. \quad (114)$$

Under the condition that:

$$\lambda + (1 - \lambda)\mathbb{E}[p] \leq \frac{\lambda(1 - r)}{1 - \lambda}, \quad (115)$$

$f(L)$ is negative, as the utility function $u(x)$ is increasing. Therefore, because of the concavity of $u(x)$, in order for the equilibrium condition (37) to hold it must be the case that $c^{NB} < c^A$.

This ends the proof. ■