

Gradual Bargaining in Decentralized Asset Markets*

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Abstract

We introduce a new approach to bargaining into a model of decentralized asset market with unrestricted portfolios. Gradual bargaining (O'Neill et al., 2004) captures the idea that portfolios of assets are sold sequentially, one unit of asset at a time. In contrast to Nash or Kalai solutions, it is both monotone and ordinal. Moreover, gradual bargaining reduces asset misallocation in decentralized markets and can implement first best with or without endogenous participation. We generalize the solution to allow for time-varying bargaining powers and time-intensive technologies to negotiate assets thereby capturing the notion of asset negotiability, i.e., some assets can take longer than others to negotiate due to their complexity. The model can explain the rate-of-return-dominance puzzle.

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1 Introduction

Both modern monetary theory and financial economics formalize asset trades in the context of economic environments where agents meet bilaterally (e.g., Duffie et al., 2005; Lagos and Wright, 2005). By being explicit about the exchange process—who trades with whom and how prices are formed—this description has deepened our understanding of liquidity. Two key determinants of liquidity are the technology through which consumers and producers are brought together and the process through which prices are formed. This paper focuses on the latter: the determination of asset prices and trade sizes in the context of bilateral monopolies.

The standard approach to price formation in decentralized asset markets consists in applying axiomatic bargaining solutions, such as the ones proposed by Nash (1950) or Kalai (1977). However, in models with unrestricted asset holdings, these bargaining solutions display features that have been largely seen as undesirable. The main criticism of the Nash solution is that it lacks monotonicity, i.e., the surplus of the asset holder can shrink when the size of the trade expands, even if gains from trade remain unexploited (e.g., Aruoba et al., 2007). As a result, were asset holdings private information, the solution would not be incentive-compatible and agents would have incentives to hide some of their portfolio. The proportional solution of Kalai (1977) avoids this issue by being strongly monotone, but it is not invariant to affine transformation of utilities. We will show that both shortcomings have important positive and normative implications. As a consequence, there is a need for a bargaining solution better suited to describe the negotiation over multidimensional portfolios—portfolios composed of multiple units of different assets.

Our contribution is to introduce a new, tractable approach to bargaining over portfolios of assets, gradual bargaining, into a model of decentralized asset market. A gradual bargaining problem, as defined by Wiener and Winter (1998) and O'Neill et al. (2004), consists in a collection of expanding bargaining sets and a disagreement point. The idea is to formalize negotiations with many assets over which to bargain. The bargaining set expands as agents put more of those assets on the negotiating table. The solution to a gradual bargaining problem proposed by O'Neill et al. (2004) satisfies five axioms. Three are common with Nash: the solution is Pareto efficient, scale invariant, and symmetric. Two axioms are added to accommodate the new definition of the bargaining problem and its solution as a path of agreements (instead of a single agreement point). The solution is required to be continuous and, crucially, time consistent. In this context, time consistency means that if one were to take any interim sale of assets, seen as final, as the new starting point of the negotiation, the bargaining path would remain the same onwards. The unique solution that satisfies these five axioms is remarkably tractable and it addresses the shortcomings of the Nash and Kalai

solutions: the gradual bargaining solution is both monotone and ordinal.

We incorporate this bargaining solution into the New-Monetarist models of Geromichalos et al. (2007) and Lagos (2010). Agents trade Lucas trees and/or fiat money in pairwise meetings because of their liquidity role and readjust their asset holdings in centralized markets (which can also be interpreted as perfectly competitive interdealer markets, as in Duffie et al., 2005). These models feature unrestricted portfolios and endogenous liquidity premia, which will allow us to illustrate the importance of the bargaining procedure for asset prices and for efficiency.

We start with an economy composed of Lucas trees that pay off a constant dividend flow (or fiat money at the limit when the dividend goes to 0), and characterize the relationship between trade size and real prices in closed form. In contrast to Nash bargaining, the solution is monotone in that the asset owner's surplus always increases with his asset holdings. Moreover, the gradual bargaining solution generates tractable, concave portfolio problems, making it straightforward to establish uniqueness of the steady-state equilibrium. In contrast to the proportional solution, gradual bargaining can generate a larger range of liquidity premia, which has direct implications for the existence of monetary equilibria.

In terms of normative results, the equilibrium under Nash bargaining is inefficient irrespective of the asset supply, as long as the asset owner does not have all the bargaining power. As the asset supply increases, the fraction held by agents with no liquidity needs increases (weakly) while agents with liquidity needs do not hold enough assets to exploit all the gains from trade. This misallocation of the asset is entirely due to the non-monotonicity of the Nash solution. In contrast, under gradual bargaining, assets are held by agents with liquidity needs, and the first best is implemented as long as the asset supply is sufficiently abundant. We also consider the case with endogenous participation in the market and derive a modified Hosios condition for an efficient allocation.

We then turn to the case with multiple assets. For the sake of illustration we focus our study on the fundamental asset pricing puzzle in monetary theory, namely, the coexistence of money and interest-bearing risk-free bonds.¹ We first generalize the gradual solution by allowing bargaining powers to vary along the negotiation path. Even if asset sales are negotiated according to an exogenous agenda—some assets are negotiated first while others are negotiated last—and the bargaining power of the asset owner erodes over time, rate-of-return equality prevails. In order to break rate-of-return equality, the bargaining power must be allowed to vary with the type of assets that is put on the negotiating table. A special case of this asymmetric gradual bargaining solution is the trading mechanism proposed by Zhu and Wallace (2007), where asset

¹It should be clear, however, that our model could also be used to address other standard asset pricing puzzles, e.g., the risk-free-rate and equity premium puzzles (e.g., Lagos, 2010) or the on-off-the-run yield spread puzzle (e.g., Vayanos and Weill, 2008).

owners have no bargaining power when they sell government bonds. We generalize this mechanism and derive implications for policy.

Finally, we propose a novel explanation of the rate-of-return dominance puzzle by adding a stochastic risk of break down for the negotiation. Moreover, the negotiation over bonds requires more time than the negotiation over units of money. For instance, assets other than money might take longer to authenticate (Lester et al. 2012, Li et al., 2012) or the transfer of ownership might be more involved. We assume that the bargaining agenda is endogenous and show that agents choose to negotiate first assets that can be sold quickly. Hence, our model generates a pecking-order theory of the liquidation of portfolios: agents choose to sell money first and bonds last. In equilibrium bonds pay interest and coexist with money. Moreover, a monetary equilibrium exists irrespective of the quantity of bonds in the economy. The model can also be used for policy. An open market sale of bonds raises the nominal interest rate and reduces output due to aggregate liquidity falling. An increase in the money growth rate, which raises the nominal rate on illiquid bonds, can simultaneously reduce the nominal rate on liquid government bonds, thereby widening the spread between liquid and illiquid assets.

Literature

Both Lagos and Wright (2005) in the monetary literature and Duffie et al. (2005) in the financial economics literature adopted generalized Nash as their bargaining solution. Aruoba et al. (2007) introduced the proportional solution of Kalai (1977). Other trading mechanisms studied in the context of these models include competitive search (Rocheteau and Wright, 2005; Lester et al., 2015), price taking (Rocheteau and Wright, 2005), auctions (Galenianos and Kircher, 2008), price posting (Jean et al., 2010), and monopolistic competition (Silva, 2017). Socially optimal mechanisms were characterized by Hu et al. (2009).

The gradual bargaining solution was developed by Wiener and Winter (1998) and O'Neill et al. (2004) for bargaining problems with an agenda, such as the sale of a portfolio of assets. Other bargaining games with an agenda include Fershtman (1990) and Bac and Raff (1996), among others. Coles and Wright (1998) describe the strategic negotiation of units of money in continuous time in the non-stationary monetary equilibria of the Shi-Trejos-Wright model. The key difference is that our solution is not strategic but axiomatic. The applicability of the gradual bargaining solution requires portfolios of assets to be unrestricted, in contrast to Shi (1995), Trejos and Wright (1995), and Duffie et al. (2005), where asset holdings belong to $\{0, 1\}$. Finally, Tsoy (2016) proposes a model with bargaining and delays in equilibrium and applies it to OTC markets.

Our description of a decentralized asset market with divisible Lucas trees is similar to Geromichalos et al. (2007) and Lagos (2010). In those papers, however, the asset owner has all the bargaining power. Rocheteau

and Wright (2013) adopt the proportional bargaining solution and endogenize participation. Relative to the models of over-the-counter markets with divisible assets of Lagos and Rocheteau (2009) and Lagos et al. (2011), our model does not assume perfect credit nor transferable utility in pairwise meetings. This makes liquidity constraints relevant.

The extension of our model with multiple assets is related to Zhu and Wallace (2007) and Nosal and Rocheteau (2013). They explain the coexistence of money and interest-bearing assets with a mechanism that treats assets asymmetrically. This mechanism is a particular case of the gradual bargaining solution. Hu and Rocheteau (2013, 2015) derive the optimal mechanism and show it features rate-of-return dominance. Other explanations of rate-of-return differences across assets include Vayanos and Weill (2008) based on increasing-returns-to-scale matching technologies; Rocheteau (2011), Li et al. (2012) and Hu (2013) based on informational asymmetries; and Lagos (2013) based on self-fulfilling beliefs in the presence of assets' extrinsic characteristics.

2 Environment

Time is discrete, continues forever, and each period is divided into two stages. There is a continuum of agents with measure two evenly divided between two types, called consumers and producers. An agent's type corresponds to his role in the first stage, where only consumers wish to consume while only producers have the technology to produce. During that stage, labeled DM (for decentralized market), a fraction α of consumers and producers are matched bilaterally. The parameter α captures both the difficulty to sell assets and the frequency of consumption opportunities. The second stage, labeled CM (for centralized market), features a centralized Walrasian market. This centralized market can be interpreted as the inter-dealer market in Duffie et al. (2005), where consumers and producers reallocate their portfolios after having traded in the DM, except that here all agents can access a dealer with probability one and dealers have no bargaining power. There is a one good in each stage and we take the CM good as numeraire.

Buyers' preferences are represented by the period utility function, $u(y) - h$, where y is DM consumption and h is the CM supply of labor. Sellers' preferences are represented by $-v(y) + c$, where y corresponds to the production of the DM good and c is the consumption of the CM good. We assume $u'(y) > 0$, $u''(y) < 0$, $u(0) = v(0) = v'(0) = 0$, $v'(y) > 0$, $v''(y) > 0$, and $v(\bar{y}) = u(\bar{y})$ for some $\bar{y} > 0$. Let y^* denote the solution to $u'(y^*) = v'(y^*)$. All agents share the same discount factor across periods, $\beta \equiv (1 + \rho)^{-1} \in (0, 1)$.

Our specification for preferences and technologies generates gains from trade across stages. However, agents cannot commit to fulfill promises across stages and their actions cannot be monitored, i.e., they

are *anonymous*. Hence, private IOUs cannot serve as means of payment as agents would default on their repayment obligations with no fear of punishment. These frictions generate an essential role for liquid assets. To play this role there is an exogenous supply of Lucas trees, A_t , that are perfectly durable, storable at no cost, and non-counterfeitable. Each Lucas tree pays off $d \geq 0$ units of numeraire in the CM. The supply grows at rate π , $A_{t+1} = (1 + \pi)A_t$, where new trees are allocated to consumers in a lump-sum fashion. In the following, we set $\pi = 0$ when $d > 0$ but we allow $\pi \in (\beta - 1, +\infty)$ when $d = 0$. The case $d = 0$ corresponds to fiat money. We denote ϕ_t the price of Lucas trees in terms of the numeraire.

In the DM we assume that it takes time to negotiate the sale of assets, e.g., it takes time to authenticate and to secure the transfer of ownership of the asset [GIVE CONCRETE EXAMPLES]. We index time within the DM by τ . The technology to negotiate the sale of assets is such that δ units of assets can be sold per unit of time. Hence, the higher δ , the more negotiable the asset. We also assume that there is a time horizon, $\bar{\tau}$, to complete the negotiation. We will consider the case where $\bar{\tau}$ is fixed and equal across all matches and the case where $\bar{\tau}$ is a random variable drawn at the beginning of the match. When describing the gradual negotiation over time we will adopt both an axiomatic approach and a strategic approach and we will show that both approaches give the same outcome.

3 Preliminary results

We first derive some preliminary results that will be useful to set up the bargaining problem in the DM. We write the Bellman equations and obtain basic properties of the value functions in the two stages. We then write down the general portfolio problem of a consumer in the CM. We restrict our attention to stationary equilibria where the price of Lucas trees is constant at ϕ and hence their gross rate of return is also constant and equal to $R = 1 + r = (\phi + d)/\phi$. We measure a consumer's asset holdings in the DM in terms of their value in the coming CM. More precisely, a units of asset in the DM are worth

$$z = (\phi + d)a.$$

The lifetime expected utility of a consumer with wealth z in the CM is

$$W^b(z) = \max_{z', h} \{-h + \beta V^b(z')\} \quad \text{s.t.} \quad z' = R(z + h + T), \quad (1)$$

where T denotes lump-sum transfers (expressed in terms of CM goods), and z' are next-period asset holdings. From (1) the consumer chooses his supply of labor and future asset holdings in order to maximize his discounted continuation value net of the disutility of work. According to the budget constraint, next-period asset holdings are equal to current asset holdings, plus savings and the transfer, everything multiplied by

the rate of return of liquid assets. Substituting h by its expression coming from the budget identity into the objective, we obtain

$$W^b(z) = z + T + \max_{z' \geq 0} \left\{ -\frac{z'}{R} + \beta V^b(z') \right\}. \quad (2)$$

As is standard, W^b is linear in wealth. By a similar reasoning, the value function of a producer is

$$W^s(z) = z + \max_{z' \geq 0} \left\{ -\frac{z'}{R} + \beta V^s(z') \right\}.$$

The lifetime expected utility of a consumer holding z assets in the DM solves

$$V^b(z) = \alpha \{ u[y(z)] + W^b[z - p(z)] \} + (1 - \alpha) W^b(z), \quad (3)$$

where $y(z)$ is the consumer's consumption and $p(z)$ is his sale of Lucas trees in the DM in terms of assets. Note that we conjecture (and verify later) that the terms of trade in a bilateral match, $[y(z), p(z)]$, only depend on the consumer's wealth. According to (3) a consumer meets a producer with probability α , in which case he enjoys $y(z)$ units of DM consumption in exchange for $p(z)$ units of real balances. With probability $1 - \alpha$ the consumer is unmatched and enters the CM with z units of asset. We now turn to the choice of asset holdings in the CM. Substituting $V^b(z)$ by its expression given by (3), the consumer's choice of asset holdings solves

$$\max_{z \geq 0} \{ -sz + \alpha \{ u[y(z)] - p(z) \} \}, \quad (4)$$

where s is the spread between the real interest rate on an illiquid asset that cannot be traded in the DM and the real rate on liquid Lucas trees,

$$s = \frac{\rho - r}{R} \geq 0. \quad (5)$$

According to (4), the consumer chooses his asset holdings in order to maximize his expected surplus from trading in the DM net of the cost of holding the asset. That cost is approximately equal to the difference between the rate of time preference and the rate of return of the asset. By a similar reasoning, the lifetime expected utility of a producer at the start of the DM solves

$$V^s(z) = \alpha \{ -v[y(z^b)] + p(z^b) \} + W^s(z),$$

where z^b are the consumer's assets in equilibrium. It is easy to check that for all $s \geq 0$ it is weakly optimal for the producer to choose $z = 0$.

4 Gradual bargaining

We now introduce the notion of gradual bargaining to determine the terms of trade in pairwise meetings. We first adopt an axiomatic approach. We define a gradual bargaining problem along the lines of O'Neill et

al. (2004) and we describe their axiomatic solution. Second, we adopt a strategic approach and describe an extensive-form game with a sequence of ultimatum-offers with alternating proposers. We show that the two approaches give the same outcome.

4.1 A gradual bargaining problem

Consider a match between a producer and a consumer holding z . Recall that δ units of assets can be sold for each unit of time, and hence, at time τ , the amount of assets that have been up for negotiation is equal to $\delta\tau$. Following Wiener and Winter (1998) and O'Neill et al. (2004) a gradual bargaining problem has as primitive a family of feasible sets indexed by time elapsed within the negotiation process.² The gradual bargaining problem is defined by this collection of bargaining sets – also called the agenda of the negotiation – and an initial pair of disagreement points.

We now define the collection of bargaining sets in the context of our model. The utility of the consumer from an interim agreement reached at time τ , $(y(\tau), p(\tau))$, according to which the seller produces $y(\tau)$ in exchange for $p(\tau)$ in liquid wealth is

$$u^b(\tau) = u[y(\tau)] + W^b[z - p(\tau)] = u[y(\tau)] - p(\tau) + u_0^b, \quad (6)$$

where $u_0^b = W^b(z)$. The consumer's surplus, $u^b - u_0^b$, is the utility of consumption net of the transfer of assets. The utility of the producer from the same interim agreement is

$$u^s(\tau) = -v[y(\tau)] + p(\tau) + u_0^s, \quad (7)$$

where $u_0^s = W^s(0)$. The producer's surplus, $u^s - u_0^s$, is the transfer of real balances net of the disutility of production. While u_0^b and u_0^s are taken as given within the bargaining problem, they are endogenously determined in equilibrium. The set of feasible utility levels associated with this problem is $S(\tau)$. From the feasibility constraint according to which a consumer cannot transfer more assets than there are on the negotiating table, $p(\tau) \leq \delta\tau$, it follows that $S(\tau) \subseteq S(\tau')$ if $\tau < \tau'$. In words, the bargaining set expands as consumers put more assets on the table. We represent the collection of bargaining sets of a gradual bargaining problem in Figure 1, where $\Delta^* = u(y^*) - v(y^*)$.

Lemma 1 *The Pareto frontier of a negotiation over τ units of assets satisfies*

$$H(u^b, u^s, \tau) = 0 \quad (8)$$

²In contrast Nash (1950) defines a bargaining problem as a single set of utility levels for the two parties and a pair of disagreement points.

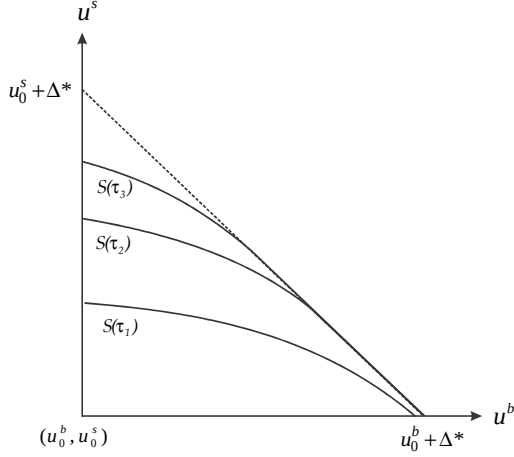


Figure 1: Bargaining sets of a gradual bargaining problem

where

$$H(u^b, u^s, \tau) = \begin{cases} u(y^*) - v(y^*) - (u^b - u_0^b) - (u^s - u_0^s) & \text{if } u^s - u_0^s \leq \delta\tau - v(y^*) \\ \delta\tau - v[u^{-1}(\delta\tau + u^b - u_0^b)] - (u^s - u_0^s) & \text{otherwise} \end{cases} \quad (9)$$

For any triple, (u^b, u^s, τ) , on the Pareto frontier, there is a corresponding output level, y , such that $u(y) - v(y) = u^b + u^s - (u_0^b + u_0^s)$. Note that the function H is continuously differentiable, increasing in τ (strictly so if $y < y^*$), decreasing in u^b and u^s . Consequently, each Pareto frontier has a negative slope:

$$\left. \frac{\partial u^s}{\partial u^b} \right|_{H(u^b, u^s, \tau)=0} = \begin{cases} -1 & \text{if } u^s - u_0^s \leq \delta\tau - v(y^*) \\ -\frac{v'(y)}{u'(y)} & \text{otherwise} \end{cases}$$

The Pareto frontier is linear when $y = y^*$. When $y < y^*$, it is strictly concave. These different properties of the Pareto frontiers are illustrated in Figure 1.

Definition 1 A gradual bargaining problem between a consumer holding z units of asset and a producer is a collection of Pareto frontiers, $\langle H(u^b, u^s, \tau) = 0, \tau \in [0, z/\delta] \rangle$ and a pair of disagreement points, (u_0^b, u_0^s) .

4.2 An axiomatic solution

A gradual agreement path is a function, $o : [0, z] \rightarrow \mathbb{R}_+ \times [0, z]$, that specifies an allocation (y, p) for all $\tau \in [0, z/\delta]$ and associated utility levels, $\langle u^b(\tau), u^s(\tau) \rangle$. The gradual Nash solution, proposed by Wiener and Winter (1998) and O'Neill et al. (2004), is the unique solution to satisfy the following five axioms: Pareto optimality, covariance with respect to positive linear transformations of utility, symmetry, directional continuity, and time-consistency. The first three axioms are axioms imposed by Nash (1950). This axiomatization does not require Nash's fourth and more controversial axiom, independence of irrelevant alternatives. The last two axioms are specific to the new definition of the bargaining problem. Directional continuity imposes

a notion of continuity for the bargaining path with respect to changes in the agenda. More importantly, the key addition is the requirement of time-consistency. It captures the idea that any temporary agreement reached at time τ is seen as final. If the negotiation were to start at time τ with that agreement being the disagreement point, the bargaining path going onward would be the same as the one obtained starting at $\tau = 0$.

While these five axioms are primitives, the solution exhibits ordinality endogenously: the solution is covariant with respect to any order-preserving transformation. This result is noteworthy because Shapley (1969) shows that in the standard Nash framework, with two players, no single-valued solution can satisfy Pareto efficiency, symmetry, and ordinality.

The gradual Nash solution satisfies the following system of differential equations:

$$u^b(\tau) = -\frac{1}{2} \frac{\partial H(u^b, u^s, \tau)/\partial \tau}{\partial H(u^b, u^s, \tau)/\partial u^b} \quad (10)$$

$$u^s(\tau) = -\frac{1}{2} \frac{\partial H(u^b, u^s, \tau)/\partial \tau}{\partial H(u^b, u^s, \tau)/\partial u^s}. \quad (11)$$

We can interpret (10) and (11) as follows. An increase in τ by one unit expands the bargaining set by $\partial H(u^b, u^s, \tau)/\partial \tau \geq 0$. The maximum increase in the consumer's utility that is consistent with the expansion of the bargaining set is

$$\Delta_b^{\max} = -\frac{\partial H(u^b, u^s, \tau)/\partial \tau}{\partial H(u^b, u^s, \tau)/\partial u^b}.$$

According to (10), since the solution is symmetric, the consumer will enjoy half of this maximum utility gain. Similarly, the maximum increase in the producer's utility that is consistent with the expansion of the bargaining set is

$$\Delta_s^{\max} = -\frac{\partial H(u^b, u^s, \tau)/\partial \tau}{\partial H(u^b, u^s, \tau)/\partial u^s}.$$

According to (11) the producer enjoys half of this maximum gain. We combine (10) and (11) to obtain the slope of the gradual agreement path:

$$\frac{\partial u^s}{\partial u^b} = \frac{\partial H(u^b, u^s, \tau)/\partial u^b}{\partial H(u^b, u^s, \tau)/\partial u^s}. \quad (12)$$

According to (12), the slope of the gradual bargaining path is equal to the opposite of the slope of the Pareto frontier, which is also the marginal rate of substitution between the utility levels of the two players along the Pareto frontier. Another geometric interpretation of the solution is that the direction of the agreement path, (H_{u^s}, H_{u^b}) , is orthogonal to the flipped gradient, $(H_{u^b}, -H_{u^s})$. We represent the solution in Figure 2.

From (9) we obtain the following definition for the gradual Nash solution in the context of our model.

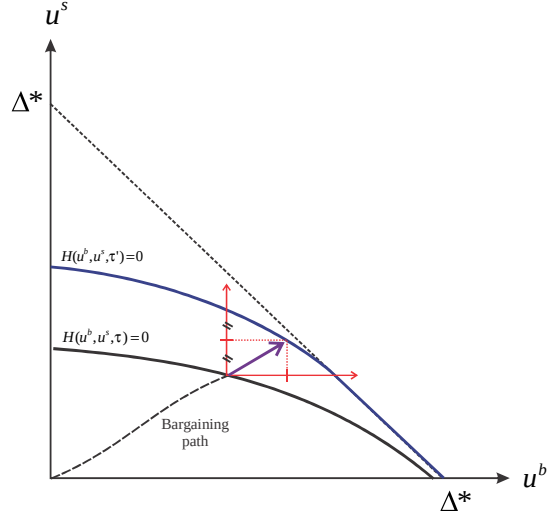


Figure 2: Solution to a gradual bargaining problem

Definition 2 Consider a pairwise meeting between a buyer holding z units of wealth and a seller. The gradual Nash solution to the bargaining problem is a pair of payoff functions, $\langle u^b(\tau), u^s(\tau) \rangle$, and an allocation function, $\langle y(\tau), p(\tau) \rangle$, solution to the following system of differential equations,

$$u^{b'}(\tau) = \delta \frac{u'(y) - v'(y)}{2v'(y)} \quad (13)$$

$$u^{s'}(\tau) = \delta \frac{u'(y) - v'(y)}{2u'(y)}, \quad (14)$$

if $\delta\tau < u^s - u_0^s + v(y^*)$ and $u^{b'}(\tau) = u^{s'}(\tau) = 0$ otherwise; the initial conditions, $u^b(\tau) = u_0^b$ and $u^s(\tau) = u_0^s$; the relationships between payoffs and allocations, (6)-(7).

From (9), when $\delta\tau \geq u^s - u_0^s + v(y^*)$, $\partial H(u^b, u^s, \tau)/\partial\tau = 0$, in which case all the gains from trade are exploited, $y = y^*$, and the Pareto frontier no longer expands as the consumer raises his asset holdings. It follows that the utilities of the buyer and seller remain constant past that point: there is no change in the agreement and the allocation. In contrast, if $\delta\tau < u^s - u_0^s + v(y^*)$, then the increase in the consumer's surplus is equal to the increase in the match surplus multiplied by a factor $1/2v'(y)$. From (13) and (14) along the gradual bargaining path, $\partial u^s / \partial u^b = v'(y)/u'(y)$.

Proposition 1 Along the gradual bargaining path, the price of the asset in terms of DM consumption is

$$\frac{y'(\tau)}{\delta} = \frac{u'(y) + v'(y)}{2v'(y)u'(y)} \quad \text{for all } y < y^*. \quad (15)$$

The overall payment for y units of consumption is

$$z(y) = \int_0^y \frac{2v'(x)u'(x)}{u'(x) + v'(x)} dx. \quad (16)$$

From (15) the quantity of DM consumption increases with time as more assets are sold. From the viewpoint of the seller one unit of DM consumption is worth $v'(y)$ units of numeraire. Hence, the price of the asset in terms of numeraire is:

$$\frac{v'(y)y'(\tau)}{\delta} = \frac{1}{2} \left(1 + \frac{v'(y)}{u'(y)} \right).$$

This price increases with the quantities of assets sold. This result captures the idea that larger trades are more expensive.

From (16) we can compute the buyer's surplus from a trade:

$$u(y) - z(y) = \int_0^y \frac{u'(x)[u'(x) - v'(x)]}{u'(x) + v'(x)} dx, \quad \text{for all } y \leq y^*.$$

The surplus increases with the quantity consumed and it is maximum when $y = y^*$. Moreover, $u'(y) - v'(y) < 0$ for all $y < y^*$. So the surplus is strictly concave, which will be analitically convenient when introducing the portfolio choice.

At this point we can compare the gradual bargaining solution to the standard Nash solution. Suppose that buyers can negotiate the sale of their assets in two ways. They can put all their assets at once on the negotiation table, in which case the outcome is given by the Nash solution, $z = z^N(y)$, where

$$z^N(y) = \frac{v'(y)u(y) + u'(y)v(y)}{u'(y) + v'(y)}.$$

The integrand of (16) coincides with $z^N(x)$ where $u(x)$ and $v(x)$ are replaced by their derivatives. Alternatively, they can negotiate their assets gradually over time, in which case the outcome is given by $z = z(y)$. Their preferred method of negotiation is given by the following proposition.

Proposition 2 *Buyers prefer to negotiate the sale of their assets sequentially over time according to the gradual solution instead of all at once according to the Nash solution, i.e., $z^N(y) > z(y)$ for all $y \leq y^*$.*

By negotiating the sale of their assets gradually over time buyers can obtain better terms of trade and can raise their surplus from trade. By selling all his assets at once the buyer must pay an additional

$$z^N(y) - z(y) = \int_0^y \left[\frac{v'(y)}{u'(y) + v'(y)} - \frac{v'(x)}{u'(x) + v'(x)} \right] [u'(x) - v'(x)] dx$$

for y units of consumption. The difference arises from the fact that under Nash bargaining the seller's share in each increment of the match surplus, $u'(x) - v'(x)$, is $v'(y)/[u'(y) + v'(y)]$ which is larger than $v'(x)/[u'(x) + v'(x)]$ for all $x < y$. In other words, the seller's share is based on the overall consumption and

not the consumption that is purchased gradually over time. This result captures the idea that selling all the assets at once has a larger negative impact on the price rather than selling them through small quantities. This result occurs in our model even though the buyer's actions are perfectly observable.

Now suppose that we compare the two solutions taking into account risk of termination when bargaining gradually. The expected surplus of the buyer is:

$$U^b(\delta; z^b) = \int_0^\infty \lambda e^{-\lambda \bar{\tau}} \{u[y(\min\{\bar{\tau}, z^b/\delta\})] - z[y(\min\{\bar{\tau}, z^b/\delta\})]\} dx \quad (17)$$

$$= \int_0^y e^{-\frac{\lambda}{\delta} z(x)} \frac{u'(x) [u'(x) - v'(x)]}{u'(x) + v'(x)} dx, \quad (18)$$

where $y = z^{-1}(z^b)$, and the derivation of the second inequality can be found in the Appendix. Note that $U^b(\delta; z^b)$ is increasing in δ : as $\delta \rightarrow 0$ it approaches 0 and as $\delta \rightarrow \infty$ it approaches the unconstrained gradual solution. Hence, we have the following corollary:

Corollary 1 *Suppose there is a risk of termination of the gradual negotiation. There exists $\bar{\delta} \in (0, +\infty)$ such that for all $\delta > \bar{\delta}$ buyers prefer gradual bargaining to all-at-once bargaining.*

4.3 Strategic foundations

We consider an extensive-form game between a buyer holding $z > 0$ real balances and a seller. The game has N rounds with two stages each. In the first stage an offer is made; in the second stage the offer is accepted or rejected. The buyer can transfer at most z/N units of real balances for some output in each round. The identity of the proposer (the buyer or the seller) alternates across rounds.

We assume that the number of rounds, N , is odd and the buyer is the one making the first offer. So the buyer is also making the last offer. These assumptions will be with no loss in generality since we will consider the outcome of the game when N becomes large.

1. In each round $n \in \{1, 3, \dots, N\}$, it is the buyer's turn to make an offer (y_n, d_n) , where $y_n \in [0, \infty)$ is the output asked in exchange for $d_n \in [0, z/N]$ real balances.
2. In each round $n \in \{1, 3, \dots, N\}$, the seller accepts or rejects the buyer's offer (y_n, d_n) .
3. In each round $n \in \{2, 4, \dots, N-1\}$, it is the seller's turn to make an offer (y_n, d_n) , where $y_n \in [0, \infty)$ is the output offered in exchange for $d_n \in [0, z/N]$ real balances.
4. In each round $n \in \{2, 4, \dots, N-1\}$, the buyer accepts or rejects the seller's offer (y_n, d_n) .

We define intermediate payoffs (U_n^b, U_n^s) as the utilities that the players would enjoy based on the agreements reached up to some round $n \in \{1, \dots, N\}$. Our main proposition is:

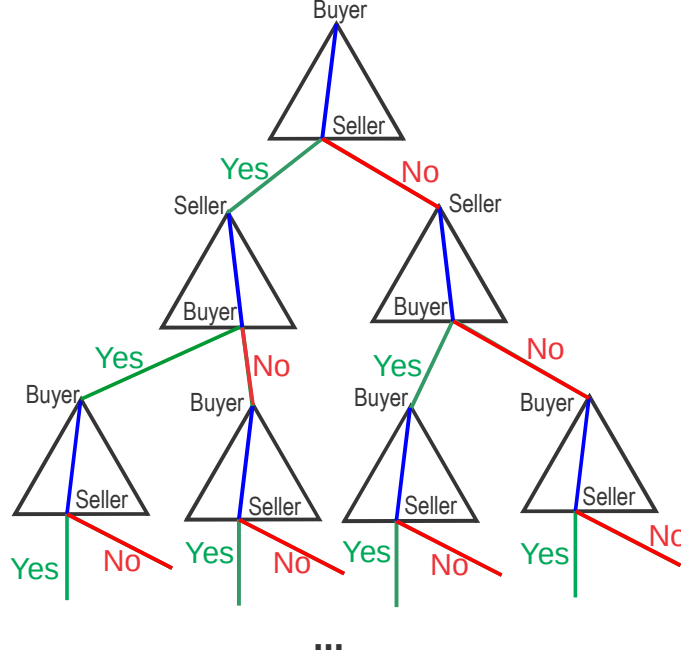


Figure 3: Game tree

Proposition 3 *There exist subgame-perfect equilibria of the alternating-ultimatum-proposals games such that the sequence of intermediate payoffs $\{(U_n^b, U_n^s)\}_{n=1,2,\dots,N}$ converges to the ordinal solution $(u^b(\tau), u^s(\tau))$ as N approaches ∞ .*

5 Asset prices

We now move to the general equilibrium implications of the gradual bargaining protocol for asset prices, allocations, and welfare. We study first the pricing of Lucas trees. We then discuss succinctly the case of fiat money and make comparisons to alternative bargaining protocols.

5.1 Lucas trees

We consider the case of Lucas trees where $d > 0$ and $\pi = 0$. We study alternative assumptions regarding the horizon of the negotiation, $\bar{\tau}$.

Deterministic negotiating time If $s > 0$ then the rate of return of assets is less than the rate of time preference, in which case buyers do not hold more assets than what they intend to spend in the DM. We can re-express the consumer's choice of asset holdings, (4), as a choice of DM consumption:

$$\max_{y \in [0, y^*]} \{-sz(y) + \alpha[u(y) - z(y)]\} \quad \text{s.t.} \quad z(y) \leq \delta \bar{\tau}. \quad (19)$$

From (19) the consumer chooses asset holdings, and hence DM output, to maximize his expected surplus from trade, $\alpha[u(y) - z]$, net of the cost of holding liquid assets, sz . The constraint, $z(y) \leq \delta\bar{\tau}$, captures the assumption that δ units of wealth can be negotiated per unit of time and the time horizon of the negotiation is $\bar{\tau}$. Hence, the upper bound on the amount of assets that can be sold is $\delta\bar{\tau}$. The objective function is continuous and maximized over a compact set, and it is strictly concave for all $y \in (0, y^*)$ so that the solution is unique. Let ξ denote the Lagrange multiplier associated with the constraint $z(y) \leq \delta\bar{\tau}$. The first-order condition gives

$$s = \frac{\rho\phi - d}{\phi + d} = \frac{\alpha}{2} \left[\frac{u'(y)}{v'(y)} - 1 \right] - \xi. \quad (20)$$

The spread between illiquid and liquid assets on the left side is equal to the liquidity premium on the right side. The liquidity premium is equal to the increase in the buyer's surplus from spending an additional unit of asset in the DM, assuming the negotiability constraint $z(y) \leq \delta\bar{\tau}$ does not bind, net of the shadow value of this constraint as measured by ξ . When $s > 0$, buyers hold exactly $z(y)$. Hence, by market clearing, $z(y) = (\phi + d)A$, i.e.,

$$z(y) = \frac{(1 + \rho)Ad}{\rho - s}, \quad (21)$$

where we have used that the cum-dividend price of the asset is $\phi + d = (1 + \rho)d/(\rho - s)$. The payment by the consumer is equal to the total value of his assets, $(\phi + d)A$.

If $s = 0$ then assets are not costly to hold so that buyers can hold more than what they intend to spend, and sellers could hold assets as well. The solution to (19) and market clearing imply

$$z(y) = \min\{\delta\bar{\tau}, z(y^*)\} \quad \text{and} \quad \frac{(1 + \rho)Ad}{\rho} \geq z(y). \quad (22)$$

If the negotiability constraint does not bind the buyer purchases y^* . Otherwise, he spends the maximum amount that can be sold until $\bar{\tau}$. The total supply of the asset, $(\phi + d)A$, is larger or equal than $z(y)$ since assets can also be held as a pure store of value. An equilibrium can then be defined as a pair (s, y) that solves (20)-(21) if $s > 0$ and (22) if $s = 0$.

Proposition 4 *There are four different regimes.*

1. If $z(y^*) \leq \min\{\delta\bar{\tau}, (1 + \rho)Ad/\rho\}$ then $s = 0$ and $y = y^*$.
2. If $\delta\bar{\tau} < z(y^*)$ and $Ad \geq \rho\delta\bar{\tau}/(1 + \rho)$ then $s = 0$ and $y = z^{-1}(\delta\bar{\tau}) < y^*$.
3. If $(1 + \rho)Ad/\rho < z(y^*)$ and

$$\rho - \frac{(1 + \rho)Ad}{\delta\bar{\tau}} > \frac{\alpha}{2} \left[\frac{u' \circ z^{-1}(\delta\bar{\tau})}{v' \circ z^{-1}(\delta\bar{\tau})} - 1 \right],$$

then

$$s = \frac{\alpha}{2} \left[\frac{u'(y)}{v'(y)} - 1 \right] \quad (23)$$

$$y = z^{-1} \left[\frac{(1+\rho)Ad}{\rho - s} \right]. \quad (24)$$

4. If $(1+\rho)Ad/\rho < z(y^*)$ and

$$\frac{\alpha}{2} \left[\frac{u' \circ z^{-1}(\delta\bar{\tau})}{v' \circ z^{-1}(\delta\bar{\tau})} - 1 \right] > \rho - \frac{(1+\rho)Ad}{\delta\bar{\tau}} > 0,$$

then

$$s = \rho - \frac{(1+\rho)Ad}{\delta\bar{\tau}} \quad (25)$$

$$y = z^{-1}(\delta\bar{\tau}). \quad (26)$$

In Figure 4 we represent the different types of equilibria and their associated outcomes. The first regime corresponds to the first-best outcome where $s = 0$ and $y = y^*$. It requires both an abundant asset supply and a high negotiability. The second regime is such that $s = 0$ but $y < y^*$. This regime occurs when the asset is abundant relative to the amount that can be negotiated within a match. In that case output increases with the negotiability of the asset but the asset price is independent of negotiability. The third regime is such that the asset pays a liquidity premium, $s > 0$, and agents are not time-constrained to negotiate the sale of their assets. In that case the asset is scarce relative to agents' liquidity needs. Equation (23) gives a negative relation between s and y while (24) gives a positive relation. Hence, it is easy to check that there is a unique (s, y) solution to this system. Moreover, this solution is independent of the negotiability of the asset, δ . The relevant friction for asset prices in that case is the search-matching friction, α . The fourth and last regime is such that the asset pays a liquidity premium but now agents face a binding time constraint when negotiating the sale of their assets. The spread and output increase with the negotiability of the asset, δ , but they are independent of α . So we see that the two types of delays that affect outcomes in decentralized asset markets, α and δ , matter in different regimes.

Stochastic negotiating time Suppose now that the horizon for the negotiation, $\bar{\tau}$, is a random variable realized at the beginning of a match. It is exponentially distributed with mean $1/\lambda$. This assumption captures the idea that in some matches agents have a lot of time to negotiate the sale of assets whereas in other matches they have little time. Now, plug in the expression for buyer trade surplus given by (17), we can generalize the portfolio choice problem, (19), to:

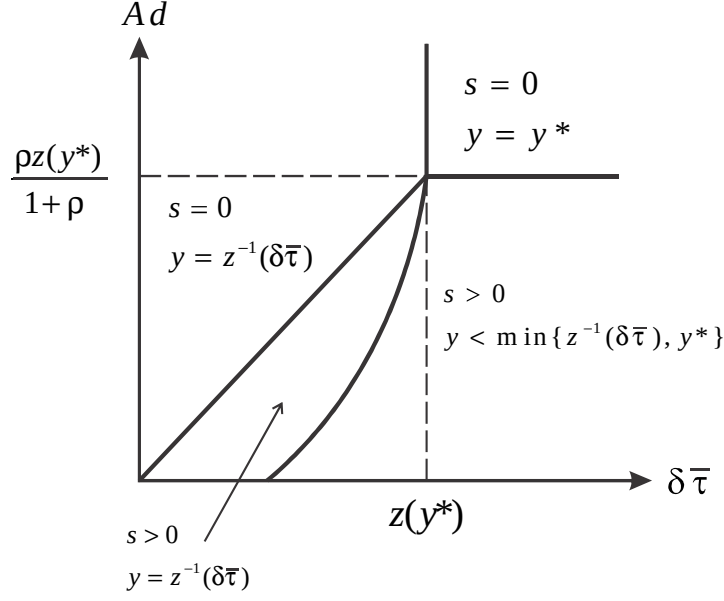


Figure 4: Asset prices and negotiability: Typology of equilibria

$$\max_{y \geq 0} \left\{ -sz(y) + \alpha \int_0^y e^{-\frac{\lambda}{\delta} z(x)} \frac{u'(x) [u'(x) - v'(x)]}{u'(x) + v'(x)} dx \right\}. \quad (27)$$

The second terms in the objective function, (27), corresponds to the buyer's expected surplus from a DM trade. With probability $e^{-\lambda z(x)/\delta}$ the negotiability constraint does not bind and the buyer can purchase x by selling his $z(x)$ units of asset. The first-order condition implies:

$$s = \frac{\alpha}{2} e^{-\frac{\lambda}{\delta} z(y)} \left[\frac{u'(y)}{v'(y)} - 1 \right]. \quad (28)$$

The spread is the product of four components: the search friction, α , the bargaining power, $1/2$, the negotiability friction, $e^{-\frac{\lambda}{\delta} z(y)}$, and the marginal value of wealth in the DM. The negotiability term captures the probability the buyer has enough time to liquidate his portfolio. By market clearing:

$$z(y) \leq \left(\frac{1+\rho}{\rho-s} \right) Ad, \quad " = " \quad \text{if } s > 0. \quad (29)$$

An equilibrium can be reduced to a pair (s, y) solution to (28) and (29). We measure social welfare as the sum of surpluses in pairwise meetings:

$$\mathcal{W} = \alpha \int_0^y e^{-\frac{\lambda}{\delta} z(x)} [u'(x) - v'(x)] dx. \quad (30)$$

Note that this measure of welfare does not take into account the output from Lucas trees, Ad . However, changes in Ad may still affect equilibrium DM trades and hence affect the welfare indirectly.

Proposition 5 Assume that the time limit for the negotiation, $\bar{\tau}$, is exponentially distributed with mean $1/\lambda$. An equilibrium exists and is unique. Moreover:

1. If $Ad \geq \rho z(y^*)/(1 + \rho)$ then $s = 0$ and y^* is implemented in a fraction $e^{-\frac{\lambda}{\delta} z(y^*)}$ of all matches. Social welfare is independent of Ad but it increases with δ and decreases with λ .
2. If $Ad < \rho z(y^*)/(1 + \rho)$ then $s > 0$ and y^* is never implemented. The asset spread, s , decreases with Ad and λ but increases with δ . Social welfare increases with Ad and δ but decreases with λ .

In the presence of stochastic termination, there are two regimes. In the first regime agents hold enough wealth to buy y^* provided that the negotiation lasts long enough to sell $z(y^*)$ units of wealth, which takes $z(y^*)/\delta$ units of time. Given that the termination time is exponentially distributed, it means that the efficient trade is implemented in a fraction $e^{-\frac{\lambda}{\delta} z(y^*)}$ of all matches. As the risk of termination decreases (λ is lower) and the asset becomes more negotiable (δ is higher) then this fraction increases and welfare increases. However, $s = 0$, so that the asset price is not affected by λ or δ . In the second regime agents hold less than $z(y^*)$ and hence trades are inefficient in all matches. In that case, the asset spread decreases with the asset supply, and it increases with both α and δ . So frictions in the matching process and frictions in the negotiations have qualitatively similar effects on asset prices, and these frictions reinforce each other.

Gradual vs all-at-once bargaining So far we have emphasized the negotiability of the asset on prices when there is a finite (possibly stochastic) horizon for the negotiation. We now want to argue that even in the absence of negotiability constraint, when $\bar{\tau}$ is large, the fact that the asset is negotiated gradually has implications for asset prices and allocations. In order to make this point we compare equilibria under Nash bargaining where portfolios are negotiated all at once and gradual Nash bargaining.

Proposition 6 (i) *Nash Bargaining.* There exists a steady-state equilibrium. If

$$Ad \geq \frac{\rho}{1 + \rho} \frac{v'(\tilde{y})u(\tilde{y}) + u'(\tilde{y})v(\tilde{y})}{u'(\tilde{y}) + v'(\tilde{y})},$$

where $\tilde{y} < y^*$ solves $u'(\tilde{y}) = z^{N'}(\tilde{y})$, then $s = 0$, $r = \rho$, and $y = \tilde{y} < y^*$. Moreover, if the inequality is strict, a fraction of the asset supply is held by producers. Otherwise, $s > 0$, $r < \rho$, $y < \tilde{y}$.

(ii) *Gradual Bargaining* ($\bar{\tau} \rightarrow +\infty$). There exists a unique steady-state equilibrium. If

$$Ad \geq \frac{\rho}{1 + \rho} \int_0^{y^*} \frac{2v'(x)u'(x)}{u'(x) + v'(x)} dx, \quad (31)$$

then $s = 0$, $r = \rho$, and $y = y^*$. Otherwise, $s > 0$, $r < \rho$, $y < y^*$, and $\partial r / \partial A > 0$.

Under gradual bargaining, if the supply of the asset is sufficiently large, then the first best is implementable and the asset is priced at its fundamental value, $\phi = \phi^* = d/\rho$ and $s = 0$. In contrast, under Nash bargaining, if liquidity is abundant consumers choose asset holdings to finance y that maximize $u(y) - z^N(y)$ and the solution is $\tilde{y} < y^*$. Even though it would be socially optimal to have consumers holding larger quantities of assets, they choose not to do so. Buyers hold $z^N(\tilde{y})$ and producers hold $Ad(1 + \rho)/\rho - z^N(\tilde{y})$. The asset is misallocated since producers hold assets even though they have no liquidity use for it and consumers are liquidity constrained. Such misallocation does not occur under gradual bargaining because consumers can extract a non-negative surplus from each unit of asset that they hold. Hence, if assets are plentiful they will trade up to y^* .

5.2 Fiat money

We now briefly consider the case where the asset is fiat money, $d = 0$ and the rate of growth of the money supply is $\pi \in (\beta - 1, \infty)$. We denote $i \equiv (1 + \rho)(1 + \pi) - 1$ the nominal interest rate on an illiquid bond. It is easy to check that y is a solution to (20) where $s = i$, i.e.,

$$i \leq \frac{\alpha}{2} \left[\frac{u'(y)}{v'(y)} - 1 \right] \quad \text{"="} \quad \text{if } z(y) \leq \delta\bar{\tau}.$$

Moreover, $z(y) = \phi A$.

Proposition 7 *If $u'(0)/v'(0) = +\infty$, then there exists a unique steady-state monetary equilibrium. If $z(y^*) \leq \delta\bar{\tau}$ then y approaches y^* as i approaches 0. If $z(y^*) > \delta\bar{\tau}$ then there exists $\underline{i} > 0$ such that for all $i \in (0, \underline{i})$, $z(y) = \delta\bar{\tau}$.*

If preferences satisfy the Inada conditions, then a monetary equilibrium exists for all inflation rates irrespective of the negotiability constraint, which is consistent with the standard Nash bargaining solution but it is in contrast with equilibria under Kalai bargaining. Indeed, under the egalitarian bargaining solution a monetary equilibrium only exists if $i < \alpha \leq 1$ (see, e.g., Rocheteau and Nosal, 2017). The consumer's choice of real balances is strictly concave and, as a result, equilibrium is unique. The same concavity property holds under Kalai bargaining. However, under Nash bargaining, the consumer's problem might not be concave and hence one cannot rule out multiple steady states.³ Under gradual bargaining, provided that $\delta\bar{\tau}$ is sufficiently large, the Friedman rule implements the first best allocation. While this result is consistent with the Kalai solution, it is in contrast with the Nash bargaining solution under which the Friedman rule implements $\tilde{y} < y^*$.

³See, however, Wright (2010) for a proof of generic uniqueness of a stationary monetary equilibrium under Nash bargaining.

If $\delta\bar{\tau}$ is smaller than $z(y^*)$, then the negotiability constraint binds for low inflation rates. In that case the optimal monetary policy corresponds to any i lower than some positive threshold, $\underline{i}$. So a positive nominal interest rate is optimal provided that it is not too large but it fails to implement the first best. As the negotiability of money increases, $\underline{i}$ decreases so that optimal monetary policy corresponds to lower interest rates.

The presence of stochastic termination would affect these results. In particular, under exponential distribution, optimal monetary policy consists only of the Friedman rule. In contrast, if the distribution has a upper bound in its support that is less than $z(y^*)/\delta$, then there is still a range of optimal monetary policies. Nevertheless, in both deterministic and stochastic cases, our model shows that negotiability can be a relevant factor in determination of optimal monetary policy, and we shall endogenize it in the next section.

6 Endogenous bargaining and search

We now endogenize the negotiability of assets and the frequency of trading opportunities. We will ask whether the negotiability of assets chosen in a decentralized equilibrium is constrained efficient and we will study how bargaining and search frictions interact. Because investment decisions in search and bargaining technologies depend on how the match surplus is shared between the buyer and the seller we first relax the symmetry assumption of the gradual bargaining solution. For expositional purposes we assume stochastic termination under exponential distribution.

6.1 Asymmetric gradual bargaining

Following Wiener and Winter (1998), we generalize the gradual bargaining solution, (10)-(11), as follows:

$$u^{b'}(\tau) = -\theta \frac{\partial H(u^b, u^s, \tau)/\partial \tau}{\partial H(u^b, u^s, \tau)/\partial u^b} \quad (32)$$

$$u^{s'}(\tau) = -(1-\theta) \frac{\partial H(u^b, u^s, \tau)/\partial \tau}{\partial H(u^b, u^s, \tau)/\partial u^s}, \quad (33)$$

where $\theta \in [0, 1]$ is interpreted as the consumer's bargaining power.⁴ Intuitively, the consumer can capture a fraction θ of the shift of the Pareto frontier induced by an increase in asset holdings.

Proposition 8 *Suppose agents bargaining according to an asymmetric gradual Nash solution where $\theta \in [0, 1]$ is the buyer's bargaining power. The DM price of assets evolves according to*

$$\frac{\partial y}{\partial \tau} = \delta \frac{\theta u'(y) + (1-\theta)v'(y)}{u'(y)v'(y)}. \quad (34)$$

⁴One could make the bargaining power a function of time, τ , or output traded, y , without affecting the results significantly.

The payment for y units of DM consumption is

$$z(y) = \int_0^y \frac{u'(x) v'(x)}{\theta u'(x) + (1 - \theta) v'(x)} dx \quad \text{for all } y \leq y^*. \quad (35)$$

The asset spread solves

$$s = \alpha \theta e^{-\frac{\lambda}{\delta} z(y)} \left[\frac{u'(y) - v'(y)}{v'(y)} \right]. \quad (36)$$

From (34) the DM price of the asset is increasing in θ . If $\theta = 1$ the consumer needs $v'(y)$ assets (valued at the cum-dividend price in the CM) to purchase one unit of DM output. If $\theta = 0$, he needs $u'(y)$ assets to buy the same quantity of DM output. From (36) the search and bargaining frictions enter the asset spread in a multiplicative fashion.⁵ We can then generalize Propositions 6 and 7.

Proposition 9 *There exists a unique stationary equilibrium. If*

$$Ad \geq \frac{\rho}{1 + \rho} \int_0^{y^*} \frac{v'(x) u'(x)}{\theta u'(x) + (1 - \theta) v'(x)} dx,$$

then $s = 0$, $r = \rho$, and $y = y^$. Otherwise, $s > 0$, $r < \rho$, $y < y^*$, and $\partial r / \partial A > 0$.*

Suppose τ approaches 0 so that agents have enough time to negotiate all their wealth. Then the first best is implemented if the asset supply is sufficiently large. Moreover, as θ increases less assets are needed to achieve y^* . This result is in contract with the outcome of the generalized Nash solution where for all $\theta < 1$ the first best is not achievable, even for large values of Ad .

6.2 Endogenous negotiability

We extend our model to endogenize the negotiability of assets, δ . Suppose as in Section ?? that the horizon for the negotiation, $\bar{\tau}$, is exponentially distributed with mean $1/\lambda$. We now assume that buyers can choose the speed at which their assets are negotiated and transferred. For instance, in financial markets trades can be conducted at different speeds, e.g., due to innovations in electronic communication networks (e.g., Pagnotta and Philippon, 2017). In the case of crypto-currencies agents can also speed up the confirmation of a transfer at some cost. In the context of our model we assume that when a match is formed, before $\bar{\tau}$ is determined, buyers can choose δ . There is a cost, $\psi(\delta)$, associated with the speed of the transaction, where $\psi(0) = \psi'(0) = 0$, $\psi'(\delta) > 0$ and $\psi''(\delta) > 0$. We can think of it as the cost of computer power to execute a trade and transfer assets safely.

The expected surplus of a buyer who meets a buyer holding z^b is:

⁵The idea that search and bargaining frictions are compounded is a common theme of models of over-the-counter markets. See, e.g., Lagos and Rocheteau (2009).

$$S^b(z^b, \delta) = \int_0^y e^{-\lambda \frac{z(x)}{\delta}} \frac{\theta u'(x) [u'(x) - v'(x)]}{\theta u'(x) + (1 - \theta)v'(x)} dx,$$

where $y = z^{-1}(z^b)$. The partial derivatives are:

$$\begin{aligned} \frac{\partial S^b(z^b, \delta)}{\partial z^b} &= e^{-\lambda \frac{z^b}{\delta}} \frac{\theta [u'(y) - v'(y)]}{v'(y)} \geq 0 \\ \frac{\partial S^b(z^b, \delta)}{\partial \delta} &= \int_0^y \frac{\lambda}{\delta^2} e^{-\lambda \frac{z(x)}{\delta}} z(x) \frac{\theta u'(x) [u'(x) - v'(x)]}{\theta u'(x) + (1 - \theta)v'(x)} dx > 0. \end{aligned}$$

The buyer's surplus increases with both asset holdings and asset negotiability. Moreover, $\partial^2 S^b(z^b, \delta) / \partial z^b \partial \delta > 0$ if $y < y^*$. So there are complementarities between the choice of asset holdings and the speed of negotiation.

The buyer's choice of asset holdings and speed of negotiation can be written compactly as:

$$\max_{z^b, \delta} \left\{ -sz^b + \alpha [-\psi(\delta) + S^b(z^b, \delta)] \right\}. \quad (37)$$

The novelty is the first term in squared brackets that represents the cost to invest in a technology to negotiate assets at speed δ . The first-order condition with respect to δ is:

$$\psi'(\delta) = \int_0^y \frac{\lambda}{\delta^2} e^{-\lambda \frac{z(x)}{\delta}} z(x) \frac{\theta u'(x) [u'(x) - v'(x)]}{\theta u'(x) + (1 - \theta)v'(x)} dx. \quad (38)$$

Despite the lack of concavity of the problem (37) its solution is characterized in the following lemma.

Lemma 2 *There exists a solution, (z^*, δ^*) , to (37). It is generically unique, upper-hemi continuous, non-increasing in s , and non-decreasing in α . If $s = 0$ then buyers hold at least $z(y^*)$ and δ^* is at its maximum. As s tends to infinity, (z^*, δ^*) goes to $(0, 0)$.*

From Lemma 2 the buyer's choice of δ and his choice of z are complements since an investment in speed to negotiate a sale of assets is more profitable when there are more assets to sell. Hence, as the cost of holding asset, s , increases buyers reduce both their asset holdings and the speed of negotiation. A reduction in search frictions raises the demand for assets and the speed of negotiation. We now turn to the general equilibrium where s is endogenous.

Proposition 10 (Equilibrium with endogenous negotiability.) *There exists a steady-state equilibrium with endogenous negotiability and it is generically unique. If $Ad \geq \rho z(y^*) / (1 + \rho)$ then $s = 0$ and δ is maximum. If $Ad < \rho z(y^*) / (1 + \rho)$ then an increase in A reduces s , but raises δ .*

An increase the asset supply reduces the spread s , which leads to a higher choice of δ . If buyers have to sell more assets, they will find it worthwhile to increase the speed at which they can negotiate those assets.

We now turn to the socially efficiency of the asset negotiability. We consider the problem of a planner who chooses the level of asset negotiability, δ , subject to the same cost as private agents, $\psi(\delta)$, and subject to the same trading protocols in the DM and CM. It means that the pricing in the DM is given by $z(y)$ and the asset spread in the CM is a market clearing price. The planner's problem solves:

$$\max_{z, \delta, s} \left\{ -\psi(\delta) + \int_0^y e^{-\lambda \frac{z(x)}{\delta}} [u'(x) - v'(x)] dx \right\} \quad (39)$$

$$\text{s.t. } z \in \arg \max_z \{ -sz + \alpha S^b(z, \delta) \} \quad (40)$$

$$z(y) \leq \left(\frac{1+\rho}{\rho-s} \right) Ad, \quad "=" \text{ if } s > 0 \quad (41)$$

According to (39) the planner maximizes the expected surplus of each match net of the negotiability cost. It subject to (40) according to which buyers choose their asset holdings optimally taking as given the negotiability of the asset and its cost (which is omitted from the buyer's objective). From (41) the spread, s , is consistent with market clearing.

Proposition 11 (*Constrained-efficient negotiability*) *Asset negotiability is constrained-efficient if and only if $Ad \geq \rho z(y^*)/(1+\rho)$ and $\theta = 1$.*

Proof. From Proposition 5, if $Ad \geq \rho z(y^*)/(1+\rho)$ then equilibrium is such that $s = 0$ and $y = y^*$ irrespective of δ . Hence, the solution to (39) is

$$\psi'(\delta) = \int_0^{y^*} \frac{\lambda}{\delta^2} z(x) e^{-\lambda \frac{z(x)}{\delta}} [u'(x) - v'(x)] dx.$$

It coincides with (38) if and only if $\theta = 1$. If $\theta < 1$ then the decentralized choice of δ is smaller than the planner's choice. For the case $Ad < \rho z(y^*)/(1+\rho)$, we proved in Proposition 5 that s increases with δ . From market clearing $z(y) = \left(\frac{1+\rho}{\rho-s} \right) Ad$, and hence y is an increasing function of s . Hence, the solution to the planner's problem is:

$$\psi'(\delta) = \int_0^y \frac{\lambda}{\delta^2} z(x) e^{-\lambda \frac{z(x)}{\delta}} [u'(x) - v'(x)] dx + e^{-\lambda \frac{z(y)}{\delta}} [u'(y) - v'(y)] \frac{\partial y}{\partial \delta}.$$

The second term on the right side captures the effect of an increase of negotiability on the spread and hence y . Even if $\theta = 1$ this condition does not coincide with (38). ■

Proposition 11 establishes that equilibrium asset negotiability is constrained efficient if and only if the asset supply is abundant, so that the spread is zero, and buyers have all the bargaining power. This result is intuitive since the costly investment in asset negotiability creates a holdup problem that can only be solved by having the ones making the investment receive the whole match surplus. However, if the asset supply is low so that $s > 0$, then the investment in δ is inefficiently low even when $\theta = 1$. This inefficiency

occurs because of a pecuniary externality according to which the demand for the asset, and hence its price, increases with δ . As the asset becomes more valuable, buyers' wealth increases, which relaxes their liquidity constraint. The planner understands this externality and hence chooses a δ larger than the one that buyers would choose even if they had all the bargaining power.

6.3 Endogenous search

So far we have endogenized the time required to negotiate the sale of assets. We now endogenize the time it takes to receive trading opportunities, $1/\alpha$. To do this, we introduce a participation decision on one side of the market. Suppose that producers of the DM good can choose to participate in the DM at some cost.⁶ The measure of producers who participate is denoted n and the measure of DM matches is $\alpha(n)$ where $\alpha(n)$ is strictly concave. Assuming an interior solution n solves

$$\kappa = \frac{\alpha(n)}{n} \int_0^y e^{-\lambda \frac{z(x)}{\delta}} \frac{(1-\theta)v'(x) [u'(x) - v'(x)]}{\theta u'(x) + (1-\theta)v'(x)} dx. \quad (42)$$

The entry cost, κ , is equal to the expected surplus of the producer, which is equal to the matching probability, $\alpha(n)/n$, times the expected surplus from the negotiation. Entry requires $\theta < 1$ and it increases with y and δ .

From the previous subsection we can derive an equilibrium condition between y and α by substituting the market clearing spread, $s(Ad, \alpha)$, into the asset demand, $z^b(s, \alpha)$, and using the fact that $z^b = z(y)$. As α increases so does y . Hence, there is a positive relationship between y and n . As n approaches 0, s tends to 0 and $z(y) = \min\{(1+\rho)Ad/\rho, z(y^*)\}$. As n tends to infinity, α approaches 1 and $z(y)$ approaches a finite limit. The second relationship between n and y is given by (42). Provided that κ is not too large, an active equilibrium exists. Moreover, the model can generate multiple steady-state equilibria. At the high equilibrium the entry curve intersects the asset demand curve by below in the space (n, y) . Hence, a reduction in κ that shifts the entry curve to the right leads to higher n and y and hence higher δ . This illustrates how search and negotiation frictions are complement.

We now compare equilibrium allocations to the constrained-efficient allocation in the absence of bargaining friction, λ goes to 0. The constrained-efficient allocation is a pair (y, n) that maximizes total welfare, $\alpha(n) [u(y) - v(y)] - \kappa n$. The solution is $y = y^*$ and $\alpha'(n) [u(y^*) - v(y^*)] = \kappa$.

Proposition 12 *Suppose there is free entry of producers and asymmetric gradual bargaining. An equilibrium*

⁶There are several ways to endogenize the measure of trades taking place within a period: through costly entry (e.g., Rocheteau and Wright, 2005), endogenous search intensity (Lagos and Rocheteau, 2005), endogenous market composition (Rocheteau and Wright, 2009), endogenous asset acceptability (e.g., Lester et al., 2012).

allocation coincides with the constrained-efficient allocation if

$$Ad \geq \frac{\rho}{1+\rho} \int_0^{y^*} \frac{u'(x) v'(x)}{\theta u'(x) + (1-\theta) v'(x)} dx \quad (43)$$

and

$$\frac{\alpha'(n)n}{\alpha(n)} = (1-\theta) \frac{\int_0^{y^*} \frac{v'(x)[u'(x)-v'(x)]}{\theta u'(x) + (1-\theta) v'(x)} dx}{\int_0^{y^*} [u'(x) - v'(x)] dx}. \quad (44)$$

The gradual Nash bargaining solution can implement the efficient allocation while the generalized bargaining solution cannot. The first best requires liquidity to be plentiful and a modified version of the Hosios condition to be satisfied. The bargaining power must be such that the effective producers' share is equal to the producers' contribution in the matching process.

7 Bargaining with multiple assets and endogenous agenda

So far we have considered an economy with a single asset, either a Lucas tree or fiat money, to study the implications of gradual bargaining and limited negotiability on asset prices. We now extend our model to have different assets in order to investigate cross-sectional differences in asset prices. We will show that our model can generate a pecking order of payments and rate-of-return differences across assets. Moreover, we will provide two applications to show that our theory has novel implications for monetary policy and for the determination of exchange rates.

We now assume that there are J types of one-period lived Lucas trees indexed by $j \in \{1, \dots, J\}$, where each Lucas tree born in $t-1$ pays off one unit of numeraire in the CM of t . The supply of each Lucas tree is denoted A_j and the new Lucas trees are distributed to buyers in a lump-sum fashion at the beginning of each CM. We index fiat money by $j = 0$ and we think of assets $j \geq 1$ as stocks, corporate and government bonds, and other financial securities. Since assets are negotiated gradually over time, a natural dimension to distinguish different assets is the time it takes to negotiate their sale, which includes assessing and authenticating the asset and securing the transfer of the ownership. The negotiability of asset j is δ_j . We rank assets according to their negotiability, $\delta_0 \geq \delta_1 \geq \delta_2 \geq \dots \geq \delta_J$. So, by assumption, fiat money is the most negotiable asset. In each pairwise meeting, the negotiation ends at time $\bar{\tau}$ where $\bar{\tau}$ is exponentially distributed with mean $1/\lambda$. The consumer's bargaining power is constant over time and equal to θ .⁷

In the presence of multiple assets one needs to specify the order according to which assets are sold. We let consumers choose the agenda of the negotiation. The amount of asset of type j up for negotiation at

⁷One could allow θ to be a function of τ , which would not affect our results qualitatively. One could also assume that θ varies with the type of asset that is currently under negotiation. Such extension would allow our theory to encompass the explanations for rate-of-return differences across assets by Zhu and Wallace (2007) and Nosal and Rocheteau (2011).

time τ is denoted $\omega_j(\tau)$ and total wealth up for negotiation is $\omega(\tau)$. They obey the following law of motion:

$$\begin{aligned}\dot{\omega}(\tau) &= \sum_{j=0}^J \dot{\omega}_j(\tau), \\ \dot{\omega}_j(\tau) &= \delta_j \sigma_j(\tau), \text{ for all } j \in \{0, 1, \dots, J\}\end{aligned}$$

where $\sigma_j(\tau) \in [0, 1]$ is the fraction of time devoted to the sale of asset j and $\sum_{j=0}^J \sigma_j(\tau) = 1$. Moreover, feasibility implies $\sigma_j(\tau) \in [0, 1]$ if $\omega_j(\tau) < a_j$ and $\sigma_j(\tau) = 0$ otherwise. In words, an agent can add asset j on the negotiation table at time τ only if he has not sold all his holdings of asset j prior to τ . From (34), the change in the consumer's consumption and the change in the overall payment over time are

$$y'(\tau) = \frac{\theta u'(y) + (1 - \theta) v'(y)}{u'(y)v'(y)} \dot{\omega} \quad (45)$$

$$p'(\tau) = \dot{\omega}, \quad (46)$$

if $y(\tau) < y^*$ and $y'(\tau) = p'(\tau) = 0$ otherwise.

The expected surplus of a consumer in a DM match with portfolio $\mathbf{a} = [a_j]_{j=0}^J$ is:

$$S(\mathbf{a}) = \int_0^{+\infty} \lambda e^{-\lambda x} \int_0^x \{u'[y(\tau)] y'(\tau) - p'(\tau)\} d\tau dx = \theta \int_0^{+\infty} e^{-\lambda \tau} \left\{ \frac{u'[y(\tau)] - v'[y(\tau)]}{v'[y(\tau)]} \right\}^+ \omega'(\tau) d\tau, \quad (47)$$

where $\{x\}^+ = \max\{x, 0\}$. Over a small time interval of length $d\tau$ the consumer raises his consumption by $y'(\tau)d\tau$, where consumption is valued according to the marginal utility $u'(y)$, and increases his payment by $p'(\tau)d\tau \leq \omega'(\tau)d\tau$. The negotiation ends at some random time x that is exponentially distributed. From (45)-(46) $y(\tau)$ and $p(\tau)$ depend on the portfolio $\mathbf{a}$ through the feasibility constraints according to which if $\omega_j(\tau) < a_j$ then $\sigma_j(\tau) = 0$. The right side of (47) is obtained by changing the order of integration in the middle term and replacing $y'(\tau)$ and $p'(\tau)$ by their expressions given by (45) and (46). It states that the consumer's surplus is the discounted sum of the marginal surpluses along the bargaining path where the discount rate is the survival rate of the negotiation, $e^{-\lambda \tau}$.

In order to characterize the optimal strategy to sell assets we denote $T_0 = 0$ and

$$T_j(\mathbf{a}) = \sum_{k=0}^{j-1} \frac{a_k}{\delta_k} \text{ for all } j \in \{1, 2, \dots, J+1\}$$

So T_j is the time that it takes to sell the first $j-1$ most negotiable assets. The following lemma characterizes the optimal choice $\boldsymbol{\sigma} = [\sigma_j^*]$.

Proposition 13 (Endogenous agenda) *For any portfolio $\mathbf{a}$, the optimal agenda is given by*

$$\sigma_j^*(\tau) = \begin{cases} 1 & \text{if } T_j \leq \tau \leq T_{j+1} \\ 0 & \text{otherwise} \end{cases}.$$

Proof. Rewrite (47) as:

$$S(\mathbf{a}) = \theta \int_0^{+\infty} e^{-\lambda\tau} \left\{ \frac{u'[y(\tau)] - v'[y(\tau)]}{v'[y(\tau)]} \right\}^+ \sum_{j=0}^J \delta_j \sigma_j(\tau) d\tau.$$

The list $[\sigma_j(\tau)]$ that maximizes $S(\mathbf{a})$ is the solution to:

$$\max_{\sigma_j(\tau)} \left\{ \sum_{j=0}^J \delta_j \sigma_j(\tau) d\tau \right\} \quad \text{s.t.} \quad \sum_{j=0}^J \sigma_j(\tau) = 1 \text{ and } \sigma_j(\tau) = 0 \text{ if } \omega_j(\tau) = a_j,$$

for all $\tau \geq 0$. This gives the result in the lemma. To obtain (48) use that $\omega'(\tau) = \sum_{j=0}^J \delta_j \sigma_j^*(\tau)$ into (47). ■

Lemma 13 shows that it is optimal to adopt a pecking order to sale of assets according to their negotiability.⁸ Consumers start paying with money because it is the asset that can be negotiated the fastest. When their money holdings are exhausted, they start selling asset 1. And so on. It follows that our theory endogenizes and generalizes cash-in-advance constraints. In a fraction $1 - e^{-\lambda T_1}$ of matches only money is used to finance consumption, where T_1 is endogenous and depends on $\mathbf{a}$. In a fraction $e^{-\lambda T_1} - e^{-\lambda T_2}$ of matches both money and type-1 Lucas trees serve as means of payments. And so on.

Given this pecking order the maximized surplus of the consumer is:

$$S(\mathbf{a}) = \theta \sum_{j=0}^J \delta_j \int_{T_j}^{T_{j+1}} e^{-\lambda\tau} \left[\frac{u'[y(\tau)] - v'[y(\tau)]}{v'[y(\tau)]} \right] d\tau. \quad (48)$$

Equation (48) has the following interpretation. Over the time interval $[T_j, T_{j+1}]$ agents negotiate asset j where the speed of the negotiation is given by δ_j . The asset owner gets a fraction θ of the surplus of the negotiation.

We now turn to the asset pricing implications of this pecking order. The portfolio problem in the CM is given by

$$\max_{\mathbf{a} \geq \mathbf{0}} \{ -\mathbf{s}\mathbf{a} + \alpha S(\mathbf{a}) \}, \quad (49)$$

where $\mathbf{s} = [s_j]$ is the vector of asset spreads, i.e., $s_j = (i - i_j) / (1 + i_j)$ where the nominal interest rate of asset j is i_j . For fiat money, $i_0 = 0$ and $s_0 = i$. According to (49) the consumer maximizes his expected DM surplus net of the costs of holding assets as measured by the spreads $[s_j]$. The FOCs of the maximization problem (49) are:

$$s_j = \alpha \frac{\partial S(\mathbf{a})}{\partial a_j}. \quad (50)$$

The left side of (50) is the opportunity cost of holding asset j . The right side is the probability α that the consumer receives an opportunity to spend, α , times the marginal liquidity value from holding asset j . The expression of this last term is given in the following lemma.

⁸For a pecking-order theory of payments based on informational asymmetries between consumers and producers, see Rocheteau (2011).

Lemma 3 *The marginal value of asset j to a consumer with portfolio $\mathbf{a}$ is*

$$\begin{aligned} \frac{\partial S(\mathbf{a})}{\partial a_j} &= \theta \lambda \sum_{k=j+1}^J \int_{T_k}^{T_{k+1}} \frac{(\delta_j - \delta_k)}{\delta_j} e^{-\lambda \tau} \left[\frac{u'[y(\tau)] - v'[y(\tau)]}{v'[y(\tau)]} \right] d\tau \\ &\quad + \theta e^{-\lambda T_{J+1}} \left[\frac{u'[y(T_{J+1})] - v'[y(T_{J+1})]}{v'[y(T_{J+1})]} \right]. \end{aligned} \quad (51)$$

Proof. We use the change of variable $\omega = \omega(\tau)$ together with the notation $\Omega_j(\mathbf{a}) = \sum_{k=0}^{j-1} a_k$ for all $j = 1, \dots, J+1$ with $\Omega_0(\mathbf{a}) = 0$, to rewrite (48) as

$$S(\mathbf{a}) = \theta \sum_{j=0}^J \int_{\Omega_j}^{\Omega_{j+1}} e^{-\lambda \left[\frac{(\omega - \Omega_j)}{\delta_j} + T_j \right]} \left[\frac{u'[z^{-1}(\omega)] - v'[z^{-1}(\omega)]}{v'[z^{-1}(\omega)]} \right] d\omega.$$

Equation (51) is obtained by taking the derivative with respect to a_j . ■

From (51), holding an additional unit of a_j has two benefits to the consumer. First, the consumer has more wealth, which relaxes his liquidity constraint and allows him to consume more if the negotiation is not terminated before the whole portfolio has been sold. This effect is captured by the last term on the right side, which is analogous to the expression for the spread in the one asset case. The novelty in a multi-asset environment is given by the first term according to which asset j speeds up the negotiation relative to less negotiable assets of types $j+k$.

By market clearing $a_j = A_j$ for all $j \geq 1$. Hence, an equilibrium can be reduced to a list $\langle a_0, \{s_j\}_{j=1}^J \rangle$ solution to (50).

Proposition 14 (The negotiability structure of asset yields.) *For all $\{A_j\}_{j=1}^J$ if $\delta_0 > \delta_1$ then there is a $\bar{t} > 0$ such that for all $i < \bar{t}$ there exists a unique steady-state monetary equilibrium with aggregate real balances $A_0(i) > 0$. If $\sum_{k=0}^j A_k < z(y^*)$ and $\delta_j > \delta_{j+1}$, then $s_j > s_{j+1}$. If $\sum_{k=0}^j A_k \geq z(y^*)$, then $s_{j+k} = 0$ for all $k \geq 0$.*

Proof. The equilibrium is solved recursively. The FOC (50) when $j = 0$ determines a_0 . Indeed, the RHS of (??) is strictly decreasing in a_0 , it is strictly positive at $a_0 = 0$ provided that $\delta_0 > \delta_1$ and equal to 0 as a_0 goes to ∞ . The threshold for the nominal interest rate below which a monetary equilibrium exists is

$$\begin{aligned} \bar{t} &= \alpha \theta \lambda \sum_{k=1}^J \frac{(\delta_0 - \delta_k)}{\delta_0} \int_{T_k}^{T_{k+1}} e^{-\lambda \tau} \left[\frac{u'[y(\tau)] - v'[y(\tau)]}{v'[y(\tau)]} \right] d\tau \\ &\quad + \alpha \theta e^{-\lambda T_{J+1}} \left[\frac{u'[y(T_{J+1})] - v'[y(T_{J+1})]}{v'[y(T_{J+1})]} \right], \end{aligned}$$

where $T_j = \sum_{k=1}^{j-1} A_k / \delta_k$ for all $j \in \{1, 2, \dots, J+1\}$. Given a_0 , the spreads $\{s_j\}_{j=1}^J$ are determined by (50).

From (50) we can compute the difference between two consecutive spreads:

$$s_j - s_{j+1} = \alpha \theta \lambda \frac{(\delta_j - \delta_{j+1})}{\delta_j} \int_{T_{j+1}}^{T_{j+2}} e^{-\lambda \tau} \left[\frac{u'[y(\tau)] - v'[y(\tau)]}{v'[y(\tau)]} \right] d\tau.$$

Hence, $s_j - s_{j+1} > 0$ requires $\delta_j - \delta_{j+1} > 0$ and $y(T_{j+1}) < y^*$, i.e., $\sum_{k=0}^j A_k < z(y^*)$. ■

Proposition 14 has several implications. First, fiat money is valued for low i irrespective of the supply of Lucas trees. Even if the capitalization of all Lucas trees, $\sum_{k=0}^j A_k$, is larger than liquidity needs, $z(y^*)$, money is still useful because it allows agents to secure some consumption when there is a risk that negotiation breaks down.

Second, even though all Lucas trees yield identical dividends, our model generates rate-of-return differences across assets. Provided that asset supplies are not too large, assets with a high negotiability will command a lower interest rate than assets with a low negotiability, i.e., $i_j < i_{j+1}$ if $\delta_j > \delta_{j+1}$. We obtain this result even though there is no informational asymmetries regarding the intrinsic values of the assets. The key components of our theory is that negotiation takes time as assets are sold gradually, and not all assets can be sold equally fast due to technological differences to authenticate and transfer assets.

Proposition 14 has drawn implications of differences in negotiability for asset prices. We now turn to implications for asset liquidity as measured by their velocity or turnover. The velocity of asset j is defined as

$$\mathcal{V}_j \equiv \frac{\alpha \int_0^{+\infty} \lambda e^{-\lambda x} \int_0^x \dot{\omega}_j(\tau) 1_{\{\omega(\tau) < z(y^*)\}} d\tau dx}{A_j}. \quad (52)$$

The numerator corresponds to the aggregate quantity of asset j sold in pairwise meetings while the denominator is the supply of the asset.

Proposition 15 (Negotiability and asset velocity) *The velocity of asset j is*

$$\mathcal{V}_j = \frac{\alpha \delta_j e^{-\lambda T_j} \left[1 - e^{-\frac{\lambda}{\delta_j} [\min\{z(y^*) - \Omega_j, A_j\}]} \right]}{A_j \lambda}, \quad (53)$$

if $z(y^*) > \Omega_j$ and $\mathcal{V}_j = 0$ otherwise. If $\delta_j > \delta_{j+1}$ and $z(y^*) > \Omega_j$, then $\mathcal{V}_j > \mathcal{V}_{j+1}$.

Proof. By changing the order of integration we can simplify it to:

$$\mathcal{V}_j = \frac{\alpha \int_0^{+\infty} e^{-\lambda \tau} \dot{\omega}_j(\tau) 1_{\{\omega(\tau) < z(y^*)\}} d\tau}{A_j}.$$

Using Lemma 13 and the fact that $\dot{\omega}_j(\tau) = \delta_j 1_{\{T_j \leq \tau < T_{j+1}\}}$ it can be rewritten as:

$$\mathcal{V}_j = \frac{\alpha \int_{T_j}^{T_{j+1}} e^{-\lambda \tau} \delta_j 1_{\{\omega(\tau) < z(y^*)\}} d\tau}{A_j}.$$

Using the expressions for T_j and T_{j+1} we distinguish three cases:

$$\mathcal{V}_j = \begin{cases} A_j^{-1} \lambda^{-1} \alpha \delta_j e^{-\lambda T_j} \left(1 - e^{-\frac{\lambda}{\delta_j} A_j} \right) & \geq \Omega_{j+1} \\ A_j^{-1} \lambda^{-1} \alpha \delta_j e^{-\lambda T_j} \left[1 - e^{-\frac{\lambda}{\delta_j} [z(y^*) - \Omega_j]} \right] & \in (\Omega_j, \Omega_{j+1}) \\ 0 & \leq \Omega_j \end{cases} \quad \text{if } z(y^*)$$

■

Proposition 15 shows that assets that are more negotiable have a higher velocity, which is a consequence of the endogenous pecking order. As a result, there is a positive correlation between velocity and asset prices.

A key assumption of our theory is that negotiation is gradual over time but the time allocated to the negotiation is limited and stochastic. We next show what happens when the expected time horizon of the negotiation becomes arbitrarily large.

Proposition 16 (*Rate of return equality*) *As λ approaches 0, $|s_j - s_{j'}|$ approaches 0 for all $j, j' \in \{0, \dots, J\}$. Asset velocity, $\mathcal{V}_j$, approaches α for all j such that $\Omega_{j+1} \leq z(y^*)$, 0 for all j such that $\Omega_j \geq z(y^*)$, and $\alpha[z(y^*) - \Omega_j]/A_j$ for j such that $z(y^*) \in (\Omega_j, \Omega_{j+1})$.*

Proof. It follows directly from (50) and the fact that:

$$s_j - s_{j+1} = \alpha\theta\lambda \frac{(\delta_j - \delta_{j+1})}{\delta_j} \int_{T_{j+1}}^{T_{j+2}} e^{-\lambda\tau} \left[\frac{u'[y(\tau)] - v'[y(\tau)]}{v'[y(\tau)]} \right] d\tau.$$

■

If there is no risk of breakdown, then the rates of return of all assets converge to the same value, i.e., there is rate of return equality. In that case the negotiability of assets, and the order according to which they are negotiated, does not affect their rates of return. Indeed, what matters for the bargaining is the consumer's total wealth and not the composition of its portfolio or the timing at which assets are sold. The order at which assets are sold only matters when there is a risk that the negotiation might end before the portfolio has been sold. The order at which assets are sold, however, matters for velocities. Indeed, only a fraction of assets are used for transactions and those assets have a maximum velocity equal to α . There is a fraction of assets that are not used for transaction so that their velocity is 0.

7.1 Application 1: Money and bonds

We now illustrate some novel comparative statics of our model regarding the effects of open-market operations (OMOs) on aggregate output. We consider the case where $J = 1$ with asset 1 being interpreted as short-term government bonds. We start with the case where $\bar{\tau}$ is deterministic, which will allow us to build intuition for the results, and we will return to the case where $\bar{\tau}$ is exponentially distributed later.

The buyer's portfolio problem in the CM is given by

$$\max_{(a_0, a_1)} -ia_0 - s_1 a_1 + \alpha\{u[y(a_0, a_1)] - z[y(a_0, a_1)]\},$$

where DM output is

$$y(a_0, a_1) = \begin{cases} z^{-1}(\delta_0 \bar{\tau}) & \leq a_0/\delta_0 \\ z^{-1}[a_0(1 - \delta_1/\delta_0) + \delta_1 \bar{\tau}] & \text{if } \bar{\tau} \in (a_0/\delta_0, a_0/\delta_0 + a_1/\delta_1] \\ z^{-1}(a_0 + a_1) & \geq a_0/\delta_0 + a_1/\delta_1 \end{cases}.$$

While $a_1 = A_1$ by market clearing, a_0 is endogenous and depends on policy through both i and A_1 . We distinguish four regimes represented in the parameter space $(\bar{\tau}, A_1)$ in Figure 5.

Regime IV: $\bar{\tau} > T_2$ In such equilibria the negotiability constraint does not bind. In that case $y(a_0, a_1) = \min\{z^{-1}(a_0 + a_1), y^*\}$. If money is valued, $a_0 > 0$, the FOCs imply rate-of-return equality between money and bonds, i.e., $s = i$ and $i_1 = 0$, bonds do not pay interest. This corresponds to the rate of return dominance puzzle in monetary theory. In this regime OMOs are ineffective. If money is not valued (the white area in Figure 5) then $y(a_1) = \min\{z^{-1}(a_1), y^*\}$ and output increases with the supply of bonds. This will be the case if A_1 is sufficiently large.

Regime I: $\bar{\tau} = T_1$ Money is the only means of payment, and the buyer holds just enough real balances to spend them all by the time the negotiation ends. In such equilibria y is determined as in the pure currency economy, $y(a_0, a_1) = z^{-1}(a_0)$, and $s = 0$. This corresponds to an endogenous "cash-in-advance" regime. A permanent change in A_1 has no effect on interest rates and output.

Regime II: $\bar{\tau} \in (T_1, T_2)$ Only a fraction of bonds can be sold before the negotiation ends. Hence, the marginal unit of bond does not serve as means of payment and $s_1 = 0$. In such equilibria $y = z^{-1}[a_0(1 - \delta_1/\delta_0) + \delta_1 \bar{\tau}]$ and from the FOCs:

$$i = \left(\frac{\delta_0 - \delta_1}{\delta_0} \right) \alpha \theta \left[\frac{u'(y)}{v'(y)} - 1 \right].$$

A change in A_1 has no effect on DM output. However, changes in negotiability matter. For instance, as δ_1 increases y decreases.

Regime III: $T_2 = \bar{\tau}$ The buyer's portfolio is sold in exactly $\bar{\tau}$ units of time. Recall that this regime is not knife-edge because a_0 is an endogenous variable. In that case, $a_0/\delta_0 + a_1/\delta_1 = \bar{\tau}$ implies

$$z(y) = \delta_0 \bar{\tau} - \left(\frac{\delta_0 - \delta_1}{\delta_1} \right) a_1.$$

The FOC gives the following expression for the asset spread:

$$s_1 = \frac{\delta_0}{\delta_1} i - \alpha \theta \left(\frac{\delta_0 - \delta_1}{\delta_1} \right) \left[\frac{u'(y)}{v'(y)} - 1 \right].$$

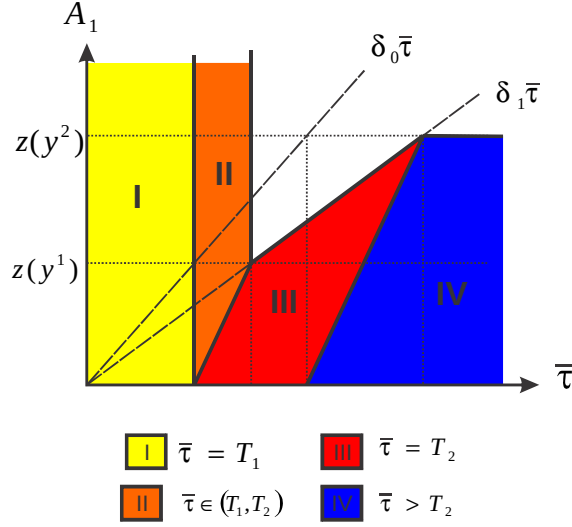


Figure 5: Typology of equilibria with money and bonds

The following proposition describes the effects of an open-market operation on output and interest rates in this regime.

Proposition 17 *A monetary equilibrium with $T_2 = \bar{\tau}$ exists if An open-market sale of bonds raises i_1 and reduces y .*

As A_1 increases buyers reduce a_0 so that they are still able to sell their whole portfolio in $\bar{\tau}$ units of time. But bonds take more time than money to be sold, and hence buyers' consumption decreases.⁹ This type of equilibrium captures the common wisdom that an open market sale of bonds reduces the overall liquidity of the economy and hence it reduces aggregate output.

We now turn to the case where τ is exponentially distributed. Asset spreads solve:

$$s_1 = \alpha \theta e^{-\lambda T_2} \left[\frac{u'[y(T_2)] - v'[y(T_2)]}{v'[y(T_2)]} \right]$$

$$i = \alpha \theta \lambda \frac{(\delta_0 - \delta_1)}{\delta_0} \int_{T_1}^{T_2} e^{-\lambda \tau} \left[\frac{u'[y(\tau)] - v'[y(\tau)]}{v'[y(\tau)]} \right] d\tau + \alpha \theta e^{-\lambda T_2} \left[\frac{u'[y(T_2)] - v'[y(T_2)]}{v'[y(T_2)]} \right],$$

where $T_1 = a_0/\delta_0$, $T_2 = a_0/\delta_0 + a_1/\delta_1$.

Proposition 18 *There is $\bar{i} > 0$ such that for all $i < \bar{i}$ there exists an equilibrium where fiat money and interest-bearing bonds coexist.*

⁹The main logic would go through if real bonds are replaced with nominal bonds that yield one unit of fiat money. The money and bond supplies grow at the same rate π and the ratio of the bond supply over the money supply is denoted μ_1 . The only equilibria where OMOs are effective are the ones such that $\bar{\tau} = T_2$. In that case $a_0(1/\delta_0 + \mu_1/\delta_1) = \bar{\tau}$. As μ_1 increases a_0 decreases. Output in DM matches is such that

$$z(y) = \left(\frac{1 + \mu_1}{\delta_1 + \delta_0 \mu_1} \right) \delta_0 \delta_1 \bar{\tau}.$$

Output increases with μ_1 .

Aggregate output is

$$Y = \alpha \left[\int_0^{T_2} \lambda e^{-\lambda x} y(x) dx + e^{-\lambda T_2} y(T_2) \right].$$

[ADD NUMERICAL EXAMPLE WHERE Y IS NON-MONOTONE WITH A_1]

7.2 Application 2: Multiple (crypto-)currencies

Our model can be applied to economies with multiple fiat monies. This application is especially topical given the development of multiple cryptocurrencies, such as Bitcoins, Litecoin, Ethereum, and others. A transaction with cryptocurrencies requires confirmation that takes time and confirmation times vary across currencies.¹⁰ Our negotiability parameter, δ , can be interpreted as proxy for these transaction times.

We now consider an economy with two currencies, currency 0 with money growth rate π_0 and currency 1 with money growth rate π_1 . Currency 0 has lower confirmation times and can be transferred faster than currency 1, i.e., $\delta_0 > \delta_1$. We focus on steady-state equilibria where the rate of return of each currency is constant. If $\pi_0 \leq \pi_1$ then agents will not want to hold the currency 1. Hence, we focus on the case where $\pi_0 > \pi_1$, i.e., currency 1 has a lower inflation rate than currency 0.

We start with the simple case where $\bar{\tau}$ is deterministic. For the two monies to coexist the equilibrium must feature $\bar{\tau} \geq T_2$ since otherwise one of the two currencies would have no utility as means of payment at the margin. If the inequality is strict, then the two currencies must have the same rate of return, which requires $\pi_0 = \pi_1$ in a steady-state equilibrium. Moreover, the exchange rate between the two currencies is indeterminate by a similar argument as in Kareken and Wallace (1981). In the following we focus on equilibria where $T_2 = \bar{\tau}$, i.e., $a_0/\delta_0 + a_1/\delta_1 = \bar{\tau}$ and $z(y) = a_0 + a_1$. The buyer's choice of consumption solves

$$\max_{y \geq 0} -(i_0 - i_1) \frac{\delta_0}{\delta_0 - \delta_1} [z(y) - \delta_1 \bar{\tau}] - i_1 z(y) + \alpha [u(y) - z(y)]. \quad (54)$$

The cost of holding $z(y)$ in the form of currency 1 is i_1 . By holding currency 0 the buyer incurs an additional cost equal to the differential of interest rates, $i_0 - i_1 > 0$, but it speeds up the negotiation by $\delta_0 - \delta_1$. The FOC gives

$$\frac{i_0 \delta_0 - i_1 \delta_1}{\delta_0 - \delta_1} = \alpha \theta \left(\frac{u'(y) - v'(y)}{v'(y)} \right). \quad (55)$$

There exists a unique y solution to (55). We define the exchange rate between the two currencies as $e_t \equiv \phi_{0,t}/\phi_{1,t}$ which is the price of currency 0 in terms of currency 1. It is easy to check that it is equal to

$$e_t = \frac{\delta_0 z(y) - \delta_1 \bar{\tau} A_{1,t}}{\delta_1 \delta_0 \bar{\tau} - z(y) A_{0,t}}. \quad (56)$$

¹⁰For instance, it takes on average 10 minutes with Bitcoins to receive a network confirmation. This transfer time is lowered to 2.5 minutes with Litecoins, 2 minutes for Monero, 14 seconds for Ethereum, and 3.5 second for Ripple.

The nominal exchange rate is the product of three terms: the ratio of the negotiability parameters; the ratio of the money supplies; and a middle term that captures the loss of purchasing power from using currency 1 only relative to the gain in purchasing power from using currency 0 only.

Proposition 19 (2-currency equilibrium.) *There is $0 < \bar{\tau}_0 < \bar{\tau}_1$ such that for all $\tau \in (\bar{\tau}_0, \bar{\tau}_1)$ there exists a unique steady-state equilibrium where both currencies 0 and 1 are valued. Inflation rates affect output according to $\partial y / \partial \pi_0 < 0$ and $\partial y / \partial \pi_1 > 0$. Moreover, currency 0 appreciates vis-a-vis currency 1 as α or θ increases.*

Proof. Using that $a_0 / \delta_0 + a_1 / \delta_1 = \bar{\tau}$ and $z(y) = a_0 + a_1$ we can solve for a_0 and a_1 :

$$\begin{aligned} a_1 &= \frac{\delta_1}{\delta_0 - \delta_1} [\delta_0 \bar{\tau} - z(y)] \\ a_0 &= \frac{\delta_0}{\delta_0 - \delta_1} [z(y) - \delta_1 \bar{\tau}]. \end{aligned}$$

The condition for the two currencies to be valued are

$$\delta_1 \bar{\tau} < z(y) < \delta_0 \bar{\tau}.$$

This condition can be rewritten as $\bar{\tau} \in (\bar{\tau}_0, \bar{\tau}_1)$ where $\bar{\tau}_0 = z(y) / \delta_0$, $\bar{\tau}_1 = z(y) / \delta_1$ and y solves (55). It is immediate from (55) that $\partial y / \partial i_0 < 0$ and $\partial y / \partial i_1 > 0$. Similarly, $\partial y / \partial \alpha > 0$ and $\partial y / \partial \theta > 0$ which from (56) gives $\partial e_t / \partial \alpha > 0$ and $\partial e_t / \partial \theta > 0$. ■

The logic is similar to the one of the OMO described earlier. As i_1 increases, agents find it optimal to reduce their holdings of currency 1 and raise their holdings of currency 0 which is more negotiable. As a result, they can buy more output over the time horizon $\bar{\tau}$.

We now turn to the case where $\bar{\tau}$ is exponentially distributed. From (50), and assuming an interior solution,

$$i_1 = \alpha \theta e^{-\lambda T_2} \left[\frac{u' [y(T_2)] - v' [y(T_2)]}{v' [y(T_2)]} \right] \quad (57)$$

$$i_0 - i_1 = \alpha \theta \lambda \frac{(\delta_0 - \delta_1)}{\delta_0} \int_{T_1}^{T_2} e^{-\lambda \tau} \left[\frac{u' [y(\tau)] - v' [y(\tau)]}{v' [y(\tau)]} \right] d\tau, \quad (58)$$

where $T_1 = a_0 / \delta_0$, $T_2 = a_0 / \delta_0 + a_1 / \delta_1$, $y(T_2) = z^{-1}(a_0 + a_1)$, and $y(\tau) = a_0 + \delta_1(\tau - a_0 / \delta_0)$.

Proposition 20 (Dual currency economy) *There exists a unique dual-currency steady-state equilibrium with $\delta_0 > \delta_1$ for all $i_1 \in (\underline{l}, i_0)$ where $\underline{l}(\delta_0, \delta_1) < i_0$. Moreover, $\partial \underline{l} / \partial \delta_0 < 0$ and $\partial \underline{l} / \partial \delta_1 > 0$.*

Proof. From the first equation we can expression $a_1 = g(a_0; i_1)$ where $g(0; i_1) = \bar{a}(i_1) > 0$, $g(a_0; i_1) = 0$ for some $a_0 < \bar{a}(i_1)$, and $g' < 0$ for all a_0 such $g(a_0; i_1) > 0$. We can then rewrite the second equation as:

$$i_0 - i_1 = \alpha \theta \lambda \frac{(\delta_0 - \delta_1)}{\delta_0} e^{-\lambda \frac{a_0}{\delta_0}} \int_0^{\frac{g(a_0; i_1)}{\delta_1}} e^{-\lambda x} \left[\frac{u' [z^{-1}(a_0 + \delta_1 x)] - v' [z^{-1}(a_0 + \delta_1 x)]}{v' [z^{-1}(a_0 + \delta_1 x)]} \right] dx.$$

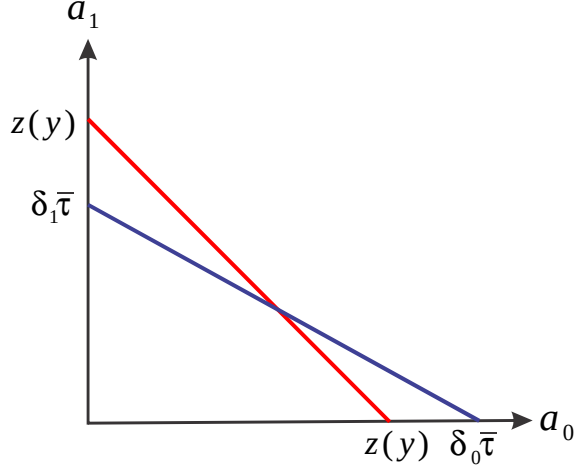


Figure 6: Equilibrium with 2 currencies

The right side is decreasing in a_0 . There is a positive solution provided that $i_1 > \underline{t}$ where $\underline{t}$ solves

$$i_0 - \underline{t} = \alpha \theta \lambda \frac{(\delta_0 - \delta_1)}{\delta_0} \int_0^{\frac{\bar{a}(\underline{t})}{\delta_1}} e^{-\lambda x} \left[\frac{u' [z^{-1}(\delta_1 x)] - v' [z^{-1}(\delta_1 x)]}{v' [z^{-1}(\delta_1 x)]} \right] dx.$$

It is easy to check that the right side is increasing in δ_0 and decreasing in δ_1 , hence $\partial \underline{t} / \partial \delta_0 < 0$ and $\partial \underline{t} / \partial \delta_1 > 0$. ■

Provided that the differential of inflation rates is not too large, there is coexistence of the two currencies. The differential consistent with a dual currency economy increases as currency 0 becomes more negotiable and decreases as currency 1 becomes more negotiable. Moreover, our model predicts that small trades are conducted with the most negotiable currency while larger trades are financed with both currencies.

Proposition 21 (Exchange rate) *In any dual-currency, steady-state equilibrium the nominal exchange rate is uniquely determined and can be expressed as $e_t = \bar{e}(\delta_0, \delta_1) A_{1,t} / A_{0,t}$ where $\bar{e}(\delta_0, \delta_1)$ increases with δ_0 but decreases with δ_1 .*

Proof. The exchange rate can be expressed as $e_t \equiv (a_0 / a_1) \times (A_{1,t} / A_{0,t})$. It can be checked from the proof of previous proposition that a_0 increases with δ_0 and decreases with δ_1 . Moreover, $a_1 = g(a_0; i_1)$ decreases with a_0 . ■

8 Conclusion

The objective of this paper was to introduce a new approach to bargaining into a model of decentralized asset market with unrestricted portfolios. Following O'Neill et al. (2004) we define the bargaining problem

between the owner of a portfolio of assets and a potential consumer as a collection of Pareto frontiers that expand with asset holdings. This definition captures the idea that the different items in a portfolio are sold sequentially, with each sale being final. In addition to standard axioms (Pareto efficiency, scale invariance, symmetry), the gradual bargaining solution is required to be continuous and time consistent. The solution that obeys these five axioms is characterized by a system of differential equations that can be solved in closed form. We show that the portfolio choice problem induced by this bargaining solution is concave, which makes the model tractable.

We showed that gradual bargaining has important positive and normative implications that distinguish it from other bargaining solutions. Relative to Nash, gradual bargaining is incentive-compatible when portfolios are private information. Moreover, gradual bargaining can implement socially-efficient outcomes while Nash cannot. Relative to Kalai bargaining, the gradual bargaining solution is not only scale invariant—as required by the axioms—but it is also ordinal. Thus, in contrast to all other solutions, the outcome is unaffected by any monotone transformation of utilities. On the positive side, gradual bargaining implements a wider range of liquidity premia, which matters for the existence of equilibria. For instance, under Kalai bargaining, monetary equilibria can break down for finite inflation rates even if the marginal utility of consumption approaches infinity when agents’ real balances approach zero. This is not the case with gradual bargaining: under the Inada conditions, a monetary equilibrium exists for all inflation rates.

We extended the gradual bargaining problem to the case of portfolios composed of different assets, to allow for time-consuming technologies to negotiate the sale of assets as well as time-varying bargaining powers. We showed that the trading mechanism used by Zhu and Wallace (2007) to explain rate-of-return dominance is a special case of a gradual bargaining solution. We generalized this mechanism to arbitrary bargaining powers and investigated the implications for policy.

We proposed an alternative explanation for the rate-of-return dominance puzzle based on the idea that the time it takes to negotiate a portfolio of assets is finite and units of money are negotiated faster than units of bonds, e.g., because they are easier to authenticate. This version of the model can generate a regime where bonds pay interest and open-market operations are effective. A sale of bonds raises their interest rate and decreases aggregate output. Moreover, an increase in the money growth rate can at the same time raise the nominal rate on illiquid bonds (a Fisher effect) and lower the nominal rate on government bonds (a liquidity effect).

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Appendix A

Proofs of Lemmas and Propositions

Proof of Lemma 1. The Pareto frontier is derived from the program

$$u^b = \max_{y, p \leq \delta\tau} \{u(y) - p + u_0^b\} \quad \text{s.t.} \quad p - v(y) + u_0^s \geq u^s.$$

The consumer chooses the terms of trade, (y, p) , to maximize his utility subject the constraint that he must guarantee some utility level u^s to the producer. If $\delta\tau \geq u^s - u_0^s + v(y^*)$, then $y = y^*$ and $p = u^s - u_0^s + v(y^*)$. Moreover, $u^b + u^s = u(y^*) - v(y^*) + u_0^b + u_0^s$. If $\delta\tau < u^s - u_0^s + v(y^*)$, then $p = \delta\tau = u^s - u_0^s + v(y)$, i.e., $y = v^{-1}(\delta\tau - u^s + u_0^s)$. ■

Proof of Proposition 1. By the definition of the consumer's utility, $u^b(\tau) = u_0^b + u[y(\tau)] - \delta\tau$, it follows that

$$u^{b'}(\tau) = u'(y) \frac{\partial y}{\partial \tau} - \delta. \quad (59)$$

The change in the consumer's utility along the gradual bargaining path is determined by the change in DM consumption as the consumer adds assets to the negotiation table. From (13) and (59), we obtain (15). The total transfer of assets is $z(y) = \int_0^y \delta \frac{\partial \tau}{\partial x} dx$ where from (15) $\partial \tau / \partial x$ coincides with $1/y'(\tau)$ evaluated at x . ■

Proof of Proposition 2.. The payment under Nash can be reexpressed as:

$$\begin{aligned} z^N(y) &= \frac{v'(y)}{u'(y) + v'(y)} \int_0^y u'(x) dx + \frac{u'(y)}{u'(y) + v'(y)} \int_0^y v'(x) dx \\ &= \int_0^y \frac{v'(y)u'(x) + u'(y)v'(x)}{u'(y) + v'(y)} dx \\ &= \int_0^y \frac{v'(y)}{u'(y) + v'(y)} [u'(x) - v'(x)] + v'(x) dx \end{aligned}$$

Using that

$$\frac{v'(x)}{u'(x) + v'(x)} < \frac{v'(y)}{u'(y) + v'(y)} \quad \forall x < y,$$

and $u'(x) - v'(x) > 0$ for all $x < y \leq y^*$, we have:

$$\begin{aligned} z^N(y) &> \int_0^y \frac{v'(x)}{u'(x) + v'(x)} [u'(x) - v'(x)] + v'(x) dx \\ &> \int_0^y \frac{2v'(x)u'(x)}{u'(x) + v'(x)} dx = z(y). \end{aligned}$$

■

Derivation of buyer surplus. The expected surplus is given by

$$U^b(\delta; z^b) = \int_0^\infty \lambda e^{-\lambda \bar{\tau}} \{u[y(\min\{\bar{\tau}, z^b/\delta\})] - z[y(\min\{\bar{\tau}, z^b/\delta\})]\} d\bar{\tau} \quad (60)$$

$$= \int_0^\infty \lambda e^{-\lambda \bar{\tau}} \int_0^{z^{-1}(\min\{\delta \bar{\tau}, z^b\})} \frac{u'(x) [u'(x) - v'(x)]}{u'(x) + v'(x)} dx d\bar{\tau} \quad (61)$$

$$= \int_0^\infty \int_0^{z^{-1}(\min\{\delta \bar{\tau}, z^b\})} \lambda e^{-\lambda \bar{\tau}} \frac{u'(x) [u'(x) - v'(x)]}{u'(x) + v'(x)} dx d\bar{\tau} \quad (62)$$

$$= \int_0^{z^{-1}(z^b)} \int_{z(x)/\delta}^\infty \lambda e^{-\lambda \bar{\tau}} \frac{u'(x) [u'(x) - v'(x)]}{u'(x) + v'(x)} d\bar{\tau} dx \quad (63)$$

$$= \int_0^y e^{-\frac{\lambda}{\delta} z(x)} \frac{u'(x) [u'(x) - v'(x)]}{u'(x) + v'(x)} dx, \quad (64)$$

where $y = z^{-1}(z^b)$. The derivation for social welfare follows exactly the same steps except for replacing $\frac{u'(x)[u'(x)-v'(x)]}{u'(x)+v'(x)}$ by $[u'(x) - v'(x)]$. ■

Proof of Proposition 4. Recall that equilibrium is determined by

$$s = \frac{\alpha}{2} \left[\frac{u'(y)}{v'(y)} - 1 \right] - \xi, \quad (65)$$

$$z(y) \leq \frac{(1+\rho)}{\rho-s} Ad, \quad (66)$$

$$z(y) \leq \delta \bar{\tau}, \quad (67)$$

where the second inequality is equality whenever $s > 0$, and $\xi \geq 0$ and $\xi[z(y) - \delta \bar{\tau}] = 0$.

(1) We verify that $s = 0$, $y = y^*$, and $\xi = 0$ constitute an equilibrium. Since $z(y^*) \leq \min\{\delta \bar{\tau}, \frac{(1+\rho)}{\rho} Ad\}$, both (66) and (67) are satisfied with $s = 0$; since $y = y^*$, (65) is satisfied with $\xi = 0$ and $s = 0$.

(2) We verify that $s = 0$, $y = z^{-1}(\delta \bar{\tau})$, and

$$\xi = \frac{\alpha}{2} \left[\frac{u'[z^{-1}(\delta \bar{\tau})]}{v'[z^{-1}(\delta \bar{\tau})]} - 1 \right] > 0 \quad (68)$$

constitute an equilibrium. (67) is satisfied by construction, and (66) is satisfied with $s = 0$ by the assumption that $\delta \bar{\tau} \leq \frac{(1+\rho)}{\rho} Ad$. Finally, $\xi > 0$ since $z^{-1}(\delta \bar{\tau}) < y^*$ as $\delta \bar{\tau} < z(y^*)$. Thus, (68) implies that (65) is satisfied with $s = 0$.

(3) Consider the equation (65) with $\xi = 0$, which defines a unique $s \geq 0$ for each $y \leq y^*$, denoted by $s^a(y)$. Moreover, $s^a(y)$ is strictly decreasing in y . Since the right-side of (66) is strictly increasing in s and since $z(y)$ is strictly increasing in s , the assumption that $\frac{(1+\rho)}{\rho} Ad < z(y^*)$ implies that there is a unique $y^a > 0$ such that

$$z(y^a) = \frac{(1+\rho)}{\rho - s^a(y^a)} Ad. \quad (69)$$

Moreover, for any $y \in [0, y^*]$, we have

$$z(y) > \frac{(1+\rho)}{\rho - s^a(y)} Ad \text{ iff } y > y^a. \quad (70)$$

We now verify that $y = y^a$, $s = s^a(y^a)$, and $\xi = 0$ constitute an equilibrium. We have already shown that (65) and (66) are satisfied. If $\delta\bar{\tau} > z(y^*)$, then (67) is satisfied. Otherwise, note that

$$s^a[z^{-1}(\delta\bar{\tau})] = \frac{\alpha}{2} \left[\frac{u'[z^{-1}(\delta\bar{\tau})]}{v'[z^{-1}(\delta\bar{\tau})]} - 1 \right] < \rho - \frac{(1+\rho)}{\delta\bar{\tau}} Ad$$

and hence

$$z[z^{-1}(\delta\bar{\tau})] = \delta\bar{\tau} > \frac{(1+\rho)}{\rho - s^a[z^{-1}(\delta\bar{\tau})]} Ad,$$

and by (70), $z^{-1}(\delta\bar{\tau}) > y^a$, that is, $z(y^a) < \delta\bar{\tau}$.

Then, the assumption that

$$\rho - \frac{(1+\rho)}{\delta\bar{\tau}} Ad = s^t > \frac{\alpha}{2} \left[\frac{u'[z^{-1}(\delta\bar{\tau})]}{v'[z^{-1}(\delta\bar{\tau})]} - 1 \right]$$

implies that $z^{-1}(\delta\bar{\tau}) > y^a(s^t)$

(4) We verify that $y = z^{-1}(\delta\bar{\tau})$, and

$$s = \rho - \frac{(1+\rho)}{\delta\bar{\tau}} Ad \text{ and} \tag{71}$$

$$\xi = \frac{\alpha}{2} \left[\frac{u'[z^{-1}(\delta\bar{\tau})]}{v'[z^{-1}(\delta\bar{\tau})]} - 1 \right] - s \tag{72}$$

constitute an equilibrium. Note that (65)-(67) are satisfied by construction. Moreover, the assumption that

$$\frac{\alpha}{2} \left[\frac{u'[z^{-1}(\delta\bar{\tau})]}{v'[z^{-1}(\delta\bar{\tau})]} - 1 \right] > \rho - \frac{(1+\rho)}{\delta\bar{\tau}} Ad > 0$$

implies that $\xi > 0$ and $s > 0$. ■

Proof of Proposition 5. For each $y \in (0, y^*]$, equation (28) gives a negative relationship between s

and y , denoted by $s = s^r(y)$, with $\lim_{y \rightarrow 0} s^r(y) = +\infty$ and $s^r(y^*) = 0$, and s^r is strictly decreasing. Given this function, equilibrium is given by y that satisfies (29). Since the left-side of (29) is strictly increasing in y and the right-side is strictly increasing in s and hence strictly decreasing in y with $s = s^r(y)$, and since the right-side of (29) is positive at $y = 0$, there is unique y that satisfies (29).

(1) Since $z(y^*) \leq (1+\rho)Ad/\rho$ and $s^r(y^*) = 0$, $y = y^*$ is the unique equilibrium. In this equilibrium, the time it takes to sell $z(y^*)$ units of wealth is $\tau^* = z(y^*)/\delta$ and the probability that $\bar{\tau} \geq \tau^*$ is $e^{-\frac{\lambda}{\delta}z(y^*)}$. From (30), social welfare is

$$\mathcal{W} = \alpha \int_0^{y^*} e^{-\frac{\lambda}{\delta}z(x)} [u'(x) - v'(x)] dx,$$

which is independent of Ad but decreasing with λ/δ .

(2) Since $z(y^*) > (1 + \rho)Ad/\rho$, the unique equilibrium features $y < y^*$ and $s > 0$. From (28) and (29) the spread is the unique $s \in (0, \rho)$ solution to

$$s = \frac{\alpha}{2} e^{-\frac{\lambda}{\delta} \left(\frac{1+\rho}{\rho-s} \right) Ad} \left[\frac{u' \circ z^{-1} \left[\left(\frac{1+\rho}{\rho-s} \right) Ad \right]}{v' \circ z^{-1} \left[\left(\frac{1+\rho}{\rho-s} \right) Ad \right]} - 1 \right].$$

The right side is decreasing in Ad and λ/δ . Hence, s decreases with Ad and λ but increases with δ . From (28) y is a decreasing function of s , hence y increases with Ad and, from (30), social welfare increases with Ad . Similarly, y decreases with λ/δ and hence $\mathcal{W}$ decreases with λ/δ . ■

Proof of Proposition 6. We only prove (ii). It is straightforward to check that when (31) holds, $s = 0$ and $y = y^*$ is a solution to (20) and (21). Suppose that (31) does not hold. The case where $\bar{\tau}$ is sufficiently large corresponds to region (3) in Proposition 4, and, as argued in the proof there, the unique equilibrium is determined by (69). Moreover, since the left-side of (69) is strictly increasing in y and the right-side is strictly decreasing, an increase in A will lead to an increase in equilibrium y , and hence, a decrease in equilibrium s . ■

Proof of Proposition 7. Recall the equilibrium condition

$$i \leq \frac{\alpha}{2} \left[\frac{u'(y)}{v'(y)} - 1 \right] \quad (73)$$

with equality whenever $z(y) \leq \delta\bar{\tau}$. By the Inada condition there exists a unique $y^m(i) \leq y^*$ that satisfies (73) with equality. Moreover, $y^m(0) = y^*$ and $y^m(i)$ is strictly decreasing in i .

(1) Suppose that $z(y^*) \leq \delta\bar{\tau}$. Then, equilibrium $y = y^m(i)$, which converges to y^* as i approaches 0.

(2) Suppose that $z(y^*) > \delta\bar{\tau}$. Then, there exists a unique $\bar{i} > 0$ such that $y^m(\bar{i}) = z^{-1}(\delta\bar{\tau})$. For all $i \leq \bar{i}$, equilibrium $y = z^{-1}(\delta\bar{\tau})$. For all $i > \bar{i}$, equilibrium $y = y^m(i)$. ■

Proof of Proposition 8. Following the same reasoning as before, (13) and (14) become

$$u^{b'}(\tau) = \theta \frac{u'(y) - v'(y)}{v'(y)} \quad (74)$$

$$u^{s'}(\tau) = (1 - \theta) \frac{u'(y) - v'(y)}{u'(y)}. \quad (75)$$

From (59), $u^{b'}(\tau) = u'(y) \partial y / \partial \tau - 1$, which gives (34). By integrating $\partial \tau / \partial y$ we can compute (35). Finally, using the same reasoning as before, we can derive the buyer surplus as

$$U^b(\delta; z^b) = \int_0^y e^{-\frac{\lambda}{\delta} z(x)} \frac{\theta u'(x) [u'(x) - v'(x)]}{\theta u'(x) + (1 - \theta) v'(x)} dx, \quad (76)$$

and hence the CM problem can be written as

$$\max_{y \geq 0} \left\{ -sz(y) + \alpha \int_0^y e^{-\frac{\lambda}{\delta} z(x)} \frac{\theta u'(x) [u'(x) - v'(x)]}{\theta u'(x) + (1 - \theta) v'(x)} dx \right\},$$

and one can obtain the FOC (36). ■

Proof of Propostion 9. For each $y \in (0, y^*]$, equation (36) gives a negative relationship between s and y , denoted by $s = s^r(y)$, with $\lim_{y \rightarrow 0} s^r(y) = +\infty$ and $s^r(y^*)$, and s^r is strictly decreasing. Given this function, equilibrium is given by y that satsifies (29). The rest of the argument is the same as in Proposition 5. ■

Proof of Lemma 2. The buyer's surplus is bounded above by $u(y^*) - v(y^*)$. Hence, it is never optimal to choose a δ larger than $\bar{\delta} = \psi^{-1}[u(y^*) - v(y^*)]$. Similarly, for all $s > 0$ it is not optimal to accumulate more than $z(y^*)$ units of assets. Hence, with no loss in generality, we restrict the maximization problem to the compact set, $[0, \bar{\delta}] \times [0, z(y^*)]$. The objective in (37) is continuous. By the Theorem of the Maximum, a solution exists and it is upper hemi-continuous in s . Denote:

$$\bar{S}(z^b) = \max_{\delta \in [0, \bar{\delta}]} \{-\psi(\delta) + S^b(z^b, \delta)\}$$

The objective function has strictly increasing differences in (z^b, δ) since $\partial S^b(z^b, \delta)/\partial z^b$ is strictly increasing in δ . By Theorem 2.8.4 in Topkis (1998) $\arg \max_{\delta \in [0, \bar{\delta}]} \{-\psi(\delta) + S^b(z^b, \delta)\}$ is increasing in $z^b < z(y^*)$. Moreover, it is strictly increasing because the FOC (38) cannot hold for two distinct $z^b < z(y^*)$ but the same δ . Consider now two spreads, $s^1 < s^2$, with associated choices of real balances z^1 and z^2 . It follows that:

$$\begin{aligned} -s^1 z^1 + \alpha \bar{S}(z^1) &\geq -s^1 z^2 + \alpha \bar{S}(z^2) \\ -s^2 z^2 + \alpha \bar{S}(z^2) &\geq -s^2 z^1 + \alpha \bar{S}(z^1). \end{aligned}$$

Rearrange these inequalities to obtain:

$$s^1 (z^1 - z^2) \leq \alpha [\bar{S}(z^1) - \bar{S}(z^2)] \leq s^2 (z^1 - z^2).$$

Using that $s^2 > s^1$ it follows that $z^1 \geq z^2$. The inequality is strict because the FOC would not hold for two distinct s but the same choice of z . The solution for z^b is generically unique by an argument analogous to the one in Gu and Wright (2015). (Alternatively, the correspondance for z^b is strictly decreasing and hence can have at most a countable number of jumps.) ■

Proof of Proposition 12. We verify that $s = 0$, $y = y^*$ and $n = n^*$ is an equilibrium. Note that (43) ensures that $s = 0$ and $y = y^*$ solve (??) and (??), regardless of n . Now, (44) ensures that n^* solves (42). ■

Proof of Proposition ??. The first-order conditions for Proposition ?? is given by

$$\begin{aligned} -(i + \alpha) z'_m(y_m) - \alpha \frac{\partial}{\partial y_m} z_b(y_m, y_b) + \alpha u'(y_m + y_b) &\leq 0, \text{ with equailty whenever } y_m > 0, \\ -(s + \alpha) \frac{\partial}{\partial y_b} z_b(y_m, y_b) + \alpha u'(y_m + y_b) &\leq 0, \text{ with equailty whenever } y_b > 0. \end{aligned}$$

Now, let $y = y_m + y_b$,

$$z'_m(y_m) = \frac{u'(y_m)v'(y_m)}{\theta(y_m)u'(y_m) + [1 - \theta(y_m)]v'(y_m)}, \quad (77)$$

$$\frac{\partial}{\partial y_m} z_b(y_m, y_b) = \frac{u'(y)v'(y)}{\theta(y)u'(y) + [1 - \theta(y)]v'(y)} - \frac{u'(y_m)v'(y_m)}{\theta(y_m)u'(y_m) + [1 - \theta(y_m)]v'(y_m)}, \quad (78)$$

$$\frac{\partial}{\partial y_b} z_b(y_m, y_b) = \frac{u'(y)v'(y)}{\theta(y)u'(y) + [1 - \theta(y)]v'(y)}. \quad (79)$$

Thus, $\frac{\partial}{\partial y_b} z_b(y_m, y_b) = \frac{\partial}{\partial y_m} z_b(y_m, y_b) + z'_m(y_m)$, and hence, the only way to satisfy the FOC's above is to have $s = i$. ■

Proof of Proposition ??. The FOC's to the portfolio choice problem (??) is given by

$$\begin{aligned} -iv'(y_m) - s[u'(y_m + y_b) - u'(y_m)] + \alpha[u'(y_m) - v'(y_m)] &= 0, \\ -su'(y_m + y_b) &\leq 0. \end{aligned}$$

The only possible equilibrium is to have $s = 0$. Note that at $s = 0$ there is multiplicity: since all agents are indifferent between holding the asset across the periods or not, any distribution of asset holding is an equilibrium. ■

Proof of Proposition ??. Assuming that $y_m + y_b > 0$, the FOC's for (??) are given by

$$\begin{aligned} -(i + \alpha) \frac{u'(y_m)v'(y_m)}{\theta_m u'(y_m) + (1 - \theta_m)v'(y_m)} + (s + \alpha) \frac{u'(y_m)v'(y_m)}{\theta_b u'(y_m) + (1 - \theta_b)v'(y_m)} \\ - (s + \alpha) \frac{u'(y_m + y_b)v'(y_m + y_b)}{\theta_b u'(y_m + y_b) + (1 - \theta_b)v'(y_m + y_b)} + \alpha u'(y_m + y_b) \leq 0, \end{aligned}$$

with equality whenever $y_m > 0$, and

$$-(s + \alpha) \frac{u'(y_m + y_b)v'(y_m + y_b)}{\theta_b u'(y_m + y_b) + (1 - \theta_b)v'(y_m + y_b)} + \alpha u'(y_m + y_b) = 0.$$

Thus, assuming that $y_m + y_b > 0$, we have the following necessary conditions for equilibrium:

$$[(s + \alpha)\theta_m - (i + \alpha)\theta_b][u'(y_m) - v'(y_m)] \leq (i - s)v'(y_m) \text{ (with equality if } y_m > 0), \quad (80)$$

$$\alpha\theta_b[u'(y_m + y_b) - v'(y_m + y_b)] = sv'(y_m + y_b). \quad (81)$$

For any $s \in [0, i]$, there is a unique y_m that satisfies (80), denoted by $y_m(s)$. There are two cases. When $\alpha\theta_m - (i + \alpha)\theta_b > 0$, $y_m(s) > 0$ for all $s \in [0, i]$ and is strictly increasing. When $\alpha\theta_m - (i + \alpha)\theta_b \leq 0$, $y_m(s) = 0$ for all $s \leq \frac{(i + \alpha)\theta_b - \alpha\theta_m}{\theta_m} < i$, and for all $s > \frac{(i + \alpha)\theta_b - \alpha\theta_m}{\theta_m}$, $y_m(s) > 0$ and is strictly increasing up to $s = i$. For any $s \in [0, i]$, there is a unique $y = y_m + y_b$ that satisfies (81), denoted by $y(s)$, which is strictly decreasing in s . Moreover, note that

$$y_m(0) < y^* = y(0), \quad y_m(i) = y^* > y(i).$$

We consider two cases.

(a) Suppose that $i < \frac{\alpha(\theta_m - \theta_b)}{\theta_b}$. Then, $y_m(0) > 0$. If

$$A_b \geq \int_{y_m(0)}^{y^*} \frac{u'(x)v'(x)}{\theta_b u'(x) + (1 - \theta_b)v'(x)} dx. \quad (82)$$

the unique equilibrium is given by $s = 0$. Note that $y_m(0)$ is implicitly a function of i , and that it is strictly decreasing in i with value y^* when i approaches 0 and approaches 0 as i approaches $\frac{\alpha(\theta_m - \theta_b)}{\theta_b}$. Let $\bar{i} \leq \frac{\alpha(\theta_m - \theta_b)}{\theta_b}$ be the highest i for which (82) holds. Then, for all $i \leq \bar{i}$, we have $s = 0$; otherwise, because $y_m(i) = y^* > y(i)$, there exists a unique $s \in (0, i)$ such that (??) holds with equality. The unique equilibrium is then given by such s . In both cases the equilibrium is monetary.

(b) Suppose that $i \geq \frac{\alpha(\theta_m - \theta_b)}{\theta_b}$. Then, $y_m(s) = 0$ for all $s \leq \tilde{s} = \frac{(i + \alpha)\theta_b - \alpha\theta_m}{\theta_m}$. First suppose that $\hat{s} > 0$. Now, $i < \frac{\alpha(\theta_m - \theta_b) + \hat{s}\theta_m}{\theta_b}$ implies that

$$\tilde{s}\theta_m = (i + \alpha)\theta_b - \alpha\theta_m < \hat{s}\theta_m.$$

Hence, by definition of $\hat{s}$ and the fact that $\hat{s} > \tilde{s}$,

$$A_b < \int_0^{y(\tilde{s})} \frac{u'(x)v'(x)}{\theta_b u'(x) + (1 - \theta_b)v'(x)} dx.$$

Thus, there exists some $s \in (\tilde{s}, \hat{s})$ such that (??) holds with equality. Note that in this case $s > 0$. In contrast, if $i \geq \frac{\alpha(\theta_m - \theta_b) + \hat{s}\theta_m}{\theta_b}$, then $\hat{s} \leq \tilde{s}$, and hence $y_m(\hat{s}) = 0$ and (??) holds with $s = \hat{s}$. There is only a nonmonetary equilibrium. ■

Proof of Lemma 13. Let portfolio (a_m, a_b) be given. For any strategy F and the corresponding τ , the final payoff for the consumer is given by

$$u[y(T)] - z[y(T)] = u[z^{-1}(\tau(T))] - \tau(T).$$

Note that this payoff depends on $\tau(T)$ only, and, since the payoff is increasing in τ , the consumer's optimal strategy is to maximize $\tau(T)$. It is straightforward to check that F^* maximizes $\tau(T)$. ■

Appendix B

Here we show how to solve the gradual bargaining problem when the bargaining power, denoted by θ , can be dependent on the current unit of asset being bargained. This gives rise to the following differential equations for the solution:

$$\begin{aligned} u^b(\tau) &= -\theta(\tau) \frac{\partial H(u^b, u^s, \tau)/\partial \tau}{\partial H(u^b, u^s, \tau)/\partial u^b}, \\ u^s(\tau) &= -[1 - \theta(\tau)] \frac{\partial H(u^b, u^s, \tau)/\partial \tau}{\partial H(u^b, u^s, \tau)/\partial u^s}, \end{aligned}$$

where $\theta(\tau)$ is interpreted as the bargaining power for consumer for the τ th unit of real balances. Following the same reasoning as before, for all $y \leq y^*$,

$$\begin{aligned} u^{b'}(\tau) &= \theta(\tau) \frac{u'(y) - v'(y)}{v'(y)}, \\ u^{s'}(\tau) &= [1 - \theta(\tau)] \frac{u'(y) - v'(y)}{u'(y)}, \end{aligned}$$

and we have $u^{b'}(\tau) = 0 = u^{s'}(\tau)$ if $y > y^*$. Again, for any candidate solution $(u^b(\tau), u^s(\tau))$, we have a corresponding solution for trades, $(y(\tau), \tau)$, that satisfies $u^{b'}(\tau) = u'(y)y'(\tau) - 1$. Hence, we can rewrite the above differential equations in terms of $y(\tau)$:

$$y'(\tau) = \frac{\theta(\tau)u'(y) + [1 - \theta(\tau)]v'(y)}{u'(y)v'(y)} \text{ for all } y \leq y^*, \quad (83)$$

with boundary conditions $y(0) = 0$ and $y'(\tau) = 0$ for all $y > y^*$. We assume that $\theta(\tau)$ is continuous over $[0, u^{-1}(y^*)]$, except for possibly a finite set of points, and require the solution to be differentiable (and to satisfy (83)) except for those points. This also implies that any solution $y(\tau)$ is continuous. To obtain an analytical solution, first we show that we can transform the above equation into another. Specifically, we consider the following equation:

$$y'(\tau) = \frac{\theta(y)u'(y) + [1 - \theta(y)]v'(y)}{u'(y)v'(y)}. \quad (84)$$

The following lemma shows that to solve (83), it is without loss of generality to solve (84).

Lemma 4 (i) Suppose that $y(\tau)$ solves (83). Then, there exists a function $\hat{\theta}(y)$ such that $y(\tau)$ solves (84) w.r.t. $\hat{\theta}(y)$.

(ii) Suppose that $y(\tau)$ solve (84). Then, there exists a function $\hat{\theta}(\tau)$ such that $y(\tau)$ solves (83) w.r.t. $\hat{\theta}(\tau)$.

Proof. (i) Let $y(\tau)$ solve (83). It then follows that $y(\tau)$ is continuous and strictly increasing until $y(\tau)$ reaches y^* , and let τ^* be such that $y(\tau^*) = y^*$. Let $\psi(y)$ be its inverse over $[0, y^*]$, i.e., $y[\psi(y)] = y$ for all $y \in [0, y^*]$. Now, for all $y \in [0, y^*]$, define $\hat{\theta}(y) = \theta[\psi(y)]$. It is then straightforward to verify that $y(\tau)$ solves (84) w.r.t. $\hat{\theta}(y)$.

(ii) Conversely, let

$$z(y) = \int_0^y \frac{u'(x)v'(x)}{\theta(x)u'(x) + [1 - \theta(x)]v'(x)} dx \text{ for all } y \leq y^*. \quad (85)$$

Then, $y(\tau) = z^{-1}(\tau)$ for all $\tau \leq z(y^*)$ and $y(\tau) = y^*$ for all $\tau > z(y^*)$ is a solution to (84). Now, let $\widehat{\theta}(\tau) \equiv \theta[y(\tau)]$ for all $\tau \leq z(y^*)$ and let $\widehat{\theta}(\tau) \equiv \theta[y^*]$ for all $\tau > z(y^*)$. Then, for all $\tau \leq z(y^*)$,

$$\begin{aligned} y'(\tau) &= \frac{1}{z'[y(\tau)]} = \frac{\theta[y(\tau)]u'[y(\tau)] + \{1 - \theta[y(\tau)]\}v'[y(\tau)]}{u'[y(\tau)]v'[y(\tau)]} \\ &= \frac{\widehat{\theta}(\tau)u'[y(\tau)] + [1 - \widehat{\theta}(\tau)]v'[y(\tau)]}{u'[y(\tau)]v'[y(\tau)]}. \end{aligned}$$

Hence, $y(\tau)$ solves (83) w.r.t. $\widehat{\theta}(\tau)$. ■

Now we solve the ODE (84) explicitly. Indeed, let

$$z(y) = \int_0^y \frac{u'(x)v'(x)}{\theta(x)u'(x) + [1 - \theta(x)]v'(x)} dx \text{ for all } y \leq y^*. \quad (86)$$

Then, $y(\tau) = z^{-1}(\tau)$ for all $\tau \leq z(y^*)$ and $y(\tau) = y^*$ for all $\tau > z(y^*)$ is a solution to (84). The above lemma then shows that a solution to (83) takes the form of (86) in general.

We illustrate the above lemma with one example that will be useful for our later purpose. Consider $\theta(\tau)$ given by a sequence $0 = \tau_0 < \tau_1 < \dots < \tau_n < \tau_{n+1} = \infty$ such that

$$\theta(\tau) = \theta_i \text{ for all } \tau \in [\tau_{i-1}, \tau_i), \quad i = 1, \dots, n+1. \quad (87)$$

The following lemma provides a full characterization for such functions. Then, there exists $k \leq n+1$ and $y_1 < \dots < y_k = y^*$ such that $y(\tau) = z^{-1}(\tau)$ for all $\tau \leq z(y^*)$ and $y(\tau) = y^*$ for all $\tau > z(y^*)$, and $z(y)$ is such that, for all $i = 1, \dots, k$ ($y_0 = 0$),

$$z(y) = z(y_{i-1}) + \int_{y_{i-1}}^y \frac{u'(x)v'(x)}{\theta_i u'(x) + (1 - \theta_i)v'(x)} dx \text{ for all } y \in [y_{i-1}, y_i). \quad (88)$$

That is, $\widehat{\theta}(y)$ takes the same form as $\theta(\tau)$. To see this, define the thresholds $y_1 < \dots < y_k$ inductively as follows. Let y_1 be the largest $y \leq y^*$ such that

$$\tau_1 \geq \int_0^{y_1} \frac{u'(x)v'(x)}{\theta_1 u'(x) + (1 - \theta_1)v'(x)} dx.$$

If $y_1 \geq y^*$, then set $k = 1$. Otherwise, continue. Suppose that we have defined y_i and have not stopped.

Then, define y_{i+1} as the largest $y \leq y^*$ such that

$$\tau_{i+1} - \tau_i \geq \int_{y_i}^{y_{i+1}} \frac{u'(x)v'(x)}{\theta_{i+1} u'(x) + (1 - \theta_{i+1})v'(x)} dx.$$

If $y_{i+1} \geq y^*$, then set $k = i+1$. Otherwise, continue. Since

$$\frac{u'(x)v'(x)}{\theta_i u'(x) + (1 - \theta_i)v'(x)} > 0 \text{ for all } x \leq y^* \text{ and for all } i,$$

both k and $y_1 < \dots < y_k$ is well-defined. Note that $\tau_{n+1} = \infty$ and hence $k \leq n$. Given such $y_1 < \dots < y_k$, define $z(y)$ as in (88). This implies that for all $i = 1, \dots, k$, $z(y_i) = \tau_i$. Finally, we show that $y(\tau)$ defined by

$y(\tau) = z^{-1}(\tau)$ for all $\tau \leq z(y^*)$ and $y(\tau) = y^*$ for all $\tau > z(y^*)$, with $z(y)$ given by (88) satisfies (83). Let $\tau \in (\tau_{i-1}, \tau_i)$. If $i - 1 \geq k$, then $y(\tau) = y^*$ and (83) is satisfied. So suppose that $i < k + 1$. Then, by (88),

$$z[y(\tau)] = z(y_{i-1}) + \int_{y_{i-1}}^{y(\tau)} \frac{u'(x)v'(x)}{\theta_{i-1}u'(x) + (1 - \theta_{i-1})v'(x)} dx,$$

and hence

$$y'(\tau) = \frac{1}{z'[y(\tau)]} = \frac{u'[y(\tau)]v'[y(\tau)]}{\theta_i u'[y(\tau)] + (1 - \theta_i)v'[y(\tau)]}.$$

The above lemma shows that when $\theta(\tau)$ is given by (87), the solution to (83) is also a solution to (84) with $\theta(y) = \theta_i$ for all $y \in [y_{i-1}, y_i]$, where the thresholds are given by the lemma. Moreover, in both cases $z(y)$ given by (86) is valid and it represents the real balances transferred from the consumer in exchange for y units of DM good from the producer.