

Contingent Convertibles with Stock Price Triggers: The Case of Perpetuities*

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Comments Welcome

Abstract

Initial proposals for bank contingent convertibles (CoCos) envisioned that these bonds would convert to equity when the bank's stock price declined to a pre-specified trigger. Subsequent research claimed that doing so causes the stock price to have multiple equilibria or no equilibrium. We show that when CoCos are perpetuities, which characterizes most actual CoCos, a unique stock price equilibrium exists except under unrealistic conditions. Unique equilibria occur when conversion favors or disfavors CoCo investors, when CoCos convert to equity or are written down, and when CoCos are callable. We also analyze a bank's choice of risk before and after conversion.

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1 Introduction

The 2008 financial crisis sparked interest in contingent convertibles (CoCos) as a means of automatically recapitalizing banks and avoiding financial distress. CoCos are bonds issued by banks that either convert to new equity shares or experience a principal write down following an adverse triggering event. Since the first CoCo by Lloyds Bank in 2009, there have been over 500 different CoCo issues that, in total, have raised \$520 billion.¹ While these CoCos are triggered if a bank's regulatory capital declines below a pre-specified level, Flannery (2005) initially proposed that conversion be triggered by a decline in the market price of the bank's stock. However, subsequent research by Sundaresan and Wang (2015) and Glasserman and Nouri (2016) uncovered potential problems with stock price-triggered CoCos. Because conversion can affect the price of the bank's stock, yet the bank's stock price determines whether conversion occurs, an endogeneity problem can imply no equilibrium stock price. Specifically, these studies find that a rational stock price equilibrium *never* exists when a CoCo's contractual terms benefit the bank's shareholders by converting CoCos to a value below their par value.

Our paper shows that this no-equilibrium result critically depends on these prior studies' assumption that CoCos have a finite maturity. Instead, if a CoCo has a perpetual maturity, then there is no equilibrium only under very unlikely conditions. We develop a structural model that provides closed-form solutions for the values of a bank's stock and its CoCo perpetuity and prove that a unique equilibrium exists for reasonable parameter values. Only if the bank's asset volatility is unrealistically *low* would an equilibrium fail to exist. Thus, unlike finite-maturity CoCos, CoCo perpetuities imply unique equilibrium

¹These amounts are for the period ending in July 2017. See Moodys Global Credit Research 25 May 2016 and 18 October 2017.

stock prices even when they convert to little or no value.

The paper’s results have practical significance for at least two reasons. First, virtually all CoCos issued to date have conversion or write-down terms that harm CoCo investors and benefit shareholders.² This is exactly the contractual terms where a stock price equilibrium exists when a stock-triggered CoCo has a perpetual, but not finite, maturity. Second, in practice most CoCos are perpetuities. Under Basel III standards, CoCos must have a perpetual maturity to qualify as “Additional Tier 1” capital, rather than Tier 2 capital (Basel Committee on Banking Supervision (2011a), page 15). Berg and Kaserer (2015) document that of the 22 different CoCos issued during the 2009 to 2013 period that they analyze, 12 had perpetual maturities while 10 had finite maturities. Similarly, Avdjiev, Bolton, Jiang, Kartasheva, and Bogdanova (2015) report that 57.5% of their sample of 187 CoCos issued from September 2009 to March 2015 have a perpetual maturity.³ Consequently, it appears that banks will have an incentive to issue CoCos as perpetuities for the foreseeable future.

Our model contributes to a recent literature that analyzes various forms of CoCos. In particular, papers such as Bulow and Klemperer (2015), Calomiris and Herring (2013), Flannery (2009), Hart and Zingales (2011), McDonald (2013), and Pennacchi, Vermaelen, and Wolff (2014) advocate CoCos with different types of market value triggers.⁴ Yet, many policymakers and academics are skeptical of market-triggered CoCos, in part due to the Sundaresan and Wang (2015) finding that they lead to multiple equilibria or no equilibrium

²See Berg and Kaserer (2015) and Avdjiev, Bolton, Jiang, Kartasheva, and Bogdanova (2015).

³See also “Investors in Asia Return to Perpetual Bonds,” *Wall Street Journal* 8 July 2014 which states that Australian bank CoCos have led the issuance of perpetual bonds in Asia. Moreover, the article “Corporate Issuance of Perpetual Debt Soars,” *Financial Times* 16 June 2015 notes that banks have increased their issuance of perpetual CoCos.

⁴Exceptions are Squam Lake Working Group (2009) and Glasserman and Nouri (2012) that consider CoCos with regulatory capital triggers.

for the bank’s stock price.⁵ The claim is that when conversion terms heavily dilute the bank’s initial shareholders by issuing a large value of new equity to CoCo investors, there are multiple equilibria for the bank’s stock price. In the opposite case where shareholders benefit by converting CoCos to less than their par value, there is no stock price equilibrium. Only for the knife-edge case where conversion is perfectly neutral does a unique stock price equilibrium exist. The Basel Committee on Banking Supervision (2010) expressed concerns with market-triggered CoCos and, along with national bank supervisory authorities, have called for further study on the appropriate design of these CoCos.⁶

Upon further study, one result in Sundaresan and Wang (2015) has been shown to be mistaken. Glasserman and Nouri (2016) assume the same continuous-time setting as in Sundaresan and Wang (2015), but prove that when conversion terms benefit CoCo investors there is a unique equilibrium for the bank’s stock price, rather than multiple equilibria. Pennacchi and Tchistyi (2017) also provide a closed-form solution for unique equilibrium stock and finite-maturity CoCo prices that is a counter-example to the claim of multiple equilibria.⁷ Yet these two papers confirm the conclusion of Sundaresan and Wang (2015) that when CoCos benefit shareholders by converting to an equity value less than par, then an equilibrium stock price never exists.

The current paper demonstrates that CoCo perpetuities are an important exception to this conclusion of no equilibrium. It proves that a unique equilibrium exists when the present value of the stock’s future dividends, which we label the “candidate” stock price, is a monotonic function of the bank’s asset value. When CoCo conversion terms benefit

⁵Examples of economists citing Sundaresan and Wang (2015) and expressing concerns over multiple stock price equilibria or no stock price equilibrium are Avdjiev, Kartasheva, and Bogdanova (2013) (Bank for International Settlements) and Leitner (2012) and Prescott (2012) (U.S. Federal Reserve).

⁶See the Basel Committee on Banking Supervision (2013) and the U.S. Financial Stability Oversight Council (2012) which both recommend further review of CoCos.

⁷Both papers identify the same error in a proof by Sundaresan and Wang (2015).

the bank’s shareholders, monotonicity occurs when this candidate stock price reflects a sufficiently large “conversion option” value. In particular, if the bank’s assets are sufficiently volatile then the stock price is raised by this option-to-convert even for asset values significantly higher than the asset level where equilibrium conversion must occur, thereby making the price function monotonic in assets. But if a CoCo has a finite maturity, this option value must vanish as the CoCo approaches maturity, so that it cannot prevent nonmonotonicity and the no equilibrium condition.

Our analysis is comparable to prior research on market-based corrective actions. Bond, Goldstein, and Prescott (2010) also relate equilibrium to a stock price’s monotonicity with respect to a firm’s assets. They study a situation where a government uses information on a firm’s stock price to decide upon an intervention that improves the firm’s fundamental value and, in turn, the value of its shareholders’ equity.⁸ Similarly, in our case when conversion terms benefit shareholders, a CoCo-issuing bank’s stock price also both determines an intervention (conversion) and reflects that intervention’s improvement in shareholders’ equity. Yet a difference is that a CoCo’s conversion does not change the firm’s fundamental value. Rather, it raises shareholders’ equity via a transfer of value from CoCo investors.

Our finding that perpetual CoCos lead to a monotonic relation between equilibrium stock prices and asset values has implications for several models that assume CoCos are triggered by the market value of a bank’s assets. This research includes Albul, Jaffee, and Tchisty (2013), Hilscher and Raviv (2014), Koziol and Lawrenz (2012), and Himmelberg and Tsyplakov (2014). A potential criticism of this work is that, in practice, such CoCos cannot be implemented since the market value of a bank’s assets is not directly observable.

⁸Birchler and Facchinetti (2007) model a similar problem of a government intervening to reduce the probability of a firm’s failure. They assume the government’s action is a continuous function while Bond, Goldstein, and Prescott (2010) assume a binary action. Both papers relate the existence of equilibrium to a security value’s monotonicity with respect to a firm’s pre-intervention fundamentals.

Yet if these papers' models are modified to permit perpetual CoCos, then asset values may be mapped to unique equilibrium stock prices that can be observed for banks with publicly-traded stock. Thus, prior work may gain practical significance based on our results.

The plan of our paper is as follows. Section 2 develops a structural model of a bank that funds its assets with senior debt, a CoCo with a stock price trigger, and shareholders' equity. The model is consistent with the continuous-time framework of Sundaresan and Wang (2015) and Glasserman and Nouri (2016) except that the CoCo is assumed to be a perpetuity. We derive the conditions for which a unique stock price equilibrium exists and show that they are likely to hold even when CoCos convert to little or no value. Section 3 considers extensions of the basic model that confirm the robustness of its results. Conditions under which a unique equilibrium exists expands if the model is extended to allow a bank to call its CoCo. Other extensions allow for direct costs of bankruptcy, for more general conditions determining bank failure, and for permitting a bank to choose its asset risk that maximizes shareholder value. Section 4 discusses recent bank regulatory events that reinforce the need for a market-triggered CoCo perpetuity. Section 5 concludes.

2 A Model of Perpetual-Maturity CoCos

This section develops a structural model of a bank that funds its assets with senior debt, a perpetual-maturity CoCo that has a stock (equity) price trigger, and shareholders' equity. It shows that a unique stock price equilibrium can exist for a broad set of conditions, including situations where conversion terms favor the bank's initial shareholders. Other than the CoCo being a perpetuity, the model's assumptions are consistent with the frameworks

of Sundaresan and Wang (2015) and Glasserman and Nouri (2016).⁹

2.1 Model Assumptions

Let there be a risk-neutral probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \in [0, T]\}, Q)$ in which the information flow $\{\mathcal{F}_t\}$ is generated by the Brownian motion z_t . Assume that a bank's assets produce a continuous cashflow equal to rate a_t per unit time at date t . This cashflow rate follows the risk-neutral process

$$da_t = \mu a_t dt + \sigma a_t dz_t \quad (1)$$

where μ and $\sigma > 0$ are constants.¹⁰ Let $r > 0$ be the constant risk-free rate of interest. Then assuming $\mu < r$, the market value of the bank's assets, A_t , equals

$$A_t = E^Q \left[\int_t^\infty e^{-r(s-t)} a_s ds \right] = \frac{a_t}{r - \mu} . \quad (2)$$

Thus, the value of assets, without reinvested cashflows, follows the risk-neutral process¹¹

$$\frac{dA_t}{A_t} = \mu dt + \sigma dz_t \quad (3)$$

At the initial date $t = 0$, the bank's liabilities consist of shareholders' equity, senior non-convertible debt, and perpetual-maturity CoCos. The initial shareholders own n shares of stock. Senior debt has a principal value of B and pays fixed coupon interest continuously

⁹Section 6.3 of Glasserman and Nouri (2016) briefly considers perpetual maturity CoCos and verifies that a unique stock price equilibrium exists when conversion terms favor CoCo investors. However, they do not explore the possibility that a unique equilibrium exists when conversion terms favor shareholders.

¹⁰The cashflow rate's actual (physical) process can be written as $da_t = \mu^P a_t dt + \sigma a_t dz_t^P$ where $\mu^P = \mu + \theta\sigma$ and θ is the market price of risk associated with the Brownian motion $dz_t^P = dz_t - \theta dt$.

¹¹Since $a_t = A_t(r - \mu)$, assets' risk-neutral returns including cashflows are $dA_t + a_t dt = r A_t dt + \sigma A_t dz_t$.

at the rate of b . Prior to conversion, CoCos also pay fixed coupon interest continuously at the rate of c and have a principal value of C . CoCos are assumed to convert to m additional shares of stock when the bank's per share stock price falls to the level $L > 0$. As will become clear, it is assumed that $A_0 > L(n + m) + bB/r$ to rule out the case where CoCos immediately convert to equity at the initial date.

Bank regulators are assumed to close the bank when its asset value first falls to the default-free value of its non-convertible debt. Specifically, the bank is declared to fail if its assets decline to bB/r , at which time senior debtholders own all of the assets and shareholders are wiped out. Consequently in this benchmark model, senior debt is default-free.¹² For tractability, senior debt is assumed to be perpetual, yet note that if it is default-free and its coupon rate $b = r$, then the model is unchanged if senior debt has a finite maturity but is rolled over such that its outstanding principal remains at B .¹³

As in Sundaresan and Wang (2015) and Glasserman and Nouri (2016), the assets' cashflows, a_t , are paid out as coupons and dividends to the bank's debtholders and shareholders. Consequently, shareholders receive a continuous dividend per share of $[(a_t - bB - cC)/n] dt$ prior to the CoCos' conversion and $[(a_t - bB)/(n + m)] dt$ following the CoCos' conversion. Define $\tau_\delta = \inf \{t \in [0, \infty) : A_t \leq bB/r\}$ to be the bank's closure date, which will be shown to occur after the date of the CoCos' conversion. Then from equation (2), the per share dividend paid just prior to closure equals $[(r - \mu) \frac{bB}{r} - bB]/(n + m) dt$, which is positive (*negative*) if μ is negative (*positive*).¹⁴

¹²An extension in Section 3 considers bankruptcy costs and default-risky senior debt.

¹³Alternatively, if the bank is closed at an asset level less than B , senior debt could have a finite maturity and be rolled-over at the rate r if it was guaranteed by explicit or implicit government deposit insurance. Leland and Toft (1996) also assumes finite-maturity debt is continuously reissued such that its coupon and principal remain constant.

¹⁴Therefore, if μ^P and θ are the cashflows' actual growth rate and the market price of risk, respectively, negative dividends can be ruled out if $\mu^P \leq \theta\sigma$.

2.2 The Equilibrium Stock Price

Similar to Glasserman and Nouri (2016), consider two hypothetical banks whose assets and senior debt are identical to the previously-discussed bank that issues CoCos. The first is referred to as a “post-conversion” bank. It has no CoCos but $n + m$ shares of equity, and its stock price per share is denoted $U_t(A_t)$. The second reference bank is called a “no-conversion” bank. It has n shares of equity and additional non-convertible debt with the same coupon interest and principal as the CoCos of the CoCo-issuing bank. This no-conversion bank’s stock price is denoted $V_t(A_t)$. As before, it is assumed that regulators close these banks whenever assets fall to the default-free value of the bank’s non-convertible debt, which for the no-conversion bank is $(bB + cC)/r$. Since both banks’ debt is default-free, the market value of their equity equals the residual asset value. The post-conversion bank’s stock price per share prior to its closure is

$$\begin{aligned}
 U_t &= \frac{1}{n+m} E_t^Q \left[\int_t^{\tau_\delta} e^{-r(s-t)} (a_s - bB) ds \right] \\
 &= \frac{1}{n+m} \left\{ E_t^Q \left[\int_t^\infty e^{-r(s-t)} (a_s - bB) ds \right] - E_t^Q \left[\int_{\tau_\delta}^\infty e^{-r(s-t)} (a_s - bB) ds \right] \right\} \\
 &= \frac{1}{n+m} \left(A_t - \frac{bB}{r} \right)
 \end{aligned} \tag{4}$$

where the last line uses the fact that regulators close the bank at date τ_δ when $A_{\tau_\delta} = bB/r$, so that the second expected value equals zero. Note that the stock price U_t always exists, is unique, and is strictly increasing in the asset level A_t .

Similar logic implies that the no-conversion bank’s stock price is

$$V_t = \frac{1}{n} \left(A_t - \frac{bB + cC}{r} \right). \tag{5}$$

Now define A_{uc} as the asset value such that the post-conversion bank's per share stock price equals L ; that is, $U(A_{uc}) = L$. Inverting (4) implies

$$A_{uc} = L(n + m) + \frac{bB}{r}. \quad (6)$$

Similarly, define A_{vc} as the asset value such that the no-conversion bank's per share stock price equals L . Then

$$A_{vc} = Ln + \frac{bB + cC}{r}. \quad (7)$$

Next let $\tau_{uc} = \inf \{t \in [0, \infty) : A_t \leq A_{uc}\}$ be the date when the post-conversion bank's stock price equals L . From (6), this date occurs prior to the bank's closure, $\tau_{uc} < \tau_\delta$.

We now return to the CoCo-issuing bank and define what constitutes a rational expectations equilibrium for its CoCo conversion and stock price. Our definition is that of Sundaresan and Wang (2015) and Glasserman and Nouri (2016), namely, that conversion occurs when the stock price first equals L and that the stock price equals the risk-neutral expectation of discounted future dividends received before and after conversion.

Definition: *A conversion time, $\hat{\tau}$, and a per-share stock value, $\hat{S}_t$, are an equilibrium if*

$$\hat{\tau} = \inf \left\{ t \in [0, \infty) : \hat{S}_t \leq L \right\}, \quad (8)$$

and the stock price per share is

$$\hat{S}_t = E_t^Q \left[\int_t^\infty e^{-r(s-t)} \left(\frac{1}{n} (a_s - bB - cC) \cdot 1_{s < \min\{\hat{\tau}, \tau_\delta\}} + \frac{1}{n+m} (a_s - bB) \cdot 1_{\hat{\tau} \leq s < \tau_\delta} \right) ds \right]. \quad (9)$$

Note that equation (9) implies that the equilibrium stock price following conversion satisfies $\hat{S}_t = U_t$ in (4) for $\hat{\tau} \leq t < \tau_\delta$. Next we show that *if* an equilibrium exists, then

the stock price must be continuous and, furthermore, conversion must occur at date τ_{uc} . The following lemma first makes the continuity argument.

Lemma 1: *If there is an equilibrium stock price, then $\hat{S}_t$ is continuous in t .*

Proof: *See the Appendix.*

The intuition for Lemma 1 can be understood by noting that information about the bank's future cashflows, and thus dividends, evolves continuously from the Brownian motion dz_t . Therefore, the stock price, which is the rational expectation of discounted future dividends, must also evolve continuously and cannot jump.¹⁵

The following proposition builds on Lemma 1 to state that if the bank's equilibrium stock price exists, then conversion must occur at date τ_{uc} , which is when the post-conversion bank's stock price equals L .

Proposition 1: *If there is an equilibrium stock price, then conversion happens when A_t falls to A_{uc} for the first time; that is,*

$$\hat{\tau} = \tau_{uc} = \inf \{t \in [0, \infty) : A_t \leq A_{uc}\} . \quad (10)$$

Proof: *See the Appendix.*

The essence of this proposition is that at the time of conversion, the CoCo-issuing bank becomes identical to the post-conversion bank. So to avoid a jump in the bank's stock price at conversion, the post-conversion bank's stock price must equal L . But from (6) that can only occur when $A_t = A_{uc}$, so that $\hat{\tau} = \tau_{uc}$.¹⁶

¹⁵A similar argument can be made if the cashflow rate, and therefore the value of assets, follows a mixed jump-diffusion process. As discussed in Glasserman and Nouri (2016), in this case an equilibrium stock price can jump only when there is a realized jump in the cashflow rate and asset value.

¹⁶The continuity requirement that conversion occur when the post-conversion stock price first equals L generalizes to a setting where stock values depend on a multiple state variables. See Glasserman and Nouri

Making the substitution $\hat{\tau} = \tau_{uc}$ in the rational pre-conversion stock price formula (9) allows us to solve in closed-form for the stock price, as well as the CoCo value, when an equilibrium exists. The result is given in the next proposition.

Proposition 2: *If there is an equilibrium stock price, then the pre-conversion date t values of the stock, S_t , and the CoCo, C_t , equal*

$$\begin{aligned} S_t(A_t) &= \frac{1}{n} \left\{ A_t - \frac{bB}{r} - \frac{cC}{r} \left[1 - \left(\frac{A_t}{A_{uc}} \right)^{-\gamma} \right] - mL \left(\frac{A_t}{A_{uc}} \right)^{-\gamma} \right\} \\ &= V_t + \varsigma(A_t) \end{aligned} \quad (11)$$

$$\begin{aligned} C_t(A_t) &= A_t - nS_t - \frac{bB}{r} = \frac{cC}{r} \left[1 - \left(\frac{A_t}{A_{uc}} \right)^{-\gamma} \right] + mL \left(\frac{A_t}{A_{uc}} \right)^{-\gamma} \\ &= \frac{cC}{r} - n\varsigma(A_t) \end{aligned} \quad (12)$$

where

$$\gamma \equiv \frac{1}{\sigma^2} \left[\mu - \frac{1}{2}\sigma^2 + \sqrt{\left(\mu - \frac{1}{2}\sigma^2 \right)^2 + 2r\sigma^2} \right] > 0, \text{ and} \quad (13)$$

$$\varsigma(A_t) \equiv \frac{1}{n} \left(\frac{cC}{r} - mL \right) \left(\frac{A_t}{A_{uc}} \right)^{-\gamma}. \quad (14)$$

Proof: *See the Appendix.*

The quantity $\varsigma(A_t)$ defined in (14) is described in more detail below and equals the value of a “conversion option.” It plays a key role in determining whether a stock price equilibrium exists. For now, we emphasize that $S_t(A_t)$ in (11) is only a “candidate” pre-

(2016). For example, if $U(X_t)$ where X_t is a vector of state variables generated by multiple Brownian motions, then conversion must occur the first time that $U(X_t) = L$. Because our model has the single state variable, A_t , and $U(A_t)$ is invertible, Proposition 1 allows us to define $A_{uc} = U^{-1}(L)$ as the unique asset value at which conversion must occur. In a more general model, equilibrium stock prices may depend on variables in addition to A_t , such as default-free interest rates or stochastic asset volatility state variables, but $\hat{S}(X_{\hat{\tau}}) = U(X_{\hat{\tau}}) = L$ is still required for an equilibrium.

conversion stock price in that it is the equilibrium price only if the equilibrium exists. Next we show that the existence of a unique equilibrium is equivalent to this candidate stock price being an increasing function of the bank's assets. The next lemma specifies the set of parameters for which $S_t(A_t)$ is strictly increasing in A_t for all $A_t \geq A_{uc}$.

Lemma 2: *If one of the following is true:*

$$(i) \quad mL \geq \frac{cC}{r} \text{ or}$$

$$(ii) \quad mL < \frac{cC}{r} \text{ and}$$

$$\sigma^2 > \frac{2(r + \mu\gamma^*)}{\gamma^*(1 + \gamma^*)}, \quad (15)$$

where

$$\gamma^* \equiv \frac{\frac{bB}{r} + L(n + m)}{\frac{cC}{r} - Lm}, \quad (16)$$

then $S_t(A_{uc}) = L$ and $S_t(A_t)$ is strictly increasing in A_t for all $A_t \geq A_{uc}$. Otherwise, $S_t(A_t) \leq L$ for some $A_t > A_{uc}$.

Proof: See the Appendix.

Note that the inequality (15) can be re-written in terms of the CoCo's trigger, L :

$$L > \frac{\gamma cC - bB}{r(n + (1 + \gamma)m)}. \quad (17)$$

Lemma 2 shows that the candidate stock price is an increasing function of the bank's assets when conversion terms benefit CoCo investors ($mL \geq cC/r$), meaning that CoCo investors obtain a value of stock, mL , that exceeds the value of their bond if it was not convertible, cC/r . Alternatively, when conversion terms benefit initial stockholders ($mL < cC/r$) so that CoCo investors are compensated with less stock than their unconverted bond's value, the parametric region under which the stock price continues to increase can

be characterized in various ways. Condition (15) is in terms of the bank's asset return variance while condition (17) is in terms of the conversion trigger, L .

Since $\mu < r$ for the value of the bank's assets to be finite, if we set $\mu = r$ in condition (15), a lower bound on σ^2 results that is sufficient, though not necessary, to guarantee the monotonicity of $S_t(A_t)$:

$$\sigma^2 > \frac{2r}{\gamma^*}. \quad (18)$$

While condition (18) is stronger than condition (15), it holds for any feasible cashflow growth rate. Substituting (16) into (18) and rearranging terms results in a bound on the conversion trigger, which can be useful for implementation purposes.

Corollary 1: *If $mL < \frac{cC}{r}$ and*

$$L > \frac{\frac{cC}{r} - \frac{\sigma^2}{2r} \frac{bB}{r}}{m + \frac{\sigma^2}{2r}(n + m)} \quad (19)$$

then $S_t(A_{uc}) = L$ and $S_t(A_t)$ is strictly increasing in A_t for all $A_t \geq A_{uc}$.

Given Lemma 2, we can now state the theorem for the existence and uniqueness of a stock price equilibrium:

Theorem 1: *If either condition (i) or (ii) of Lemma 2 is satisfied, then there exists a unique equilibrium in which the CoCo converts when the bank's asset level drops to A_{uc} for the first time and where the equilibrium stock prices per share before and after conversion are given by (11) and (4), respectively. If neither condition (i) nor (ii) in Lemma 2 is satisfied, then there is no equilibrium.*

Proof: Equation (11) sets $S_t(A_t)$ equal to the present value of the dividends per share assuming that CoCos convert when the asset level drops to A_{uc} for the first time. Moreover,

$S_t(A_{uc}) = L$, equal to the post-conversion stock price. When either condition (i) or (ii) in Lemma 2 is satisfied, then $S_t(A_t) > L$ for all $A_t > A_{uc}$, and $S_t(A_t)$ is an equilibrium price. On the other hand, when neither condition (i) nor (ii) in Lemma 2 is satisfied, $S_t(A_t) \leq L$ for some $A_t > A_{uc}$, which is inconsistent with conversion occurring the first time that $A_t = A_{uc}$, so that $S_t(A_t)$ cannot be an equilibrium stock price. Proposition 1 ensures that there is no other equilibrium.

To understand the economic intuition for Theorem 1, note that the candidate stock price in (11) is $S_t = V_t + \varsigma(A_t)$ where V_t is the no-conversion stock price and $\varsigma(A_t) \equiv \frac{1}{n} \left(\frac{cC}{r} - mL \right) \left(\frac{A_t}{A_{uc}} \right)^{-\gamma}$ is a per-share “conversion option” value. Similarly, the CoCo value in (12) is $C_t = cC/r - n\varsigma(A_t)$, equal to its no-conversion value minus the same total conversion option value. We refer to $\varsigma(A_t)$ loosely as an “option” though its payoff to shareholders (*CoCo investors*) is positive only when conversion terms benefit shareholders (*CoCo investors*). It has the property of a “digital” or “binary” barrier option that pays the fixed amount $\frac{1}{n} (cC/r - mL)$ when assets fall to the exercise price A_{uc} . Note that $\text{sign}[\partial \varsigma(A_t) / \partial A_t] = -\text{sign}[(\frac{cC}{r} - mL)]$, $\lim_{A_t \rightarrow \infty} \varsigma(A_t) = 0$, and $\varsigma(A_{uc}) = U_t(A_{uc}) - V_t(A_{uc})$. This last property ensures that $S_t = V_t + \varsigma(A_t)$ equals the post-conversion stock price, U_t , when $A_t = A_{uc}$.

First consider the case of conversion terms that benefit CoCo investors, $mL \geq cC/r$, which is condition (i) of Lemma 2. Here both $\partial V_t / \partial A_t > 0$ and $\partial \varsigma(A_t) / \partial A_t > 0$, so that $\partial S_t / \partial A_t$ is strictly increasing in assets. Intuitively, as the bank’s assets increase, the stock price increases due to the usual increase in equity’s residual asset value (V_t) and due to the lower likelihood of conversion that harms shareholders. With both of these effects being positive, monotonicity is assured and an equilibrium stock price always exists.

The opposite case of conversion terms that benefit shareholders, $cC/r > mL$, is more

complex. As the bank's assets increase, one effect is that the no-conversion price increases at a constant rate ($\partial V_t / \partial A_t = 1/n$) reflecting shareholders' residual claim on the assets. But a second effect, due to the lower likelihood of a conversion that benefits shareholders, lowers the stock price: $\partial \varsigma(A_t) / \partial A_t < 0$. If this second effect is stronger than the first for some asset levels, the candidate stock price becomes non-monotonic and no equilibrium exists. This can happen only when the asset volatility σ is very low.

To see this, recall that at the conversion threshold, A_{uc} , the conversion option value equals its maximum of $\varsigma(A_{uc}) = \frac{1}{n} \left(\frac{cC}{r} - mL \right)$. If volatility is very low, a small increase in the asset level from A_{uc} drastically reduces the chance of conversion. As a result, $\varsigma(A_t)$ declines so quickly that the overall candidate stock price decreases with the asset level in the right neighborhood of A_{uc} . On the other hand, if volatility is not very low, a small increase in the asset level does not significantly reduce the likelihood of conversion. As a result, the first effect ($\partial V_t / \partial A_t = 1/n > 0$) offsets the second ($\partial \varsigma(A_t) / \partial A_t < 0$) and the candidate stock price is monotonically increasing in the asset level.

Figure 1 illustrates this case of conversion terms that benefit shareholders, $cC/r > mL$. Each of the three panels in Figure 1 graphs stock prices as a function of the bank's asset level and shows identical stock triggers (L : short dashed line), no-conversion stock prices (V_t : long dashed line), and post-conversion stock prices (U_t : dotted line). Since $cC/r > mL$, the post-conversion price U_t hits the trigger level L at a lower asset value than that where the no-conversion price V_t hits the trigger level L ; that is, $A_{uc} < A_{vc}$.

Because the pre-conversion candidate stock price equals $S_t = V_t + \varsigma(A_t)$ and the option value $\varsigma(A_t) > 0$ when conversion benefits shareholders, the candidate stock price must be above the no-conversion price. This is indicated by the solid line, S_t , in each of the three panels, with the difference between S_t and V_t equal to the option value $\varsigma(A_t)$. The upper-

most panel graphs S_t when σ is very close to zero ($\sigma \approx 0$), which makes the option value very close to zero ($\varsigma \approx 0$) except for asset values in the immediate right neighborhood of A_{uc} where the option value must rise very quickly to equal $U_t(A_{uc}) - V_t(A_{uc}) = \frac{1}{n} \left(\frac{cC}{r} - mL \right)$. A consequence of this very low asset volatility is that S_t effectively coincides with the no-conversion stock price, V_t , for $A_t > A_{uc}$ so that the candidate stock price first equals the trigger level at $A_{vc} > A_{uc}$. This is clearly a situation where condition (ii) of Lemma 2 fails, so that S_t is non-monotonic and no stock price equilibrium exists.

The middle panel shows another instance where the bank's asset volatility, σ , while not zero is still smaller than required to satisfy condition (ii) of Lemma 2. The option value, $\varsigma(A_t) = S_t - V_t$, is too small for asset values greater than A_{uc} to prevent the candidate stock price from breaching the trigger before A_{uc} . Consequently, the candidate stock price remains non-monotonic and no equilibrium exists.

Finally, the bottom panel of Figure 1 shows the candidate stock price for an asset volatility that we later argue is more typical for banks. Here, σ satisfies condition (ii) of Lemma 2 so that the conversion option value $\varsigma(A_t)$ is large enough to keep the candidate stock price above L for all asset values greater than A_{uc} . The candidate stock price is a monotonic function of the bank's assets, so that it is the unique equilibrium stock price.

Our discussion illustrated in Figure 1 has been in terms of Lemma 2 condition (ii)'s inequality (15) relating to σ . Yet the equivalent inequality (17) relates to the trigger level, L . A higher L reduces the conversion benefit to shareholders, $\left(\frac{cC}{r} - mL \right)$, relative to no conversion. Thus, as the bank's asset value declines to the threshold A_{uc} , there is less upward pressure on the stock price due to the smaller conversion benefit. Consequently, the conversion option effect will be dominated by the opposite effect stemming from the decline in equity's residual asset value (V_t). Therefore, a sufficiently high trigger also

ensures monotonicity and a unique equilibrium stock price.

Based on assumptions similar to those of this paper’s model, Pennacchi and Tchisty (2017) show that if CoCos have a finite maturity and conversion terms benefit shareholders, the candidate stock price is not monotone in the asset level for *any* volatility, so that an equilibrium stock price never exists.¹⁷ The intuition is that as time t approaches the maturity date T , the conversion option’s total asset volatility until maturity, $\sigma\sqrt{T-t}$, collapses to zero for any σ . As a result, the candidate stock price is non-monotonic in the right neighborhood of A_{uc} , since a small change in the asset level can make the conversion option either very valuable or worthless. Hence, an equilibrium stock price never exists.

2.3 Principal Write Down

CoCos that have their principal written down have become more common.¹⁸ This section considers a CoCo perpetuity where the bank’s stock price triggers an automatic principal write-down instead of a conversion to new equity.¹⁹ As before, senior debt pays fixed coupon interest of b and has a principal value of B . A CoCo bond is now contingent debt that pays fixed coupon interest of c and has a principal value of C as long as the stock price remains above L , but has its principal reduced to αC when the stock price falls to L for the first time. We assume senior debt and post-write-down contingent debt are default-free, since bank regulators close the bank whenever assets fall to $bB/r + \alpha cC/r$, the default-free value of the bank’s debt. As we demonstrate below, this setting is virtually

¹⁷Corollary 5.1 in Glasserman and Nouri (2016) proves this result for finite-maturity CoCos under more general modeling assumptions.

¹⁸Boermans, Petrescu, and Vlahu (2014) state that CoCos with principal write downs have become more popular with European banks. Avdjiev, Bolton, Jiang, Kartasheva, and Bogdanova (2015) report that from 2009 to 2015 the proportion of CoCos that have a principal write down is 67% in terms of number of issues and 56% in terms of dollar volume of issues.

¹⁹Section 5.2 of Glasserman and Nouri (2016) considers finite-maturity CoCos with a principal write-down and shows that an equilibrium stock price never exists.

identical to a setting with conversion-to-equity CoCos having $m = \alpha \frac{cC}{rL}$.

The stock price per share of a “post-write-down” bank prior to its closure equals

$$U_t^w = \frac{1}{n} \left(A_t - \frac{bB}{r} - \alpha \frac{cC}{r} \right) , \quad (20)$$

which allows us to define A_{uc}^w as the asset value such that this stock price equals L :

$$A_{uc}^w = nL + \frac{bB}{r} + \alpha \frac{cC}{r} . \quad (21)$$

When a CoCo has its principal written down, the definition of a rational expectations equilibrium is the same as the previous one for a conversion-to-equity CoCo except that the pre-conversion stock price (9) is replaced with the pre-write-down stock price

$$\hat{S}_t = \frac{1}{n} E_t^Q \left[\int_t^\infty e^{-r(s-t)} \left((a_s - bB - cC) \cdot 1_{s < \min\{\hat{\tau}, \tau_\delta\}} + (a_s - bB - c\alpha C) \cdot 1_{\hat{\tau} \leq s < \tau_\delta} \right) ds \right] . \quad (22)$$

Based on logic nearly identical to that in Lemma 1 and Propositions 1 and 2, the “candidate” equilibrium stock price S_t^w before the write-down is derived to equal

$$\begin{aligned} S_t^w(A_t) &= \frac{1}{n} \left\{ A_t - \frac{bB}{r} - \frac{cC}{r} \left[1 - \left(\frac{A_t}{A_{uc}^w} \right)^{-\gamma} \right] - \alpha \frac{cC}{r} \left(\frac{A_t}{A_{uc}^w} \right)^{-\gamma} \right\} \\ &= V_t + \varsigma^w(A_t) \end{aligned} \quad (23)$$

where $\varsigma^w(A_t) \equiv \frac{1}{n} \left(\frac{cC}{r} \right) (1 - \alpha) \left(\frac{A_t}{A_{uc}^w} \right)^{-\gamma}$. Comparing equations (21) with (6) and (23) with (11), one sees that $A_{uc}^w = A_{uc}$ and $S_t^w(A_t) = S_t(A_t)$ when $\alpha \frac{cC}{r} = mL$. Therefore, the results of Lemma 2 and Theorem 1 can be translated to this setting of principal write-down CoCos. While the principal “write-up” case of $\alpha > 1$ is not practically meaningful, it is

included for completeness.²⁰

Lemma 3: *If one of the following is true:*

(i) $\alpha \geq 1$ or

(ii) $\alpha < 1$ and

$$L > \frac{\gamma cC - bB}{r \left(n + (1 + \gamma) \alpha \frac{cC}{rL} \right)} \quad (24)$$

then $S_t^w(A_{uc}^w) = L$ and $S_t^w(A_t)$ is strictly increasing in A_t for all $A_t \geq A_{uc}^w$. Otherwise, $S_t^w(A_t) \leq L$ for some $A_t > A_{uc}^w$.

By solving (24) for α , condition (ii) of Lemma 3 can be re-written as

$$(ii) \quad 1 > \alpha > \frac{\gamma cC - bB - nrL}{cC(1 + \gamma)} . \quad (25)$$

Theorem 2: *If either condition (i) or (ii) of Lemma 3 is satisfied, then there exists a unique equilibrium in which CoCos are written down when the bank's asset level drops to A_{uc}^w for the first time and where the equilibrium stock prices per share before and after the write-down are given by (23) and (20), respectively. If neither condition (i) nor (ii) of Lemma 3 is satisfied, there is no equilibrium.*

Note from condition (25) that a unique equilibrium stock price exists for any non-negative value of α when $\gamma cC < (bB + nrL)$. Furthermore, the logic of Lemma 2 with $m = \alpha \frac{cC}{rL}$ can be applied to show that a unique equilibrium exists whenever

$$\sigma^2 > \frac{2(r + \mu\gamma^{**})}{\gamma^{**}(1 + \gamma^{**})}, \quad (26)$$

²⁰The proofs of Lemma 3 and Theorem 2 are nearly identical to those of Lemma 2 and Theorem 1.

where

$$\gamma^{**} \equiv \frac{\frac{bB}{r} + \alpha \frac{cC}{r} + Ln}{\frac{cC}{r} (1 - \alpha)}. \quad (27)$$

2.4 Numerical Examples

This section presents numerical examples that show a unique stock price equilibrium exists for realistic parameter values. The following are benchmark parameter values:

<u>Parameter</u>	<u>Value</u>	<u>Parameter</u>	<u>Value</u>
Senior Debt Principal, B	90	CoCo Conversion Shares, m	1
Senior Debt Coupon Rate, b	3.2%	Trigger Price, L	4
CoCo Principal, C	5	Risk-neutral Cashflow Growth, μ	0.0%
CoCo Coupon Rate, c	3.6%	Volatility of Asset Returns, σ	4.0%
Initial Equityholder Shares, n	1	Default-free Interest Rate, r	3.0%

These parameters imply that the value of senior debt equals $bB/r = 96$, which is also the value of assets at which regulators close the bank. Non-convertible debt with the same principal and coupon as CoCos would be worth $cC/r = 6$. If CoCos do convert, they receive 50% of the post conversion total shares ($m = n = 1$). The volatility of asset returns, $\sigma = 4\%$, equals what Pennacchi, Vermaelen, and Wolff (2014) estimate to be the average asset return volatility for Bank of America, Citigroup, and JPMorgan Chase over the period 2003 to 2012.²¹ The risk-neutral cashflow growth rate of $\mu = 0$ implies that dividends paid to equity holders decline to equal zero at the time that regulators close the bank. Given these values of μ , σ , and $r = 3\%$, the implied value of γ from (13) is 5.62.

²¹The model in Pennacchi, Vermaelen, and Wolff (2014) assumes assets follow a jump-diffusion process. Their estimate of a total return volatility of 4% is broken down between a diffusion component volatility of 3% and a jump component volatility of 1%. The asset return volatility estimates for individual banks are 4.2%, 4.4%, and 3.3% for Bank of America, Citigroup, and JPMorgan Chase, respectively. Berg and Gider (2017) estimate an average asset volatility of 3.5% for all banks in Compustat during 1965 to 2013.

2.4.1 Conversion to Equity

Since Theorem 1 shows that a unique equilibrium always exists if conversion terms benefit CoCo investors ($mL \geq cC/r$), for brevity our examples focus on the opposite case of a conversion option that benefits shareholders so that $\varsigma(A_t) \equiv \frac{1}{n} \left(\frac{cC}{r} - mL \right) \left(\frac{A_t}{A_{uc}} \right)^{-\gamma} > 0$. Thus, we assume a low stock price trigger of $L = 4$ so that $\left(\frac{cC}{r} - mL \right) = (6 - 4) = 2$. The case of the benchmark parameters is displayed in Figure 2 Panel A which graphs stock prices as a function of the bank's asset values.

The dotted line in Figure 2 is the stock price of the hypothetical post-conversion bank, U_t , in equation (4). The asset value at which $U_t = L = 4$ is $A_{uc} = L(n + m) + \frac{bB}{r} = 104$. The long dashed line in Figure 2 is the stock price of the no-conversion bank, V_t , in equation (5). The asset value at which $V_t = L = 4$ is $A_{vc} = Ln + \frac{bB+cC}{r} = 106$. The solid line is the “candidate” pre-conversion equilibrium stock price, S_t , in equation (11) for $A_t \geq A_{uc}$. Note that the benchmark parameter values satisfy condition (ii) of Lemma 2 since $\sigma = 4.0\% > \sqrt{2(r + \mu\gamma^*) / [\gamma^*(1 + \gamma^*)]} = 0.47\%$.²² Thus, as Theorem 1 guarantees and the figure shows, S_t is a monotonically increasing function of A_t and is an equilibrium pre-conversion stock price. For assets less than $A_{uc} = 104$, the equilibrium stock price equals the post-conversion price, U_t .

Condition (ii) of Lemma 2 implies that when $L = 4$, the minimum asset volatility for a unique equilibrium is $\sigma > \sqrt{2(r + \mu\gamma^*) / [\gamma^*(1 + \gamma^*)]} = 0.47\%$, and this minimum volatility case is illustrated in Figure 2 Panel B. Consequently, for there to be no equilibrium, a bank's asset volatility would need to be less than 12% of the typical volatility of 4%. Of course one might question whether a trigger of $L = 4$ is realistic, and maybe a lower trigger could easily lead to no stock price equilibrium. Yet when condition (ii) of Lemma 2 is

²²When $L = 4$, equation (16) gives $\gamma^* = 52$.

rewritten as a minimum trigger price, the benchmark parameters imply $L > \frac{\gamma cC - bB}{r(n + (1 + \gamma)m)} = -8.17$. Consequently, even a trigger of $L = 0$, which implies that conversion occurs at the bank closure date when $A_{uc} = L(n + m) + \frac{bB}{r} = 96$, would ensure a unique equilibrium stock price. This zero stock price trigger is illustrated in Figure 2 Panel C and represents a polar case where CoCos become subordinated debt.²³

As discussed in the previous section and illustrated in Figure 1, no equilibrium exists when the conversion option $\varsigma(A_t) \equiv \frac{1}{n} \left(\frac{cC}{r} - mL \right) \left(\frac{A_{uc}}{A_t} \right)^\gamma$ declines rapidly for assets in the right neighborhood of A_{uc} . Since $\partial \varsigma(A_t) / \partial A_t|_{A_t=A_{uc}} = -\gamma \frac{1}{n} \left(\frac{cC}{r} - mL \right) / A_{uc}$, the decline is more rapid the larger is γ given by equation (13). Besides raising γ by lowering the asset volatility σ , γ can be raised by increasing the bank's risk-neutral cashflow growth rate, μ . Yet, there are reasonable limits to doing so. First, to prevent shareholders from receiving negative dividends prior to the bank's closure, one needs $\mu \leq 0$. Negative dividends, or requiring equityholders to contribute cash to the bank, is inconsistent with shareholders' limited liability. Second, even if one accepts negative dividends, equation (2) requires that $\mu < r$ for a finite bank asset value. Based on Corollary 1's exercise of setting $\mu = r$, when $L = 4$ an equilibrium exists when σ exceeds $\sqrt{2r/\gamma^*} = \sqrt{0.06/52} = 3.4\%$, which is still less than the $\sigma = 4\%$ that was estimated for large banks.²⁴ Thus, only for this highly unrealistic upper bound on cashflow growth does the implied asset volatility start to appear realistic. Consequently, it is not easy to find reasonable parameter values that imply no stock price equilibrium for our CoCo perpetuity model.

²³As will be discussed, CoCos issued by the Spanish bank Banco Popular converted to worthless equity when it was closed by regulators.

²⁴Empirical evidence finds that in times of stress when bank assets decline, volatility rises above average. Moreover, Nagel and Purnanandam (2016) model a bank's assets as a portfolio of default-risky loans and show, theoretically, that asset volatility rises following market declines. While the current paper's model assumes that asset volatility is constant, a more general model where volatility rises as assets decline toward the conversion trigger would likely raise the conversion option's value, thereby making the existence of a stock price equilibrium more likely.

2.4.2 Principal Write Down

This section illustrates stock prices for a bank issuing a CoCo whose principal is written down. Using the previous benchmark parameter values, Figure 3 Panel A illustrates the case of conversion terms that are most beneficial to shareholders: $\alpha = 0$, implying a 100% principal write-down. In this case an equilibrium write-down must occur when the asset value is $A_{uc}^w = nL + (bB + \alpha cC) / r = 100$. Since condition (ii) of Lemma 3 indicates that a unique stock price equilibrium exists for $\alpha > (\gamma cC - bB - nrL) / (cC(1 + \gamma)) = -1.67$, the figure verifies that the candidate stock price, S_t^w , is indeed monotonic and, therefore, is an equilibrium.²⁵

With a trigger of $L = 4$ and a complete write down ($\alpha = 0$), condition (26) states that a unique stock price equilibrium exists whenever $\sigma > \sqrt{2(r + \mu\gamma^{**}) / [\gamma^{**}(1 + \gamma^{**})]} = 1.43\%$. Figure 3 Panel B illustrates the case where asset volatility equals the minimum of 1.43%, a level that is less than 36% of a typical large bank's 4% asset volatility.

3 Extensions

This section considers several extensions of the perpetual maturity model that investigate the robustness of stock price equilibria.

²⁵Indeed, for the benchmark parameter values, a unique stock equilibrium exists for a CoCo with a 100% writedown that has any stock price trigger, L . For example, with $L = 0$, condition (ii) of Lemma 3 requires $\alpha > -1.57$. This case of a write-down CoCo with a zero stock price trigger is the same as an equity conversion CoCo with a zero trigger and is illustrated in Figure 2 Panel C.

3.1 Callable CoCos

In practice, many CoCo perpetuities are callable by their issuing bank.²⁶ As happens with a finite-maturity CoCo at its maturity, CoCo investors receive a final principal payment when a callable CoCo perpetuity is redeemed. Thus, a concern might be that permitting banks to call CoCo perpetuities could eliminate stock price equilibria that would exist if the CoCo was noncallable. This section shows that, in fact, permitting a CoCo perpetuity to be called weakly expands the set of parameters for which a unique equilibrium exists. Giving shareholders a call option can increase the value of the candidate stock price relative to that of a noncallable CoCo, thereby making it more likely that the candidate stock price is monotonically increasing in the bank's assets.

So consider an extension of the basic model where perpetual-maturity CoCos can be redeemed by their issuer. As before, assume CoCos automatically convert to m additional shares when the bank's per share stock price falls to L . In addition, the bank has the right, but not the obligation, to buy back CoCos at their principal (par) value, C , at any time prior to conversion. As a tie-breaking rule, we assume that when the stock price drops to L , the bank has the right to call CoCos before they are converted.²⁷ Also, because calling bonds requires a lump sum redemption payment, we assume that the bank finances the payment by issuing default-free debt with principal C and coupon r , priced at par.²⁸ As

²⁶Martijn Boermans of the De Nederlandsche Bank finds that at least 28 of 42 CoCo perpetuities issued from 2011 to 2014 were callable.

²⁷This rule is without loss of generality, since the bank can always call CoCos when the stock price is equal to $L + \varepsilon$, where ε is an arbitrarily small positive amount.

²⁸Bank regulators are assumed to close the bank when the value of its assets drops to $\frac{bB}{r} + C$. The results are unchanged if the redemption is funded by new shareholders' equity. As long as the new securities are fairly priced and do not affect the bank's prior (default-free) senior debt, the value of initial shareholders' equity is independent of the type of security issued.

a result, the post-redemption stock price is

$$W_t = \frac{1}{n} \left(A_t - \frac{bB}{r} - C \right). \quad (28)$$

Definition: *A conversion time, $\hat{\tau}$, a redemption time, τ_r , and a per-share stock value, $\dot{S}_t$, are an equilibrium if*

$$\begin{aligned} \dot{S}_t = & E_t^Q \left[\int_t^\infty e^{-r(s-t)} \left(\frac{1}{n} (a_s - bB - cC) \cdot 1_{s < \min\{\hat{\tau}, \tau_r, \tau_\delta\}} + \frac{1}{n+m} (a_s - bB) \cdot 1_{\hat{\tau} \leq s < \tau_\delta} \cdot 1_{\hat{\tau} < \tau_r} \right. \right. \\ & \left. \left. + \frac{1}{n} (a_s - bB - rC) \cdot 1_{\tau_r \leq s < \tau_\delta} \cdot 1_{\tau_r \leq \hat{\tau}} \right) ds \right], \end{aligned} \quad (29)$$

$$\hat{\tau} = \inf \left\{ t \in [0, \infty) : \dot{S}_t \leq L \right\}, \quad (30)$$

and τ_r is a stopping time with respect to $\{\mathcal{F}_t, t \in [0, \infty)\}$ that maximizes the stock value.

Note that (29) implies that the post-conversion and post-redemption stock prices per share are given by U_t in (4) and W_t in (28), respectively. As a tie-breaking rule, we assume without loss of generality that when the shareholders are indifferent between CoCo conversion and redemption, they choose conversion.

Since the default-free interest rate, r , is assumed to be constant, we are not analyzing a stochastic interest rate motive for calling CoCos. Rather, the bank's call option has value because the CoCos' value can vary with changes in the bank's underlying asset value. The bank is assumed to follow the CoCo call policy that maximizes the value of its initial shareholders' equity. There are four qualitatively different cases regarding how this call policy affects the existence of a unique equilibrium stock price. The Appendix provides detailed analysis of these cases which are summarized here.

Perhaps the least interesting case is where $C \leq mL$ and $c > r$, so that the CoCo's

conversion value exceeds par and its coupon exceeds the default-free rate. In this case the bank will immediately redeem the CoCo since it is paying a high coupon and shareholders do not benefit from conversion. A second case is where $C \leq mL$ and $c \leq r$. Here, the bank is paying a low coupon which makes it optimal to delay calling its CoCo until just before the stock price declines to L in order to pay off the CoCo at its par value rather than convert it to a higher value of equity.²⁹

A third case is $C > mL$ and $c \leq r$, so that conversion is at less than par and the CoCo coupon is less than the default-free rate. It is never optimal to redeem the CoCo, so its value and stock price are exactly the same as when the bank issues an equivalent noncallable CoCo. The fourth case is $C > mL$ and $c > r$ so that CoCos are converted at less than par but their coupon exceeds the default-free rate. It is optimal to allow the CoCos to convert at the same asset level, A_{uc} , as in the basic model but to redeem them at a higher asset level determined by the condition that this higher asset boundary maximizes shareholders' equity.³⁰

An implication of the call policies for these four cases is the following theorem.

Theorem 3: *When CoCos are callable, there exists a unique stock price equilibrium when either $C \leq mL$ or $C > mL$ and either condition (i) or (ii) in Lemma 2 is satisfied.*

Proof: *See the Appendix which also contains formulas for the equilibrium values of the stock and callable CoCos.*

Theorem 3 gives sufficient, though not necessary, conditions for a unique equilibrium.

²⁹These call policies when $C \leq mL$ may explain why few, if any, actual CoCos have conversion terms that benefit CoCo investors. The bank has the incentive to redeem CoCos immediately ($c > r$) or redeem them just before conversion ($c \leq r$). The latter case might concern regulators because a large asset payout to CoCo investors just prior to conversion makes the bank's failure more likely compared to a conversion or write down.

³⁰Proposition 3 in the Appendix gives the condition that this redemption boundary satisfies and a closed-form solution for the candidate equilibrium stock price.

For the fourth case, a higher stock value due to the redemption option could lead to monotonicity of the stock price for asset values exceeding A_{uc} even when condition (ii) of Lemma 2 fails. Hence, relative to a noncallable CoCo, the set of parameters for which a unique stock price equilibrium exists expands for callable CoCos.

3.2 Risky Senior Debt

Our basic model assumed that there were no direct costs to resolving a bank's failure and senior debt was default-free. Now consider a modification that incorporates bankruptcy costs and a range of closure times. Let ω , where $0 \leq \omega < 1$, denote the proportion of asset value that is lost to bankruptcy costs at the time that regulators close the bank, which is the first time that its assets drop to the level A_B . We assume that $A_L < A_B \leq A_H$, where $A_L = \frac{\gamma bB}{(1+\gamma)r}$ and $A_H = \frac{bB}{(1-\omega)r}$.³¹ The lower bound A_L ensures that the bank's equity value remains nonnegative,³² while the upper bound A_H is the asset level at which senior debt starts to be default-free. We refer to $A_B \in (A_L, \frac{bB}{r})$ as *late* closure and $A_B \in [\frac{bB}{r}, A_H]$ as *early* closure. When closure is late, senior debt bears the entire ex-post cost of bankruptcy while for early closure shareholders bear some of these costs.

Let the superscript “*b*” denote security values in this risky debt setting. The next proposition derives security values similar to Propositions 1 and 2.

Proposition 3: *With default-risky senior debt, if there is an equilibrium stock price then the date t value of senior debt B_t^b , the pre-conversion CoCo value C_t^b , the pre-*

³¹Note that since $\gamma > 0$ and $\omega \geq 0$, then $A_L < A_H$. A_B is assumed to be known. A more general model where regulators imperfectly observe the bank's assets might assume that A_B is a random variable distributed over the range $(A_L, A_H]$.

³²See Duffie (2001) Section 11.C for more details.

conversion stock price S_t^b , and the post-conversion stock price U_t^b are given by

$$B_t^b = \frac{bB}{r} - \left(\frac{bB}{r} - (1 - \omega)A_B \right) \left(\frac{A_t}{A_B} \right)^{-\gamma}, \quad (31)$$

$$C_t^b = \frac{cC}{r} - \left(\frac{cC}{r} - mL \right) \left(\frac{A_t}{A_{uc}^b} \right)^{-\gamma}, \quad (32)$$

$$S_t^b = \frac{1}{n} \left[A_t - \frac{bB}{r} - \frac{cC}{r} + \left(\frac{cC}{r} - mL \right) \left(\frac{A_t}{A_{uc}^b} \right)^{-\gamma} + \left(\frac{bB}{r} - A_B \right) \left(\frac{A_t}{A_B} \right)^{-\gamma} \right], \quad (33)$$

$$U_t^b = \frac{1}{n+m} \left[A_t - \frac{bB}{r} + \left(\frac{bB}{r} - A_B \right) \left(\frac{A_t}{A_B} \right)^{-\gamma} \right]. \quad (34)$$

CoCo conversion happens when A_t falls to A_{uc}^b for the first time; that is,

$$\tau_{uc}^b = \inf \{ t \in [0, \infty) : A_t \leq A_{uc}^b \}, \quad (35)$$

where the conversion boundary A_{uc}^b is the solution to the equation

$$A_{uc}^b = L(n+m) + \frac{bB}{r} - \left(\frac{bB}{r} - A_B \right) \left(\frac{A_B}{A_{uc}^b} \right)^{-\gamma}, \quad (36)$$

such that $A_{uc}^b > A_B$.

Equations (33), (34) and (36) are analogous to (11), (4) and (6), with the main difference being the term $\left(\frac{bB}{r} - A_B \right) \left(\frac{A_t}{A_B} \right)^{-\gamma}$ that can be interpreted as the shareholders' "default option" value.³³ For the case of late bank closure, $A_B < \frac{bB}{r}$, this option value is positive; while for the case of early bank closure, $A_B > \frac{bB}{r}$, the default option value is negative. Note that if $A_B = \frac{bB}{r}$, then $A_{uc}^b = A_{uc}$, $U_t^b = U_t$, and $S_t^b = S_t$. The next lemma characterizes the post-conversion stock price U_t^b .

³³Specifically, this "option" is contingent on the bank's closure when assets equal A_B and is relative to shareholder value under unlimited liability. Unlike a typical option, its value can be positive or negative.

Lemma 4: *The post-conversion stock price, U_t^b , is strictly increasing in the asset level A_t and strictly decreasing in the closure boundary A_B . The conversion boundary A_{uc}^b is unique and strictly increasing in the closure boundary A_B .*

Lemma 4 states that earlier bank closure (a higher A_B) lowers the post-conversion stock price so that equilibrium conversion occurs earlier (a higher A_{uc}^b). The next theorem states necessary and sufficient conditions for an equilibrium. Interestingly, the set of parameters for which a unique equilibrium stock price exists expands with earlier closure.

Theorem 4: *If one of the following is true:*

- (i) $mL \geq \frac{cC}{r} - \left(A_B - \frac{bB}{r}\right) \left(\frac{A_B}{A_{uc}^b}\right)^\gamma$ or
- (ii) $mL < \frac{cC}{r} - \left(A_B - \frac{bB}{r}\right) \left(\frac{A_B}{A_{uc}^b}\right)^\gamma$ and

$$(1 + \gamma)A_{uc}^b > Ln + \frac{bB}{r} + \frac{cC}{r}, \quad (37)$$

then there exists a unique equilibrium in which the CoCo converts when the bank's asset level drops to A_{uc}^b for the first time and where the equilibrium stock prices per share before and after conversion are given by (33) and (34), respectively. If neither condition (i) nor (ii) is satisfied, then there is no equilibrium. Moreover, the earlier that a bank is closed (higher A_B), the larger is the set of parameters for which an equilibrium exists.

Conditions (i) and (ii) in Theorem 4 coincide with conditions (i) and (ii) in Lemma 2 when $A_B = \frac{bB}{r}$, even when $\omega > 0$. Compared to the benchmark case with default-free senior debt, the set of parameters for which a unique stock price equilibrium exists is the same for $A_B = \frac{bB}{r}$, larger for early bank closures, $A_B > \frac{bB}{r}$, and smaller for late bank closure, $A_B < \frac{bB}{r}$. In general, Theorem 4 states that this set expands as A_B increases.

3.3 CoCos and Risk-Taking Incentives

This section maintains the assumption of default-risky senior debt and considers a final extension that allows the bank to choose its asset risk. We permit the bank to continuously select an asset volatility $\sigma_t \in [\sigma_L, \sigma_H]$ to maximize the value of its shareholders' equity. To simplify our analysis, we assume that there is an equilibrium stock price for any volatility process σ_t that is adapted to $\{\mathcal{F}_t\}$ and bounded below and above by σ_L and $\sigma_H > \sigma_L$, respectively.³⁴ As a tie-breaking rule, we assume that if the bank is indifferent between the asset volatility level σ_L and any other level, it chooses σ_L .³⁵

We first consider the bank's shareholder value-maximizing volatility policy following conversion and then, given this policy, solve for its policy prior to conversion. Suppose, for the moment, that the choice of volatility after conversion is a constant, σ .³⁶ Then, the post-conversion stock price equals

$$U^b(A_t, \sigma) = \frac{1}{n+m} \left[A_t - \frac{bB}{r} + \left(\frac{bB}{r} - A_B \right) \left(\frac{A_t}{A_B} \right)^{-\gamma(\sigma)} \right], \quad (38)$$

where

$$\gamma(\sigma) = \frac{1}{\sigma^2} \left[\mu - \frac{1}{2}\sigma^2 + \sqrt{\left(\mu - \frac{1}{2}\sigma^2 \right)^2 + 2r\sigma^2} \right]. \quad (39)$$

When bank closure is late, $A_B < \frac{bB}{r}$, the default option value $\left(\frac{bB}{r} - A_B \right) \left(\frac{A_t}{A_B} \right)^{-\gamma(\sigma)}$ is positive and $U^b(A_t, \sigma)$ is convex in the asset level A_t . As a result, shareholders always prefer the highest volatility σ_H since it maximizes this default option value. When bank closure is early, $A_B > \frac{bB}{r}$, the "option" value is negative and $U^b(A_t, \sigma)$ is concave in the

³⁴Our previous analysis suggests that this should be the case when condition (i) or (ii) of Theorem 4 is met when the bank's asset volatility is σ_L .

³⁵One could assume there is a trivially small regulatory cost of choosing a volatility greater than σ_L .

³⁶We verify that this is indeed the case in the proof of Lemma 5.

asset level A_t . Consequently, shareholders always prefer the lowest volatility σ_L . This is the intuition for the proof of the following lemma.

Lemma 5: *The post-conversion volatility level is constant and equal to*

$$\sigma_2 = \begin{cases} \sigma_H & \text{if } A_B < \frac{bB}{r} \\ \sigma_L & \text{if } A_B \geq \frac{bB}{r} \end{cases}, \quad (40)$$

and the post-conversion value of the stock equals $U^b(A, \sigma_2)$ in (38).

Since CoCo conversion must occur when the post-conversion stock price equals L , the conversion boundary $A_{uc}^b(\sigma_2)$ satisfies the following equation:

$$A_{uc}^b(\sigma_2) + \left(\frac{bB}{r} - A_B \right) \left(\frac{A_{uc}^b(\sigma_2)}{A_B} \right)^{-\gamma(\sigma_2)} = L(n + m) + \frac{bB}{r}. \quad (41)$$

We now employ similar logic to analyze the pre-conversion volatility policy. Let us assume (and later verify) that the preferred volatility prior to conversion remains constant, say equal to σ . Then the pre-conversion equilibrium stock price equals

$$S^b(A_t, \sigma) = \frac{1}{n} \left[A_t - \frac{bB}{r} - \frac{cC}{r} + \left(\frac{cC}{r} - mL + D(A_{uc}^b(\sigma_2)) \right) \left(\frac{A_t}{A_{uc}^b(\sigma_2)} \right)^{-\gamma(\sigma)} \right]. \quad (42)$$

where

$$D(A_{uc}^b(\sigma_2)) \equiv \left(\frac{bB}{r} - A_B \right) \left(\frac{A_{uc}^b(\sigma_2)}{A_B} \right)^{-\gamma(\sigma_2)} \quad (43)$$

The first three terms in the brackets of (42), $A_t - \frac{bB}{r} - \frac{cC}{r}$, is the total equity value without conversion and default options. The last term is the combined value of the conversion and default options. Indeed, at the time of conversion, $\frac{cC}{r} - mL$ is the value of the conversion option, $D(A_{uc}^b(\sigma_2))$ given by (43) is the value of the default option, while $\left(\frac{A_t}{A_{uc}^b(\sigma_2)} \right)^{-\gamma(\sigma)}$ is

the market value of a unit payment at conversion.

When the combined value of the conversion and default options is positive, $\frac{cC}{r} - mL + D(A_{uc}^b(\sigma_2)) > 0$, $S^b(A, \sigma)$ is convex in A_t and shareholders prefer the highest volatility σ_H prior to conversion. Instead, shareholders prefer the lowest volatility σ_L prior to conversion when $\frac{cC}{r} - mL + D(A_{uc}^b(\sigma^p)) < 0$ since $S^b(A, \sigma)$ is concave. The following theorem verifies these observations.

Theorem 5: *The pre-conversion volatility is constant until conversion and equal to*

$$\sigma_1 = \begin{cases} \sigma_H & \text{if } \frac{cC}{r} - mL + D(A_{uc}^b(\sigma_2)) > 0 \\ \sigma_L & \text{if } \frac{cC}{r} - mL + D(A_{uc}^b(\sigma_2)) \leq 0 \end{cases}, \quad (44)$$

and the pre-conversion value of the stock equals $S^b(A_t, \sigma_1)$ in (42).

These results suggest that appropriately-designed CoCos can be important for determining a bank's pre-conversion choice of risk, even if the amount of CoCos issued is considerably smaller than that of its senior debt. For example, suppose that regulators close banks late such that $A_B < bB/r$ and, therefore, the default option is positive. Lemma 5 shows that banks without CoCos, including the post-conversion bank, always prefer high volatility. Yet note from Theorem 5 that the conversion option at the time of conversion, $\frac{cC}{r} - mL$, is declining in the trigger level, L . Moreover, so is the default option: higher L increases $A_{uc}^b(\sigma_2)$ and, in turn, $D(A_{uc}^b(\sigma_2))$ in (43) declines. The implication is that with a sufficiently high trigger, L , the sum of the values of conversion and default options can be made negative, even if the unconverted value of CoCos, cC/r , is small relative to the default-free value of senior debt, bB/r . Consequently, high-trigger CoCos may encourage low risk-taking even if their amount issued is relatively small.

Finally, our analysis tentatively suggests that the bank’s pre-conversion risk choice aligns with a greater likelihood that an equilibrium exists. A straightforward extension of the proof of Theorem 4 shows that $S^b(A_t, \sigma_1)$ in (42) is strictly increasing for all $A_t \geq A_{uc}^b(\sigma_2)$ when

$$\frac{\gamma(\sigma_1)}{A_{uc}^b(\sigma_2)} \left[\frac{cC}{r} - mL + D(A_{uc}^b(\sigma_2)) \right] < 1. \quad (45)$$

Note that the terms in brackets in (45) are the sum of the values of the conversion and default options. Theorem 5 states that the bank chooses $\sigma_1 = \sigma_L$ when this sum is negative, a situation where the inequality (45) holds for *any* asset risk so that an equilibrium stock price always exists. Instead, when the sum of option values is positive, the bank optimally chooses $\sigma_1 = \sigma_H$. Since $\partial\gamma(\sigma_1)/\partial\sigma_1 < 0$, choosing the highest asset risk maximizes the chance that the inequality (45) is satisfied. So endogenizing the choice of risk appears to make the existence a unique equilibrium stock price more likely.³⁷

4 Policy Importance

Both academics and policymakers intend for CoCos to recapitalize a bank while it is a “going concern,” making CoCos distinct from “bail-in” debt which absorbs losses after a bank is a “gone concern” (Flannery (2014)). Basel III sets qualifications for CoCos to count as going-concern “Additional Tier 1” (AT1) capital.³⁸ In addition to being perpetuities, CoCos must have a regulatory (book value) capital ratio trigger that, when breached, requires a conversion or write-down. Specifically, AT1 CoCos must contain a common equity

³⁷A formal proof of this statement may require additional assumptions that specify shareholders’ payoffs when no equilibrium exists.

³⁸Avdjiev, Bolton, Jiang, Kartasheva, and Bogdanova (2015) document that for banks in advanced economies during the period 2009 to 2015, the proportions of CoCos qualifying as AT1 capital are 59% based on the number of issues and 67% based on dollar volumes.

Tier 1 (CET1) to risk-weighted assets ratio trigger of at least 5.125% (Basel Committee on Banking Supervision (2011b)). Basel III’s focus on a regulatory capital trigger may partly explain why no CoCos issued thus far contain market equity value trigger.

Unfortunately, empirical evidence indicates that a regulatory capital ratio is unlikely to trigger a CoCo at an early stage of bank distress. Haldane (2011) categorized large international banks as “crisis” banks versus “no-crisis” banks, where the former failed or required public assistance following the Lehman Brothers bankruptcy whereas the latter did not. The average regulatory capital ratios of these two groups of banks were indistinguishable prior to the crisis and, indeed, the average regulatory capital ratio of the crisis banks actually surpassed that of the no-crisis banks shortly before Lehman’s failure. This odd behavior is consistent with Mariathasan and Merrouche (2014), Begley, Purnanandam, and Zheng (2017), and Plosser and Santos (2015) who find that distressed banks manipulate upward their regulatory capital ratios. In contrast, Haldane shows that the crisis banks’ average market value of equity to debt ratio declined much before the Lehman failure, falling far below the more stable average market equity ratio of the no-crisis banks. Thus, market equity values were more accurate and timely indicators of actual distress and potentially better for triggering a CoCo while a bank is a going concern.

Many CoCos contain discretionary triggers that, in principle, give bank regulators the power to force a timely CoCo conversion or write down. Indeed, our model shows that when a stock price equilibrium exists, there is a one-to-one mapping of a bank’s stock price to the market value of its assets. If regulators observed the market value of assets, they could promptly trigger a CoCo, making a market value (stock price) trigger unnecessary.³⁹ Yet Glasserman and Perotti (2017) cite evidence of regulatory forbearance

³⁹However, regulators may imperfectly observe the market value of bank assets. Stock prices may contain additional information on a bank’s assets by aggregating the knowledge of many investors as in

regarding CoCos, specifically in early 2016 when Deutsche Bank was in distress and suffered a severe decline in its stock price. To calm investor fears, the European Commission announced a regulatory ruling that reduced the chance that Deutsche Bank’s CoCos would absorb losses.

Indeed, as of this writing, there has been only one instance of a CoCo being converted or written down. That occurred on June 7, 2017 for equity-conversion CoCos issued by Banco Popular, Spain’s sixth largest bank. However, conversion occurred the day after the European Central Bank declared Banco Popular “failing or likely to fail” and Europe’s Single Resolution Board arranged its acquisition by Santander. The CoCos converted to worthless equity since the resolution wiped out the bank’s common shareholders and also its Tier 2 subordinated bondholders.⁴⁰ Banco Popular had two issues of AT1 CoCos, one with a “low” 5.125 CET1 ratio trigger and another with a “high” 7.0% CET1 ratio trigger. Neither converted while the bank was a going concern since its last reported CET1 ratio was over 10%. Yet, investors long-recognized Banco Popular’s distress: its share price fell by over 75% during the first two of the three years prior to its closure.

Banco Popular is a stark example of the inadequacy of triggers based on regulatory capital ratios or regulatory discretion. It is increasingly evident that truly going-concern CoCos require a market-based trigger. This paper’s results provide critical guidance in designing market-triggered CoCos by showing that perpetuities are consistent with a well-defined stock price equilibrium.

Hayek (1945) and Grossman (1976). Regulators might use both their private and the bank’s stock price information to determine when a CoCo should be triggered. See Flannery (1998), Bond, Goldstein, and Prescott (2010), Birchler and Facchinetti (2007), and Glasserman and Perotti (2017) on this issue.

⁴⁰Senior bondholders suffered no losses as Santander acquired these liabilities and Banco Popular’s assets.

5 Conclusion

CoCos can enhance a bank's safety and soundness by pre-committing it to convert debt to equity at the onset of financial stress, thereby automatically recapitalizing the bank. Given the inadequate response of regulatory capital ratios and a history of regulatory forbearance, only market-based triggers are likely to convert CoCos in a timely fashion. But prior research has cast doubt on the viability of triggers linked to a market value, such as the bank's stock price: when finite-maturity CoCos have conversion terms that favor shareholders, an equilibrium stock price never exists.

In contrast, we show that when CoCos have a perpetual maturity, a feature of most actual CoCos, unique stock price equilibria exist for a wide variety of realistic conditions. By developing a structural model that leads to closed-form solutions for a bank's stock price and CoCo value, we are able to clearly specify the necessary and sufficient conditions for the existence of a unique equilibrium. We show that situations where a stock price equilibrium does not exist require an unrealistically low level of bank asset risk. Unique stock price equilibria are likely to exist when CoCos are converted to equity or written down, when conversion favors either CoCo investors or shareholders, when CoCos are callable, when there are direct costs of bankruptcy, and when a bank chooses its asset risk.

By showing that an ill-defined stock price is unlikely when CoCos are perpetuities, our results have practical implications for CoCo design. Fortunately, Basel III capital regulations provide incentives to issue CoCos with perpetual maturities. A natural next step would be that regulation encourage banks to choose CoCo perpetuities with market-based triggers. Doing so would restore the original vision of CoCos as instruments that preserve banks as going concerns and reduce the likelihood of financial crises.

6 Appendix

6.1 Proof of Lemma 1

We have assumed the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \in [0, \infty)\}, Q)$ where the information flow $\{\mathcal{F}_t, t \in [0, \infty)\}$ is generated by the Brownian motion z_t and all processes are adapted to this information flow. Consider the following quantity, Y , which equals the discounted dividends paid to shareholders starting at date 0:

$$Y = \int_0^\infty e^{-rs} \left(\frac{1}{n} (a_s - bB - cC) \cdot 1_{s < \min\{\hat{\tau}, \tau_\delta\}} + \frac{1}{n+m} (a_s - bB) \cdot 1_{\hat{\tau} \leq s < \tau_\delta} \right) ds \quad (\text{A.1})$$

Since Y is a realized stream of cashflows discounted as of date 0, it does not depend on time t but it is a random variable for all times t prior to the bank's closure. Define $X_t \equiv E_t^Q[Y]$ where E_t^Q is the expectation conditional on $\mathcal{F}_t$. By the law of iterated expectations, X_t is a martingale adapted to the filtration generated by z_t . Hence, X_t is a continuous process since all martingales adapted to a Brownian filtration are continuous.

Comparing X_t to $\hat{S}_t$ in equation (9) yields

$$X_t = \int_0^t e^{-rs} \left(\frac{1}{n} (a_s - bB - cC) \cdot 1_{s < \min\{\hat{\tau}, \tau_\delta\}} + \frac{1}{n+m} (a_s - bB) \cdot 1_{\hat{\tau} \leq s < \tau_\delta} \right) ds + e^{-rt} \hat{S}_t, \quad (\text{A.2})$$

which can be rewritten as

$$\hat{S}_t = e^{rt} \left[X_t - \int_0^t e^{-rs} \left(\frac{1}{n} (a_s - bB - cC) \cdot 1_{s < \min\{\hat{\tau}, \tau_\delta\}} + \frac{1}{n+m} (a_s - bB) \cdot 1_{\hat{\tau} \leq s < \tau_\delta} \right) ds \right]. \quad (\text{A.3})$$

Since X_t and the time integral are continuous in t , $\hat{S}_t$ must be continuous in t .

6.2 Proof of Proposition 1

Note that the CoCo-issuing bank becomes identical to the post-conversion bank at $\hat{\tau}$. Since the post-conversion stock price U_t in (4) is strictly monotone in A_t , conversion at any asset level other than A_{uc} defined in (6) would lead to a jump in the stock price, which cannot be an equilibrium according to Lemma 1.

To finish the proof we need to show that conversion must happen when A_t falls to A_{uc} for the first time. Our argument is that if it did not, it would lead to a contradiction. So, suppose that conversion did not occur the first time $A_t = A_{uc}$. Define the stopping time

$$\theta^\varepsilon = \inf \{t \in [0, \infty) : A_t \leq A_{uc} - \varepsilon\} \quad (\text{A.4})$$

for some $\varepsilon > 0$. Due to the properties of a geometric Brownian motion, at time θ^ε there is a positive probability that the asset process $\{A_{\theta^\varepsilon+t}, t > 0\}$ will fall further to bB/r before returning to A_{uc} . However, by assumption, the bank is closed by regulators when $A_{\tau_\delta} = bB/r$ so that the bankruptcy date, τ_δ , could occur before the conversion date, $\hat{\tau}$.⁴¹ From the definition of an equilibrium stock price in (9), $\hat{S}_{\tau_\delta} = 0$ since at bankruptcy stockholders lose all claims on the bank's assets. In this case conversion must occur when A_t is strictly below A_{uc} , which leads to a contradiction. Consequently, in equilibrium conversion must happen before time θ^ε for any $\varepsilon > 0$. Taking the limit $\varepsilon \rightarrow 0$ yields that conversion must happen at $\tau_{uc} = \inf \{t \in [0, \infty) : A_t \leq A_{uc}\}$.

⁴¹While this closure boundary is specific to our model, any sensible model of a levered firm leads to bankruptcy when A_t hits some critical lower bound.

6.3 Proof of Proposition 2

From Proposition 1 we know that if an equilibrium stock price exists, then conversion happens when $\hat{\tau} = \tau_{uc}$. Substituting this result into (9), one obtains

$$\begin{aligned}
S_t(A_t) &= \frac{1}{n} E_t^Q \left[\int_t^{\tau_{uc}} e^{-r(s-t)} (a_s - bB - cC) ds \right] + \frac{1}{n+m} E_t^Q \left[\int_{\tau_{uc}}^{\tau_\delta} e^{-r(s-t)} (a_s - bB) ds \right] \\
&= \frac{1}{n} \left\{ E_t^Q \left[\int_t^{\tau_\delta} e^{-r(s-t)} (a_s - bB) ds \right] - E_t^Q \left[\int_t^{\tau_{uc}} e^{-r(s-t)} cC ds \right] \right. \\
&\quad \left. - \frac{m}{n+m} E_t^Q \left[\int_{\tau_{uc}}^{\tau_\delta} e^{-r(s-t)} (a_s - bB) ds \right] \right\} \\
&= \frac{1}{n} \left\{ E_t^Q \left[\int_t^{\tau_\delta} e^{-r(s-t)} (a_s - bB) ds \right] - E_t^Q \left[\int_t^{\tau_{uc}} e^{-r(s-t)} cC ds \right] - E_t^Q \left[e^{-r(\tau_{uc}-t)} mL \right] \right\}
\end{aligned} \tag{A.5}$$

The first line of (A.5) equates the value of equity to the value of dividends received per share before and after the conversion date τ_{uc} . The second line uses equivalent cashflow accounting, but where the first term inside the curly brackets is the value of total cashflows per n shares after subtracting the value of coupons paid to senior debt, which occurs until the bank's closure date when assets equal the value of senior debt. The second term subtracts the value of coupons paid to CoCo investors until the conversion date τ_{uc} , while the third term subtracts the proportion of total dividends paid to CoCos investors after conversion. The last equality in (A.5) follows from the fact that

$$\begin{aligned}
\frac{m}{n+m} E_t^Q \left[\int_{\tau_{uc}}^{\tau_\delta} e^{-r(s-t)} (a_s - bB) ds \right] &= \frac{m}{n+m} E_t^Q \left[E_{\tau_{uc}}^Q \left[\int_{\tau_{uc}}^{\tau_\delta} e^{-r(s-t)} (a_s - bB) ds \right] \right] \\
&= \frac{m}{n+m} E_t^Q \left[e^{-r(\tau_{uc}-t)} E_{\tau_{uc}}^Q \left[\int_{\tau_{uc}}^{\tau_\delta} e^{-r(s-\tau_{uc})} (a_s - bB) ds \right] \right] \\
&= m E_t^Q \left[e^{-r(\tau_{uc}-t)} L \right]
\end{aligned} \tag{A.6}$$

since $\frac{1}{n+m} E_{\tau_{uc}}^Q \left[\int_{\tau_{uc}}^{\tau_\delta} e^{-r(s-\tau_{uc})} (a_s - bB) ds \right]$ is the stock price immediately after conversion and is equal to L . Evaluating the three terms in the last line of (A.5), one obtains the first

line in equation (11) where we use the fact that

$$E_t^Q [e^{-r(\tau_{uc}-t)}] = \left(\frac{A_t}{A_{uc}} \right)^{-\gamma} \quad (\text{A.7})$$

and where γ is given by the formula in (13).⁴²

6.4 Proof of Lemma 2

One can easily verify from (11) and (6) that $S_t(A_{uc}) = L$. Also from (11) for $t < \tau_{uc}$

$$\frac{\partial S_t}{\partial A_t} = \frac{1}{n} \left\{ 1 - \frac{\gamma}{A_{uc}} \left(\frac{cC}{r} - mL \right) \left(\frac{A_{uc}}{A_t} \right)^{1+\gamma} \right\}. \quad (\text{A.8})$$

From (A.8) it is clear that when $\frac{cC}{r} \leq mL$, $\frac{\partial S_t}{\partial A_t} > 0$ so that S_t is increasing in A_t . When $\frac{cC}{r} > mL$, note from (A.8) that $\frac{\partial S_t}{\partial A_t}$ is increasing in A_t . As a result, when $\frac{cC}{r} > mL$, S_t is strictly increasing in A_t for all $A_t \geq A_{uc}$ if and only if $\frac{\partial S_t}{\partial A_t} \Big|_{A_t=A_{uc}} > 0$. Since

$$\begin{aligned} \frac{\partial S_t}{\partial A_t} \Big|_{A_t=A_{uc}} &= \frac{1}{n} \left\{ 1 - \frac{\gamma}{A_{uc}} \left(\frac{cC}{r} - mL \right) \right\} \\ &= \frac{1}{n} \left\{ 1 - \gamma \frac{\frac{cC}{r} - mL}{L(n+m) + \frac{bB}{r}} \right\}, \end{aligned} \quad (\text{A.9})$$

one can see that $\frac{\partial S_t}{\partial A_t} \Big|_{A_t=A_{uc}} > 0$ is equivalent to (17). On the other hand, when (17) does not hold, $\frac{\partial S_t}{\partial A_t} \Big|_{A_t=A_{uc}} \leq 0$, and $S_t(A_t) \leq L$ in the right neighborhood of A_{uc} .

According to equation (13), γ is the positive root of the quadratic equation $\frac{1}{2}\sigma^2\gamma^2 -$

⁴²See Back (2017) Chapter 19 for a derivation of equation (A.7).

$(\mu - \frac{1}{2}\sigma^2)\gamma - r = 0$. On the other hand, inequality (17) can be rewritten as

$$\gamma < \frac{\frac{bB}{r} + L(n+m)}{\frac{cC}{r} - Lm} = \gamma^* . \quad (\text{A.10})$$

Since $\gamma^* > 0$, $\gamma < \gamma^*$ is equivalent to $\frac{1}{2}\sigma^2(\gamma^*)^2 - (\mu - \frac{1}{2}\sigma^2)\gamma^* - r > 0$, which gives (15).

6.5 Proof of Theorem 3

As in Section 2.2, the stock price process is adapted to the Brownian filtration, and dividends are paid continuously. Hence, the stock price process must be continuous over time. As a result, we have Lemma 1A, which is analogous to Lemma 1.

Lemma 1A: *If there is an equilibrium stock price with callable CoCos, then $\dot{S}_t$ is continuous in t .*

Proof: *Consider the following quantity, $\dot{Y}$, which equals the discounted dividends paid to shareholders starting at date 0:*

$$\begin{aligned} \dot{Y} &= \int_0^\infty e^{-rs} \left(\frac{1}{n} (a_s - bB - cC) \cdot 1_{s < \min\{\hat{\tau}, \tau_r, \tau_\delta\}} + \frac{1}{n+m} (a_s - bB) \cdot 1_{\hat{\tau} \leq s < \tau_\delta} \cdot 1_{\hat{\tau} < \tau_r} \right. \\ &\quad \left. + \frac{1}{n} (a_s - bB - rC) \cdot 1_{\tau_r \leq s < \tau_\delta} \cdot 1_{\tau_r \leq \hat{\tau}} \right) ds \end{aligned} \quad (\text{A.11})$$

Define $\dot{X}_t \equiv E_t^Q [\dot{Y}]$. By the law of iterated expectations, $\dot{X}_t$ is a martingale adapted to the filtration generated by z_t and is a continuous process since all martingales adapted to a Brownian filtration are continuous. Comparing $\dot{X}_t$ to $\dot{S}_t$ in (29) yields

$$\begin{aligned} \dot{X}_t &= \int_0^t e^{-rs} \left(\frac{1}{n} (a_s - bB - cC) \cdot 1_{s < \min\{\hat{\tau}, \tau_r, \tau_\delta\}} + \frac{1}{n+m} (a_s - bB) \cdot 1_{\hat{\tau} \leq s < \tau_\delta} \cdot 1_{\hat{\tau} < \tau_r} \right. \\ &\quad \left. + \frac{1}{n} (a_s - bB - rC) \cdot 1_{\tau_r \leq s < \tau_\delta} \cdot 1_{\tau_r \leq \hat{\tau}} \right) ds + e^{-rt} \dot{S}_t, \end{aligned} \quad (\text{A.12})$$

which can be rewritten as

$$\begin{aligned} \dot{S}_t = & e^{rt} \left[\dot{X}_t - \int_0^t e^{-rs} \left(\frac{1}{n} (a_s - bB - cC) \cdot 1_{s < \min\{\bar{\tau}, \tau_r, \tau_\delta\}} + \right. \right. \\ & \left. \left. + \frac{1}{n+m} (a_s - bB) \cdot 1_{\bar{\tau} \leq s < \tau_\delta} \cdot 1_{\bar{\tau} < \tau_r} + \frac{1}{n} (a_s - bB - rC) \cdot 1_{\tau_r \leq s < \tau_\delta} \cdot 1_{\tau_r \leq \bar{\tau}} \right) ds \right] \end{aligned} \quad (\text{A.13})$$

Since $\dot{X}_t$ and the time integral are continuous in t , $\dot{S}_t$ must be continuous in t .

Lemma 1A states there can be no jump in an equilibrium stock price when CoCos are called or converted. In particular, it implies that if an equilibrium stock price exists, then conversion must occur at the asset level A_{uc} for which the post-conversion stock price equals L . As a consequence, CoCo investors' payoff at conversion must equal mL . Therefore, the callable CoCo's value, $\dot{C}_t$, equals

$$\dot{C}_t = E_t^Q \left[\int_t^{\min\{\bar{\tau}, \tau_r\}} e^{-r(s-t)} cC ds + e^{-r(\bar{\tau}-t)} mL \cdot 1_{\bar{\tau} < \tau_r} + e^{-r(\tau_r-t)} C \cdot 1_{\bar{\tau} \geq \tau_r} \right] \quad (\text{A.14})$$

Since senior debt is default-free, the equilibrium stock price, $\dot{S}_t$, prior to conversion or redemption must be

$$\dot{S}_t = \frac{1}{n} \left(A_t - \frac{bB}{r} - \dot{C}_t \right). \quad (\text{A.15})$$

The bank chooses to redeem its CoCos at a time τ_r that maximizes the value of shareholders' equity or, equivalently, minimizes the CoCo value.

Proposition 1A: *When $C < mL$, callable CoCos are never converted in equilibrium. When $C \geq mL$, if there is an equilibrium stock price then conversion must occur when A_t falls to A_{uc} for the first time, provided the CoCos were not redeemed earlier.*

Proof: Lemma 1A requires that CoCo investors receive a payoff equal to mL at conversion. When $C < mL$, conversion cannot occur in an equilibrium since it is optimal for

shareholders to avoid conversion by redeeming the CoCos when the stock price falls to L .

When $C \geq mL$, one can verify that the post-redemption stock price $W_t < L$ for all $A_t < A_{uc}$. From this fact and Lemma 1A, CoCo conversion or redemption cannot occur in equilibrium when A_t is strictly below A_{uc} , as it would lead to a downward jump in the stock price from above to strictly below the trigger L . The rest of the proof for the case of $C \geq mL$ is analogous to the proof of Proposition 1.

We now consider four cases based on the parameters values C , mL , c , and r :

Case 1: $C < mL$ and $c > r$

Because the CoCo conversion value mL is greater than the face value C , CoCos will never be converted since they can be redeemed at C when the stock price drops to L . However, since $c > r$, it is optimal to redeem the CoCos immediately. Indeed, if CoCos were not redeemed at the initial date $t = 0$, then their value must be strictly above C . To see this, let ρ be a (random) time of CoCo conversion or redemption, and let X_ρ be the payoff to the CoCo investors at time ρ . Note that $X_\rho \geq \min\{C, mL\} = C$, since $C \leq mL$. If $\rho > 0$, a realized payoff to the CoCo investors is given by

$$\int_0^\rho e^{-rs} c C ds + e^{-r\rho} X_\rho > \int_0^\rho e^{-rs} r C ds + e^{-r\rho} C = C. \quad (\text{A.16})$$

Inequality (A.16) shows that if CoCos were not redeemed at date $t = 0$, the realized payoff for CoCo investors is strictly greater than C since $c > r$. Consequently, shareholders' optimal call policy is to redeem them immediately. As a result this case leads to a unique equilibrium in which the CoCo value equals C and the stock price equals $\frac{1}{n} \left(A_t - \frac{bB}{r} - C \right)$.

Case 2: $C < mL$ and $c \leq r$

Since $c \leq r$, CoCo redemption is not optimal prior to the stock price falling to L .

However, when the stock price first equals L , CoCos are redeemed to avoid conversion since $C < mL$. This will occur when the bank's assets fall to a level $A_t = A_r$ such that the post-redemption stock price equals L . Specifically:

$$\frac{1}{n} \left(A_r - \frac{bB}{r} - C \right) = L, \quad (\text{A.17})$$

which gives

$$A_r = nL + \frac{bB}{r} + C. \quad (\text{A.18})$$

Note that $A_r < A_{uc}$ since $C < mL$. The intuition is that, *ceteris paribus*, the stock price is more valuable compared to the case of a non-callable CoCo since redemption at the lower value C substitutes for conversion at the higher value mL , which means that the stock price first equals L at a lower bank asset value. Indeed, this callable CoCo's payoff is equivalent to a noncallable CoCo that converts to $m_r \equiv \frac{C}{L}$ additional shares when the bank's per share stock price falls to L . Consequently, there always exists a unique stock price equilibrium since condition (i) of Lemma 2 is satisfied. According to (12), the callable CoCo value equals

$$\dot{C}_t = \frac{cC}{r} \left[1 - \left(\frac{A_t}{A_r} \right)^{-\gamma} \right] + C \left(\frac{A_t}{A_r} \right)^{-\gamma}. \quad (\text{A.19})$$

Case 3: $C \geq mL$ and $c \leq r$

CoCo redemption is never optimal since the coupon rate is below the default-free rate and the conversion value is less than the par value. As a result, this callable CoCo is equivalent to the noncallable CoCo with the same conversion parameters. Therefore, Theorem 1 on the existence and uniqueness of a stock price equilibrium applies so that if condition (i) or (ii) of Lemma 2 holds, there is a unique equilibrium stock price and the

CoCo value equals equation (12).

Case 4: $C \geq mL$ and $c > r$

Since $C \geq mL$, Proposition 1A requires that conversion occur when A_t falls to A_{uc} for the first time provided CoCos were not redeemed earlier. Note that at A_{uc} , the bank's per share stock price equals L whether or not its CoCo is callable. However, when the CoCo is callable its pre-conversion candidate stock price must be at least as high as that of a non-callable CoCo since it reflects the shareholders' additional redemption option. Thus, if condition (ii) of Lemma 2 holds, the candidate stock price must remain above the conversion trigger for $A_t > A_{uc}$ and a unique stock price equilibrium exists.

While it is not optimal to call CoCos when A_t is close to A_{uc} since with $C \geq mL$ conversion is preferred, due to the high coupon rate $c > r$ it can be optimal for shareholders to redeem CoCos when A_t is high. Since the current asset level A_t is a sufficient statistic for future cash flows and asset levels, it is optimal for shareholders to base their CoCo redemption decision on the most recent asset value. Thus, CoCo payoffs can be described by two asset boundaries: the conversion boundary A_{uc} and the redemption boundary that maximizes shareholder value, A_r . Thus, we can state the following proposition.

Proposition 2A: *If $C \geq mL$, $c > r$, and condition (ii) of Lemma 2 holds, the value of the callable CoCo for $A_t \in [A_{uc}, A_r]$ is given by*

$$\dot{C}_t = \frac{cC}{r} + \left(mL - \frac{cC}{r}\right) \left(\frac{A_t}{A_{uc}}\right)^{-\gamma} \frac{1 - \left(\frac{A_r}{A_t}\right)^{-(\gamma-\gamma_1)}}{1 - \left(\frac{A_r}{A_{uc}}\right)^{-(\gamma-\gamma_1)}} + \left(C - \frac{cC}{r}\right) \left(\frac{A_r}{A_t}\right)^{\gamma_1} \frac{1 - \left(\frac{A_t}{A_{uc}}\right)^{-(\gamma-\gamma_1)}}{1 - \left(\frac{A_r}{A_{uc}}\right)^{-(\gamma-\gamma_1)}}, \quad (\text{A.20})$$

where

$$\gamma_1 \equiv \frac{1}{\sigma^2} \left[\mu - \frac{1}{2}\sigma^2 - \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2r\sigma^2} \right], \quad (\text{A.21})$$

and the unique optimal redemption boundary $A_r \geq A_{uc}$ solves the following equation:

$$\gamma \left(\frac{A_r}{A_{uc}} \right)^{\gamma_1} - \gamma_1 \left(\frac{A_r}{A_{uc}} \right)^{\gamma} = (\gamma - \gamma_1) \frac{\frac{cC}{r} - mL}{\frac{cC}{r} - C}. \quad (\text{A.22})$$

Proof: Given the conditions stated in the proposition, the CoCo's value, $\dot{C}(A_t)$, is a function of the bank's asset value and satisfies the ordinary differential equation.⁴³

$$\frac{1}{2}\sigma^2 A^2 \dot{C}''(A) + \mu A \dot{C}'(A) + cC = r \dot{C}(A) \quad (\text{A.23})$$

with the boundary conditions $\dot{C}(A_{uc}) = mL$ and $\dot{C}(A_r) = C$. The general solution is

$$\dot{C}(A) = \frac{cC}{r} + K_1 \left(\frac{A}{A_{uc}} \right)^{-\gamma} + K_2 \left(\frac{A}{A_r} \right)^{-\gamma_1} \quad (\text{A.24})$$

where the constants K_1 and K_2 are determined by the boundary conditions:

$$\begin{aligned} mL &= \frac{cC}{r} + K_1 + K_2 \left(\frac{A_{uc}}{A_r} \right)^{-\gamma_1}, \\ C &= \frac{cC}{r} + K_1 \left(\frac{A_r}{A_{uc}} \right)^{-\gamma} + K_2. \end{aligned} \quad (\text{A.25})$$

Solving the system of the linear equations gives

$$\begin{aligned} K_1 &= \frac{\left(mL - \frac{cC}{r} \right) + \left(\frac{cC}{r} - C \right) \left(\frac{A_r}{A_{uc}} \right)^{\gamma_1}}{1 - \left(\frac{A_r}{A_{uc}} \right)^{-(\gamma - \gamma_1)}} \\ K_2 &= - \frac{\left(\frac{cC}{r} - C \right) + \left(mL - \frac{cC}{r} \right) \left(\frac{A_r}{A_{uc}} \right)^{-\gamma}}{1 - \left(\frac{A_r}{A_{uc}} \right)^{-(\gamma - \gamma_1)}} \end{aligned} \quad (\text{A.26})$$

⁴³See Dumas (1991) for more details.

Substituting K_1 and K_2 into (A.24) and rearranging terms yields (A.20).

The optimal redemption boundary A_r is unique and can be determined from the smooth-pasting condition.⁴⁴ Differentiating (A.20) gives

$$\left. \frac{\partial \dot{C}}{\partial A} \right|_{A=A_r} = \frac{-(\gamma - \gamma_1) \left(mL - \frac{cC}{r} \right) \left(\frac{A_r}{A_{uc}} \right)^{-\gamma} + \left(C - \frac{cC}{r} \right) \left(-\gamma_1 + \gamma \left(\frac{A_r}{A_{uc}} \right)^{-(\gamma - \gamma_1)} \right)}{1 - \left(\frac{A_r}{A_{uc}} \right)^{-(\gamma - \gamma_1)}} = 0 \quad (\text{A.27})$$

and equation (A.22) follows.

Finally, we verify that there is a unique $A_r \geq A_{uc}$ that solves equation (A.22). This equation can be rewritten as $F(A_r) = 0$, where

$$F(A_r) = \gamma \left(\frac{A_r}{A_{uc}} \right)^{\gamma_1} - \gamma_1 \left(\frac{A_r}{A_{uc}} \right)^{\gamma} - (\gamma - \gamma_1) \frac{\frac{cC}{r} - mL}{\frac{cC}{r} - C}. \quad (\text{A.28})$$

Since $C \geq mL$, $c > r$, $\gamma > 0$, and $\gamma_1 < 0$, one can verify that

$$\begin{aligned} F(A_{uc}) &= (\gamma - \gamma_1) \frac{mL - C}{\frac{cC}{r} - C} \leq 0, \\ F'(A_r) &= \frac{\gamma \gamma_1}{A_r} \left(\left(\frac{A_r}{A_{uc}} \right)^{\gamma_1} - \left(\frac{A_r}{A_{uc}} \right)^{\gamma} \right) > 0 \text{ for } A_r > A_{uc}, \\ \lim_{A_r \rightarrow \infty} F(A_r) &> 0, \end{aligned} \quad (\text{A.29})$$

which implies that equation $F(A_r) = 0$ has a unique solution such that $A_r \geq A_{uc}$.

Given the pre-conversion, pre-redemption CoCo value in (A.20), the corresponding unique equilibrium stock price equals (A.15). Also note that Proposition 2A implies that

⁴⁴The smooth-pasting condition $\left. \frac{\partial \dot{C}(A, A_r)}{\partial A} \right|_{A=A_r} = 0$ used in the proof is equivalent to the first order optimization condition $\frac{\partial \dot{C}(A, A_r)}{\partial A_r} = 0$. Indeed, one can verify that $\frac{\partial \dot{C}(A, A_r)}{\partial A_r} = 0$ yields equation (A.22). For more information on smooth-pasting conditions see Dumas (1991) or Back (2017) Chapter 19.

the CoCo will be redeemed immediately if $A_t > A_r$.

6.6 Proof of Proposition 3

Let $\tau_\delta^b = \inf \{t \in [0, \infty) : A_t \leq A_B\}$ denote the time of the bank's closure. The present value of bankruptcy costs is $\omega A_B (A_t/A_B)^{-\gamma}$ due to the fact that $E_t^Q \left[e^{-r(\tau_\delta^b - t)} \right] = (A_t/A_B)^{-\gamma}$.

Thus, the value of the senior debt equals

$$\begin{aligned} B_t^b &= E_t^Q \left[\int_t^{\tau_\delta^b} e^{-r(s-t)} b B ds + e^{-r(\tau_\delta^b - t)} (1 - \omega) A_B \right] \\ &= E_t^Q \left[\frac{bB}{r} \left(1 - e^{-r(\tau_\delta^b - t)} \right) + e^{-r(\tau_\delta^b - t)} (1 - \omega) A_B \right] \end{aligned} \quad (\text{A.30})$$

which simplifies to (31). The value of the post-conversion stock is

$$U_t^b = \frac{1}{n + m} \left[A_t - B_t^b - \omega A_B \left(\frac{A_t}{A_B} \right)^{-\gamma} \right] \quad (\text{A.31})$$

which simplifies to (34). Lemma 1 and Proposition 1 hold when senior debt is default-risky since their proofs do not depend on a specific closure boundary. Thus, due to equilibrium stock price continuity, CoCo conversion must occur when $U_t^b = L$, which yields equation (36) for the conversion boundary A_{uc}^b . Therefore, the pre-conversion CoCo value is

$$\begin{aligned} C_t^b &= E_t^Q \left[\int_t^{\tau_{uc}^b} e^{-r(s-t)} c C ds + e^{-r(\tau_{uc}^b - t)} m L \right] \\ &= E_t^Q \left[\frac{cC}{r} \left(1 - e^{-r(\tau_{uc}^b - t)} \right) + e^{-r(\tau_{uc}^b - t)} m L \right] \end{aligned} \quad (\text{A.32})$$

which simplifies to (32). Finally, equation (33) follows from the fact that

$$S_t^b = \frac{1}{n} \left[A_t - B_t^b - C_t^b - \omega A_B \left(\frac{A_t}{A_B} \right)^{-\gamma} \right]. \quad (\text{A.33})$$

6.7 Proof of Lemma 4

We first verify that U_t^b is strictly increasing in A_t . Denote $U(A_t, A_B) \equiv U_t^b$ in (34). Then

$$\frac{\partial U(A_t, A_B)}{\partial A_t} = \frac{1}{n+m} \left[1 - \frac{1}{A_t} \gamma \left(\frac{bB}{r} - A_B \right) \left(\frac{A_t}{A_B} \right)^{-\gamma} \right]. \quad (\text{A.34})$$

Since $A_B > A_L = \frac{\gamma bB}{r(1+\gamma)}$,

$$\gamma \left(\frac{bB}{r} - A_B \right) < \gamma \left(\frac{bB}{r} - \frac{\gamma bB}{r(1+\gamma)} \right) = \gamma \frac{bB}{r} \frac{1}{1+\gamma} = A_L. \quad (\text{A.35})$$

Thus,

$$\frac{\partial U(A_t, A_B)}{\partial A_t} > \frac{1}{n+m} \left[1 - \left(\frac{A_B}{A_t} \right)^\gamma \frac{A_L}{A_t} \right] > 0, \quad (\text{A.36})$$

since $A_t \geq A_B > A_L$.

Next, we show that U_t^b is strictly decreasing in A_B .

$$\begin{aligned} \frac{\partial U(A_t, A_B)}{\partial A_B} &= \frac{1}{n+m} \left[\frac{1}{A_B} \gamma \left(\frac{bB}{r} - A_B \right) \left(\frac{A_t}{A_B} \right)^{-\gamma} - \left(\frac{A_t}{A_B} \right)^{-\gamma} \right] \\ &< \frac{1}{n+m} \left[\frac{A_L}{A_B} \left(\frac{A_t}{A_B} \right)^{-\gamma} - \left(\frac{A_t}{A_B} \right)^{-\gamma} \right] \\ &= \frac{1}{n+m} \left(\frac{A_t}{A_B} \right)^{-\gamma} \left[\frac{A_L}{A_B} - 1 \right] < 0. \end{aligned} \quad (\text{A.37})$$

Finally, recall that A_{uc}^b is determined by the equation $U(A_{uc}^b, A_B) = L$, where $U(A_t, A_B)$

is strictly increasing in A_t , $U(A_B, A_B) = 0$, and $U(A_t, A_B) \rightarrow \infty$ as $A_t \rightarrow \infty$. Hence, there is a unique $A_{uc}^b > A_B$ that solves $U(A_{uc}^b, A_B) = L$. Moreover, by the implicit function theorem,

$$\frac{\partial A_{uc}^b}{\partial A_B} = - \frac{\frac{\partial U(A_t, A_B)}{\partial A_B} \Big|_{A_t=A_{uc}^b}}{\frac{\partial U(A_t, A_B)}{\partial A_t} \Big|_{A_t=A_{uc}^b}} > 0, \quad (\text{A.38})$$

where the inequality follows from (A.36) and (A.37). Thus, A_{uc}^b is increasing in A_B .

6.8 Proof of Theorem 4

First, we show that the candidate stock price (33) is monotone in A_t if and only if either condition (i) or (ii) is satisfied. One can verify from (33) and (36) that $S_t^b(A_{uc}) = L$. Also from (33) for $t < \tau_{uc}^b$

$$\frac{\partial S_t^b}{\partial A_t} = \frac{1}{n} \left\{ 1 + \gamma(A_t)^{-(\gamma+1)} \left[(A_{uc}^b)^\gamma \left(mL - \frac{cC}{r} \right) + (A_B)^\gamma \left(A_B - \frac{bB}{r} \right) \right] \right\}. \quad (\text{A.39})$$

From (A.39) it is clear that when $(A_{uc}^b)^\gamma \left(mL - \frac{cC}{r} \right) + (A_B)^\gamma \left(A_B - \frac{bB}{r} \right) \geq 0$, which is equivalent to condition (i), $\frac{\partial S_t^b}{\partial A_t}$ is always positive so that S_t^b is increasing in A_t . When $(A_{uc}^b)^\gamma \left(mL - \frac{cC}{r} \right) + (A_B)^\gamma \left(A_B - \frac{bB}{r} \right) < 0$, note from (A.39) that $\frac{\partial S_t^b}{\partial A_t}$ is increasing in A_t , meaning that S_t^b is strictly increasing in A_t for all $A_t \geq A_{uc}^b$ if and only if $\frac{\partial S_t^b}{\partial A_t} \Big|_{A_t=A_{uc}^b} > 0$.

$$\begin{aligned} \frac{\partial S_t^b}{\partial A_t} \Big|_{A_t=A_{uc}^b} &= \frac{1}{n} \left\{ 1 + \frac{\gamma}{A_{uc}^b} \left[\left(mL - \frac{cC}{r} \right) + \left(A_B - \frac{bB}{r} \right) \left(\frac{A_B}{A_{uc}^b} \right)^\gamma \right] \right\} \\ &= \frac{1}{n} \left\{ 1 + \frac{\gamma}{A_{uc}^b} \left[A_{uc}^b - \frac{cC}{r} - \frac{bB}{r} - Ln \right] \right\}, \end{aligned} \quad (\text{A.40})$$

where the last equality follows from (36). One can see that $\frac{\partial S_t^b}{\partial A_t} \Big|_{A_t=A_{uc}^b} > 0$ is equivalent to (37). On the other hand, when (37) does not hold, $\frac{\partial S_t^b}{\partial A_t} \Big|_{A_t=A_{uc}^b} \leq 0$, and $S_t^b(A_t) \leq L$ in

the right neighborhood of A_{uc}^b . Thus, the candidate stock price (33) is monotone in A_t if and only if either condition (i) or (ii) is satisfied.

If either condition (i) or (ii) is satisfied, then $S_t^b(A_t) > L$ for all $A_t > A_{uc}^b$ and $S_t^b(A_t)$ is an equilibrium price. If not, $S_t^b(A_t) \leq L$ for some $A_t > A_{uc}^b$, which is inconsistent with conversion occurring the first time that $A_t = A_{uc}^b$ so that $S_t^b(A_t)$ cannot be an equilibrium stock price. Proposition 3 ensures that there is no other equilibrium.

Finally, using (36), one can rewrite condition (i) as

$$mL \geq \frac{cC}{r} + \frac{bB}{r} + L(n+m) - A_{uc}^b. \quad (\text{A.41})$$

From Lemma 4, a higher A_B raises A_{uc}^b , thereby relaxing conditions (i) and (ii). Thus, earlier bank closure expands the set of parameters for which an equilibrium exists.

6.9 Proof of Lemma 5

Let $U(A_t)$ be the post-conversion stock price given optimal choices of the volatility levels. Because at the optimum the shareholders should earn an instantaneous total return equal to the risk-free rate, r , we have the following Hamilton–Jacobi–Bellman (HJB) equation

$$rU(A_t) = \max_{\sigma \in [\sigma_L, \sigma_H]} \left\{ \frac{(r - \mu)A_t - bB}{n + m} + \mu A_t \frac{\partial U}{\partial A_t} + \frac{1}{2} \sigma^2 A_t^2 \frac{\partial^2 U}{\partial A_t^2} \right\}, \quad (\text{A.42})$$

where $\frac{(r-\mu)A_t-bB}{n+m}$ is the expected dividend per share and $\mu A_t \frac{\partial U}{\partial A_t} + \frac{1}{2}\sigma^2 A_t^2 \frac{\partial^2 U}{\partial A_t^2}$ is the expected capital gain. $U(A)$ must satisfy the following boundary conditions:

$$\begin{aligned} U(A_B) &= 0, \\ \lim_{A_t \rightarrow \infty} U(A_t) &= \frac{1}{n+m} \left(A_t - \frac{bB}{r} \right). \end{aligned} \quad (\text{A.43})$$

The HJB equation implies that shareholders optimally choose: σ_H when $U(A_t)$ is convex, i.e., $\partial^2 U / \partial A_t^2 > 0$; and σ_L when $U(A)$ is concave, i.e., $\partial^2 U / \partial A_t^2 < 0$. Substituting $U^b(A_t, \sigma_2)$ given by (38) into (A.42) verifies that it solves the HJB equation and its boundary conditions.

While σ is allowed to depend on A_t , note that the sign of the second derivative of $U^b(A_t, \sigma_2)$ in (38) does not depend on A_t since it is determined by $\frac{bB}{r} - A_B$. As a result, the optimal post-conversion volatility remains constant and is given by (40).

6.10 Proof of Theorem 5

Let $S(A_t)$ be the pre-conversion stock price given optimal choices of the volatility levels. Then, $S(A_t)$ must satisfy the following Hamilton–Jacobi–Bellman (HJB) equation:

$$rS(A_t) = \max_{\sigma \in [\sigma_L, \sigma_H]} \left\{ \frac{(r-\mu)A_t - bB - cC}{n} + \mu A_t \frac{\partial S}{\partial A_t} + \frac{1}{2}\sigma^2 A_t^2 \frac{\partial^2 S}{\partial A_t^2} \right\}, \quad (\text{A.44a})$$

with the boundary conditions:

$$\begin{aligned} S(A_{uc}^b(\sigma_2)) &= L, \\ \lim_{A_t \rightarrow \infty} S(A_t) &= \frac{1}{n} \left(A_t - \frac{bB}{r} - \frac{cC}{r} \right). \end{aligned} \quad (\text{A.45})$$

The HJB equation implies that the shareholders optimally choose: σ_H when $\partial^2 S / \partial A_t^2 > 0$; and σ_L when $\partial^2 S / \partial A_t^2 < 0$. One can verify that

$$S(A_t) = \frac{1}{n} \left[A_t - \frac{bB}{r} - \frac{cC}{r} + \left(\frac{bB}{r} + \frac{cC}{r} + nL - A_{uc}^b(\sigma_2) \right) \left(\frac{A_t}{A_{uc}^b(\sigma_2)} \right)^{-\gamma(\sigma_1)} \right] \quad (\text{A.46})$$

solves the HJB equation and its boundary conditions. Note that (A.46) can be rewritten as $S^b(A_t, \sigma_1)$ in (42) since equation (41) implies that

$$\frac{bB}{r} + \frac{cC}{r} + nL - A_{uc}^b(\sigma_2) = \frac{cC}{r} - mL + D(A_{uc}^b(\sigma_2)). \quad (\text{A.47})$$

Finally, note that the sign of the second derivative of $S(A_t)$ is determined by $\frac{cC}{r} - mL + D(A_{uc}^b(\sigma_2))$, which does not depend on A_t . As a result, the optimal pre-conversion volatility remains constant until conversion and is given by (44).

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Figure 1 Existence of Equilibrium

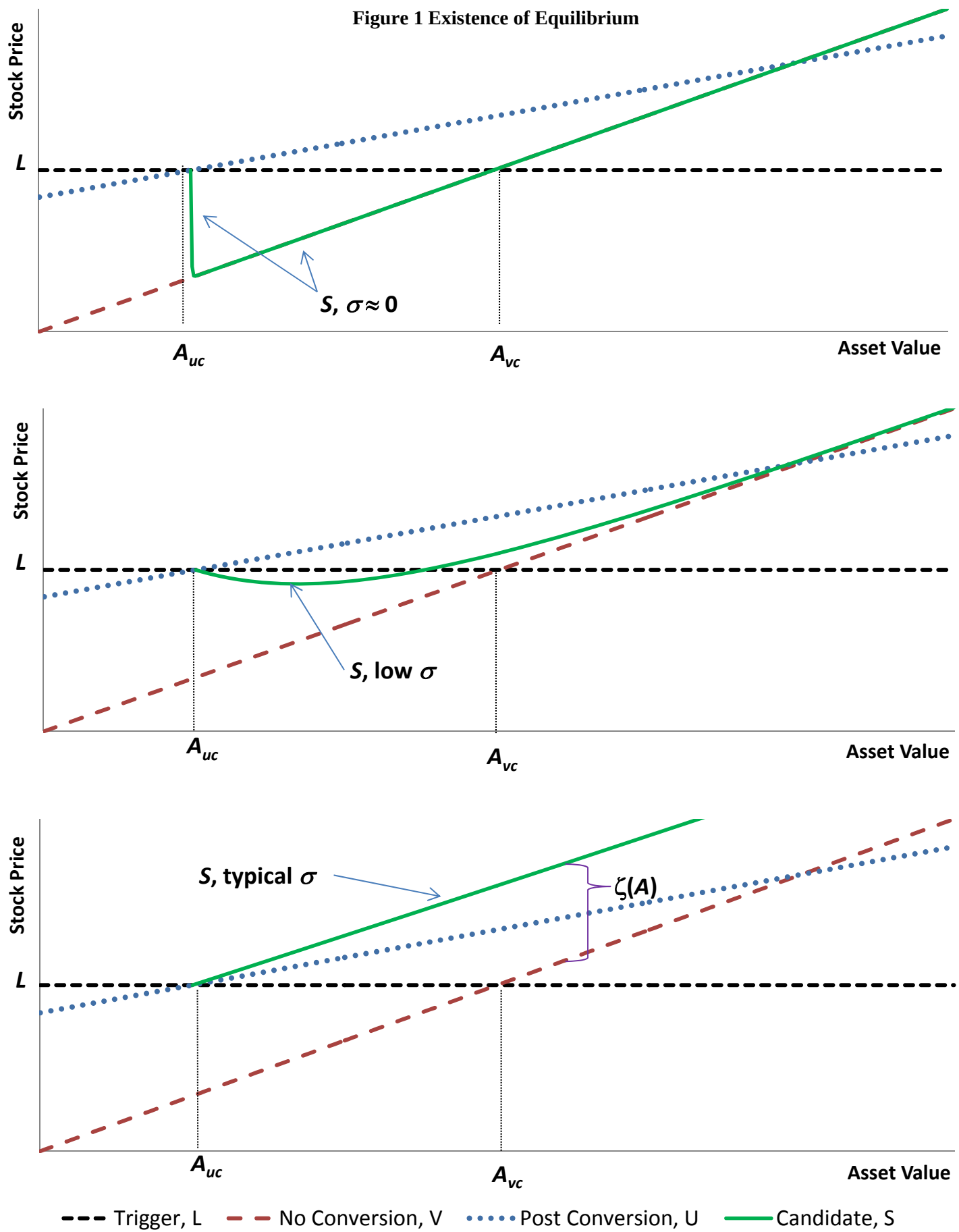
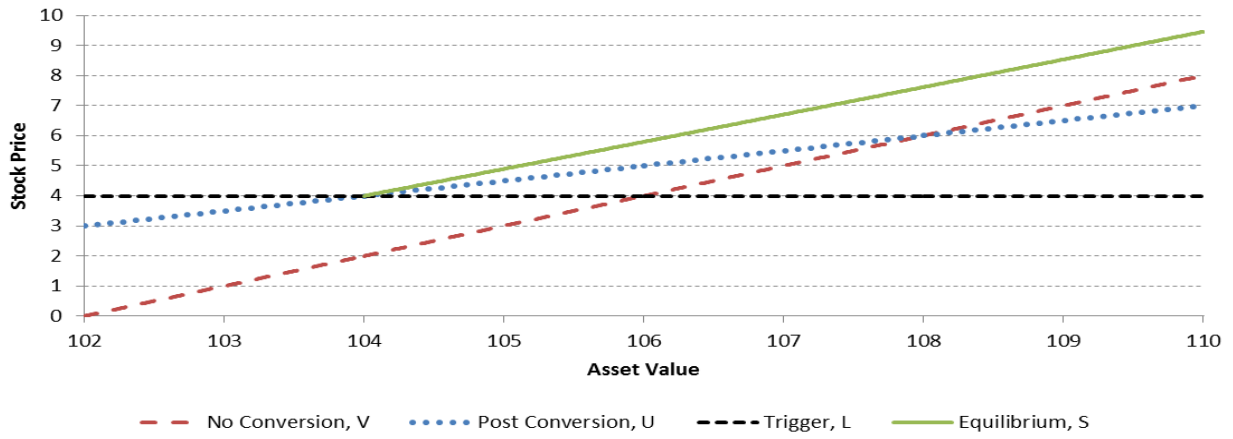
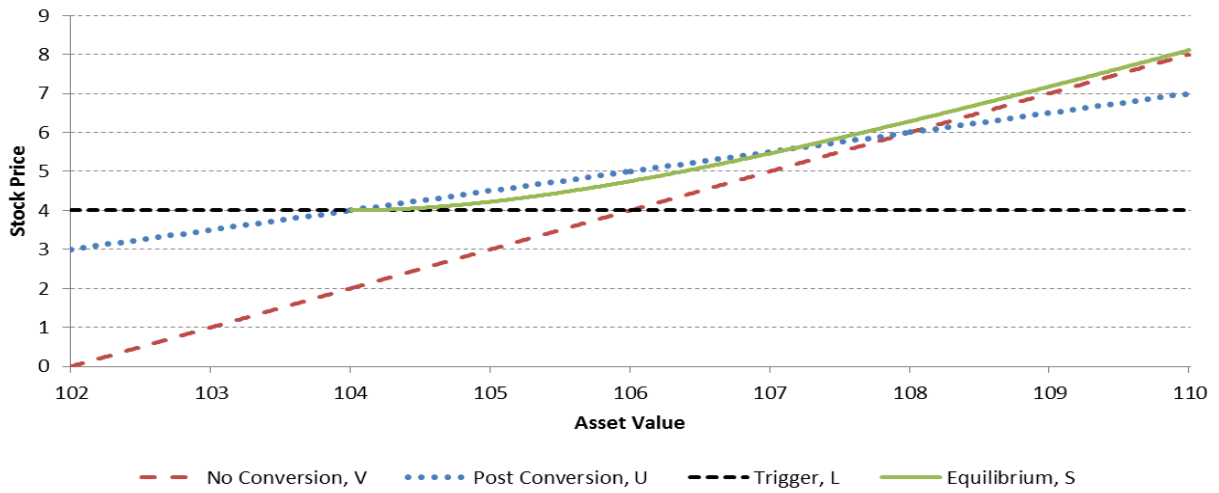


Figure 2 Equity Conversion CoCos

Panel A Benchmark Parameters



Panel B Benchmark Parameters except Volatility $\sigma = 0.47\%$



Panel C Benchmark Parameters except Trigger $L = 0$

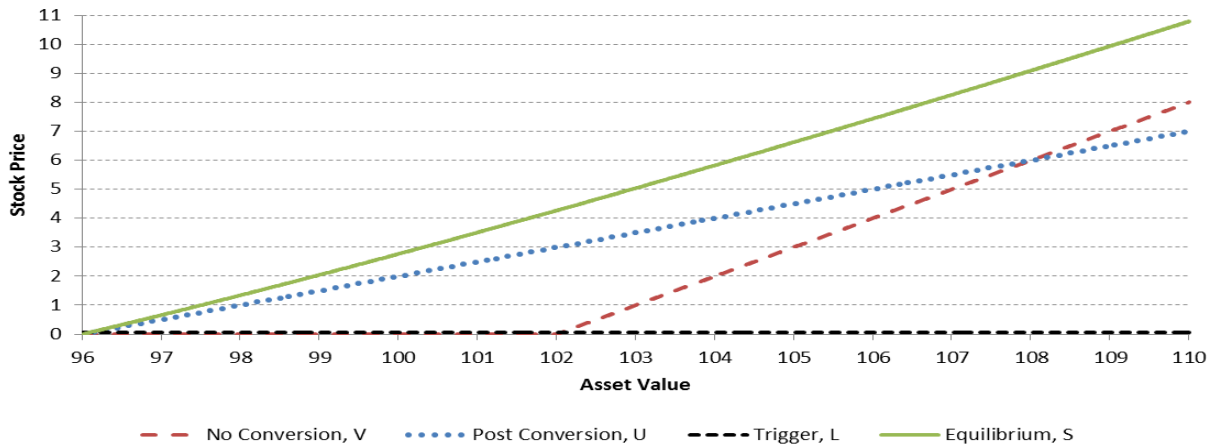
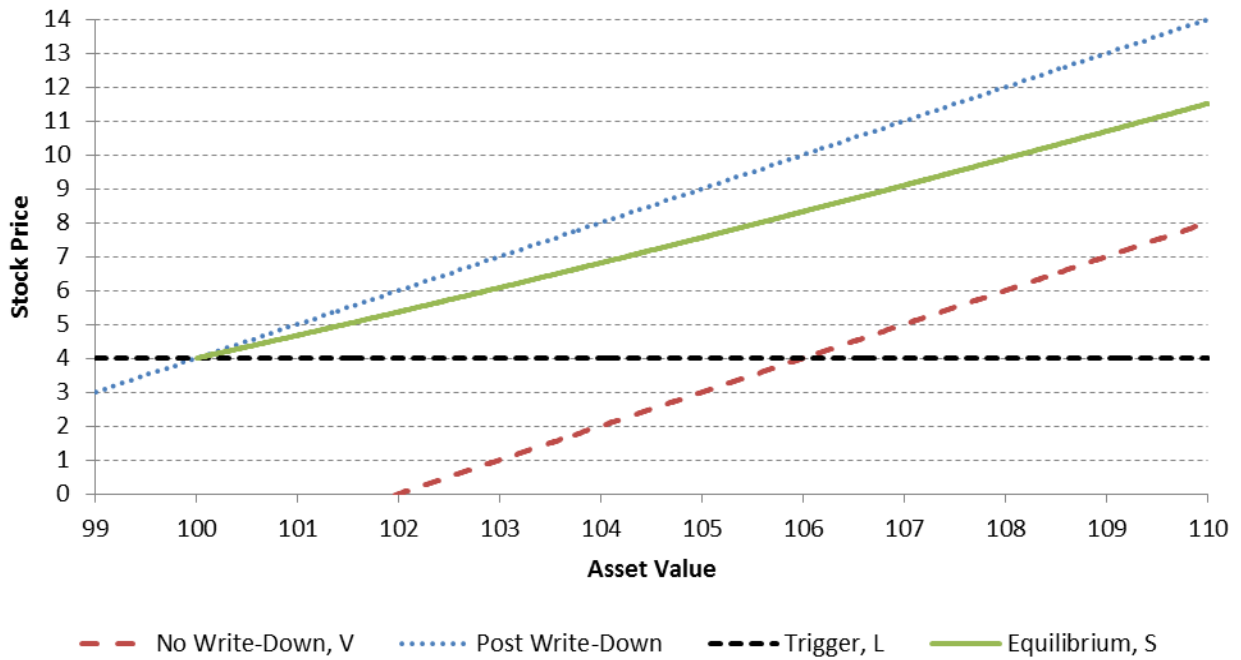


Figure 3 Write-Down CoCos

Panel A 100% Write Down ($\alpha = 0$) and Benchmark Parameters



Panel B 100% Write Down ($\alpha = 0$) and Benchmark Parameters except Volatility $\sigma=1.43\%$

