

Flight-to-Quality Debt Crises*

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Abstract

I characterize risk averse lenders' optimal bond holdings under which flight-to-quality crises can arise when there are large differences in borrowing countries' future default risks, and do this within a dynamic, stochastic general equilibrium model. In this paper, there is a substitution effect between bonds issued by different countries because they are competing with each other on international borrowing. The relative fundamentals, rather than the absolute fundamentals, determine the magnitude of the substitution effect, and thus the direction of lenders' cross-border capital movements. Specifically, when the difference in countries' fundamentals is large enough, international lenders would like to move toward countries with relatively low future default risks, which improves these countries' borrowing conditions and deteriorates other countries'. Furthermore, the safer countries accommodate lenders' capital movements by issuing more debt, which reduces the borrowing resources available to other countries, further intensifies the difficulties faced by countries with deteriorated borrowing conditions, and may finally force these countries to default. Such forces were quantitatively important in explaining the empirical evidence from the recent European Debt Crisis: European peripheries had difficulty raising funds in international markets, while in countries such as Germany, and the United States, the yields declined and the debt positions rose since 2010.

JEL Classification: F34, G01, G15

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1 Introduction

The Greek recession around late 2009 has caused sharply different impacts on sovereign borrowers since 2010. As Figure 1 shows, debt yields in European peripheries increased dramatically since 2010, which is the so-called European debt crisis. However, debt yields in Germany, and the United States declined at the outset of the crisis¹. That is, besides being immune to a crisis, these two countries² had significant improvements³ in their borrowing conditions during the recent crisis. The declined yields in Germany, and the United States are even more puzzling given the fact that these two countries had increased their new bond issuance since 2010: As Figure 2 shows, debt position in these two countries increase steadily since 2007. The theoretical work on sovereign debt crises has successfully explained how and why European peripheries have a sovereign debt crisis, and how the crisis that originated in Greece was transmitted to other countries. Yet, it says nothing about why Germany, and the United States had such different experiences during the recent crisis and how such a phenomenon relates to the occurrence of a crisis in European peripheries.

This paper develops a multicountry model in which sovereign debt crises occur in some countries, and improved borrowing conditions and increased debt positions occur in other countries, and both are the consequence of international lenders' optimal portfolio choices. Specifically, there exists a substitution effect between bonds issued by different borrowing countries, which motivates lenders to adjust their bond holdings across borders if borrowing countries' fundamentals are relatively different. When international lenders are changing their bond positions across borders, countries with relatively low future default risks can have improved borrowing conditions. Furthermore, these countries also accommodate the cross-border capital movements by increasing their new bond issuance, which will intensify the difficulties faced by countries with deteriorated borrowing conditions, and may finally force these countries to default. Such forces are quantitatively important to match observed debt yield dynamics during the recent European debt crises: from the perspective of one group of countries, both yields and default probabilities are lower when the other group of countries have a lower endowment realization, the debt positions increase as a result of the better borrowing conditions, and the correlations of yields among these two groups of countries are lower when one group of countries is defaulting.

¹Figure 1 plots only the average of yields for these two groups in order to demonstrate the pattern of change clearly. Yields in individual countries within the same group change in exactly the same way as the group average does. For example, yields in Germany, and the United States are hardly different since 2010.

²The list of countries that got benefited is longer than we have here and includes, e.g., Japan.

³Given that the inflation rate is around 2%, the real return is around 0 and can even be negative for more than 5 years, meaning that international lenders are paying for the privilege of holding safe assets.

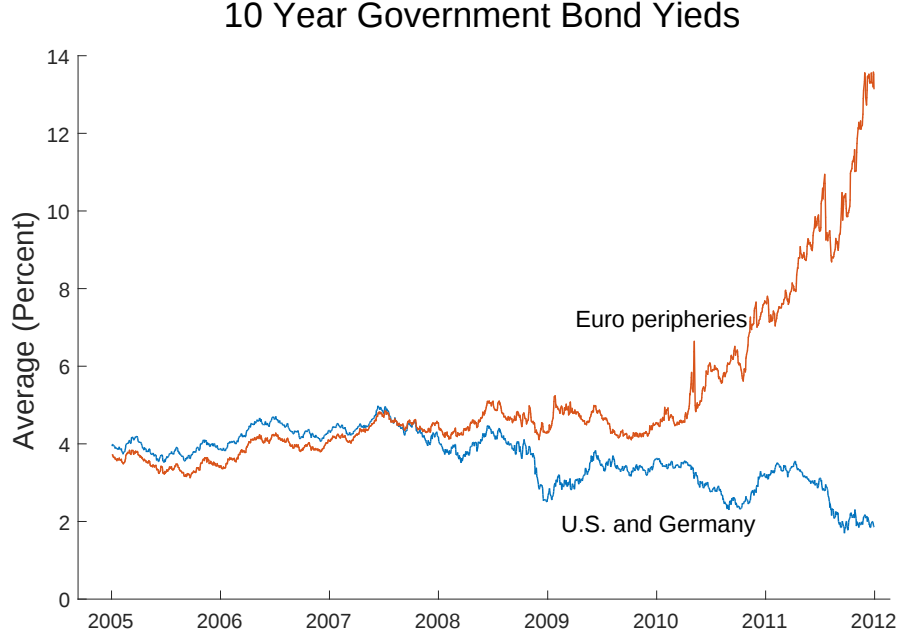


Figure 1: Debt Yields

My framework builds on a sovereign default model with risk-averse lenders and multiple borrowing countries, in the manner of [Arellano and Bai \(2013\)](#) and [Lizarazo \(2013\)](#). In the model, two groups of small countries are borrowing from competitive risk-averse lenders by issuing bonds with default options. Borrowing countries in both groups are connected in the sense that they are competing with each other for international borrowing. Lenders are investing in a portfolio composed of bonds issued by different countries, and the concave payoff makes them care about both the risk structure and the expected return. Hence, lenders choose their bond positions in according to the relative soundness of countries' fundamentals. If the difference in fundamentals between these two groups is large, lenders would like to change their bond positions across borders so that one group of countries have flight-to-quality crises and the other group of countries have improved borrowing conditions. Furthermore, the other group of countries accommodate the lenders' capital flight by borrowing more, and finally have a combination of lower yields and higher debt positions. By doing so, the model fully characterizes the conditions under which countries will experience different patterns of capital flows, and to what extent some of them may default due to the capital outflows.

To understand the mechanism of the model, consider a scenario in which GDP in one group of countries (say, European peripheries) unexpectedly declines. Given the fundamentals deteriorate, lenders would like to cut their bond holdings in European peripheries, and

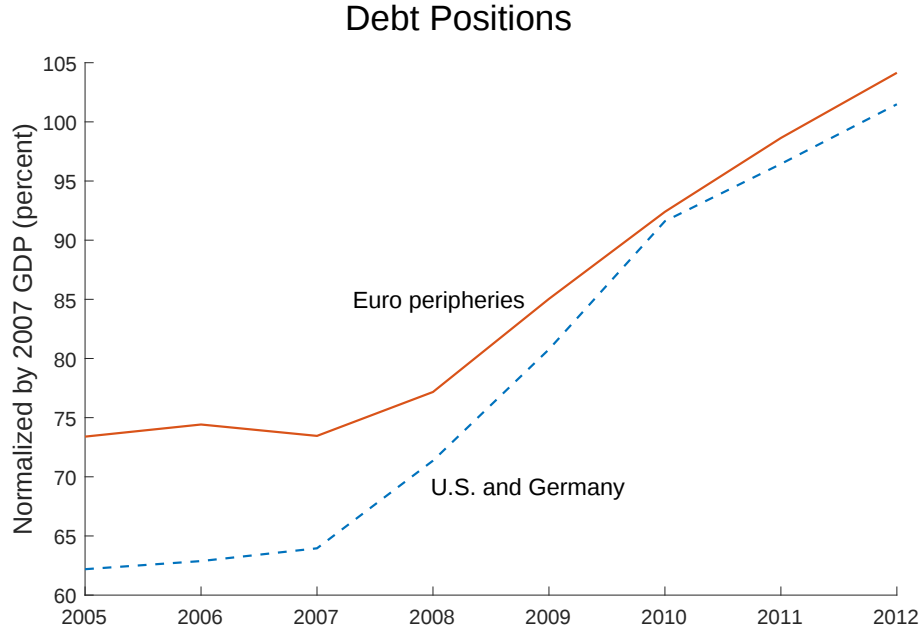


Figure 2: Debt Positions

thus impose a downward pressure on these countries' borrowing conditions. If European peripheries are the only borrowers in the market, because the lenders lack of other means to save, the magnitude of such a reduction may not be large enough to force European peripheries to default. If instead, there is another group of countries (say, Germany, and the United States) in the market, and more importantly, fundamentals in these countries signal low enough future default risks relative to the European peripheries. With better alternatives to save, lenders would have greater incentives to substitute from European peripheries to Germany, and the United States. That is, because fundamentals in Germany, and the United States are relatively strong, lenders' capital retrenchment from European peripheries has been intensified to the extent that it may bring crises to these countries. Instead of the absolute fundamentals in each group, such a behavior is driven by the relative difference between these two groups of countries' fundamentals.

Meanwhile, Germany, and the United States' endogenous responses to the capital inflows also contribute to the crises happening in European peripheries. For instance, if Germany, and the United States cannot increase the amount of bond issuance, the increased demand of bond in these countries is fully reflected by decreases in their bond yields, which will in turn limit the lenders' incentives to shift toward these countries. In contrast, Germany, and the United States in this paper accommodate lenders' capital flight by increasing their debt issuance, meaning that in addition to relatively low future default risks, bond supplies in these

countries also increase to meet the increased demand by international lenders. Hence, the lenders' incentive to adjust their bond positions have been intensified, and it may become large enough to trigger sovereign debt crises in European peripheries. I call this type of sovereign debt crises as flight-to-quality debt crises, because lenders are retrenching out of European peripheries due to the fact that they want to move toward Germany, and the United States, which have relatively low future default risks and higher debt capacities.

The model is calibrated to Greece. Both the values of lender and borrower's risk aversion are borrowed from the standard international business cycle literature. The lender's discount factor is calibrated to match the average risk-free interest rate at 4%. The borrower's discount factor is calibrated to the average default probability at 5%. The market reentry probability is calibrated to match the average market exclusion after default to be around 4 years, as in [Cruces and Trebesch \(2013\)](#). The stochastic structure of endowment realization is borrowed from [Arellano and Bai \(2013\)](#), with the persistence as 0.88 and the shock as 0.03. Other parameters are calibrated to match observed average yields in Greece, which is around 5% before the crisis.

The numerical exercise shows that borrowing condition (as well as the default probability) in one group of countries deteriorates due to the presence of the other group of countries. In particular, the overall mean of yield is 6.0% when the other group of countries repay, and it is 5.2% when the other group of countries default. Because lenders have an alternative to save, the fact that the other group of countries repay will impose an additional downward pressure on the bond price. The higher yield when the other group of countries repay also forces countries to be more likely to default: The default probability is 5.8% when the other group of countries repay, higher than a value of 5.3% when the other group of countries default. Furthermore, countries have been forced to borrow less under the deteriorated borrowing conditions: The debt to GDP ratio decreases from 4.5% to 4.3% when the other group countries repay.

I then examine how the yield in one group of countries co-moves with the endowment realization in the other group of countries. Specifically, when the endowment realization in the other group of countries changes from a high value to a low value, the mean of yield declines from 6.3% to 5.3%. That is, countries' borrowing condition improves when the other group of countries are struggling with a low endowment realization, which motivates the lenders to flee into countries with better relative fundamentals. Since the substitution effect is mainly driven through the channel of lenders' pricing kernel, the decrease in yield mentioned above can also be justified by looking into the implied risk free interest rate: It decreases from 4.4% to 4.0% at the same time. That is, even if the bond is riskless, the borrowing cost declines

if the bond issued by the other group of countries becomes more risky, and thus lenders are searching for safe heavens.

To show that how the substitution effect can leave countries with different changes in their bond prices and debt positions during a crisis, I then conduct an experiment in which endowment in one group of countries has been held constant endowment in the other group of countries drops unexpectedly. Results from the experiment show that both the bond price and the debt to GDP ratio increase when the other group of countries' endowment declines. Furthermore, the lenders' shift toward safe heavens, as well as the safe heavens' increased bond issuance have both intensified the difficulties faced by countries with deteriorated fundamentals, which is reflected by the fact that yield in these countries increase dramatically. In summary, the numerical results show that the model in this paper is able to explain why Germany, and the United States have increased bond price and debt position simultaneously when there is a crisis in European peripheries.

Finally, I also consider the case of lenders with a higher risk aversion or a lower income, either of which makes lenders more sensitive to changes in the risk structure of their portfolios. Consequently, lenders are more eagerly to reallocate their bond holdings when countries' relative fundamentals are different. The numerical results show that these arguments are valid: When the lenders' risk aversion doubles, the yield declines by an amount of 4.6% when the other group of countries' endowment declines from a high value to a low value, which is larger than the amount of 1.0% decline in benchmark model. Because the higher risk aversion also increases the the overall mean of yield, the change in default probability is even more dramatic: It increases by an amount of 2.6% as group B countries' endowment declines, compared to the 0.7% increase in the benchmark model. Moreover, The correlation of yields across these two groups of countries becomes more dispersed. All of these reflect the fact that the substitution effect is intensified by a higher risk aversion, as of a consequence of lenders are more actively changing their bond positions across borders. Similar numerical results can be obtained when lenders are having a lower income. Because international lenders lost a large fraction of their investment due to the Greek default, they became more risk-averse and more sensitive to the relative difference between countries' fundamentals. In this sense, the model in this paper has the potential to explain why crisis originated in Greece transmitted and it is only transmitted to other European peripheries, but not to Germany, and the United States.

1.1 Literature review

This work builds on the benchmark model of sovereign default analyzed by [Eaton and Gersovitz \(1981\)](#), [Aguiar and Gopinath \(2007\)](#) and [Arellano \(2008\)](#). The models presented in these papers focus on sovereign default in the case of a single borrowing country trades with risk-neutral lenders. [Lizarazo \(2009, 2013\)](#) and [Arellano and Bai \(2013\)](#) study the case of risk-averse lenders' trade with multiple sovereign countries. They show that default (or renegotiation) in one country makes the lenders poorer, and thus increases the bond premium requested from the other country, which will force the other country to default as well. Through the channel of wealth effect, they are able to generate defaults and renegotiation in tandem, which is a feature of sovereign defaults in the data. However, the wealth effect in their paper fails to explain why Germany, and the United States have seen improved borrowing conditions, when there are crises in European peripheries. This paper studies a model in which risk-averse lenders trade with two groups of small borrowing countries. The small borrower setting in this paper reduces the magnitude of wealth effect, because the strategic interactions that coordinate the borrowers' decisions on default (renegotiation) no longer exist. Instead, the substitution effect in this paper, which captures the lenders' incentive to distinguish bonds issued by different countries, becomes more prominent to determine borrowing conditions in different countries. In particular, I show that the substitution effect is able to motivate lenders to commit flight-to-quality behavior, and thus leaves countries with sharply different experiences in the international borrowing market.

Another branch of sovereign debt literature examines cross-border capital flows during a crisis, such as [Reinhart and Calvo \(2000\)](#), [Aguiar and Gopinath \(2006, 2007\)](#), [Mendoza and Yue \(2012\)](#), etc. In particular, [Reinhart and Calvo \(2000\)](#) document that the crisis country's capital inflow changes from positive before the crisis to negative after. Although cross-border capital outflows have long been viewed as the reason a sovereign country becomes insolvent⁴, there is a lack of theoretical studies to analyze why lenders choose to dramatically retrench out of a country, and how such changes would affect other countries. In this paper, since lenders care about the chance to get repayments in addition to the expected return, they tend to cut their bond positions in a country if the country has relative weak fundamentals compared to other countries. Moreover, because the opposite side of the capital outflow in one country is that other countries will experience capital inflow, this paper also predicts simultaneous improvements in borrowing conditions in the other countries.

More generally, this paper also contributes to sovereign debt literature on modeling multiple borrowers and the interactions between them. The multiple big players setting in previous

⁴See, for example, [Arellano \(2008\)](#), [Calvo \(1988\)](#), [Cole and Kehoe \(2000\)](#) and [Lorenzoni \(2014\)](#).

literature feature strategic interactions between agents, which expose the model to the risk of no pure strategy equilibrium. In this paper, risk-averse lenders trade with two groups of small borrowing countries, each with a total mass of 1, which is similar to the setting in large-game literature, including [Schmeidler \(1973\)](#), [Mas-Colell \(1984\)](#) and [Rath \(1992\)](#). As it has been proved, the existence of a pure strategy equilibrium is guaranteed in such a setting. Specifically, each borrowing country in this paper identifies itself as a small player in the market, and takes the average response of other agents as given in making its own decisions. Therefore, strategic interactions are shut off and the pure strategy equilibrium exists.

A recent wave of literature has examined why the flight-to-quality phenomenon emerges and how it relates to the financial crises. [Caballero and Krishnamurthy \(2008\)](#) develop a model in which the flight-to-quality episode arises when investors' uncertainty increases. In their paper, investors respond to increased uncertainty by moving toward safe assets, which leaves the economy in an amplified downturn. [Bemanke et al. \(1996\)](#) show that by tightening the firm's borrowing constraint, investors' flight-to-quality amplifies initial negative shocks and, finally, results in a financial accelerator. However, neither of these papers looks into the interplay between the flight-to-quality and default incentives. More recently, [He et al. \(2015, 2016\)](#) examine a problem similar to mine from the angle of a global game between lenders. In their paper, lenders are faced with a coordination problem on rollover of one of the two countries' debt. Similar to this paper, they also find that relative fundamentals and relative debt capacities determine whether a country's bond will be chosen as the safe asset. However, by focusing on a static model of default, their model fails to generate a decline in yield over time. In their paper, a country's yield is lower in the sense of comparing to the other country. Moreover, lenders in their model are assumed to choose only one country's bond in which to store the value, which contradicts common practice among international investors.

Lastly, the model in this paper is also related to the literature on how a supplier of a safe asset would be affected during episodes of flight-to-quality. [Krishnamurthy and Vissing-Jorgensen \(2012\)](#) find that the US Treasury yields declined by 73 basis points, on average, from 1926 to 2008 because investors valued the liquidity and safety of the US Treasury. [Caballero and Krishnamurthy \(2009\)](#) develop a model to study the relationship between flight-to-quality and global imbalance. They find that the already large net external liabilities of the United States are likely to continue, because international investors value the safety attributes of the US treasury. However, they argue that the low interest rate could turn out to be toxic, because it encourages borrowing and increases the leverage level in domestic institutions of the United States. [He et al. \(2015, 2016\)](#) show that yield in the United States drops because

international lenders are choosing the bond issued by the United States as a mean to store the value. Similar to this paper, safe countries such as Germany, and the United States in their study accommodate the increased demand of their bonds by issuing more bonds, and finally have a combination of improved borrowing conditions and higher debt positions.

2 A Two-Period Example

The full model specified in the last section is too complicated to illustrate the basic intuitions. In this section, I construct a two-periods example to proceed the arguments on how an improvement in group f 's endowment realization (and thus their borrowing conditions) deteriorates borrowing conditions in group h , and thus may force them to default. I call it the *substitution effect*, as it is due to the fact that these two groups are competing on international borrowing, and to distinguish it from the *wealth effect* emphasized in [Arellano and Bai \(2013\)](#). The key difference between this paper and [Arellano and Bai \(2013\)](#) is in their paper, group f countries are able to default on their debt. As a consequence, an improvement in group f 's endowment realization increases the lender's chance to get bond repayments from group f countries today, which in turn helps improving borrowing conditions in group h countries. In contrast, the model here captures the idea that group h countries are facing severe competitions from group f countries in the international borrowing market. In particular, the lenders are taking the non-default-able group f countries bond as a safe asset. Furthermore, a higher endowment realization in group f countries induces them to borrow more against future, which increases the supply of safe bonds to the lenders, and reduces the borrowing resource available for group h countries.

Consider a two-periods version of my model in which endowment realizations in both groups of countries are stochastic in period 1, (y_h, y_f) , but both are fixed in period 2 once realized in period 1, that is

$$y'_h = y_h, y'_f = y_f$$

In other words, the endowment process is perfectly persistent and the only source of uncertainty stems from period 1 realizations. By doing so, I am able to focus on how the period 1 endowment realization in group f countries would affect borrowing conditions in group h countries, and thus the default decisions. To make the problem meaningful, in period 1, both groups are in the international borrowing market with positive matured debt levels $b_h \geq 0$ and $b_f \geq 0$. For simplicity, I assume $\theta = 0$ so that I can ignore the possibility of market reentry in period 2. In addition, countries under market exclusion are assumed to suffer from an output cost so that its endowment process changes to be y_h^d as I've defined

before. In contrast, if a country chooses to repay, then it is allowed to choose the new bond issuance, b'_h in period 1. The group f countries are again assumed to repay their debt in any circumstance⁵.

2.1 Group f Borrower's Decisions

Because group f countries cannot default and because y'_f is deterministic, they are solving a standard consumption smoothing problem. That is, given the bond price schedule, q'_f , a group f government is choosing b'_f to smooth consumption in both periods, thus we have

$$u'(y_f + q'_f b'_f - b_f) = \beta(1 + r_f) u'(y_f - b'_f) \quad (1)$$

where

$$\frac{1}{1 + r_f} = q'_f = \frac{\delta \psi'(c'_L)}{\psi'(c_L)} \quad (2)$$

Assumption 1. β is sufficiently low such that $\beta(1 + r_f) < 1$ always holds.

The assumption states that countries are impatient such that they would like to shirk the debt repayment duty into next period. Since $\beta < \delta$, this assumption is true if there exists only the lenders and the group f borrowers, as can be found in the classical consumption smoothing literature such as Aiyagari (1994); Huggett (1993). The implication of it is that group f countries would gradually accumulate debt across time without borrowing limits. This is somewhat consistent with the reality in the sense that countries like Germany and United States had accumulated a huge amount of debt over time. Therefore, the assumption of $\beta(1 + r_f) < 1$ is innocuous for my analysis of group h countries' default decisions in this paper.

Lemma 1. *Group f countries' net repayment in period 1, $b_f - q'_f b'_f$ is strictly lower than their net repayment in period 2, b'_f . Furthermore, the period 1 new bond issuance in group f countries, b'_f is strictly increasing in y_f .*

Proof. See Appendix A.1. □

⁵Even if group f countries are allowed to default, this can be easily achieved when b_f is small enough so that group f countries are very unlikely to default in period 1. For example, if b_f is smaller than $\bar{b}$ defined in the manner of Cole and Kehoe (2000). Furthermore, because period 2 is the last period and the endowment process is assumed to be fixed, group f countries would never be allowed to borrow to an amount such that they would default in period 2. Thus, the lender doesn't need to worry about defaults in group f countries if b_f is sufficiently small. In a limiting case, if $b_f = 0$, then group f countries would never default in both periods.

The results of Lemma 1 comes directly from the assumptions of $\beta(1 + r_f) < 0$ and y_h is fixed in period 2. Even if $b_f = 0$, group f countries are impatient such that they would like to borrow against future to have a higher current consumption, $b'_f > 0$. An increase in y_f not just increases the resource available today, but also increases the ability to repay tomorrow. Because $\beta(1 + r_f) < 0$, the second effect dominates and thus group f countries would like to borrow more.

2.2 Group h Borrower's Decisions

Because the problem only has two periods, I can solve group h 's default and borrowing decisions by backward inductions. In period 2, the country only chooses to repay the debt b'_h in period 2 if

$$y_h - b'_h \geq y_h^d$$

which implies

$$b'_h \leq y_h - y_h^d$$

It thus imposes a borrowing limit on group h countries in period 1, which can be denoted as $\bar{b}'_h = y_h - y_h^d$. In essential, because there is no further uncertainty in the country's period 2 endowment realization, equilibrium default can not exist in period 2. However, the option to default in period 2 still imposes a restriction on period 1 borrowing such that the bond price equals to 0 if new bond issuance exceeds such a limit. I further assume that β is sufficiently low so that the country always want to borrow up to the borrowing limit in period 1 if it already chose to repay b_h , $b'_h = \bar{b}'_h$.

In period 1, the country chooses to repay only if

$$u(y_h + q'_h b'_h - b_h) + \beta u(y_h - b'_h) \geq (1 + \beta) u(y_h^d) \quad (3)$$

From equation (3), it is easy to see that the interdependence between countries' decisions is through the channel of bond price. In particular, endowment realizations in group f countries affect the lender's consumption, and thus can affect the borrowing conditions in group h countries.

Lemma 2. *There exists a cutoff on y_h , $y^*(\mathbf{B}, y_f, b_h)$ such that a group h country repays if $y_h \geq y^*(\mathbf{B}, y_f, b_h)$ and defaults otherwise.*

Proof. See Appendix A.2. □

Lemma 3. *For $y_h = y^*(\mathbf{B}, y_f, b_h)$, the net repayments in both periods by a group h country are the same.*

Proof. See Appendix A.3. □

Lemma 4. *When a group h country chooses to repay, the net repayment in period 1 is strictly decreasing in y_h , while the net repayment in period 2 is strictly increasing in y_h .*

Proof. See Appendix A.4. □

In summary, because a higher endowment increases the borrowing limit in period 1, and hence the new bond issuance in period 1, period 2 net repayment by a group h country increases. As a consequence, given group h countries choose to repay, a higher endowment realization in these countries shifts lenders' consumption from current period to the next period.

Lemma 5. *For values of y_h strictly greater than $y^*(\mathbf{B}, y_f, b_h)$, net repayment in period 2 is strictly higher than the net repayment in period 1.*

Proof. See Appendix A.5. □

2.3 Substitution Effect

One feature that distinguishes this paper from others is the model here admits *substitution effect*. In particular, given group f countries are in the market, substitution effect states that a higher endowment realization in group f countries makes their bond a better saving apparatus for the lenders, and thus would reduce the lender's demand of group h bond. In essential, risk averse lenders in this paper make both groups of countries compete with each other in the international borrowing market. Thus, a higher endowment makes group f countries more competent in the international market, and thus their bond becomes a better substitute to the bond issued by group h countries.

I use the Lemma 6 to show that a higher endowment realization in group f countries induces these countries to borrow more, which in turn tightens borrowing conditions in group h countries, and I call it the *substitution effect*.

Lemma 6. *Given group h countries repay, their bond price, q'_h is strictly decreasing in group f 's endowment realization, y_f .*

Proof. See Appendix A.6. □

Lemma 6 shows that a higher y_f deteriorates borrowing conditions in group h countries. Because the value of repaying decreases, it is easy to see that the group h countries are more likely to default with such a downward pressure on their bond price schedule, q'_h . Thus, a higher y_f forces group h countries to be more likely to default, and I call it the substitution effect.

In this special case, a higher y_f induces group f countries to increase their supply of safe assets (to the lenders), which in turn reduces the lenders' demand of group h bonds. Because bond supply in group h countries are inelastic, such a reduction is fully reflected at declines in group h countries' bond prices, and thus forces group h countries to be more likely to default. Intuitively, the increased supply of safe bonds in group f countries crowds out the demand of risky bonds issued by group h countries.

2.4 Wealth Effect

The two-periods example in this section also admits the existence of *wealth effect* mentioned in Arellano and Bai (2013). To demonstrate this point, I then need to show what happens to the bond price in group h countries when group f countries are (conceptually) not in the market, compared to the price when group f countries are in the market. Then, I will be able to show that the existence of group f countries help improving group h countries' borrowing conditions in some circumstances. However, as I've argued in the last section, once group f countries are in the market, they chooses their new bond issuance in a manner such that it tilts the lenders' consumption against future. To disentangle the wealth effect from the substitution effect, I thus assume $b'_f = b_f - q'_f b'_f$ through my whole analysis of the wealth effect in order to isolate such consumption tilts.

If group f countries are conceptually not in the market, bond price in group h countries is

$$\tilde{q}'_h = \frac{\delta \psi'(y_L + b'_h)}{\psi'(y_L + b_h - q'_h b'_h)} \quad (4)$$

On the other hand, if group f countries are in the market, bond price in group h countries is

$$\hat{q}'_h = \frac{\delta \psi'(y_L + b'_h + b'_f)}{\psi'(y_L + b_h - q'_h b'_h + b_f - q'_f b'_f)} \quad (5)$$

For $y_h = y^*(\mathbf{B}, y_f, b_h)$, Lemma 3 states that $b'_h = b_h - q'_h b'_h$, and we have $\tilde{q}'_h = \delta$. Also

because I am consider the case of $b'_f = b_f - q'_f b'_f$, then $\hat{q}'_h = \delta$. That is, the wealth effect doesn't exist when $y_h = y^*(\mathbf{B}, y_f, b_h)$.

When group h countries strictly prefer to repay their debt, $y_h > y^*(\mathbf{B}, y_f, b_h)$, we have $b'_h > b_h - q'_h b'_h$. As a consequence, the concavity of $\psi(\cdot)$ implies

$$\frac{\psi'(y_L + b'_h + b'_f)}{\psi'(y_L + b_h - q'_h b'_h + b_f - q'_f b'_f)} > \frac{\psi'(y_L + b'_h)}{\psi'(y_L + b_h - q'_h b'_h)}$$

and thus $\hat{q}'_h > \tilde{q}'_h$ if $y_h > y^*(\mathbf{B}, y_f, b_h)$. Notice that the inequality above is driven by the fact that $\psi'(y_L + b'_h) < \psi'(y_L + b_h - q'_h b'_h)$ and $\psi(\cdot)$ is strictly concave. Reciprocally, we have $\hat{q}'_h < \tilde{q}'_h$ if $y_h < y^*(\mathbf{B}, y_f, b_h)$. In sum, the existence of group f countries improves borrowing conditions in group h countries when these countries would like to repay and deteriorates their borrowing conditions when they would like to default.

To understand the wealth effect, it is good to think about the financial resource available for the lenders besides debt repayments by group h countries. Specifically, it is $y_L + b_f - q'_f b'_f$ (or $y_L + b'_f$ in period 2) when group f countries are in the market while it is y_L otherwise. The existence of group f countries provides to the lenders a tool to save and thus enjoy a higher level of wealth in general. Because their wealth increases, the risk averse lenders would in general be able to lend to group h countries, which is reflected at a higher bond price in these countries. In this sense, the existence of group f countries in the market help improving the borrowing conditions in group h countries, and I document it as the wealth effect.

2.5 Total Influence

I already show that in considering the impacts on bond price brought by group f countries, there exist wealth effect and substitution effect, and they are in different directions. I then need to discuss on the total impacts on group h countries' borrowing conditions because the magnitudes of them vary across endowment realizations.

As a easy starting point, I already show that there is no wealth effect when $y_h = y^*(\mathbf{B}, y_f, b_h)$. Thus, given the substitution effect and continuity of the problem, the total impact is for sure negative at the neighborhood this point. However, the positive wealth effect emerges for any y_h that is strictly above the level of $y^*(\mathbf{B}, y_f, b_h)$, and it could complicate my analysis of the total effect.

Lemma 7. *The substitution effect strictly dominates when $y_h = y^*(\mathbf{B}, y_f, b_h)$. That is, the existence of group f countries in the market imposes a downward pressure on group*

h countries' bond price. Furthermore, the conclusion also holds for some values of $y_h > y^*(\mathbf{B}, y_f, b_h)$.

Proof. See Appendix A.7. □

Notice that the magnitude of the total effect also depends on the initial debt level in group f countries. Specifically, a higher initial debt level in group f countries implies higher debt repayments in both periods, which intensifies the wealth effect imposed on the borrowing condition in group h countries. As a consequence, the substitution effect is less likely to dominate. To illustrate this point intuitively, consider an extreme case of $b_f = 0$. Equation (1) and assumption 1 imply

$$b'_f > 0 > b_f - q'_f b'_f$$

That is, when $b_f = 0$, borrowing behavior in group f countries tilts lender's consumption towards future period for sure, which will unambiguously reduce the bond price in group h countries. Or in other words, the substitution effect strongly dominates. Furthermore, Lemma 1 states that the difference between b'_f and $b_f - q'_f b'_f$ is increasing in y_f . That is, group f countries tilt the lender's consumption more when their endowment realization is higher. Thus, the substitution effect is increasing in y_f .

On the other hand, the wealth effect is also related to the amount of b_f , e.g., if b_f is sufficiently large. The concavity of the utility function implies that the motive to smooth the burden across periods is higher than the motive to consume against future period. Therefore, compared to tilting the lender's consumption, the existence of group f countries contributes more in increasing the wealth available to the lender, the so-called wealth effect.

2.6 Flight-to-Quality Debt Crises

Figure 3 depicts group h countries' default decisions in two cases: when group f countries are in the market, $y^*(\mathbf{B}, y_f, b_h)$ and when group f countries are (hypothetically) not in the market, $\tilde{y}(\mathbf{B}_f, b_h)$. Notice that in Lemma 3, I find that substitution effect always dominates for $y_h = y^*(\mathbf{B}, y_f, b_h)$ and it is intensified as y_f increases. Therefore, in figure 3, $y^*(\mathbf{B}, y_f, b_h)$ is at the right hand side of $\tilde{y}(\mathbf{B}_f, b_h)$ and it has a positive slope. The shadow area, which contains the endowment realizations such that group h countries repays if group f countries don't exist in the market. Reciprocally, the existence of group f countries imposes a downwards pressure on the borrowing conditions in group h countries, and thus forces them to default. I denote such a type of crises as *flight-to-quality debt crises*.

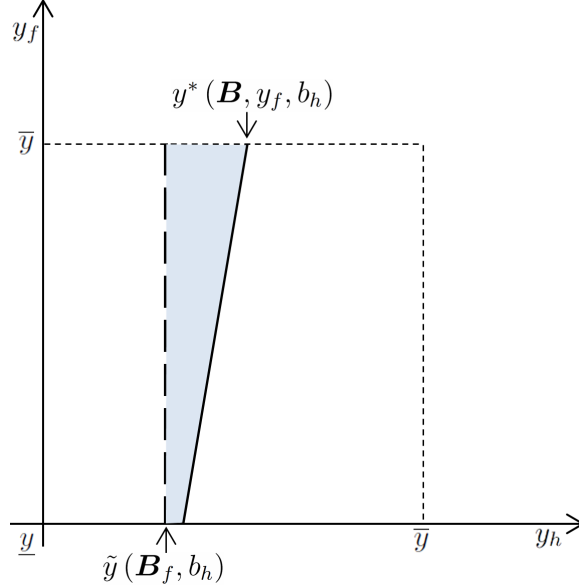


Figure 3: A Group h Country's Best Response

As I've argued before, the existence of group f countries brings two effects on the borrowing conditions in group h countries. First, the existence of group f countries in the market improves the borrowing conditions in group h countries through the wealth effect. In particular, with a sufficiently high b_f , group f countries makes net repayments in both periods, which makes the lenders wealthier and be more willing to hold risky assets, namely, group h bonds. Intuitively, the lender uses a portfolio composed of bonds from these two groups of countries as a way to save for the next period. Compared to the case when group f countries are not in the market, the existence of them with a sufficiently high b_f implies that the lenders are having more wealth. With a higher wealth, lenders will increase their bond holdings in both groups of countries, which increases the demand of bond in group h countries. As a consequence, the resulted higher bond price also encourages group h countries to repay. I call such an effect as *wealth effect*, as it has been widely discussed by [Arellano and Bai \(2013\)](#), [Kyle and Xiong \(2001\)](#) and [Xiong \(2001\)](#).

On the other hand, group f countries also affects the borrowing conditions in group h countries in the opposite direction, by decreasing its bond price through the substitution effect. The reason is because in addition to bring a positive impact on the lender's wealth, group f countries also tend to tilt the lender's consumption toward future. With higher incomes in future periods compared to the current periods, the lender is less willing to save more, and thus would reduce their demand of risky bonds issued by group h countries. In this sense, the lender is substitute risky bonds from group h countries by safe bonds from group f countries. More importantly, discussions in section 3.1 through 3.5, as well as figure 3 also

demonstrate that the magnitude of substitution effect is increasing in y_f . Specifically, because the endowment process is fully persistent, a higher y_f induces group f countries to borrow more. Given the lender's intertemporal substitution (or saving schedule), the increased supply of safe assets by group f countries induces the lenders to re-balance the composition of their portfolio toward the group f bonds, which is similar to the mechanism addressed by He et al. (2015, 2016). Such a portfolio re-balance reduces lender's demand of bonds in group h countries, and thus tightens the bond price schedule. Intuitively, because both groups of countries are borrowing from the common international lenders, an improvement in group f 's endowment realization implies that group h countries are less competent in the international borrowing market, which makes their bond becomes less appealing to the lenders. I call such an impact the *substitution effect*, which is the main mechanism through which this model can generate flight-to-quality behavior by the lenders.

In this model, the total impact is a combination of the wealth effect and the substitution effect. If the wealth effect dominates, the presence of group f countries in the market benefits group h countries by improving their borrowing conditions. On the other hand, if the substitution effect dominates, it implies that borrowing conditions in group h countries deteriorate because lenders are withdrawing their money from these countries and, at the same time, moving toward group f countries. Notice that under the endowment realization under where group h countries are different between default and repay, the substitution effect dominates for sure, which is also demonstrated in Figure 3. Thus, the main conclusion is that the existence of group f countries would force group h countries to be more likely to default. Furthermore, group h countries become more likely to default as group f countries are having a higher endowment realization, y_f , as it is also demonstrated in Figure 3.

Despite the two-periods example in this section is successful in illustrating the basic intuition, both the two-periods setting and the strict assumptions on endowment process impede further investigations. More importantly, since the lenders don't need to care about the role of group f bonds as a way for insurance, the two period example here fails to generate the improved borrowing conditions in group f countries when group h countries are facing difficulties in international borrowing. To incorporate these effects, and thus to explain the empirical findings in safe assets, I then need to analyze the full model specified in section 2, which is done in section 4 and 5.

During the recent Euro Zone Crisis, Germany, and the United States experienced improved borrowing conditions while, some the European peripheries continued to have difficulties in borrowing abroad. In this paper, I argue that the substitution effect provides a reason for the decreased debt–yield rate observed in those countries during the European debt crisis. As

I’ve argued above, the substitution effect indicates lenders to reduce their bond holdings in group h countries and, at the same time, increase their bond holdings in group f countries. Other things being equal, capital flight toward group f countries would push up the bond prices or, similarly, push down the bond yield rate in these countries. Reciprocally, the cross border capital movements also force group h countries to default. These conclusions are consistent with the findings in [He et al. \(2015, 2016\)](#), where they use a global game framework to discuss about the substitution between bonds issued by different countries. Similar to this paper, they find that how the increased supply of safe assets would increase the yields of risky bonds. In their model, two countries are compete in obtaining the funding from international lenders. As a consequence, an increase in bond supply in one country signals to the lenders a sounded fundamental and thus the bond issued by it would be recognized as the safe asset. Reciprocally, the bond issued by the other country would bear a higher yield since it fails in the competition. Although their paper is succeed in having the substitution between different bonds, the static setting prohibit their model from generating bond yield dynamics as well as debt positions across different countries. In contrast, the quantitative sovereign debt framework in this paper is able to generate those business cycle dynamics, and thus bears richer economics implications.

3 General Model

The model consists of agents who live for infinite periods, and there is a single good in each period. There are two groups of countries, denoted by $j \in \{h, f\}$. Each group contains a continuum with measure one of identical small open economies⁶. There are three types of agents in this model: households, governments, and international lenders. There is a government and a representative household in each country. For simplicity, I will mainly focus on the representative country in group h in laying out the model, because the problems for both these two groups are somewhat symmetric.

The household’s preferences are

$$\mathbb{E} \left[\sum_{t=0}^{\infty} \beta^t u(c_{h,t}) \right]$$

where $c_{h,t}$ denotes its consumption at period t . For simplicity, the subscript t is omitted in

⁶The main purpose of this setting is to address the similarities shared by the same type of countries in the sovereign debt market—for example, emerging market economies versus developed economies. The assumption can possibly be loosened by allowing idiosyncratic risk for every country. However, it will greatly complicate my analysis—without bringing any key insights—once idiosyncratic risk is not big enough to overturn the similarities shared by countries within the same group.

laying out the model in the following sections. The periodic utility function $u(\cdot)$ is continuously differentiable, strictly increasing, and strictly concave. Each period, the household receives a stochastic endowment, y_h , which is public information and has a bounded support of $[y_h, \bar{y}_h] \in \mathbb{R}$. Notice that endowment realizations are assumed to be the same across countries within one group. Thus, the only difference in endowment realizations is across different groups. Finally, y_h follows a Markov process with transition function $f(y'_h | y_h)$ ⁷.

The government in each country trades with international lenders on uncollateralized bonds. It is benevolent, in the sense that its goal is to maximize its domestic households' utility. I further assume that group f countries always choose to repay their debt while group h countries may choose to default. That is, the model here captures the idea that bonds issued by group f countries are recognized by lenders as safe assets while bonds issued by group h countries are recognized as risky assets. By doing so, I can focus on the question of how the changes in supply of safe assets would affect the yields of risky assets.

Each period, after y_h realizes, the government chooses whether to default on the matured debt, d_h

$$d_h = \begin{cases} 0 & \text{repay} \\ 1 & \text{default} \end{cases}$$

The default decision is public knowledge once it has been made. Thus, all governments and lenders have perfect information about default decisions before taking any further actions. Once a group h government chooses to default, it is immediately excluded from international borrowing. The defaulting country also suffers an output cost similar to [Chatterjee and Eyigungor \(2012\)](#); that is, the endowment process changes to y_h^d after the government defaults

$$y_h^d = \min \{y_h - d_0 y_h - d_1 y_h^2, y_h\}, d_1 \geq 0$$

As argued by [Chatterjee and Eyigungor \(2012\)](#), when $d_0 < 0$ and $d_1 > 0$, the output cost defined above resembles the one defined in [Arellano \(2008\)](#). Furthermore, it is smoother than the one in [Arellano \(2008\)](#) so that it alleviates the possibility of non-convergence in quantitative analysis.

If the government repays, then given the bond price schedule q'_h , it is allowed to choose the new bond issuance, b'_h . The new bond issuance is assumed to have a finite support, $[\underline{b}, \bar{b}]$ ⁸.

⁷I don't impose any restrictions of symmetry among these two groups of countries. Indeed, the endowment process in these two groups are allowed to be different. However, the model does require the endowment process to be identical among countries within the same group to simplify the analysis.

⁸Given that the endowment process is bounded and the debt is default-able, this assumption is innocuous if $\underline{b}$ is sufficiently low and $\bar{b}$ is sufficiently high.

The budget constraint can thus be summarized as

$$c_h + (1 - d_h) b_h = (1 - d_h) (y_h + q'_h b'_h) + d_h y_h^d$$

Furthermore, the country that defaults at period t will have a probability of $\theta \in (0, 1)$ to re-access the market since period $t + 1$, as in [Arellano \(2008\)](#) and [Chatterjee and Eyigungor \(2012\)](#).

The group h governments are the only strategic agents in this model. Similar to the one-borrower setting in [Cole and Kehoe \(2000\)](#) and [Arellano \(2008\)](#). In particular, when a group h government makes its decision, it takes into account the decision's influence on the price of its own debt. Furthermore, since all countries borrow from the same lenders, there is a linkage between borrowing countries. However, the linkage in this paper differs from the one in [Arellano and Bai \(2013\)](#), who consider only two countries while both can default on their debts, and the government in each country is a big player in the international borrowing market. In their paper, there are strategic interactions between these two big borrowing countries, which could possibly impede the existence of a pure strategy equilibrium. In contrast, the model in this paper assumes a continuum of countries in each group, and each government recognizes that any single country is ignorable compared to the sum of other countries in the international borrowing market. Therefore, the government ignores the impacts on the international borrowing market brought by its own actions and takes into account only the average responses of other governments instead of any specific (other) country's. Such a large-game setting was firstly formulated by [Schmeidler \(1973\)](#), then developed by [Mas-Colell \(1984\)](#). It has been proved by [Rath \(1992\)](#) that the existence of a pure strategy equilibrium is guaranteed under such a setting. Intuitively, because any single government ignores its own influence on other countries' borrowing conditions, the model here doesn't have the strategic interactions that impedes the existence of a pure strategy equilibrium.

There is a continuum with measure one of identical international lenders who are competitive. The representative lender is risk averse and has a preference as

$$\mathbb{E} [\delta^t \psi(c_{L,t})]$$

where $c_{L,t}$ is the lender's consumption at period t . $\psi(\cdot)$ is continuously differentiable, strictly increasing, and strictly concave. Each lender trades with each country on uncontingent bonds. The lender's discount factor is assumed to be greater than β , $0 < \beta < \delta < 1$, in order to guarantee borrowings in equilibrium. Each period, the lender receives a fixed amount of endowment, y_L .

The arguments before demonstrates that besides individual debt level, a typical country should keep track of the average indebtedness in both groups as well as endowment realizations to solve the problem. To formulate the problem, define $\mathbf{B}$ and $\mathbf{y}$ as

$$\mathbf{B} = (B_h, B_f), \mathbf{y} = (y_h, y_f)$$

where B_j is the average indebtedness of group $j \in \{h, f\}$ countries:

$$B_j = \int_0^1 b_j(i) di$$

Furthermore, D_h is the ratio of default in group h :

$$D_h = \int_0^1 d_h(i) di$$

In equilibrium, since countries within the same group are assumed to receive the same endowment realization each period, it will be non-optimal for a single country to deviate by choosing a different default decision or a different debt level. Thus, in equilibrium, group h countries would all default or all repay, $D_h \in \{0, 1\}$. Moreover, the individual debt level equals the average indebtedness within the same group

$$b_j(i) = B_j, \forall i$$

Therefore, the net repayment from group h countries can be written as

$$\begin{aligned} NR_h &= \int_0^1 [1 - d_h(i)] \{b_h(i) - q'_h[b'_h(i)] b'_h(i)\} di \\ &= (1 - D_h) [B_h - q'_h(s, B'_h) B'_h] \end{aligned}$$

while the net repayment from group f countries are

$$\begin{aligned} NR_f &= \int_0^1 \{b_f(i) - q'_f[b'_f(i)] b'_f(i)\} di \\ &= B_f - q'_f(s, B'_f) B'_f \end{aligned}$$

The arguments above imply that instead of the whole distribution of initial debt level in each group, an individual country only needs to keep track of the average indebtedness and the default ratio within each group to formulate the problem. Each period, with the initial aggregate state $s = (\mathbf{B}, \mathbf{y})$, the timing of actions is similar to [Arellano \(2008\)](#) and can be summarized as:

1. (y_h, y_f) realizes, then given (s, b_h) , the group h government decides whether to default or not, d_h .

2. If the government defaults, it simply consumes y_h^d and will reenter the market with probability θ since next period.
3. If the government repays, taking $q'_h = q(s, b'_h)$ as given, it chooses how much to borrow, b'_h .
4. Given (s, b_f) , group f government repays b_f and chooses how much to borrow, b'_f .
5. International lenders, taking the bond price schedule as given, choose how much to lend, $\tilde{b}'_h$ and $\tilde{b}'_f$.
6. The equilibrium condition requires $b'_h = \tilde{b}'_h$ and $b'_f = \tilde{b}'_f$.

3.1 The Borrower's and Lender's Problems

In formulating its problem in a recursive way, the government takes as given the rule on the evolution on the aggregate state, s , which can be summarized as

$$s' = \mathcal{H}(s)$$

I firstly consider the problem at the second stage faced by a government that has already made a default decision. Then a group h government's value function, given it repays, can be written as

$$V_h^{nd}(s, b_h) = \max_{b'_h} \{u(y_h + q'_h b'_h - b_h) + \beta \mathbb{E}[V_h(s', b'_h)]\} \quad (6)$$

Similarly, the value for the government that decided to default can be written as

$$V_h^d(s) = u(y_h^d) + \beta \mathbb{E}[\theta V_h^{nd}(s', 0) + (1 - \theta) V_h^d(s')]$$

Back to the first stage, given the value on default or repay, the government's problem at the default decision can be written as

$$V_h(s, b_h) = \max_{d_h} [(1 - d_h) V_h^{nd}(s, b_h) + d_h V_h^d(s)] \quad (7)$$

The equilibrium outcome can be summarized by the government's default decision and new bond issuance, $d(s, b_h)$ and $b'(s, b_h)$, as well as consumption $c(s, b_h)$, and values $V(s, b_h)$, $V^{nd}(s, b_h)$, and $V^d(s)$.

The problem for group f government is simpler given the fact that they cannot default, which is

$$V_f(s, b_f) = \max_{b'_f} \{u(y_f + q'_f b'_f - b_f) + \beta \mathbb{E}[V_f(s', b'_f)]\} \quad (8)$$

On the other hand, the lender takes as given the evolution of aggregate state $s' = \mathcal{H}(s)$. Also, taking as given the bond price schedule $Q(s, \tilde{b}'_j)$, the lender decides how much to lend to each country in group j , $\tilde{b}'_j$, and how much to consume. The lender's budget constraints can be summarized as

$$\begin{aligned} c_L &= y_L + \sum_{j \in \{h, f\}} N R_j \\ &= y_L + \int_0^1 [1 - d_h(i)] [\tilde{b}_h(i) - Q'_h(i) \tilde{b}'_h(i)] di \\ &\quad + \int_0^1 [\tilde{b}_f(i) - Q'_f(i) \tilde{b}'_f(i)] di \end{aligned} \quad (9)$$

where $i \in [0, 1]$ denotes an individual country within one group and $j \in \{h, f\}$ denotes the group. The equilibrium condition requires $\tilde{b}'_j(i) = b_j(i), \forall i, j$. Thus, the lender's value function is

$$V_L(s) = \max_{\{\tilde{b}'_h(i), \tilde{b}'_f(i), c_L\}} \{\psi(c_L) + \delta \mathbb{E}[V_L(s')]\} \quad (10)$$

where c_L is defined in equation (9). From the lender's problem, we can solve for the bond price function as

$$q'_h = q_h(s, \tilde{b}'_j) = \mathbb{E} \left[\frac{\delta \psi'(c'_L)}{\psi'(c_L)} (1 - d'_h) \right] \quad (11)$$

$$q'_f = q_f(s, \tilde{b}'_f) = \mathbb{E} \left[\frac{\delta \psi'(c'_L)}{\psi'(c_L)} \right] \quad (12)$$

3.2 Equilibrium

I focus on a recursive Markov equilibrium in which all decision rules are solely the functions of state variables (s, b_h) .

Definition 1. A recursive Markov equilibrium for this economy consists of (i) group h countries' policy functions for repayment, borrowing, and consumption, $\{d_h(s, b_h), b'_h(s, b_h), c_h(s, b_h)\}$, and values $V_h(s, b_h)$, $V_h^{nd}(s, b_n)$, and $V_h^d(s)$; (ii) group f countries' policy functions for borrowing, and consumption, $\{b'_f(s, b_f), c_f(s, b_h)\}$, and value function $V_f(s, b_f)$; (iii) lenders' policy functions for lending choices and consumption $\{\tilde{b}'_h(s), \tilde{b}'_f(s), c_L(s)\}$ and value function $V_L(s)$; (iv) the functions for bond price $\{q_h(s, b'_h), q_f(s, b'_f)\}$ for borrowers; (v) equilibrium prices of lending $\{Q_h(s, \tilde{b}'_h), Q_f(s, \tilde{b}'_f)\}$; and (vi) the evolution of the aggregate state $\mathcal{H}(s)$ such that, given that the initial debt levels are the same across countries within each group, $b_{j,0} = \tilde{b}_{j,0}, \forall j$:

1. Taking as given the evolution of the aggregate state $\mathcal{H}(s)$ and the bond price functions $\{q_h(s, b'_h), q_f(s, b'_f)\}$, the group h countries' policy and value functions solve their problem in equation (7) while group f countries' policy and value functions solve their problem in equation (8).
2. Taking as given the bond prices $\{Q_h(s, \tilde{b}'_h), Q_f(s, \tilde{b}'_f)\}$ and evolution of the aggregate states $\mathcal{H}(s)$, the policy functions and value functions for the lenders $\{\tilde{b}'_h(s), \tilde{b}'_f(s), c_L(s)\}$ solve the lender's problem in equation (10).
3. Taking as given countries' policy and value functions, the bond price function satisfies equation (11).
4. The prices of debt $\{Q_h(s, \tilde{b}'_h), Q_f(s, \tilde{b}'_f)\}$ clear the bond market for every country, $b_j(i) = \tilde{b}_j(i)$ and $q_j(s, b'_j) = Q_j(s, \tilde{b}'_j), \forall j \in \{h, f\}$.
5. The goods market clears, $c_h + c_f + c_L = y_h + y_f + y_L$.
6. The law of motion for the evolution aggregate state $s' = \mathcal{H}(s)$ is consistent with countries' decision rules and shocks.

3.3 Case of i.i.d. Shocks

In this section, I consider the case of i.i.d. shocks in order to see the interdependence between these two groups of countries, I then map the default decisions into the space of endowment realizations. I also assume $\theta = 0$ and no output cost after default. That is, after default, a country will be sent to permanent financial autarky. From equation (6), it is easy to see that the value of non-default is strictly increasing in y_h . Hence, there exists a threshold $y^*(\mathbf{B}, y_f, b_h)$ such that the country chooses to repay only if its endowment realization is higher than $y^*(\mathbf{B}, y_f, b_h)$. In this sense, $y^*(\mathbf{B}, y_f, b_h)$ is also the best response function with regard to a group h country's default decision.

Proposition 1. *If for b_h , there exists some level of y_h such that a group h country finds it optimal to default, then for any $\{q'_h, b'_h\}$, it has to make a positive net repayment, $b_h - q'_h b'_h > 0$.*

Proof. See Appendix A.8. □

The country would like to choose to default only if all available options require a positive net repayment. To understand this point, suppose there exists an option of negative net

repayment to the country. Then the country would strictly prefer the alternative of choosing to rollover the debt (and hence enjoys a positive capital inflow), and postpone the default into at least the next periods. By doing so, the borrower is strictly better off due to the fact that market exclusion is permanent and there is no output cost under market exclusion. However, such a plan is not optimal to the lender and should not be provided in equilibrium.

Proposition 2. *There exist a function $y_h^*(\mathbf{B}, y_f, b_h) \in [\underline{y}_h, \bar{y}_h]$ such that*

$$d(\mathbf{B}, y_f, b_h) = \begin{cases} 0 & y_h \geq y_h^*(\mathbf{B}, y_f, b_h) \\ 1 & y_h < y_h^*(\mathbf{B}, y_f, b_h) \end{cases} \quad (13)$$

Proof. See Appendix A.9. □

Equation (13) specifies the best response by a group h country, based on the value of $(\mathbf{B}, y_f)$ as they affects the borrowing condition and the value of b_h as it indicates the amount to repay. The best response specified here is similar to the one I depicted in figure 3. Similar to the two-periods example, the best response function also depends on the value of B_f and y_f through their impacts on the borrowing conditions in group h countries, q'_h . However, the full model is too complicated to be analyzed. Instead, I would mainly focus on heuristically discussions in this section and rely on the numerical exercise in next section to proceed my analysis.

From the lender's perspective, debt repayments from group f countries can be perceived as a source of income, which increases her capacity of holding risky assets issued by group h countries. For example, if the lender's periodic utility function is CRRA, the lender's absolute risk aversion is decreasing in the level of her wealth. Or in other words, decreasing in the average net repayment by group f countries. With a lower absolute risk aversion, the lender's willingness to hold group h bond increases, and thus borrowing conditions in group h countries improves, the so-called wealth effect I mentioned above.

On the other hand, the relative impatience of group f countries encourages them to borrow against future in general, which tilts the lender's consumption towards future. Specifically, if endowment process in group f countries is persistent, the setting I will adopt in the section of quantitative analysis, then a higher endowment today signals higher endowments in the future. With a higher income, group f countries would like to borrow more and thus tilt the lender's consumption towards future. Since the lender is investing on bonds issued by both groups, the increased supply of group f bonds would definitely reduce the demand of risky issued by group h countries, and thus deteriorate their borrowing conditions, the so-called substitution effect.

Different from the two-periods example above, under the full model, it is difficult to discuss on the total impact on group h borrowing conditons by group f countries. However, it is easy to imagine that compared to the case when fundamentals in group h countries are sound, if the debt level and endowment realizaton in group h countries indicate a higher probability of default in the future, the lender are more likely to rebalance her bond holdings towards group f countries. Such a flight-to-quality behavior by lenders could serve as a driving factor for defaults in group h countries. In the next section, I will rely on a quantitative analysis to discuss on the total impact, and thus the occurence of flight-to-qaulity debt crises.

4 Quantitative Analysis

I solve the model numerically and analyze the interactions between these two groups of countries in the international borrowing market. Specifically, I analyze how the fundamentals and choices in one group of countries would affect bond prices and default decisions in the other group. I show that the substitution effect is quantitatively important, especially when one group of countries are on the brink of defaulting.

4.1 Calibration

The periodic utility function for the borrower is CRRA,

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

Following common business cycle studies, I set the intertemporal elasticity of substitution (IES), $1/\sigma$ to $1/2$. The periodic utility function for the lender is logarithmic⁹,

$$\psi(c_L) = \log c_L$$

The length of a period is one year. The stochastic process for endowment is independent across these two groups and follows a lognormal AR(1) process,

$$\log(y_{t+1}) = \rho \log(y_t) + \varepsilon_{t+1}$$

with $\mathbb{E}(\varepsilon^2) = \eta^2$. The endowment process in one group of countries is assumed to be independent of the endowment process in the other group of countries. I use the method

⁹The lender's utility function would be loosened to be CRRA in the section of sensitivity analysis.

	Value	Target
Borrower's IES	$1/\sigma = 1/2$	Standard Value
Stochastic structure for shocks	$\rho = 0.88, \eta = 0.03$	Arellano and Bai (2013)
Output cost after default	$\lambda = 0.016$	Arellano (2008)
Reentry probability	$\theta = 0.282$	average Market Exclusion, 4 years
Borrower's discount factor	$\beta = 0.90$	Mean Default Probability, 5%
Lender's discount factor	$\delta = 0.96$	Mean Risk Free Interest Rate, 4%
Lender's endowment	$y_L = 1.4$	Arellano and Bai (2013)

Table 1: Parameter Values

proposed by [Tauchen \(1986\)](#) to discretize the shock into a Markov Chain. The endowment space $[\underline{y}, \bar{y}]$ is the set of $[0.83, 1.21]$ with an unconditional mean equals to 1.04, and it has discretized into 11 grids. The debt space $[\underline{b}, \bar{b}]$ is discretized into 31 grids. The numerical algorithm is explained in detail in Appendix [B](#).

Eight parameters are calibrated: I follow [Arellano and Bai \(2013\)](#) to set up the stochastic structure for borrower's endowment process and the lender's fixed endowment. The reentry probability θ is set to be 0.282 to capture the fact that, on average, the duration of market exclusion is around 4 years; see, for example, [Cruces and Trebesch \(2013\)](#). The value of default cost λ is borrowed from [Arellano \(2008\)](#), which has been set to be 0.016. On the other hand, the lender's discount factor, δ , is set to match with the average risk-free interest rate, which equals to 4%. Finally, the borrower's discount rate β , is set to match with two moments: the average default probability, 5%; and thus the mean of yield rate in Greece, 5.6%. Calibrated parameters and their matching moments are summarized in Table [1](#).

4.2 Main Results

The model is simulated for 2,500 times and statistics summarizing the debt markets in group h countries are reported. Table [2](#) reports the yield, the default probability, the debt to GDP ratio, and the implied risk-free rate for three cases: the overall mean, when group f countries repay, when group f countries default. The yield is defined as $1/q - 1$, while the implied risk-free rate is calculated by

$$r_f = \frac{1}{\mathbb{E} \left[\frac{\delta \psi'(c'_L)}{\psi'(c_L)} \right]} - 1$$

Table 2 shows that the overall mean of yield is 5.4%, the overall mean of the default probability is 4.5%, the overall mean of the debt to GDP ratio is 4.4%¹⁰, and the overall mean of the risk-free interest rate is 4%. These four statistics, however, are affected by the default decisions in group f countries. When group f countries default, the yield in group h countries decreases to 5.2%. This reflects the fact that default by group f countries makes the bond issued by group h countries the only asset to save for the future, and thus lenders would prefer to pay, in general, a higher price for the group h bond. The better borrowing conditions encourage group h countries to repay, as the default probability in group h countries decreases to 4.0%. Furthermore, the better borrowing conditions also encourage group h countries to borrow more: The debt to GDP ratio is 4.5% when group f countries default, higher than the value of 4.3% when group f countries repay. The pattern of changes in yield can also be demonstrated by looking into the implied risk-free rate, which decreases from 4.1% to 3.9%. That is, when group f countries default, lenders in general would request a lower interest rate from any other forms of borrowing.

The arguments in the previous section show that the combination of the wealth effect and the substitution effect is not monotonic across y_f values. In particular, for medium levels of y_f , the wealth effect dominates and it improves borrowing conditions in group h countries. On the other hand, for high levels of y_f , the substitution effect dominates, which will impose a downward pressure on the group h bond price. Table 3 reports the same statistics as Table 2 conditional on endowment realization in group f countries. Specifically, the endowment realization in group f countries has been separated into three categories—low: $y_f < 0.963$; medium: $0.963 \leq y_f \leq 1.04$; and high: $y_f > 1.04$.

Table 3 shows that compared to the case when group f countries have a low endowment realization, the group h bond yield is lower when the endowment realization in group f countries takes a medium value and it is higher when group f countries' endowment realizes to be high. Furthermore, when the endowment realization in group f countries takes a high value, the group h bond yield is at its highest level among these three cases. That is, although a medium level of y_f benefits group h countries through the fact that the wealth effect dominates, the further increases in y_f would raise the importance of the substitution effect and finally motivate the lenders to request a higher premium from group h countries. In this case, group h countries respond to the worst borrowing conditions by being more likely to default: Default probability increases from 3.9% to 5.0%. Similar to the discussions before, the changing borrowing conditions also change the debt positions in group h countries: with the worst borrowing conditions when group f countries have a high endowment

¹⁰The model in this paper shares the same difficulties with quantitative default model in generating the right magnitude of debt to GDP ratio, see, for example, [Lizarazo \(2013\)](#).

Group h countries	Overall Mean	Group f repays	Group f defaults
Yield (%)	5.4	6.0	5.2
Default Probability (%)	4.5	4.8	4.0
Debt to GDP (%)	4.4	4.3	4.5
Risk-Free Rate (%)	4.0	4.1	3.9

Table 2: Statistics (Default Decision)

realization, the debt to GDP ratio is 4.1%, lower than the value of 4.6% when the wealth effect dominates. Such a pattern of changes in group h debt yield can also be justified by looking into the risk-free rate, which is also at its highest value when group f countries have a high endowment realization. In particular, with a higher endowment realization in group f countries, lenders are more heavily investing on the group f bond and are more confident about getting payments from group f countries in the next period. As a consequence, lenders would thus request a higher interest rate on any form of borrowings (even if they are risk-free), given that their payoffs at the next period are expected to be higher.

The results in both tables show that the model in this paper is able to explain the empirical evidence from the recent European debt crisis. In particular, the results in Table 3 show that if the endowment realization in the European peripheries change from a high value to a low value, the yield in Germany, and the United States is reduced by 1.0% and the debt to GDP ratio in these two countries increases by 0.3%. Because the low value of endowment realization in European peripheries makes the bond issued by these countries more risky, lenders would like to shift toward Germany, and the United States. In addition, results in Table 2 show that the presence of Germany, and the United States in the borrowing market also intensifies the difficulties faced by European peripheries. Compared to the case in which these two countries default, the yield in European peripheries is higher by 0.8%. That is, the borrowing conditions in European peripheries are further deteriorated given the fact that international lenders are substitution from their bonds to the bond issued by Germany, and the United States. In summary, the results in both table match with the empirical evidence from the recent crisis: The bond price in Germany, and the United States increases, which worsens the difficulties faced by European peripheries.

Another interesting fact about the results in Table 2 and Table 3 is that the changes in the risk-free rate are always smaller than the changes in the yield. For example, when y_f increases from a low value to a high value, the risk-free rate increases by 0.4%, while the increase in yield is 1.0%. Such a smaller change in risk-free interest rate can be attributed to the group h countries' endogenous response to an improved borrowing condition. Notice

Group h countries	Low y_f	Medium y_f	High y_f
Yield (%)	5.3	5.0	6.3
Default Probability (%)	4.3	3.9	5.0
Debt to GDP (%)	4.4	4.6	4.1
Risk-Free Rate (%)	4.0	3.9	4.4

Table 3: Statistics (Endowment)

that when calculating the risk-free interest rate, the issuer has been implicitly assumed to always repay their debt. The non-defaultable nature makes the supply of risk-free bond more elastic. That is, the risk-free bond issuer accommodates lenders' shift toward them by issuing more debt, rather than lowering the yield, and thus would have the largest impact on the demand of bond in group f countries. In my model, if group h countries respond to the increased demand only by issuing more bond, their default risk increase dramatically. Hence, the group h countries' endogenous response is more moderate compared to the case of risk-free bond: When facing with improved borrowing conditions, group h countries would choose to have a combination of more debt issuance and a lower debt yield, which implies the impact on group f countries is less dramatic compared to the case of risk-free bond.

4.3 Bond Price Schedule

Results in both Table 2 and Table 3 demonstrate the interdependence of bond price function across countries by examining the average statistics. In this section, I further study how the bond price schedule in group h countries have been affected by endowment realization in group f countries. In particular, Figure 4 plots the bond price schedule, q'_h , as a function of new bond issuance, b'_h . The schedules are for a group h endowment realization as $y_h = 1.04$ and initial debt levels in both groups, $B_h = B_f = 0.09$. I plot the bond price schedules at three different levels of endowment realization in group f countries, $y_f = 0.96$, $y_f = 1.04$, and $y_f = 1.12$. The bond price is a decreasing function of new bond issuance, because a higher new bond issuance implies a higher default probability in the next period, which is consistent with the theoretical discussions before.

The bond price schedule in the case of $y_f = 0.96$ is depicted as the solid line in Figure 4. When y_f changes to its unconditional mean, 1.04, the bond price schedule shifts up (the dashed line), which implies that borrowing conditions in group h countries improve, as a consequence of the wealth effect. Specifically, when y_f increases to 1.04, instead of defaulting, group f countries are more likely to choose to repay, but do not pay down the

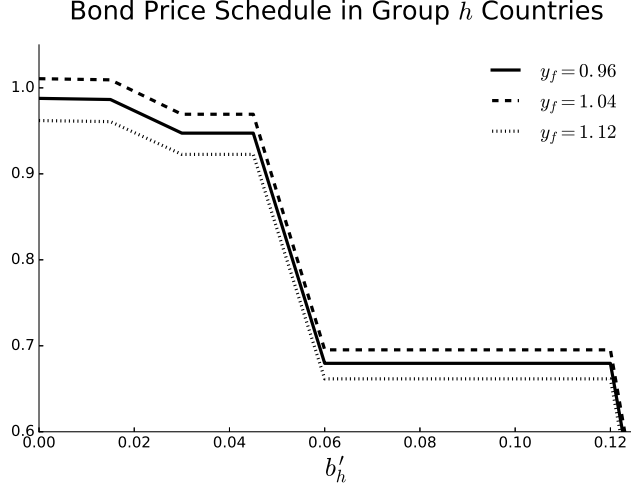


Figure 4: Bond price schedule

debt by a large amount. As a consequence, the group f bond serves as a complement to the group h bond, and thus lenders would request a lower premium from group h countries on borrowing. On the other hand, when y_f increases further to 1.12, the bond price schedule (the dotted line) becomes tighter, and it is tightest among these three cases. In this case, further increases in y_f make the group f bond a safe asset for the lenders, and also as a good substitute for the group h bond. As a consequence, the lenders would like to flee to group f countries, and thus imposes a downward pressure on the group h bond price. Hence, it leaves group h countries with the worst borrowing conditions of the three cases.

To further investigate how the bond price in group h countries varies with the endowment realization in group f countries, Figure 5 plots bond price in group h countries as a function of y_f in two cases: Group f countries are in the market and group f countries are not in the market. Similar to the Figure 4, I plot the bond price in group h countries under the case of $y_h = 1.04$, $B_h = 0.09$, and $B_f = 0.09$ if group f countries are in the market. The new bond issuance B'_h is obtained by using the optimal rule by group h countries. The bond price in the case of group f countries are under market exclusions is depicted as the solid line in Figure 5. It is slightly increasing in y_f because under the market exclusion, group f countries still have an exogenous possibility to reenter the market, which will affect the borrowing conditions in group h countries. On the other hand, if group f countries are in the market with a debt level $B_f = 0.09$, the bond price in group h countries is highly related to y_f , as it has been depicted as the dashed line in Figure 5. Specifically, for low values of y_f , both these two curves overlap with each other because with a low value of endowment realization, group f countries would choose to default even if they are in the market. For intermediate values of y_f , q'_h is increasing in y_f because of the wealth effect I've discussed

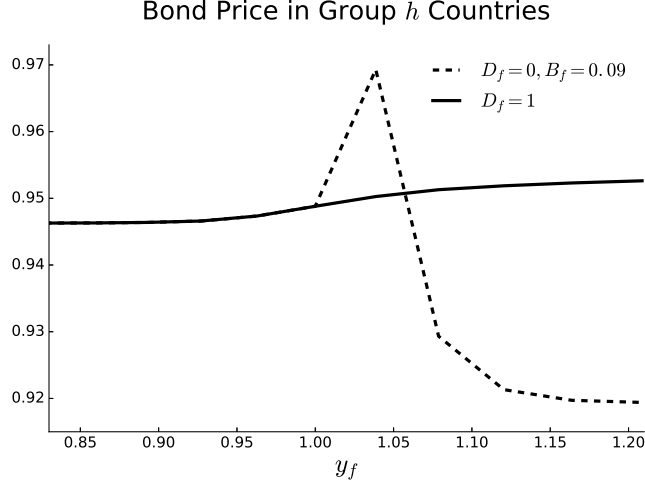


Figure 5: Bond price in group h countries

before, which makes the dashed line lies above the solid line. On the other hand, for high values of y_f , q'_h is decreasing in y_f because of the substitution effect, which makes the dashed line below the solid line. By comparing the Figure 4 to the Figure ??, it is easy to see that the numerical results here are consistent with my previous discussions of how the combination of the wealth effect and the substitution effect would affect the borrowing conditions in group h countries.

4.4 A Sudden Drop

One main goal of this paper is to see how the debt yields and debt positions in both groups countries have been affected when there is a crisis in one group. In particular, both the debt position and the bond price in group h countries increase as a response to a crisis happening in group f countries, and the lenders are shifting toward group h countries. However, such a mechanism is hard to show in a full simulation because it is difficult to disentangle the changes due to lenders' flight-to-quality behavior from the changes due to fluctuations in group h countries' endowment realization. I consider an experiment in which the endowment realization in group h countries has been held constant. Specifically, both y_h and y_f equal to 1.12 initially and remain unchanged for the first 50 periods. The initial debt levels at both groups of countries are set to be 0.105. At period 50, y_f is assumed to change be 1.04 unexpectedly, and y_h is held constant at 1.12.

Figure 6 plots the dynamics of bond prices and bond positions in both groups. In the initial period, given $y_h = y_f = 1.12$ and $B_h = B_f = 0.105$, the optimal new bond issuance in both

groups are equalized, $B'_h = B'_f = 0.105$. That is, given the endowment realization in both groups, each country will simply pay the interest and maintain the same debt level. Hence, the bond prices in both groups are equalized and have a value of 0.932. Given endowment realizations in both group are set to be constant for the first 50 periods, the bond positions and the bond prices in both groups are constant and equalized for the first 50 periods. When y_f decreases unexpectedly at period 50, the spread of debt yields between these two groups increases. Specifically, with a lower endowment realization relative to group h countries, the bond price in group f countries declines from 0.932 to 0.914. On the other hand, the bond price in group h countries increases from 0.932 to 0.95 because their relatively higher endowment realization, and the lenders are shifting toward them. Both the lower endowment realization and the lower bond price make group f countries to borrow less: The bond position in group f countries decreases from 0.105 to 0.045. Although the endowment realization in group h has been held constant, the higher bond price induces them to borrow more: The bond position in group h countries increases from 0.105 to 0.12.

The results in Figure 6 show that the relative fundamentals, rather than absolute fundamentals, are important to motivate lenders' flight-to-quality behavior, and thus are important to determine the bond prices across countries. Specifically, the decline in y_f at period 50 makes the bond issued by group h countries more appealing to the bond issued by group f countries. Although the endowment realization in group h countries doesn't change, the bond price in these countries increases because international lenders are substituting from the group f bond to the group h bond.

Another interesting aspect of Figure 6 is that the short-run changes in bond price is different from the long-run changes. In particular, there are overshootings in the group h bond price, and the group f bond price increases initially despite it declines in the long run. This is because the sudden drop in y_f would firstly be reflected in changes in the bond prices before both groups of countries had optimally responded to it by changing their debt positions. When y_f decreases at period 50, the difference in relative fundamentals is large and will be firstly reflected by the group h bond price. After the group h countries increase their debt issuance, the group h bond price would drop, but it is still higher than the bond price before the change. This implies that the supply of bond in group h countries is not perfectly elastic: Group h countries accommodate the improved borrowing conditions by issuing more debt, as can be seen from Figure 6, but they will not increase the debt issuance enough to exhaust all the increase in the demand of their bond, and still leave rooms for their bond price to increase. Furthermore, because the group h countries are accommodating the lenders' flight-to-quality behavior, an increase in group h bond issuance implies the group f countries are now facing with decreased demand of their bond, which will finally drive down the bond

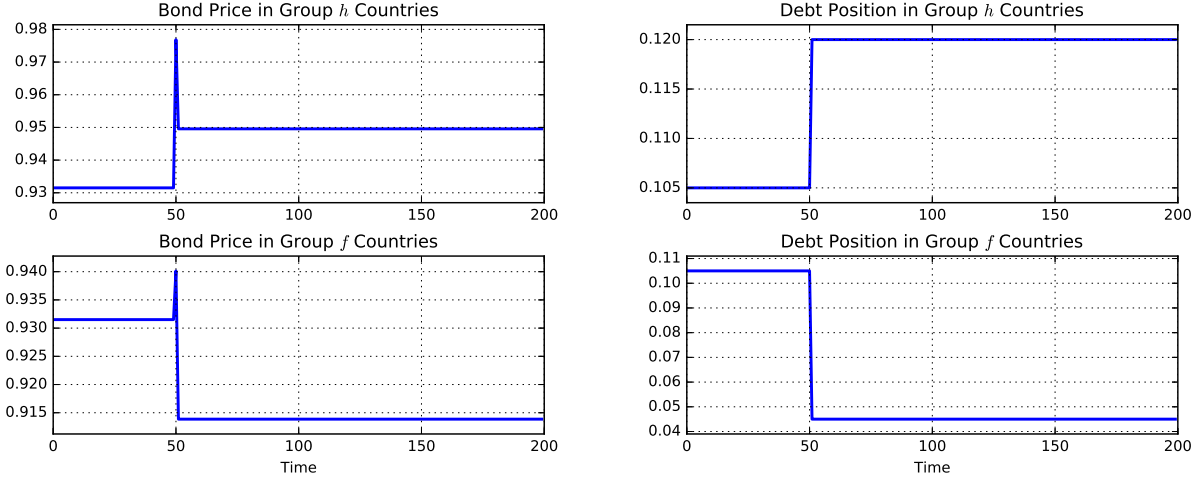


Figure 6: A Sudden Drop in Group f Countries' Endowment

price in group f countries and force these countries to reduce their debt positions, as it is shown in Figure 6.

The experiment here is able to explain the dynamics in debt yields and debt positions during the recent crisis. Specifically, the numerical results predict that because the GDP in European peripheries dropped at late 2009, international lenders shifted toward Germany, and the United States. The international lenders' flight-to-quality behavior left Germany, and the United States with increased demand of their bonds, while reduced the demand of bonds issued by European peripheries. On the other hand, these two safer countries accommodated such a change by increasing their bond issuance, which finally resulted at a combination of increased bond prices and debt positions in these two countries, and, at the same time, the decreased bond prices and difficulties in raising funds abroad in European peripheries. In this sense, the bonds issued by Germany, and the United States served as the global safe asset for the international lenders, which is similar to the results in He et al. (2015, 2016).

4.5 Sensitivity

The results in previous sections show that the substitution effect is quantitatively important for generating different changes in debt yields and debt positions across countries during the recent crisis. Because the flight-to-quality behavior is a consequence of lenders' willingness to avoid a loss, its magnitude is affected by how risk averse and how wealthy the lenders

are. In particular, the more risk averse the lenders are, or the fewer periodic income they have, the more likely they would like to rebalance their portfolios, when there is a difference between countries' relative fundamentals.

In this section, I show that the above arguments are consistent with results from the numerical exercise by changing the lender's risk aversion or periodic fixed endowment. In this exercise, the lenders' utility function has been changed to be CRRA,

$$u(c_L) = \frac{c_L^{1-\gamma}}{1-\gamma}$$

I consider different values of lender's risk aversion or periodic fixed endowment: a linear utility function ($\gamma = 0$), a lower value of γ , a higher value of γ , or a lower value of y_L . The first four rows of Table 4 report results under these four variations, as well as the benchmark model. First, the overall mean of debt yield and default probability are increasing in the value of γ : The more risk averse the lenders are, the higher the premium they would request from borrowing countries, which will force borrowing countries to be more likely to default. Second, both these two statistics increase in the case of $y_L = 1.0$. Because the absolute risk aversion is decreasing in the level of wealth under a CRRA utility function, a lower periodic income in this model makes the lenders charge a higher premium in general. Furthermore, the higher yield associated with the higher risk aversion (or the lower income) also forces borrowing countries to borrow less: The overall mean of debt to GDP ratio drops, but the magnitude is lower than the change in debt yield. Finally, the overall mean of risk-free interest rate remains unchanged, because it is governed by lenders' discount rate, δ , which also remains unchanged.

Table 4 also reports results that showing how changes in lenders' risk attitude and endowment determine how borrowing countries have been treated differently in the international borrowing market. First, I examine the difference in these four statistics in group h countries between the case of a high y_f and the case of a low y_f . The results are consistent with the economic intuition: The differences in all these four statistics are bigger when the lenders either become more risk averse or have a lower periodic fixed endowment. That is, international lenders are more likely to commit flight-to-quality behavior when they become more risk averse. Second, I also examine the correlations of the yield rate between these two groups of countries by separating y_f into three categories in a similar way as Table 3. It is easy to notice that correlations become more diverse if lenders are more risk averse or have a lower endowment, which again demonstrates that lenders with a higher risk aversion are more likely to shift across borders when the relative fundamentals are different across countries.

Group h countries	Benchmark ($\gamma = 1$)	Linear: $\gamma = 0$	$\gamma = 0.5$	$\gamma = 2$	$y_L = 1.0$
Mean					
Yield (%)	5.4	4.6	4.8	7.2	5.8
Default Probability (%)	4.5	4.2	4.3	8.8	4.8
Debt to GDP (%)	4.4	5.3	5.1	4.0	4.2
Risk-Free Rate (%)	4.0	4.0	4.0	4.0	4.0
Difference, High y_f - Low y_f					
Yield (%)	1.0	0.0	0.3	4.6	1.2
Default Probability (%)	0.7	0.0	0.2	2.6	1.1
Debt to GDP (%)	0.3	0.0	0.1	0.5	0.4
Risk-Free Rate (%)	0.4	0.0	0.2	0.7	0.5
Corr. of Yields					
Low y_f	0.52	-0.01	0.34	0.53	0.63
Medium y_f	0.58	0.00	0.46	0.78	0.65
High y_f	0.42	0.00	0.18	0.41	0.44

Table 4: Sensitivity Analyses

5 Discussion

The previous sections have demonstrated that the substitution effect is important in inducing lenders' flight-to-quality behavior, and finally results in different changes in borrowing conditions across countries, which is consistent with empirical evidence from the recent crisis. Although the numerical study is successful in matching the general pattern of changes in yields and debt positions in Germany, and the United States, it fails to generate the increase in yield in European peripheries with the same magnitude. In this paper, both groups of countries are symmetric, and thus the changes in yields would have same magnitude. However, countries such as Germany, and the United States are also big in their economic sizes, compared to European peripheries. As [He et al. \(2015, 2016\)](#) argue, these countries' bigger economic size implies that they have bigger debt capacities to absorb capital flights from European peripheries. By taking country size into account, I expect that the model will generate the decrease in yields in European peripheries with the correct magnitude, because the bigger debt capacities in Germany, and the United States would further motivate lenders' flight-to-quality behavior. However, such an experiment is left for future research.

Another important feature of sovereign debt crises is that they occur in tandem, e.g., the

Asian financial crisis in 1997, the recent Euro Zone Crisis, etc. Prior studies address the question of how shocks in one country are transmitted to another country. [Schinasi and Smith \(2001\)](#) first noticed that the contagion of sovereign debt crisis can be explained by the basic principles of portfolio theory. [Kyle and Xiong \(2001\)](#) describe crisis contagion as a phenomenon caused by the wealth effect: when negative shocks in one particular asset brings the investor a loss, she tends to retrench her investment from the whole portfolio due to the lower level of wealth. In a numerical exercise, [Arellano and Bai \(2013\)](#) show that home country's default probability increases dramatically when foreign country defaults due to the wealth effect.

However, if the recent crisis in European peripheries is taken as a contagion of the Greek crisis, it is easy to see that a crisis is not propagated to every country without any discrimination. [Broner et al. \(2006\)](#) show that investors who suffer a loss due to a crisis are more likely to retrench their funds from countries (and thus the contagion of crisis) that share overexposed investors with the crisis country. However, their model implies that whether a country suffers from the crisis contagion is purely determined by lenders' past behavior. By inducing lenders to treat borrowing countries differently, the substitution effect (and in turn the lenders' flight-to-quality behavior) examined in this paper has the potential to explain such a pattern of the crisis propagation. I will provide a case study below to address the importance of the substitution effect in explaining why some countries suffer from contagion during the crisis, but others don't. In particular, the following analysis focuses on answering the question of what characteristics contaminated countries share with the crisis country that makes them targets for lenders to retrench their investment.

Suppose that in addition to these two groups of countries, lenders also trade with another country—say country z —which has been expected to always repay the debt. Suppose country z defaults unexpectedly, which causes lenders to lose a fixed fraction of their periodic endowment, $\mu y_L, \mu \in (0, 1)$. Lenders' lower income implies that they would charge a higher premium in general on international lending, which shifts group h countries' best response functions to the right. Moreover, as it has been demonstrated in the sensitivity analysis in section 4.5, the substitution effect has been intensified, because lenders are more risk averse due to the lower level of wealth. As a consequence, $\hat{y}(\mathbf{B}, y_f, b_h)|_{D_f=0}$ shifts to the right more than $\hat{y}(\mathbf{B}, y_f, b_h)|_{D_f=1}$.

Figure 7 plots group h 's best response functions before and after the change: $\hat{y}_0(\mathbf{B}, y_f, b_h)|_{D_f=0}$ and $\hat{y}_0(\mathbf{B}, y_f, b_h)|_{D_f=1}$ are best response functions before the change, while $\hat{y}_1(\mathbf{B}, y_f, b_h)|_{D_f=0}$ and $\hat{y}_1(\mathbf{B}, y_f, b_h)|_{D_f=1}$ are best response functions after the change. Consider an output realization at point a in Figure 7: Group h countries don't default if country z doesn't default,

but they will be forced to default after country z defaults. This is because when y_L decreases, lenders are motivated to reexamine their portfolios due to the increases in their absolute risk aversion. When the endowment realization happens to be point a , lenders notice that that group h countries' future default risks are relatively high compared to group f countries'. This property of group h bond becomes more unfavorable to the lenders under the new level of wealth, and thus lenders would like to retrench from group h countries and increase the proportion of group f bonds in their portfolios. Decreased demand will again force group h countries to default immediately. On the other hand, group f countries have better borrowing conditions, because they are experiencing capital inflows. That is, the lenders in this model are retrenching from countries which share similar risk profiles as the crisis country, instead of similar portfolio exposures as in [Broner et al. \(2006\)](#).

The example in the last paragraph can be mapped onto the recent European debt crisis. Specifically, after Greece defaults, international lenders with a lower level of wealth are more sensitive to the future default risks and will reexamine the risk structure of their portfolios. Because European peripheries have a relatively higher future default risk compared to Germany, and the United States, they are the first group of countries from which lenders want to withdraw their money. At the same time, the larger debt capacities in Germany, and the United States provide the lenders safe assets to store their value. As a consequence, the lenders' flight-to-quality behavior leaves other European peripheries with the contagion of the Greek crisis, and at the same time, improved borrowing conditions and increased debt positions in Germany, and the United States, because they are relatively safe. Such a pattern of sovereign debt crisis contagion is more consistent with empirical evidence from the recent crisis.

6 Conclusion

In this paper, I develop a model of sovereign default with two groups of borrowing countries and one continuum of lenders. In the model, sovereign debt crises happens in one group of countries, and improved borrowing conditions and increased debt positions in the other group of countries coexist when lenders rebalance their portfolios dramatically. A substitution effect arises between different bonds, because borrowing countries are competing with each other for international borrowing. In particular, the substitution effect indicates international lenders to reduce their bond holdings in one group of countries if countries' relative fundamentals are different, and the future default risks in these countries are relatively high. When the difference between these two groups' fundamentals is big enough, lenders' flight-to-quality

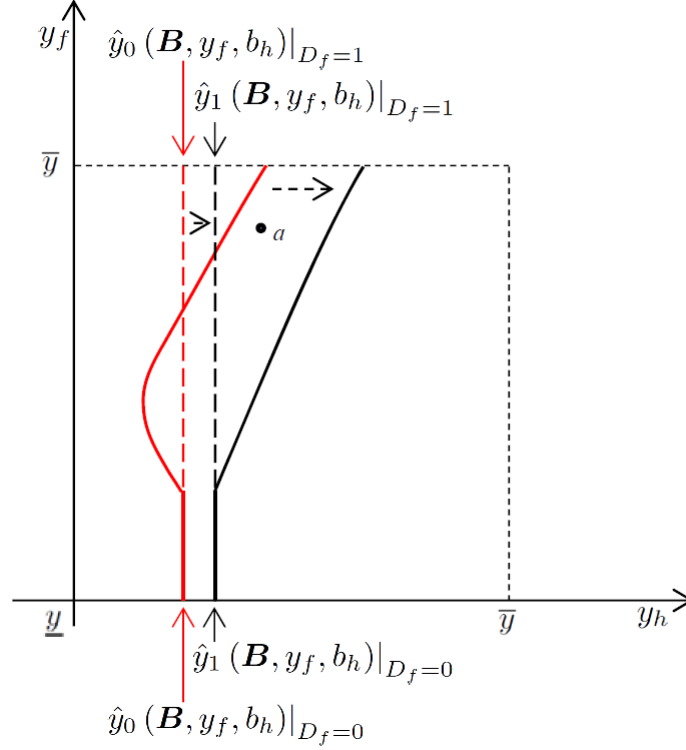


Figure 7: Contagion of Crisis

behavior is dramatic enough to leave one group of countries with flight-to-quality debt crises, and at the same time, the other group of countries with improved borrowing conditions and increased debt positions, because they are experiencing capital inflows. In this sense, the model is able to explain the debt yield dynamics across countries during the recent crisis, especially the drop in debt yield in Germany, and the United States. Furthermore, because Germany, and the United States accommodate the lenders' capital flight by issuing more debt, the model also predicts that the difficulties faced by European peripheries have been intensified. Lastly, the substitution effect emphasized in this paper predicts a crisis is only proved to be contagious to countries which are likely to share similar problems as the crisis country in servicing their debt. In this way, the model in this paper is also able to explain why a crisis originating in Greece only spreads to other European peripheries, but not to Germany, and the United States.

References

Aguiar, M. and Gopinath, G. Defaultable debt, interest rates and the current account. *Journal of International Economics*, 69(1):64–83, 2006.

- Aguiar, M. and Gopinath, G. Emerging market business cycles: the cycle is the trend. *Journal of Political Economy*, 115(1):69–102, 2007.
- Aiyagari, S. R. Uninsured idiosyncratic risk and aggregate saving. *The Quarterly Journal of Economics*, 109(3):659–684, 1994.
- Arellano, C. Default risk and income fluctuations in emerging economies. *The American Economic Review*, pages 690–712, 2008.
- Arellano, C. and Bai, Y. Linkages across sovereign debt markets. 2013.
- Bemanke, B., Gertler, M., and Gilchrist, S. The financial accelerator and the flight to quality. *The Review of Economics and Statistics*, 78(1):1–15, 1996.
- Broner, F. A., Gaston Gelos, R., and Reinhart, C. M. When in peril, retrench: Testing the portfolio channel of contagion. *Journal of International Economics*, 69(1):203–230, 2006.
- Caballero, R. J. and Krishnamurthy, A. Collective risk management in a flight to quality episode. *The Journal of Finance*, 63(5):2195–2230, 2008.
- Caballero, R. J. and Krishnamurthy, A. Global imbalances and financial fragility. *American Economic Review*, 99(2):584–88, May 2009.
- Calvo, G. Servicing the public debt: The role of expectations. *The American Economic Review*, 78(4):647–661, 1988.
- Chatterjee, S. and Eyigungor, B. Maturity, indebtedness, and default risk. *American Economic Review*, 102(6):2674–99, 2012.
- Cole, H. L. and Kehoe, T. J. Self-fulfilling debt crises. *The Review of Economic Studies*, 67(1):91–116, 2000.
- Cruces, J. J. and Trebesch, C. Sovereign defaults: The price of haircuts. *American economic Journal: macroeconomics*, 5(3):85–117, 2013.
- Eaton, J. and Gersovitz, M. Debt with potential repudiation: Theoretical and empirical analysis. *Review of Economic Studies*, 48(2):289–309, 1981.
- He, Z., Krishnamurthy, A., Milbradt, K., et al. A model of the reserve asset. 2015.
- He, Z., Krishnamurthy, A., and Milbradt, K. What makes us government bonds safe assets? *The American Economic Review*, 106(5):519–523, 2016.

- Huggett, M. The risk-free rate in heterogeneous-agent incomplete-insurance economies. *Journal of economic Dynamics and Control*, 17(5):953–969, 1993.
- Krishnamurthy, A. and Vissing-Jorgensen, A. The aggregate demand for treasury debt. *Journal of Political Economy*, 120(2):233–267, 2012.
- Kyle, A. S. and Xiong, W. Contagion as a wealth effect. *The Journal of Finance*, 56(4):1401–1440, 2001.
- Lizarazo, S. V. Contagion of financial crises in sovereign debt markets. 2009.
- Lizarazo, S. V. Default risk and risk averse international investors. *Journal of International Economics*, 89(2):317–330, 2013.
- Lorenzoni, G. International financial crises. In Helpman, E., Rogoff, K., and Gopinath, G., editors, *Handbook of International Economics*, volume 4, pages 689–740. Elsevier, 2014.
- Mas-Colell, A. On a theorem of schmeidler. *Journal of Mathematical Economics*, 13(3):201–206, 1984.
- Mendoza, E. G. and Yue, V. Z. A general equilibrium model of sovereign default and business cycles. *The Quarterly Journal of Economics*, 127(2):889–946, 2012.
- Rath, K. P. A direct proof of the existence of pure strategy equilibria in games with a continuum of players. *Economic Theory*, 2(3):427–433, 1992.
- Reinhart, C. M. and Calvo, G. A. When capital inflows come to a sudden stop: Consequences and policy options. In Kenen, P. B. and Swoboda, A. K., editors, *Reforming the International Monetary and Financial System*, pages 175–201. International Monetary Fund, 2000.
- Schinasi, G. J. and Smith, R. T. Portfolio diversification, leverage, and financial contagion. In Claessens, S. and Forbes, K. J., editors, *International Financial Contagion*, pages 187–221. Springer, 2001.
- Schmeidler, D. Equilibrium points of nonatomic games. *Journal of Statistical Physics*, 7(4):295–300, 1973.
- Tauchen, G. Finite state markov-chain approximations to univariate and vector autoregressions. *Economics letters*, 20(2):177–181, 1986.
- Xiong, W. Convergence trading with wealth effects: an amplification mechanism in financial markets. *Journal of Financial Economics*, 62(2):247–292, 2001.

A Proof

A.1 Proof of Lemma 1

Proof. Under assumption 1, equation (1) implies

$$u'(y_f + q'_f b'_f - b_f) < u'(y_f - b'_f)$$

then by the concavity of $u(\cdot)$

$$q'_f b'_f - b_f > -b'_f$$

and thus

$$b_f - q'_f b'_f < b'_f$$

Since group f countries can not default, an increase in y_f will increase both c_f and c'_f by the same amount. If group f countries don't change their new bond issuance after such an increase, strict concavity of $u(\cdot)$ implies

$$u'(y_f + q'_f b'_f - b_f) > \beta(1 + r_f) u'(y_f - b'_f)$$

To equalize the equation above, a group f country has to increase c_f and reduce c'_f . Notice that q'_f is defined in equation (2), which is independent of b'_f . Thus, $q'_f b'_f$ is strictly increasing in b'_f . Therefore, group f countries will increase b'_f to equalize the equation. That is, b'_f is strictly increasing in y_f . $\square$

A.2 Proof of Lemma 2

Proof. The value of repay and default, $V^{nd}(s, b_h)$ and $V^d(s)$ can be written as

$$\begin{aligned} V_h^{nd}(s, b_h) &= u(y_h + q'_h b'_h - b_h) + \beta u(y_h - b'_h) \\ V_h^d(s, b_h) &= (1 + \beta) u(y_h^d) \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} V_h^{nd}(s, b_h) - V_h^d(s, b_h) &= u(y_h + q'_h b'_h - b_h) + \beta u(y_h - b'_h) - (1 + \beta) u(y_h^d) \\ &= u(y_h + q'_h b'_h - b_h) - u(y_h^d) \end{aligned}$$

where the second equality comes from the result that $b'_h = \bar{b}'_h = y_h - y_h^d$. The country chooses to repay if $V_h^{nd}(s, b_h) - V_h^d(s, b_h) \geq 0$. Consider firstly the case of $y_h \leq (1 - \lambda) \mathbb{E}(y_h)$, then the definition of y_h^d implies $b'_h = 0$ and $y_h = y_h^d$, then

$$V_h^{nd}(s, b_h) - V_h^d(s, b_h) = u(y_h - b_h) - u(y_h^d) < 0$$

if $b_h > 0$. Reciprocally, the country is only likely to repay the debt if the default output cost is strictly positive, $y_h > (1 - \lambda) \mathbb{E}(y_h)$. By definition, $u(y_h^d)$ is constant for values of $y_h > (1 - \lambda) \mathbb{E}(y_h)$. Notice that $q'_h b'_h$ is increasing in b'_h because otherwise the country can improve both periods' consumption by issuing less debt. By definition, $b'_h = \bar{b}'_h$ is strictly increasing in y_h , which implies that $u(y_h + q'_h b'_h - b_h)$ is strictly increasing in y_h . Hence, $V_h^{nd}(s, b_h) - V_h^d(s, b_h)$ is strictly increasing in y_h . Therefore, there exists a cutoff function on y_h , $y^*(\mathbf{B}, y_f, b_h)$ such that the country repays if $y_h \geq y^*(\mathbf{B}, y_f, b_h)$ and defaults otherwise. $\square$

A.3 Proof of Lemma 3

Proof. By definition, $y_h = y^*(\mathbf{B}, y_f, b_h)$ is the lowest endowment realization such that countries would choose to repay, thus

$$u(y_h + q'_h b'_h - b_h) + \beta u(y_h - b'_h) = (1 + \beta) u(y_h^d) \quad (14)$$

by assumption, we have

$$u(y_h - b'_h) = u(y_h^d)$$

Hence, equation (14) changes to be

$$u(y_h + q'_h b'_h - b_h) = u(y_h^d)$$

Because $u(\cdot)$ is strictly increasing, the equation above implies

$$b_h - q'_h b'_h = y_h - y_h^d = b'_h$$

when $y_h = y^*(\mathbf{B}, y_f, b_h)$. That is, with an endowment equals $y^*(\mathbf{B}, y_f, b_h)$, countries would choose to equally spread the burden of debt repayments into both periods. $\square$

A.4 Proof of Lemma 4

Proof. Consider two possible endowment realizations, $\hat{y}_h$ and $\tilde{y}_h$ such that $\hat{y}_h, \tilde{y}_h \in [\underline{y}, \bar{y}]$ and $\hat{y}_h > \tilde{y}_h \geq y^*(\mathbf{B}, y_f, b_h)$. As I've argued before, a country only chooses to repay its debt when the output cost of default is strictly positive. Thus, we have $\hat{y}_h > \tilde{y}_h > (1 - \lambda) \mathbb{E}(y_h)$. Let $\hat{b}'_h$ and $\tilde{b}'_h$ be the associated period 1 new bond issuance such that

$$\hat{b}'_h = \hat{y}_h - \hat{y}_h^d, \tilde{b}'_h = \tilde{y}_h - \tilde{y}_h^d$$

The property of y_h^d implies that $y_h - y_h^d$ is strictly increasing in y_h for all $y_h > (1 - \lambda) \mathbb{E}(y_h)$, and thus

$$\hat{b}'_h > \tilde{b}'_h > 0$$

That is, with a higher endowment realization in period 1, countries borrow more due to the fact that the borrowing limit is strictly increasing in period 1 endowment realization. Furthermore, $q'_h b'_h$ is strictly increasing in b'_h because otherwise countries can get a higher net resource transfer from the lenders by issuing less debt. Hence,

$$\hat{q}'_h \hat{b}'_h > \tilde{q}'_h \tilde{b}'_h$$

which implies

$$b_h - \hat{q}'_h \hat{b}'_h < b_h - \tilde{q}'_h \tilde{b}'_h$$

□

A.5 Proof of Lemma 5

Proof. Lemma 3 predicts that net repayments in both periods are equalized when countries are indifferent between repay and default. On the other hand, Lemma 4 predicts that period 1 net repayment is strictly decreasing in y_h , while period 2 net repayment is strictly increasing in the endowment realization. Therefore, when countries strictly prefer to repay in period 1, the net repayment in period 2 is strictly higher than the net repayment in period 1

$$b'_h > b_h - q'_h b'_h$$

□

A.6 Proof of Lemma 6

Proof. Consider two possible endowment realizations in group f countries, $\hat{y}_f$ and $\tilde{y}_f$ such that $\hat{y}_f, \tilde{y}_f \in [\underline{y}, \bar{y}]$ and $\tilde{y}_f > \hat{y}_f$. Lemma 1 implies

$$\begin{aligned} \hat{b}'_f &> \tilde{b}'_f \\ b_f - \hat{q}'_f \hat{b}'_f &< b_f - \tilde{q}'_f \tilde{b}'_f \end{aligned}$$

then given (q'_h, b'_h) , that means

$$\begin{aligned} \psi' \left(y_L + b'_h + \hat{b}'_f \right) &< \psi' \left(y_L + b'_h + \tilde{b}'_f \right) \\ \psi' \left(y_L + b_h - q'_h b'_h + b_f - \hat{q}'_f \hat{b}'_f \right) &> \psi' \left(y_L + b_h - q'_h b'_h + b_f - \tilde{q}'_f \tilde{b}'_f \right) \end{aligned}$$

and thus

$$\frac{\psi'(y_L + b'_h + \hat{b}'_f)}{\psi'(y_L + b_h - \hat{q}'_h b'_h + b_f - \hat{q}'_f \hat{b}'_f)} < \frac{\psi'(y_L + b'_h + \tilde{b}'_f)}{\psi'(y_L + b_h - \tilde{q}'_h b'_h + b_f - \tilde{q}'_f \tilde{b}'_f)}$$

However, group h countries also response to the change in their bond price by borrowing less, which will counteract the effect on its bond price brought by a higher. Assuming that such a second order effect will not induce group h countries to reduce their period 1 new bond issuance too much to overturn the whole result¹¹ so that

$$\frac{\psi'(y_L + \hat{b}'_h + \hat{b}'_f)}{\psi'(y_L + \hat{b}_h - \hat{q}'_h \hat{b}'_h + b_f - \hat{q}'_f \hat{b}'_f)} < \frac{\psi'(y_L + \tilde{b}'_h + \tilde{b}'_f)}{\psi'(y_L + \tilde{b}_h - \tilde{q}'_h \tilde{b}'_h + \tilde{b}_f - \tilde{q}'_f \tilde{b}'_f)}$$

That is, $\hat{q}'_h < \tilde{q}'_h$. □

A.7 Proof of Lemma 7

Proof. For $y_h = y^*(\mathbf{B}, y_f, b_h)$, by Lemma 3 and Lemma 4:

$$b'_h = b_h - q'_h b'_h$$

which implies

$$\psi'(y_L + b'_h) = \psi'(y_L + b_h - q'_h b'_h)$$

By Assumption 1, I have

$$b'_f > b_f - q'_f b'_f$$

Then, by comparing the bond price in group h countries when group f countries are in the market to the one when group f countries don't exist, it is easy to see that the existence of group f countries will unambiguously reduce the bond price in group h countries. Or in other words, the substitution effect dominates. By 1, $b'_f - (b_f - q'_f b'_f)$ is strictly increasing in y_f , and thus will make q'_h even smaller. That is, the substitution effect is increasing in y_f , so as the downward pressure on group h countries' bond price imposed by the existence of group f countries. Assume for contradiction that for all $y_h > y^*(\mathbf{B}, y_f, b_h)$, the substitution effect is dominated. That is, for $y_h = y^*(\mathbf{B}, y_f, b_h) + \varepsilon, \varepsilon > 0$

$$\frac{\psi'(y_L + b'_h + b'_f)}{\psi'(y_L + b_h - q'_h b'_h + b_f - q'_f b'_f)} \geq \frac{\psi'(y_L + b'_h)}{\psi'(y_L + b_h - q'_h b'_h)}$$

¹¹This is true if both b_h and y_h are sufficiently small, and thus b'_h is sufficiently small.

Since for $y_h = y^*(\mathbf{B}, y_f, b_h) + \varepsilon, \varepsilon > 0$, the borrower always would like to choose to repay, then due to the continuity of the problem, it implies the inequality above still holds as $\varepsilon \rightarrow 0$, a contradiction to my previous arguments when $y_h = y^*(\mathbf{B}, y_f, b_h)$. Thus, the substitution effect also dominates for some $y_h > y^*(\mathbf{B}, y_f, b_h)$. $\square$

A.8 Proof of Proposition 1

Proof. Suppose that there exist some $\{q'_h, b'_h\}$ such that a group country would have $q_h b'_h - b_h > 0$, but the country chooses $\hat{b}'_h$ such that $\hat{q}'_h \hat{b}'_h - \hat{b}_h < 0$, then it would strictly prefer to default because

$$u(y_h) + \sum_{s=1}^{+\infty} \beta^s \mathbb{E}[u(y_{s,h})] > u(y_h + \hat{q}'_h \hat{b}'_h - \hat{b}_h) + \beta \mathbb{E}[V(s', \hat{b}'_h)]$$

However, by choosing $\{q'_h, b'_h\}$ that country would not choose to default because

$$\beta \mathbb{E}[V(s', b'_h)] \geq \sum_{s=1}^{+\infty} \beta^s \mathbb{E}[u(y_{s,h})]$$

and

$$u(y_h + q_h b'_h - b_h) > u(y_h)$$

Thus, $\hat{b}'_h$ cannot be optimal new bond issuance, as it leads to a default by the country. As a consequence, for any b_h , if there exists some level of y_h such that a group h country finds it optimal to default, then for any $\{q'_h, b'_h\}$, it has to make a positive net repayment, $b_h - q'_h b'_h > 0$. $\square$

A.9 Proof of Proposition 2

Proof. With the assumptions of i.i.d. shocks and permanent market exclusion, the borrower's problem

$$V(s, b_h) = \max_{d_h} [(1 - d_h) V^{nd}(s, b_h) + d_h V^d(s)]$$

where V^{nd} is the value of repay and V^d is the value of default

$$\begin{aligned} V^{nd}(s, b_h) &= \max_{b'_h} \{u(y_h + q'_h b'_h - b_h) + \beta \mathbb{E}[V(s', b'_h)]\} \\ V^d(s) &= u(y_h) + \beta \mathbb{E}[V^d(s)] = u(y_h^d) + \sum_{s=1}^{+\infty} \beta^s \mathbb{E}[u(y_{s,h})] \end{aligned}$$

The borrower's decision rule can be written as

$$d_h = \begin{cases} 0 & V^{nd}(s, b_h) \geq V^d(s) \\ 1 & V^{nd}(s, b_h) < V^d(s) \end{cases}$$

In essential we would like to prove that if the country decides to default at a particular level of $\hat{y}_h$, then it always defaults for endowment realizations lower than that value, $\tilde{y}_h \leq \hat{y}_h$. Then by definition $u(\hat{y}_h) + \sum_{s=1}^{+\infty} \beta^s \mathbb{E}[u(y_{s,h}^d)] > u(\hat{y}_h + \hat{q}'_h \hat{b}'_h - b_h) + \beta \mathbb{E}[V(s', \hat{b}'_h)]$. Then I need to prove

$$\begin{aligned} & u(\hat{y}_h + \hat{q}'_h \hat{b}'_h - b_h) + \beta \mathbb{E}[V(s', \hat{b}'_h)] - u(\tilde{y}_h + \tilde{q}'_h \tilde{b}'_h - b_h) - \beta \mathbb{E}[V(s', \tilde{b}'_h)] \\ & > u(\hat{y}_h) + \sum_{s=1}^{+\infty} \beta^s \mathbb{E}[u(y_{s,h}^d)] - u(\tilde{y}_h) - \sum_{s=1}^{+\infty} \beta^s \mathbb{E}[u(y_{s,h}^d)] \\ & = u(\hat{y}_h) - u(\tilde{y}_h) \end{aligned} \tag{15}$$

where the equality in the third line comes from the fact that shocks are i.i.d. and market exclusion is permanent after default. $\square$

Because the shocks are i.i.d., optimality of the borrower's problem then implies

$$u(\hat{y}_h + \hat{q}'_h \hat{b}'_h - b_h) + \beta \mathbb{E}[V(s', \hat{b}'_h)] > u(\hat{y}_h + \tilde{q}'_h \tilde{b}'_h - b_h) + \beta \mathbb{E}[V(s', \tilde{b}'_h)]$$

Thus, (15) holds if I can prove

$$\begin{aligned} & u(\hat{y}_h + \tilde{q}'_h \tilde{b}'_h - b_h) + \beta \mathbb{E}[V(s', \tilde{b}'_h)] - u(\tilde{y}_h + \tilde{q}'_h \tilde{b}'_h - b_h) - \beta \mathbb{E}[V(s', \tilde{b}'_h)] \\ & = u(\hat{y}_h + \tilde{q}'_h \tilde{b}'_h - b_h) - u(\tilde{y}_h + \tilde{q}'_h \tilde{b}'_h - b_h) \\ & > u(\hat{y}_h) - u(\tilde{y}_h) \end{aligned}$$

where the equality in the second line comes from the fact that shocks are i.i.d.. Proposition 1 implies $\tilde{q}'_h \tilde{b}'_h - b_h < 0$, then concavity of $u(\cdot)$ implies that the inequality above holds, and so does the inequality in equation (15).

In summary, I have proved a similar result to the one in [Arellano \(2008\)](#). That is, given $(\mathbf{B}, y_f, b_h)$, there exist $y^*(\mathbf{B}, y_f, b_h) \in [y_h, \bar{y}_h]$ such that

$$d_h = \begin{cases} 0 & y_h \geq y^*(\mathbf{B}, y_f, b_h) \\ 1 & y_h < y^*(\mathbf{B}, y_f, b_h) \end{cases}$$

B Algorithm

The endowment space $[\underline{y}, \bar{y}]$ is discretized into 11 grids using [Tauchen \(1986\)](#)'s method. Specifically, $\underline{y}$ equals to 0.83 and $\bar{y}$ equals to 1.21. The endowment process has an unconditional mean of 1.04. The debt space $[\underline{b}, \bar{b}]$ take the value of $[-0.45, 0.45]$, and it is discretized into 31 grids. The computing algorithm is as follows:

1. Make an initial guess of the borrower's value function, $V(s, b_h)$, $V^{nd}(s, b_h)$, and $V^d(s, b_h)$.
2. Make an initial guess on the equilibrium price function $q_h(s, b'_h)$.
3. Solve the borrower's problem to find her value function $\tilde{V}(s, b_h)$, $\tilde{V}^{nd}(s, b_h)$, and $\tilde{V}^d(s, b_h)$; optimal default decision, $d(s, b_h)$; optimal policy function, $b'(s, b_h)$; and the new equilibrium bond price function, $\tilde{q}_h(s, b'_h)$. This maximization involves the following sub steps:

- (a) Taking $q_h(s, b'_h)$ as given, solve the borrower's maximization problem

$$\tilde{V}(s, b_h) = \max_{d_h} \left[(1 - d_h) \tilde{V}^{nd}(s, b_h) + d_h \tilde{V}^d(s, b_h) \right]$$

where $\tilde{V}^{nd}(s, b_h)$ is the value of repay and $\tilde{V}^d(s, b_h)$ is the value of default

$$\begin{aligned} \tilde{V}^{nd}(s, b_h) &= \max_{b'_h} \left\{ \begin{aligned} &u(y_h + q'_h b'_h - b_h) \\ &+ \beta \int_{\underline{y}}^{\bar{y}} \int_{\underline{y}}^{\bar{y}} \hat{y}(\mathbf{B}', y'_f, b'_h) V^d(s') f(y'_h | y_h) f(y'_f | y_f) dy'_h dy'_f \\ &\beta \int_{\underline{y}}^{\bar{y}} \int_{\hat{y}(\mathbf{B}', y'_f, b'_h)}^{\bar{y}} V^{nd}(s', b'_h) f(y'_h | y_h) f(y'_f | y_f) dy'_h dy'_f \end{aligned} \right\} \\ \tilde{V}^d(s, b_h) &= u(y_h^d) + \beta \theta \int_{\underline{y}}^{\bar{y}} \int_{\underline{y}}^{\bar{y}} V^{nd}(s', 0) f(y'_h | y_h) f(y'_f | y_f) dy'_h dy'_f \\ &\quad + \beta (1 - \theta) \int_{\underline{y}}^{\bar{y}} \int_{\underline{y}}^{\bar{y}} V^d(s') f(y'_h | y_h) f(y'_f | y_f) dy'_h dy'_f \end{aligned}$$

The borrower's policy functions $d(s, b_h)$ and $b'(s, b_h)$ are obtained by solving the problem above.

- (b) Given $b'(s, b_h)$ and $d(s, b_h)$, compute the new equilibrium bond price function

$$\tilde{q}_h(s, b'_h) = \int_{\underline{y}}^{\bar{y}} \int_{\hat{y}(\mathbf{B}', y'_f, b'_h)}^{\bar{y}} \frac{\delta \psi'(c'_L)}{\psi'(c_L)} f(y'_h | y_h) f(y'_f | y_f) dy'_h dy'_f \quad (16)$$

where

$$\begin{aligned} c_L(s) &= y_L + \sum_{j=h,f} NR_j(s) \\ NR_j(s) &= (1 - D_j) [B_j - q'_j(B'_j) B'_j] \end{aligned}$$

- (c) Compute the distance between $q_h(\cdot)$ and $\tilde{q}_h(\cdot)$. If it is bigger than the critical value, then update the guess on $q_h(\cdot)$ by using $\tilde{q}_h(\cdot)$ and repeat steps (b) and (c). Otherwise, proceed to step 4¹².
4. Compute the distance between $V(\cdot)$ and $\tilde{V}(\cdot)$. If it is bigger than the critical value, then update the guess on $V(\cdot)$, $V^{nd}(\cdot)$ and $V^d(\cdot)$ by using $\tilde{V}(\cdot)$, $\tilde{V}^{nd}(\cdot)$, and $\tilde{V}^d(\cdot)$. Repeat step 1 to step3. Otherwise, stop.

¹²The computing speed of step (b) to step (c) can be greatly enhanced by solving a fixed point problem of equation (16).