

Recursive equilibria in dynamic economies with bounded rationality

Runjie Geng
DBF, University of Zurich
and Swiss Finance Institute
runjie.geng@bf.uzh.ch

Abstract

I provide a new way to model bounded rationality and show the existence of recursive equilibria with bounded rational agents. The existence proof applies to dynamic stochastic general equilibrium models with infinitely lived heterogeneous agents and incomplete markets. In this type of models, recursive methods are widely used to compute equilibria, yet recursive equilibria do not exist generically with rational agents. I change the rational expectation assumption and model bounded rationality as follows. Different from a rational agent, a bounded rational agent does not know the true Markov transition of the state space of the economy. In order to make decisions, the bounded rational agent would try to compute a stationary distribution of the state space using a numerical method and then use the Markov transition associated with it to maximize utility. For a certain distribution of the current period, given other agents' strategies, the agent would get its next-period transition: the distribution of the state space in the next period that results from the competitive equilibrium in the next period. However, if a distribution stays "closer" to its next-period transition than the minimum error the numerical method can observe, the agent would consider it as computational stationary. In equilibrium, each agent maximizes utility with a computational stationary distribution and markets clear. I use the Kantorovich-Rubinshtein norm to characterize the distance between distributions of the state space. With this set up, usual convergence criteria used in the literature can be incorporated and thus many computed equilibria in the literature using recursive methods can be categorized as bounded rational recursive equilibria in the sense of this paper.

Keywords: Recursive equilibrium; bounded rationality; heterogeneous agents; incomplete markets.

*The author thank Felix Kubler for helpful discussions and comments.

1. Introduction

Dynamic stochastic general equilibrium models with infinitely lived heterogeneous agents and incomplete financial markets are extensively studied in the literature of economics and finance. In this paper, I focus on recursive equilibria (or stationary Markov equilibria in the terminology of stochastic games) of these models. Recursive equilibria are characterized by a pair of functions (characteristic functions): a transition function mapping this period’s “state” into probability distributions over next period’s state, and a “policy function” mapping the current state into current prices and choices. The state space is set to be the current exogenous shock and the beginning-of-period distribution of assets across individuals. This way of restricting the state space is often considered “minimal” or “natural” and widely used in the literature for both philosophical and practical reasons as follows. The philosophical reasons are the same as Maskin and Tirole’s (2001) reasons to consider Markov perfect equilibria (MPE). That is, the “natural” state space recursive equilibria “prescribe the simplest form of behavior that is consistent with rationality”¹, it “captures the notion that ‘bygones are bygones’ ”, and it “embodies the principle that ‘minor causes should have minor effects’ ”. On the practical side, recursive methods can be used to approximate stationary Markov equilibria numerically. Heaton and Lucas (1996), Krusell and Smith (1998), and Kubler and Schmedders (2003) are early examples of papers that approximate stationary Markov equilibria in models with infinitely lived, heterogeneous agents.

Despite the importance of the recursive equilibria described above, there are cases when recursive equilibrium does not exist for models with infinitely lived agents and incomplete financial markets under standard assumptions. This problem was first illustrated by Hellwig (1983) and since then has been demonstrated in different contexts. Kubler and Schmedders (2002) give an example showing the nonexistence of stationary Markov equilibria in models with incomplete asset markets and infinitely lived agents. Santos (2002) provides examples of nonexistence for economies with externalities. Kubler and Polemarchakis (2004) present such examples for overlapping generations (OLG) models. Brumm et al. (2017) further provide one simple example demonstrating the possibility of nonexistence.

The nonexistence problem has been explained (dealt with) in the literature from different angles. Hellwig (1983) ascribed the reason of nonexistence to the “the simultaneous determination of prices for different periods”. He stated that “The concept of a rational expectations equilibrium involves the simultaneous determination of prices for different periods. This si-

¹In this case consistent with bounded rationality (which is specified later), since equilibria with rational expectation assumption fail to exist generically.

multaneity conflicts with the sequential nature of the temporary equilibrium process, so that rational expectations equilibria can only be implemented if the auctioneer in each period is informed of past expectations and uses them to select his temporary equilibria.”

Duffie et al. (1994) showed existence of competitive equilibria for general Markovian exchange economies with a different state space. The authors also proved that the equilibrium process is a stationary Markov process. However, we follow the well established terminology in dynamic games and do not refer to these equilibria as stationary Markov equilibria, because the state space also contains consumption choices and prices from the previous period.

Citanna and Siconolfi (2010 and 2012) provide sufficient conditions for the generic existence of stationary Markov equilibria in OLG models. However, their arguments cannot be extended to models with infinitely lived agents or to models with occasionally binding constraints on agents choices, and for their argument to work in their OLG framework they need to assume a very large number of heterogeneous agents within each generation.

Brumm et al. (2017) provide existence for stochastic economies with stochastic productions by assuming that there are two atomless shocks that are stochastically independent (conditional on a possible third shock that can be arbitrary). This construction is provided first by Dugan (2012), who proved the existence of Markov Perfect Equilibrium in noisy stochastic games.

The literature showing the existence of recursive equilibria usually take the way of specifying the model set up such that although the agents do not make decisions recursively, the optimal strategies take a recursive form in equilibrium. I differ from the literature by reexamining the rational expectation assumption. Looking back at why we are interested in recursive equilibria in the first place, rational expectation assumption is actually inconsistent with the reasons: Philosophically, a rational agent would not put any restrictions on the form of the strategy and would consider the entire history to maximize utility; practically, the rational agent does not need an efficient algorithm to compute the equilibria or to approximate equilibria since she knows the analytical solution to the maximization problem. Simply put, when considering large complex economic systems, it is unrealistic to solve or compute rational expectation strategies. So it is natural and necessary to consider bounded rational behavior with recursive strategy that results in recursive equilibria.

Since the notion of bounded rationality had been brought up by Herbert A. Simon, it has been modeled in various ways. To cater to the current problem and to keep the set of assumptions as weak as possible, my way of modeling bounded rationality can be simply summarized as: The bounded rational agent would use numerical method to compute (ap-

proximate) a “computational stationary equilibrium” instead of knowing the exact stationary equilibrium (if it exists). To explain more specifically, the agent tries to find a computational stationary distribution of the state space of the economy in the following way. For a certain distribution of the current period, given other agents’ strategies, the agent would get its next-period transition: the distribution of the state space in the next period that results from the competitive equilibrium in the current period. Then if a distribution stays “closer” to its next-period transition than the minimum error the numerical method can distinguish, the agent would consider it as computational stationary. In equilibrium, each agent maximizes utility with a computational stationary distribution and markets clear. With this set up, usual convergence criteria used in the literature can be incorporated and thus many computed equilibria in the literature using recursive methods can be categorized as bounded rational recursive equilibria in the sense of this paper. Potentially, this paper can provide a foundation for computational recursive equilibria in the literature.

The rest of the paper is organized as follows: Section 2 describes the model set up. Section 3 defines the bounded rational recursive equilibrium. Section 4 proves existence of bounded rational equilibrium. Section 5 concludes the paper. Detailed proofs can be found in the appendix.

2. The Economic Model

In this section we describe the model set up. The markets structure is adapted from the Lucas (1978) asset pricing model. With this framework, we specify how a bounded rational agent maximizes utility given a computed distribution of the state space of the economy and an information set.

2.1. The Physical Economy

Time is discrete and denoted by $t \in \mathbb{N}_0$. The exogenous shocks denoted by z realize from a Euclidean space $\mathbf{Z}$, and follow a first-order Markov process with transition probability $\mathbb{P}(\cdot \mid z)$ defined on the Borel σ -algebra $\mathcal{Z}$ on $\mathbf{Z}$, $\mathbb{P} : \mathbf{Z} \times \mathcal{Z} \rightarrow [0, 1]$. Let $(z_t)_{t=0}^\infty$, or in short (z_t) , denote the stochastic process and let $(\mathcal{F}_t)$ denote its natural filtration. A history of shocks up to some date t is denoted by $z^t = (z_0, z_1, \dots, z_t)$.

There are H types of agents, $h \in \mathbf{H} = \{1, \dots, H\}$. There is one perishable commodity and J Lucas trees, $j \in \mathbf{J} = \{1, \dots, J\}$, denote the prices normalized to a simplex by $p = (p_c, p_J) \in \Delta^J$. The Lucas trees are long-lived assets in unit net supply that pay exogenous positive dividends in terms of a single consumption good $d : \mathbf{Z} \rightarrow \mathbb{R}_+^J$. The agent h ’s

endowment is denoted by $e_h : \mathbf{Z} \rightarrow \mathbb{R}_+$. The dividends and endowments are time-invariant and measurable functions of the current shock. The agent h 's consumption for period t is $x_{h,t} \in \mathbf{X}_h$; her holding of financial assets $\alpha_{h,t} \in \Xi_h$. The budget set is denoted as $\Gamma_h : \mathbf{Z} \times \Xi_h \times \Delta^J \rightrightarrows \mathbf{X}_h \times \Xi_h$. The consumption vector of the economy is $x \in \mathbf{X}$, the asset holdings vector is $\alpha \in \Xi$. $\mathbf{X}, \Xi$ are Euclidean spaces. There is no short selling on trees, and we restrict the asset holdings vector with financial markets clearing conditions, $\Xi = (\Delta^{H-1})^J$.

Denote the minimal state space as $\mathbf{S} = \mathbf{Z} \times \Xi$ and associated Borel algebra as $\mathcal{S}$. Also, we specify an augmented state space that contains the exogenous shock, the beginning-of-period asset holdings, the end-of-period asset holdings, and the prices similar as Duffie et al. (1994). Denote the augmented state space as $\tilde{\mathbf{S}} = \mathbf{Z} \times (\Xi \times \Xi \times \Delta^J)$ and associated Borel algebra as $\tilde{\mathcal{S}}$.

The characteristic transition of the economy is

$$F : \mathbf{S} \rightarrow \mathcal{P}(\Xi \times \Delta^J)$$

In the sense of Aliprantis and Border (2006) Chapter 19.2, we denote the associated transition operator as $\mathbf{F}$, the adjoint operator $\mathbf{F}'$, and the kernel k_F .

Then the Markov transition of the augmented state space $\tilde{F} : \tilde{\mathbf{S}} \rightarrow \mathcal{P}(\tilde{\mathbf{S}})$ generated by $F, \mathbb{P}$ is defined as follows, for any $\tilde{s}_0 = [z_0, (\alpha_0^-, \alpha_0, p_0)]$ and letting $\mu = \tilde{F}(s_0)$, we have:

- (i) the marginal on $\mathbf{Z}$ denoted as $\mu_{\mathbf{Z}} = \mathbb{P}(\cdot \mid z_0)$, and the marginal on Ξ^- denoted as $\mu_{\Xi^-} = \delta_{\alpha_0^-}$ almost surely;
- (ii) for any $z'_0 \in \text{supp}(\mathbb{P}(\cdot \mid z_0))$, the conditional probability $\mu(\cdot \mid (z'_0, \alpha_0)) = F(z'_0, \alpha_0)$.

Similarly, we denote the associated transition operator as $\tilde{\mathbf{F}}$, the adjoint operator $\tilde{\mathbf{F}}'$, and the kernel $k_{\tilde{F}}$.

2.2. Utility Maximization of Agents

Given the economy described above, the agent h believe that the economy is in stationary equilibrium and the stationary distribution is $\mu_h \in \mathcal{P}(\tilde{\mathbf{S}})$. Then she would maximize utility with an information set $I_h \in \mathbf{I}_h$, where $\mathbf{I}_h$ is a finite dimensional Euclidean space. We first specify the structure of the information set, and then introduce the utility maximization problem.

2.2.1. Information structure

The information set of agent $I_h \in \mathbf{I}_h$ consists of two parts, the price $p \in \Delta^J$ and the gathered information of asset holdings of other agents $f_I^h : \Xi_{-h} \times \Xi_{-h} \rightarrow \mathbf{I}_{h,\alpha}$.

Assumption 2.1. 1. $\mathbf{I}_{h,\alpha}$ is compact and convex for all $h \in \mathbf{H}$;

2. f_I^h is continuous for all $h \in \mathbf{H}$.

This set up is general and can incorporate many forms of information structures used in the literature. We specify three special cases that satisfy Assumption 2.1 to illustrate this.

Special Case 1. $f_I^h(\alpha_{-h}^-, \alpha_{-h})$ is a subset of $(\alpha_{-h}^-, \alpha_{-h})$ for all $h \in \mathbf{H}$. The agents would have partial information of asset holdings of the economy. This setting is the same as the “sparse max” agents used by Gabaix (2012), when agents do not put any weight on the omitted information.

Special Case 2. $f_I^h(\alpha_{-h}^-, \alpha_{-h})$ is the first several moments of α_{-h}^- . The agents would use the match of moments to describe the distribution of asset holdings among agents. This setting is the same as the bounded rational agents used by Krusell and Smith (1998). Although Krusell and Smith (1998) assumed a continuum of agents in the model, the computation part had to use a large number of agents.

Special Case 3. $f_I^h(\alpha_{-h}^-, \alpha_{-h}) = (\alpha_{-h}^-, \alpha_{-h})$ for all $h \in \mathbf{H}$. The agents would have full information of asset holdings of the economy. This setting also shows that Assumption 2.1 is harmless in the sense that it incorporate the full information setting. It will be shown that with this special case, many computed equilibria with recursive method in the literature with rational expectation assumption are in fact bounded rational recursive equilibria as I will describe in the next section. One example is the computed equilibria by Kubler and Schmedders (2003).

2.2.2. Utility function given an equilibrium distribution

Given probability space $(\tilde{\mathbf{S}}, \tilde{\mathcal{S}}, \mu_h)$, each agent h choose an arbitrary random variable $T_h : \tilde{\mathbf{S}} \rightarrow \tilde{\mathbf{S}}$, and denote the regular conditional probability measure of μ_h on A associated with T_h as $\mu_h(\cdot | A)$. Denote the marginal of $\mu_h(\cdot | A)$ on b as $\mu_{h,b}(\cdot | A)$. Then the prediction of the distribution of the next period’s information set $\mathbb{Q}_h[I'_h | (z', I_h)]$ can be written as follows:

$$\mathbb{Q}_h[I'_h | (z', I_h)] = \int_{\alpha \in \Xi} \mu_I[I'_h | (\alpha, z')] \mu_\alpha[d\alpha | I_h]$$

With $\mathbb{P}$ and $\mathbb{Q}_h$, we get the overall predicted transition probability as

$$\mathbb{R}_h[(z', I'_h) \mid (z, I_h)] = \mathbb{Q}_h[I'_h \mid (z', I_h)]\mathbb{P}(z' \mid z)$$

Easy to check that $\mathbb{R}_h[(z', I'_h) \mid (z, I_h)]$ is a transition probability. Define $\mu_h^t, t \geq 0$ in the same way as in 8.2 of Stokey (1989):

$$\begin{aligned} & \mu_h^t(A_1 \times \dots \times A_t \mid (z_0, I_{h,0})) \\ &= \int_{A_1} \dots \int_{A_{t-1}} \int_{A_t} \mathbb{R}(d(z_t, I_{h,t}) \mid (z_{t-1}, I_{h,t-1})) \mathbb{R}(d(z_{t-1}, I_{h,t-1}) \mid (d(z_{t-2}, I_{h,t-2}))) \\ & \dots \mathbb{R}(d(z_1, I_{h,1}) \mid (z_0, I_{h,0})), \text{ where } A_1, \dots, A_t \subset \mathbf{Z} \times \mathbf{I}_h. \end{aligned}$$

With the single period utility function $u_h : \mathbf{Z} \times \mathbf{X}_h \rightarrow \mathbb{R}$, each agent maximizes a time-separable expected utility function. For a given sequence $(x_{h,t}(z_t, I_{h,t}), \alpha_{h,t}(z_t, I_{h,t}))_{t=0}^\infty$,

$$\begin{aligned} & \mathbb{E}_{\mu_h} \left[\sum_{t=0}^{\infty} \delta^t u_h(z_t, x_{h,t}(z_t, I_{h,t})) \right] \\ &= u_h(z_0, x_{h,0}(I_{h,0})) + \sum_{t=1}^{\infty} \int_{(z, I_h)^t} \delta^t u(z_t, x_{h,t}(z_t, I_{h,t})) \mu^t(d(z, I_h)^t \mid (z_0, I_{h,0})) \end{aligned}$$

Each agent maximize the total utility subject to the budget constraint.

$$\begin{aligned} U_h^{\mu_h}(z_0, \alpha_{h,0}^-, I_{h,0}) &= \max_{x_{h,t}, \alpha_{h,t}} \mathbb{E}_{\mu_h} \left[\sum_{t=0}^{\infty} \delta^t u_h(z_t, x_{h,t}(z_t, I_{h,t})) \right] \\ (x_{h,t}(z_t, I_{h,t}), \alpha_{h,t}(z_t, I_{h,t})) &\in \Gamma(z_t, \alpha_{h,t}^-, p_t) \end{aligned}$$

To compare this utility to rational expectation utility, consider the special case 3 mentioned above, where the agents have full information about asset holdings in the economy. If the computed distribution μ_h is the “true” stationary equilibrium (if it exists), then the utility function coincides with rational expectation utility. In later sections, I will provide a μ_h that is “close to be stationary” with an arbitrarily small error term.

2.3. Assumptions

To ensure the existence, a few further assumptions are needed.

Assumption 2.2. *$\mathbf{Z}$ is compact and convex. For all $z \in \mathbf{Z}$, $\mathbb{P}(\cdot \mid z)$ is absolutely continuous with respect to Lebesgue measure with density $p(\cdot \mid z)$.*

Assumption 2.3. 1. e_h, d are bounded measurable functions. Further, there are $\underline{e}, \bar{e}, \bar{d} \in \mathbb{R}_{++}$ such that for all z, h, j

$$e_h(z) \geq \underline{e}, \sum_{h \in \mathbf{H}} e_h(z) \leq \bar{e}, 0 < d_j(z) < \bar{d}.$$

2. The agents' discount factor satisfies $\delta \in (0, 1)$.

3. The utility functions $u_h : \mathbf{Z} \times \mathbf{X} \rightarrow \mathbb{R}$ are measurable in z , continuous, strictly increasing and concave in x .

The two assumptions above are standard in the literature, and we relaxed the assumption that the utility functions are differentiable. Another assumption is needed to keep the consumption and asset holding choices bounded is stated as follows.

Assumption 2.4. The agent cannot consume more than the aggregate amount of the commodity and cannot hold more than one unit of any tree—that is, $(x_h, \alpha_h) \in \mathbf{B}$. Where

$$\mathbf{B} := \{(x, \alpha) : 0 \leq x \leq \bar{e} + J\bar{d}; 0 \leq \alpha_j \leq 1, \forall j \in \mathbf{J}\}.$$

This assumption is needed because the bounded rational agent perceives the future differently from the true distribution. Combined with Assumption 2.3, the budget set is

$$\Gamma(z, \alpha_h^-, p_t) = \{p_c(x_h + d(z)\alpha_h^- - e_h(z)) + p_J(\alpha_h^- - \alpha_h) \leq 0\} \cap \mathbf{B}.$$

3. Bounded Rational Recursive Equilibrium

With the model set up above, I go on to define the bounded rational recursive equilibrium in this section. First, I introduce the concept of computational stationary measure. And then I define a bounded rational competitive equilibrium given computed distribution. Last I define the bounded rational recursive equilibrium.

3.1. Computational Stationary Measure

Consider a state space $(\mathbf{A}, \mathcal{A})$, where $\mathbf{A}$ is a compact convex subset of a finite dimensional Euclidean space and $\mathcal{A}$ is the Borel algebra. Denote a Markov transition as a Borel measurable function $P : \mathbf{A} \rightarrow \mathcal{P}(\mathbf{A})$. Denote the associated Markov transition operator and its adjoint operator as $\mathbf{P}$ and $\mathbf{P}'$, and denote the associated Markov kernel in the sense of Aliprantis and Border's (2006) Definition 19.11 as $k_P : \mathbf{A} \times \mathcal{A} \rightarrow [0, 1]$.

To characterize distance between measures, we equip the space $\mathcal{P}(\mathbf{A})$ with the following Kantorovich-Rubinshtein norm:

$$\|\mu\|_0 = \sup\left\{\int_{\mathbf{A}} f d\mu : f \in \text{Lip}_1(\mathbf{A}), \|f\|_{\infty} \leq 1\right\},$$

where

$$\text{Lip}_1(\mathbf{A}) := \{f : \mathbf{S} \rightarrow \mathbb{R}, |f(x) - f(y)| \leq |x - y|, \forall x, y \in \mathbf{A}\}.$$

Denote the space of those test functions as $\mathbf{L}(\mathbf{A}) = \{f \in \text{Lip}_1(\mathbf{A}) : \|f\|_{\infty} \leq 1\}$. By theorem 8.3.2 of Billingsley(2013), the topology generated by $\|\cdot\|_0$ coincides with the weak topology on $\mathcal{P}(\mathbf{A})$.

The agent is associated with a parameter arbitrary small $\epsilon > 0$. The agent's rationality is bounded in the sense that she would consider two measures "close" enough if the Kantorovich-Rubinshtein distance is within ϵ . We call a measure ϵ -computational stationary with respect to P if it is "close" enough to its transition under P .

Definition 1 (Computational stationary measure). *Measure u is said to be ϵ -computational stationary with respect to P if $\|u - \mathbf{P}'u\|_0 \leq \epsilon$.*

3.2. Definition of Recursive Equilibrium

3.2.1. μ_h -Competitive Equilibrium

Different from rational expectation models, the agents perceive the future differently. So the maximization problems need to be restated for each time period. To put it more formally, we define the bounded rational competitive equilibrium as follows:

a μ_h -competitive equilibrium given initial conditions $z \in \mathbf{Z}, \alpha_h^- \in \Xi$ consists of prices and choices,

$$(p, (x_h, \alpha_h)_{h \in \mathbf{H}})$$

such that markets clear and each agent h optimize utility with respect to μ_h —that is to say, (A), (B), and (C) hold.

(A) Commodity market clearing:

$$\sum_{h \in \mathbf{H}} x_h = \sum_{h \in \mathbf{H}} e_h(z) + \sum_{j \in \mathbf{J}} d_j(z).$$

(B) Financial markets clearing:

$$\sum_{h \in \mathbf{H}} \alpha_h = \mathbf{1}.$$

(C) Utility maximization:

$$(x_{h,t}, \alpha_{h,t})_0^\infty \in \arg \max \mathbb{E}_{\mu_h} \left[\sum_0^\infty \delta^s u_h(z_t, x_{h,t}(z_t, I_{h,t})) \right],$$

where the budget set is

$$\mathbf{\Gamma}(z_t, \alpha_{h,t}^-, p_t) = \{p_{c,t}(x_{h,t} + d(z_t)\alpha_{h,t}^- - e_h(z_t)) + p_{J,t}(\alpha_{h,t}^- - \alpha_{h,t}) \leq 0\} \cap \mathbf{B},$$

and

$$(x_h, \alpha_h) = (x_{h,0}, \alpha_{h,0}), I_{h,0} = (p, f_h^I(\alpha_{-h}^-, \alpha_{-h})).$$

With strictly increasing utility functions, this equilibrium can be represented by $((\alpha_h)_{h \in \mathbf{H}}, p)$ using the budget constraint.

3.2.2. Bounded Rational Recursive Equilibrium

A ϵ -bounded rational recursive equilibrium $(F, (\mu_h)_{h \in \mathbf{H}})$ consists of a characteristic transition

$$F : \mathbf{S} \rightarrow \mathcal{P}(\mathbf{\Xi} \times \mathbf{\Delta}^J)$$

and computational invariant measures μ_h for all $h \in \mathbf{H}$

$$\| \mu_h - \tilde{\mathbf{F}}' \mu_h \|_0 \leq \epsilon,$$

such that for all initial conditions $s_0 \in \mathbf{S}$, $((\alpha_h)_{h \in \mathbf{H}}, p)$ is a μ_h -competitive equilibrium if $((\alpha_h)_{h \in \mathbf{H}}, p) \in \text{supp} F(s_0)$.

Again, recall the special case 3, if $\epsilon = 0$, we will have a recursive equilibrium with rational agents. However, a positive ϵ is needed to construct a continuous value function in the later proof.

4. Existence of Bounded Rational Recursive Equilibrium

4.1. Perceivable Model Constructed by Bounded Rational Agents

There are two parts in this section. In the first part, I show that for any characteristic Markov transition, there exists a computational invariant distribution with a continuous density function. In the second part, I define a characteristic perceivable model as a function

that maps a Markov transition to a computational invariant continuous density function. And further, I show existence of a continuous perceivable model: a continuous function that maps a Markov transition with its weak topology to a continuous density function with the sup norm.

4.1.1. Continuous Density Functions

Let $(\mathbf{N}(\mathbf{A}), \|\cdot\|_\infty)$ be the subset of $(\mathbf{C}_b(\mathbf{A}), \|\cdot\|_\infty)$ such that for any $f \in \mathbf{N}(\mathbf{A})$, we have $f \geq 0$, $\int_{\mathbf{A}} f d\lambda = 1$ so that the elements are all density functions. Denote μ_f as the measure with density function f . Then we define a computational invariant density function for agent h as follows.

Definition 2 (computational invariant continuous density). *A function f is said to be a computational invariant continuous density function with respect to P if $f \in \mathbf{N}(\mathbf{A})$ and $\|\mu_f - \mathbf{P}'\mu_f\|_0 \leq \epsilon$.*

Theorem 4.1. *For any $\epsilon > 0$, the set of computational invariant continuous density functions $\Upsilon(\epsilon, P)$ is non-empty, convex, and closed.*

4.1.2. Continuous Perceivable Model

Denote the space of λ -equivalent class of Markov kernels $k_P : \mathbf{A} \times \mathcal{A} \rightarrow [0, 1]$ as $\mathbf{K}_A$, and equip it with the weak topology $\tau(\lambda)$ defined as in 2.2 of Häusler, Erich, and Harald Luschgy (2015):

$$K_\alpha \rightarrow K \Leftrightarrow \int f \otimes h d\lambda \otimes K_\alpha \rightarrow \int f \otimes h d\lambda \otimes K$$

for every $f \in L^1$ and $h \in \mathbf{C}_b$.

Then by Theorem 2.7 of Häusler, Erich, and Harald Luschgy (2015), we have $\mathbf{K}_A$ with $\tau(\lambda)$ is a non-empty, weakly compact and convex subset of a locally convex Hausdorff space. We call a function mapping $\mathbf{K}_A$ into $\mathcal{P}(\mathbf{A})$ a perceivable model. The agent gets a computed distribution given any Markov transition. Then we define a continuous perceivable model as follows.

Definition 3 (continuous perceivable model). *A continuous perceivable model is a function $M : \mathbf{K}_A \rightarrow \mathbf{N}(\mathbf{A})$ that is continuous with respect to the $\tau(\lambda)$ weak topology and the $\|\cdot\|_\infty$ uniform norm.*

Theorem 4.2. *For any $\epsilon > 0$, there exists a continuous perceivable model M such that for all $k_p \in \mathbf{K}_A$, $\|\mu_{M(k_p)} - \mathbf{P}'\mu_{M(k_p)}\|_0 \leq \epsilon$.*

Corollary 1. *For any $\epsilon > 0$, there exists a continuous perceivable model M with a positive lower bound $\underline{b} > 0$ such that for all $k_p \in \mathbf{K}_A$, $\|\mu_{M(k_p)} - \mathbf{P}'\mu_{M(k_p)}\|_0 \leq \epsilon$, and $\inf_{k_p} M(k_p) \geq \underline{b}$.*

Denote the set of all continuous perceivable models satisfying conditions in Corollary 1 as $\mathbf{M}_\epsilon(\mathbf{A})$, and denote the space of continuous density functions bounded below by $\underline{b}$ as $\mathbf{N}_{\underline{b}}(\mathbf{A})$.

4.2. Existence of u.h.c. Policy Functions

In this section, we provide sufficient conditions under which the following functional equation is equivalent as the utility function,

$$V_h^{\mu_h}(z, \alpha_h^-, I_h) = \max_{\alpha_h, x_h} u_h(z, x_h) + \delta \int V_h^{\mu_h}(z', \alpha_h, I_h') \mathbb{Q}(d(I_h') \mid (z', I_h)) \mathbb{P}(z' \mid z)$$

With the budget set

$$\alpha_h, x_h \in \Gamma(z, \alpha_h^-, p)$$

And the corresponding policy correspondence attains the maximum utility level.

Assumption 4.1. $\mathbf{Z}, \mathbf{X}, \Xi$ are compact and convex.

Assumption 4.2. $\Gamma_h : \mathbf{Z} \times \Xi_h \times \Delta^J \rightrightarrows \mathbf{X}_h \times \Xi_h$ is weakly measurable, non empty, compact valued, and continuous in $\Xi_h \times \Delta^J$ for all h .

Assumption 4.3. $u_h : \mathbf{Z} \times \mathbf{X} \rightarrow \mathbb{R}$ is a bounded Caratheodry function for all h , and is strictly increasing, and concave in x .

Define a continuous pricing density function as follows, and denote the Banach space of all those functions as $(\mathbf{Q}(\mathbf{I}_h \times (\mathbf{Z} \times \mathbf{I}_h)), \|\cdot\|_\infty)$.

Definition 4 (continuous pricing density function). $q : (\mathbf{I}_h \times (\mathbf{Z} \times \mathbf{I}_h)) \rightarrow \mathbb{R}$ is a continuous pricing density function if

1. q is jointly continuous in all arguments;
2. For all $s \in (\mathbf{Z} \times \mathbf{I}_h)$, $q(\cdot \mid s)$ is a continuous probability density function (that is $\int_{\mathbf{I}_h} q(i \mid s) di = 1, q(\cdot \mid s) \geq 0$).

Assumption 4.4. $\mathbb{Q}_h$ is absolutely continuous with respect to Lebesgue measure with density function q_{μ_h} . And $q_{\mu_h} \in \mathbf{Q}$.

Theorem 4.3. *Under Assumption 4.1- 4.4, for all $q_{\mu_h} \in \mathbf{Q}$, there exists a unique bounded function $V_h^{q_{\mu_h}} : \mathbf{Z} \times \Xi_h \times \mathbf{I}_h \rightarrow \mathbb{R}$ such that $V_h^{q_{\mu_h}}(z, \alpha_h^-, I_h) = U_h^{\mu_h}(z, \alpha_h^-, I_h)$ is jointly continuous in $(\alpha_h^-, I_h, q_{\mu_h})$, strictly increasing, and concave in α_h^- , and is measurable in z . The policy correspondence $G_h^{q_{\mu_h}}(z, \alpha_h^-, I_h)$ is u.h.c., and convex valued in (z, α_h^-, I_h) . Furthermore, $G_h^{q_{\mu_h}}(z, \alpha_h^-, I_h)$ is measurable in z and admits a measurable selector.*

The proof has a similar structure as Theorem 9.2 in Stokey(1989) and is given in detail in the appendix.

4.3. The Main Theorem

Recall the compact set $\mathbf{A}$. Let $\mathbf{A} = \tilde{\mathbf{S}}$ for Corollary 1 and denote the corresponding lower bound as $\underline{b}'$. Denote $\mathbf{N}_{\underline{b}'}(\tilde{\mathbf{S}})$ as the space of continuous density functions bounded below by $\underline{b}'$. Then we can find a continuous mapping from a continuous density function to a continuous pricing density function.

Lemma 4.1. *Under Assumption 2.2, for any $f \in \mathbf{N}_{\underline{b}'}(\tilde{\mathbf{S}})$ with associated μ_f , q_{μ_f} is continuous in all its arguments and uniformly continuous in f .*

With this Lemma and Theorem 4.3, we can provide existence of a bounded rational competitive equilibrium.

Lemma 4.2. *Under Assumption 2.1- 2.4, for any function $f_h \in \mathbf{N}_{\underline{b}'}(\tilde{\mathbf{S}})$ and initial conditions $z_t \in \mathbf{Z}, \alpha_{h,t}^- \in \Xi$, there exists a μ_{f_h} -competitive equilibrium, where μ_{f_h} is the measure with density function f_h .*

Denote the space of λ -equivalent class of Markov kernels $k : \mathbf{S} \times \mathcal{S} \rightarrow [0, 1]$ as $\mathbf{K}$, and the space of λ -equivalent class of Markov kernels $k : \tilde{\mathbf{S}} \times \tilde{\mathcal{S}} \rightarrow [0, 1]$ as $\mathbf{K}_{\tilde{F}}$. Equip both of them with the $\tau(\lambda)$ weak topology, and recall the definition of $k_{\tilde{F}}$, we have the following lemma.

Lemma 4.3. *Under Assumption 2.2, $k_{\tilde{F}}$ is continuous in k_F under weak topologies.*

Theorem 4.4. *Under Assumption 2.1- 2.4, for arbitrarily small $\epsilon > 0$, ϵ -bounded rational recursive equilibrium exists.*

By Corollary 1, choose a continuous perceivable model $M_h \in \mathbf{M}_{\epsilon}(\tilde{\mathbf{S}})$ for each agent h . Then by Lemma 4.3, $k_{\tilde{F}}$ is continuous in k_F . So we can find for each agent a continuous mapping $N_h : \mathbf{K} \rightarrow \mathbf{N}_{\underline{b}'}(\tilde{\mathbf{S}})$, where $\mathbf{K}$ is equipped with $\tau(\lambda)$ and $\mathbf{N}_{\underline{b}'}(\tilde{\mathbf{S}})$ with $\|\cdot\|_{\infty}$.

For any $k_F \in \mathbf{K}$, given $s \in \tilde{\mathbf{S}}$, choose $f_h = N_h(k_F)$. Denote the set of μ_{f_h} -competitive equilibria as $\mathbf{R}_F(s)$. Let $\mathbf{R}(k_F)$ be the set of (equivalence classes of) measurable selections

of $\mathbf{R}_F$. For all $Y \in \mathbf{R}(k_F)$, we embed $Y(\tilde{\mathbf{S}})$ into $\mathcal{P}(\tilde{\mathbf{S}})$ via $Y(s) \mapsto \delta_{Y(s)}$, and denote δ_Y as the transition kernel generated by the embedding. Since Y is measurable by definition, we get $\delta_Y \in \mathbf{K}$. Denote the set $\mathbf{K}'(k_F) := \{\delta_Y : Y \in \mathbf{R}(k_F)\}$, and the closure of the convex hull of $\mathbf{K}'(k_F)$ as $\bar{\text{co}}\mathbf{K}'(k_F)$. Clearly, $\bar{\text{co}}\mathbf{K}'(k_F) \subset \mathbf{K}$. We show in the appendix that the map $\mathbf{K} \rightarrow \bar{\text{co}}\mathbf{K}'(k_F)$ has a fixed point and generate the bounded rational recursive equilibrium from the fixed point.

5. Conclusion

I provide existence of the recursive equilibrium in a dynamic stochastic model with bounded rational agents. This set up is realistic when the economy is large and complex, which is a fact for a lot of markets. A large and complex economy would effect the agents in two ways: first, each agent would rely on recursive methods to optimize utility; second, each agent cannot solve for the stationary equilibrium analytically and has to approximate a equilibrium distribution.

The entire part of constructing a continuous value function does not put extra assumptions on the model, but it provide one possible interpretation of what the rationality is bounded by – that is, the agent’s choices are continuous in the information she observes. This is consistent with the philosophical principles stated in Maskin and Tirole (2001). Also, the continuous value function is analogous to the continuity assumption of the utility function in the static exchange economy models.

Although relatively abstract, this model has the potential to explore dynamic economic models further in two main ways. First, it may provide foundation for many algorithms computing general equilibrium models. As we stated before, examples like the computational part of Krusell and Smith (1998) and Kubler and Schmedders (2003) can be incorporated into my set up. Second, it may generate interesting economic processes from standard set up. As we can see from the proof, although the equilibrium is recursive, the Markov process of the whole economy may not be ergodic. This may provide a new angle to study business cycles and financial crisis from a standard model set up.

Appendix A. Proofs of Section 4.1

A.1. Proof of Theorem 4.1

Clearly, $(\mathbf{N}(\mathbf{A}), \|\cdot\|_\infty)$ is a convex, Hausdorff locally convex topological vector space. The very first task is to find a compact subset of $\mathbf{N}(\mathbf{A})$. Consider the space of Hölder continuous functions $\mathbf{C}^\alpha(\mathbf{A})$, $\alpha \in (0, 1)$ with the norm

$$\|f\|_\alpha = \|f\|_\infty + |f|_\alpha, \text{ where } |f|_\alpha = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

Denote the subset of $\mathbf{C}^\alpha(\mathbf{A})$ uniformly bounded in $\|\cdot\|_\alpha$ by M as $\mathbf{C}_M^\alpha(\mathbf{A})$, it is compact. Further we have $\mathbf{N}_M(\mathbf{A}) := \mathbf{N}(\mathbf{A}) \cap \mathbf{C}_M^\alpha(\mathbf{A})$ is compact for any $M \in (0, +\infty)$.

For any Markov transition P with associated adjoint operator $\mathbf{P}'$, define a correspondence G_M as:

$$G_M(f) \Rightarrow \{f' \in \mathbf{N}_M(\mathbf{A}) : \|\mu_{f'} - \mathbf{P}'\mu_f\|_0 \leq \epsilon\}, \forall f \in \mathbf{N}(\mathbf{A}).$$

To prove the theorem, it is equivalent to show that for a large enough M , there exists a fixed point of G_M . The proof is structured as follows.

First we show that for large enough M , G_M is non-empty, compact, convex valued. Equip $\mathcal{P}(\mathbf{A})$ with the weak* topology $\sigma(\mathcal{P}, \mathbf{C}_b)$. Since $\mathbf{A}$ is compact, this coincide with the vague topology $\sigma(\mathcal{P}, \mathbf{C}_c)$. And let $L^\infty(\mathbf{A}, \mathcal{A}, \lambda)$ be the space of essentially bounded measurable (equivalent classes of) functions equipped with the weak* topology $\sigma(L^\infty, L^1)$. By Theorem 19.9(2) of Aliprantis and Border (2006), for all $f \in \mathbf{N}(\mathbf{A})$, $\mathbf{P}'\mu_f \in \mathcal{P}(\mathbf{A})$. The following lemma provides a function $\hat{f} \in \mathbf{N}_M(\mathbf{A})$ as an approximate density function for $\mathbf{P}'\mu_f$.

Lemma A.1. *Given $\epsilon > 0$, there exists an $N < +\infty$ such that $\forall M \geq N, f \in \mathbf{N}(\mathbf{A})$, $G_M(f)$ is non-empty valued.*

Proof: Consider $s \in \mathbf{A}$ with the embedding $\delta_s \in \mathcal{P}(\mathbf{A})$, it is easy to construct a sequence of nascent delta function $\eta_N(\cdot | s) \in \mathbf{N}_N(\mathbf{A})$ such that $\lim_{N \rightarrow +\infty} \eta_N(\cdot | s) \rightarrow \delta_s$ in the sense of vague topology (which coincides with the weak*-topology on $\mathbf{A}$). As an example, consider the following mollifier:

$$\eta_N(x | s) = N^{\dim(\mathbf{A})} \varphi(Nx | s),$$

where

$$\varphi(x | s) = \begin{cases} e^{-1/(1-|x-s|^2)} / \text{Const} & \text{if } |x - s| < 1 \\ 0 & \text{if } |x - s| \geq 1 \end{cases}.$$

Given the norm $\|\cdot\|_0$ generates weak*-topology, there exists $N < +\infty$, such that

$$\sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} \eta_N(u | s) f_L(u) d\lambda(u) - f_L(s) \right| \leq \epsilon.$$

Then by the symmetry of set $\mathbf{A}$, we can easily construct a set of functions $\hat{\eta}_s \in \mathbf{N}_N(\mathbf{A})$ such that for all $s \in \mathbf{A}$,

$$\sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(u) \hat{\eta}_s(u) d\lambda(u) - f_L(s) \right| \leq \epsilon.$$

This means we found approximate density functions for all $\delta_s, s \in \mathbf{A}$, namely all extreme points in $\mathcal{P}(\mathbf{A})$. By theorem 15.10 of Aliprantis and Border (2006), $\mathcal{P}(\mathbf{A}) \subset \bar{\text{co}}\mathbf{A}$. Take the closure of convex hull of $\hat{\eta}_s$ accordingly. Since the convex hull of $\hat{\eta}_s$ is uniformly bounded by N in the $\|\cdot\|_\alpha$, the limit points also lies in $\mathbf{N}_N(\mathbf{A})$. Then we can find $\hat{f} \in \mathbf{N}_N(\mathbf{A})$ for all $\mu \in \mathcal{P}(\mathbf{A})$ such that

$$\sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) \hat{f}(s) d\lambda(s) - \int_{\mathbf{A}} f_L(s) d\mu(s) \right| \leq \epsilon.$$

Combined with the fact that $\mathbf{P}'\mu_f \in \mathcal{P}(\mathbf{A})$, we have $G_M(f)$ is not empty. $\square$

Lemma A.2. *For any $f \in \mathbf{N}(\mathbf{A})$, $G_M(f)$ is convex and close valued.*

Proof: For any $f'_1, f'_2 \in G_M(f)$, consider the convex combination $f' = \beta f'_1 + (1-\beta)f'_2, \beta \in (0, 1)$,

$$\begin{aligned}
& \| \mu_{f'} - \mathbf{P}'\mu_f \|_0 \\
&= \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f'(s) d\lambda(s) - \int_{\mathbf{A}} f_L(s) d\mathbf{P}'\mu_f(s) \right| \\
&= \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) [\beta f'_1(s) + (1-\beta) f'_2(s)] d\lambda(s) - \int_{\mathbf{A}} f_L(s) d\mathbf{P}'\mu_f(s) \right| \\
&= \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \beta \left[\int_{\mathbf{A}} f_L(s) f'_1(s) d\lambda(s) - \int_{\mathbf{A}} f_L(s) d\mathbf{P}'\mu_f(s) \right] \right. \\
&\quad \left. + (1-\beta) \left[\int_{\mathbf{A}} f_L(s) f'_2(s) d\lambda(s) - \int_{\mathbf{A}} f_L(s) d\mathbf{P}'\mu_f(s) \right] \right| \\
&\leq \beta \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f'_1(s) d\lambda(s) - \int_{\mathbf{A}} f_L(s) d\mathbf{P}'\mu_f(s) \right| \\
&\quad + (1-\beta) \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f'_2(s) d\lambda(s) - \int_{\mathbf{A}} f_L(s) d\mathbf{P}'\mu_f(s) \right| \\
&\leq \beta\epsilon + (1-\beta)\epsilon = \epsilon.
\end{aligned}$$

So $f' \in G_M(f)$, it is convex. Next, consider $f'_n \xrightarrow{\text{unif.}} f', f'_n \in G_M(f)$. Notice first $f' \in \mathbf{N}(\mathbf{A})$ under uniform convergence. Using prove by contradiction, assume that $\| \mu_{f'} - \mathbf{P}'\mu_f \|_0 = \epsilon' > \epsilon$,

$$\begin{aligned}
& \| \mu_{f'} - \mathbf{P}'\mu_f \|_0 \\
&= \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f'(s) d\lambda(s) - \int_{\mathbf{A}} f_L(s) d\mathbf{P}'\mu_f(s) \right| \\
&\leq \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f'(s) d\lambda(s) - \int_{\mathbf{A}} f_L(s) f'_n(s) d\lambda(s) \right| \\
&\quad + \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f'_n(s) d\lambda(s) - \int_{\mathbf{A}} f_L(s) d\mathbf{P}'\mu_f(s) \right| \\
&\leq \int_{\mathbf{A}} |f'(s) - f'_n(s)| d\lambda(s) + \epsilon.
\end{aligned}$$

This means $\int_{\mathbf{A}} |f'(s) - f'_n(s)| d\lambda(s) \geq \epsilon' - \epsilon > 0$ for all n , which contradicts uniform convergence of f'_n . So $G_M(\cdot)$ is close valued. $\square$

Now we have G_M is non-empty, compact, convex valued. Then we show that it is also *u.h.c.*, and then the Kakutani-Glicksberg-Fan theorem applies.

Lemma A.3. $G_M(\cdot)$ is *u.h.c.*.

Proof: Now we characterize the integral with $\mathbf{P}$. $\int_{\mathbf{A}} f_L(s) d\mathbf{P}'\mu_f(s) = \int_{\mathbf{A}} (\mathbf{P}f_L)(s) d\mu_f(s)$ by the way $\mathbf{P}, \mathbf{P}'$ are defined. By Theorem 19.7(4) of Aliprantis and Border (2006), $\mathbf{P}f_L \in \mathbf{B}_b(\mathbf{A})$. It is also easy to check that for any $f_L \in \mathbf{L}(\mathbf{A})$, $\mathbf{P}f_L \in \mathbf{B}_b(\mathbf{A})$ is bounded by 1. So for all $f \in \mathbf{N}(\mathbf{A})$, $f' \in \mathbf{N}_M(\mathbf{A})$,

$$\begin{aligned} & \| \mu_{f'} - \mathbf{P}'\mu_f \|_0 \\ &= \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f'(s) d\lambda(s) - \int_{\mathbf{A}} (\mathbf{P}f_L)(s) d\mu_f(s) \right| \\ &= \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f'(s) d\lambda(s) - \int_{\mathbf{A}} (\mathbf{P}f_L)(s) f(s) d\lambda(s) \right|. \end{aligned}$$

Now consider a sequence of functions $f_n \xrightarrow{\text{unif.}} f$, and let $f'_n \xrightarrow{\text{unif.}} f'$ where $f'_n \in G_M(f_n)$, we go on to show that $f' \in G_M(f)$.

$$\begin{aligned} & \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f'(s) d\lambda(s) - \int_{\mathbf{A}} (\mathbf{P}f_L)(s) f(s) d\lambda(s) \right| \\ & \leq \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f'_n(s) d\lambda(s) - \int_{\mathbf{A}} f_L(s) f'(s) d\lambda(s) \right| \\ & \quad + \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f'_n(s) d\lambda(s) - \int_{\mathbf{A}} (\mathbf{P}f_L)(s) f_n(s) d\lambda(s) \right| \\ & \quad + \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} (\mathbf{P}f_L)(s) f_n(s) d\lambda(s) - \int_{\mathbf{A}} (\mathbf{P}f_L)(s) f(s) d\lambda(s) \right| \\ & \leq \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f'_n(s) d\lambda(s) - \int_{\mathbf{A}} (\mathbf{P}f_L)(s) f_n(s) d\lambda(s) \right| \\ & \quad + \int_{\mathbf{A}} |f_n - f| d\lambda + \int_{\mathbf{A}} |f'_n - f'| d\lambda \\ & \leq \epsilon + \int_{\mathbf{A}} |f_n - f| d\lambda + \int_{\mathbf{A}} |f'_n - f'| d\lambda \rightarrow \epsilon. \end{aligned}$$

So we have $f' \in G_M(f)$. $\square$

Now choose $M > N$ as in Lemma A.1. By the Kakutani-Glicksberg-Fan theorem, the set of fixed points of $G_M : \mathbf{N}_M(\mathbf{A}) \rightrightarrows \mathbf{N}_M(\mathbf{A})$ is non-empty. So the set of computational invariant density functions $\mathfrak{T}(\epsilon, P)$ is non-empty. The final part of the proof is to show that it is also convex and close valued. It is analogous to Lemma A.2 and concludes the proof of the theorem.

A.2. Proof of Theorem 4.2

To simplify notations, redefine $\Upsilon(\epsilon, P)$ in a slightly different way: Define a correspondence H as:

$$H(k_P) \Rightarrow \{f \in \mathbf{N}(\mathbf{A}) : \|\mu_f - \mathbf{P}'\mu_f\|_0 \leq \epsilon\}, \forall k_P \in \mathbf{K}_A.$$

Lemma A.4. *H is l.h.c..*

Proof: By definition, we need to show $\forall k_{P_m} \rightarrow k_P, \forall f \in H(k_P)$, there exists a subsequence of $k_{P_{m_k}}$ such that $\exists f_k \in H(k_{P_{m_k}}), f_k \rightarrow f$.

First, $k_{P_m} \rightarrow k_P$, for all $f_c \in \mathbf{C}_b(\mathbf{A})$,

$$\int_{\mathbf{A}} \int_{\mathbf{A}} f_c(u) k_{P_m}(s, du) f(s) ds \rightarrow \int_{\mathbf{A}} \int_{\mathbf{A}} f_c(u) k_P(s, du) f(s) ds.$$

By the definitions of $k_P, \mathbf{P}'$, this is equivalent to

$$\int_{\mathbf{A}} f_c(u) d\mathbf{P}'_m \mu_f(u) \rightarrow \int_{\mathbf{A}} f_c(u) d\mathbf{P}' \mu_f(u).$$

Then further we have,

$$\|\mu_f - \mathbf{P}'_m \mu_f\|_0 \rightarrow \|\mu_f - \mathbf{P}' \mu_f\|_0.$$

So if $f \in H(k_P)$ and $\|\mu_f - \mathbf{P}' \mu_f\|_0 < \epsilon$, there exists m_1 such that for $m \geq m_1$, $\|\mu_f - \mathbf{P}'_m \mu_f\|_0 < \epsilon$. Choose $m_k = m_1 + k$ and $f_k = f$, we get $f_k \in H(k_{P_{m_k}}), f_k \xrightarrow{\text{unif.}} f$.

Otherwise, if $f \in H(k_P)$ and $\|\mu_f - \mathbf{P}' \mu_f\|_0 = \epsilon$. By theorem 4.1, there is also $f' \in H(k_P)$ such that $\|v_3 - v_3 \Pi\| \leq \epsilon/2 < \epsilon$ (By choosing $\epsilon/2$ as the lower bound). By the convexity of $\Upsilon(\epsilon, P)$, there exists a sequence $f_k \xrightarrow{\text{unif.}} f$ such that $\|\mu_{f_k} - \mathbf{P}' \mu_{f_k}\|_0 < \epsilon$. So by similar argument as the first scenario, there exists $m_{(k)}$ such that for $m \geq m_{(k)}$, $\|\mu_{f_k} - \mathbf{P}'_m \mu_{f_k}\|_0 < \epsilon$. Then let $m_1 = m_{(1)}$ and $m_k = \max\{m_{(k)}, m_{k-1} + 1\}, k \geq 2$. And we have $f_k \in H(k_{P_{m_k}}), f_k \xrightarrow{\text{unif.}} f$. $\square$

Given in Theorem 4.1 that the correspondence is convex, close valued, by Michael selection theorem, there exists a continuous selection.

A.3. Proof of Corollary 1

The proof is analogous to Lemma A.3 and can be provided upon request.

Appendix B. Proofs of Section 4.2 (Theorem 4.3)

First, we define an operator $M_{q_\mu}f$ that projects bounded, Caratheodry functions to bounded, Caratheodry functions for all bounded, Caratheodry pricing density $q_\mu \in \mathbf{Q}$. And $M_{q_\mu}f$ is jointly continuous in $((\alpha_h, I_h), q_\mu)$. $M_{q_\mu}f$ is defined by

$$(M_{q_\mu}f)(z, \alpha_h, I_h) = \int_{z' \in \mathbf{Z}} \int_{I'_h} f(z', \alpha_h, I'_h) q_\mu(I'_h | (z', I_h)) d(I'_h) p(z' | z) dz$$

Given $z' \in \mathbf{Z}$ and q_μ is continuous with compact support. So q_μ is bounded. Then by Theorem 19.7 of Aliprantis and Border (2006), $M_{q_\mu}f$ is measurable in z . Now fix z and consider a sequence $\{(\alpha_h^m, I_h^m, q_\mu^m)\}$ converging to (α_h, I_h, q_μ) . Then we have

$$\begin{aligned} & | M_{q_\mu^m}f(z, \alpha_h^m, I_h^m) - M_{q_\mu}f(z, \alpha_h, I_h) | \\ & \leq | M_{q_\mu^m}f(z, \alpha_h^m, I_h^m) - M_{q_\mu^m}f(z, \alpha_h, I_h) | \\ & \quad + | M_{q_\mu^m}f(z, \alpha_h, I_h) - M_{q_\mu}f(z, \alpha_h, I_h) | \\ & = | \int \int [f(z', \alpha_h^m, I'_h) q_\mu(I'_h | (z', I_h^m)) - f(z', \alpha_h, I'_h) q_\mu(I'_h | (z', I_h))] d(I'_h) p(z' | z) dz | \\ & \quad + | \int \int [q_\mu^m(I'_h | (z', I_h)) - q_\mu(I'_h | (z', I_h))] f(z', \alpha_h, I'_h) d(I'_h) p(z' | z) dz | \\ & \rightarrow 0 \end{aligned}$$

The first term converges to 0 because f and q_μ are continuous in (α_h, I_h) ; the second term converges because of the uniform convergence of q_μ^m . Then we define another operator $T_{q_\mu}f$ as

$$\begin{aligned} (T_{q_\mu}f)(z, \alpha_h^-, I_h) &= \max_{\alpha_h, x_h} u_h(z, x_h) + \delta(M_{q_\mu}f)(z, \alpha_h, I_h) \\ \alpha_h, x_h &\in \mathbf{\Gamma}(z, \alpha_h^-, p) \end{aligned}$$

For given (α_h^-, I_h, q_μ) , by Theorem 18.19 in Aliprantis and Border (2006), $T_{q_\mu}f$ is measurable in z . Fix z and consider a sequence $\{(\alpha_h^{-m}, I_h^m, q_\mu^m)\}$ converging to (α_h^-, I_h, q_μ) . Then we have

$$\begin{aligned}
& | T_{q_\mu^m} f(z, \alpha_h^{-m}, I_h^m) - T_{q_\mu} f(z, \alpha_h^-, I_h) | \\
& \leq | T_{q_\mu^m} f(z, \alpha_h^{-m}, I_h^m) - T_{q_\mu^m} f(z, \alpha_h^-, I_h) | \\
& \quad + | T_{q_\mu^m} f(z, \alpha_h^-, I_h) - T_{q_\mu} f(z, \alpha_h^-, I_h) | \rightarrow 0
\end{aligned}$$

The first term converges to 0 because $T_{q_\mu^m} f(z, \cdot)$ is continuous according to maximum theorem. For the second term, since f is bounded, for a given z , $u_h(z, x_h) + \delta(M_{q_\mu} f)(z, \alpha_h, I_h)$ is uniformly continuous in q_μ . So the maximum over the same compact budget set also converges to 0.

Then by induction with the same argument as above, we have $T_{q_\mu^n} f$ is measurable in z and jointly continuous in (α_h^-, I_h, q_μ) for all $n \in \mathbb{N}_{++}$

Easy to check that Blackwell's sufficient conditions are satisfied for T_{q_μ} for all $q_\mu \in \mathbf{Q}$. Then take any Caratheodry function f_0 , and we have the value function $V^{q_\mu} = T_{q_\mu}^\infty f_0$. Obviously, V^{q_μ} is measurable in z and jointly continuous in α_h^-, I_h, q_μ .

More specifically, choose a Caratheodry function $\hat{f}_0$ that is strictly increasing, and concave in α_h^- . Analogously, we get $V^{q_\mu} = T_{q_\mu}^\infty \hat{f}_0$ is measurable in z and jointly continuous in α_h^-, I_h, q_μ and strictly increasing, and concave in α_h^- . The equivalence of the functional equation and the utility function is directly implied. To describe the policy correspondence, notice

$$\begin{aligned}
V^{q_\mu}(z, \alpha_h^-, I_h) &= \max_{\alpha_h, x_h} u_h(z, x_h) + \delta \int_{z' \in \mathbf{Z}} \int_{I_h'} V^{q_\mu}(z', \alpha_h, I_h') q_\mu(I_h' | (z', I_h)) d(I_h') p(z' | z) dz \\
&\quad \alpha_h, x_h \in \Gamma(z, \alpha_h^-, I_h).
\end{aligned}$$

Fix z , by maximum theorem, we have $G_h^{q_{\mu_h}}(z, \alpha_h^-, I_h)$ is u.h.c., and convex valued in $(\alpha_h^-, I_h, q_{\mu_h})$. And fix $(\alpha_h^-, I_h, q_{\mu_h})$, by Theorem 18.19 of Aliprantis and Border (2006), $G_h^{q_{\mu_h}}(z, \alpha_h^-, I_h)$ is measurable in z and admits a measurable selector. $\square$

Appendix C. Proofs in Section 4.3

C.1. Proof of Lemma 4.1

As described in Section 3, we can write q_{μ_f} as follows, when f is a continuous density function:

$$q_{\mu_f}(I_h' | (z', I_h)) = \int_{\alpha} f_I(I_h' | (z', \alpha)) f_{\alpha}(\alpha | I_h) d\lambda(\alpha), \quad (1)$$

f_I and f_α are defined as follows:

$$f_\alpha[\alpha \mid I_h] = \frac{\int_{s \in S_{\alpha, I_h}} f(s) ds}{\int_{s \in S_{I_h}} f(s) ds}$$

where $S_{\alpha, I_h} = \{s : z(s) = \alpha, I_h(s) = I_h, \alpha(s) = \alpha\}$ and $S_{I_h} = \{s : z(s) = z, I_h(s) = I_h\}$; Similarly, we have

$$f_I[I'_h \mid (z', \alpha)] = \frac{\int_{s \in S_{I'_h, (z', \alpha)}} f(s) ds}{\int_{s \in S_{(z', \alpha)}} f(s) ds}$$

where $S_{I'_h, (z', \alpha)} = \{s : z(s) = z', \alpha^-(s) = \alpha, I_h(s) = I'_h\}$ and $S_{(z', \alpha)} = \{s : z(s) = z', \alpha^-(s) = \alpha\}$.

Easy to check, the four integrals in the definitions of f_I, f_α are bounded linear transformations of f . Hence, they are all uniformly continuous in f . Further, if $f \in \mathbf{N}_{J\bar{J}}(\mathbf{S})$, all these four integrals are bounded away from 0. Then we have f_I, f_α are uniformly continuous in f and hence $f_I \cdot f_\alpha$ is uniformly continuous in f . Next, notice the integration in (3) is also a bounded linear operator. So we have q_{μ_f} is uniformly continuous in f . And it is trivial to check that q_{μ_f} is jointly continuous in all its arguments. $\square$

C.2. Proof of Lemma 4.3

To show that $k_{\tilde{F}}$ is continuous in k_F , consider a sequence $k_{F_n} \rightarrow k_F$ in $\tau(\lambda)$, so we have for all $f \in L^1(\mathbf{S}), g \in \mathbf{C}_b(\Xi \times \Delta^J)$,

$$\int_{\mathbf{S}} \int_{\Xi \times \Delta^J} f(s) g(a) dF_{n,s}(a) d\lambda(s) \rightarrow \int_{\mathbf{S}} \int_{\Xi \times \Delta^J} f(s) g(a) dF_s(a) d\lambda(s). \quad (2)$$

By Theorem 2.6 of Häusler, Erich, and Harald Luschgy (2015), this is equivalent to for all measurable, bounded function $h : (\mathbf{S}, \Xi \times \Delta^J) \rightarrow \mathbb{R}$ such that $h(s, \cdot) \in \mathbf{C}_b$,

$$\int_{\mathbf{S}} \int_{\Xi \times \Delta^J} h(s, a) dF_{n,s}(a) d\lambda(s) \rightarrow \int_{\mathbf{S}} \int_{\Xi \times \Delta^J} h(s, a) dF_s(a) d\lambda(s). \quad (3)$$

We go on to show that for all $f' \in L^1(\tilde{\mathbf{S}}), g' \in \mathbf{C}_b(\tilde{\mathbf{S}})$

$$\int_{\tilde{\mathbf{S}}} \int_{\tilde{\mathbf{S}}} f'(s) g'(a) d\tilde{F}_{n,s}(a) d\lambda(s) \rightarrow \int_{\tilde{\mathbf{S}}} \int_{\tilde{\mathbf{S}}} f'(s) g'(a) d\tilde{F}_s(a) d\lambda(s)$$

where

$$\int_{\tilde{\mathbf{S}}} \int_{\tilde{\mathbf{S}}} f'(s) g'(a) d\tilde{F}_{n,s}(a) d\lambda(s) = \int_{\tilde{\mathbf{S}}} \int_{\mathbf{Z}} \int_{\Xi \times \Delta^J} f'(s) g'(a) d\mathbb{P}(z' | z) dF_{n,(z',\alpha)}(a) d\lambda(s).$$

First, for all z, α , letting $h := p(z' | z) g'(z', \alpha, \alpha', p')$, from (2) we have

$$\begin{aligned} & \int_{\mathbf{Z}} \int_{\Xi \times \Delta^J} p(z' | z) g'(z', \alpha, \alpha', p') dF_{n,(z',\alpha)}(\alpha', p') d\lambda(z') \rightarrow \\ & \int_{\mathbf{Z}} \int_{\Xi \times \Delta^J} p(z' | z) g'(z', \alpha, \alpha', p') dF_{(z',\alpha)}(\alpha', p') d\lambda(z'). \end{aligned}$$

Since $g' \in \mathbf{C}_b$, g' is bounded. Denote the bound as B , we have

$$\int_{\mathbf{Z}} \int_{\Xi \times \Delta^J} p(z' | z) g'(z', \alpha, \alpha', p') dF_{n,(z',\alpha)}(\alpha', p') d\lambda(z') \leq \int_{\mathbf{Z}} p(z' | z) B d\lambda(z') = B.$$

So by dominated convergence theorem, we have

$$\int_{\tilde{\mathbf{S}}} \int_{\tilde{\mathbf{S}}} f'(s) g'(a) d\tilde{F}_{n,s}(a) d\lambda(s) \rightarrow \int_{\tilde{\mathbf{S}}} \int_{\tilde{\mathbf{S}}} f'(s) g'(a) d\tilde{F}_s(a) d\lambda(s).$$

□

C.3. Proof of Lemma 4.2

Given $f \in \mathbf{N}_{\underline{b}'}(\tilde{\mathbf{S}})$, by Lemma 3.2, we have $q_{\mu_f} \in \mathbf{Q}$. Combined with Assumptions 4.1-4.3, all the assumptions in section 3 are satisfied. So the equilibrium conditions are equivalent to the following conditions:

(A) Commodity market clearing:

$$\sum_{h \in \mathbf{H}} x_{h,t} = \sum_{h \in \mathbf{H}} e_h(z_t) + \sum_{j \in \mathbf{J}} d_j(z_t)$$

(B) Financial market clearing:

$$\sum_{h \in \mathbf{H}} \alpha_{h,t} = \mathbf{1}$$

(C') Utility maximization:

$$x_{h,t}, \alpha_{h,t} \in G_h^{q_{\mu_h}}(z_t, \alpha_{h,t}^-, I_{h,t})$$

Define a price player's best response $\psi : \mathbf{X} \times \Xi \rightrightarrows \Delta^J$ by

$$\psi((x_{h,t}, \alpha_{h,t})_{h \in \mathbf{H}}) = \arg \max_{p \in \Delta^J} \{p_{c,s} \sum_h (x_{h,s} + d(z_s) \alpha_{h,s}^- - e_h(z_s)) + p_{J,s} (\sum_h \alpha_{h,s} - 1)\}$$

Then the standard existence proof of a static exchange economy applies. See, for example, Lemma 3 of Brumm et al. (2017). $\square$

C.4. Proof of Theorem 4.4

First, notice that $s \rightrightarrows \mathbf{R}_F(s)$ is measurable. This is shown in the proof of Lemma 4 of Brumm et al.(2017). Then by theorem 18.13 of Aliprantis and Border(2006), $\mathbf{R}_F$ has a measurable selector. So the map $k_F \rightrightarrows \mathbf{R}(k_F)$ is non-empty valued. Then in the same way as the second part of the proof of Lemma 4 of Brumm et al.(2017), we can get $k_F \rightrightarrows \mathbf{R}(k_F)$ has a closed graph. Then by the definition of $\mathbf{K}'$ as a subset of $\mathbf{K}$ equipped with $\tau(\lambda)$, we have $k_F \rightrightarrows \mathbf{K}'(k_F)$ has a closed graph. Then theorem 17.35(1) in Aliprantis and Border(2006) implies that $k_F \rightrightarrows \bar{\text{co}}\mathbf{K}'(k_F)$ has a closed graph. Apply Kakutani-Glicksberg-Fan theorem, we have a fixed point $k_F^* \in \bar{\text{co}}\mathbf{K}'(k_F^*)$.

If $k_F^* \in \text{co}\mathbf{K}'(k_F^*)$, choose $f_h^* = N_h(k_F^*)$. Then $(F^*, (\mu_{f_h^*})_{h \in \mathbf{H}})$ is a ϵ -bounded rational equilibrium.

If $k_F^* \notin \text{co}\mathbf{K}'(k_F^*)$, we make use of the fact that ϵ is arbitrarily small, and repeat the procedure above for $\epsilon' = \epsilon/2$. And we find another fixed point $k_F' \in \bar{\text{co}}\mathbf{K}'(k_F')$. If $k_F' \notin \text{co}\mathbf{K}'(k_F')$, then we can choose $k_F'' \in \text{co}\mathbf{K}'(k_F')$ within an arbitrary ϵ -neighbourhood of k_F' , and $f_h' = N_h(k_F')$ such that $(F'', (\mu_{f_h'})_{h \in \mathbf{H}})$ is a ϵ -bounded rational equilibrium. $\square$

References

- [1] Aliprantis, C. D., and K. C. Border. "Infinite dimensional analysis: a hitchhiker's guide." *Studies in Economic Theory* 4 (1999).
- [2] Billingsley, Patrick. *Convergence of probability measures*. John Wiley & Sons, 2013.
- [3] Brumm, Johannes, Dominika Kryczka, and Felix Kubler. Recursive equilibria in dynamic economies with stochastic production. *Econometrica* forthcoming, 2017.
- [4] Citanna, Alessandro, and Paolo Siconolfi. "Recursive equilibrium in stochastic OLG economies: Incomplete markets." *Journal of Mathematical Economics* 48.5 (2012): 322-337.
- [5] Citanna, Alessandro, and Paolo Siconolfi. "Recursive Equilibrium in Stochastic Overlapping Generations Economies." *Econometrica* 78.1 (2010): 309-347.
- [6] Duffie, Darrell, et al. "Stationary markov equilibria." *Econometrica: Journal of the Econometric Society* (1994): 745-781.
- [7] Duggan, John. "Noisy stochastic games." *Econometrica* 80.5 (2012): 2017-2045.
- [8] Dynkin, Evgenii Borisovich, and Igor Vyacheslavovich Evstigneev. "Regular conditional expectations of correspondences." *Theory of Probability Its Applications* 21.2 (1977): 325-338.
- [9] Häusler, Erich, and Harald Luschgy. *Stable convergence and stable limit theorems*. Vol. 74. Berlin: Springer, 2015.
- [10] He, Wei, and Yeneng Sun. "Stationary Markov perfect equilibria in discounted stochastic games." *Journal of Economic Theory* 169 (2017): 35-61.
- [11] Heaton, John, and Deborah J. Lucas. "Evaluating the effects of incomplete markets on risk sharing and asset pricing." *Journal of political Economy* 104.3 (1996): 443-487.
- [12] Hellwig, Martin F. "A note on the implementation of rational expectations equilibria." *Economics Letters* 11.1-2 (1983): 1-8.
- [13] Krusell, Per, and Anthony A. Smith, Jr. "Income and wealth heterogeneity in the macroeconomy." *Journal of political Economy* 106.5 (1998): 867-896.
- [14] Kubler, Felix, and Herakles Polemarchakis. "Stationary Markov equilibria for overlapping generations." *Economic Theory* 24.3 (2004): 623-643.

- [15] Kubler, Felix, and Karl Schmedders. "Recursive equilibria in economies with incomplete markets." *Macroeconomic dynamics* 6.2 (2002): 284-306.
- [16] Kubler, Felix, and Karl Schmedders. "Stationary Equilibria in AssetPricing Models with Incomplete Markets and Collateral." *Econometrica* 71.6 (2003): 1767-1793.
- [17] Maskin, Eric, and Jean Tirole. "Markov perfect equilibrium: I. Observable actions." *Journal of Economic Theory* 100.2 (2001): 191-219.
- [18] Nowak, Andrzej S., and T. E. S. Raghavan. "Existence of stationary correlated equilibria with symmetric information for discounted stochastic games." *Mathematics of Operations Research* 17.3 (1992): 519-526.
- [19] Rubinstein, Ariel. *Modeling bounded rationality*. MIT press, 1998.
- [20] Santos, Manuel S. "On non-existence of Markov equilibria in competitive-market economies." *Journal of Economic Theory* 105.1 (2002): 73-98.
- [21] Stokey, Nancy L. *Recursive methods in economic dynamics*. Harvard University Press, 1989.