Misallocation and intersectoral linkages*

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Abstract

We study the effects of distortions in the use of primary and intermediate inputs on aggregate productivity. We show analytically how the aggregate productivity loss from distortions depends on the input-output structure of the economy and the degree of complementarity between inputs. We find that the production network does not systematically amplify the aggregate productivity loss, and that higher complementarity between inputs reduce the effects of distortions. Then, we apply our framework to quantify the effects of distortions caused by market power. Calibrated on Mexican industry-level data, the model suggests that reducing industry-level markups in Mexico to the US levels could raise aggregate TFP by as much as 15 percent. The TFP gain is as much as 5 times larger than without input-output linkages.

Keywords: Input-output, Misallocation, CES production function, imperfect competition.

JEL Classifications: D57, D61, O41, O47

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1 Introduction

The differences in aggregate total factor productivity (TFP) across countries, which are key to understanding the cross-country variation in income per capita, are not only due to differences in technology. As documented in the growing literature stemming from Hsieh and Klenow (2009), differences in the efficiency of the allocation of inputs across firms also play an important role. In this paper, we study how distortions in the use of inputs propagate along the production chain. For example, distortions in the use of inputs in the agricultural sector will lead to changes in the input choices of the processed food industry, which in turn will affect the choices made by the wholesale and retail distributors, and eventually will lead to changes in the price of food items in stores. The channels through which allocative distortions reduce aggregate TFP depend on the nature of the intersectoral linkages of production. The interconnections across sectors – for example, how intensively does the processed food industry use agricultural products? – but also the substitutability of inputs in the production process – can agricultural products be easily substituted for other inputs? – appear key to understanding the effects of allocative distortions.

We study the effects of distortions and the role of the production structure using a static multi-sector model with intermediate inputs linkages which builds on Jones (2011) and Jones (2013). The model is very similar to Long and Plosser (1983) but allows for more flexible production functions. Each sector combines primary inputs with intermediate inputs using a nested CES production function, with possibly non-unitary elasticity of substitution between the primary input and intermediates, and between the intermediate inputs. We adopt a rich input-output structure in which each sector may use the various inputs in different proportions. We introduce distortions that affect the use of primary and intermediate inputs in a reduced-form manner. The distortions, which could result from contract enforcement frictions, distortionary regulation or imperfect competition, are modeled as a wedge that leads firms to deviate from the frictionless allocation of inputs following Restuccia and Rogerson (2008) and Hsieh and Klenow (2009). We study, in this framework, how the intersectoral linkages propagate the distortions throughout the economy and characterize the input-output structures that amplify the effects of distortions. The model can be used to assess the aggregate implications of any of the frictions that distort the firms' primary and intermediate inputs decisions.

In the applied section of the paper, we focus on the distortions coming from imperfect competition and apply the model to quantify the consequences of market power on aggregate productivity. We show that intersectoral linkages amplify the TFP loss caused by market power, which gives rise to an additional channel through which markups reduce welfare.

We are able to characterize analytically the effect of distortions on aggregate productivity in several cases. We show theoretically that when distortions are random, input-output linkages do not systematically amplify the effects of distortions. When the random distortions affect only primary inputs, input-output linkages can amplify or dampen the TFP loss, depending on the value of the elasticity of substitution between inputs. We show, for a simple input-output economy, that the TFP loss increases with the share of intermediate inputs when inputs are substitutes, or when the distortions affect intermediate inputs. When intermediate inputs decisions are distorted, input-output linkages amplify the effect of distortions, whatever the value of the elasticity of substitution between inputs.

Our second theoretical result concerns the degree of complementarity between inputs. We show that higher complementarity between inputs mitigates the effects of distortions on the efficiency of resource allocation. This result may seem counter-intuitive: with less flexibility to substitute from distorted to less-distorted inputs, the firms' output should decline more, and one could expect distortions to have a larger impact on aggregate productivity. On the contrary we find that, with higher complementarity, distortions have a smaller impact on aggregate productivity. While higher complementarity amplifies the effect of a change in input on the firms' output, it also reduces the effect of the distortions on the firms' input decision. Firms respond less to the change in the effective relative prices caused by distortions when the elasticity of substitution is low. The second effect dominates, and higher complementarity leads to a smaller aggregate productivity loss. This result is in stark contrast with the consequences of productivity shocks. We show, in line with previous work in the literature, that a negative productivity shock has more severe consequences on aggregate productivity when inputs are complements. This result suggests that, with input complementarity, market inefficiencies that raise the real resource cost of production have a larger effect than inefficiencies that distort relative prices.

The theoretical analysis also allows us to isolate the sectoral characteristics that lead to a larger impact of the distortion on aggregate productivity. We find that the impact of the distortions depends on the importance of the input that is distorted (when not all inputs are) as well as on the importance of the sector in the economy. In our simple input-output economy, the importance of the sector depends on the importance of the sector as a supplier of final goods and as a supplier of intermediate inputs. Hence, when the distortions affect intermediate inputs, both the importance of the sector as an input purchaser and as an input supplier matter.

We also show how distortions affect gross output and consumption of sectors upstream and downstream from the distorted sector. We find that for any elasticity of substitution, final consumption of upstream sectors increases, a result in contrast with the case for productivity shocks analyzed by Acemoglu et al. (2016). Also, since the prices of intermediate goods are higher downstream of the distortion, low elasticity of substitution implies that the intermediate shares for sectors downstream of the distortion will be *higher*.

We then use the model to quantify the consequences of the distortions caused by imperfect competition. We first estimate industry-level markups across countries from the WIOD dataset and find that markups tend to be higher in developing countries. To which extent do input-output linkages amplify the negative impact of markups on aggregate productivity? To give some insight on this question, we quantify the effects of markups in Mexico where the median price-cost margin is about twice the US level. We calibrate the model on Mexican industry-level data, accounting for the presence of markups when estimating the parameters of the production function, and then assess the TFP gain from lowering the markups to the US level. We find that the TFP gain could be as high as 15%, and even higher when accounting for the degree of substitution in final goods. The TFP gain is between 1.5 and 5 times higher than without input-output linkages. The quantitative analysis confirms our theoretical on the role of the elasticity of substitution. We find that the TFP gain is lower when the degree of complementarity is higher.

Our paper contributes to the literature on the consequences of allocative distortions on aggregate TFP. In this literature, the focus is on the allocation of labor and capital, and intermediate inputs are usually not explicitly modeled (see for example Restuccia and Rogerson (2008), Hsieh and Klenow (2009) or Osotimehin (2016))[1]. In addition to constituting an additional source of misallocation, intermediate inputs create linkages between firms, which modify the mechanism through which distortions reduce aggregate TFP. Our paper is closely related to Jones (2013) and several related papers such as Bartelme and Gorodnichenko (2015), Fadinger et al. (2015) and Bigio and La'o (2016) that also investigate the effects of distortions in the presence of intersectoral linkages.[2] We complement these papers by characterizing analytically the effects of the input-output structure and the role of input complementarity.[3]

[1]For a more extensive description of the literature on misallocation, see the surveys by Hopenhayn (2014) or Restuccia and Rogerson (2017).

[2]Bartelme and Gorodnichenko (2015) estimate industry-level distortions and compute the implications for aggregate TFP while Fadinger et al. (2015) investigates how the productivity and tax rates across sectors interact with the input-output structure of the economy ; Bigio and La'o (2016) study how financial constraints are amplified by input-output linkages.

[3]Other recent papers that emphasize non-unitary elasticity of substitution between inputs include Atalay (2017), Carvalho et al. (2015), Miranda-Pinto (2018), Miranda-Pinto and Young (2018), and Baqaee and Farhi (2017*b*); all these papers focus on the propagation of productivity shocks.

In contrast with Jones (2011), we show that complementarity actually *reduces* the effect of distortions on aggregate TFP. Jones (2011) considers the effect of higher dispersion in productivity and in distortions jointly. We find that the degree of complementarity has opposite implications for productivity and distortion, and analyzing them jointly may mask the mitigating effect of complementarity on distortions. Recent papers by Caliendo et al. (2017) and Baqaee and Farhi (2017*a*) study the effects of distortions in the presence of input complementarity, but with a different focus than ours. Caliendo et al. (2017) estimate the effects of domestic and external distortions on the world GDP, while Baqaee and Farhi (2017*a*) focus on decomposing aggregate productivity changes in the presence of inefficiencies. By contrast, our objective is to study analytically how distortions propagate through the production network and to quantify the effect of the distortions caused by markups on aggregate productivity.[4]

We also contribute to the literature on the macroeconomic implications of market power. Basu (1995) highlights how markups can reduce aggregate productivity in the presence of intermediate inputs, and discusses the implications for the cyclicality of aggregate productivity. More recently, several papers have studied the role of market power on macroeconomic outcomes (Epifani and Gancia, 2011; Edmond et al., 2015; Peters, 2014; de Locker and Eeckout, 2017), but none of these models include input-output linkages. In these models, the average level of markups has no static effect on TFP and only the dispersion of markups matters. In contrast, in our framework the level of markups also matters for aggregate TFP.

2 Model

The production structure of the model is a generalization of Long and Plosser (1983). There are n sectors in the economy, with a representative firm in each sector. The model features intersectoral linkages that arise from firms purchasing and selling intermediate goods to each other. In each sector, the representative firm produces goods using labor and intermediate goods with the production function[5]

$$Q_i = A_i \left[(1-\alpha_i)^{1-\sigma}(B_i L_i)^{\sigma} + \alpha_i^{1-\sigma} X_i^{\sigma}\right]^{\frac{1}{\sigma}}, \tag{1}$$

[4] Baqaee and Farhi (2017*a*) also focus on markups, but rely on a first-order approximation to illustrate the potential TFP gains from eliminating markups. We instead undertake a full quantitative evaluation using the numerical solution of the model.

[5] Note that the effects of distortions on intermediate inputs we emphasize in the paper would similarly affect the firms' capital decision. To simplify the analysis we choose to abstract from this dimension and focus here on intermediate inputs only.

where B_i is the labor-augmenting productivity component and A_i is the Hicks-neutral productivity component, L_i is the labor input and X_i is the intermediate input bundle used in sector i which is given by

$$X_i = \left(\sum_j v_{ij}^{1-\rho} X_{ij}^{\rho} \right)^{\frac{1}{\rho}}, \tag{2}$$

and X_{ij} is the quantity of intermediate goods from sector j used by sector i. Sectors are allowed to differ in both their overall use of intermediate inputs and their mix of intermediate inputs. We impose $\alpha_i \in [0,1]$, $v_{ij} \in [0,1]$ and $\sum_{j=1}^{n} v_{ij} = 1$ for all $i = 1, ..., n$. The structure of intersectoral linkages in the model is governed by these two sets of parameters, collected in the vector $\alpha = (\alpha_1, ..., \alpha_n)'$ and in the matrix V with element (i,j) given by the intermediate input intensities v_{ij}. The other key parameters of our analysis are the degree of substitution between inputs. The parameter ρ is related to the elasticity of substitution between different intermediate goods and σ governs the elasticity of substitution between the composite intermediate good and labor. The corresponding elasticities can be computed as $\varepsilon_\rho \equiv 1/(1-\rho)$ and $\varepsilon_\sigma \equiv 1/(1-\sigma)$.[6] Note also that in our specification, we have $\alpha_i^{1-\sigma}$ and $(1-\alpha_i)^{1-\sigma}$, rather than α_i and $1-\alpha_i$ and similarly $v_{ij}^{1-\rho}$ instead of v_{ij}. This specification implies that measuring inputs at different levels of aggregation has no artificial effect on productivity.

Many different combinations of A_i, B_i, α_i, and v_{ij} represent the same production function. We will exploit this multiplicity later to derive a useful normalization.

The final good sector aggregates the inputs of the different sectors:

$$Y = \prod_{i=1}^{n} \beta_i^{-\beta_i} \prod_{i=1}^{n} C_i^{\beta_i}, \tag{3}$$

with $\beta_i \in [0,1]$, $\sum_{i=1}^{n} \beta_i = 1$. The first term is a convenient normalization that simplifies some of the expressions without affecting the results. This production function can be easily generalized to the constant elasticity of substitution class.

The production possibilities of the economy are determined by equations (1)- (3) and by the $n+1$ resource constraints:

$$\sum_{i=1}^{n} X_{ij} + C_j = Q_j, \forall j = 1, ..., n \tag{4}$$

[6] If the elasticity of either of the two nested CES functions is one, we get the Cobb-Douglas case with $Q_i = A_i(1-\alpha_i)^{-(1-\alpha_i)}(\alpha_i)^{-\alpha_i}(B_i L_i)^{1-\alpha_i} X_i^{\alpha_i}$ or $X_i = \prod_{j=1}^{n} v_{ij}^{-v_{ij}} X_{ij}^{v_{ij}}$.

$$\sum_{i=1}^{n} L_i = \bar{L}. \tag{5}$$

Equation (4) shows that the gross output of every sector must be equal to the sum of all its uses as intermediates (including by itself) and of its use in the production of the final good. Equation (5) is the market-clearing condition for labor.

The inputs choices may be distorted by market frictions. Following Restuccia and Rogerson (2008) and Hsieh and Klenow (2009), we model the distortions as wedges in the first-order conditions, which make inputs to deviate from the frictionless allocation. We define the distortions as deviations from the social planner solution. We denote the Lagrange multiplier associated with the resource constraint of product i by λ_i and that of labor by η. The allocation of inputs is characterized by the following distorted first-order conditions[7]:

$$\begin{aligned}
\beta_i \frac{Y}{C_i} &= \lambda_i &(6)\\
\lambda_i A_i^{\sigma} \left(\frac{\alpha_i Q_i}{X_i}\right)^{1-\sigma} \left(\frac{\upsilon_{ij} X_i}{X_{ij}}\right)^{1-\rho} &= \lambda_j (1+\tau_{ij}) &(7)\\
\lambda_i A_i^{\sigma} B_i^{\sigma} \left(\frac{(1-\alpha_i) Q_i}{L_i}\right)^{1-\sigma} &= \eta (1+\tau_{iL})(1+\tau_L) &(8)
\end{aligned}$$

Production efficiency is achieved whenever $\tau_{ij} = \tau_{iL} = 0$ for all $i, j = 1, ..., n$. The distortions τ_{ij}, τ_{iL} and τ_L modify the perceived marginal cost of inputs and hence distort the allocation of inputs across sectors. This framework is sufficiently flexible that any allocation that satisfies the production and resource constraints of our environment can be rationalized by some collection of distortions.

These distortions encompass various sources of frictions. We will focus in the second part of the paper on distortions caused by market power, in which case the distortions reflects the markup rate and depends only on the purchasing industry, that is $\tau_{ij} = \tau_{iL} = \tau_i \forall j =)1, ..., n$. The distortion $\tau_{ij} > 0$ could also be the result of contract enforcement frictions, expropriation or corruption, while $\tau_{ij} < 0$ corresponds to preferential treatment. Note that we allow distortions on intermediate inputs to depend both on the purchasing industry i and the supplying industry j. For example, distortions due to contract enforcement may vary across sectors supplying intermediates depending on how contract-intensive the goods or services are.

[7] We show in Appendix A.1 that the absence of distortion in equation (6) is without loss of generality: for any allocation characterized by distortions in equation (6), we can find a transformation of the distortions and multipliers that allows us to rewrite the distorted first-order conditions in the form of equations (6) to (8). We also show that there exists a unique set of distortions (up to the mean of the labor distortions) that can rationalize a feasible allocation.

2.1 Solution of the model

Our approach is to solve the constrained planner's problem. Aggregate output is the maximum output that can be obtained given the resource constraints and the distortions.

The Constant Elasticity of Substitution production function does not satisfy the Inada conditions, which implies that an equilibrium does not exist for all parameter values, even in an economy without frictions. In particular, if labor and intermediates are complements, output can be zero (since the marginal product of intermediates converges to a finite value as intermediates tend to zero); if labor and intermediates are substitutes, output can be infinite (since the marginal product of intermediates converges to a strictly positive value as intermediates tend to infinity).[8] We will not concern ourselves with the issue of equilibrium existence in general, but we will provide conditions for existence in a simple special case. Moreover, as we shall see, in an economy without frictions a sufficient condition for existence will be that $\max\{|A_i - 1|\}$ be not too large.

The distorted first-order conditions (equations (6) to (8)) and the resource constraints (equations (4) and (5)) pin down the economy's allocation. The following proposition gives the aggregate production function and aggregate productivity of the distorted economy.

Proposition 1 *Suppose that a solution to the constrained planner's problem exists. Then final output is given by* $Y = TFP\ \bar{L}$, *aggregate productivity is* $TFP = [d'(I - M')^{-1}c]^{-1}$, *the vector of gross output is given by* $Q = (I - M')^{-1}cY$, *final consumption of good* i *is* $C_i = c_i Y$, *labor inputs are given by* $L_i = d_i Q_i$ *and intermediate inputs by* $X_{ij} = m_{ij} Q_i$, *where* I *denotes the identity matrix and*

i. d *is a column vector with*

$$d_i = (1 - \alpha_i)(A_i B_i)^{\frac{\sigma}{1-\sigma}} \left(\frac{\lambda_i}{(1 + \tau_{iL})\eta^\star} \right)^{\frac{1}{1-\sigma}}.$$

ii. M *is a matrix with elements*

$$m_{ij} = A_i^{\frac{\sigma}{1-\sigma}} \alpha_i v_{ij} \left[\sum_j v_{ij} \left(\frac{\lambda_i}{(1 + \tau_{ij})\lambda_j} \right)^{\frac{\rho}{1-\rho}} \right]^{\frac{\sigma-\rho}{\rho(1-\sigma)}} \left(\frac{\lambda_i}{(1 + \tau_{ij})\lambda_j} \right)^{\frac{1}{1-\rho}}.$$

iii. c *is a column vector with* $c_i = \beta_i / \lambda_i$

[8] If we regard intermediates as capital goods with 100% depreciation, then the outcome of the model can be thought of as the steady state of a growth model with no discounting. Then, for some parameter values, complements imply that output converges to zero, while substitutes may lead to unbounded growth.

iv. λ *is a column vector that contains the Lagrange multiplier and which is given by* $\lambda = \exp(\log(\lambda/\eta^\star) - \beta' \log(\lambda/\eta^\star))$

v. The Lagrange multipliers solve

$$(\lambda_i/\eta^\star) = A_i^{-1}\left[(1-\alpha_i)B_i^{\frac{\sigma}{1-\sigma}}(1+\tau_{iL})^{-\frac{\sigma}{1-\sigma}} + \alpha_i\left[\sum_{j=1}^{n} v_{ij}\left((1+\tau_{ij})\lambda_j/\eta^\star\right)^{-\frac{\rho}{1-\rho}}\right]^{\frac{1-\rho}{\rho}\frac{\sigma}{1-\sigma}}\right]^{-\frac{1-\sigma}{\sigma}}$$

Proof. See Appendix A.2 ■

In the proposition, $\eta^\star = (1+\tau_L)\eta$ is the employer price of labor and $\lambda_i/\eta^\star$ is the price of good i in terms of labor. The distortions affect aggregate productivity through several channels. In addition to the well known effects on the allocation of labor (or any primary inputs) across firms, distortions also modify the allocation of intermediate inputs across firms both directly and indirectly by modifying the relative shadow price of goods. The change in relative price also leads to changes in the final goods consumption. The relative price equation (see (v)) shows that the common component of the labor wedge, τ_L has no effect on relative prices and (as we shall see) allocation and output, so without loss of generality, we can set it to an arbitrary value, most often zero.

Before we proceed with the theoretical analysis, we present a useful normalization which will be useful to interpret and map the model to the data.

Proposition 2 *Suppose that* $\alpha_i A_i^{\frac{\sigma}{1-\sigma}} < 1, \forall i$ *is satisfied. Then there exists a set of positive constants* $\{k_i\}$ *such that* $\bar{Q}_i = k_i Q_i$ *has production function in the form of (1) with some parameters* $\{\bar{A}_i, \bar{B}_i, \bar{\alpha}_i\}_i$ *and* $\{\bar{v}_{ij}\}_{i,j}$ *that satisfy the constraints on these parameters and* $\bar{B}_i = \bar{A}_i = 1, \forall i = 1, ..., n$.

In the equilibrium for the normalized parameters without distortions, $\lambda_i = 1, \forall i$, $\tilde{\alpha}_i = \sum_j \lambda_j \bar{X}_{ij}/(\lambda_i \bar{Q}_i)$, $\bar{v}_{ij} = \lambda_j \bar{X}_{ij}/\sum_k \lambda_k \bar{X}_{ik}$.

Proof. In Appendix A.4 ■

This proposition implies that, in an economy without distortions, we can choose units so that physical output is equal to the nominal value of output. In that case, α_i is equal to the cost share of intermediate goods in output and v_{ij} is equal to the cost share of a particular variety of intermediates. This proposition also shows that A_i and B_i should be interpreted as deviations from some baseline.

While we will use the solution to the general model to quantify the effects of market power, we need to restrict the elasticities of substitution to derive our theoretical results. In our theoretical analysis, we will assume that the elasticity between different varieties of intermediates is equal to the elasticity between intermediates and labor, that is $\sigma = \rho$. With this simplification, we obtain a closed form solution for output. We present here the solution of the model when $\sigma = \rho \neq 0$ and describe the Cobb-Douglas production function ($\sigma = \rho = 0$) case in Appendix A.3.

Proposition 3 *Suppose that $\sigma = \rho$ and $\sigma \neq 0$. Then aggregate output is given by*

$$\log Y = \log \bar{L} + \log TFP,$$

with aggregate productivity given by

$$\log TFP = \frac{1-\sigma}{\sigma}\beta' \log(\hat{\lambda}) - \log(\hat{c}'[I - \tilde{\alpha}\hat{A}(\Delta^{Xq} \circ V)]^{-1}\hat{A}\hat{B}\Delta^{Lq}(\mathbf{1} - \alpha)) \tag{9}$$

and the vector of Lagrange multipliers is given by

$$\hat{\lambda} = [I - \tilde{\alpha}\hat{A}(\Delta^{Xp} \circ V)]^{-1}\hat{A}\hat{B}\Delta^{Lp}(\mathbf{1} - \alpha), \tag{10}$$

where V is the matrix of intermediate input intensities with elements v_{ij}, $\hat{A}, \hat{B}, \Delta^{Xq}, \Delta^{Xp}$, Δ^{Lp} , Δ^{Lq} are square matrices and $\hat{\lambda}, \hat{c}$ are column vectors and $\hat{A}$ is a matrix with diagonal elements $\hat{A}_{ii} = A_i^{\frac{\sigma}{1-\sigma}}$ and 0 otherwise and $\hat{B}$ defined similarly; $\hat{\lambda}_i = (\lambda_i/\eta^*)^{-\frac{\sigma}{1-\sigma}}$, $\hat{c}_i = \beta_i(\lambda_i/\eta^*)^{\frac{\sigma}{1-\sigma}}$ and the following matrices of distortions are $\Delta^{Xp}_{ij} = (1+\tau_{ij})^{-\frac{\sigma}{1-\sigma}}$, $\Delta^{Xq}_{ij} = (1+\tau_{ij})^{-\frac{1}{1-\sigma}}$, $\Delta^{Lp}_{ii} = (1+\tau_{iL})^{-\frac{\sigma}{1-\sigma}}$, $\Delta^{Lp}_{ij} = 0, i \neq j$; $\Delta^{Lq}_{ii} = (1+\tau_{iL})^{-\frac{1}{1-\sigma}}$, $\Delta^{Lq}_{ij} = 0, i \neq j$, and $\circ$ denotes the Hadamard (entrywise) product.

Proof. In Appendix A.3. ■

In the absence of distortions, Proposition 3 implies that aggregate productivity is then

$$\log TFP = \frac{1-\sigma}{\sigma}\beta' \log((1-\alpha)[I - \tilde{\alpha}V]^{-1}\hat{B}),$$

where the Hicks neutral productivities are normalized to 1. As we will see, several indicators based on the undistorted economy will be useful to characterize analytically the effect of distortions on aggregate output. We define those indicators below.

Definition 1

1. *the Leontief inverse is* $\Omega \equiv [I - \tilde{\alpha} V]^{-1}$*, with its elements denoted* ω_{ij}*.*

2. *the importance of the sector is* $s_i \equiv \sum_k \beta_k \omega_{ki}$*.*

3. *the dispersion in the use of the sector as a supplier is* $s_{ii} \equiv \sum_k \beta_k \omega_{ki}^2$*.*

4. *the correlation between the use of sector* i *and* j *as suppliers is* $s_{ij} \equiv \sum_k \beta_k \omega_{ki} \omega_{kj}$ *for* $i \neq j$*.*

The Leontief inverse Ω, which appears commonly in input-output analysis, captures the direct and indirect linkages between sectors. The element ω_{ki} of the Leontief inverse governs the effect of a change in sector i's productivity on the gross output of sector k, including the effect of sector i on the suppliers of sector k, and the suppliers of its suppliers, and so on. A higher ω_{ki} therefore captures the importance of sector i as an input supplier of sector k. The importance of a sector s_i can then be computed as the sum of the sector's importance for each sector k of the economy, weighted by each sector's weight in final consumption β_k. The dispersion of the use of the sector as an input supplier s_{ii} reflects the variation in the importance of i for the different sectors of the economy whereas the correlation between input suppliers s_{ij} captures to which extent two sectors are supplying intermediate inputs to the same sectors.

3 Theoretical Analysis

In this section, we derive the consequences of distortions on aggregate productivity. We study how the effect of distortions depends on the production process, focusing in particular on the structure of intersectoral linkages and the elasticity of substitution between inputs. We first study how intermediate inputs matter for the size of the aggregate productivity loss from distortions. Then, we analyze the role of input substituability. Finally, we consider a specific input-output matrix to further study how the nature of the input-output network determines the effect of distortions on aggregate productivity. Throughout this section we rely on the assumption that the two elasticities of substitution are equal, $\sigma = \rho$ and we consider distortions on intermediate inputs that are specific to the purchasing industry, that is $\tau_{ij} = \tau_i, \forall j = 1, ..., n$.

3.1 Aggregate implications of distortions in the presence of input-output linkages

To better understand how input-output linkages shape the aggregate TFP loss from distortions, we start by studying the role of intermediate inputs. We first consider separately the implications of labor distortions and intermediate input distortions and highlight the crucial differences in how the two types of distortions propagate across sectors. We then derive results for distortions that affect all inputs.

Let us first consider the case of labor distortions. We approximate TFP around the non distorted equilibrium in the presence of labor distortions.

Proposition 4 *Suppose that, $\sigma = \rho$, $\alpha_i \in (0, 1)$ for all i, $A_i = B_i = 1$ and $\tau_{ij} = 0, \forall i, j$. For small distortions, aggregate TFP is approximatively equal to*

$$\begin{aligned}\log TFP \;\; &\approx \;\; \frac{1}{2}\left(\sum_i s_i(1-\alpha_i)\tau_{iL}\right)^2 + \frac{1}{2}\frac{\sigma}{1-\sigma}\sum_i\sum_j s_{ij}(1-\alpha_i)(1-\alpha_j)\tau_{iL}\tau_{jL} \\ &-\frac{1}{2}\frac{1}{1-\sigma}\sum_i s_i(1-\alpha_i)\tau_{iL}^2\end{aligned}$$

where $s_i = \sum_k \beta_k\omega_{ki}$, $s_{ij} = \sum_k \beta_k\omega_{ki}\omega_{kj}$ and ω_{ki} is the (k, i) element of the undistorted Leontief inverse Ω.

Proof. In Appendix A.6. ■

The proposition shows that the effect of labor distortions depends on the share of labor in the production process $(1 - \alpha_i)_{1,..,n}$ and on the structure of input-output linkages. The key input-output indicator in Proposition 4 is s_i, which reflects the importance of the sector in the economy. As explained in the previous section, the indicator s_i includes the direct and indirect supply of the sector's product to all the sectors of the economy.

To investigate whether the presence of input-output linkages leads to a larger reduction in aggregate productivity, we compare the economy to an economy otherwise identical but without any with input-output linkages. Using the approximation of Proposition 4, with the assumption that distortions are identically and independently distributed across sectors with mean τ_L and cross-sectional variance $V(\tau_{iL})$, but also independently distributed from any other model parameters (in particular α_i and any parameters affecting s_i and s_{ij}), we get

$$\log TFP \approx -\frac{1}{2}V(\tau_{iL}) - \frac{1}{2}\frac{\sigma}{1-\sigma}\left(1 - \sum_i s_{ii}(1-\alpha_i)^2)\right)V(\tau_{iL}), \qquad (11)$$

where $V(\tau_{iL})$ denotes the cross-sectional variance of τ_{iL} and $\left(1 - \sum_i s_{ii}(1-\alpha_i)^2)\right) \geq 0$. In the economy without input-output linkages, $\alpha_i = 0, \forall i = 1, .., n$, and we can show that the second term is equal to zero and aggregate TFP is simply

$$\log TFP \approx -\frac{1}{2} V(\tau_{iL}). \tag{11'}$$

The details of the derivation of these two expressions is given in Appendix A.6. As is well known in the literature, an increase in the dispersion of distortion reduces aggregate productivity whereas the average distortions τ_L has no effect on aggregate productivity. The allocation of primary inputs is undistorted when distortions are identical across firms.[9] We focus here on the role of input-output linkages. Comparing equations (11) and (11'), it appears clearly that, contrary to intuition, input-output linkages do not always amplify the impact of distortions. Whether or not input-output linkages amplify the effect of labor distortions crucially depends on the elasticity of substitution. In particular, if the production function is Cobb-Douglas, $\sigma = 0$, equation (11) simplifies to $\log TFP \approx -(1/2)V(\tau_{iL})$, and aggregate TFP is therefore identical to that of the economy without input-output linkages, given in equation (11'). In fact, the presence of intermediate inputs generates two counteracting effects. On the one hand, input-output linkages propagate distortions across sectors which tends to amplify the effect of distortions. The labor distortions distort the production of each sector, and as the products are in turn used as intermediate inputs this further distorts the production process. On the other hand, the use of intermediate inputs in production reduces the effect of labor distortions, as labor then represents a smaller share of the firms' inputs. These two offsetting effects cancel out exactly when the production function is Cobb-Douglas, but in general input-output linkages could amplify or dampen the effects of distortions. Equation (11) shows that when inputs are complement ($\sigma < 0$), intermediate inputs dampen the effects of labor distortions. Conversely, the TFP loss is larger than the economy without intermediate inputs if inputs are substitutable ($\sigma > 0$). The aggregate implications of primary input distortions hence depend crucially on the substitutability between inputs. As we will see, this result holds only when distortions are on primary input only.

We now examine the effects of intermediate inputs distortions. Naturally in that case, the TFP loss crucially depends on the presence of intermediate inputs. We approximate aggregate TFP in the presence of intermediate input distortions around the non-distorted equilibrium.

[9] See for example Osotimehin (2016).

Proposition 5 *Suppose that, $\sigma = \rho$, $\alpha_i \in (0,1)$ for all i, $A_i = B_i = 1$ and $\tau_{iL} = 0, \forall i$ and $\tau_{ij} = \tau_i, \forall j$. For small distortions, aggregate TFP is approximatively equal to*

$$\begin{aligned}\log TFP &\approx -\frac{1}{2}\frac{1}{1-\sigma}\sum_i\sum_j \left(s_i\omega_{ij}\alpha_j + s_j\omega_{ji}\alpha_i\right)\tau_i\tau_j \\ &\quad +\frac{1}{2}\frac{\sigma}{1-\sigma}\sum_i\sum_j \alpha_i\alpha_j s_{ij}\tau_i\tau_j + \frac{1}{2}\left(\sum_i \alpha_i s_i \tau_i\right)^2 + \frac{1}{2}\frac{1}{1-\sigma}\sum_i \alpha_i s_i \tau_i^2,\end{aligned}$$

where $s_i = \sum_k \beta_k\omega_{ki}$, $s_{ij} = \sum_k \beta_k\omega_{ki}\omega_{kj}$ and ω_{ki} is the (k,i) element of the undistorted Leontief inverse $\Omega = (I - \tilde{\alpha}V)^{-1}$.

As for the case of labor distortions, the proposition shows that the effect of intermediate input distortions depends on the share of intermediate inputs in the production process $(\alpha_i)_{1,..n}$ and on the importance of the sectors $(s_i)_{1,..,n}$. Using again the assumption that distortions are identically and independently distributed across sectors, and independently distributed from the sector parameters, with mean $\bar{\tau}$ and cross-sectional dispersion $V(\tau_i)$, we obtain

$$\log TFP \approx -\psi_\sigma(\Omega,\alpha)\bar{\tau} - \chi(\Omega,\alpha)V(\tau_i) - \frac{\sigma}{1-\sigma}\left(\chi(\Omega,\alpha) - \sum_i \alpha_i^2 s_{ii}\right)V(\tau_i), \quad (12)$$

where $\chi(\Omega,\alpha) \geq 0$ and $\chi(\Omega,\alpha) - \sum_i \alpha_i^2 s_{ii} \geq 0$. The derivation is in Appendix A.7.

When intermediate inputs are distorted, the average level of distortions matters and reduces aggregate productivity. This result, also highlighted in Jones (2011), contrasts with the labor distortions case for which only the dispersion of distortions reduces aggregate productivity. In addition to distorting the allocation of goods across sectors, distortions to intermediate inputs modify the allocation of goods between final consumption and intermediate inputs, whether or not distortions are heterogeneous across sectors. Furthermore, we find that the role of input-output linkages crucially depends on the nature of the distortion. Input-output linkages shape the TFP loss from intermediate input distortions, whatever the value of the elasticity of substitution. By contrast with the labor distortions case, the intersectoral linkages determine the size of the TFP loss even when the production function is Cobb-Douglas.

We have studied the cases where either labor or intermediate inputs are distorted. Many sources of distortions, among which imperfect competition which, we will focus on in Section 4, would, however, affect all inputs at the same time. The next proposition gives the approximation of TFP when all inputs are distorted.

Proposition 6 *Suppose that, $\sigma = \rho$, $\alpha_i \in (0,1)$ for all i, $A_i = B_i = 1$ and $\tau_{Xij} = \tau_{Li} = \tau_i, \forall i, j$. For small distortions, aggregate TFP is approximatively equal to*

$$\log TFP \approx \frac{1}{2}\sum_i \sum_j \left[\frac{\sigma}{1-\sigma}s_{ij} + s_i s_j - \frac{1}{1-\sigma}(s_i\omega_{ij} + s_j\omega_{ji})\right]\tau_i\tau_j + \frac{1}{2}\frac{1}{1-\sigma}\sum_i s_i\tau_i^2$$

where $s_i = \sum_k \beta_k \omega_{ki}$, $s_{ij} = \sum_k \beta_k \omega_{ki}\omega_{kj}$ and ω_{ki} is the (k,i) element of the undistorted Leontief inverse Ω.

Similarly to the case where only intermediate inputs are distorted, the nature of input-output linkages matter for the TFP loss from distortions.

3.2 Aggregate implications of distortions and input substitutability

As shown in the previous section, the degree of input substitutability plays an important role in how input-output linkages interact with distortions. We now study in more detail how input substitutability affects the aggregate productivity loss from distortions.

When only labor is distorted, and when distortions are independent from the sector's position in the network, we can show from equation (11) that

$$\frac{d\log TFP}{d\sigma} = -\frac{1}{1-\sigma}\left(1 - \sum_i (1-\alpha_i)^2 s_{ii}\right)\mathrm{var}(\tau_{Li}) \leq 0. \tag{13}$$

When intermediate inputs only are distorted, we can show using equation (12) that

$$\frac{d\log TFP}{d\sigma} = -\frac{1}{1-\sigma}\left(\chi(\Omega,\alpha) - \sum_i \alpha_i^2 s_{ii}\right)\mathrm{var}(\tau_{Xi}) \leq 0. \tag{14}$$

As shown in the two equations above, a higher elasticity of substitution, that is a higher σ, amplifies the effects of distortions whereas stronger complementarity dampens the effect of distortions on aggregate productivity. This holds when either labor or intermediates are distorted, and we expect this to hold when distortions affect both labor and intermediate input decisions. At first, it could seem counterintuitive that distortions have less impact when firms cannot substitute away from distorted inputs. But distortions actually matter less for the firms' input decision when inputs cannot be easily substituted. When the elasticity of substitution is low, firms respond less to the distorted prices, hence the firms' input decision does not deviate much from the first-best outcome. At the limit, with perfectly complementary inputs, the input decision is unaffected by relative price distortions.[10] Smaller changes in the allocation lead to smaller changes in labor used directly and indirectly for the production of the sectoral good and hence smaller decline in aggregate productivity. This is not, however, the only mechanism at play. A lower elasticity of substitution tends to amplify the consequences that an inefficient mix of inputs can bring about on production, thus amplifying the TFP loss. We find that for moderately positive distortions the first effect dominates, and a higher degree of complementary leads to a smaller TFP loss.

To complement the results of equations (13) and (14), which rely on the law of large numbers, we show that a higher degree of complementarity reduces the impact of distortions for any number of sectors. This result, described in the following proposition, relies on symmetric distortions.

Proposition 7 *Suppose that, $\sigma = \rho$, $\alpha_i \in (0,1)$ for all i, $A_i = B_i, \forall i$ and $\tau_{ij} = \tau, \forall i,j$ and $\tau_{iL} = \tau_L = 0$. Also suppose that $\sigma_1 < \sigma_2$ and the conditions for the existence of undistorted equilibrium hold for both values. Then there exists some $\underline{\tau} < 0 < \bar{\tau}$ such that $Y(\tau, \sigma_1) > Y(\tau, \sigma_2)$ for $\tau \in (\underline{\tau}, 0) \cup (0, \bar{\tau})$.*

Proof. In Appendix A.8 ■

This result relies on small distortions, but we will see in practice that for plausible values of the elasticity of substitution, the result appears to hold for high level of distortions as well.

The effect of input complementarity on the consequences of distortions contrasts sharply with its effect on the consequences of sectoral productivity shocks. Whereas input complementarity dampens the consequences of distortions, it amplifies the consequences of sectoral productivity shocks. When the elasticity of substitution is low, the relative use of intermediate goods does not change much, but more resources need to be devoted to the relatively less productive sector, which reinforces the decline in aggregate productivity. We formalize this intuition in the following proposition.

Proposition 8 *Suppose that, $\sigma = \rho$, $\alpha_i \in (0,1)$ for all i, and set $A_i = 1$. Let $Y(B, \sigma)$ to be the output of an economy without distortions with parameter σ , and productivity vector B Suppose that $\sigma_2 > \sigma_1$ and let $x \in \mathbf{R}^n$, $x \neq c\mathbf{1}$, $c \in \mathbf{R}$. Then there exist some $\bar{\xi} > 0$, such that $Y(\mathbf{1} + \xi x, \sigma_2) > Y(\mathbf{1} + \xi x, \sigma_1)$ for all $\xi \in (0, \bar{\xi}]$.*

Proof. In Appendix A.8 ■

[10] See for example Rotemberg and Woodford (1993).

Proposition 7 and Proposition 8 highlight an interesting contrast between the effects of different inefficiencies. Frictions that distort the relative prices of goods will be less harmful in economies with more complementarities; on the other hand, any friction that causes more resources to be expended will have larger effects in economies with complementarities.

3.3 Aggregate implication of distortions and the input-output structure

In this section, we delve further into the role of input-output linkages and study the effect of the importance of intermediate inputs in the production process and the structure of the production network. To gain more insight, we focus here on a specific input-output structure. We consider a simple input-output structure in which sectors differ both in their use of intermediate inputs and in how much they are used by other sectors as intermediate inputs.

3.3.1 A simple input-output economy

In this simple input-output economy, sectors may differ in their overall use of intermediate inputs but not in their relative use of intermediate inputs from each sector. Every sector uses intermediates from sector j with intensity v_j. Sectors may also differ in how intensively their goods are used as final goods. This structure is simple enough to derive the Leontief inverse analytically, yet complex enough to capture the three essential dimensions in which sectors differ in the production network: their importance as a final good supplier, as an intermediate input supplier, and as an intermediate input purchaser. The production network of the simple input-output economy is characterized by the following parameters:

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \dots \\ \beta_n \end{pmatrix}, \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \dots \\ \alpha_n \end{pmatrix} \text{ and } V = \begin{pmatrix} v_1 & v_2 & \dots & v_n \\ v_1 & v_2 & \dots & v_n \\ \dots & \dots & \dots & \dots \\ v_1 & v_2 & \dots & v_n \end{pmatrix}$$

To ease the exposition, we will focus here on the Cobb Douglas production function, that is $\sigma = \rho = 0$. We show in Appendix A that the main conclusions presented here hold when we relax this assumption and allow $\sigma = \rho \neq 0$.[11]

[11] To be added.

3.3.2 The role of the production network when distortions are random

What are the characteristics of a production network that amplify the effects of distortions on aggregate productivity? To answer this question, we first consider the case of random distortions. We assume that distortions are identically and independently distributed across sectors, and independent from any other sectoral parameter. As shown in Section 3.1, the details of the input-output matrix do not matter for the effect of labor distortions when the production function is Cobb-Douglas, as is assumed here. We therefore focus here on intermediate input distortions.

For simplicity, we abstract from the effect of average distortions and assume $\bar{\tau}_X = 0$. Applying equation (12) to the simple input-output economy, we have $\log TFP = -\chi(\Omega, \alpha) V(\tau_{Xi})$, with

$$\chi(\Omega, \alpha) \quad = \quad \bar{\alpha}_\beta + \frac{1}{1-\bar{\alpha}_v} \sum_i \alpha_i v_i (\bar{\alpha}_\beta + 2\alpha_i \beta_i) + \frac{2\bar{\alpha}_\beta}{(1-\bar{\alpha}_v)^2} \sum_i \alpha_i^2 v_i^2, \qquad (15)$$

where $\bar{\alpha}_v \equiv \sum_k v_k \alpha_k$ and $\bar{\alpha}_\beta \equiv \sum_k \beta_k \alpha_k$. The simple input-output economy illustrates well how the structure of input-output linkages can amplify the effects of distortions. Equation (15) suggests that the negative effects of random distortions are amplified when the production network is characterized by a high covariance between α_i, β_i and v_i. All else equal, aggregate TFP is lower when a sector whose products are used intensively as intermediate inputs, are also intensively used as final goods. Aggregate TFP is further reduced when the sector also uses intermediate inputs intensively. Moreover, aggregate TFP is further reduced when sectors are more heterogenous along any of these three dimensions.

The simple input-output economy also allows us to study the amplification effect coming from the use of intermediate inputs. To study the effect of the intensity of intermediate inputs usage, we assume $\alpha_i = \bar{\alpha}$. Rewriting equation (15) under that assumption, we can show that[12]

$$\frac{d\chi}{d\bar{\alpha}}(\Omega, \bar{\alpha}) \geq 0.$$

A higher intermediate input share amplifies the TFP loss from intermediate input distortions.

[12]See Appendix A.9 for the derivation.

3.3.3 The key sectoral characteristics that amplify distortions

We now consider how the correlation between the distortions and the position of the sector in the production network affects the TFP loss from distortions. We start by investigating the sectoral characteristics that amplify the effect of a sectoral distortion.

Using Proposition 4 and 5, we compute the effect of a labor distortion and an intermediate input distortion *in one sector only*:

$$\begin{aligned}\frac{d\log TFP}{d\tau_{Li}} &\approx -(1-\alpha_i)s_i(1-(1-\alpha_i)s_i)\tau_{Li},\\ \frac{d\log TFP}{d\tau_{Xi}} &\approx -\alpha_i s_i(1-\alpha_i s_i+2\alpha_i g_{ii})\tau_{Xi}.\end{aligned}$$

Note that now that distortions are not random, the details of the network also matter for the effect of labor distortion when the production function is Cobb-Douglas. Using the two equations, we can show that the aggregate implication of a sectoral distortion increases with how intensively the sector uses the distorted input, captured by α_i or $(1-\alpha_i)$, as well as with how important the sector is in the economy, captured by s_i.[13]. In the simple input-output economy, the importance of the sector is given by

$$s_i = \beta_i + \frac{\bar{\alpha}_\beta}{1-\bar{\alpha}_v}v_i,$$

where, as in the previous section, $\bar{\alpha}_v \equiv \sum_k v_k\alpha_k$ and $\bar{\alpha}_\beta \equiv \sum_k \beta_k\alpha_k$. A distortion has a more detrimental impact on aggregate TFP if it affects a sector which accounts for a large share of final output (high β_i) or for a large share of intermediate inputs (high v_i). When $\alpha_i = \alpha$, the effect of a sectoral distortion increases with the average of its importance as a supplier of final goods and of intermediate goods, weighted respectively by the intensity of labor and intermediates in the production function.

3.3.4 Correlation between distortions and the importance as an input supplier

To fully characterize the effect of the correlation between distortions and the position of the sector in the production network, we now consider the effect of distortions *in all sectors*. We focus here on the correlation between the distortion and the importance as an input supplier.

We use the following specification. We assume α is identical across sectors, β is randomly distributed and v_i and τ_i are jointly distributed with correlation coefficient δ. More

[13] The derivation of these results are shown in Appendix A.

specifically, they are characterized by

$$
\begin{aligned}
v_i &= w_i / \sum w_i, \\
w_i &= \bar{w} + \sigma_v u_{vi}, \\
\tau_i &= \bar{\tau} + \sigma_\tau z_i, \\
z_i &= \delta u_{vi} + \sqrt{1-\delta^2} u_{\tau i},
\end{aligned}
$$

where $E(u_{vi}) = E(u_{\tau i}) = E(u_{vi} u_{\tau i}) = 0$, $Var(u_{vi}) = Var(u_{\tau i}) = 1$ and we normalize $\bar{w} = 1$. This specification ensures that $\sum_i v_i = 1$ and that varying δ affects only the correlation and not the variance of the distortions. The mean of the distortions is τ, the variance is σ_τ^2 and the variance of v_i increases with σ_v^2.

Using Proposition 5, together with the law of large numbers, we obtain, when only labor is distorted

$$
\log TFP \approx -\frac{1}{2}\sigma_\tau^2 - \frac{1}{2}\alpha\delta^2\sigma_\tau^2\sigma_v \left[E(u_{vi}^3) - \sigma_v\right], \tag{16}
$$

And when only intermediate inputs are distorted

$$
\log TFP \approx -a_0(\alpha)\bar{\tau}^2 - \sigma_\tau^2 a_1(\alpha)\sigma_\tau^2 - a_2(\alpha)\delta\sigma_\tau\sigma_v - a_3(\alpha)\delta^2\sigma_\tau^2\sigma_v \left[E(u_{vi}^3) + a_4(\alpha)\sigma_v\right], \tag{17}
$$

with $a_0(\alpha), a_1(\alpha), a_2(\alpha), a_3(\alpha), a_4(\alpha) \geq 0$.

We find that a higher correlation between v_i and the distortion amplifies the TFP loss only if the distribution has a sufficiently high skewness when distortions are only on labor. However, when distortions are on intermediate inputs, a higher correlation always amplifies the TFP loss whatever the shape of the distribution. A more positively skewed distribution reinforces the effect of the correlation.

3.4 Upstream and downstream propagation

Do distortions on sectors early in the production chain matter more or less? As we move down the production process, substitution possibilities should attenuate the effect of distortions, but on the other hand, they have effects on multiple sectors. Acemoglu et al. (2016) tackle this question in the context of productivity shocks in a Cobb-Douglas framework. They find that productivity shocks travel downstream, while demand shocks travel upstream.

In the same spirit, we consider an interesting special case in which we can separate the production network into upstream and downstream components. We define sector i to be a purchaser of sector j if $[\tilde{\alpha}V]_{ij}^{(n)} > 0$ for some n, that is, if sector j is a direct or indirect supplier of sector i. Sector k is upstream of j it is not a purchaser of sector ℓ.[14] It is trivial to relabel the sectors in such a way that the upstream sectors are numbered $1, 2 \dots \ell - 1$, the sector in question to be ℓ and the purchaser sectors are $\ell + 1 \dots n$.

Lemma 1 *A necessary and condition for sectors* $1, \dots \ell - 1$ *to be upstream from* ℓ *is that* $\alpha_i v_{ij} = 0$, $\forall i \leq \ell - 1, j \geq \ell$.

Our definition of upstreamness is ordinal and cannot necessarily compare any two sectors, but when it does, it agrees with the measure that Antràs et al. (2012) propose.

Given that lemma and proposition 3, we can characterize the effect of imposing distortions on sector ℓ.

Proposition 9 *Let* x *be some sector* k *such that* $\alpha_\ell v_{\ell k} > 0$ *or the labor input (if* $1 - \alpha_\ell > 0$*). Then:*

1. $\frac{\partial}{\partial \tau_{\ell,x}} \frac{\lambda_i}{\eta *} = 0$ *if* $i < \ell$ *and* $\frac{\partial}{\partial \tau_{\ell,x}} \frac{\lambda_i}{\eta *} > 0$ *if* $i \geq \ell$.
2. $\frac{\partial}{\partial \tau_{\ell,x}} \lambda_i < 0$ *if* $i < \ell$.
3. $\frac{\partial}{\partial \tau_{\ell,x}} \frac{C_i}{Y} > 0$ *if* $i < \ell$ *and* $\frac{\partial}{\partial \tau_{\ell,x}} C_i > 0$ *if* $i < \ell$.
4. $\frac{\partial}{\partial \tau_{\ell,x}} \frac{\sum_j (1+\tau_{ij}) \lambda_j X_{ij}}{\lambda_i Q_i} = 0$ *if* $i < \ell$; $\frac{\partial}{\partial \tau_{\ell,x}} \frac{\sum_j \lambda_j X_{ij}}{\lambda_i Q_i} > 0$ *if* $i > \ell$ *and* $\sigma < 0$; $\frac{\partial}{\partial \tau_{\ell,xx}} \frac{\sum_j \lambda_j X_{ij}}{\lambda_i Q_i} < 0$ *if* $i > \ell$ *and* $\sigma > 0$.

The division of labor in the upstream sectors is unaffected by the distortion and hence their price in terms of labor is unaffected. Since imposing distortions lowers the real wage, this implies that upstream industries become cheaper and expand in a relative and absolute sense. This conclusion does not depend on the elasticity of substitution, and is in stark contrast with the predictions for the productivity shocks case. Another interesting implication of this special case is that, except for the sector directly affected by the distortion, it leads to *higher* intermediate shares in downstream sectors if the elasticity of substitution is low. The reason is that the distortion has mainly price effects. Therefore, high intermediate shares are consistent with output loss and distortion on intermediate goods.

[14] Sector ℓ can itself be a purchaser of a purchases sector, hence we do not use the term "downstream" sector.

4 Quantifying the TFP loss from imperfect competition

In this section, we use the model to quantify the effects of market power on aggregate productivity. We first estimate markups using cross-country industry-level data and document that lower income countries tend to have higher markups. We calibrate the model on Mexican data and compute the TFP gains from reducing markups in Mexico to the levels observed in the US. The model highlights the additional productivity gains coming from input-output linkages. Finally, we compute the aggregate productivity gains from reducing markups in a selection of lower income countries (In progress).

4.1 Measuring the degree of imperfect competition with industry-level data

We measure the degree of imperfect competition at the industry level using price-cost margins, defined as sales minus costs over sales. We use two alternative measures. In our first measure, pcm_1, we abstract from capital costs, which are challenging to measure, and compute the price-cost margin as

$$\text{pcm}_1 = (\text{sales} - \text{wage bill} - \text{intermediate input cost})/\text{sales}.$$

This measure may overstate the difference in markups between low income and high income countries if capital costs are higher in low income countries. For our second measure, pcm_2, we subtract capital costs

$$\text{pcm}_2 = (\text{sales} - \text{wage bill} - \text{intermediate input cost} - \text{capital costs})/\text{sales}.$$

As will be shown in section 4.2.2, price-cost margins provide a measure of the firms' markups, provided that other distortions average out at the industry level. While the first measure of price-cost margin pcm_1 is likely to overstate the level of markups, it gives a useful proxy of how markups differ across countries. The two measures are computed using industry-level data from the World Input-Output Socio-Economic Accounts. The dataset gives industry-level data on sales, labor compensation, intermediate input costs, and the capital stock for 40 countries and 30 sectors over the period 1995-2011.[15] More details on both the data sources and the construction of the variables are given in Appendix B.2.

[15] The data are publicly available at: http://www.wiod.org/home. The data are classified in 35 industries, but we drop Public Admin and Defence; Compulsory Social Security Education Health and Social Work, Other Community, Social and Personal Services, Private Households with Employed Persons

Table 1: Markups across countries

	(1)	(2)	(3)	(4)
	ln(pcm)	pcm	ln(pcm)	ln(pcm)
ln(Y/L)	-0.290***	-0.0451***	-0.0399***	-0.0129*
	(-15.10)	(-15.82)	(-3.90)	(-2.47)
country f-e	No	No	No	Yes
N	534	534	33	534

Note: Notes: Estimated on panel except column (3) where year=2000. $ln(Y/L)$ denotes the log of gdp per capita, and ln(pcm) is the log of the country median price-cost margin pcm_1. t statistics in parentheses. * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

We first document how the price-cost margin varies across countries. We use data on income per capita, which are obtained from the Penn World table, as well as our first measure of price-cost margin pcm_1, which abstracts from capital costs, to have the largest sample possible. Using the cross-country panel, we find a systematic link between income per capita and the median price-cost margin. As shown in Table 1, a higher income per capita is associated with a lower median price-cost margin. The first two columns show the estimate of the OLS regression on the cross-country panel. The third column reports results of the regression estimated only on the cross-section, and the fourth column reports results for the full panel but with country fixed-effects. This relation holds in all four specifications, across countries as well as within countries.

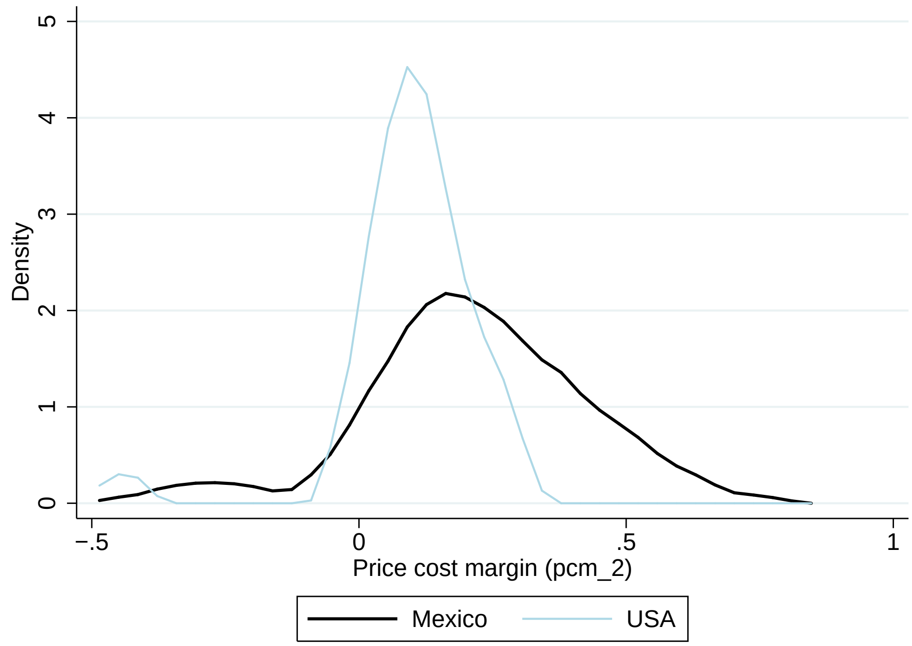
4.2 The aggregate productivity cost of imperfect competition in Mexico

We quantify the consequences of imperfect competition on aggregate productivity in Mexico, which has a high level of markups relative to more developed economies but also relative to other countries with a similar level of development. We estimate the median price-cost margin in Mexico to be about twice as large as in the US. As shown in Figure 1, price-cost margins in Mexico are not only higher but also more dispersed than in the US. In this section, we use our model to quantify the effects of imperfection competition on aggregate productivity in Mexico.

4.2.1 Incorporating markups into the model

We use the general model described in section 2 to study the aggregate implication of market power. Markups may vary across sectors due to differences in the elasticity of substitution *within* each sector or differences in the market structure, regulation or the number of firms.

Figure 1: The distribution of markups across sectors: Mexico vs the US



Our model is consistent with any of these sources of market power but does not specify the source of imperfect competition. We adopt a parsimonious modeling strategy and consider markups as exogenous. With a particular distortion in mind, we can now be explicit about the market prices. We assume that each firm is committed to selling unlimited amounts at a constant markup over marginal cost, and therefore prices are

$$p_i = \mu_i mc_i,$$

where μ_i is the sector-specific markup, and mc_i is the marginal cost of production. Under this assumption, markups are isomorphic to sector-specific distortions. We show in the appendix that firm cost minimization implies that the allocation satisfies the equations (7) and (8) with $\tau_{ij} = \tau_{iL} = \mu_i - 1$, with $mc_i = \lambda_i$ given by (1) and the solution of the model with markups is thus given by Proposition 1. The solution can easily be computed numerically by iterating on the Lagrange multipliers.

4.2.2 Calibration

The model is calibrated on Mexican industry-level data on sales and input costs for the year 2005. The data are obtained from World Input-Output Database (WIOD), the WIOD Socio Economic accounts, and the OECD. We describe here the calibration strategy and report in Appendix B.2 details on the data sources and the construction of the variables.

Markups. We estimate markups from our conservative price-cost margin measure pcm_2. The alternative measure pcm_1 abstracts from capital costs and is therefore likely to overestimate the mean and possibly the dispersion of markups. To see the link between the price-cost margin and the markup, let us write the expression of the price-cost margin in terms of the model variables

$$\text{pcm}_2 = \left(p_i Q_i - \eta L_i - \sum p_j X_{ij} - (r+\delta) q K_i\right) / (p_i Q_i),$$

where r is the interest rate, δ is the depreciation rate and q is the price of capital. Note that while the model described in section 2 does not have capital explicitly, we do take into account the capital cost when measuring the price cost margin. Using the first order conditions (equations (7) and (8) with $\tau_{Li} = \tau_{ij}$) and the corresponding equation for capital, it follows from Euler's theorem that $\eta L_i - \sum_j p_j X_{ij} - (r+\delta) q K_i = (p_i Q_i)/\mu_i$, and hence

$$\mu_i = \frac{1}{1 - \text{pcm}_2}.$$

Implicit here is the assumption that other distortions average out at the industry level. We report in Appendix B.3 the results when we instead use pcm_1 instead to measure the markups.

Final demand. The final expenditure shares can be used to pin down the parameter β_i. From equation (6), we obtain

$$\beta_i = p_i C_i / (\sum_j p_j C_j).$$

Technology. The parameters of the production function $\{\alpha_i, v_{ij}\}_{i,j=1...n}$ cannot be estimated directly from intermediate input cost shares since the markups create a wedge between the cost shares and the technology parameters. We use the first-order conditions of the firms to derive an estimate for these parameters given the elasticity of substitution and the industry-level markups. From Proposition 1 and the fact that $p_i = mc_i \mu_i = \lambda_i \mu_i$, we have[16]

$$n_{ij} \equiv \alpha v_{ij} = \frac{p_j X_{ij}}{p_i Q_i} \left(\sum_j v_{ij} [p_i/p_j]^{\frac{\rho}{1-\rho}} \right)^{-\frac{\sigma-\rho}{\rho(1-\sigma)}} \left(\frac{p_i}{p_j} \right)^{-\frac{\rho}{1-\rho}} \mu_i^{\frac{1}{1-\sigma}},$$

where $(p_j X_{ij})/(p_i Q_i)$ is the cost share and the price vector p is itself a function of the parameters v_{ij} and α_i. Hence the calibration involves finding a fixed point.[17] We find that a simple iterative procedure finds the fixed point quickly and is robust to different initial guesses. With a solution for $\{n_{ij}\}$, we obtain $\alpha_i = \sum_j n_{ij}$ and then $v_{ij} = n_{ij}/\alpha_i$.

The remaining two parameters, ρ and σ which govern the elasticity of substitution between intermediate and primary inputs $\varepsilon_\rho \equiv 1/(1-\rho)$ and that between intermediate inputs $\varepsilon_\sigma \equiv 1/(1-\sigma)$, are let to vary within a range. A few empirical papers have estimated these elasticities of substitution in the short-run. The elasticity between primary inputs and the intermediate input bundle has been estimated to be between 0.4 and 0.9. Rotemberg and Woodford (1996) find a value of 0.7, Oberfield and Raval (2014) find a range between 0.6 and 0.9 and Atalay (2017) finds estimates ranging between 0.4 and 0.8. Atalay (2017) also estimates the elasticity of substitution between intermediates and finds an estimate close to zero. Our focus is the effect of distortions on *long run* outcomes, and we view these *short-run* elasticities as plausible lower bounds for the long run elasticities. Most likely, the elasticities of substitution are higher when the horizon considered is longer. We report the results for the values of the elasticities between 0 and 1 (σ and ρ are let to vary in the range $(-\infty, 0]$).[18] Note that the technology parameters $\{\alpha_i, v_{ij}\}_{i,j=1\ldots n}$ need to be re-calibrated whenever the elasticities of substitution are let to vary.

4.2.3 Results

We consider the aggregate productivity gains from lowering markups in Mexico to US levels. Markups are both on average higher and more dispersed in Mexico than in the US. The median price-cost margin is 0.22 in Mexico and 0.10 in the US, and the standard deviation of the price-cost margin is 0.15 in Mexico and 0.08 in the US. We first compute, using the general solution described in Proposition 1, output and aggregate productivity when

[16] As justified by Proposition 2, we can choose the units in such a way to ensure $A_i = 1$, $B_i = 1$ for all i.

[17] In Appendix B.2 we plot the estimated α in Mexico for one of our measures of markups and the 'naive' estimate $\sum_j \frac{p_j X_{ij}}{p_i Q_i}$ (see figure 2). As we might expect, the estimated parameters are higher than the observed shares because of the presence of markups. In a few instances, however, markups cause the price of intermediates in some sectors to increase so much that the observed intermediate share is higher than the underlying technical parameter.

[18] In a recent paper, Miranda-Pinto and Young (2018) show that the elasticity of substitution vary across sectors, with higher elasticities in services than in manufacturing industries. The implications of this sectoral heterogeneity are left for further research.

markups are equal to the values estimated in Mexico. We then compute the TFP gain from replacing each sector's markup to its US value. We report the results for different values of the elasticity of substitution between intermediates, and between labor and the intermediate input bundle, recalibrating the model every time we change the value of one of the elasticities of substitution.

Table 2: TFP gain from lowering markups (percents)

		Baseline			No-dispersion			Prop. reduction		
		ε_σ			ε_σ			ε_σ		
		0.01	0.7	1	0.01	0.7	1	0.01	0.7	1
ε_ρ	0.01	2.8	9.3	12.8	1.4	5.1	6.9	2.2	7.9	10.9
	0.7	3.6	10.6	14.5	1.9	5.8	7.7	2.8	8.8	12.0
	1	3.9	11.2	15.3	2.1	6.1	8.1	3.1	9.2	12.4
No IO linkages		2.3			0.0			1.5		

Notes: Baseline experiment refers to lowering each sector markup from its level in Mexico to the US level. The no-dispersion experiment refers to setting all the markups to the Mexican average, and then reducing it to the US average. Markups are measured using pcm_2 and no linkages economy is obtained by setting $\alpha_i = 0$ for all sectors. Proportional Reduction experiment refers to multiplying all Mexican markups by a scalar to reduce their mean level to the US level.

Table 2 reports the aggregate TFP gain from lowering the markups to US levels. In the baseline experiment, we calibrate the markup in each sector to its level in Mexico and lower it to the US level. The results are reported in the left panel of the table. We find that reducing Mexican markups would yield sizable TFP gains, between 3% to 15% whereas the TFP gains would be about 2% in the absence of input-output linkages.[19] Ignoring input-output linkages would thus lead us to underestimate the cost of markups.[20] The allocation of inputs across sectors is more distorted in the presence of input-output linkages.

We find that the TFP gain increases with the elasticity of substitution between the primary inputs intermediate input bundle, and between intermediate inputs. While the results shown here are consistent with what we obtained in the theoretical analysis of section 3, they are not directly comparable since the model parameters are re-calibrated whenever the value of the elasticities of substitution is modified. We confirm in Appendix B, that

[19]We report in Appendix B.3 the results obtained from using the alternative price-cost margin measure (pcm_1). Note that with both measures, reducing Mexican markups to US levels implies dividing the price-cost margin by two. The fact that the overall effect of TFP is substantially larger with pcm_1, despite a comparable change in price-cost margin, indicates strong non linearities. The TFP gain from dividing the price-cost margin by two is increasing in the level of the price-cost margin.

[20]Our exercise likely shows a lower bound for the welfare effect of lowering markups. Aghion et al. (2005) show that a high degree of monopoly power is detrimental for entrepreneurship and innovation. Hence accounting for the link between markups and innovation would lead to even larger welfare losses.

the TFP gain increases with the two elasticities of substitution when the model parameters (other than the elasticities of substitution) are kept constant.

With the numerical solution, we no longer have to restrict the two elasticities to be equal. Interestingly, the results show that the gain in aggregate TFP is more sensitive to the value of the elasticity of substitution between labor and the intermediate input bundle, that to the elasticity between intermediate inputs. We find that a unit elasticity of substitution between labor and the intermediate input bundle multiplies the TFP gain by 4, relative to a low elasticity of substitution of 0.01. The TFP gain increases by a factor of 1.4 for the same increase in the elasticity of substitution between intermediates.

Markups in Mexico are both higher and more dispersed that in the US. To disentangle the effect of lowering average markups from that of reducing the dispersion of markups, we run a second experiment in which we set all markups to the Mexican average and compute the TFP gain obtained from lowering this markup level to the US average. As shown in the right panel of Table 2, the TFP gain is still sizable. For the midpoint value of the elasticities of substitution, the TFP gain is equal to 6%, which is more than half of the overall effect. The average level of markups hence accounts for a substantial share of the overall misallocation. This effect is absent in standard models of misallocation where input-output linkages are not explicitly modeled.

Finally, we consider the role of the elasticity of substitution between final goods. Our baseline functional form is Cobb-Douglas, which implies an elasticity of substitution between final goods of one. Markups distort the relative price of goods, which distort final goods choices all the more so when the elasticity of substitution is high. In table 3 and 4, we present the welfare gain of lowering Mexican markups to the US level for three final demand elasticities of substitution: 0 (Leontief), 1 (Cobb-Douglas) and 3. As expected, a higher elasticity of substitution in final goods amplifies the TFP gain, which can be as high as 31%. We also find substantial gains from lowering markups even when the intermediate input mix is fixed by technology.

5 Conclusion

In this paper, we explore how input-output linkages propagate distortions to the allocation of inputs across sectors. We have introduced reduced-form distortions into a model with rich structure of intermediate input linkages. A crucial component of our approach is that we allow for different degrees of substitution between types of intermediates and also between intermediates and labor (or, more generally, primary inputs), as well as for different

Table 3: TFP gain from lowering markups to US level (percents)

		Final Good Elasticity of Substitution								
		0			1			3		
		ε_σ			ε_σ			ε_σ		
		0.01	0.7	1	0.01	0.7	1	0.01	0.7	1
ε_ρ	0.01	0.1	6.1	9.3	2.8	9.3	12.8	16.1	24.1	28.5
	0.7	0.8	7.3	10.8	3.6	10.6	14.5	17.0	25.7	30.7
	1	1.1	7.8	11.5	3.9	11.2	15.3	17.3	26.4	31.5
No IO Linkages		0			2.3			17.8		

Table 4: TFP gain from lowering the average markup to average US level (identical markups) (percents)

		Final Good Elasticity of Substitution								
		0			1			3		
		ε_σ			ε_σ			ε_σ		
		0.01	0.7	1	0.01	0.7	1	0.01	0.7	1
ε_ρ	0.01	0.1	3.6	5.3	1.4	5.1	6.9	4.1	8.3	10.4
	0.7	0.5	4.2	6.0	1.9	5.8	7.7	4.8	9.2	11.4
	1	0.7	4.5	6.3	2.1	6.1	8.1	5.0	9.6	11.9
No IO Linkages		0			0			0		

intermediate input and labor usage across sectors.

In our theoretical analysis, we show that distortions affect output in two distinct ways: first, the inefficient use of intermediates increases the cost (in terms of the primary input, labor) of producing different varieties of goods; second, the relative prices deviate from the costs of production, which exacerbates output losses. Our first theoretical result is that distortions are not systematically amplified by intermediate input linkages. When only the primary input decision is distorted and when inputs are complement, input-output linkages actually dampen the TFP loss from distortions. Our second theoretical result concerns the elasticity of substitution. We find that a higher degree of substitution *worsens* misallocation. The reason is that distortions modify the relative prices that producers face, but since these relative prices do not reflect actual resource costs, it is optimal for them to be ignored.

We apply the model to study the consequences of market power for aggregate productivity. Our framework highlights an additional channel through which imperfect competition reduces welfare. We show how markups distort the allocation of inputs across sectors and reduce aggregate productivity. In our quantitative exercise, we focus on Mexico which has both high average markups and a large dispersion of markups across sectors. We calibrate the model on Mexican sectoral-level data and compute the increase in aggregate TFP obtained from lowering markups to the level observed in the US. In line with our theoretical analysis, we find that the value of the elasticities of substitution is an important determinant of the size of the increase in aggregate productivity. Our quantitative results indicate that, for plausible value of the elasticity of substitution between inputs, reducing markups could boost aggregate productivity in Mexico by 15 percent.

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A Calculation details

A.1 Allocation with frictions

If a planner maximizes output (and welfare) subject to the production functions (1)- (3) and resource constraints (4) - (5), the optimal allocation will satisfy:

$$\frac{dY}{dC_i} = \lambda_i$$

$$\lambda_i \frac{dQ_i}{dX_{ij}} = \lambda_j$$

$$\lambda_i \frac{dQ_i}{dL_i} = \eta.$$

This implies that conditions (6)- (8) can be considered as deviations from the frictionless allocation.

There may be distortions that don't occur during the production process but change the consumption mix; for example consumption taxes or subsidies. Therefore, we may want to modify (6) as follows:

$$\frac{dY}{dC_i} = \lambda_i(1 + \tau_{iC}),$$

with τ_{iC} being the consumption distortion.

With a change of variables we can normalize $\tau_{iC} = 0$ as follows. Define $\lambda'_i = \lambda_i(1 + \tau_{iC})$. Then the conditions can be rewritten as:

$$\frac{dY}{dC_i} = \lambda'_i$$

$$\lambda'_i \frac{dQ_i}{dX_{ij}} = \lambda'_j \frac{1 + \tau_{iC}}{1 + \tau_{jC}}(1 + \tau_{ij})$$

$$\lambda'_i \frac{dQ_i}{dL_i} = \eta(1 + \tau_{iL})(1 + \tau_{iC}).$$

Define $\tau'_{ij} = \frac{1+\tau_{iC}}{1+\tau_{jC}}(1 + \tau_{ij}) - 1$ and $\tau'_{iL} = (1 + \tau_{iL})(1 + \tau_{iC}) - 1$. Then $\lambda', \tau'_{ij}\tau'_{iL}$ and the original allocation satisfy satisfies (6)- (8).

Proposition A.1 *Suppose that the allocation $\{C_i, Q_i, X_{ij}, L_i\}$ satisfies* (1))- (5). *Then there exist $\{\lambda_i, \tau_{ij}, \tau_{iL}\}$ such that the allocation satisfies* (6)- (8).

Proof. Define λ_i by (6). Define

$$1 + \tau_{ij} = \frac{dQ_i}{dX_{ij}} \frac{dY/dC_i}{dY/dC_j}$$

and

$$\frac{1 + \tau_{iL}}{1 + \tau_{1L}} = \frac{dY/dC_i \frac{dQ_i}{dL_i}}{dY/dC_1 \frac{dQ_1}{dL_1}}.$$

We can set τ_{i1} at an arbitrary value greater than -1. ∎

A.2 Solution of the input-output model, general case

Proof of Proposition 1.

Part (i), (ii) and (iii) of the proposition give the expressions of m_{ij}, d_i and c_i, which are defined as $m_{ij} = X_{ij}/Q_i$, $d_i = L_i/Q_i$ and $c_i = C_i/Y$. There expressions are obtained from the first-order conditions. Equation (6) implies expression (iii). Equation (7) implies:

$$X_{ij} = A_i^{\frac{\sigma}{1-\rho}} v_{ij} X_i \left[\frac{\alpha_i Q_i}{X_i}\right]^{\frac{1-\sigma}{1-\rho}} \left(\frac{\lambda_i}{(1+\tau_{ij})\lambda_j}\right)^{\frac{1}{1-\rho}}.$$

Then plugging in the expression for X_i and solving,

$$X_i = A_i^{\frac{\sigma}{1-\sigma}} \alpha_i Q_i \left[\sum_j v_{ij} \left(\frac{\lambda_i}{(1+\tau_{ij})\lambda_j}\right)^{\frac{\rho}{1-\rho}}\right]^{\frac{1-\rho}{\rho}\frac{1}{1-\sigma}}. \tag{A.1}$$

Combining the two equations gives the expression (ii). Expression (i) is obtained from equation (8):

$$L_i = (A_i B_i)^{\frac{\sigma}{1-\sigma}} (1-\alpha_i) Q_i \left(\frac{\lambda_i}{(1+\tau_{iL})(1+\tau_L)\eta}\right)^{\frac{1}{1-\sigma}}. \tag{A.2}$$

Then plugging equations (A.1) and (A.2) in the production function (1) yields

$$Q_i = Q_i A_i^{\frac{1}{1-\sigma}} \lambda_i^{\frac{1}{1-\sigma}} \left[(1-\alpha_i) B_i^{\frac{\sigma}{1-\sigma}} [(1+\tau_{iL})(1+\tau_L)\eta]^{-\frac{\sigma}{1-\sigma}} + \alpha_i \left[\sum_j v_{ij} \left(\frac{1}{(1+\tau_{ij})\lambda_j}\right)^{\frac{\rho}{1-\rho}}\right]^{\frac{1-\rho}{\rho}\frac{\sigma}{1-\sigma}}\right]^{\frac{1}{\sigma}},$$

which gives the expression in part v of the proposition.

From equation (6), $\log C_i = \log \beta_i + \log Y - \log \lambda_i$ and from equation (3), $\log Y = \sum_i \beta_i \log C_i - \sum_i \beta_i \log \beta_i$. Combining the two equations, we get $\sum_i \beta_i \log \lambda_i = 0$ and hence $\log \eta^* = -\sum_i \beta_i \log(\lambda_i/\eta^*)$, therefore

$$\log \lambda_i = \log(\lambda_i/\eta^*) - \sum_i \beta_i \log(\lambda_i/\eta^*),$$

which yields expression (iv) of the Proposition.

The resource constraint for good i can be written as

$$\sum_j m_{ji} q_j + c_i = q_i,$$

where $q_i = Q_i/Y$. In matrix form, this is $M'q + c = q$, so $q = (I - M')^{-1}c$. Since $L_i = d_i Q_i = d_i q_i Y$, the resource constraint on labor can be written as $d'(I - M')^{-1}cY = \bar{L}$, which implies the proposition.

■

A.3 Solution of the input-output model with equal elasticities of substitution ($\sigma = \rho$)

Proof of Proposition 3. Substituting $\sigma = \rho$ in (A.1) and (A.2), we get that $X_{ij} = \alpha_i v_{ij} A_i^{\frac{\sigma}{1-\sigma}} [\lambda_i/((1+\tau_{ij})\lambda_j)]^{\frac{1}{1-\sigma}} Q_i$ and $L_i = (1-\alpha_i)(A_i B_i)^{\frac{\sigma}{1-\sigma}} (\lambda_i/[(1+\tau_{ij})(1+\tau_L)\eta))^{\frac{1}{1-\sigma}} Q_i$. Then substituting in the production function for sector i, we get:

$$\left(\frac{\lambda_i}{\eta^*}\right)^{-\frac{\sigma}{1-\sigma}} = (1-\alpha_i)(A_i B_i)^{\frac{\sigma}{1-\sigma}} (1+\tau_{iL})^{-\frac{\sigma}{1-\sigma}} + \alpha_i A_i^{\frac{\sigma}{1-\sigma}} \sum_{j=1}^{n} v_{ij} (1+\tau_{ij})^{-\frac{\sigma}{1-\sigma}} \left(\frac{\lambda_j}{\eta}\right)^{-\frac{\sigma}{1-\sigma}}$$

Expressing this in matrix notation,

$$\hat{\lambda} = \hat{A}\hat{B}\Delta^{Lp}(I - \tilde{\alpha}) + \tilde{\alpha}\hat{A}(\Delta^{Xp} \circ V)\hat{\lambda},$$

which implies (10). The resource constraint (4) and m_{ij} imply that

$$[\tilde{\alpha}\hat{A}(\Delta^{Xq} \circ V)]'\hat{q} + \hat{c}\eta^{*\frac{\sigma}{1-\sigma}} = \hat{q},$$

where $\hat{q}_i = Q_i \lambda_i^{\frac{1}{1-\sigma}}/Y$ and the rest of the notation is defined in the proposition. So,

$$\hat{q} = [I - (\tilde{\alpha}\hat{A}(\Delta^{Xq} \circ V))']^{-1} \hat{c}\eta^{*\frac{\sigma}{1-\sigma}}$$

Finally, the resource constraint on labor can be expressed as $(\mathbf{1} - \alpha)'\hat{A}\hat{B}\Delta^d \hat{q}\eta^{*-\frac{1}{1-\sigma}} Y = \bar{L}$. Then substituting and using the fact that $\log \eta^* = -\beta' \log(\lambda/\eta^*)$ we get the result. ■

A further specialization of our model is the Cobb-Douglas production function, which is a limiting case of our model when $\varepsilon_\sigma \to 1, \varepsilon_\rho \to 1$. Some important studies that utilize Cobb-Douglas production in the context of networks include Long and Plosser (1983), Jones (2013), Bigio and La'o (2016), Acemoglu et al. (2012).

Proposition A.2 *Suppose that $\sigma = \rho = 0$. Then λ_i/η^* is given by*

$$\begin{aligned} \log(\lambda_i/\eta^*) &= -(I - \tilde{\alpha}V)^{-1}((I - \tilde{\alpha})\log B + \log A) + \\ &\quad (I - \tilde{\alpha})V)^{-1}[(I - \tilde{\alpha})\Delta^{Lp} + \tilde{\alpha}(V \circ \Delta^{Xp})]\mathbf{1}, \end{aligned} \tag{A.3}$$

and output by

$$\begin{aligned} \log Y &= \log \bar{L} + \beta'(I - \tilde{\alpha}V)^{-1}((I - \tilde{\alpha})\log B + \log A) - \beta'(I - \tilde{\alpha}V)^{-1}[(I - \tilde{\alpha})\Delta^{Lp} + \tilde{\alpha}(V \circ \Delta^{Xp})]\mathbf{1} \\ &\quad - \log(\beta'(I - \tilde{\alpha}(V \circ \Delta^{Xq}))^{-1}\Delta^{Lq}(\mathbf{1} - \alpha)), \end{aligned}$$

where $\Delta^{Xp}_{ij} = \log(1+\tau_{ij})$, $\Delta^{\ell}_{ii} = \log(1+\tau_{iL})$; $\Delta^{\ell}_{ij} = 0, i \neq j$ *and* $\tilde{\alpha}$ *is a square matrix with* $\tilde{\alpha}_{ii} = \alpha_i$ *and* $\tilde{\alpha}_{ij} = 0$ *if* $i \neq j$ *and* $\Delta^{Xq}_{ij} = (1+\tau_{ij})^{-1}$; $\Delta^{d}_{ii} = (1+\tau_{iL})^{-1}, \Delta^{d}_{ij} = 0, i \neq j$ *are matrices of distortions, and* $\circ$ *denotes the Hadamard (entrywise) product.*

Proof of Proposition A.2.

Substituting $\rho = \sigma = 0$ in the expressions above, we get that $m_{ij} = \alpha_i v_{ij} \lambda_i / ((1+\tau_{ij})\lambda_j)$ and $d_i = (1-\alpha_i)\lambda_i / [\eta(1+\tau_{iL})$.

Then substituting in the resource constraint for sector i implies

$$\sum_k \alpha_k \frac{v_{ki}}{1+\tau_{ki}} \frac{\lambda_k Q_k}{Y} + \beta_i = \frac{\lambda_i Q_i}{Y}$$

$$\hat{q}' = \beta'(I - \tilde{\alpha}(V \circ \Delta^{Xq}))^{-1},$$

where $\hat{q}_i = \lambda_i Q_i / Y$. Since $\sum_i ((1-\alpha_i)/[(1+\tau_{iL})\eta^*])\hat{q}_i Y = \bar{L}$, we get:

$$\log Y = \log \eta^* - \log(\beta'(I - \tilde{\alpha}(V \circ \Delta^{Xq}))^{-1} \Delta^{Lq}(\mathbf{1} - \alpha) + \log \bar{L}.$$

Since $\eta^* = -\beta' \log(\lambda/\eta^*)$, the proposition follows. ■

A.4 A useful normalization

Proof of Proposition 2. First, we show that we can always reparameterize the production function in such a way to have $B_i = 1$ for all i. Let $\{A_i, B_i, \alpha_i, v_{ij}\}_{i,j}$ be some arbitrary parameters of the production function.

Then for all i set

$$B'_i = 1$$

$$\alpha'_i = 1 - \frac{(1-\alpha_i)B_i^{\frac{\sigma}{1-\sigma}}}{(1-\alpha_i)B_i^{\frac{\sigma}{1-\sigma}} + \alpha_i}$$

$$A'_i = A_i[(1-\alpha_i)B_i^{\frac{\sigma}{1-\sigma}} + \alpha_i]^{\frac{1-\sigma}{\sigma}}$$

Inspection shows that for all i and $L_i > 0, X_{ij} > 0$,

$$A_i \left[(1-\alpha_i)^{1-\sigma}(B_i L_i)^{\sigma} + \alpha_i^{1-\sigma}\left(\sum_j v_{ij}^{1-\rho} X_{ij}^{\rho}\right)^{\frac{\sigma}{\rho}}\right]^{\frac{1}{\sigma}} = A'_i \left[(1-\alpha'_i)^{1-\sigma} L_i^{\sigma} + \alpha_i'^{1-\sigma}\left(\sum_j v_{ij}^{1-\rho} X_{ij}^{\rho}\right)^{\frac{\sigma}{\rho}}\right]^{\frac{1}{\sigma}}.$$

Then B'_i, α'_i and A'_i represent the same production function as B_i, α_i and A_i. Hence without loss of generality, we can assume $B_i = 1, \forall i$.

Next, we show that by choosing the units appropriately, we can have $A_i = 1$ for all i. Let $k \in \mathbf{R}^n_{++}$ be an arbitrary vector of unit changes. Define $\bar{Q}_i = k_i Q_i$. Then $X_{ij} = \bar{X}_{ij}/k_j$. Substituting in the production function we obtain:

$$\bar{Q}_i = k_i A_i \left[(1-\alpha_i)^{1-\sigma} L_i^\sigma + \alpha_i^{1-\sigma} \left(\sum_j v_{ij}^{1-\rho} k_j^{-\rho} \bar{X}_{ij}^\rho \right)^{\frac{\sigma}{\rho}} \right]^{\frac{1}{\sigma}}$$

The right-hand side of this expression looks the original production function with v_{ij}/k_j^ρ playing the role of the parameters in the intermediate good aggregator. However, in general $\sum_j v_{ij}/k_j^\rho \neq 1$. To get around this problem, define the new parameters $v_{ij}(k) = \frac{v_{ij} k_j^{-\frac{\rho}{1-\rho}}}{\sum_j v_{ij} k_j^{-\frac{\rho}{1-\rho}}}$, $m_i(k) = \left(\sum_j v_{ij} k_j^{-\frac{\rho}{1-\rho}}\right)^{-\frac{1-\rho}{\rho}}$, $\alpha_i(k) = 1 - \frac{1-\alpha_i}{1-\alpha_i+\alpha_i m_i(k)^{-\frac{\sigma}{1-\sigma}}}$ and

$$A_i(k) = k_i A_i \left[1 - \alpha_i + \alpha_i m_i(k)^{-\frac{\sigma}{1-\sigma}}\right]^{\frac{1-\sigma}{\sigma}}.$$

This parameterization satisfies all the constraints for the production function ($A_i(k) > 0, \alpha_i(k) \in [0,1], v_{ij}(k) \in [0,1], \sum_j v_{ij}(k) = 1$). Again we can verify that

$$k_i A_i \left[(1-\alpha_i)^{1-\sigma} L_i^\sigma + \alpha_i^{1-\sigma} \left(\sum_j v_{ij}^{1-\rho} k_j^{-\rho} \bar{X}_{ij}^\rho \right)^{\frac{\sigma}{\rho}} \right]^{\frac{1}{\sigma}} =$$
$$A_i(k) \left[(1-\alpha_i(k))^{1-\sigma} L_i^\sigma + \alpha_i(k)^{1-\sigma} \left(\sum_j v_{ij}(k)^{1-\rho} \bar{X}_{ij}^\rho \right)^{\frac{\sigma}{\rho}} \right]^{\frac{1}{\sigma}}$$

Hence, the new parameters represent the same production function.

The proposition is equivalent to showing that there exists a vector k such that $A_i(k) = 1 \; \forall i$. Then given the expression of $A_i(k)$ this is:

$$k_i = A_i^{-1} \left[(1-\alpha_i) + \alpha_i m_i(k)^{-\frac{\sigma}{1-\sigma}}\right]^{-\frac{1-\sigma}{\sigma}}, \; i = 1, \dots n \tag{A.4}$$

Denote the mapping defined by the right-hand side of equation (A.4) as $g(k)$, where k and g are $n \times 1$ vectors. Then (A.4) is simply a fixed point. Since g is monotone and continuous, a sufficient condition for it to have fixed points is that if $k \leq \bar{k}$ then $g(k) \leq \bar{k}$ and if $k \geq \underline{k}$ then $g(k) \geq \underline{k}$ for some positive vectors $\underline{k}, \bar{k}$, $\underline{k} \leq \bar{k}$ and the inequalities are elementwise.

If $k(s) = (s, s, \dots s)'$, then $g_i(k(s))/s = A_i^{-1}\left[(1-\alpha_i)s^{\frac{\sigma}{1-\sigma}} + \alpha_i\right]^{-\frac{1-\sigma}{\sigma}}$.

There are two cases to consider.

1. $\sigma < 0$. Since $\lim_{s \to 0} g_i(k(s))/s = \infty$, $g(k(\underline{s})) > \underline{s}$ for some sufficiently small $\underline{s}$. Set $\underline{k} = (\underline{s}, ...\underline{s})'$.

Therefore $g(\underline{k}) > \underline{k}$. Let $k \geq \underline{k}$. Then

$$g(k) \geq g(\min_i k_i, \min_i k_i, \dots) \geq g(\underline{k}) > \underline{k}$$

On the other hand, $\lim_{s\to\infty} g_i(k(s))/s = A_i^{-1}\left[\alpha_i\right]^{-\frac{1-\sigma}{\sigma}}$, so if $A_i > \alpha_i^{-\frac{1-\sigma}{\sigma}}$ for all i, $g(k(s)) < k(s)$ for s sufficiently large. Then set $\bar{k} = (s, \dots s)'$ for s sufficiently large. Showing that $k \leq \bar{k}$ implies $g(k) < \bar{k}$ is the same as the case above.

2. $\sigma > 0$. $\lim_{s\to 0} g_i(k(s))/s = A_i^{-1}\alpha_i^{-\frac{1-\sigma}{\sigma}}$, so if $A_i < \alpha_i^{-\frac{1-\sigma}{\sigma}}$ for all i, $g(k(s)) > k(s)$ for all $s < \underline{s}$ for some $\underline{s}$ sufficiently small. Set $\underline{k} = (\underline{s}, \dots, \underline{s})'$

   On the other hand, $\lim_{s\to\infty} g_i(k(s))/s = 0$, so if $s \leq \bar{s}$, $g(k(s)) < k(s)$ for some $\bar{s}$ sufficiently large. Set $\bar{k} = (\bar{s}, \dots, \bar{s})'$

   Showing that $\underline{k} \leq k \leq \bar{k}$ implies $\underline{k} < g(k) < \bar{k}$ is the same as before.

We can unify the conditions for the two cases as $A_i^{-\frac{\sigma}{1-\sigma}} > \alpha_i$, $i = 1, \dots n$. Let k^* be a fixed point. Then we can set $\bar{A}_i = A_i(k^*), \bar{\alpha}_i(k^*), \bar{v}_{ij} = v_{ij}(k^*)$.

For the redefined variables $\bar{A}_i = B_i = 1$. It is immediate that $\lambda_i/\eta = 1$ satisfy v of Proposition 1. Hence (iv) of proposition 1 implies that $\lambda_i = 1$ for all i. The last two statements of the proposition follow from condition ii of Proposition 1. ■

A.5 Second-Order Approximation of $\log Y$

Lemma A.1 *Suppose that the distortions on intermediate goods are sector-specific, so $\tau_{ij} = \tau_{ik}, \forall i, j, k$. We will denote the vector of intermediate-good distortions by τ^X; similarly τ^L denote the distortions on labor, with τ the vector of all distortions. Suppose that $A = B = \mathbf{1}$. Let $\mathbf{0}$ be a matrix of zeros with dimensions clear from context. Let $s_i \equiv \sum_k \beta_k\omega_{ki}; s_{ij} \equiv \sum_k \beta_k\omega_{ki}\omega_{kj}$. Then:*

$$\frac{d^2}{d\tau_i^L\tau_j^L}\log Y(\mathbf{0}) = \left(\frac{\sigma}{1-\sigma}s_{ij} + s_is_j\right)(1-\alpha_i)(1-\alpha_j), \quad i \neq j$$

$$\frac{d^2}{d\tau_i^L\tau_i^L}\log Y(\mathbf{0}) = \left(\frac{\sigma}{1-\sigma}s_{ii} + s_i^2\right)(1-\alpha_i)^2 - \frac{1}{1-\sigma}s_i(1-\alpha_i)$$

$$\frac{d^2}{d\tau_i^X\tau_j^L}\log Y(\mathbf{0}) = \left[\frac{\sigma}{1-\sigma}s_{ij} + s_is_j - \frac{1}{1-\sigma}s_i\left(\sum_s v_{is}\omega_{sj}\right)\right]\alpha_i(1-\alpha_j)$$

$$\frac{d^2}{d\tau_i^X\tau_j^X}\log Y(\mathbf{0}) = \left[\frac{\sigma}{1-\sigma}s_{ij} + s_is_j - \frac{1}{1-\sigma}\left(s_i\sum_s v_{is}\omega_{sj} + s_j\sum_s v_{js}\omega_{si}\right)\right]\alpha_i\alpha_j$$

$$\frac{d^2}{d\tau_i^X\tau_j^X}\log Y(\mathbf{0}) = \left[\frac{\sigma}{1-\sigma}s_{ii} + s_i^2 - \frac{2}{1-\sigma}s_i\left(\sum_s v_{is}\omega_{si}\right)\right]\alpha_i^2 - \frac{1}{1-\sigma}s_i\alpha_i$$

Proof. If there is no superscript on the distortion, we will mean that it can be either on labor or on intermediates.

Proposition 3 and the assumption that intermediate goods distortions are purchaser-specific implies that if $A_i = B_i = 1$,

$$\hat{\lambda} = [I - \tilde{\alpha}\Delta^{Xp}V)]^{-1}\Delta^{Lp}(\mathbf{1}-\alpha)$$

and

$$\log Y - \log \bar{L} = \frac{1-\sigma}{\sigma}\beta' \log(\hat{\lambda}) - \log(\hat{c}'[I - \tilde{\alpha}(\Delta^{Xq}V)]^{-1}\Delta^{Lq}(\mathbf{1}-\alpha)),$$

where Ξ^{Xp} is a square $n \times n$ matrix with $\Xi^{Xp}ii = (1+\tau_i^X)^{-\frac{\sigma}{1-\sigma}}$ and 0 otherwise; Ξ^{Xq} is defined similarly, but $\Xi^{Xq}ii = (1+\tau_i^X)^{-\frac{1}{1-\sigma}}$ and all the other notations are as in the rest of the paper.

Denote the first term $M(\tau)$ and the second term $N(\tau)$. Call $[I - \tilde{\alpha}(\Delta^{Xp} \circ V)]^{-1} = \Omega_p(\tau)$ and $[I - \tilde{\alpha}(\Delta^{Xq} \circ V)]^{-1} = \Omega_q(\tau)$. Note that $\Omega_p(\mathbf{0}) = \Omega_q(\mathbf{0}) = \Omega$.

For ease of exposition, we derive the expressions in a sequence of claims by 13 steps.

1. For any distortions, τ_i, τ_j, we have that:

$$\frac{d}{d\tau_i}M = \frac{1-\sigma}{\sigma}\sum_k \beta_k \frac{\hat{\lambda}_{k\tau_i}}{\hat{\lambda}_k} \tag{A.5}$$

and

$$\frac{d^2}{d\tau_i d\tau_j}M = \frac{1-\sigma}{\sigma}\sum_k \beta_k \left[\frac{\hat{\lambda}_{k\tau_i\tau_j}}{\hat{\lambda}_k} - \frac{\hat{\lambda}_{k\tau_i}\hat{\lambda}_{k\tau_j}}{\hat{\lambda}_k^2}\right] \tag{A.6}$$

Proof This follows from the definitions.

2. For any distortions τ_i, τ_j:

$$\frac{d}{d\tau_i}N = -\frac{\hat{c}'_{\tau_i}\Omega_q\Delta^{Lq}(\mathbf{1}-\alpha) + \hat{c}'\Omega_{q,\tau_i}\Delta^{Lq}(\mathbf{1}-\alpha) + \hat{c}'\Omega_q\Delta^{Lq}_{\tau_i}(\mathbf{1}-\alpha)}{G}$$

$$\begin{aligned}\frac{d^2}{d\tau_i d\tau_j}N \quad = \quad & -\frac{\hat{c}'_{\tau_i\tau_j}\Omega_q\Delta^{Lq} + \hat{c}'\Omega_{q,\tau_i\tau_j}\Delta^{Lq} + \hat{c}'\Omega_q\Delta^{Lq}_{\tau_i\tau_j}}{G}(\mathbf{1}-\alpha)\\ & -\frac{\hat{c}'_{\tau_i}\Omega_{q,\tau_j}\Delta^{Lq} + \hat{c}'_{\tau_i}\Omega_q\Delta^{Lq}_{\tau_j} + \hat{c}'_{\tau_j}\Omega_{q,\tau_i}\Delta^{Lq}}{G}(\mathbf{1}-\alpha)\\ & -\frac{\hat{c}'\Omega_{q,\tau_i}\Delta^{Lq}_{\tau_j} + \hat{c}'_{\tau_j}\Omega_q\Delta^{Lq}_{\tau_i} + \hat{c}'\Omega_{q,\tau_j}\Delta^{Lq}_{\tau_i}}{G}(\mathbf{1}-\alpha) + \frac{d}{d\tau_i}N\frac{d}{d\tau_j}N\end{aligned}$$

Proof This follows from the fact

$$N_{\tau_i} = -\frac{G_{\tau_i}}{G}$$

$$N_{\tau_i\tau_j} = -\frac{G_{\tau_i\tau_j}}{G} + \frac{G_{\tau_i}G_{\tau_j}}{G^2},$$

and the definition of $G = \hat{c}'\Omega_q\Delta^{Lq}(\mathbf{1}-\alpha)$.

3. Denote $Z^i = \tilde{\alpha}I^iV$, where I^i is a square matrix with $I^i_{ii} = 1$ and 0 elsewhere. Then

$$\frac{d}{d\tau_i^X}\Omega_p(\tau) = -\frac{\sigma}{1-\sigma}(1+\tau_i^X)^{-\sigma/(1-\sigma)-1}\Omega_p Z^i \Omega_p,$$

$$\frac{d^2}{d\tau_i^X d\tau_j^X}\Omega_p(\tau) = \left(\frac{\sigma}{1-\sigma}\right)^2 [(1+\tau_i^X)(1+\tau_j^X)]^{-\frac{\sigma}{1-\sigma}-1}[\Omega_p Z^i \Omega_p Z^j \Omega_p + \Omega_p Z^j \Omega_p Z^i \Omega_p].$$

$$\frac{d^2}{d\tau_i^X d\tau_i^X}\Omega_p(\tau) = \frac{\sigma}{(1-\sigma)^2}(1+\tau_i^X)^{-\frac{\sigma}{1-\sigma}-2}\Omega_p Z^i \Omega_p + 2\left(\frac{\sigma}{1-\sigma}\right)^2 (1+\tau_i^X)^{-\frac{2\sigma}{1-\sigma}-2}\Omega_p Z^i \Omega_p Z^i \Omega_p.$$

Similarly,

$$\frac{d}{d\tau_i^X}\Omega_q(\tau) = -\frac{1}{1-\sigma}(1+\tau_i^X)^{-1/(1-\sigma)-1}\Omega_q Z^i \Omega_q,$$

$$\frac{d^2}{d\tau_i^X d\tau_j^X}\Omega_q(\tau) = \left(\frac{1}{1-\sigma}\right)^2 [(1+\tau_i^X)(1+\tau_j^X)]^{-\frac{1}{1-\sigma}-1}[\Omega_q Z^i \Omega_q Z^j \Omega_q + \Omega_q Z^j \Omega_q Z^i \Omega_q].$$

$$\frac{d^2}{d\tau_i^X d\tau_i^X}\Omega_q(\tau) = \frac{2-\sigma}{(1-\sigma)^2}(1+\tau_i^X)^{-\frac{\sigma}{1-\sigma}-2}\Omega_q Z^i \Omega_q + 2\left(\frac{1}{1-\sigma}\right)^2 (1+\tau_i^X)^{-\frac{2}{1-\sigma}-2}\Omega_q Z^i \Omega_q Z^i \Omega_q.$$

Proof. Let $F(x)$ be a square matrix that depends on the parameter x and is differentiable in x. Then standard linear algebra implies that on the interior of the set where $F(x)$ is invertible, we have that

$$\frac{d}{dx}F^{-1}(x) = -F^{-1}(x)\frac{d}{dx}F(x)F^{-1}(x).$$

This fact and the definitions of Ω_p, Ω_q implies the results.

4. Let $I^i(\mathbf{1}-\alpha) \equiv (\mathbf{1}-\alpha)^i$, that is a column vector that is zero everywhere, except for the i-th row which is $1-\alpha_i$. We have

$$\hat{\lambda}_{\tau_i^X} = -\frac{\sigma}{1-\sigma}(1+\tau_i^X)^{-\frac{\sigma}{1-\sigma}-1}\Omega_p Z^i \Omega_p \Delta^{Lp}(\mathbf{1}-\alpha)$$

$$\hat{\lambda}_{\tau_j^L} = -\frac{\sigma}{1-\sigma}(1+\tau_j^L)^{-\frac{\sigma}{1-\sigma}-1}\Omega_p(\mathbf{1}-\alpha)^j,$$

$$\hat{\lambda}_{\tau_j^L \tau_s^L} = \mathbf{0}, \text{ if } s \neq j$$

$$\hat{\lambda}_{\tau_j^L \tau_j^L} = \frac{\sigma}{(1-\sigma)^2}(1+\tau_j^L)^{-\frac{\sigma}{1-\sigma}-2}\Omega_p(\mathbf{1}-\alpha)^j,$$

$$\hat{\lambda}_{\tau_i^X \tau_j^L} = \left(\frac{\sigma}{1-\sigma}\right)^2 (1+\tau_j^L)^{-\frac{\sigma}{1-\sigma}-1}(1+\tau_i^X)^{-\frac{\sigma}{1-\sigma}-1}\Omega_p Z^i \Omega_p(\mathbf{1}-\alpha)^j.$$

$$\hat{\lambda}_{\tau_i^X \tau_j^X} = \left(\frac{\sigma}{1-\sigma}\right)^2 (1+\tau_j^X)^{-\frac{\sigma}{1-\sigma}-1}(1+\tau_i^X)^{-\frac{\sigma}{1-\sigma}-1}[\Omega_p Z^i \Omega_p Z^j + \Omega_p Z^j \Omega_p Z^i]\Omega_p \Delta^{Lp}(\mathbf{1}-\alpha), \text{ if } i \neq j$$

$$\begin{aligned}\hat{\lambda}_{\tau_i^X \tau_i^X} &= \frac{\sigma}{(1-\sigma)^2}(1+\tau_i^X)^{-\frac{\sigma}{1-\sigma}-2}\Omega_p Z^i \Omega_p \Delta^{Lp}(\mathbf{1}-\alpha) \\ &\quad +2\left(\frac{\sigma}{1-\sigma}\right)^2 (1+\tau_i^X)^{-\frac{2\sigma}{1-\sigma}-2}\Omega_p Z^i \Omega_p Z^i \Omega_p \Delta^{Lp}(\mathbf{1}-\alpha).\end{aligned}$$

Proof These expressions follow from the fact that $\hat{\lambda} = \Omega_p \Delta^{Lp}(\mathbf{1} - \alpha)$ and the derivatives of the matrix Ω_p derived in Step 3.

5. Define $\Omega Z^i \Omega(\mathbf{1}-\alpha) = C^i, \Omega Z^i \Omega Z^j \Omega(\mathbf{1}-\alpha) = D^{ij}, \Omega(\mathbf{1}-\alpha)^i = E^i, \Omega Z^i \Omega(\mathbf{1}-\alpha)^j = F^{ij}$. (Note that they are all column vectors.) Then:

$$\hat{\lambda}_{\tau_i^X}(\mathbf{0}) = -\frac{\sigma}{1-\sigma} C^i$$

$$\hat{\lambda}_{\tau_j^L}(\mathbf{0}) = -\frac{\sigma}{1-\sigma} E^j,$$

$$\hat{\lambda}_{\tau_j^L \tau_s^L}(\mathbf{0}) = \mathbf{0}, \text{ if } s \neq j$$

$$\hat{\lambda}_{\tau_j^L \tau_j^L}(\mathbf{0}) = \frac{\sigma}{(1-\sigma)^2} E^j,$$

$$\hat{\lambda}_{\tau_i^X \tau_j^L}(\mathbf{0}) = \left(\frac{\sigma}{1-\sigma}\right)^2 F^{ij}.$$

$$\hat{\lambda}_{\tau_i^X \tau_j^X}(\mathbf{0}) = \left(\frac{\sigma}{1-\sigma}\right)^2 [D^{ij} + D^{ji}] \text{ if } i \neq j$$

$$\hat{\lambda}_{\tau_i^X \tau_i^X}(\mathbf{0}) = \frac{\sigma}{(1-\sigma)^2} C^i + 2\left(\frac{\sigma}{1-\sigma}\right)^2 D^{ii}.$$

Proof This follows from the fact that for $\tau = \mathbf{0}$, $\Delta^{Lp} = I$, $\Omega_p = \Omega$ and the expressions found in Step 4.

6.

$$\hat{c}_{\tau_i^X}(\mathbf{0}) = \frac{\sigma}{1-\sigma} \beta \circ C^i$$

$$\hat{c}_{\tau_j^L}(\mathbf{0}) = \frac{\sigma}{1-\sigma} \beta \circ E^j,$$

$$\hat{c}_{\tau_i^L \tau_j^L}(\mathbf{0}) = 2\left(\frac{\sigma}{1-\sigma}\right)^2 \beta \circ E^i \circ E^j, \text{ if } s \neq j$$

$$\hat{c}_{\tau_j^L \tau_j^L}(\mathbf{0}) = -\frac{\sigma}{(1-\sigma)^2} \beta \circ E^j + 2\left(\frac{\sigma}{1-\sigma}\right)^2 \beta \circ E^j \circ E^j$$

$$\hat{c}_{\tau_i^X \tau_j^L}(\mathbf{0}) = -\left(\frac{\sigma}{1-\sigma}\right)^2 \beta \circ F^{ij} + 2\left(\frac{\sigma}{1-\sigma}\right)^2 \beta \circ C^i \circ E^j$$

$$\hat{c}_{\tau_i^X \tau_j^X}(\mathbf{0}) = -\left(\frac{\sigma}{1-\sigma}\right)^2 \beta \circ [D^{ij} + D^{ji}] + 2\left(\frac{\sigma}{1-\sigma}\right)^2 \beta \circ C^i \circ C^j \text{ if } i \neq j$$

$$\hat{c}_{\tau_i^X \tau_i^X}(\mathbf{0}) = -\frac{\sigma}{(1-\sigma)^2} \beta \circ C^i - 2\left(\frac{\sigma}{1-\sigma}\right)^2 \beta \circ D^{ii} + 2\left(\frac{\sigma}{1-\sigma}\right)^2 \beta \circ C^i \circ C^i.$$

Proof By definition, $c_k(\tau) = \beta_k / \hat{\lambda}_k$. Then $\hat{c}_{k,\tau_i} = -\beta_k \hat{\lambda}_{k,\tau_i} / \hat{\lambda}_k^2; \hat{c}_{k,\tau_i \tau_j} = -\beta_k \hat{\lambda}_{k,\tau_i \tau_j} / \hat{\lambda}_k^2 + 2\beta_k \hat{\lambda}_{k,\tau_i} \hat{\lambda}_{k,\tau_j} / \hat{\lambda}_k^3$. Then the fact that $\hat{\lambda}(\mathbf{0}) = \mathbf{1}$ and the expressions in Step 5 imply the result.

7.
$$\Delta^{Lq}_{\tau_i^L}(\mathbf{0}) = -\frac{1}{1-\sigma} I^i$$
$$\Delta^{Lq}_{\tau_i^L,\tau_j^L}(\mathbf{0}) = \mathbf{0}, \text{ if } i \neq j$$
$$\Delta^{Lq}_{\tau_i^L\tau_i^L}(\mathbf{0}) = \frac{2-\sigma}{(1-\sigma)^2} I^i$$

   and all derivatives involving τ_i^X are zero matrices.

   Proof Follows directly from the definition.

8. $C_k^i = \alpha_i \omega_{ki}$; $D_k^{ij} = \omega_{ki} \sum_s v_{is} \omega_{sj} \alpha_i \alpha_j$; $E_k^i = (1-\alpha_i)\omega_{ki}$; $F_k^{ij} = \omega_{ki} \sum_s v_{is}\omega_{sj}\alpha_i(1-\alpha_j)$

   Proof This follows from the definition of these vectors and the facts that $\Omega(\mathbf{1}-\alpha) = \mathbf{1}, V\mathbf{1} = \mathbf{1}$.

9. Find $\frac{d}{d\tau_i^L \tau_j^L} \log Y(\mathbf{0}),\ i \neq j$.

   From Step 1 and Step 4, we have

$$\frac{d^2}{d\tau_i^L d\tau_j^L} M = -\frac{\sigma}{1-\sigma} \sum_k \beta_k E_k^j E_k^i = -\frac{\sigma}{1-\sigma}(1-\alpha_i)(1-\alpha_j)s_{ij}$$

   Since $\Omega_q(\mathbf{0}) = \Omega, G(\mathbf{0}) = 1, \Omega(\mathbf{1}-\alpha) = \mathbf{1}$ and some of the terms in the expression for $\frac{d^2}{d\tau_i d\tau_j} N$ are zero matrices, Step 2 implies

$$\begin{aligned} \frac{d^2}{d\tau_i^L d\tau_j^L} N(\mathbf{0}) &= -\left(\hat{c}'_{\tau_i^L\tau_j^L}\Omega + \hat{c}'_{\tau_i^L}\Omega\Delta^{Lq}_{\tau_j^L} + \hat{c}'_{\tau_j^L}\Omega_q\Delta^{Lq}_{\tau_i^L}\right)(\mathbf{1}-\alpha) \\ &\quad + \left[\left(\hat{c}'_{\tau_i^L}\Omega + \hat{c}'\Omega\Delta^{Lq}_{\tau_i^L}\right)(\mathbf{1}-\alpha)\right]\left[\left(\hat{c}'_{\tau_j^L}\Omega + \hat{c}'\Omega\Delta^{Lq}_{\tau_j^L}\right)(\mathbf{1}-\alpha)\right] \end{aligned}$$

   Then substituting the various expressions, we get

$$\begin{aligned} \frac{d^2}{d\tau_i^L d\tau_j^L} N(\mathbf{0}) &= -2\left(\frac{\sigma}{1-\sigma}\right)^2 \sum_k \beta_k E_k^i E_k^j + 2\frac{\sigma}{(1-\sigma)^2}\sum_k \beta_k E_k^i E_k^j \\ &\quad + \left(\sum_k \beta_k E_k^i\right)\left(\sum_k \beta_k E_k^j\right) \\ &= \frac{2\sigma}{1-\sigma} s_{ij}(1-\alpha_i)(1-\alpha_j) + s_i s_j (1-\alpha_i)(1-\alpha_j) \end{aligned}$$

   Then summing the derivatives of M and N, we get that

$$\frac{d^2}{d\tau_i^L \tau_j^L} \log Y(\mathbf{0}) = \left(\frac{\sigma}{1-\sigma} s_{ij} + s_i s_j\right)(1-\alpha_i)(1-\alpha_j)$$

   This proves the first statement of the lemma.

10. Find $\frac{d}{d\tau_i^L \tau_i^L} \log Y(\mathbf{0})$.

In the same way as Step 9, we see that

$$\frac{d^2}{d\tau_i^L d\tau_j^L} M = \frac{1}{1-\sigma}\sum_k \beta_k E_k^i - \frac{\sigma}{1-\sigma}\sum_k \beta_k E_k^i E_k^i = \frac{1}{1-\sigma} s_i(1-\alpha_i) - \frac{\sigma}{1-\sigma}(1-\alpha_i)^2 s_{ii}$$

$$\begin{aligned}
\frac{d^2}{d\tau_i^L d\tau_i^L} N(\mathbf{0}) &= -\left(\hat{c}'_{\tau_i^L\tau_i^L}\Omega + 2\hat{c}'_{\tau_i^L}\Omega\Delta^{Lq}_{\tau_i^L} + \hat{c}'\Omega\Delta^{Lq}_{\tau_i\tau_i}\right)(\mathbf{1}-\alpha) + \left[\left(\hat{c}'_{\tau_i^L}\Omega + \hat{c}'\Omega\Delta^{Lq}_{\tau_i^L}\right)(\mathbf{1}-\alpha)\right]^2 \\
&= \frac{\sigma}{(1-\sigma)^2}\sum_k \beta_k E_k^i - 2\left(\frac{\sigma}{1-\sigma}\right)^2 \sum_k \beta_k E_k^i E_k^i + 2\frac{\sigma}{(1-\sigma)^2}\sum_k \beta_k E_k^i E_k^i \\
&\quad - \frac{2-\sigma}{(1-\sigma)^2}\sum_k \beta_k E_k^i + s_i^2(1-\alpha_i)^2 \\
&= \frac{2\sigma}{1-\sigma} s_{ii}(1-\alpha_i)^2 + s_i^2(1-\alpha_i)^2 - \frac{2}{1-\sigma} s_i(1-\alpha_i) + s_i^2(1-\alpha_i)^2
\end{aligned}$$

Then summing the derivatives of M and N, we get that

$$\frac{d^2}{d\tau_i^L \tau_i^L}\log Y(\mathbf{0}) = \left(\frac{\sigma}{1-\sigma} s_{ii} + s_i^2\right)(1-\alpha_i)^2 - \frac{1}{1-\sigma} s_i(1-\alpha_i)$$

This proves the second statement of the lemma.

11. Find $\frac{d^2}{d\tau_i^X d\tau_j^L}\log Y(\mathbf{0})$.

In the same way as Step 9, we see that

$$\frac{d^2}{d\tau_i^X d\tau_j^L} M(\mathbf{0}) = \frac{\sigma}{1-\sigma}\sum_k \beta_k F_k^{ij} - \frac{\sigma}{1-\sigma}\sum_k \beta_k C_k^i E_k^j = \frac{\sigma}{1-\sigma}\alpha_i(1-\alpha_j)[s_i \sum_s v_{is}\omega_{sj} - s_{ij}]$$

Taking into account the zero terms,

$$\begin{aligned}
\frac{d^2}{d\tau_i^X d\tau_j^L} N(\mathbf{0}) &= -\left[\hat{c}'_{\tau_i^X\tau_j^L}\Omega + \hat{c}'_{\tau_i^X}\Omega\Delta^{Lq}_{\tau_j^L} + \hat{c}'_{\tau_j^L}\Omega_{q,\tau_i^X} + \hat{c}'\Omega_{q,\tau_i^X}\Delta^{Lq}_{\tau_j^L}\right](\mathbf{1}-\alpha) \\
&\quad + \left[\hat{c}'_{\tau_i^X}\Omega + \hat{c}'\Omega_{q,\tau_i^X}\right](\mathbf{1}-\alpha)\left[\hat{c}'_{\tau_j^L}\Omega + \hat{c}'\Omega\Delta^{Lq}_{\tau_j^L}\right](\mathbf{1}-\alpha) \\
&= \left(\frac{\sigma}{1-\sigma}\right)^2 \sum_k \beta_k F_k^{ij} - 2\left(\frac{\sigma}{1-\sigma}\right)^2 \sum_k \beta_k C_k^i E_k^j \\
&\quad + \frac{\sigma}{(1-\sigma)^2}\sum_k \beta_k C_k^i E_k^j + \frac{\sigma}{(1-\sigma)^2}\sum_k \beta_k C_k^i E_k^j \\
&\quad - \frac{1}{(1-\sigma)^2}\sum_k \beta_k F_k^{ij} + \left(\sum_k \beta_k C_k^i\right)\left(\sum_k \beta_k E_k^j\right) \\
&= \left\{-\frac{1+\sigma}{1-\sigma} s_i\left(\sum_s v_{is}\omega_{sj}\right) + \frac{2\sigma}{1-\sigma} s_{ij} + s_i s_j\right\}\alpha_i(1-\alpha_j)
\end{aligned}$$

Then summing the derivatives of M and N, we get that

$$\frac{d^2}{d\tau_i^X \tau_j^L} \log Y(\mathbf{0}) = \left[\frac{\sigma}{1-\sigma} s_{ij} + s_i s_j - \frac{1}{1-\sigma} s_i \left(\sum_s v_{is}\omega_{sj}\right)\right] \alpha_i(1-\alpha_j)$$

This proves the third statement of the lemma.

12. Find $\frac{d^2}{d\tau_i^X \tau_j^X} \log Y(\mathbf{0})$. In the same way as Step 9, we see that

$$\begin{aligned}
\frac{d^2}{d\tau_i^X d\tau_j^X} M(\mathbf{0}) &= \frac{\sigma}{1-\sigma} \sum_k \beta_k [D_k^{ij} + D_k^{ji}] - \frac{\sigma}{1-\sigma} \sum_k \beta_k C_k^i C_k^j \\
&= \frac{\sigma}{1-\sigma} \left(s_i \sum_s v_{is}\omega_{sj} + s_j \sum_s v_{js}\omega_{si} - s_{ij}\right) \alpha_i \alpha_j
\end{aligned}$$

Taking into account the zero terms,

$$\begin{aligned}
\frac{d^2}{d\tau_i^X d\tau_j^X} N(\mathbf{0}) &= -\left[\hat{c}'_{\tau_i^X \tau_j^X} \Omega + \hat{c}' \Omega_{q,\tau_i^X \tau_j^X} + \hat{c}'_{\tau_i^X} \Omega_{q,\tau_j^X} + \hat{c}'_{\tau_j^X} \Omega_{q,\tau_i^X}\right] (\mathbf{1} - \alpha) \\
&\quad + \left(\hat{c}'_{\tau_i^X} \Omega + \hat{c}' \Omega_{q,\tau_i^X}\right) (\mathbf{1} - \alpha) \left(\hat{c}'_{\tau_j^X} \Omega + \hat{c}' \Omega_{q,\tau_j^X}\right) (\mathbf{1} - \alpha) \\
&= \left(\frac{\sigma}{1-\sigma}\right)^2 \sum_k \beta_k [D_k^{ij} + D_k^{ji}] - 2\left(\frac{\sigma}{1-\sigma}\right)^2 \sum_k \beta_k C_k^i C_k^j \\
&\quad - \left(\frac{1}{1-\sigma}\right)^2 \sum_k \beta_k [D_k^{ij} + D_k^{ji}] + \frac{\sigma}{(1-\sigma)^2} \sum_k \beta_k C_k^i C_k^j + \frac{\sigma}{(1-\sigma)^2} \sum_k \beta_k C_k^i C_k^j \\
&\quad + \left(\sum_k \beta_k C_k^i\right) \left(\sum_k \beta_k C_k^j\right) \\
&= \left[-\frac{1+\sigma}{1-\sigma} \left(s_i \sum_s v_{is}\omega_{sj} + s_j \sum_s v_{js}\omega_{si}\right) + \frac{2\sigma}{1-\sigma} s_{ij} + s_i s_j\right] \alpha_i \alpha_j
\end{aligned}$$

Then summing the derivatives of M and N, we get that

$$\frac{d^2}{d\tau_i^X \tau_j^X} \log Y(\mathbf{0}) = \left[\frac{\sigma}{1-\sigma} s_{ij} + s_i s_j - \frac{1}{1-\sigma} \left(s_i \sum_s v_{is}\omega_{sj} + s_j \sum_s v_{js}\omega_{si}\right)\right] \alpha_i \alpha_j$$

This proves the fourth statement of the lemma.

13. Find $\frac{d^2}{d\tau_i^X d\tau_i^X} \log Y(\mathbf{0})$.

In the same way as Step 9, we see that

$$\begin{aligned}
\frac{d^2}{d\tau_i^X d\tau_i^X} M(\mathbf{0}) &= \frac{1}{1-\sigma} \sum_k \beta_k C_k^i + \frac{2\sigma}{1-\sigma} \sum_k \beta_k D_k^{ii} - \frac{\sigma}{1-\sigma} \sum_k \beta_k C_k^i C_k^i \\
&= \frac{\sigma}{1-\sigma} \left(2 s_i \sum_s v_{is}\omega_{si} - s_{ij}\right) \alpha_i^2 + \frac{1}{1-\sigma} s_i
\end{aligned}$$

Taking into account the zero terms,

$$
\begin{aligned}
\frac{d^2}{d\tau_i^X d\tau_j^X} N(\mathbf{0}) &= -\left[\hat{c}'_{\tau_i^X \tau_i^X}\Omega + \hat{c}'\Omega_{q,\tau_i^X \tau_i^X} + 2\hat{c}'_{\tau_i^X}\Omega_{q,\tau_i^X} + \right](\mathbf{1}-\alpha) + \left[\left(\hat{c}'_{\tau_i^X}\Omega + \hat{c}'\Omega_{q,\tau_i^X}\right)(\mathbf{1}-\alpha)\right]^2 \\
&= \frac{\sigma}{(1-\sigma)^2}\sum_k \beta_k C_k^i + 2\left(\frac{\sigma}{1-\sigma}\right)^2 \sum_k \beta_k D_k^{ii} - 2\left(\frac{\sigma}{1-\sigma}\right)^2 \sum_k \beta_k C_k^i C_k^i \\
&\quad -2\left(\frac{1}{1-\sigma}\right)^2 \sum_k \beta_k D_k^{ii} - \frac{2-\sigma}{(1-\sigma)^2}\sum_k \beta_k C_k^i \\
&\quad + \frac{2\sigma}{(1-\sigma)^2}\sum_k \beta_k C_k^i C_k^i + \left(\sum_k \beta_k C_k^i\right)^2 \\
&= \left[-2\frac{1+\sigma}{1-\sigma} s_i \left(\sum_s v_{is}\omega_{si}\right) + \frac{2\sigma}{1-\sigma} s_{ii} + s_i^2\right]\alpha_i^2 - \frac{2}{1-\sigma} s_i \alpha_i
\end{aligned}
$$

Then summing the derivatives of M and N, we get that

$$
\frac{d^2}{d\tau_i^X \tau_j^X} \log Y(\mathbf{0}) = \left[\frac{\sigma}{1-\sigma} s_{ii} + s_i^2 - \frac{2}{1-\sigma} s_i \left(\sum_s v_{is}\omega_{si}\right)\right]\alpha_i^2 - \frac{1}{1-\sigma} s_i \alpha_i
$$

This concludes the proof.

■

Lemma A.2 *Let* $g_{ij} \equiv \sum_r v_{ir}\omega_{rj}$. *Then* $\alpha_i g_{ij} = \omega_{ij}, \alpha_i g_{ii} = \omega_{ii} - 1$.

Proof. Let $G = (\alpha_i g_{ij})$. . Direct inspection shows that $G = \tilde{\alpha} V \Omega$. Since $\Omega = (I - \tilde{\alpha} V)^{-1}$, we have that $I = (I - \tilde{\alpha} V)\Omega = \Omega - G$, so $G = \Omega - I$, which proves the claim. ■

A.6 Characterization of aggregate TFP in the presence of labor distortions

Proof of Proposition 4. From lemma A.1 and the fact that $Y = 1$ when there are no distortions, Taylor's expansion implies that

$$
\log TFP \approx \frac{1}{2}\left(\sum_i s_i(1-\alpha_i)\tau_{iL}\right)^2 + \frac{1}{2}\frac{\sigma}{1-\sigma}\sum_i \sum_j s_{ij}(1-\alpha_i)(1-\alpha_j)\tau_{iL}\tau_{jL} - \frac{1}{2}\frac{1}{1-\sigma}\sum_i s_i(1-\alpha_i)\tau_{iL}^2,
$$

which concludes the proof. ■

Derivation of the role of the cross-sectional variance of distortions (Equation (11)).

We can rewrite the approximation derived above as

$$
\begin{aligned}
\log TFP \quad \approx \quad & \frac{1}{2}\left(n\frac{1}{n}\sum_i s_i(1-\alpha_i)\tau_{iL}\right)^2 + \frac{1}{2}\frac{\sigma}{1-\sigma}\left(n\frac{1}{n}\sum_i\sum_{j\neq i} s_{ij}(1-\alpha_i)(1-\alpha_j)\tau_{iL}\tau_{jL} + n\frac{1}{n}\sum_i s_{ii}(1-\alpha_i)^2\tau_{iL}^2\right) \\
& -\frac{1}{2}\frac{1}{1-\sigma}n\frac{1}{n}\sum_i s_i(1-\alpha_i)\tau_{iL}^2
\end{aligned}
$$

With the assumptions that the distortions are independent of any other sector parameters, and are independently distributed with mean $\bar{\tau_L}$ and cross-sectional variance $V(\tau_{iL})$, and using the Law of large numbers, we have

$$
\begin{aligned}
\log TFP \quad \approx \quad & \frac{1}{2}\left((\sum_i s_i(1-\alpha_i))\bar{\tau}_L)\right)^2 + \frac{1}{2}\frac{\sigma}{1-\sigma}\left(\sum_i\sum_{j\neq i} s_{ij}(1-\alpha_i)(1-\alpha_j)\bar{\tau}_L^2 + \sum_i s_{ii}(1-\alpha_i)^2 E(\tau_{iL}^2)\right) \\
& -\frac{1}{2}\frac{1}{1-\sigma}\sum_i s_i(1-\alpha_i)E(\tau_{iL}^2)
\end{aligned}
$$

Which can be rewritten as

$$
\begin{aligned}
\log TFP \quad \approx \quad & \frac{1}{2}\bar{\tau}_L^2\left(\sum_i s_i(1-\alpha_i)\right)^2 - \frac{1}{2}\frac{1}{1-\sigma}E(\tau_{iL}^2)\sum_i s_i(1-\alpha_i) \\
& +\frac{1}{2}\frac{\sigma}{1-\sigma}\left(\bar{\tau}_L^2\sum_i\sum_j s_{ij}(1-\alpha_i)(1-\alpha_j) - \bar{\tau}_L^2\sum_i s_{ii}(1-\alpha_i)^2 + E(\tau_{iL}^2)\sum_i s_{ii}(1-\alpha_i)^2\right)
\end{aligned}
$$

After re-arranging the terms, we obtain

$$
\begin{aligned}
\log TFP \quad \approx \quad & \frac{1}{2}\bar{\tau}_L^2\left(\sum_i s_i(1-\alpha_i)\right)^2 - \frac{1}{2}\sum_i s_i(1-\alpha_i)E(\tau_{iL}^2) - \frac{1}{2}\frac{\sigma}{1-\sigma}E(\tau_{iL}^2)\sum_i s_i(1-\alpha_i) \\
& +\frac{1}{2}\frac{\sigma}{1-\sigma}\left(\bar{\tau}_L^2\sum_i\sum_j s_{ij}(1-\alpha_i)(1-\alpha_j) - \bar{\tau}_L^2\sum_i s_{ii}(1-\alpha_i)^2 + E(\tau_{iL}^2)\sum_i s_{ii}(1-\alpha_i)^2\right)
\end{aligned}
$$

Finally, using $\sum_i s_i(1-\alpha_i) = 1$, and noticing that $\sum_i\sum_j s_{ij}(1-\alpha_i)(1-\alpha_j) = \sum_i s_i(1-\alpha_i) = 1$, we have

$$
\log TFP \quad \approx \quad -\frac{1}{2}V(\tau_{iL}) - \frac{1}{2}\frac{\sigma}{1-\sigma}\left(1-\sum_i s_{ii}(1-\alpha_i)^2\right)V(\tau_{iL}),
$$

with $1-\sum_i s_{ii}(1-\alpha_i)^2 \geq 0$. To see this, recall that $\sum_i \omega_{ki}(1-\alpha_i) = 1$. Therefore, $\omega_{ki}(1-\alpha_i) \leq 1$ for all i, and $\omega_{ki}^2(1-\alpha_i)^2 \leq \omega_{ki}(1-\alpha_i)$ which gives $\sum_i \omega_{ki}^2(1-\alpha_i)^2 \leq 1$, hence $\sum_i s_{ii}(1-\alpha_i)^2 \equiv \sum_k \beta_k \sum_i \omega_{ki}^2(1-\alpha_i)^2 \leq 1$.

A.7 Characterization of aggregate TFP in the presence of intermediate input distortions

Proof of Proposition 5. The proof follows directly from Taylor expansion, the second-order derivatives found in Lemma A.1 and the definition $g_{ij} \equiv \sum_r v_{ir}\omega_{rj}$. ■

As we will show below we can write $\alpha_i g_{ij} = \omega_{ij}$ for $i \neq j$ and $\alpha_i g_{ii} = \omega_{ii} - 1$.

Proof of equation (12)**.**

Let us now write TFP as a function of the variance of distortions. We have

$$\begin{aligned}
\log TFP \approx & -\frac{1}{2}\frac{1}{1-\sigma}n\frac{1}{n}\sum_i\sum_{j\neq i}\alpha_i\alpha_j\left(s_i g_{ij}+s_j g_{ji}\right)\tau_i\tau_j - \frac{1}{1-\sigma}n\frac{1}{n}\sum_i \alpha_i^2 s_i g_{ii}\tau_i^2 \\
& +\frac{1}{2}\frac{\sigma}{1-\sigma}n\frac{1}{n}\sum_i\sum_{j\neq i}\alpha_i\alpha_j s_{ij}\tau_i\tau_j + \frac{1}{2}\frac{\sigma}{1-\sigma}n\frac{1}{n}\sum_i \alpha_i^2 s_{ii}\tau_i^2 \\
& +\frac{1}{2}\left(n\frac{1}{n}\sum_i \alpha_i s_i \tau_i\right)^2 - \frac{1}{2}\frac{1}{1-\sigma}n\frac{1}{n}\sum_i \alpha_i s_i \tau_i^2
\end{aligned}$$

With the assumptions that the distortions are independent of any other sector parameters, and are independently distributed, and using the Law of large numbers, we have

$$\begin{aligned}
\log TFP \approx & -\frac{1}{2}\frac{1}{1-\sigma}\left(\sum_i\sum_j \alpha_i\alpha_j(s_i g_{ij}+s_j g_{ji})\right)\bar{\tau}^2 - \frac{1}{1-\sigma}\left(\sum_i \alpha_i^2 s_i g_{ii}\right)\left(E(\tau_i^2)-\bar{\tau}^2\right) \\
& +\frac{1}{2}\frac{\sigma}{1-\sigma}\left(\sum_i\sum_j \alpha_i\alpha_j s_{ij}\right)\bar{\tau}^2 + \frac{1}{2}\frac{\sigma}{1-\sigma}\left(\sum_i \alpha_i^2 s_{ii}\right)\left(E(\tau_i^2)-\bar{\tau}^2\right) + \frac{1}{2}\left(\sum_i \alpha_i s_i\right)^2\bar{\tau}^2 \\
& -\frac{1}{2}\frac{1}{1-\sigma}\left(\sum_i \alpha_i s_i\right)\left(E(\tau_i^2)-\bar{\tau}^2\right) - \frac{1}{2}\frac{1}{1-\sigma}\left(\sum_i \alpha_i s_i\right)\bar{\tau}^2
\end{aligned}$$

After re-arranging, we obtain

$$\begin{aligned}
\log TFP \approx & -\frac{1}{2}\left[\sum_i\sum_j \alpha_i\alpha_j(s_i g_{ij}+s_j g_{ji}) + \sum_i \alpha_i s_i - \left(\sum_i \alpha_i s_i\right)^2\right]\bar{\tau}^2 \\
& -\frac{1}{2}\frac{\sigma}{1-\sigma}\left(\sum_i\sum_j \alpha_i\alpha_j(s_i g_{ij}+s_j g_{ji}) + \sum_i \alpha_i s_i - \sum_i\sum_j \alpha_i\alpha_j s_{ij}\right)\bar{\tau}^2 \\
& -\frac{1}{2}\left(\sum_i \alpha_i s_i + 2\alpha_i^2 s_i g_{ii}\right)V(\tau_i) - \frac{1}{2}\frac{\sigma}{1-\sigma}\left(\sum_i \alpha_i s_i + 2\alpha_i^2 s_i g_{ii} - \alpha_i^2 s_{ii}\right)V(\tau_i)
\end{aligned}$$

We now show that $\sum_i \alpha_i s_i + 2\alpha_i^2 s_i g_{ii} - \alpha_i^2 s_{ii} \geq 0$

$$\begin{aligned}
\sum_i \alpha_i s_i + 2\alpha_i^2 s_i g_{ii} - \alpha_i^2 s_{ii} &= \sum_i \alpha_i \sum_k \beta_k \omega_{ki} + 2\alpha_i \sum_k \beta_k \omega_{ki}(\omega_{ii} - 1) - \alpha_i^2 \sum_k \beta_k \omega_{ki}^2 \\
&= \sum_i \sum_k \beta_k \alpha_i \omega_{ki}(1 + 2(\omega_{ii} - 1) - \alpha_i \omega_{ki}) \\
&= \sum_i \sum_k \beta_k \alpha_i \omega_{ki}[(\omega_{ii} - 1) + (\omega_{ii} - \alpha_i \omega_{ki})]
\end{aligned}$$

The expression is positive since $\omega_{ii} > 1$ and $\omega_{ki} < \omega_{ii}$. ■

Proof of Proposition 6. Since $\tau_i^X = \tau_i^L = \tau_i$, we have that for any i, j (including $i = j$)

$$\frac{d^2}{d\tau_i d\tau_j} \log TFP = \frac{d^2}{d\tau_i^X d\tau_j^X} \log TFP + \frac{d^2}{d\tau_i^X d\tau_j^L} \log TFP + \frac{d^2}{d\tau_i^L d\tau_j^X} \log TFP + \frac{d^2}{d\tau_i^L d\tau_j^L} \log TFP.$$

Then applying lemma A.1 and grouping terms, we obtain for $i \neq j$:

$$\begin{aligned}
\frac{d^2}{d\tau_i d\tau_j} \log TFP(\mathbf{1}) &= \frac{\sigma}{1-\sigma} s_{ij} + s_i s_j - \frac{1}{1-\sigma}[s_i \alpha_i g_{ij} + s_j \alpha_j g_{ji}] \\
&= \frac{\sigma}{1-\sigma} s_{ij} + s_i s_j - \frac{1}{1-\sigma}[s_i \omega_{ij} + s_j \omega_{ji}],
\end{aligned}$$

where we used lemma A.2 in the second line.

Similarly, if $i = j$

$$\begin{aligned}
\frac{d^2}{d\tau_i d\tau_i} \log TFP &= \frac{\sigma}{1-\sigma} s_{ii} + s_i^2 - 2\frac{1}{1-\sigma} s_i \alpha_i g_{ii} - \frac{1}{1-\sigma} s_i \\
&= \frac{\sigma}{1-\sigma} s_{ii} + s_i^2 - 2\frac{1}{1-\sigma} s_i \omega_{ii} + \frac{1}{1-\sigma} s_i,
\end{aligned}$$

where we used that $\alpha_i g_{ii} = \omega_{ii} - 1$ (lemma A.2) in the second line.

Then the result follows from Taylor's expansion around $\tau_i = 0, \forall i$. ■

A.8 Proofs of Proposition 7 and 8

Lemma A.3 $(I - \tilde{\alpha} V)^{-1}(\mathbf{1} - \alpha) = \mathbf{1}$.

Proof. Since $V\mathbf{1} = \mathbf{1}$, and $\alpha = \tilde{\alpha}\mathbf{1} = \tilde{\alpha} V \mathbf{1}$ we have that

$$\mathbf{1} - \alpha = I\mathbf{1} - \tilde{\alpha} V \mathbf{1} = (I - \tilde{\alpha} V)\mathbf{1}.$$

Premultiplying the equality above by $(I - \tilde{\alpha} V)^{-1}$ implies the result. ■

Proof of Proposition 7. Denote $\tilde{\alpha}\hat{A}V = Z$ and $[I - \tilde{\alpha} z_i(\tau) V]^{-1} = H_i(\tau)$. Then by Proposition 3, we can express the equilibrium as:

$$\hat{\lambda}(\tau) = H_1(\tau)\hat{A}\hat{B}(\mathbf{1}-\alpha)$$

$$\log Y(\tau) = \log \bar{L} + \frac{1-\sigma}{\sigma}\beta' \log \hat{\lambda}(\tau) - \log(\hat{c}'(\tau)H_2(\tau)\hat{A}\hat{B}(\mathbf{1}-\alpha)),$$

$z_1(\tau) = (1+\tau)^{-\frac{\sigma}{1-\sigma}}$ and $z_2(\tau) = (1+\tau)^{-\frac{1}{1-\sigma}}$ and $\hat{c}_i = \beta_i/\hat{\lambda}_i$.

Let $M(\tau) = \frac{1-\sigma}{\sigma}\beta' \log \hat{\lambda}(\tau)$ and $N(\tau) = -\log(G(\tau)) = -\log(\hat{c}'(\tau)H_2(\tau)\hat{A}\hat{B}(\mathbf{1}-\alpha))$. Then with this notation,

$$\log Y(\tau) = \log \bar{L} + M(\tau) + N(\tau).$$

We will work on differentiating $M(\tau)$ and $N(\tau)$.

Lemma A.3 implies that $\frac{d}{d\tau}H_i(\tau) = ZH_i(\tau)^2 z_{i\tau}(\tau)$ and $\frac{d^2}{d\tau^2}H_i(\tau) = ZH_i(\tau)^2 z_{i\tau\tau}(\tau) + 2Z^2 H_i(\tau)^3 z_{i\tau}(\tau)^2$. So,

$$\lambda_\tau(\tau) = H_{1\tau}(\tau)\hat{A}\hat{B}(\mathbf{1}-\alpha) = ZH_1\hat{\lambda}z_{1\tau}$$

$$\lambda_{\tau\tau}(\tau) = H_{1\tau\tau}(\tau)\hat{A}\hat{B}(\mathbf{1}-\alpha) = ZH_1(\tau)\hat{\lambda}z_{1\tau\tau} + 2Z^2H_1(\tau)^2\hat{\lambda}z_{1\tau}(\tau)^2$$

$$\frac{d}{d\tau}M(\tau) = \frac{1-\sigma}{\sigma}\sum_i \beta_i \frac{\hat{\lambda}_{i\tau}}{\hat{\lambda}_i} = \frac{1-\sigma}{\sigma}z_{1\tau}(\tau)\sum_i \beta_i \frac{(ZH_1(\tau)\hat{\lambda})_i}{\hat{\lambda}_i}$$

$$\frac{d^2}{d\tau^2}M(\tau) = \frac{1-\sigma}{\sigma}\sum_i \beta_i \left[\frac{\hat{\lambda}_{i\tau\tau}}{\hat{\lambda}_i} - \frac{\hat{\lambda}_{i\tau}^2}{\hat{\lambda}_i^2}\right]$$

Note that $H_1(0) = H_2(0) = H$. Denote $ZH\mathbf{1} = a$ and $2Z^2H^2\mathbf{1} = b$. Then if we assume $A = B = \mathbf{1}$ and evaluating the above expressions at $\tau = 0$, we get

$$\lambda_\tau = z_{1\tau}a, \quad \lambda_{\tau\tau} = z_{1\tau\tau}a + z_{1\tau}^2 b$$

$$\frac{d}{d\tau}M(\tau) = \frac{1-\sigma}{\sigma}z_{1\tau}(0)\sum_i \beta_i a_i$$

$$\frac{d^2}{d\tau^2}M(\tau) = \frac{1-\sigma}{\sigma}\sum \beta_i[z_{1\tau\tau}a_i + z_{1\tau}^2 b_i - z_{1\tau}^2 a_i^2]$$

Next, we turn to N.

$$N_\tau = -\frac{G_\tau}{G}$$

$$N_{\tau\tau} = -\frac{G_{\tau\tau}}{G} + \frac{G_\tau^2}{G^2}$$

$$\hat{c}_{i\tau}(\tau) = -\beta_i \frac{\hat{\lambda}_{i\tau}(\tau)}{\hat{\lambda}_i^2}, \quad \hat{c}_{i\tau}(0) = -\beta_i a_i z_{1\tau}(0).$$

$$\hat{c}_{i\tau\tau}(\tau) = -\beta_i \frac{\hat{\lambda}_{i\tau\tau}(\tau)}{\hat{\lambda}_i^2} + 2\beta_i \frac{(\hat{\lambda}_{i\tau}(\tau))^2}{\hat{\lambda}_i^3}, \quad \hat{c}_{i\tau\tau}(0) = -\beta_i z_{1\tau\tau}a_i - \beta_i z_{i\tau}^2 b_i + 2\beta_i z_{1\tau}^2 a_i^2$$

$$G_\tau = \hat{c}'_\tau H_2(\mathbf{1}-\alpha) + \hat{c}' H_{2\tau}(\mathbf{1}-\alpha)$$

$$G_{\tau\tau} = \hat{c}'_{\tau\tau} H_2(\mathbf{1}-\alpha) + 2\hat{c}'_\tau H_{2\tau}(\mathbf{1}-\alpha) + \hat{c} H_{2\tau\tau}(\mathbf{1}-\alpha)$$

Again, evaluating at $\tau = 0$,

$$G(0) = 1$$

$$G_\tau = -\sum_i \beta_i a_i z_{1\tau} + \sum_i \beta_i a_i z_{2\tau} = [z_{2\tau} - z_{1\tau}] \sum_i \beta_i a_i$$

$$G_{\tau\tau} = \sum_i \beta_i [-z_{1\tau\tau} a_i - z_{1\tau}^2 b_i + 2 z_{1\tau}^2 a_i^2] + 2 z_{1\tau} z_{2\tau} \sum_i \beta_i [-a_i^2] + \sum_i \beta_i [a_i z_{2\tau\tau} + b_i z_{2\tau}^2]$$

$$G_{\tau\tau} = [z_{2\tau\tau} - z_{1\tau\tau}] \sum_i \beta_i a_i + [z_{2\tau}^2 - z_{1\tau}^2] \sum_i \beta_i b_i + [2 z_{1\tau}^2 - 2 z_{1\tau} z_{2\tau}] \sum_i \beta_i a_i^2$$

Then

$$\frac{d}{d\tau} \log Y(\tau) = M_\tau - G_\tau = \frac{1-\sigma}{\sigma} z_{1\tau} \sum_i \beta_i a_i + \sum_i \beta_i a_i z_{1\tau} - \sum_i \beta_i a_i z_{2\tau} = (\frac{1}{\sigma} z_{1\tau} - z_{2\tau}) \sum_i \beta_i a_i = 0$$

Next,

$$\begin{aligned} \frac{d^2}{d\tau^2} \log Y(0) &= M_{\tau\tau}(0) - G_{\tau\tau}(0) + G_\tau(0)^2 \\ &= -\frac{1}{1-\sigma} \sum_i \beta_i (a_i + b_i) + \frac{\sigma}{1-\sigma} \sum_i \beta_i a_i^2 + \left(\sum_i \beta_i a_i \right)^2 \end{aligned}$$

This implies that

$$\frac{d}{d\sigma} \left(\frac{d^2}{d\tau^2 d} \log Y(0) \right) = \frac{1}{(1-\sigma)^2} \left(-\sum_i \beta_i (a_i + b_i) + \sum_i \beta_i a_i^2 \right).$$

If we show that $\frac{d}{d\sigma} \left(\frac{d^2}{d\tau^2 d} \log Y(0) \right) < 0$, then by the same argument as in the proof of Proposition 8, the result will follow.

A sufficient condition for this claim would be $-(a_i + b_i) + a_i^2 < 0$ for all i. That's what we prove next.

$$a = ZH\mathbf{1} = \left(\sum_{\ell=1}^{\infty} Z^\ell \right) \mathbf{1}$$

$$a_i = \sum_{\ell=1}^{\infty} \left(\sum_{k=1}^{n} Z_{ik}^\ell \right) = \sum_{\ell=1}^{\infty} s_{i,\ell},$$

where $s_{i,\ell} = \sum_{k=1}^n Z_{ik}^\ell$.

$$b = 2Z^2 H^2 \mathbf{1} = 2 \left(\sum_{\ell=1}^{\infty} Z^\ell \right)^2 \mathbf{1} = 2 \left(\sum_{\ell=2}^{\infty} (\ell - 1) Z^\ell \right) \mathbf{1},$$

therefore $b_i = 2\sum_{\ell=2}^{\infty}(\ell-1)s_{i,\ell}$.

Mathematical induction shows immediately that $Z_{ik}^{\ell} \in [0,1)$ and $s_{i,\ell} \leq (\max_i\{\alpha_i\})^{\ell} < 1$. Then the series above converge absolutely and

$$a_i^2 = \sum_{\ell=1}^{\infty} s_{i,\ell}^2 + 2\sum_{\ell=2}^{\infty}\left(\sum_{k=1}^{\ell-1} s_{i,k}s_{i,\ell}\right) < \sum_{\ell=1}^{\infty} s_{i,\ell} + 2\sum_{\ell=2}^{\infty}(\ell-1)s_{i,\ell} = a_i + b_i,$$

which establishes the claim and concludes the proof. ■

Proof of Proposition 8. Let $\Omega = [I - \tilde{\alpha}V]^{-1}$. Then the equilibrium can be expressed as:

$$\hat{\lambda} = \Omega\hat{B}(\mathbf{1}-\alpha)$$

$$\log Y = \log\bar{L} + \frac{1-\sigma}{\sigma}\beta'\log\hat{\lambda} - \log(\hat{c}'F\hat{B}(\mathbf{1}-\alpha)$$

Using the definition of $\hat{c}$ and the fact that $\Omega\hat{B}(\mathbf{1}-\alpha) = \hat{\lambda}$, we get that

$$\log(\hat{c}'\Omega\hat{B}(\mathbf{1}-\alpha)) = \log\left(\sum_i \beta_i\hat{\lambda}_i^{-1}\hat{\lambda}_i\right) = 0.$$

Therefore, $\log Y = \log\bar{L} + \frac{1-\sigma}{\sigma}\beta'\log\hat{\lambda}$.

From the expression for $\hat{\lambda}$,

$$\frac{\partial}{\partial B_i}\hat{\lambda}_k = \omega_{ki}(1-\alpha_i)\frac{\sigma}{1-\sigma}B_i^{\frac{\sigma}{1-\sigma}-1},$$

which implies that

$$\frac{\partial^2}{\partial B_i^2}\hat{\lambda}_k = \omega_{ki}(1-\alpha_i)\frac{\sigma}{1-\sigma}\left(\frac{\sigma}{1-\sigma}-1\right)B_i^{\frac{\sigma}{1-\sigma}-2},$$

$$\frac{\partial^2}{\partial B_i\partial B_j}\hat{\lambda}_k = 0, \quad \text{if } i \neq j$$

We have that

$$\frac{\partial}{\partial B_i}\log Y(z,\sigma) = \frac{1-\sigma}{\sigma}\sum_k \beta_k\frac{\frac{\partial}{\partial B_i}\hat{\lambda}_k}{\hat{\lambda}_k}$$

$$\frac{\partial^2}{\partial B_i\partial B_j}\log Y(z,\sigma) = \frac{1-\sigma}{\sigma}\sum_k \beta_k\left[\frac{\frac{\partial^2}{\partial B_i\partial B_j}\hat{\lambda}_k}{\hat{\lambda}_k} - \frac{\frac{\partial}{\partial B_i}\hat{\lambda}_k\frac{\partial}{\partial B_j}\hat{\lambda}_k}{\hat{\lambda}_k^2}\right].$$

At $B = \mathbf{1}$, $\hat{\lambda}_k = 1$ for all k. Then

$$\frac{\partial}{\partial B_i}\log Y(\mathbf{1},\sigma) = \frac{1-\sigma}{\sigma}\sum_k \beta_k\frac{\frac{\partial}{\partial B_i}\hat{\lambda}_k}{\hat{\lambda}_k} = \sum_k \beta_k\omega_{ki}(1-\alpha_i).$$

So, the first-order effect does not depend on σ! Substituting in the expressions above and evaluating at $\mathbf{1}$,

$$\frac{\partial^2}{\partial B_i \partial B_j} \log Y(\mathbf{1},\sigma) = \begin{cases} \sum_k \beta_k \left[\left(\frac{\sigma}{1-\sigma} - 1 \right) \omega_{ki}(1-\alpha_i) - \frac{\sigma}{1-\sigma} \omega_{ki}^2 (1-\alpha_i)^2 \right] = C_{ij} + \frac{1}{1-\sigma} \sum_k \beta_k [(1-\alpha_i)\omega_{ki} - (1-\alpha_i)^2 \omega_{k}^2 \\ \sum_k \beta_k \left[-\frac{\sigma}{1-\sigma} (1-\alpha_i)(1-\alpha_j) \omega_{ki}\omega_{kj} \right] = C_{ij} + \frac{1}{1-\sigma} \sum_k \left[-\beta_k (1-\alpha_i)(1-\alpha_j)\omega_{ki}\omega_{kj} \right] \end{cases}$$

where C_{ij} does not depend on σ. Rewriting, the Hessian evaluated at $\mathbf{1}$ is given by:

$$\frac{\partial^2}{\partial B \partial B} \log Y(\mathbf{1},\sigma) = C + \frac{1}{1-\sigma} \sum_k \beta_k H^k, \tag{A.7}$$

where $H_{ii}^k = (1-\alpha_i)\omega_{ki} - (1-\alpha_i)^2 \Omega_{ki}^2$, $H_{ij}^k = -(1-\alpha_i)(1-\alpha_j)\omega_{ki}\omega_{kj}$. Next, we show that for any $z \in \mathbf{R}^n$, $z' H^k z \geq 0$ (H^k are positive semi-definite matrices) and $z' H^k z > 0$ if $z_i \neq z_j$ for some i, j. From the definitions, it follows that

$$z' H^k z = \sum_i (1-\alpha_i)\omega_{ki} z_i^2 - \left(\sum_i (1-\alpha_i)\omega_{ki} z_i \right)^2.$$

Since $f(z) = z^2$ is strictly convex, $(1-\alpha_i)\omega_{ki} > 0$, $\sum_i (1-\alpha_i)\omega_{ki} = 1$ (lemma A.3), Jensen's inequality proves both results.

This result and the expression for the Hessian (A.7) in turn implies that for any $z \in \mathbf{R}^n$, $z_i \neq x_j$ for some i, j, we have

$$z' \frac{\partial^2}{\partial B \partial B} \log Y(\mathbf{1},\sigma_2) z > z' \frac{\partial^2}{\partial B \partial B} \log Y(\mathbf{1},\sigma_1) z. \tag{A.8}$$

Now we can prove the result. For ξ small enough $\mathbf{1} + \xi x > \mathbf{0}$. By second order Taylor expansion around $\mathbf{1}$,

$$\log Y(\mathbf{1}+\xi x,\sigma) = \sum_i \frac{\partial}{\partial B_i} \log Y(\mathbf{1},\sigma) \xi x_i + \frac{1}{2} \xi^2 x' \frac{\partial^2}{\partial B \partial B} \log Y(\mathbf{1}+\xi' x,\sigma) x,$$

where $\xi' \in [0,\xi]$. Then,

$$\log Y(\mathbf{1}+\xi x,\sigma_2) - \log Y(\mathbf{1}+\xi x,\sigma_1) = \frac{1}{2}\xi^2 \left(x' \frac{\partial^2}{\partial B \partial B} \log Y(\mathbf{1}+\xi_2' x,\sigma_2) x - x' \frac{\partial^2}{\partial B \partial B} \log Y(\mathbf{1}+\xi_1' x,\sigma_1) x \right),$$

$\xi_1', \xi_2' \in [0,\xi]$. Inequality (A.8) and continuity implies that for ξ small enough, $\log Y(\mathbf{1}+\xi x,\sigma_2) > \log Y(\mathbf{1}+\xi x,\sigma_1)$. ■

A.9 Proofs and derivations of Section 3.3

Proof of equation (15)**.**

Direct inspection shows that for the simple IO economy, we have the following

$$(I - \tilde{\alpha}V)^{-1} = \Omega = \begin{pmatrix} 1 + \frac{\alpha_1}{1-\bar{\alpha}}v_1 & \frac{\alpha_1}{1-\bar{\alpha}}v_2 & \dots & \frac{\alpha_1}{1-\bar{\alpha}}v_n \\ \frac{\alpha_2}{1-\bar{\alpha}}v_1 & 1 + \frac{\alpha_2}{1-\bar{\alpha}}v_2 & \dots & \frac{\alpha_2}{1-\bar{\alpha}}v_n \\ \dots & \dots & \dots & \dots \\ \frac{\alpha_n}{1-\bar{\alpha}}v_1 & \frac{\alpha_n}{1-\bar{\alpha}}v_2 & \dots & 1 + \frac{\alpha_n}{1-\bar{\alpha}}v_n \end{pmatrix}$$

with with $\bar{\alpha} \equiv \sum_j \alpha_j v_j$. Applying the definitions directly implies that $s_i = \beta_i + v_i\bar{\alpha}_\beta/(1-\bar{\alpha})$ with $\bar{\alpha} = \sum_i \alpha_i\beta_i$.

Equation (12) for the case $\bar{\tau}_x = 0$ and $\sigma = 0$ implies that $\log TFP \approx -\chi(\Omega,\alpha)V(\tau_i)$ with $\chi(\Omega,\alpha) = \sum_i[\alpha_i s_i + 2\alpha_i^2 s_i g_{ii}] = \sum_i[\alpha_i s_i + 2\alpha_i s_i(\omega_{ii}-1)]$.

Then

$$\begin{aligned}
\chi(\Omega,\alpha) &= \sum_i \alpha_i s_i + 2\alpha_i^2 s_i g_{ii} \\
&= \sum_i \alpha_i[s_i + s_i(\omega_{ii}-1) \\
&= \sum_i \alpha_i \left(\beta_i + \frac{\bar{\alpha}_\beta}{1-\bar{\alpha}_v}v_i\right) + 2\alpha_i\left(\beta_i + \frac{\bar{\alpha}_\beta}{1-\bar{\alpha}_v}v_i\right)\frac{\alpha_i}{1-\bar{\alpha}_v}v_i \\
&= \sum_i \alpha_i\beta_i + \frac{1}{1-\bar{\alpha}_v}\sum_i \alpha_i v_i(\bar{\alpha}_\beta + 2\alpha_i\beta_i) + \frac{2\bar{\alpha}_\beta}{(1-\bar{\alpha}_v)^2}\sum_i \alpha_i^2 v_i^2
\end{aligned}$$

which concludes the proof. ■

In the case of $\alpha_i = \alpha$, substituting implies that

$$\chi(\Omega,\alpha) = \alpha + \frac{\alpha^2}{1-\alpha} + 2\frac{\alpha^2}{1-\alpha}\sum_i \beta_i v_i + 2\frac{\alpha^3}{(1-\alpha)^2}\sum v_i^2$$

It is easy to check that $d\chi(\Omega,\alpha)/d\alpha \geq 0$.

Next, we extend these results for the general case. Recall that:

$$\log TFP \approx -\psi_\sigma(\Omega,\alpha)\bar{\tau} - \chi(\Omega,\alpha)V(\tau_i) - \frac{\sigma}{1-\sigma}\left(\chi(\Omega,\alpha) - \sum_i \alpha_i^2 s_{ii}\right)V(\tau_i)$$

Let's do the σ term:

$$\begin{aligned}
\chi(\Omega,\alpha) - \sum_i \alpha_i^2 s_{ii} &= \chi(\Omega,\alpha) - \sum_i \alpha_i^2\left(\sum_k \beta_k \frac{\alpha_k^2}{(1-\bar{\alpha}_v)^2}v_i^2 + \beta_i(1 + \frac{2}{1-\bar{\alpha}_v}\alpha_i v_i)\right) \\
&= \sum_i \alpha_i\beta_i(1-\alpha_i) + \frac{1}{1-\bar{\alpha}_v}\sum_i \alpha_i v_i(\bar{\alpha}_\beta + 2\alpha_i\beta_i - 2\alpha_i^2\beta_i) + \frac{2\bar{\alpha}_\beta - \bar{\alpha}_{2\beta}}{(1-\bar{\alpha}_v)^2}\sum_i \alpha_i^2 v_i^2
\end{aligned}$$

with $\bar{\alpha}_v \equiv \sum_k v_k\alpha_k$ and $\bar{\alpha}_\beta \equiv \sum_k \beta_k\alpha_k$ and $\bar{\alpha}_{2\beta} \equiv \sum_k \beta_k\alpha_k^2$.

Similarly, for $\alpha_i = \alpha$, the σ term becomes,

$$\begin{aligned}
\chi(\Omega, \alpha) + \sum_i \alpha_i^2 s_{ii} &= \chi(\Omega, \alpha) - \sum_i \alpha^2 \beta_i \left(1 + 2\frac{\alpha}{1-\alpha} v_i\right) - \alpha^2 \sum_i \frac{\alpha^2}{(1-\alpha)^2} v_i^2 \\
&= \chi(\Omega, \alpha) - \alpha^2 - 2\frac{\alpha^3}{1-\alpha} \sum_i \beta_i v_i - \frac{\alpha^4}{(1-\alpha)^2} \sum_i v_i^2 \\
&= \alpha + \frac{\alpha^3}{1-\alpha} + 2\alpha^2 \sum_i \beta_i v_i + \frac{\alpha^3(2-\alpha)}{(1-\alpha)^2} \sum v_i^2
\end{aligned}$$

Proof of equation (17)**.** Proposition 5 implies that in the case of Cobb-Douglas:

$$\log TFP \approx -\frac{1}{2} \sum_i \sum_j \alpha_i \alpha_j \left(s_i g_{ij} + s_j g_{ji}\right) \tau_i \tau_j - \frac{1}{2} \sum_i \alpha_i s_i \tau_i^2 + \frac{1}{2} \left(\sum_i \alpha_i s_i \tau_i\right)^2$$

Using the notation $a = \alpha/(1-\alpha)$, and the expression for $s_i = \beta_i + a v_i$ and $\alpha_i g_{ij} = \omega_{ij} = a v_j$ $\forall j$ (even $j = i$ since $\alpha_i g_{ii} = \omega_{ii} - 1 = a v_i$) we get

$$\begin{aligned}
\sum_i \alpha_i s_i \tau_i &= \alpha \sum_i (\beta_i + a v_i) \tau_i \\
&= \alpha \sum_i \beta_i \tau_i + \frac{a\alpha}{n\bar{V}} \sum_i V_i \tau_i \\
\sum_i \alpha_i s_i \tau_i^2 &= \alpha \sum_i (\beta_i + a v_i) \tau_i^2 \\
&= \alpha \sum_i \beta_i \tau_i^2 + \frac{a\alpha}{n\bar{V}} \sum_i V_i \tau_i^2 \\
\sum_i \sum_j \alpha_i \alpha_j \left(s_i g_{ij} + s_j g_{ji}\right) \tau_i \tau_j &= \alpha \sum_i \sum_j [(\beta_i + a v_i) a v_j + (\beta_j + a v_j) a v_i] \tau_i \tau_j \\
&= 2\alpha \sum_i \sum_j (\beta_i + a v_i) a v_j \tau_i \tau_j \\
&= 2\alpha a [\sum_i \sum_j \beta_i v_j \tau_i \tau_j + \sum_i \sum_j a v_i v_j \tau_i \tau_j] \\
&= 2\alpha a [\bar{\tau}^2 + \bar{\tau} \sigma_\tau \frac{\sigma_V}{\bar{V}} \delta + a \bar{\tau}^2 + 2a\bar{\tau}\sigma_\tau \delta \frac{\sigma_V}{\bar{V}} + a \sigma_\tau^2 (\delta \frac{\sigma_V}{\bar{V}})^2] \\
&= 2\alpha a(1+a)\bar{\tau}^2 + 2\alpha a(1+2a)\bar{\tau}\sigma_\tau \frac{\sigma_V}{\bar{V}} \delta + 2\alpha a^2 \sigma_\tau^2 (\delta \frac{\sigma_V}{\bar{V}})^2
\end{aligned}$$

where we have used

$$
\begin{aligned}
\sum_i \sum_j \beta_i v_j \tau_i \tau_j &= \sum_i \sum_j \beta_i v_j (\bar{\tau}^2 + \bar{\tau}\sigma_\tau (z_j + z_i) + \sigma_\tau^2 z_j z_i) \\
&= \bar{\tau}^2 + \bar{\tau}\sigma_\tau [(\sum_i \beta_i)(\sum_j v_j z_j) + (\sum_j z_j)(\sum_i \beta_i z_i) + \sigma_\tau^2 (\sum_i \beta_i z_i)(\sum_j v_j z_j) \\
&= \bar{\tau}^2 + \bar{\tau}\sigma_\tau (\sum_j v_j z_j) + 0 + 0 \\
&= \bar{\tau}^2 + \bar{\tau}\sigma_\tau \frac{1}{n\bar{V}} (\sum_j (\bar{V} + \sigma_V u_{vj})(\delta u_{vj} + \sqrt{1-\delta^2} u_{\tau j})) \\
&= \bar{\tau}^2 + \bar{\tau}\sigma_\tau \frac{\sigma_V}{\bar{V}} \delta \\
\sum_i \sum_j a v_i v_j \tau_i \tau_j &= \sum_i \sum_j a v_i v_j (\bar{\tau}^2 + \bar{\tau}\sigma_\tau (z_j + z_i) + \sigma_\tau^2 z_j z_i) \\
&= a\bar{\tau}^2 + 2a\bar{\tau}\sigma_\tau \sum_i v_i \sum_j v_j z_j + a\sigma_\tau^2 (\sum_i v_i z_i)^2 \\
&= a\bar{\tau}^2 + 2a\bar{\tau}\sigma_\tau \delta \frac{\sigma_V}{\bar{V}} + a\sigma_\tau^2 (\delta \frac{\sigma_V}{\bar{V}})^2
\end{aligned}
$$

Let us compute $\sum_i V_i \tau_i$ and $\sum_i V_i \tau_i^2$ and $\sum_i V_i^2 \tau_i^2$ (we don't need the last one).

$$
\begin{aligned}
\frac{1}{n}\sum_i V_i \tau_i &= \frac{1}{n}\sum_i (\bar{V} + \sigma_V u_{vi})[\bar{\tau} + \sigma_\tau (\delta u_{vi} + \sqrt{1-\delta^2} u_{\tau i})] \\
&= \bar{V}\bar{\tau} + \delta \sigma_V \sigma_\tau \\
\frac{1}{n}\sum_i V_i \tau_i^2 &= \frac{1}{n}\sum_i (\bar{V} + \sigma_V u_{vi})[\bar{\tau}^2 + 2\sigma_\tau \bar{\tau}(\delta u_{vi} + \sqrt{1-\delta^2} u_{\tau i}) + \sigma_\tau^2 (\delta u_{vi} + \sqrt{1-\delta^2} u_{\tau i})^2] \\
&= \bar{V}\bar{\tau}^2 + \bar{V}\sigma_\tau^2 + 2\bar{\tau}\delta\sigma_\tau\sigma_V + \delta^2 \sigma_\tau^2 \sigma_V E(u_{vi}^3) \\
\frac{1}{n}\sum_i V_i^2 \tau_i^2 &= \frac{1}{n}\sum_i (\bar{V} + \sigma_V u_{vi})^2[\bar{\tau}^2 + 2\sigma_\tau \bar{\tau}(\delta u_{vi} + \sqrt{1-\delta^2} u_{\tau i}) + \sigma_\tau^2 (\delta u_{vi} + \sqrt{1-\delta^2} u_{\tau i})^2] \\
&= \bar{V}^2\bar{\tau}^2 + \bar{V}^2\sigma_\tau^2 + 4\bar{V}\bar{\tau}\delta\sigma_\tau\sigma_V + 2\bar{V}\delta^2\sigma_\tau^2\sigma_V E(u_{vi}^3) + \bar{\tau}^2\sigma_V^2 + 2\bar{\tau}\delta\sigma_V^2\sigma_\tau E(u_{vi}^3) + \delta^2\sigma_V^2\sigma_\tau^2 E(u_{vi}^4)
\end{aligned}
$$

To simplify, let's take $\bar{V} = 1$. We then have

$$
\begin{aligned}
\left(\sum_i \alpha_i s_i \tau_i\right)^2 &= [\alpha\bar{\tau} + \alpha a(\bar{\tau} + \delta\sigma_v\sigma_\tau)]^2 \\
&= \alpha^2(1+a)^2\bar{\tau}^2 + 2\alpha^2\bar{\tau}(1+a)a\delta\sigma_v\sigma_\tau + (a\alpha\delta\sigma_V\sigma_\tau)^2 \\
\sum_i \alpha_i s_i \tau_i^2 &= \alpha(\sigma_\tau^2 + \bar{\tau}^2) + \alpha a(\bar{\tau}^2 + \sigma_\tau^2 + 2\bar{\tau}\delta\sigma_\tau\sigma_V + \delta^2\sigma_\tau^2\sigma_v E(u_{vi}^3)) \\
&= \alpha(1+a)(\bar{\tau}^2 + \sigma_\tau^2) + 2\alpha a\bar{\tau}\delta\sigma_\tau\sigma_v + \alpha a\delta^2\sigma_\tau^2\sigma_v E(u_{vi}^3) \\
\sum_i \sum_j \alpha_i \alpha_j \left(s_i g_{ij} + s_j g_{ji}\right) \tau_i \tau_j &= 2\alpha a(1+a)\bar{\tau}^2 + 2\alpha a(1+2a)\bar{\tau}\sigma_\tau\sigma_v\delta + 2\alpha a^2\delta^2\sigma_\tau^2\sigma_v^2
\end{aligned}
$$

So finally,

$$\log TFP \approx -\frac{1}{2}\left[a(1+a)\bar{\tau}^2 + a\sigma_\tau^2 + 2\alpha a(2+a)\bar{\tau}\delta\sigma_v\sigma_\tau + \alpha a\delta^2\sigma_\tau^2\sigma_v(E(u_{vi}^3) + a(2-\alpha)\sigma_v)\right]$$

Where we used $\alpha(1+a) = a$. ■

Proof of equation (16)**.** The proof is similar to the proof of equation (17) and we use the fact that for labor distortions and Cobb-Douglas production functions the following approximation holds.

$$\log TFP \approx \frac{1}{2}\left(\sum_i s_i(1-\alpha_i)\tau_{iL}\right)^2 - \frac{1}{2}\frac{1}{1-\sigma}\sum_i s_i(1-\alpha_i)\tau_{iL}^2$$

■

A.10 upstream and downstream propagation

Proof of Lemma 1. Suppose that two arbitrary matrices F and G are of the form in the hypothesis of the lemma. Then from the definition of matrix multiplication, FG is of the same form. This fact implies (by mathematical induction) that $[\tilde{\alpha}V]^{(n)}$ is of the same form for all n, hence sectors $1,\ldots,\ell-1$ are upstream from ℓ.

Next we tackle the necessity condition. Suppose that for some $i < \ell, j \geq \ell$, $\alpha_i v_{ij} > 0$. If $j = \ell$, then i is a purchaser of ℓ by definition. Suppose not. For some n, $[\tilde{\alpha}V]_{j\ell}^{(n)} > 0$. Then $[\tilde{\alpha}V]_{i\ell}^{(n+1)} \geq \alpha_i v_{ij}[\tilde{\alpha}V]_{j\ell}^{(n)} > 0$, which is a contradiction. ■

Proof of Proposition 9.

1. Define $H \equiv \tilde{\alpha}\hat{A}(\Delta^{Xp} \circ V)]$ and $\Omega^p = [I-H]^{-1}$. It is immediate that $H_{ij}^{(n)} > 0$ if and only if $[\tilde{\alpha}V]^{(n)} > 0$. Moreover, $\Omega^p = I + H + H^2 + \ldots$, so $\Omega_{ij}^p > 0$, $i \neq j$ if and only $[\tilde{\alpha}V]_{ij}^{(n)} > 0$ for some n. This implies that if $1,\ldots \ell-1$ are upstream from ℓ and $\ell+1\ldots n$ are purchasers of ℓ, we have that $\Omega_{i\ell}^p = 0$ for $i < \ell$ and $\Omega_{i\ell}^p > 0$ for $i \geq \ell$.

   Proposition 3 implies that
   $$\frac{d(\lambda/\eta)}{d\tau_{\ell L}} = \tilde{\lambda} \circ (\Omega^p\hat{A}\hat{B}\Delta^1(\mathbf{1}-\alpha)),$$
   where $\tilde{\lambda}$ is a $n \times 1$ vector with $\tilde{\lambda}_i = (\lambda_i/\eta)^{1/(1-\sigma)}$ and $\Delta_\ell^1 = (1+\tau_{\ell L})^{-\frac{\sigma}{1-\sigma}-1}$ and $\Delta_{iL}^1 = 0$ for $i \neq \ell$. Then the fact that $\Omega_{i\ell}^p = 0$ for $i < \ell$ and $\Omega_{i\ell}^p > 0$ for $i \geq \ell$ implies the result about the sign of $\frac{d(\lambda_i/\eta)}{d\tau_{\ell L}}$.

   In a similar fashion,
   $$\frac{d(\lambda/\eta)}{d\tau_{\ell k}} = -\frac{1-\sigma}{\sigma}\tilde{\lambda} \circ (\frac{d}{d\tau_{\ell k}}\Omega^p\hat{A}\hat{B}\Delta^{Lp}(\mathbf{1}-\alpha)).$$

   From the definition of Ω^p, it follows that
   $$\frac{d}{d\tau_{\ell k}}\Omega^p = -\frac{\sigma}{1-\sigma}\Omega^p M\Omega^p,$$

where $M_{\ell k} = \alpha_\ell A_\ell^{\frac{\sigma}{1-\sigma}}(1+\tau_{\ell k})^{-\frac{1}{1-\sigma}} v_{\ell k} > 0$ and $M_{ij} = 0$ otherwise. Therefore,

$$\frac{d(\lambda/\eta)}{d\tau_{\ell k}} = \tilde{\lambda} \circ (\Omega^p M \hat{\lambda}).$$

From the structure of Ω^p and M it follows that $(\Omega^p M)_{i\ell} > 0$ if $i \geq \ell$ and $\Omega^p_{ij} = 0$ otherwise. This implies the first result.

2. Next, since $\log \eta = -\beta' \log(\lambda/\eta)$, $d\eta/d\tau_{\ell,k} < 0$. Then since $d\lambda_i/dx = d(\eta\lambda_i/\eta)/dx = (\lambda_i/\eta)d\eta/dx + \eta d(\lambda_i/\eta)/dx$, result 1 implies the result.

3. Since $C_i/Y = \beta_i/\lambda_i$, result 2 implies the first statement. Second, $\log C_i = \log \beta_i - \log \lambda_i + \log Y = \log \beta_i - \log \eta - \log(\lambda_i/\eta) + \log Y = \log \beta_i + \beta' \log(\lambda/\eta) - \log(\lambda_i/\eta) + \log Y$. Then proposition 3 implies that

$$\log C_i = \log \beta_i - \log(\lambda_i/\eta) - \log\left(\beta'[I - \tilde{\alpha}\hat{A}(\Delta^{Xq} \circ V)]^{-1}\hat{A}\hat{B}\Delta^{Lq}(\mathbf{1} - \alpha)\right).$$

Let $E = [I - \tilde{\alpha}\hat{A}(\Delta^{Xq} \circ V)]^{-1}$. The statement is equivalent to

$$\frac{d}{dx}[\beta' E\hat{A}\hat{B}\Delta^{Lq}(\mathbf{1} - \alpha)] < 0,$$

where $x = \tau_{\ell,L}$ or $x = \tau_{\ell,k}$. So,

$$\frac{d}{d\tau_{\ell,L}}[\beta' E\hat{A}\hat{B}\Delta^{Lq}(\mathbf{1} - \alpha)] = -\frac{1}{1-\sigma}\beta' E\hat{A}\hat{B}\Delta^2(\mathbf{1} - \alpha) < 0,$$

where $\Delta^2_\ell = (1 + \tau_{iL})^{-\frac{1}{1-\sigma}-1}$ and zero elsewhere. Next we turn to $\tau_{\ell,k}$.

$$\frac{d}{d\tau_{\ell k}}E = -\frac{1}{1-\sigma}ENE,$$

where $N_{\ell k} = \alpha_\ell A_\ell^{\frac{\sigma}{1-\sigma}}(1+\tau_{\ell k})^{-\frac{1}{1-\sigma}-1} v_{\ell k} > 0$ and $N_{ij} = 0$ otherwise. Therefore,

$$\frac{d}{d\tau_{\ell,k}}[\beta' E\hat{A}\hat{B}\Delta^{Lq}(\mathbf{1} - \alpha)] = -\frac{1}{1-\sigma}\beta' ENE\hat{A}\hat{B}\Delta^{Lq}(\mathbf{1} - \alpha) < 0,$$

which concludes the proof of this result.

4. Finally, the zero-profit condition implies that

$$\frac{\sum_j (1+\tau_{ij})\lambda_j X_{ij}}{\lambda_i Q_i} = 1 - \frac{(1+\tau_{iL})\eta L_i}{\lambda_i Q_i} = 1 - (A_i B_i)^{\frac{\sigma}{1-\sigma}}(1-\alpha_i)\left(\frac{\lambda_i}{(1+\tau_{iL}\eta}\right)^{\frac{\sigma}{1-\sigma}}.$$

Then result 1 implies the rest.

■

B Application to markups

B.1 Model with markups

Given the prices of all sectors, the cost minimization conditions for the firms will remain the same as before without distortions:

$$\chi_i A_i^\sigma \left(\frac{\alpha_i Q_i}{X_i}\right)^{1-\sigma} \left(\frac{v_{ij} X_i}{x_{ij}}\right)^{1-\rho} = \lambda_j \tag{B.1}$$

$$\chi_i A_i^\sigma B_i^\sigma \left(\frac{(1-\alpha_i) Q_i}{L_i}\right)^{1-\sigma} = \eta, \tag{B.2}$$

where χ_i is the multiplier to the constraint $Q_i \geq 1$, which is the same as mc_i. The fact that $\chi_i = mc_i = \lambda_i/\mu_i$, shows directly that the allocation satisfies (7) and (8) with $\tau_{ij} = \tau_{iL} = \mu_i - 1$.

Manipulating these conditions, we get that

$$mc_i = A_i^{-1}\left[(1-\alpha_i) B_i^{\frac{\sigma}{1-\sigma}} \eta^{-\frac{\sigma}{1-\sigma}} + \alpha_i \left(\sum_j v_{ij} \lambda_j^{-\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}\frac{\sigma}{1-\sigma}}\right]^{-\frac{1-\sigma}{\sigma}}. \tag{B.3}$$

Given the expression for mc_i, and hence λ_i, Proposition 1 pins down the equilibrium.

B.2 Data

The World Input-Output Database Socio economic accounts gives industry-level data on sales, labor compensation, capital and intermediate input purchases for 40 countries and 35 sectors over the period 1995-2011.[21] We remove the sectors of public administration, education, health and social work, social services and private households services (sectors c31-c35).

Data for sales, wage bill and intermediate input costs are available directly from the dataset. We now describe how we construct the remaining variables.

We compute capital costs as $(r+\delta)qK$, where r is the real interest rate, δ is the depreciation rate and qK is the capital stock in current prices. We estimate r as the 10-year government bond yield minus the country's CPI inflation rate, both obtained from the OECD Main Economic indicators database. We assume a depreciation rate identical across countries, of 7%.

We compute the intermediate input shares as follows

$$\frac{\lambda_j X_{ij}}{\lambda_i Q_i} = \frac{\text{interm-input}_{ij}}{\text{sales}_i - \sum_j \text{imported-interm-input}_{ij}}$$

[21] The data are publicly available at: http://www.wiod.org/home. We use the 2013 release.For more information on how those variables are constructed:....

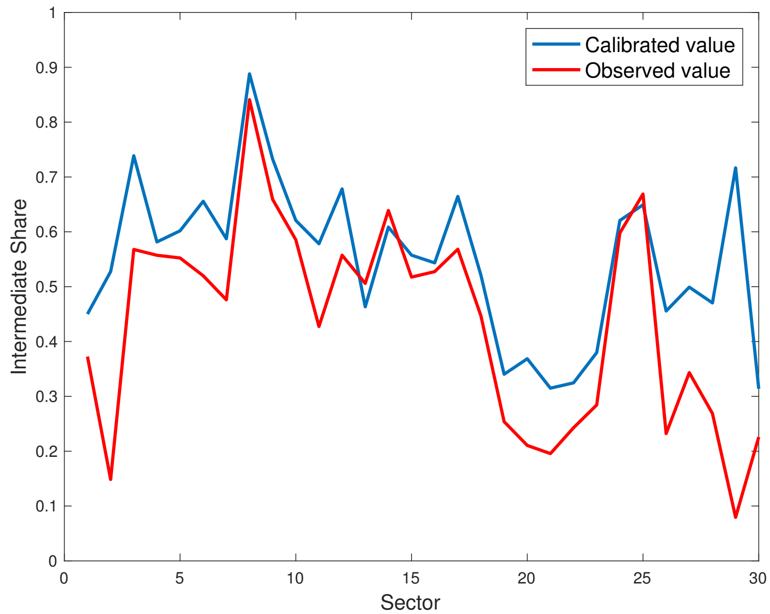

Figure 2: Estimated intermediate share

Note: The estimates are for $\epsilon_\rho = \epsilon_\sigma = 0.7$.

We divide the value of intermediate inputs purchased by sector i by its sales adjusted by the total intermediate inputs imported by the sector. This adjustment is required to be consistent with the model in which we abstract from international trade.

Final expenditures are computed as

$$\lambda_i C_i = \text{sales}_i - \sum_k \text{interm-input}_{ki}$$

B.3 Results with Alternative Markup Measures

Here we report robustness checks on our results. First, we redo the exercises, reported in Table 2 using an alternative measure of markups – PCM1.

Table 5: TFP gain from lowering markups to US levels (percents) – PCM1 measure

		ε_ρ		
		0.01	0.7	1
ε_σ	0.01	13.0	32.1	44.9
	0.7	15.4	37.4	52.9
	1	16.7	40.4	57.7
No IO linkages		11.7		

Notes: markups are measured using pcm_1 and no linkages economy is obtained by setting $\alpha_i = 0$ for all sectors.

Table 6: TFP gain from lowering average markups to average US level (percents)) – PCM1 measure

		ε_ρ		
		0.01	0.7	1
ε_σ	0.01	5.0	13.6	18.2
	0.7	7.2	16.7	21.8
	1	8.3	18.2	23.7
No IO linkages		0		

C Imported intermediate goods

About 26 % of the value all intermediate goods in Mexico and 11 % in the USA are imported. In order to simplify the analysis, we have considered closed economies in our theoretical discussion. In this extension we allow for some intermediate goods to be imported.

Here we extend the model to allow for imported intermediate goods. We abstract from imports for final consumption, so the production function for the final good remains the same. However, now foreign, as well as domestic, intermediates can be used in production. Since foreign and domestic varieties of the same good are closer substitutes than different goods, the intermediate good ij is composed of domestic X_{ij} and foreign I_{ij} components as follows $\left(u_{ij}^{1-\zeta}X_{ij}^{\zeta} + (1-u_{ij})^{1-\zeta}I_{ij}^{\zeta}\right)^{\frac{1}{\zeta}}$.

Then, the production function for sector i is modified as follows:

$$Q_i = A_i \left[(1-\mu_i)^{1-\sigma} (B_i L_i)^\sigma + \mu_i^{1-\sigma} \left(\sum_{j=1}^{n} v_{ij}^{1-\rho} \left(u_{ij}^{1-\zeta} X_{ij}^{\zeta} + (1-u_{ij})^{1-\zeta} I_{ij}^{\zeta} \right)^{\frac{\rho}{\zeta}} \right)^{\frac{\sigma}{\rho}} \right]^{\frac{1}{\sigma}} .$$

If the parameters $\rho = \zeta$, then foreign intermediates are on the same standing as domestic ones with weight $v_{ij}(1-u_{ij})$.

The foreign intermediates are paid for with the final consumption good at some exogenous prices r_j, so GDP is simply

$$O = Y - \sum_{i=1}^{n} \sum_{j=1}^{m} I_{ij} r_j .$$

(We follow Jones (2013) in this approach.) We assume a small open economy: the prices of the foreign intermediate goods are exogenous and invariant to any domestic reforms. Hence by the same construction as in Proposition 2 , we can normalize the prices of the foreign intermediates to 1. This model can be used to study the effect of tariffs and other trade barriers on productivity, but we will not pursue this avenue here.

Following the same approach as in the closed economy model, we find that the prices satisfy:

$$\lambda_i = A_i^{-1} \left\{ (1-\alpha_i) B_i^{\frac{\sigma}{1-\sigma}} ((1+\tau_{iL})\eta)^{-\frac{\sigma}{1-\sigma}} + \alpha_i \left[\sum_{j=1}^{n} v_{ij} \bar{\lambda}_{ij}^{-\frac{\rho}{1-\rho}} \right]^{\frac{1-\rho}{\rho} \frac{\sigma}{1-\sigma}} \right\}^{-\frac{1-\sigma}{\sigma}} \tag{C.1}$$

with $\bar{\lambda}_{ij}$ the price of the foreign-domestic composite ij intermediate good,

$$\bar{\lambda}_{ij} = \left(u_{ij} ((1+\tau_{ij})\lambda_j)^{-\frac{\zeta}{1-\zeta}} + 1 - u_{ij} \right)^{-\frac{1-\zeta}{\zeta}} .$$

As in the basic model, we have that

$$\beta' \log \lambda = 0 \tag{C.2}$$

for the Cobb-Douglas final good or

$$\left(\sum_i \beta_i \lambda_i^{-\frac{\chi}{1-\chi}} \right)^{-\frac{1-\chi}{\chi}} = 1 \tag{C.3}$$

for the CES case. These $n+1$ equations pin down the prices. Given the prices, the flows of domestic intermediate goods are described by the M matrix with

$$m_{ij} = A_i^{\frac{\sigma}{1-\sigma}} \alpha_i v_{ij} u_{ij} \left[\sum_j v_{ij} (\lambda_i / \bar{\lambda}_{ij})^{\frac{\rho}{1-\rho}} \right]^{\frac{\sigma-\rho}{\rho(1-\sigma)}} \left(\frac{\lambda_i}{\bar{\lambda}_{ij}} \right)^{\frac{1}{1-\rho}} \left(\frac{\bar{\lambda}_{ij}}{(1+\tau_{ij})\lambda_j} \right)^{\frac{1}{1-\zeta}} ,$$

Similarly, the flow of imported intermediates is given by the matrix M^*

$$m^*_{ij} = A_i^{\frac{\sigma}{1-\sigma}} \alpha_i v_{ij}(1-u_{ij}) \left[\sum_j v_{ij} (\lambda_i/\bar{\lambda}_{ij})^{\frac{\rho}{1-\rho}} \right]^{\frac{\sigma-\rho}{\rho(1-\sigma)}} \left(\frac{\lambda_i}{\bar{\lambda}_{ij}} \right)^{\frac{1}{1-\rho}} \bar{\lambda}_{ij}^{\frac{1}{1-\zeta}}.$$

Then domestic production Y is determined by the exact same way as before:

$$Y = [d'(I-M')^{-1}c]^{-1}\bar{L},$$

with d and c defined as before. The vector of imported intermediates is then $M^{*\prime}Q = M^{*\prime}(I - M')^{-1}cY$. Since all the prices of imported intermediates are 1, we have:

$$O = (1 - \mathbf{1}' M^{*\prime}(I-M')^{-1}c)Y$$

C.1 Markups and calibration

Next, we show how to specialize and calibrate the model with a particular distortion: markups. Again, we proceed in a manner very similar to the model without trade. The price of sector i is then $p_i = mc_i \mu_i$. Then equation (C.1) can be rewritten as:

$$p_i = A_i^{-1} \left\{ (1-\alpha_i) B_i^{\frac{\sigma}{1-\sigma}} \eta^{-\frac{\sigma}{1-\sigma}} + \alpha_i \left[\sum_{j=1}^n v_{ij} \bar{p}_{ij}^{-\frac{\rho}{1-\rho}} \right]^{\frac{1-\rho}{\rho}\frac{\sigma}{1-\sigma}} \right\}^{-\frac{1-\sigma}{\sigma}} \mu_i \tag{C.4}$$

with $\bar{p}_{ij}$ the price of the foreign-domestic composite ij intermediate good,

$$\bar{p}_{ij} = \left(u_{ij} p_j^{-\frac{\zeta}{1-\zeta}} + 1 - u_{ij} \right)^{-\frac{1-\zeta}{\zeta}}.$$

Then we have the following:

$$n_{ij} \equiv \alpha_i v_{ij} u_{ij} = \frac{p_j X_{ij}}{p_i Q_i} A_i^{-\frac{\sigma}{1-\sigma}} \left[\sum_j v_{ij} (p_i/\bar{p}_{ij})^{-\frac{\rho}{1-\rho}} \right]^{-\frac{\sigma-\rho}{\rho(1-\sigma)}} \left(\frac{p_i}{\bar{p}_{ij}} \right)^{-\frac{1}{1-\rho}} \left(\frac{\bar{p}_{ij}}{p_j} \right)^{-\frac{1}{1-\zeta}} \frac{p_i}{p_j} \mu_i^{\frac{1}{1-\sigma}}$$

and

$$n^*_{ij} \equiv \alpha_i v_{ij}(1-u_{ij}) = \frac{I_{ij}}{p_i Q_i} A_i^{-\frac{\sigma}{1-\sigma}} \left[\sum_j v_{ij} (p_i/\bar{p}_{ij})^{-\frac{\rho}{1-\rho}} \right]^{-\frac{\sigma-\rho}{\rho(1-\sigma)}} \left(\frac{p_i}{\bar{p}_{ij}} \right)^{-\frac{1}{1-\rho}} \bar{p}_{ij}^{-\frac{1}{1-\zeta}} p_i \mu_i^{\frac{1}{1-\sigma}}$$

The term $\frac{p_j X_{ij}}{p_i Q_i}$ is the observed share of domestic inputs j relative to industry i sales; $\frac{I_{ij}}{p_i Q_i}$ is the corresponding share for imported intermediates. Given n_{ij} and n^*_{ij}, we can set $u_{ij} = n_{ij}/(n_{ij} + n^*_{ij}), \alpha_i = \sum_j [n_{ij} + n^*_{ij}]$ and $v_{ij} = (n_{ij} + n^*_{ij})/\alpha_i$. Since the right-hand side of the equations above depends on the structural parameters, we need to solve for a fixed point.

Now an additional elasticity parameter that we need to pick is the elasticity of substitution $1/(1-\zeta)$ between foreign and domestic varieties of inputs within the same sectoral classification. In the international trade literature this is known as the macro elasticity. The estimates of this elasticity vary in a wide range and depend on the exact specification of aggregations and on econometric procedure. Feenstra et al. (2014) uses GMM to find a macro elasticity that varies by sector and is between 1 and 2 for the majority of sectors. On the other hand, a survey by Anderson and van Wincoop (2004) suggests a number between 5 and 10. Atkeson and Burstein (2008) suggest using a value of 10. We report estimates for a wide range of elasticities and find that that the elasticity ε_ζ has very minor effect on the results.

We repeat the same experiments as in the closed economy model: what is the TFP gain from lowering markups in Mexico to the US level? Secondly, we show that the results do not stem solely from reduction in variance. We find that our results are robust and the TFP gains from lowering markups increase by about 30% when imported intermediates are taken into account.

Table 7: TFP gain from lowering markups to US levels (percents)

	$\varepsilon_\zeta = 1$			$\varepsilon_\zeta = 1.5$			$\varepsilon_\zeta = 2$			$\varepsilon_\zeta = 2.5$		
	ε_ρ			ε_ρ			ε_ρ			ε_ρ		
	0.01	0.7	1	0.01	0.7	1	0.01	0.7	1	0.01	0.7	1
$\epsilon_\sigma = 0.01$	3.43	4.20	4.51	3.48	4.25	4.57	3.53	4.31	4.62	3.57	4.36	4.67
$\epsilon_\sigma = 0.7$	12.39	13.89	14.50	12.48	13.99	14.60	12.57	14.09	14.71	12.65	14.19	14.81
$\epsilon_\sigma = 1$	17.41	19.45	20.28	17.52	19.59	20.42	17.63	19.72	20.57	17.74	19.85	20.71

Table 8: TFP gain from lowering average markups to average US levels (percents)

	$\varepsilon_\zeta = 1$			$\varepsilon_\zeta = 1.5$			$\varepsilon_\zeta = 2$			$\varepsilon_\zeta = 2.5$		
	ε_ρ			ε_ρ			ε_ρ			ε_ρ		
	0.01	0.7	1	0.01	0.7	1	0.01	0.7	1	0.01	0.7	1
$\epsilon_\sigma = 0.01$	2.70	3.27	3.52	2.81	3.39	3.65	2.93	3.52	3.78	3.05	3.66	3.92
$\epsilon_\sigma = 0.7$	7.03	7.80	8.14	7.16	7.95	8.30	7.30	8.11	8.47	7.45	8.28	8.64
$\epsilon_\sigma = 1$	9.18	10.07	10.46	9.33	10.24	10.64	9.49	10.41	10.82	9.65	10.59	11.01

Finally, we perform an experiment that combines some of the features of the two preceding one. We find the TFP gain from lowering the average level of of Mexican markups to the level in the US as follows:

$$\mu'_{i,mex} = \mu_{i,mex}\frac{\text{Mean } \mu_{us}}{\text{Mean } \mu_{mex}},$$

with Mean μ_a being the relevant country average markup with the the country sectoral gross outputs as weights.

Table 9: TFP gain from proportional lowering of markups

	ε_ρ		
	0.01	0.7	1
$\epsilon_\sigma = 0.01$	2.65	3.21	3.44
$\epsilon_\sigma =$	10.21	11.08	11.44
$\epsilon_\sigma = 1$	14.28	15.35	15.79