LIMITED COMMITMENT AND THE IMPLEMENTATION OF MONETARY POLICY

1. Abstract

Trade volumes in the federal funds market have remained low since the recent financial crisis. While some argue that this derives from a flood of money in the system, we propose an alternate explanation based on the distinguishing characteristic of the fed funds market: unlike other markets, whose volumes have recovered, fed funds loans are unsecured. In a model of a money market with unsecured loans subject to endogenous borrowing constraints where the central bank pays interest on reserves (IOR), we show that high levels of IOR reduce trade by tightening credit limits. Friedman's dictum that the central bank should pay a market rate fails to deliver efficiency because IOR decreases the opportunity cost of money holdings relative to credit, so decreases both the profits from lending and the value of borrowing. This results in a tightening of the endogenous borrowing constraint. When IOR exceeds the growth rate of money, credit limits plummet to zero, and no borrowing occurs despite an extant need for funds. Alternate rate schedules besides constant IOR, such as the limited deposit regime employed by Norges Bank in response to dissatisfaction with the functioning of their floor system provide additional incentives to trade which relax credit constraints and improve welfare.

2. Introduction

In several jurisdictions, central bank actions in the aftermath of the financial crisis left the banking system awash in funds. Dramatic liquidity, together with interest payments designed to support its value, was hoped to, among other goals, ease the symptoms of stress in short term funding markets. In many jurisdictions operating a so-called floor system, there remains little trade. This study suggest that this lack of trade, instead of representing an efficient outcome, may derive from previously unnoticed incentive effects of a floor system. Indeed, if the disease was stress in the interbank market leading to unmet needs for liquidity, this study concludes that the medicine – drowning markets in reserves and paying interest on them – may indeed have made things worse.

More generally, this paper makes clear that the method of monetary policy implementation matters. Most authors ignore implementation – the details and microstructure of central bank actions – instead focusing on the relationship between endogenous variables, such as inflation and output. When one ignores the implementation of policy, considering only its potential outcomes, one implicitly assumes invariance in the structure of equilibrium. Certain frictions in the economy may lead this invariance to fail, whereupon models based on these implicit assumptions will deliver false conclusions. As the set of tools employed by central banks expands, assumptions about the structure of equilibrium grow increasingly tenuous.

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Here, we consider a particularly pernicious example, but the importance of such an approach holds more broadly.

In an attempt to support rates in the presence of large balances, central banks pay interest to their depositors. This deposit rate acts as a floor on interbank rates, and so an implementation framework where the central bank pushes interbank rates down to the deposit rate are termed "floor systems." Proponents of floor systems cite both microeconomic and macroeconomic arguments (Goodfriend 2002 and Keister, Martin, and McAndrews 2008). On the one hand, large balances allow banks to settle payments earlier in the day without careful regard for inflowing payments, moreover floor systems require little central bank intervention reducing staffing and other costs. On the other hand, in many models, the Friedman rule is implemented if the central bank pays the market rate on money. This study takes no position on the microeconomic arguments. Also, in our study, the efficient outcome is implemented if the return to money equals the real rate, but this does not equal the money market rate because of frictions.

Despite recovery in other money markets, there remains little unsecured interbank trade in jurisdictions which have implemented floor systems, such as the US. Indeed, there remains very little activity in the federal funds market among domestic depository institutions. One explanation of this behavior interprets a lack of trade as evidence of satiation in liquidity. That is, no one borrows or lends because no one has need of funds at the prevailing interest rate. This interpretation depends on a well functioning market. A lack of trade only indicates a lack of gains from trade if markets are frictionless.

Myriad frictions might plague overnight funding markets. Search and matching frictions may prevent counterparties from finding beneficial trading opportunities (Afonso and Lagos, 2016). Various costs of trade and regulations may preempt otherwise profitable opportunities (Armenter and Lester, 2017). This study will focus on limited commitment frictions – the fact that institutions can not credibly promise to repay loans if it is not profitable to do so.

Whether a borrower will choose to repay depends on the consequences of default. Here, as is usual in the literature on limited commitment, defaulters lose access to credit and so must carry greater money balances. A floor system, in paying interest on money holdings, makes this outside option more palatable. By decreasing the potential cost of default, interest on reserves decreases endogenous borrowing limits. A core result of the paper is that, if the real supply of money is large enough to push interbank rates down to the floor, this effect wholly dominates and completely eliminates any repayment incentives, thus precluding credit of any size, leading to a collapse in overnight lending and borrowing. Crucially, this collapse derives from changes in repayment constraints, but in no way represents a satiation of the banking system with respect to liquidity. Banks want to borrow to fund real activity; no one will lend to them because of a rational concern over default.

While banks' most stable sources of funding are deposits, both retail and wholesale, when facing unforeseen liquidity needs banks rely heavily on interbank markets and similar sources of short-term funding (e.g. repurchase agreement (repos)). Interbank markets in the US are unsecured, thus repayment is enforced by reputational concerns. Defaults on interbank loans are not necessarily the outcome of bankruptcy filings, as late repayments –known as settlement fails– are, effectively, defaults on the terms of the loans. Financial markets acting as a significant source of funding for banks, such as repo markets, are typically secured by a variety of assets, but they are rarely fully collateralized. There is indeed evidence that the *proliferation of settlement fails following the insolvency of Lehman Brothers Holdings Inc.*

in September 2008 led the Treasury Market Practices Group (TMPG) – a group of market professionals committed to supporting the integrity and efficiency of the U.S. Treasury market – to promote [...] the introduction of the fails charge, and explicit penalty for failure to deliver the securities on a scheduled settlement date.[1] Therefore, limited commitment constitutes an important friction in credit markets, for the purpose of our study.

We consider two potential policy interventions which ameliorate the problems created by a floor system: a quota system and a voluntary reserve targeting system. Both systems work on the principle of hurting defaulted agents more than non-defaulted agents. The primary differences between the two system involve the degree of information required for their implementation by a central bank.

A quota system mirrors the policy implemented in 2011 by the Norges Bank. Instead of a full allotment deposit facility, under a quota system the central bank sets a quota for each bank and only pays interest on deposits up to the quota. If quotas are set carefully, defaulted agents without access to credit and who consequently want to carry greater real balances find this costly as unspent balances may not receive interest. By punishing autarkic agents but not others, carefully set quotas can revive lending.

A voluntary reserve targeting framework solicits targets from banks. Deposits up to the target earn a high interest rate. Deposits over the target earn a low rate. A fee is charged in proportion to the target itself. Such a schedule incentivizes banks to report targets depending on their true expected holdings. Hence, a central bank without sufficient information, or who might be barred from basing compensation based on such information, to achieve similar effects to the quota system.

3. Environment

The environment follows Berentsen, Marchesiani, and Waller (2014) with two important adjustments. First, the only asset in the economy is fiat money, there are no bonds that can collateralize loans. Second, agents can not commit to repay loans, and this limited commitment bounds the size of loans. The details of this infinite horizon model are as follows.

3.1. **Agents and Preferences.**

The economy is populated with a unit measure each of two types of agents, buyers and sellers. Buyers analogize to a consolidation of payers and their depository institutions. Sellers serve to provide a need for cash and to endogenize the price of goods, but might be thought of as a consolidation of all payees and their depositories during a period.

A non-strategic central bank supplies fiat currency that all agents can recognize but can not counterfeit. This is the only asset in the economy. The central bank can also operate a deposit facility, whose properties will differ across periods. The deposit facility should be thought of as redemption and reisssue: at an appointed time each day, agents costlessly present their currency to the central bank and are returned a different quantity of a new vintage of currency. The policy experiments considered below differ in redemption rates. We consider only equilibria where agents believe that no one will accept old vintages of currency in trade. In equilibria of this type, the central bank can pay negative rates without maintaining any records or accounts, and all agents remain anonymous to the central bank.

[1]See *The Introduction of the TMPG Fails Charge for U.S. Treasury Securities*, FRBNY Economic Policy Review, October 2010

Besides money, agents can trade their labor and two differnt kinds of goods. An early good is consumed by sellers and produced by buyers' labor, with linear utility and a linear technology, respectively. The late good is produced by sellers' labor with a linear disutility and consumed by buyers who enjoy a concave utility function. This will be the source of curvature in the model.[2]

3.2. **Markets.** Three markets occur in sequence each period. The first is a settlement market, where obligations from previous periods are resolved, including principal and interest owed by the central bank and private agents, and agents rebalance their money balances with the production or consumption of the early good. The second market is a money market where, after learning their preference shock, buyers borrow and lend money with each other subject to limited commitment. The third and final market is the late goods market where buyers purchase consumption from sellers.

Each market clears and is priced competitively, subject to some frictions. The settlement market is frictionless. In the money market, borrowers can only borrow up to an endogenous limit given by repayment incentives. The late goods market requires cash up front because of anonymity.

In the money market, buyers can borrow and lend subject to limited commitment frictions following Kehoe and Levine (1993). If having previously defaulted, agents are permanently excluded from future borrowing. We assume that agents can, however, still access all other markets. Specifically, a defaulted agent can still work to secure money, use money to buy goods, and redeem money at the central bank. One might suppose that the punishment for default of a bank should be more stringent. But technical defaults with respect to overnight loans, while perhaps incurring some penalty rates, may be largely unreported beyond the cabal of banks. In such an environment, the central bank or other agents may not be able to tell the difference between a default having been granted an extension and a simple rollover of a loan. Our model depends on such a default being reported to the interbank community.[3] We assume it is not, however, made generally public and so, specifically, is unknown to the central bank. We also assume that, since no loan would be made that would result in default, there are no agents in autarky along the equilibrium path.

3.3. **Preferences.** Sellers:

$$\sum_t \beta^t \left(x_t - h_t + X_t - H_t\right)$$

Buyers:

$$\mathbb{E}\left[\sum_t \beta^t \left(X_t - H_t + \tilde{\epsilon}_t u(q_t)\right)\right]$$

[2]Berentsen and Rocheteau (2002, 2003) and Lagos and Rocheteau (2005) use a similar specification.

[3]An interesting notion, however, might involve side payments to a creditor from a defaulting agent lacking liquidity but not solvency in order to protect reputation and, specifically, access to central bank facilities which would be unavailable to a defaulting agent.

4. Floor system: payment of interest on reserves

Consider now an economy where upon presenting d units of the current vintage of cash at the end of the goods market, the central bank returns $R(d) = \frac{d}{\rho_E}$ units of the new vintage. As it eases calculations, this is written as a discount, but can equivalently be expressed as gross interest $1 + r_E = 1/\rho_E$. We now proceed to solve for value functions and choice variables. Sellers have trivial choices, so we derive these first. We then consider Buyers and Autarks.

4.1. **Sellers.** The value of sellers in the goods' market is

$$W^S = \max_{\{d,h\}} -h + \beta V^S(d)$$
$$\text{s.t. } d = ph$$

where $V^S(d) = \phi' R(d) = \phi' \frac{d}{\rho_E}$. Thus

$$W^S = \max_h -h + \frac{\beta\phi'}{\rho_E} ph$$

the solution satisfies:

$$h \in \begin{cases} 0 & \text{if } \frac{p\beta\phi'}{\rho_E} < 1, \\ [0,\infty] & \text{if } \frac{p\beta\phi'}{\rho_E} = 1, \\ \infty & \text{if } \frac{p\beta\phi'}{\rho_E} > 1. \end{cases} \tag{1}$$

We then have the following result:

Lemma 1. For an equilibrium with positive output in the goods market, the following must hold: $p\beta\phi^+/\rho_E = 1$.

4.2. **Buyers.** Buyers' decision problem in the settlement market is

$$V^i(\hat{m}, l, d) = \max_{(X,H,m^+)} X - H + W^i(m^+)$$

$$\text{s.t.} \quad X + \phi m^+ = \phi\hat{m} + H - \phi l/\rho_l + \phi R(d) - \phi\tau M$$

Buyers' value of entering the money and goods market is $W^B(m) = \mathbb{E}_\epsilon W^B(m,\epsilon)$ where:

$$W^B(m,\epsilon) = \max_{(q_\epsilon, l_\epsilon, d_\epsilon)} \epsilon u(q_\epsilon) + \beta V^B(\hat{m}', l_\epsilon, d_\epsilon)$$

$$\text{s.t.} \quad m + l_\epsilon - pq_\epsilon - d_\epsilon - \hat{m}' \geq 0, \quad \text{and} \quad \rho_l B - l_\epsilon \geq 0.$$

The first order conditions to this problem are the same as in the economy, characterized in section 6.3, except that $T = 0$ and that $R(d) = \frac{1}{\rho_E}$. The following lemma characterizes buyer's optimal choices.

Lemma 2. Suppose $\rho_L < \rho_E$. Then no buyer deposits at the central bank and there existo numbers ϵ_L, ϵ_B such that the following holds: Buyers with $\epsilon < \epsilon_L$ lend. Buyers with $\epsilon \in (\epsilon_L, \epsilon_B)$ borrow but are not constrained. Buyers with $\epsilon > \epsilon_B$ borrow and are borrowing constrained. These values solve:

$$\epsilon_L = \frac{\beta\phi' m}{\rho_l} \quad \epsilon_B = \frac{\beta\phi'}{\rho_l}(m + \rho_l B)$$

Proof.

4.2.1. *Region $\epsilon \leq \epsilon_L$.* As buyers in this region are lenders, $\lambda_B = 0$, $l_\epsilon < 0$ and $d_\epsilon = 0$ since $\rho_L < \rho_E$. Also, assuming away the Friedman rule (i.e. $\gamma\rho_T > \beta$) then $\hat{m}' = 0$ for all buyers. The first order conditions for q_ϵ and l_ϵ are:

$$pq_\epsilon\beta\phi'\frac{1}{\rho_L} = \epsilon \quad \Rightarrow q_\epsilon = \frac{\epsilon\rho_l}{\rho_E}$$

$$\beta\phi'(-\frac{1}{\rho_l} + \lambda_\epsilon - \lambda_B) = 0$$

Thus $\lambda_\epsilon = \frac{1}{\rho_L}$ and from the budget constraint $l_\epsilon = pq_\epsilon - m = \frac{\epsilon\rho_l}{\beta\phi'} - m$. The marginal buyer who lends in the money market is one with sufficiently large shock that he chooses $l_\epsilon = 0$. The budget constraint, using $l_\epsilon = 0$, is $pq_\epsilon = m$ and the first order condition for q_ϵ then becomes $m\beta\phi'\frac{1}{\rho_L} = \epsilon$ which then defines $\epsilon_L = \frac{m\beta\phi'}{\rho_l}$.

4.2.2. *Region $\epsilon_L < \epsilon \leq \epsilon_B$.* Buyers in this region borrow but are not borrowing constrained. Thus $\lambda_B = 0$, $l_\epsilon > 0$ and $\lambda_\epsilon = \frac{1}{\rho_l}$. The first order condition for q_ϵ is $\epsilon = pq_\epsilon\beta\phi'\lambda_\epsilon$, which yields $q_\epsilon = \frac{\rho_l\epsilon}{\rho_E}$, where we substituted out $p\beta\phi' = \rho_E$. The budget constraint, $m + l_\epsilon = pq_\epsilon$, then yields $l_\epsilon = \frac{\rho_l\epsilon}{\beta\phi'} - m$. The marginal buyer who is not borrowing constrained is one with sufficiently large shock that $l_\epsilon = \rho_l B$. Thus the budget constraint is $pq_\epsilon = m + \rho_l B$ and the first order condition for q_ϵ defines the shock of this marginal buyer: $\epsilon_B = \frac{\beta\phi'(m+\rho_l B)}{\rho_l}$.

4.2.3. *Region $\epsilon > \epsilon_B$.* Buyers in this region are borrowing constrained, thus $\lambda_B > 0$, $l_\epsilon = \rho_l B$. The budget constraint implies $pq_\epsilon = m + \rho_l B$, which we can rearrange using $p\beta\phi' = \rho_E$ as $q_\epsilon = \frac{(m+\rho_l B)\beta\phi'}{\rho_E}$, and the first order condition for q_ϵ yields $\lambda_\epsilon = \frac{\epsilon}{\beta\phi'(m+\rho_l B)} = \frac{\epsilon}{\epsilon_B\rho_l}$.

□

We summarize the results from the previous lemma in the following table, where we also use $p\beta\phi' = \rho_E$ and the definition of ϵ_B.

$q_\epsilon =$	$l_\epsilon =$	$d_\epsilon =$	$\lambda_\epsilon =$	
$\frac{\epsilon\rho_l}{\rho_E}$	$\frac{\epsilon\rho_l}{\beta\phi'} - m$	0	$\frac{1}{\rho_l}$	if $\epsilon < \epsilon_L$
$\frac{\rho_l\epsilon}{\rho_E}$	$\frac{\rho_l\epsilon}{\beta\phi'} - m$	0	$\frac{1}{\rho_l}$	if $\epsilon_L < \epsilon < \epsilon_B$
$\frac{\rho_l\epsilon_B}{\rho_E}$	$\rho_l B$	0	$\frac{\epsilon}{\epsilon_B\rho_l}$	if $\epsilon_B < \epsilon$.

4.3. **Autarky.** The value of buyers in autarky in the goods' market is

$$W^A(m,\epsilon) = \max_{\{q_\epsilon, d_\epsilon\}} \epsilon u(q_\epsilon) + \beta V^A(\hat{m}'d_\epsilon)$$

$$\text{s.t.} \quad m - pq_\epsilon - d_\epsilon - \hat{m}' \geq 0, \quad \text{and} \quad \geq 0.$$

Let λ_ϵ^A denote the Lagrange multiplier on the budget constraint. Then the first order conditions are

$$-\lambda_\epsilon^A + \frac{1}{\rho_E} \leq 0$$

$$\epsilon u'(q_\epsilon) - p\beta\phi'\lambda_\epsilon^A \leq 0$$

The following lemma characterizes the optimal choices of buyers in autarky.

Lemma 3. There exists $\epsilon_E^A \in \mathbb{R}_+$ such that the following holds: Autarkic Buyers with $\epsilon < \epsilon_L$ deposit at the central bank. Autarkic Buyers with $\epsilon \in (\epsilon_L, \epsilon_B)$ do not deposit and consume all their money. This value solves: $\frac{\epsilon}{\beta\phi' m^A}$.

Proof.

4.3.1. *Region $\epsilon < \epsilon_E^A$.* In this region $d_\epsilon^A > 0$, implying $\lambda_\epsilon = \frac{1}{\rho_E}$, which, combined with the first order condition for q_ϵ yields $pq_\epsilon^A = \frac{\epsilon\rho_E}{\beta\phi'}$ and, using $p\beta\phi' = \rho_E$, $q_\epsilon^A = \epsilon$. The budget constraint then implies $d_\epsilon^A = m^A - p\epsilon$. The marginal agent who deposits at the central bank is one with a sufficiently large shock to choose $d_\epsilon^A = 0$. In this case the budget constraint implies $pq_\epsilon^A = m$, and the first order condition for q_ϵ implies $\epsilon_E^A = \frac{\beta\phi' m^A}{\rho_E}$

4.3.2. *Region $\epsilon > \epsilon_E^A$.* In this region $d_\epsilon^A = 0$. Then the budget constraint implies $q_\epsilon^A = \frac{m}{p} = \frac{\beta\phi' m^A}{\rho_E} = \epsilon_E^A$, and the first order condition for q_ϵ implies $\lambda_\epsilon^A = \frac{\epsilon}{p\beta\phi' q_\epsilon^A} = \frac{\epsilon}{\beta\phi' m^A}$. □

4.4. **Equilibrium conditions.** Consider first the first order condition for money holdings for buyers and autarkic buyers respectively: $-\phi + \frac{\partial W^i}{\partial m^+} = 0$ that can be rearranged as $\frac{\gamma}{\beta} = \mathbb{E}_\epsilon \lambda_\epsilon^i$.

For buyers we have then

$$\frac{\gamma}{\beta} = \int_0^{\epsilon_B} \frac{1}{\rho_l} dF(\epsilon) + \int_{\epsilon_B}^{\infty} \frac{\epsilon}{\beta\phi'(m + \rho_l B')} dF(\epsilon)$$

that can be simplified to

$$\frac{\gamma}{\beta} = \frac{1}{\rho_l}\Big[F(\epsilon_B) + \int_{\epsilon_B}^{\infty} \frac{\epsilon}{\epsilon_B} dF(\epsilon)\Big]$$

where

$$\epsilon_B = \frac{\beta\phi'(m + \rho_l B')}{\rho_l} = \frac{\beta}{\gamma\rho_l}\phi m + \beta\phi' B'$$

For autarkic buyers we have

$$\frac{\gamma}{\beta} = \int_0^{\epsilon_E^A} \frac{1}{\rho_E} dF(\epsilon) + \int_{\epsilon_E^A}^{\infty} \frac{\epsilon}{\beta\phi' m^A} dF(\epsilon)$$

$$\frac{\gamma}{\beta} = \frac{1}{\rho_E}\Big[F(\epsilon_E^A) + \int_{\epsilon_E^A}^{\infty} \frac{\epsilon}{\epsilon_E^A} dF(\epsilon)\Big]$$

4.4.1. *Money market.* Lenders here are only buyers with $\epsilon < \epsilon_L$, thus

$$S(\rho_l) = \int_0^{\epsilon_L} \Big(m - \frac{\epsilon\rho_l}{\beta\phi'}\Big) dF(\epsilon)$$

Analogously, borrowers are all the buyers with shocks larger than ϵ_L. Therefore

$$D(\rho_l) = \int_{\epsilon_L}^{\epsilon_B} \Big(\frac{\rho_l\epsilon}{\beta\phi'} - m\Big) dF(\epsilon) + \int_{\epsilon_B}^{\infty} \rho_l B' dF(\epsilon)$$

Thus the market clearing condition is

$$0 = \int_0^{\epsilon_B} \Big(\frac{\rho_l\epsilon}{\beta\phi'} - m\Big) dF(\epsilon) + \rho_l B'(1 - F(\epsilon_B))$$

multiplying by ϕ', the market clearing condition is

$$0 = \int_0^{\epsilon_B} \Big(\frac{\rho_l}{\beta}\epsilon - \phi' m\Big) dF(\epsilon) + \rho_l \phi' B'(1 - F(\epsilon_B))$$

which we can rearrange as

$$0 = \int_0^{\epsilon_B} \Big(\frac{\rho_l}{\beta}\epsilon - \frac{\phi m}{\gamma}\Big) dF(\epsilon) + \rho_l \phi' B'(1 - F(\epsilon_B))$$

4.4.2. *Borrowing limit.* A buyer pays his loan if $-\phi\frac{l_\epsilon}{\rho_l} + V^B > V^A$, where V^B, V^A denote the value of a buyer and an autarkic buyer at the settlement market after repayment of previous period's debt and before acquiring money holdings to carry into the money and goods' market. Using the definition of agents' values in the settlement market we then have

$$\begin{aligned}(2)\qquad \Big(V^B - V^A\Big) &= \phi(m^{A+} - m^+) + \mathbb{E}_\epsilon\Big[\epsilon u(q_\epsilon) - \epsilon u(q_\epsilon^A)\Big] \\ &\quad + \beta\phi'\mathbb{E}_\epsilon\Big[-\frac{l_\epsilon}{\rho_l} + R(d_\epsilon) - R(d_\epsilon^A)\Big] + \beta\Big(V^{B'} - V^{A'}\Big)\end{aligned}$$

Notice that the money market clearing condition implies that $\mathbb{E}_\epsilon l_\epsilon = 0$. Notice also that lemma 2 implies that $d_\epsilon = 0, \forall \epsilon$. Therefore, substituting $\phi B = V^B - V^A$ yields:[4]

$$(3)\qquad \phi B = \phi(m^{A+} - m^+) + \mathbb{E}_\epsilon\Big[\epsilon u(q_\epsilon) - \epsilon u(q_\epsilon^A) - \beta\phi' R(d_\epsilon^A) + \beta\phi' B'\Big]$$

Therefore, using the characterization of the solution for autarkic buyers in lemma 3, we have:

$$\begin{aligned}\mathbb{E}_\epsilon - \beta\phi' R(d_\epsilon^A) &= -\frac{\beta\phi'}{\rho_E}\int_0^{\epsilon_E^A}(m^A - p\epsilon)dF(\epsilon) \\ &= \int_0^{\epsilon_E^A} -\frac{\beta\phi'}{\rho_E\phi}\phi m^A dF(\epsilon) + \int_0^{\epsilon_E^A}\frac{\beta\phi' p\epsilon}{\rho_E}dF(\epsilon) \\ &= -\frac{\beta}{\gamma\rho_E}\phi m^A F(\epsilon_E^A) + \int_0^{\epsilon_E^A}\epsilon dF(\epsilon)\end{aligned}$$

Thus the endogenous borrowing limit is

$$(4)\phi B = \phi(m^{A+} - m^+) + \mathbb{E}_\epsilon\Big[\epsilon u(q_\epsilon) - \epsilon u(q_\epsilon^A) - \frac{\beta}{\gamma\rho_E}\phi m^A F(\epsilon_E^A) + \int_0^{\epsilon_E^A}\epsilon dF(\epsilon)\Big] + \beta\phi' B'$$

4.4.3. *Goods' market and Money supply.* In this equilibrium with $\rho_l < \rho_E$ only sellers deposit: $d = ph$ where

$$ph = \frac{\rho_E}{\beta\phi'}\int_0^\infty q_\epsilon dF(\epsilon)$$

Substituting out from the buyers' decision problem we have

$$ph = \frac{\rho_E}{\beta\phi'}\Big\{\int_0^{\epsilon_L}\frac{\rho_l\epsilon}{p\beta\phi'}dF(\epsilon) + \int_{\epsilon_L}^{\epsilon_B}\frac{\rho_l\epsilon}{p\beta\phi'}dF(\epsilon) + \int_{\epsilon_B}^\infty\frac{(m + \rho_l B)}{p}dF(\epsilon)\Big\}$$

[4]Even in the corner case of $\rho_l = \rho_E$ the equation for the borrowing limit is the same as this one, since the market clearing condition for loans would be $\mathbb{E}_\epsilon[-\frac{l_\epsilon}{\rho_l} + R(d_\epsilon)] = 0$, since buyers with $\epsilon < \epsilon_L$ are now indifferent between depositing and lending.

which we can rearrange using $p\beta\phi' = \rho_E$ and the definition of ϵ_B, as:

$$ph = \frac{\rho_E}{\beta\phi'}\Big\{\frac{\rho_l}{\rho_E}\int_0^{\epsilon_B}\epsilon dF(\epsilon) + \frac{\beta\phi'}{\rho_E}\int_{\epsilon_B}^{\infty}\frac{\rho_l\epsilon_B}{\beta\phi'}dF(\epsilon)\Big\} \tag{5}$$

The resulting money supply satisfies:

$$M^+ = M + (\frac{1}{\rho_E} - 1)\Big(\int_0^\infty d_\epsilon + d^S\Big)$$

where the term d^S denote sellers' deposit at the central bank. Lemma 2 implies that $\int_0^\infty d_\epsilon = 0$, while $d^S = ph$ can be substituted out form equation (5).

4.5. **Equilibrium under a Floor System: $\rho_E = \rho_l$.** Consider the model described in the previous section and assume that the central bank remunerates reserves at the same rate as that on interbank loans in the money market.

Rewrite the value of a buyer after earning interest on previous period's deposits and having paid previous period's loan, as

$$V_0^B = -\phi m^+ + \int_0^\infty \epsilon u(q_\epsilon) + \beta\phi'\Big(R(d_\epsilon) - \frac{l_\epsilon}{\rho_l}\Big)dF(\epsilon) + \beta V_0^{B'}$$

Similarly, for autarks:

$$V_0^A = -\phi m^{A+} + \int_0^\infty \epsilon u(q_\epsilon^A) + \beta\phi' R(d_\epsilon^A)dF(\epsilon) + \beta V_0^{A'}$$

Using the result that $q_\epsilon = q_\epsilon^A$, and substituting equilibrium quantities from Lemma 2 and $\rho_l = \rho_E$, we have that

$$\begin{aligned} V_0^B - V_0^A &= \phi(m^{A+} - m^+) + \frac{\beta\phi'}{\rho_E}\Big\{\int_0^{\epsilon_L}(m - \frac{\epsilon\rho_E}{\beta\phi'})dF(\epsilon) - \int_{\epsilon_L}^{\epsilon_B}(\frac{\epsilon\rho_E}{\beta\phi'} - m)dF(\epsilon) \\ &\quad - \int_{\epsilon_B}^\infty \rho_E B' dF(\epsilon) - \int_0^{\epsilon_E^A} R(d_\epsilon^A)dF(\epsilon)\Big\} + \beta(V_0^{B'} - V_0^{A'}) \end{aligned}$$

where $\phi B = V_0^B - V_0^A$ and $\phi' B' = V_0^{B'} - V_0^{A'}$. Notice that $\epsilon_E^A = \epsilon_B$. [5] From $\epsilon_E^A = \epsilon_B$ it follows that $m^{A+} = m^+ + \rho_E B'$. Moreover, substituting d_ϵ^A from Lemma 2 yields:

$$\phi B = \phi\rho_E B' + \frac{\beta\phi'}{\rho_E}\Big\{\int_0^{\epsilon_L}(m^+ - m^{A+})dF(\epsilon) + \int_{\epsilon_L}^{\epsilon_B}\Big(-\frac{\rho_E\epsilon}{\beta\phi'} + m^+ - m^{A+} + \frac{\rho_E\epsilon}{\beta\phi'}\Big)dF(\epsilon) - \int_{\epsilon_B}^\infty \rho_E B'\Big\} + \beta\phi' B'$$

which, using $m^{A+} = m^+ + \rho_E B'$, can be simplified to

$$\begin{aligned} \phi B &= \phi\rho_E B' - \frac{\beta\phi'}{\rho_E}\rho_E B' + \beta\phi' B' \\ &= \gamma\rho_E\phi' B' \end{aligned}$$

In any steady state equilibrium (i.e. all real variables are stationary, including $\phi B = \phi' B'$) we then have:

$$\phi B(1 - \gamma\rho_E) = 0.$$

[5]This follows from the first order condition for money holdings for buyers and autarks. Combining them, and substituting $\rho_l = \rho_E$, yields

$$F(\epsilon_B) + \int_{\epsilon_B}^\infty \frac{\epsilon}{\epsilon_B}dF(\epsilon) = F(\epsilon_E^A) + \int_{\epsilon_E^A}^\infty \frac{\epsilon}{\epsilon_E^A}dF(\epsilon).$$

Therefore, the equilibrium borrowing limit is $B = 0$ and there is no borrowing and lending in the money market.

Rewrite intuition here Intuitively, when the central banks remunerates reserves at a rate $\rho_E = \rho_l$, an autarkic buyer can do as well as a buyer who has access to the money market because the reward for carrying money is the same as the opportunity cost of borrowing (i.e. $\frac{1}{\rho_E}$). This implies that the real value of the liquidity brought in the goods' market by buyers (i.e. $\phi m + \rho_l \phi' B'$) is the same as that brought by autarks (i.e. ϕm^A). As a consequence, the consumption allocation of buyers and autarks is the same for each ϵ. Furthermore, notice that the market clearing condition for loans requires zero excess demand at $\rho_l = \rho_E$. As some lenders may start depositing at the central bank their unspent money rather than lending it, an example in which $B = 0$ occurs if all lenders switch from participating to the loan market to depositing at the central bank. Then, with no lenders, the demand for loan must equal zero ($B = 0$) for the equilibrium interest rate to satisfy $\rho_l = \rho_E$. In terms of endogenous variables, we can rewrite it as $\int_0^\infty d_\epsilon dF(\epsilon) = \int_0^\infty l_\epsilon dF(\epsilon)$.

Thus, the only difference between the value of a buyer and that of an autark must arise from 1) buyers saving on the cost of acquiring money in the settlement market; and 2) the difference in the central bank's remuneration of reserves. However, because autarks' money holdings equal the entire buyers' portfolio of money holdings plus loans (i.e. $m^A = m + \rho_l B'$), then both 1) the difference in labor cost in the settlement market and 2) the difference in reserves' remuneration are simply a function of the loans. This causes the net value of having access to the money market, $V^B - V^A$, which caps the size of the loans, to depend exclusively on the future value of having access to the money market, $V^{B'} - V^{A'}$, that is the cap on the size of future loans. In a stationary equilibrium, however, this is feasibly if and only if there is no difference between the value of having access to the money market and not having such access.

In other words, when the central bank remunerates reserves at the money market rate, the option of trading with money is remunerated too highly for buyers to value their good credit standing. As a result, it is not feasible to have positive borrowing limit.

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5. Voluntary Reserve Targets

This section considers an alternate central bank deposit facility termed "Voluntary Reserve Targets" which asks agents to report a targeted deposit level in the morning and pays a high rate on balances up to the target, a low rate on balances over the target, and charges a fee for shortages relative to the target. This differentially affects buyers who can borrow and those who can not because the former have the ability to adjust their balances through borrowing and lending, so hit their targets more precisely.

5.1. **Central Bank Deposit Facility under VRT.** The central bank offers a deposit facility to all agents. At the beginning of the settlement market all agents report a target to the CB. Given a target, T, the deposit facility offers remuneration according to a three part schedule. Balances up to the target earn a rate r_T, balances in excess of the target earn a rate r_E, and shortages are charged a penalty r_P. We assume that deposits are voluntary and can be costlessly withdrawn as cash. Hence, if the central bank payed negative rates, the deposit facility would not be used. Hence, we assume $r_i \geq 0$ for $i \in \{E, T, P\}$. Moreover, assume $r_T \geq r_E$ and $r_T \geq r_P$. One can write this as

$$\tilde{R}(d|T) = \begin{cases} (1+r_T)d - r_p(T-d) & \text{if } T > d \\ (1+r_T)T + (1+r_E)(d-T) & \text{if } d \geq T \end{cases}$$

If one writes $\rho_P = (1+r_T+r_P)^{-1}$, $\rho_T = (1+r_T)^{-1}$, and $\rho_E = (1+r_E)^{-1}$, then this schedule can be written in a form that will prove more convenient:[6] $\tilde{R}(d|T) = T/\rho_T + R(d-T)$ where

$$R(d-T) = \begin{cases} (d-T)/\rho_P & \text{if } d-T < 0 \\ (d-T)/\rho_E & \text{if } d-T \geq 0. \end{cases}$$

6. Household Decisions

6.1. **Settlement Market.** Agents enter the settlement market with outstanding loan obligations l (so that $l > 0$ indicates borrowings and $l < 0$ indicates lending), money m, and CB deposits d with last period's target T. They choose labor effort h, consumption x, new balances m^+, and arrange a new target with the CB T^+. Write V^i for the value of entering this market for agents $i \in \{B, S, A\}$. As we assume that the deposit facility pays positive marginal rates, no agents will carry money into the settlement market, preferring CB deposits. Hence, we do not include m in the arguments of V. The problem differs between buyers, autarkic buyers, and sellers in this market only in their continuation value going into the money and goods market which we write as W^i.

$$V^i(\hat{m}, l, d|T) = \max_{(X,H,m^+,T^+)} X - H + W^i(m^+|T^+)$$

$$\text{s.t.} \quad X + \phi m^+ = \phi \hat{m} + H - \phi l/\rho_l + \phi(T/\rho_T + R(d-T)) - \phi \tau M$$

where $\hat{m}$ denotes, money holdings brought into the settlement market from the previous goods' market. In equilibrium, in economies on which we will focus (indexed by restrictions on parameters $\rho_E > \rho_l > \rho_P$), there will be no money holdings brought into the settlement market from the previous goods' market as any left over money at the end of the goods' market will be deposited at the central bank. In general, however, buyers could bring money into the next settlement market.

Substituting for $X - H$ gives

$$V^i(\hat{m}, l, d|T) = \phi\left(\hat{m} + T/\rho_T + R(d-T) - l/\rho_l - \tau M\right) + \max_{(m^+,T^+)} \left\{-\phi m^+ + W^i(m^+|T^+)\right\}.$$

The first order conditions with respect to m^+ and T^+, respectively, give

$$W^i_{m^+}(m^+|T^+) \leq \phi \quad (= \text{ if } m^+ > 0) \quad \text{and} \quad W^i_{T^+}(m^+|T^+) \leq 0 \quad (= \text{ if } T^+ > 0). \tag{6}$$

Linear disutility of labor makes $-\phi$ the marginal cost of balances which must equal the marginal value of carrying those balances into the next market. Agents are free to choose any positive target.

[6]Notice that $\rho_E < 1$ is a feasible policy. This relates to our discussion of negative interest rates with an exemption threshold that can be analyzed within this framework (thus connecting the framework with some of the actual frameworks currently adopted by central banks.

Envelope conditions give

$$V^i_{\hat{m}} = \phi; \quad V^i_l = -\phi/\rho_l; \quad V^i_d = \begin{cases} \phi/\rho_P & \text{if } d < T, \\ \phi/\rho_E & \text{if } d > T; \end{cases} \text{ and } V^i_T = \phi/\rho_T - \begin{cases} \phi/\rho_P & \text{if } d < T, \\ \phi/\rho_E & \text{if } d > T. \end{cases} \tag{7}$$

6.2. **Money and Goods Markets.** While we imagine the money and goods market to be separate, for the purposes of characterization they can be consolidated without loss of generality as sellers do not have use of money outside of the settlement market, so do not interact with the money market. Further, sellers and buyers both deposit all receipts in the CB deposit facility.

6.2.1. *Sellers.* Here, we characterize the seller's problem. As Sellers do not participate in the money market, they carry no loans, $l = 0$, so we drop this argument from V^S. [7]

$$W^S(m|T) = \max_h -h + \beta V^S(d|T) \quad \text{s.t.} \quad d = ph.$$

Given the envelope on V above, we get the following supply curve:

$$h \in \begin{cases} \infty & \text{if } p > \rho_E/\beta\phi^+, \\ [T/p, \infty] & \text{if } p = \rho_E/\beta\phi^+, \\ T/p & \text{if } \rho_P/\beta\phi^+ < p < \rho_E/\beta\phi^+, \\ [0, T/p] & \text{if } p = \rho_P/\beta\phi^+, \text{ and} \\ 0 & \text{if } p < \rho_P/\beta\phi^+. \end{cases} \tag{8}$$

The maximized value is

$$\begin{cases} \infty & \text{if } p > \rho_E/\beta\phi^+, \\ T\left[\beta\phi^+/\rho_T - 1/p\right] & \text{if } \rho_P/\beta\phi^+ < p \le \rho_E/\beta\phi^+, \\ -T\beta\phi^+ r_P & \text{if } p \le \rho_P/\beta\phi^+. \end{cases}$$

This gives our first result:

Lemma 4. For a positive and finite target choice by sellers, the following must hold:

$$p\beta\phi^+/\rho_T = 1, \text{ and } T^s = pD(p)$$

where $D(p)$ is the demand for goods from buyers and T^s is the choice of sellers' target.

[7] That sellers do not participate in the money market is a result, which follows from the comparison of their first order condition for m:

$$-\phi + \beta\phi^+(1 + i_l) \le 0$$

with that of a buyer:

$$-\phi + \beta\phi^+\mathbb{E}_\epsilon W^B_m(m|T, \epsilon) = 0$$

where $W^B_m(m|T, \epsilon)$ denotes the partial derivative of $W^B(m|T, \epsilon)$ with respect to m. Intuitively, this says that buyers have a higher expected marginal utility of money, as they might be borrowing constrained in the money market, and, because of that, the price of money is higher in the settlement market than what sellers would be willing to pay just to lend out the funds in the money market.

6.3. **Buyers.** After the settlement market, buyers draw ϵ. Hence, write $W^B(m|T,\epsilon)$ for the value conditional on ϵ, so that $W^B(m|T) = \mathbb{E}_\epsilon\left[W^B(m|T,\epsilon)\right]$. Buyers borrow or lend funds which they use to buy goods or make deposits at the CB. Borrowers are subject to a repayment constraint B which will be endogenized later. Write r_l as the market rate on loans and $\rho_l = (1+r_l)^{-1}$. The value of entering the money and goods market is

$$W^B(m|T,\epsilon) = \max_{(q_\epsilon, l_\epsilon, d_\epsilon)} \epsilon u(q_\epsilon) + \beta V^B(\hat{m}', l_\epsilon, d_\epsilon|T)$$

$$\text{s.t.} \quad m + l_\epsilon - pq_\epsilon - d_\epsilon - \hat{m}' \geq 0, \quad \text{and} \quad \rho_l B - l_\epsilon \geq 0.$$

where $\hat{m}'$ denotes money holdings carried over to the next settlement market rather than spend on the goods' market or deposited at the central bank. Write $\beta\phi^+\lambda_\epsilon$ and $\beta\phi^+\lambda_B$ as the constraints on the budget and repayment constraints, respectively. The first order conditions with respect to $\hat{m}'$, q_ϵ, d_ϵ, and l_ϵ are, respectively,[8]

$$-\beta\phi\lambda_\epsilon + \beta V^B_{\hat{m}'}(\hat{m}', l_\epsilon, d_\epsilon|T) \leq 0 \tag{9}$$

$$\epsilon u'(q_\epsilon) - p\beta\phi^+\lambda_\epsilon = 0, \tag{10}$$

$$\beta V^B_{d_\epsilon}(\hat{m}', l_\epsilon, d_\epsilon|T) - \beta\phi'\lambda_\epsilon \leq 0 \tag{11}$$

with "=" if $d_\epsilon > 0$.

$$\beta V^B_{l_\epsilon}(\hat{m}', l_\epsilon, d_\epsilon|T) + \beta\phi'\lambda_\epsilon - \beta\phi'\lambda_B = 0 \tag{12}$$

with "=" if $l_\epsilon > 0$. Notice that $l_\epsilon < 0$ denotes lending.

Using $V^B_{d_\epsilon}(\hat{m}', l_\epsilon, d_\epsilon|T) = \phi' R'_d$ and $V^B_{l_\epsilon}(\hat{m}', l_\epsilon, d_\epsilon|T) = -\frac{\phi}{\rho_l}$, the first order condition for d_ϵ can be rearranged as:

$$0 \geq -\lambda_\epsilon + \begin{cases} 1/\rho_P & \text{if } 0 < d < T \\ 1/\rho_E & \text{if } d > T \end{cases} \tag{13}$$

and the first order condition for l_ϵ can be rearranged as:

$$-1/\rho_l + \lambda_\epsilon - \lambda_B = 0 \tag{14}$$

The first order condition for $\hat{m}'$ can be simplified to

$$\beta\phi'(1-\lambda_\epsilon) \leq 0$$

with "=" if $\hat{m}' > 0$. Thus $\lambda_\epsilon > 1$ is a necessary and sufficient condition for $\hat{m}' = 0$.

[8] These follow from rewriting the Lagrangian as:

$$L(z) = \epsilon u(q_\epsilon) + \beta V^B(\hat{m}', l_\epsilon, d_\epsilon|T) + \beta\phi'\lambda_\epsilon(m + l_\epsilon - d_\epsilon - pq_\epsilon - \hat{m}') + \beta\phi'\lambda_B(\rho_l B - l_\epsilon) + \beta\phi'\lambda_d d_\epsilon + \beta\phi'\lambda_q q_\epsilon$$

where $z \in \mathbb{R}^4$, $z = (\hat{m}', q_\epsilon, l_\epsilon, d_\epsilon)$ with $\hat{m}'$ denoting money holdings not spent or deposited but carried over to the next settlement market, so that the budget constraint in the goods market is $pq_\epsilon + d_\epsilon + \hat{m}' \leq m + l_\epsilon$. Also, $\beta\phi'\lambda_B$ and $\beta\phi'\lambda_\epsilon$ denotes the Lagrange multiplier on the borrowing constraint and the budget constraint in the goods' market, and $\beta\phi'\lambda_d$, $\beta\phi'\lambda_q$ denote the Lagrange multipliers on non negativity constraints on d_ϵ and q_ϵ respectively.

Lemma 5. Suppose $\rho_E > \rho_l > \rho_P$. There exist numbers $\epsilon_l < \epsilon_B < \epsilon_T < \epsilon_M$ such that the following hold: Buyers with $\epsilon < \epsilon_l$ lend. Buyers with $\epsilon_l < \epsilon < \epsilon_B$ borrow and fully fund their targets. Buyers with $\epsilon_B < \epsilon < \epsilon_T$ are borrowing constrained and fully fund their target. Buyers with $\epsilon_T < \epsilon < \epsilon_M$ are borrowing constrained and partially fund their target. Buyers with $\epsilon_M < \epsilon$ are borrowing constrained, and make no deposits. These values solve

$$\epsilon_l = \beta\phi^+(m-T)/\rho_l, \quad \epsilon_B = \beta\phi^+(m-T)/\rho_l + \beta\phi^+ B,$$
$$\epsilon_T = \epsilon_B\rho_l/\rho_P = \beta\phi^+(m-T)/\rho_P + \beta\phi^+ B\rho_l/\rho_P, \quad \text{and} \quad \epsilon_M = \beta\phi^+ m/\rho_P + \beta\phi^+ B\rho_l/\rho_P.$$

Proof. See Appendix A. □

The result derives from the assumption on the ordering of prices: $\rho_P < \rho_l < \rho_E$. For low levels of ϵ, a buyer fills their target and borrows or lends. Because of the gap between ρ_l and ρ_P, a region of buyers consume a constant amount until a critical threshold is reached such that consumption is more valuable than funding the target. This is followed by a region where targets are partially funded and then a final region where the deposit facility is not used, and a borrowing constrained buyer spends all its funds on consumption.

6.4. **Autarky.** Buyers can keep records amongst themselves in the settlement and money markets. Assume a defaulting agent, or anyone lending to a defaulting agent, is barred from the money market in the future. The central bank is assumed, however, not to be party to the borrowers' records. A bank's business depends on its reputation. Public default or bankruptcy is disastrous. We assume banks share information with each other, but not generally. The notion is that a defaulting bank would eventually make their lender whole, and that this technical default is not broadly disclosed. Buyers in autarky effectively face $\rho_l = 0$ and $B = 0$. This alters behavior in the money and goods market. As above, write $W^A(m|T) = \mathbb{E}_\epsilon\left[W^A(m|T,\epsilon)\right]$.

$$W^A(m|T,\epsilon) = \max_{q^A_\epsilon, d^A_\epsilon} \epsilon u(q^A_\epsilon) + \beta V^A(d_\epsilon|T)$$
$$\text{s.t.} \quad m - pq^A_\epsilon - d^A_\epsilon \geq 0.$$

Writing $\beta\phi^+\lambda_A$ as the constraint on the budget, the first order conditions are as follows:

$$\epsilon u'(q_\epsilon) - p\beta\phi^+\lambda_\epsilon = 0,$$

$$0 \geq -\lambda_\epsilon + \begin{cases} 1/\rho_P & \text{if } 0 < d < T \\ 1/\rho_E & \text{if } d > T. \end{cases}$$

Lemma 6. There exist $\epsilon^A_E, \epsilon^A_P, \epsilon^A_M$ such that autarkic borrowers with $\epsilon < \epsilon^A_E$ deposit more than their target, those with $\epsilon^A_E < \epsilon < \epsilon^A_P$ just meet their target, those with $\epsilon^A_P < \epsilon < \epsilon^A_M$ partially meet their targets, and those with $\epsilon > \epsilon^A_M$ make no deposits. These thresholds satisfy

$$\epsilon^A_E = \frac{\beta\phi^+(m-T)}{\rho_E}, \quad \epsilon^A_P = \frac{\beta\phi^+(m-T)}{\rho_P}, \quad \epsilon^A_M = \frac{\beta\phi^+ m}{\rho_P}$$

Proof. See Appendix A. □

6.5. **Choice of Targets.** Targets will be interior only if $\rho_P < \rho_T < \rho_E$. If $\rho_T = \rho_E$, then there is no cost of exceeding one's target, so agents will set $T = 0$. Alternately, if $\rho_P = \rho_T$, there is no cost of falling below one's target, and agents set $T = \infty$. While we will return to these observations, suppose for now that the inequalities hold strictly. We proceed to solve for each type's targets in turn.

6.5.1. *Sellers.* Sellers face no uncertainty over their end of period deposits. For fixed deposits, the remuneration schedule is maximized by setting the target equal to deposits. This is the intuition behind Lemma 4 which ensures that $T^S = pD(p)$.

6.5.2. *Buyers.* From the settlement market problem, voluntary targets must satisfy $W_T^i = 0$. Because R is not differentiable in T, a simple envelope theorem will not hold. Alternatively, one can substitute the maximized values of q_ϵ, l_ϵ, and d_ϵ and differentiate this. The following lemma formalizes this result for both buyers and autarks

Lemma 7. A voluntary target for buyers, T^B, must satisfy

$$\frac{1}{\rho_T} = \frac{1}{\rho_l}\left(\int_0^{\epsilon_B} dF(\epsilon) + \int_{\epsilon_B}^{\epsilon_T}\left(\frac{\epsilon}{\epsilon_B}\right) dF(\epsilon)\right) + \frac{1}{\rho_P}\int_{\epsilon_T}^{\infty} dF(\epsilon) \tag{15}$$

which implies $\rho_l \in [\rho_P, \rho_T]$.

A voluntary target for autarkic buyers, T^A, must satisfy

$$\frac{1}{\rho_T} = \frac{1}{\rho_E}\left(\int_0^{\epsilon_E^A} dF(\epsilon) + \int_{\epsilon_E^A}^{\epsilon_P^A} \frac{\epsilon}{\epsilon_E^A} dF(\epsilon)\right) + \frac{1}{\rho_P}\int_{\epsilon_P^A}^{\infty} dF(\epsilon) \tag{16}$$

Proof. See Appendix A. □

7. VRT Equilibrium

Symmetric stationary equilibrium requires all agents maximize and all markets clear and that all real quantities are constant over time and across agents of the same type. Policy variables are τ, γ, r_T, r_E and r_P. Endogenous variables are the target choices T^B, T^S, and T^A; prices ρ_l and p; preference cutoffs $\epsilon_l, \epsilon_B, \epsilon_T, \epsilon_M, \epsilon_E^A, \epsilon_P^A$, and ϵ_T^A; along with the borrowing constraint B.

7.1. **Endogenizing the borrowing constraint.** Consider a buyer who repays his money market loan from the previous period. Because his repayment is publicly recorded then his continuation value at the end of the settlement market, before choosing target and money holdings, is $V^B(0,0,0|0)$, as defined in (45) in the appendix. To save on notation, let $V^B = V^B(0,0,0|0)$. Consider now a buyer who plans to default on his money market loan from the previous period. Analogously to a buyer, his continuation value in the settlement market, before choosing target and money holdings, is $V^A(0,0|0) = \max_{\{m^{A+},T^{A+}\}} -\phi m^{A+} + W^A(m^{A+}|T^{A+})$. To save on notation, let $V^A = V^A(0,0|0)$. The payoff to a buyer who repays in the settlement market is $-\phi\frac{l_\epsilon}{\rho_\epsilon} + V^B$, since the loan was in terms of money, while the payoff to a buyer who defaults is V^A. Therefore, the repayment constraint can be rewritten as

$$l_\epsilon \;\leq\; \frac{\rho_l}{\phi}(V^B - V^A) \tag{17}$$

which, as shown in Appendix A.5 can be rearranged as:

$$
\begin{aligned}
\phi B &= \phi(m^{A+} - m^{+}) + \mathbb{E}_\epsilon[\epsilon u(q_\epsilon) - \epsilon u(q_\epsilon^A)] \\
&\quad + \mathbb{E}_\epsilon \beta\phi'\Big\{(\frac{T^{+} - T^{A+}}{\rho_T}) - \frac{l_\epsilon}{\rho_l} + R(d_\epsilon - T^{+}) - R(d_\epsilon^A - T^{A+})\Big\} + \beta\phi' B'
\end{aligned} \tag{18}
$$

Notice that none of the autarky terms depend on B, while ρ_l, T^+ and the allocation of consumption and deposits do.

Definition. Given policy variables $\tau, \rho_P, \rho_T, \rho_E$, an equilibrium is:

- an allocation for sellers h, d and target choice T^S
- an allocation for buyers $m^+, \hat{m}', q_\epsilon, d_\epsilon, l_\epsilon$ and target choice T^+
- an allocation for autarkic buyers $m^{A+}, \hat{m}^{A'}, q_\epsilon^A, d_\epsilon^A$ and target choice T^{A+}
- prices p, ρ_l
- money growth rate $\gamma = \frac{M^+}{M}$
- value functions for buyers and autarkic buyers V^B, W^B and V^A, W^A
- endogenous borrowing constraint $\phi B = V^B - V^A$

such that: i) taking prices as given the allocations solve the respective agents' decision problem and targets solve (15) for buyers and (16) for autarkic buyers, ii) markets clear and iii) B satisfies (55).

Market clearing conditions are characterized below for each market.

7.2. **Goods Market.** Equilibrium in the goods market is given by a price p where

$$q^S = \int q_\epsilon dF(\epsilon).$$

Sellers' optimal targets, from Lemma 4, imply that $p = \rho_T/\beta\phi^+$ and that $T^S = p\int q_\epsilon dF(\epsilon)$.

7.3. **Money Market.** When entering the money market, only buyers with sufficiently low preference shock lend. All other buyers borrow, with buyers experiencing a sufficiently high shock being borrowing constrained. The following lemma characterizes the money market clearing condition and the money market rate: ρ_l.

Lemma 8. Supply of funds is given by

$$S(\rho_l) = \int_0^{\epsilon_L} \frac{\rho_l}{\rho_T} p(\epsilon - \epsilon_L) dF(\epsilon). \tag{19}$$

Demand for funds is given by

$$D(\rho_l) = \int_{\epsilon_l}^{\epsilon_B} \frac{\rho_l}{\rho_T} p(\epsilon - \epsilon_L) dF(\epsilon) + \int_{\epsilon_B}^{\infty} \rho_l B dF(\epsilon) \tag{20}$$

The money market interest rate satisfies:

$$0 = \frac{\rho_l \gamma}{\beta}\int_0^{\epsilon_B} \epsilon dF(\epsilon) - [\rho_l\phi B + \phi m^+ - \phi T^+]F(\epsilon_B) + \rho_l \phi B$$

Proof. See Appendix A. □

7.4. **Settlement Market.** Here we characterize the buyers' choice for settlement period balances. From the settlement period problem, we have the first order condition on money balances which guarantees that $\phi = W_M^B(m|T)$. $W_M^B(m|T)$ is just the expectation of $W_M^B(m|T,\epsilon)$ which is just $\beta\phi^+\lambda_\epsilon$. This final quantity can be derived from the FOC for q: $\epsilon u'(q) - p\beta\phi^+\lambda_\epsilon = 0$. Putting this all together gives[9]

$$W_M^B = \frac{\rho_T}{p}\left(\int_0^{\epsilon_B}\frac{1}{\rho_l} + \int_{\epsilon_B}^{\epsilon_T}\frac{1}{\rho_l}\frac{\epsilon}{\epsilon_B}dF(\epsilon) + \int_{\epsilon_T}^{\epsilon_M}\frac{1}{\rho_P}dF(\epsilon) + \int_{\epsilon_M}^{\infty}\frac{1}{\rho_P}\frac{\epsilon}{\epsilon_M}dF(\epsilon)\right).$$

Setting this equal to ϕ and given that $\rho_t/p = \beta\phi^+$ with $\phi/\phi^+ = \gamma$ we get

$$\frac{\gamma}{\beta} = \int_0^{\epsilon_B}\frac{1}{\rho_l} + \int_{\epsilon_B}^{\epsilon_T}\frac{1}{\rho_l}\frac{\epsilon}{\epsilon_B}dF(\epsilon) + \int_{\epsilon_T}^{\epsilon_M}\frac{1}{\rho_P}dF(\epsilon) + \int_{\epsilon_M}^{\infty}\frac{1}{\rho_P}\frac{\epsilon}{\epsilon_M}dF(\epsilon)$$

Similarly, the choice of money holdings for autarkic buyers satisfies $-\phi + \beta\phi'\mathbb{E}_\epsilon\lambda_\epsilon \leq 0$, that, for $m^{A+} > 0$, is

$$F(\epsilon_E^A)\frac{1}{\rho_E} + \int_{\epsilon_E^A}^{\epsilon_P^A}\frac{\epsilon}{\rho_E\epsilon_E^A}dF(\epsilon) + \int_{\epsilon_P^A}^{\epsilon_M^A}\frac{1}{\rho_P} + \int_{\epsilon_M^A}^{\infty}\frac{\epsilon}{\epsilon_E^A\rho_P} \quad = \quad \phi \tag{21}$$

8. Three types economy with Voluntary Targets

Maintain all the assumptions of the general model, but modify the distribution of buyers' types as follows: let $E = \{\epsilon_1, \epsilon_2, \epsilon_3\}$ denote the support of $\tilde{\epsilon}$, with $\epsilon_1 < \epsilon_2 < \epsilon_3$, and let $p_i = \Pr(\epsilon = \epsilon_i)$, $i = 1, 2, 3$. The law of large numbers implies that p_i also denotes the measure of buyers who experience shock ϵ_i.

8.1. **Sellers.** Sellers are exactly the same as in the full model, so all the results derived for the full model still apply. In particular, in equilibrium $p\beta\phi' = \rho_T$.

8.2. **Buyers.** Assuming, as in the full model, that $u(c) = \log(c)$, and assuming for now an interior choice of target (i.e. $T > 0, T^A > 0$), the first order conditions to the buyers' problem imply the characterization below. Consistently with our findings from the full model, we construct an equilibrium where ϵ_1 agents lend and deposit reserves at the central bank up to the target; ϵ_2 agents borrow in order to deposit sufficient reserves at the central bank to meet the target; ϵ_3 agents borrow to consume, and deposit nothing at the central bank, thus they miss their target. The full equilibrium characterization mirrors the one in the benchmark model with continuous $F(\varepsilon)$ and is delegated to Appendix **??**. Here we summarize our results.

Lemma 9. Assume that sellers are not allowed to set targets at the central bank and that all of their deposits are remunerated at the rate above the target. Specifically, assume that $R\left(d^S\right) = \frac{d^S}{\rho_E}$. Then prices in the goods market satisfy $p\beta\phi' = \rho_E$.

Proof. It follows from substituting ρ_E for ρ_T in sellers' problem. □

The results from the previous lemma will simply facilitate the comparison between the flat rate framework and the VRT under the restriction that the rate on excess reserves in the VRT equals the interest on reserves in the flat rate regime. Otherwise we could maintain the assumptions that sellers can choose targets and restric instead ρ_T in the VRT to equal

[9] See Appendix B.

the the interest on reserves in the flat rate regime. This will also make the comparison more straightforward.

8.3. **Choice of money holdings and targets.** As in the benchmark model with continuous $F(\varepsilon)$, the choice of money holdings for buyers and autarks satisfies :

$$-\phi + \mathbb{E}_\epsilon W_m(m|T) = 0 \tag{22}$$

$$-\phi + \mathbb{E}_\epsilon W_m^A(m|T) = 0 \tag{23}$$

As we show in Appendix **??**, these can be rearranged as:

$$\frac{\gamma}{\beta} = \frac{(p_1+p_2)}{\rho_l} + \frac{\frac{\gamma}{\beta}p_3\epsilon_3}{(\phi m + \rho_l\gamma\phi' B')}. \tag{24}$$

$$\frac{\gamma}{\beta} = \frac{p_1}{\rho_E} + \frac{p_2}{\rho_P} + \frac{\frac{\gamma}{\beta}p_3\epsilon_3}{\phi m^A}. \tag{25}$$

The choice of target satisfies $\frac{\partial \mathbb{E}_\epsilon W(m|T)}{\partial T} = 0$. Substituting the solution to the buyers' decision problem, as characterized above, we obtain the following results.

Lemma 10. If $\frac{1}{\rho_T} - \frac{p_3}{\rho_P} = \frac{(p_1+p_2)}{\rho_l} < \frac{p_1}{\rho_E} + \frac{p_2}{\rho_P}$ then $T > 0 = T^A$.

Proof. It follows combining $\frac{\partial \mathbb{E}_\epsilon W(m|T)}{\partial T} = 0$ with $\frac{\partial \mathbb{E}_\epsilon W^A\left(m^A|T^A\right)}{\partial T^A} = 0$. □

Notice that Lemma 10 implies that the equilibrium money market rate is:

$$\rho_l = \frac{(p_1+p_2)}{\frac{1}{\rho_T} - \frac{p_3}{\rho_P}} \tag{26}$$

As we continue the characterization of the equilibrium, it will be possible to describe targets in more detail.

8.4. **Money market clearing.** Maintaining the assumptions in Lemma 10. We aim at constructing an equilibrium where ϵ_1 buyers lend, and ϵ_2,ϵ_3 buyers borrow (the former to meet their target and the latter to consume), thus the supply of funds in the money market is $p_1 l_1 = p_1\left(m - T - \frac{\epsilon_1\rho_l}{\beta\phi'}\right)$ while the demand of funds is $p_2 l_2 + p_3 l_3 = p_2\left(T + \frac{\epsilon_2\rho_l}{\beta\phi'} - m\right) + p_3\rho_l B'$. The market clearing condition is then:

$$\begin{aligned}
p_1\left(m - T - \frac{\epsilon_1\rho_l}{\beta\phi'}\right) &= p_2\left(T + \frac{\epsilon_2\rho_l}{\beta\phi'} - m\right) + p_3\rho_l B' \\
(p_1+p_2)(m-T) &= \rho_l\left(p_3 B' + \frac{p_1\epsilon_1 + p_2\epsilon_2}{\beta\phi'}\right) \\
(p_1+p_2)(\phi m - \phi T) &= \rho_l\left(p_3\gamma\phi' B' + \frac{\gamma}{\beta}(p_1\epsilon_1 + p_2\epsilon_2)\right) \qquad (27)
\end{aligned}$$

Thus we have ϕT as a function of $\rho_l, \phi m, \phi' B'$.

Furthermore, with $T^A = 0$, as implied by Lemma 10, equation (**??**) which characterizes the choice of m^A becomes:

$$\frac{\gamma}{\beta} = \frac{(p_1+p_2)}{\rho_E} + \frac{\frac{\gamma}{\beta}p_3\epsilon_3}{\phi m^A} \tag{28}$$

as the relevant remuneration rate for reserves is always ρ_E, even when agents experience shock ϵ_2 and, if they were choosing a positive target, they would need to borrow to meet the target.

Lemma 11. Maintain the assumptions in Lemma 10. If $\rho_E > \rho_l > \rho_P$ then $\phi m^A < \phi m + \rho_l \gamma \phi' B'$.

Proof. Combining the equations for money holdings, (24) and (**??**), yields:

$$\frac{(p_1+p_2)}{\rho_l} + \frac{\frac{\gamma}{\beta} p_3 \epsilon_3}{(\phi m + \rho_l \gamma \phi' B')} = \frac{(p_1+p_2)}{\rho_E} + \frac{\frac{\gamma}{\beta} p_3 \epsilon_3}{\phi m^A}.$$

The assumption $\rho_E > \rho_l$ then implies $\phi m^A < \phi m + \rho_l \gamma \phi' B'$. □

Notice that this result is not inconsistent with our results from the benchmark model where $F(\varepsilon)$ is continuous, as in that case we derived our results under the assumption of interior targets. [10] In this three types economy, we are constructing instead an equilibrium where autarks set zero targets.

8.5. **Borrowing limit.** As in the full model:

$$\begin{aligned}
\phi B &= V^B - V^A \\
&= \phi\left(m^A - m\right) + \mathbb{E}_\epsilon \epsilon\left[u\left(q_\epsilon\right) - u\left(q_\epsilon^A\right)\right] + \\
&\quad \mathbb{E}_\epsilon \beta \phi' \left[\frac{\left(T - T^A\right)}{\rho_T} - \frac{l_\epsilon}{\rho_l} + R\left(d_\epsilon - T\right) - R\left(d_\epsilon^A - T^A\right)\right] + \beta \phi' B'
\end{aligned}$$

Combining this equation with the money market clearing condition (27) and with the first order condition for money holdings (24) yields:

$$\phi B = \rho_l \gamma \phi' B' + \mathbb{E}_\epsilon \epsilon\left[u\left(q_\epsilon\right) - u\left(q_\epsilon^A\right)\right]$$

where q_ε depends on $\phi' B'$. We derive the full characterization of the above result in Appendix **??**, here we focus on a stationary equilibrium, implying that the borrowing limit satisfies:

$$\frac{\phi B}{\frac{1}{\rho_T} - \frac{p_3}{\rho_P}}\left(\frac{1}{\rho_T} - \frac{p_3}{\rho_P} - \gamma(p_1+p_2)\right) = \mathbb{E}_\epsilon \epsilon\left[u\left(q_\epsilon\right) - u\left(q_\epsilon^A\right)\right] \tag{29}$$

and the real value of targets satisfies:

$$\phi T \frac{\beta}{\gamma}\left(\frac{1}{\rho_T} - \frac{p_3}{\rho_P}\right) = \frac{\left(\frac{1}{\rho_T} - \frac{p_3}{\rho_P}\right)\mathbb{E}_\epsilon \epsilon}{\left(\frac{\gamma}{\beta} - \frac{1}{\rho_T} + \frac{p_3}{\rho_P}\right)} - \frac{\frac{\gamma}{\beta}\left(p_1 \epsilon_1 + p_2 \epsilon_2\right)}{\left(\frac{\gamma}{\beta} - \frac{1}{\rho_T} + \frac{p_3}{\rho_P}\right)} - \beta \phi' B' \tag{30}$$

We can then state the following result.

Proposition 1. The borrowing limit is positive in a stationary equilibrium if and only if $\mathbb{E}_\epsilon \epsilon\left[u\left(q_\epsilon\right) - u\left(q_\epsilon^A\right)\right] > 0$ and $\frac{1}{\rho_T} - \frac{p_3}{\rho_P} > (p_1 + p_2)\gamma$.

Proof. See Appendix **??**. □

[10] In the benchmark model we had $\beta\phi'(m + \rho_l B') = \beta\phi' m^A$.

Combining (29) with (30) yields:

$$\phi T=\frac{\frac{\gamma}{\beta}}{\frac{\gamma}{\beta}-\frac{1}{\rho_T}+\frac{p_3}{\rho_P}}\left(\mathbb{E}\varepsilon-\frac{\frac{\gamma}{\beta}(p_1\varepsilon_1+p_2\varepsilon_2)}{\frac{1}{\rho_T}-\frac{p_3}{\rho_P}}\right)-\frac{\gamma\mathbb{E}_\epsilon\epsilon\left[u\left(q_\epsilon\right)-u\left(q_\epsilon^A\right)\right]}{\frac{1}{\rho_T}-\frac{p_3}{\rho_P}-\gamma(p_1+p_2)} \tag{31}$$

Thus, verifying that $T>0$, requires the right hand side of (31) to be strictly positive.

8.6. **Equilibrium.** Below we summarize the necessary and sufficient conditions for an equilibrium to exists where 1) buyers who experience shock ε_1 are lenders, buyers who experience shock ε_2 borrow to meet their target and are not borrowing constrained, and buyers who experience shock ε_3 borrow to consume and are borrowing constrained; 2) autarks who experience shock ε_1 make deposits at the central bank in excess of their target, buyers who experience shock ε_2 fail to meet their target and thus deposit everything in excess of their consumption, and buyers who experience shock ε_3 spend their entire money holdings on consumption and make no deposits at the central bank.

Proposition 2. Assume $\frac{\gamma}{\beta}>\frac{1}{\rho_T}$. Then the equilibrium characterized in the previous sections exists if and only if:

$$\begin{aligned}&-\frac{1}{\rho_T}+\min\left(\frac{\gamma}{\beta}(p_1+p_2),\frac{p_1}{\rho_E}+\frac{p_2}{\rho_P}\right)>-\frac{p_3}{\rho_P}>\\ &\max\left(-\frac{1}{\rho_T}+(p_1+p_2)\max(\gamma,\frac{1}{\rho_E}),\frac{p_1}{\rho_E}+\frac{p_2}{\rho_P}-\frac{\gamma}{\beta}\right)\end{aligned} \tag{32}$$

Proof. See Appendix **??**. □

9. Three types economy with Flat Rate

In the flat rate economy the central bank remunerates deposits at a flat rate. Thus this is analogous to the VRT where $T=T^A=0$. Let $R\left(d\right)$ denote the remuneration of deposits at the central bank, specifically let $R\left(d\right)=\frac{d}{\rho_E}$.

9.1. **Sellers.** Sellers are exactly the same as in the economy with VRT, so all the results derived for the full model still apply. In particular, in equilibrium $p\beta\phi'=\rho_E$.

9.2. **Buyers.** Assuming, as in the full model, that utility is log, the first order conditions to the buyers' problem imply the characterization below. Consistently with our equilibrium construction in the economy with VRT, we construct an equilibrium where ϵ_1 and ϵ_2 agents lend and deposit no reserves at the central bank; ϵ_3 agents are borrowing constrained, borrow to consume and deposit no reserves at the central bank.

Appendix **??** provides a full characterization of the equilibrium, while here we summarize our results.

9.3. **Money market clearing.** Let us assume (and then verify in equilibrium) that $\rho_E>\rho_l$, so that potential lenders will prefer to lend than to deposit at the central bank. We aim at constructing an equilibrium where ϵ_1 and ϵ_2 buyers lend, ϵ_3 buyers borrow to consume, thus the supply of funds in the money market is $p_1l_1+p_2l_2=p_1\left(m-\frac{\epsilon_1\rho_l}{\beta\phi'}\right)+p_2\left(m-\frac{\epsilon_2\rho_l}{\beta\phi'}\right)$ while the demand of funds is $p_3l_3=p_3\rho_lB'$. The market clearing condition is then:

$$\left(p_1+p_2\right)m-\frac{\rho_l}{\beta\phi'}\left(p_1\epsilon_1+p_2\epsilon_2\right)\quad=\quad p_3\rho_lB'$$

which yields: [11]

$$\rho_l = \frac{(p_1+p_2)\,\phi m}{p_3\gamma\phi' B' + \frac{\gamma}{\beta}(p_1\epsilon_1 + p_2\epsilon_2)} \tag{33}$$

9.4. **Borrowing limit.** As in the full model:

$$\begin{aligned}\phi B &= V^B - V^A \\ &= \phi\left(m^A - m\right) + \mathbb{E}_\epsilon \epsilon\left[u(q_\epsilon) - u\left(q_\epsilon^A\right)\right] + \\ &\quad \mathbb{E}_\epsilon \beta\phi'\left[-\frac{l_\epsilon}{\rho_l} + R(d_\epsilon) - R\left(d_\epsilon^A\right)\right] + \beta\phi' B'\end{aligned}$$

Appendix **??** provides full details of the characterization of the real borrowing limit, here we focus on a stationary equilibrium where $\phi B = \phi' B'$, which yields:

(34)

$$\phi B\left[1 - (p_1+p_2)\beta\right] = (p_1+p_2)\,\mathbb{E}_\epsilon\epsilon - \frac{p_1\epsilon_1 + p_2\epsilon_2}{\beta} + \left[1 + \frac{p_1\epsilon_1 + p_2\epsilon_2}{\beta\phi B}\right]\mathbb{E}_\epsilon\epsilon\left[u(q_\epsilon) - u\left(q_\epsilon^A\right)\right]$$

where q_ϵ is also a function of ϕB: directly through q_3 and indirectly through ρ_l in q_1, q_2.

9.5. **Equilibrium.** Necessary and sufficient conditions for the existence of an equilibrium where buyers who receive shocks ε_1 and ε_2 lend while buyers who receive shocks ε_3 are borrowing constrained and borrow to consume making no deposits at the central bank, are simply conditions guaranteeing that buyers and autarks in state ε_3 spend the real value of their entire portfolio on consumption. Specifically, buyers in state ε_3 are borrowing constrained, while autarks in state ε_3 are deposit constrained.

As shown in Appendix **??**, the assumption $\gamma\rho_l > \beta$ is necessary and sufficient for the existence of an equilibrium of the type we are interested in.

10. VRT versus Flat Rate

The goal of this section is to show that under some conditions the VRT relaxes the borrowing constraint with respect to the flat rate remuneration mechanism. If the autark's value is the same under the two mechanisms, this also implies that welfare is higher uner the VRT. Recall that we are assuming that 1) lump sum taxes are available to the monetary authority/government and 2) the monetary authority cannot run the Friedman rule.

From equation (**??**) we know that with the VRT framework $\phi B\left[1 - \gamma\rho_l\right] = \mathbb{E}_\epsilon\epsilon\left[u(q_\epsilon) - u\left(q_\epsilon^A\right)\right]$.

Similarly, we can rearrange the equation for the borrowing limit in the flat rate framework as

$$\begin{aligned}\phi B &= \phi\left(m^A - m\right) + \mathbb{E}_\epsilon\epsilon\left[u(q_\epsilon) - u\left(q_\epsilon^A\right)\right] + \\ &\quad -\mathbb{E}_\epsilon\beta\phi' R\left(d_\epsilon^A\right) + \beta\phi' B'\end{aligned}$$

after substituting for the market clearing condition and $d_\epsilon = 0$ for all ϵ. Following Subsection **??**, we can use

$$\phi m^A - \mathbb{E}_\epsilon\beta\phi' R\left(d_\epsilon^A\right) = \phi m^A\left[1 - \frac{(p_1+p_2)}{\rho_E}\frac{\beta}{\gamma}\right] + p_1\epsilon_1 + p_2\epsilon_2$$

[11] Appendix **??** provides the detailed derivation.

and (??) to get

$$\phi m^A - \mathbb{E}_\epsilon \beta \phi' R\left(d_\epsilon^A\right) = \frac{p_3\epsilon_3}{1-\frac{(p_1+p_2)}{\rho_E}\frac{\beta}{\gamma}}\left[1-\frac{(p_1+p_2)}{\rho_E}\frac{\beta}{\gamma}\right]+p_1\epsilon_1+p_2\epsilon_2 = \mathbb{E}_\epsilon \epsilon$$

Thus, we are left with

$$\phi B = \mathbb{E}_\epsilon\epsilon - \phi m + \mathbb{E}_\epsilon \epsilon\left[u\left(q_\epsilon\right)-u\left(q_\epsilon^A\right)\right]+\beta\phi' B'$$

Using then (??) yields

$$\phi B = \mathbb{E}_\epsilon\epsilon - \frac{p_3\epsilon_3}{1-\frac{(p_1+p_2)}{\rho_l}\frac{\beta}{\gamma}}+\rho_l\gamma\phi' B' + \mathbb{E}_\epsilon \epsilon\left[u\left(q_\epsilon\right)-u\left(q_\epsilon^A\right)\right]+\beta\phi' B'$$

and substituting out for ρ_l from (??) further yields

$$\begin{aligned}\phi B &= \mathbb{E}_\epsilon\epsilon - \frac{p_3\epsilon_3}{1-\frac{\left(\gamma\phi' B'+\frac{\gamma}{\beta}(p_1\epsilon_1+p_2\epsilon_2)\right)}{\beta\phi' B'+\mathbb{E}_\epsilon\epsilon}\frac{\beta}{\gamma}} + \mathbb{E}_\epsilon \epsilon\left[u\left(q_\epsilon\right)-u\left(q_\epsilon^A\right)\right]+\left(\beta+\rho_l\gamma\right)\phi' B' \\ &= \mathbb{E}_\epsilon\epsilon - \frac{p_3\epsilon_3\left(\gamma\phi' B'+\frac{\gamma}{\beta}\mathbb{E}_\epsilon\epsilon\right)}{\gamma\phi' B'+\frac{\gamma}{\beta}\mathbb{E}_\epsilon\epsilon-\gamma\phi' B'-\frac{\gamma}{\beta}\left(p_1\epsilon_1+p_2\epsilon_2\right)} + \mathbb{E}_\epsilon \epsilon\left[u\left(q_\epsilon\right)-u\left(q_\epsilon^A\right)\right]+\left(\beta+\rho_l\gamma\right)\phi' B' \\ &= \mathbb{E}_\epsilon\epsilon - \left(\beta\phi' B'+\mathbb{E}_\epsilon\epsilon\right) + \mathbb{E}_\epsilon \epsilon\left[u\left(q_\epsilon\right)-u\left(q_\epsilon^A\right)\right]+\left(\beta+\rho_l\gamma\right)\phi' B' \\ &= \mathbb{E}_\epsilon \epsilon\left[u\left(q_\epsilon\right)-u\left(q_\epsilon^A\right)\right]+\rho_l\gamma\phi' B'\end{aligned}$$

Thus the two equation for the real borrowing limit are exactly the same under both frameworks. Any difference in equilibrium outcome must then stem from the money market rate ρ_l and the consumption allocation q_ϵ, q_ϵ^A. Under both frameworks agents who receive shocks ϵ_1 and ϵ_3 behave the same in the money market (i.e. lend and borrow respectively). However, agents who receive shocks ϵ_2 borrow under the VRT framework and lend under the flat rate framework. So, intuitively, the money market rate is higher under the VRT framework due to larger net demand of funds. Regarding ρ_l equations (??) and (??) imply that $\rho_l^{VRT} < \rho_l^{FR}$ if and only if:

$$\frac{1}{\frac{1}{\rho_T}-\frac{p_3}{\rho_P}} < \frac{\beta\left(\phi' B'\right)|_{FR}+\mathbb{E}_\epsilon\epsilon}{\gamma\left(\phi' B'\right)|_{FR}+\frac{\gamma}{\beta}\left(p_1\epsilon_1+p_2\epsilon_2\right)}$$

That $\rho_l^{VRT} < \rho_l^{FR}$ has implications on the consumption allocation q_ϵ, q_ϵ^A, as the consumption of agents who receive shocks ϵ_1 and ϵ_2 is distorted by the the relative rate of return on the money market and on reserves deposited at the central bank, $\frac{\rho_l}{\rho_T}$ (or $\frac{\rho_l}{\rho_E}$ if sellers are not allowed to set targets). Because $\rho_E > \rho_l$ autarks' consumption is always less distorted with respect to the first best than the consumption of buyers, when agents eceive shocks ϵ_1 and ϵ_2. [12]. We can summerize the consumption allocation in the following table:

Regarding q_ϵ, q_ϵ^A we have that:

	VRT Buyers	VRT Autarks	FR Buyers	FR Autarks
ϵ_1	$\frac{\epsilon_1\rho_l}{\rho_T}$	$\frac{\epsilon_1\rho_E}{\rho_T}$	$\frac{\epsilon_1\rho_l}{\rho_E}$	ϵ_1
ϵ_2	$\frac{\epsilon_2\rho_l}{\rho_T}$	$\frac{\epsilon_2\rho_E}{\rho_T}$	$\frac{\epsilon_2\rho_l}{\rho_E}$	ϵ_2
ϵ_3	$\frac{\beta\phi'(m+\rho_l B')}{\rho_T}$	$\frac{\beta\phi' m^A}{\rho_T}$	$\frac{\beta\phi'(m+\rho_l B')}{\rho_E}$	$\frac{\beta\phi' m^A}{\rho_E}$

[12] The first best with log utility and linear disutility of labor in the goods market, is simply $q_\epsilon = \epsilon$.

If sellers are not allowed to set targets this becomes

	VRT Buyers	VRT Autarks	FR Buyers	FR Autarks
ϵ_1	$\frac{\epsilon_1 \rho_l}{\rho_E}$	ϵ_1	$\frac{\epsilon_1 \rho_l}{\rho_E}$	ϵ_1
ϵ_2	$\frac{\epsilon_2 \rho_l}{\rho_E}$	ϵ_2	$\frac{\epsilon_2 \rho_l}{\rho_E}$	ϵ_2
ϵ_3	$\frac{\beta\phi'(m+\rho_l B')}{\rho_E}$	$\frac{\beta\phi' m^A}{\rho_E}$	$\frac{\beta\phi'(m+\rho_l B')}{\rho_E}$	$\frac{\beta\phi' m^A}{\rho_E}$

where, from (24) and (**??**) which are both $\phi m = \frac{p_3\epsilon_3}{1-\frac{(p_1+p_2)}{\rho_l}\frac{\beta}{\gamma}} - \rho_l\gamma\phi' B'$, it follows that that real money holdings are decreasing in ρ_l and $\phi' B'$. Therefore, if in equilibrium $\rho_l^{VRT} < \rho_l^{FR}$ and $(\phi B)^{VRT} > (\phi B)^{FR}$, the effect on real money holdings is ambiguous.

Overall, in order for the net gain in expected utility (i.e. $\mathbb{E}_\epsilon \epsilon \left[u\left(q_\epsilon\right) - u\left(q_\epsilon^A\right)\right]$) to be larger under the VRT than the flat rate, it is necessary that the utility from consumption of agents who receive shocks ϵ_3 is sufficiently larger under the VRT than the flat rate, compensating for 1) a lower money market rate, and 2) the loss in utility from consumption when the realized shock is either shock ϵ_1 or ϵ_2. This can be achieved if, for example, the marginal utility of consumption in state ϵ_3 is sufficiently large, or if the probability of state ϵ_3 occurring is sufficiently large.

To facilitate the comparison between the two frameworks, one can make assumptions so that the in both economies the value of autarky is the same (i.e. $V^A|VRT = V^A|FR$), so that we can conclude that welfare is higher in the economy with higher borrowing limit, as $B = \frac{V^B - V^A}{\phi}$. The value of autarky is determined by utility from consumption and disutility from labor net of interest on deposits at the central bank. If we fix the inflation rate, γ, to be the same in the two economies, and we assume that the interest rate in excess of the target in the VRT is the same as the interest rate on reserves in the flat rate economy (i.e. $\rho_E^{VRT} = \rho_E^{FR}$), then the money holdings chosen by autarks are the same in the two economies. Moreover, assuming that sellers can make deposits at the central bank, which are remunerated at rate ρ_E^{VRT}, but cannot choose targets, implies that autarks' consumption is the same in the two economies.

Assumption 1. Let sellers be able to make deposits at the central bank, and restrict policy rates in the VRT and flat rate economies so that $\rho_E^{VRT} = \rho_E^{FR}$.

Then, focusing on stationary equilibria, comparing (**??**) with (34) allows us to compare welfare in the two economies.

Appendix A. Proofs

A.1. **Proof of Lemma 5.**

Proof.

A.1.1. *Region $\epsilon \leq \epsilon_L$.* In this region, maintaining the assumption $\rho_l \in (\rho_P, \rho_E)$, buyers are lending and fully fund their target: $l_\epsilon < 0$, $\lambda_B = 0$, $d_\epsilon = T$. Optimality conditions in this case imply:

$$-\frac{1}{\rho_l} + \lambda_\epsilon = < 0$$

$$1/\rho_l = \lambda_\epsilon \geq R'_d \geq \frac{1}{\rho_E}$$

$$\frac{\epsilon}{q_\epsilon} = p\beta\phi'\lambda_\epsilon$$

Thus: $\frac{1}{\rho_E} < \frac{1}{\rho_l}$, and $\lambda_\epsilon = \frac{1}{\rho_l}$, $l_\epsilon = T - m + pq_\epsilon$ and $pq_\epsilon = \frac{\epsilon\rho_l}{\beta\phi'}$. The largest value of ϵ consistent with this solution is $\epsilon_L = \{\epsilon \in [\underline{\epsilon}, \bar{\epsilon}] : l_\epsilon = 0\}$ (i.e. the marginal buyer stops lending in the money market). This implies $pq_\epsilon = m - T$ and we can define $\epsilon_L = (m-T)\beta\phi'\lambda_\epsilon = (m-T)\frac{\beta\phi'}{\rho_E}$. In this case, the first order condition for l_ϵ evaluated at l_{ϵ_L} implies $-\frac{1}{\rho_l} + \lambda_\epsilon = 0$. Thus we can rewrite

$$\epsilon_L = (m - T)\beta\phi'\lambda_\epsilon = (m - T)\frac{\beta\phi'}{\rho_l}. \tag{35}$$

To summarize, the allocation in this region satisfies:

$$\begin{aligned}
d_\epsilon &= T \\
\lambda_\epsilon &= \frac{1}{\rho_l} \\
l_\epsilon &= pq_\epsilon + T - m < 0 \\
pq_\epsilon &= \frac{\epsilon\rho_l}{\beta\phi'}
\end{aligned}$$

where we can substitute $p\beta\phi' = \rho_T$ to get $q_\epsilon = \frac{\epsilon\rho_l}{\rho_T}$.

A.1.2. *Region $\epsilon_L < \epsilon \leq \epsilon_B$.* In this region a buyer borrows in the money market, but is not borrowing constrained, and fully funds his target. Therefore $l_\epsilon > 0$, $\lambda_B = 0$, $d_\epsilon = T$. Optimality conditions in this case imply:

$$\lambda_\epsilon = \frac{1}{\rho_l}$$

$$\lambda_\epsilon = \frac{1}{\rho_l} \geq R'_d \geq \frac{1}{\rho_E}$$

So $\lambda_\epsilon = \frac{1}{\rho_l}$. Furthermore:

$$\frac{\epsilon}{q_\epsilon} = p\beta\phi'\lambda_\epsilon \quad l_\epsilon = pq_\epsilon + T - m$$

where $pq_\epsilon = \frac{\epsilon}{\beta\phi'\lambda_\epsilon} = \frac{\epsilon\rho_l}{\beta\phi'}$ The largest value of ϵ consistent with this solution is $\epsilon_B = \{\epsilon \in [\underline{\epsilon}, \overline{\epsilon}] : l_\epsilon = \rho_l B\}$ (i.e. the marginal buyer starts being borrowing constrained in the money market). Thus, combining the first order condition for q_ϵ with the budget constraint we have

$$\epsilon_B = \beta\phi'\lambda_\epsilon(\rho_l B + m - T) = \frac{\beta\phi'}{\rho_l}(m - T) + \beta\phi' B. \tag{36}$$

To summarize, the allocation in this region satisfies:

$$\begin{aligned} d_\epsilon &= T \\ \lambda_\epsilon &= \frac{1}{\rho_l} \\ l_\epsilon &= T - m + pq_\epsilon > 0 \\ pq_\epsilon &= \frac{\epsilon\rho_l}{\beta\phi'} \end{aligned}$$

where we can substitute $p\beta\phi' = \rho_T$ to get $q_\epsilon = \frac{\epsilon\rho_l}{\rho_T}$. Notice that using (35), which can be rearranged as $\epsilon_L\rho_l = (m - T)\beta\phi'$, we can rewrite $l_\epsilon = \frac{\rho_l}{\beta\phi'}(\epsilon - \epsilon_L)$.

A.1.3. *Region $\epsilon_B < \epsilon \leq \epsilon_T$.* In this region a buyer borrows in the money market and is borrowing constrained, but fully funds his target because the marginal return on a deposit before meeting the target (i.e. $\frac{1}{\rho_P}$) exceeds the marginal utility of consumption. Therefore $l_\epsilon > 0$, $\lambda_B > 0$, $d_\epsilon = T$. Optimality conditions in this case imply:[13]

$$-\frac{1}{\rho_l} + \lambda_\epsilon = \lambda_B$$

And from the binding borrowing constraint we have $l_\epsilon = \rho_l B$. Furthermore:

$$\frac{\epsilon}{q_\epsilon} = p\beta\phi'\lambda_\epsilon$$

where q_ϵ is pinned down by the budget constraint in the goods' market: $pq_\epsilon = m + \rho_l B - T$. The largest value of ϵ consistent with this solution is $\epsilon_T = \{\epsilon \in [\underline{\epsilon}, \overline{\epsilon}] : d_\epsilon < T\}$ (i.e. the marginal buyer starts funding his target only partially). Thus, the first order condition for d_ϵ evaluated at ϵ_T yields $R'_d = \frac{1}{\rho_P}$ so that $\lambda_{\epsilon_T} = \frac{1}{\rho_P}$ and, combining the first order condition for q_ϵ with the budget constraint, we have $\lambda_B = \frac{1}{\rho_P} - \frac{1}{\rho_l}$ and

$$\epsilon_T = (m + \rho_l B - T)\beta\phi'(\frac{1}{\rho_P}) \tag{37}$$

To summarize, the allocation in this region satisfies:

$$\begin{aligned} d_\epsilon &= T \\ l_\epsilon &= \rho_l B \\ pq_\epsilon &= m + \rho_l B - T \end{aligned}$$

Using then (36) and substituting out $p\beta\phi' = \rho_T$, we can rewrite the last equation as

$$\frac{\rho_T}{\beta\phi'}q_\epsilon = \frac{\rho_l\epsilon_B}{\beta\phi'}$$

[13]Recall R'_d is not differentiable at $d = T$.

which simplifies to

$$q_\epsilon = \frac{\rho_l \epsilon_B}{\rho_T}.$$

Notice that consumption in this region is constant with respect to ϵ: this is due to the kink in the remuneration of reserves at the central bank, which makes the value function not differentiable at the value of d_ϵ where the kink occurs. Indeed in this region the first order condition for q_ϵ holds with inequality because the marginal value of a unit of deposit when $d_\epsilon = T$ is simply $\frac{1}{\rho_E}$ which is smaller than the marginal utility evaluated at q_ϵ.

$$\frac{\epsilon}{q_\epsilon} > p\beta\phi' \frac{1}{\rho_E}$$

However the same first order condition for q_ϵ holds with the inequality reversed if the marginal value of a unit of deposit when $d_\epsilon < T$ is $\frac{1}{\rho_P}$:

$$\frac{\epsilon}{q_\epsilon} < p\beta\phi' \frac{1}{\rho_P}$$

In order to pin down λ_ϵ, we combine the first order condition for q_ϵ with the budget constraint, to obtain:

$$\frac{\epsilon}{\frac{\rho_l \epsilon_B}{\rho_T}} = p\beta\phi' \lambda_\epsilon = \rho_T \lambda_\epsilon$$

where the last equation follows from lemma 4, which implies $p\beta\phi' = \rho_T$. Thus

$$\lambda_\epsilon = \frac{\epsilon}{\epsilon_B \rho_l}$$

A.1.4. *Region $\epsilon_T < \epsilon \leq \epsilon_M$.* In this region a buyer is borrowing constrained and partially funds his target: $l_\epsilon = \rho_l B$, $\lambda_B > 0$, $d_\epsilon < T$. Optimality conditions in this case imply:

$$-\frac{1}{\rho_l} + \lambda_\epsilon = \lambda_B$$

$$\lambda_\epsilon = R'_d = \frac{1}{\rho_P}$$

So that $\frac{1}{\rho_P} - \frac{1}{\rho_l} = \lambda_B > 0$. Furthermore:

$$\frac{\epsilon}{q_\epsilon} = p\beta\phi' \lambda_\epsilon = p\beta\phi' \frac{1}{\rho_P}$$

$$pq_\epsilon + d_\epsilon = \rho_l B + m$$

The largest value of ϵ consistent with this solution is $\epsilon_M = \{\epsilon \in [\underline{\epsilon}, \overline{\epsilon}] : d_\epsilon = 0\}$ (i.e. the marginal buyer can no longer fund his target at all). Thus, from the budget constraint we have $pq_\epsilon = \rho_l B + m$ and

$$\epsilon_M = \frac{\beta\phi'}{\rho_P}(\rho_l B + m) \tag{38}$$

so that $d_\epsilon = \rho_l B + m - \frac{\epsilon\rho_P}{\beta\phi'}$ To summarize, the allocation in this region satisfies:

$$d_\epsilon = \frac{p\rho_P}{\rho_T}(\epsilon_M - \epsilon) < T \tag{39}$$

$$l_\epsilon = \rho_l B \tag{40}$$

$$q_\epsilon = \frac{\epsilon\rho_P}{\rho_T} \tag{41}$$

which we obtain by rearranging the first order condition for q_ϵ as

$$\epsilon\rho_P = p\beta\phi' q_\epsilon \Rightarrow q_\epsilon = \frac{\epsilon\rho_P}{\rho_T}$$

and by rearranging the first order condition for d_ϵ using the definition of ϵ_M, (38), as follows:

$$d_\epsilon = (m + \rho_l B) - pq_\epsilon = \frac{\rho_P\epsilon_M}{\beta\phi'} - \frac{\rho_P\epsilon}{\beta\phi'}$$

$$d_\epsilon = \frac{p\rho_P}{\rho_T}(\epsilon_M - \epsilon)$$

A.1.5. *Region $\epsilon_M < \epsilon$.* In this region a buyer is borrowing constrained and makes no deposit at all, entirely missing his target: $l_\epsilon = \rho_l B$, $\lambda_B > 0$, $d_\epsilon = 0$. Optimality conditions in this case imply:

$$-\frac{1}{\rho_l} + \lambda_\epsilon = \lambda_B$$

$$\lambda_\epsilon > R'_d = \frac{1}{\rho_P}$$

and from the budget constraint $pq_\epsilon = m + \rho_l B$. Using the definition of ϵ_M, (38), we can rearrange this as $pq_\epsilon = \frac{\epsilon_M\rho_P}{\beta\phi'}$, which, substituting the equilibrium value of p, implies $q_\epsilon = \frac{\epsilon_M\rho_P}{\rho_T}$. We then obtain λ_ϵ combining the first order condition for q_ϵ with the budget constraint:

$$\frac{\epsilon}{\frac{\epsilon_M\rho_P}{\rho_T}} = \rho_T\lambda_\epsilon \quad \Rightarrow \lambda_\epsilon = \frac{\epsilon}{\epsilon_M\rho_P}$$

where $p\beta\phi' = \rho_T$ has been substituted out.

Consumption, loans, deposits, and the envelope condition satisfy

$q_\epsilon =$	$l_\epsilon =$	$d_\epsilon =$	$W_m^B(m\vert T,\epsilon) =$	
$\epsilon\rho_l/\rho_T$	$\frac{\rho_l}{\rho_T}p(\epsilon - \epsilon_l)$	T	$(1/p)(\rho_T/\rho_l)$	if $\epsilon < \epsilon_l$
$\epsilon\rho_l/\rho_T$	$\frac{\rho_l}{\rho_T}p(\epsilon - \epsilon_l)$	T	$(1/p)(\rho_T/\rho_l)$	if $\epsilon_l < \epsilon < \epsilon_B$
$\epsilon_B\rho_l/\rho_T$	$\rho_l B$	T	$\frac{1}{p}\frac{\epsilon}{\epsilon_B}\frac{\rho_T}{\rho_l}$	if $\epsilon_B < \epsilon < \epsilon_T$
$\epsilon\rho_P/\rho_T$	$\rho_l B$	$p\frac{\rho_P}{\rho_T}[\epsilon_M - \epsilon]$	$(1/p)(\rho_T/\rho_P)$	if $\epsilon_T < \epsilon < \epsilon_M$
$\epsilon_M\frac{\rho_P}{\rho_T}$	$\rho_l B$	0	$\frac{1}{p}\frac{\epsilon}{\epsilon_M}\frac{\rho_T}{\rho_P}$	if $\epsilon > \epsilon_M$.

□

A.2. **Proof of Lemma 6.**

Proof.

A.2.1. *Region where $\epsilon_E^A \geq \epsilon$.* In this region buyers in autarky deposit more than their target: $d_\epsilon > T$. Optimality conditions imply

$$R_d' = \frac{1}{\rho_E}, \quad \lambda_\epsilon = \frac{1}{\rho_E}, \quad \epsilon = pq_\epsilon \beta\phi'\lambda_\epsilon = pq_\epsilon\beta\phi'(\frac{1}{\rho_E})$$

Finally the budget constraint implies $d_\epsilon = m - pq_\epsilon = m - \frac{\epsilon\rho_E}{\beta\phi'}$ The largest value of ϵ consistent with this solution is $\epsilon_E^A = \{\max_{\{\epsilon\in[\underline{\epsilon},\bar{\epsilon}]\}} \epsilon : d_\epsilon = T\}$ (i.e. the marginal buyer can just fund his target). This implies that $pq_\epsilon = m - T$ and

$$\epsilon_E^A = (m-T)\beta\phi'\lambda_\epsilon = (m-T)\frac{\beta\phi'}{\rho_E}. \tag{42}$$

so that

$$q_\epsilon = \frac{\epsilon\rho_E}{p\beta\phi'} = \frac{\epsilon\rho_E}{\rho_T}$$

And using the budget constraint and the definition of ϵ_E^A in (42), we have:

$$\begin{aligned} d_\epsilon &= \frac{\rho_E\epsilon_E^A}{\beta\phi'} + T - pq_\epsilon \\ &= \frac{\rho_E\epsilon_E^A}{\beta\phi'} + T\frac{\rho_E\epsilon}{\beta\phi'} \\ &= \frac{\rho_E(\epsilon_E^A - \epsilon)}{\beta\phi'} + T \end{aligned}$$

A.2.2. *Region where $\epsilon_E^A < \epsilon \leq \epsilon_P^A$.* In this region autarkic buyer deposit just to fund their target: $d_\epsilon = T$ so that from the budget constraint we have $d_\epsilon = T = m - pq_\epsilon$, which, combined with the definition of ϵ_E^A in (42) and muptiplying both sides of the budget constraint by $\beta\phi'$ yields $\beta\phi' pq_\epsilon = \beta\phi'(m-T)$, which can be rearranged as:

$$\rho_T q_\epsilon = \epsilon_E^A \rho_E$$

which is constant in ϵ in this region. Notice in fact that, similarly to the case of non autarkic buyers, the first order condition for q_ϵ satisfies:

$$\frac{p\beta\phi'}{\rho_P} > \frac{\epsilon}{q_\epsilon} > \frac{p\beta\phi'}{\rho_E}$$

as the marginal utility of consumption is larger than the marginal return on an additional unit of deposits at the central bank when deposits are above the target, but it is smaller than the marginal return on an additional unit of deposits at the central bank when deposits are below the target.

The largest value of ϵ consistent with this solution is $\epsilon_P^A = \{\max_{\epsilon\in[\underline{\epsilon},\bar{\epsilon}]} \epsilon : d_\epsilon = T\}$ (i.e. the marginal buyer who chooses to fully fund his target). This implies that $\lambda_\epsilon = \frac{1}{\rho_P}$ and, as $d_\epsilon = T$, that $pq_\epsilon = m - T$ and

$$\epsilon_P^A = (m-T)\beta\phi'\lambda_\epsilon = (m-T)\frac{\beta\phi'}{\rho_P}. \tag{43}$$

To summarize, in this region $d_\epsilon = T$, $q_\epsilon = \frac{\epsilon_E^A\rho_E}{\rho_T}$. The first order condition for q_ϵ then implies:

$$\frac{\epsilon}{\frac{\epsilon_E^A\rho_E}{\rho_T}} = \rho_T\lambda_\epsilon$$

where $p\beta\phi' = \rho_T$ has been substituted out, to yield $\lambda_\epsilon = \frac{\epsilon}{\rho_E \epsilon_E^A} = \frac{\epsilon}{\rho_P \epsilon_P^A}$.

A.2.3. *Region where* $\epsilon_P^A < \epsilon \leq \epsilon_M^A$. In this region autarkic buyer only partially fund their target: $0 < d_\epsilon < T$ so that from the first order condition for d_ϵ we have $R_d' = \frac{1}{\rho_P} = \lambda_\epsilon$. The first order condition for q_ϵ yields: $\epsilon = pq_\epsilon \beta\phi' \frac{1}{\rho_P}$. Thus, since lemma 4 implies that $p\beta\phi' = \rho_T$, we have $q_\epsilon = \frac{\epsilon\rho_P}{\rho_T}$. From the budget constraint we have $d_\epsilon = m - pq_\epsilon$, which, combined with the first order condition for q_ϵ, yields $d_\epsilon = m - \frac{\epsilon\rho_P}{\beta\phi'}$. The largest value of ϵ consistent with this solution is $\epsilon_M^A = \{\epsilon \in [\underline{\epsilon}, \bar{\epsilon}] : d_\epsilon = 0\}$ (i.e. the marginal buyer can no longer fund his target at all). This implies that $pq_\epsilon = m$ and

$$\epsilon_M^A = m\frac{\beta\phi'}{\rho_P}. \tag{44}$$

Using (44) we can then rewrite deposits as $d_\epsilon = \frac{\beta\phi' m - \epsilon\rho_P}{\beta\phi'} = \frac{\rho_P(\epsilon_M^A - \epsilon)}{\beta\phi'}$. To summarize, in this region we have $q_\epsilon = \frac{\epsilon\rho_P}{\rho_T}$, $d_\epsilon = \frac{\rho_P(\epsilon_M^A - \epsilon)}{\beta\phi'}$ and $\lambda_\epsilon = \frac{1}{\rho_P}$.

A.2.4. *Region where* $\epsilon > \epsilon_M^A$. In this region autarkic buyers do not deposit at all: $d_\epsilon = 0$ so that from the first order condition for d_ϵ implies $\lambda_\epsilon > R_d' = \frac{1}{\rho_P}$, and the budget constraint implies that $pq_\epsilon = m$. We can then use the definition of ϵ_M^A in (44) to get

$$\beta\phi' pq_\epsilon = \rho_T q_\epsilon \quad = \quad \rho_P \epsilon_M^A = \beta\phi' m$$

yielding $q_\epsilon = \frac{\rho_P \epsilon_M^A}{\rho_T}$. The first order condition for q_ϵ is

$$pq_\epsilon \beta\phi' \frac{1}{\rho_P} < pq_\epsilon \beta\phi' \lambda_\epsilon = \epsilon$$

as buyers in this region are constrained by the non negativity of deposits and, as a consequence, $\lambda_\epsilon > \frac{1}{\rho_P} = R_d'$.[14] Thus, combining the first order condition for q_ϵ with the budget constraint we get $\lambda_\epsilon = \frac{\epsilon}{\rho_T q_\epsilon} = \frac{\epsilon}{\rho_P \epsilon_M^A}$.

□

Consumption, deposits, and the envelope condition satisfy

$q_\epsilon =$	$d_\epsilon =$	$W_m^A(m\|T,\epsilon) =$	
$\epsilon\rho_E/\rho_T$	$T + \frac{\rho_E}{\rho_T}p(\epsilon_E^A - \epsilon)$	$(1/p)(\rho_T/\rho_E)$	if $\epsilon < \epsilon_E^A$
$\epsilon_E^A \rho_E/\rho_T$	T	$\frac{1}{p}\frac{\epsilon}{\epsilon_E^A}\frac{\rho_T}{\rho_E}$	if $\epsilon_E^A < \epsilon < \epsilon_P^A$
$\epsilon\rho_P/\rho_T$	$\frac{\rho_P}{\rho_T}p(\epsilon_M^A - \epsilon)$	$(1/p)(\rho_T/\rho_P)$	if $\epsilon_P^A < \epsilon < \epsilon_M^A$
$\epsilon_M^A \frac{\rho_P}{\rho_T}$	0	$\frac{1}{p}\frac{\epsilon}{\epsilon_M^A}$	if $\epsilon_M^A < \epsilon$.

[14]Referring to the Lagrangean in section 6.3 the first order condition for d_ϵ is $\beta V_d^B - \beta\phi'\lambda_\epsilon + \beta\phi'\lambda_d = 0$, with $\beta\phi'\lambda_d$ denoting the multiplier on the non negativity constraint on d_ϵ. With $\beta\phi'\lambda_d > 0$ the first first order condition for d_ϵ becomes $\frac{1}{\rho_P} + \lambda_d = \lambda_\epsilon$.

A.3. **Proof of Lemma 7.**

Proof. In order to solve for the marginal value on T, strip beginning of period money balances from the settlement function and bring them back to the previous period's money and goods markets. That is, because we can write

$$V^B(l,d|T) = \phi\left(T/\rho_T + R(d-T) - l/\rho_l\right) + V^B(0,0|0),$$

we can also write a buyer's value in the money and goods market as

$$W^B(m|T) = \epsilon u(q_\epsilon) + \beta\phi^+\left(T/\rho_T + R(d_\epsilon - T) - l_\epsilon/\rho_l\right) + \beta V^B(0,0|0)$$

That is, using the definition of a buyer's value in the settlement market $V^B(\hat{m}, l_\epsilon, d_\epsilon|T)$, we can define

$$V^B(\hat{m},0,0|0) = \max_{\{m^+,T^+\}}\{\phi m^+ + W^B(m^+|T^+)\} \tag{45}$$

so that

$$V^B(\hat{m}, l_\epsilon, d_\epsilon|T) = \phi\left(\frac{T}{\rho_T} + R(d_\epsilon - T) - \frac{l_\epsilon}{\rho_l}\right) + V^B(\hat{m},0,0|0),$$

and the buyer's value in the money and goods' market as

$$W^B(m|T) = \epsilon u(q_\epsilon) + \beta\phi^+\left(\frac{T}{\rho_T} + R(d_\epsilon - T) - \frac{l_\epsilon}{\rho_l} + (\hat{m}' - \tau M)\right) + \beta V^B(\hat{m}',0,0|0).$$

Substituting for q_ϵ, l_ϵ, and d_ϵ we get $W^B(m|T,\epsilon) - \beta V^B(\hat{m},0,0|0) =$

Substituting for q_ϵ, l_ϵ, and d_ϵ we get $W^B(m|T,\epsilon) - \beta V^B(0,0|0) =$

$$\begin{cases} \epsilon\log(\epsilon\rho_l/\rho_T) + \beta\phi^+\left(T/\rho_T + R(T-T) - \frac{\rho_l}{\rho_T}p\left(\epsilon - \epsilon_l\right)\rho_l\right) & \text{if } \epsilon < \epsilon_B, \\ \epsilon\log(\epsilon_B\rho_l/\rho_T) + \beta\phi^+\left(T/\rho_T + R(T-T) - \rho_l B/\rho_l\right) & \text{if } \epsilon_B < \epsilon < \epsilon_T, \\ \epsilon\log(\epsilon\rho_P/\rho_T) + \beta\phi^+\left(T/\rho_T + R\left(p\left(\epsilon_M - \epsilon\frac{\rho_P}{\rho_l}\right) - T\right) - \rho_l B/\rho_l\right) & \text{if } \epsilon_T < \epsilon < \epsilon_M, \\ \epsilon\log(\epsilon_M) + \beta\phi^+\left(T/\rho_T + R(-T) - \rho_l B/\rho_l\right) & \text{if } \epsilon > \epsilon_M. \end{cases}$$

Substituting in for ϵ_i and simplifying gives $W^B(m|T,\epsilon) - \beta V^B(0,0|0) =$

$$\begin{cases} \epsilon\log(\epsilon\rho_l/\rho_T) + \beta\phi^+\left(T/\rho_T - \frac{p}{\rho_T}\epsilon + (m-T)/\rho_l\right) & \text{if } \epsilon < \epsilon_B, \\ \epsilon\log\left(\frac{m-T+B\rho_l}{p}\right) + \beta\phi^+\left(T/\rho_T - B\right) & \text{if } \epsilon_B < \epsilon < \epsilon_T, \\ \epsilon\log\left(\epsilon\rho_P/\rho_T\right) + \beta\phi^+\left(T/\rho_T - B + (m + \rho_l B - T)/\rho_P\right) - \epsilon & \text{if } \epsilon_T < \epsilon < \epsilon_M, \\ \epsilon\log\left(\frac{m+B\rho_l}{p}\right) + \beta\phi^+(T/\rho_T - B - T/\rho_P) & \text{if } \epsilon_M \le \epsilon. \end{cases} \tag{46}$$

I get a few different values in the above tables, which I rewrote as follows:

$$\begin{cases} \epsilon\log(\epsilon\rho_l/\rho_T) + \beta\phi^+\left(T/\rho_T + R(T-T) - \frac{p}{\rho_T}\left(\epsilon - \epsilon_l\right)\right) & \text{if } \epsilon < \epsilon_B, \\ \epsilon\log(\epsilon_B\rho_l/\rho_T) + \beta\phi^+\left(T/\rho_T + R(T-T) - B\right) & \text{if } \epsilon_B < \epsilon < \epsilon_T, \\ \epsilon\log(\epsilon\rho_P/\rho_T) + \beta\phi^+\left(T/\rho_T + R\left(p\frac{\rho_P}{\rho_T}\left(\epsilon_M - \epsilon\right) - T\right) - B\right) & \text{if } \epsilon_T < \epsilon < \epsilon_M, \\ \epsilon\log(\epsilon_M\frac{\rho_P}{\rho_T}) + \beta\phi^+\left(T/\rho_T + R(-T) - \rho_l B/\rho_l\right) & \text{if } \epsilon > \epsilon_M. \end{cases}$$

Substituting in for ϵ_i and simplifying gives

$$W^B(m|T,\epsilon) - \beta V^B(0,0|0) =$$

$$\begin{cases} \epsilon\log(\epsilon\rho_l/\rho_T) + \beta\phi^+\left(T/\rho_T - \frac{p}{\rho_T}\epsilon + (m-T)/\rho_l\right) & \text{if } \epsilon < \epsilon_B, \\ \epsilon\log\left(\frac{m-T+B\rho_l}{p}\right) + \beta\phi^+\left(T/\rho_T - B\right) & \text{if } \epsilon_B < \epsilon < \epsilon_T, \\ \epsilon\log\left(\epsilon\rho_P/\rho_T\right) + \beta\phi^+\left(T(\frac{1}{\rho_T} - \frac{1}{\rho_P}) + \frac{(m+B\rho_l)}{\rho_P} - \frac{\epsilon}{\beta\phi'} - B\right) & \text{if } \epsilon_T < \epsilon < \epsilon_M, \\ \epsilon\log\left(\frac{m+B\rho_l}{p}\right) + \beta\phi^+(T/\rho_T - B - T/\rho_P) & \text{if } \epsilon_M \leq \epsilon. \end{cases} \tag{47}$$

To see this consider each case in turn:

(1) $\epsilon < \epsilon_B$

$$W^B(m|T) = \epsilon u(q_\epsilon) + \beta\phi'[\frac{T}{\rho_T} - \frac{p}{\rho_T}(\epsilon - \frac{\beta\phi'(m-T)}{\rho_l})]$$

where $\epsilon_l = \frac{\beta\phi'(m-T)}{\rho_l}$.

(2) $\epsilon \in (\epsilon_B, \epsilon_T)$

$$W^B(m|T) = \epsilon u(\frac{\rho_l}{\rho_T}[\beta\phi'\frac{(m-T)}{\rho_l} + \beta\phi' B]) + \beta\phi'[\frac{T}{\rho_T} - B]$$

that we can rearrange as

$$W^B(m|T) = \epsilon u(\frac{(m-T+\rho_l B)}{p}) + \beta\phi'[\frac{T}{\rho_T} - B]$$

where we can substitute $\epsilon_B = \beta\phi'[\frac{(m-T+\rho_l B)}{\rho_l})]$ and $p\beta\phi' = \rho_T$ to get

$$W^B(m|T) = \epsilon u(\frac{\rho_l}{\rho_T}\epsilon) + \beta\phi'[\frac{T}{\rho_T} - B]$$

(3) $\epsilon \in (\epsilon_T, \epsilon_M)$

$$W^B(m|T) = \epsilon u(\frac{\epsilon\rho_P}{\rho_T}) + \beta\phi'[\frac{T}{\rho_T} - \frac{1}{\rho_P}(\frac{p\rho_P}{\rho_T}(\epsilon_M\epsilon) - T) - B)]$$

which we can rearrange as

$$W^B(m|T) = \epsilon u(\frac{\epsilon\rho_P}{\rho_T}) + \beta\phi'[T(\frac{1}{\rho_T} - \frac{1}{\rho_P}) + \frac{(m+B\rho_l)}{\rho_P} - \frac{\epsilon}{\beta\phi'} - B)]$$

and, using $\epsilon_M = \frac{\beta\phi'}{\rho_P}(m+B\rho_l)$, as

$$W^B(m|T) = \epsilon\log(\epsilon\rho_P/\rho_T) + \beta\phi^+\left(T/\rho_T + R\left(p\frac{\rho_P}{\rho_T}(\epsilon_M - \epsilon) - T\right) - B\right)$$

(4) $\epsilon > \epsilon_M$

$$W^B(m|T) = \epsilon\log(\epsilon_M\frac{\rho_P}{\rho_T}) + \beta\phi^+\left(T/\rho_T + R(-T) - \rho_l B/\rho_l\right)$$

That can be rearranged as

$$W^B(m|T) = \epsilon\log(\epsilon_M\frac{\rho_P}{\rho_T}) + \beta\phi^+\left(T(\frac{1}{\rho_T} - \frac{1}{\rho_P}) - B\right)$$

This, finally, gives us our expression for the marginal value of a target

$$W_T^B(m|T) = \beta\phi^+ \left[\frac{1}{\rho_T} - \frac{1}{\rho_l}\left(F(\epsilon_B) + \int_{\epsilon_B}^{\epsilon_T}\left(\frac{\epsilon}{\epsilon_B}\right)dF(\epsilon)\right) - \frac{1}{\rho_P}\left(1 - F(\epsilon_T)\right)\right].$$

Setting $W_T^B(m|T) = 0$ gives the equation (15). That $\rho_l \in [\rho_P, \rho_T]$ follows from the fact that $\epsilon/\epsilon_B \in [\rho_P^{-1}, 1]$ as $\epsilon \in [\epsilon_B, \epsilon_T]$. Indeed, so long as F has unbounded support, we must have $\rho_l > \rho_P$. Similarly, if 0 is in the support of F, then $\rho_l < \rho_T$.

Consider now autarks. Recall that

$$W^A(m|T) = \epsilon u(q_\epsilon) + \beta\phi'[\frac{T}{\rho_T} + R(d_\epsilon - T)] + \beta V^A(\hat{m}', 0|0)$$

Then, case by case we derive the marginal value of a target for an autarkic buyer:

(1) $\epsilon < \epsilon_E^A$.
In this case $\lambda_\epsilon = \frac{1}{\rho_E}$ so that $\beta\phi'\lambda_\epsilon = \frac{\rho_T}{p\rho_E}$, $q_\epsilon = \frac{\epsilon\rho_E}{\rho_T}$ and $d_\epsilon = T + \frac{\rho_E}{\beta\phi'}(\epsilon_E^A - \epsilon)$. Thus

$$\begin{aligned} W^A(m|T) - \beta V^A(\hat{m}', 0|0) &= \epsilon u(\frac{\epsilon\rho_E}{\rho_T}) + \beta\phi'[\frac{T}{\rho_T} + \frac{\epsilon_E^A - \epsilon}{\beta\phi'}] \\ &= \epsilon u(\frac{\epsilon\rho_E}{\rho_T}) + \frac{T}{p} + \epsilon_E^A - \epsilon \end{aligned}$$

So that

$$\frac{\partial W^A}{\partial T} = \frac{1}{p} - \frac{\beta\phi'}{\rho_E} = \frac{1}{p}(1 - \frac{\rho_T}{\rho_E})$$

(2) $\epsilon \in (\epsilon_E^A, \epsilon_P^A)$.
In this case $q_\epsilon = \epsilon_E^A\frac{\rho_E}{\rho_T}$, $d_\epsilon = T$, $\lambda_\epsilon = \frac{\epsilon}{\rho_E\epsilon_E^A}$, where the last equation is derived from the first order condition for q_ϵ, which, with log utility, is $\frac{\epsilon}{q_\epsilon} = p\beta\phi'\lambda_\epsilon$. Then

$$\begin{aligned} W^A(m|T) - \beta V^A(\hat{m}', 0|0) &= \epsilon u(\epsilon_E^A\frac{\rho_E}{\rho_T}) + \beta\phi'[\frac{T}{\rho_T} + R(0)] \\ &= \epsilon u(\frac{m-T}{p}) + \beta\phi'\frac{T}{\rho_T} \end{aligned}$$

where the last equation uses the definition of $\epsilon_E^A = \frac{(m-T)\beta\phi'}{\rho_E}$. So that

$$\begin{aligned} \frac{\partial W^A}{\partial T} &= \epsilon(-\frac{1}{p})u'(\frac{m-T}{p}) + \frac{1}{p} \\ &= \frac{1}{p}(1 - \epsilon u'(\frac{m-T}{p})) \\ &= \frac{1}{p}[1 - \frac{\rho_T\epsilon}{\rho_E\epsilon_E^A}] \end{aligned}$$

where, in the last equation we have rewritten $\epsilon u'(\frac{m-T}{p})$ using the first order condition for q_ϵ as

$$\epsilon u'(q_\epsilon) = p\beta\phi'\lambda_\epsilon = \rho_T\frac{\epsilon}{\rho_E\epsilon_E^A}$$

(3) $\epsilon \in (\epsilon_P^A, \epsilon_M^A)$.
In this case $d_\epsilon = m - \frac{\epsilon\rho_P}{\beta\phi'} < T$, $q_\epsilon = \epsilon\frac{\rho_P}{\rho_T}$, $\lambda_\epsilon = \frac{1}{\rho_P}$. Then

$$\begin{aligned} W^A(m|T) - \beta V^A(\hat{m}', 0|0) &= \epsilon u(\epsilon\frac{\rho_P}{\rho_T}) + \beta\phi'[\frac{T}{\rho_T} + R(m - \frac{\epsilon\rho_P}{\beta\phi'} - T)] \\ &= \epsilon u(\epsilon\frac{\rho_P}{\rho_T}) + \beta\phi'[\frac{T}{\rho_T} + \frac{1}{\rho_P}(m - \frac{\epsilon\rho_P}{\beta\phi'} - T)] \\ &= \epsilon u(\epsilon\frac{\rho_P}{\rho_T}) + \beta\phi' T[\frac{1}{\rho_T} - \frac{1}{\rho_P}] + \epsilon_E^A - \epsilon \end{aligned}$$

where the last equation follows from substituting

$$m - \frac{\epsilon\rho_P}{\beta\phi'} = \frac{1}{\beta\phi'}(\rho_P\epsilon_E^A - \epsilon\rho_P)$$

So that

$$\frac{\partial W^A}{\partial T} = \beta\phi'(\frac{1}{\rho_T} - \frac{1}{\rho_P}) = \frac{\rho_T}{p}(\frac{1}{\rho_T} - \frac{1}{\rho_P})$$

(4) $\epsilon > \epsilon_M^A$.
In this case $d_\epsilon = 0$, $\lambda_\epsilon = \frac{\epsilon}{\rho_P\epsilon_E^A} < \frac{1}{\rho_P}$ and $q_\epsilon = \frac{\rho_P}{\rho_T}\epsilon_T^A$. Then

$$\begin{aligned} W^A(m|T) - \beta V^A(\hat{m}', 0|0) &= \epsilon u(\frac{\rho_P}{\rho_T}\epsilon_T^A) + \beta\phi'[\frac{T}{\rho_T} + R(-T)] \\ &= \epsilon u(\frac{\rho_P}{\rho_T}\epsilon_T^A) + \beta\phi' T[\frac{1}{\rho_T} - \frac{1}{\rho_P}] \end{aligned}$$

So that

$$\frac{\partial W^A}{\partial T} = \beta\phi'(\frac{1}{\rho_T} - \frac{1}{\rho_P}) = \frac{\rho_T}{p}(\frac{1}{\rho_T} - \frac{1}{\rho_P})$$

Combining $\frac{\partial W^A}{\partial T}$ from all cases, yields:

$$\begin{aligned} \frac{\partial W^A}{\partial T} &= F(\epsilon_E^A)\frac{1}{p}(1 - \frac{\rho_T}{\rho_E}) + \int_{\epsilon_E^A}^{\epsilon_P^A} \frac{1}{p}[1 - \epsilon u'(\frac{\epsilon_E^A\rho_E}{\rho_T})]dF(\epsilon) + \\ &\quad + \int_{\epsilon_P^A}^{\epsilon_M^A} \frac{\rho_T}{p}(\frac{1}{\rho_T} - \frac{1}{\rho_P})dF(\epsilon) + [1 - F(\epsilon_M^A)]\frac{\rho_T}{p}(\frac{1}{\rho_T} - \frac{1}{\rho_P}) \\ &= \frac{1}{p}\Big\{F(\epsilon_E^A)(1 - \frac{\rho_T}{\rho_E}) + \int_{\epsilon_E^A}^{\epsilon_P^A}(1 - \frac{\rho_T\epsilon}{\rho_E\epsilon_E^A})dF(\epsilon) + (1 - \frac{\rho_T}{\rho_P})(1 - F(\epsilon_P^A))\Big\} \\ &= \frac{1}{p}\Big\{1 - F(\epsilon_E^A)\frac{\rho_T}{\rho_E} - \int_{\epsilon_E^A}^{\epsilon_P^A}\frac{\rho_T\epsilon}{\rho_E\epsilon_E^A}dF(\epsilon) - \frac{\rho_T}{\rho_P}(1 - F(\epsilon_P^A))\Big\} \end{aligned}$$

so that $\frac{\partial W^A}{\partial T} = 0$ if and only if

$$\frac{1}{\rho_T} = \frac{F(\epsilon_E^A)}{\rho_E} + \int_{\epsilon_E^A}^{\epsilon_P^A}\frac{\epsilon}{\rho_E\epsilon_E^A}dF(\epsilon) + \frac{1 - F(\epsilon_P^A)}{\rho_P}$$

$\square$

A.4. **Proof of Lemma 8.**

Proof. Consider supply of funds first. Using the characterization in lemma 5 it follows that $l_\epsilon < 0$ if and only if a buyer receives shock $\epsilon \leq \epsilon_L$. In this case $pq_\epsilon = \frac{\epsilon\rho_l}{\beta\phi'}$ and $l_\epsilon = pq_\epsilon + T - m$. Using also the definition of $\epsilon_L = (m - T\frac{\beta\phi'}{\rho_l})$ yields

$$l_\epsilon = \frac{\epsilon\rho_l}{\beta\phi'} - \frac{\epsilon_L\rho_l}{\beta\phi'} = (\epsilon - \epsilon_L)\frac{p\rho_l}{\rho_T}$$

where the last equation follows from lemma 4, which implies $p\beta\phi' = \rho_T$. This yields (19). Consider demand for funds. Using the characterization in lemma 5 it follows that buyers with a shock $\epsilon \geq \epsilon_B$ are borrowing constrained, so they borrow $l_\epsilon = \rho_l B$. Buyers with $\epsilon \in (\epsilon_L, \epsilon_B)$ borrow $l_\epsilon = (\epsilon - \epsilon_L)\frac{p\rho_l}{\rho_T}$. This yields (20). Thus the money market clearing condition, pinning down ρ_l is

$$\begin{aligned} 0 &= \int_0^{\epsilon_B} \frac{\rho_l}{\rho_T} p(\epsilon - \epsilon_L) dF(\epsilon) + \int_{\epsilon_B}^{\infty} \rho_l B dF(\epsilon) \\ 0 &= \frac{\rho_l\gamma}{\beta}\int_0^{\epsilon_B} \epsilon dF(\epsilon) - [\phi m^+ - \phi T^+]F(\epsilon_B) + \rho_l\phi B(1 - F(\epsilon_B)) \\ 0 &= \frac{\rho_l\gamma}{\beta}\int_0^{\epsilon_B} \epsilon dF(\epsilon) - [\rho_l\phi B + \phi m^+ - \phi T^+]F(\epsilon_B) + \rho_l\phi B \end{aligned} \tag{48}$$

□

A.5. **Endogenous borrowing limit with VRT.** Consider a buyer who repays his money market loan from the previous period. Because his repayment is publicly recorded then his continuation value at the end of the settlement market, before choosing target and money holdings, is $V^B(0,0,0|0)$, as defined in (45) in the appendix. To save on notation, let $V^B = V^B(0,0,0|0)$. Consider now a buyer who plans to default on his money market loan from the previous period. Analogously to a buyer, his continuation value in the settlement market, before choosing target and money holdings, is $V^A(0,0|0) = \max_{\{m^{A+},T^{A+}\}} -\phi m^{A+} + W^A(m^{A+}|T^{A+})$. To save on notation, let $V^A = V^A(0,0|0)$. The payoff to a buyer who repays in the settlement market is $-\phi\frac{l_\epsilon}{\rho_\epsilon} + V^B$, since the loan was in terms of money, while the payoff to a buyer who defaults is V^A. Therefore, the repayment constraint can be rewritten as

$$l_\epsilon \leq \frac{\rho_l}{\phi}(V^B - V^A) \tag{49}$$

More specifically:

$$V^B = \max_{\{m^+,T^+\}} -\phi m^+ + W^B(m^+|T^+) + \beta V^{B'} \tag{50}$$

$$= -\phi m^+ + \mathbb{E}_\epsilon\Big\{\epsilon\, u(q_\epsilon) + \beta V^B(\hat{m}', l_\epsilon, d_\epsilon|T_\epsilon)\} + \beta V^{B'} \tag{51}$$

$$= -\phi m^+ + \mathbb{E}_\epsilon\{\epsilon\, u(q_\epsilon) + \beta\phi^+[\hat{m}' + \frac{T^+}{\rho_T} + R(d_\epsilon - T^+) - \frac{l_\epsilon}{\rho_l} - \tau M]\Big\} + \beta V^{B'} \tag{52}$$

Since in equilibrium $\hat{m} = \hat{m}' = 0$ this boils down, for buyers and autarkic buyers respectively, to:

$$(53)\ V^B = -\phi m^+ + \mathbb{E}_\epsilon\Big\{\epsilon\ u(q_\epsilon) + \beta\phi^+[\hat{m}' + \frac{T^+}{\rho_T} + R(d_\epsilon - T^+) - \frac{l_\epsilon}{\rho_l} - \tau M]\Big\} + \beta V^{B'}$$

$$(54)\ V^A = -\phi m^{A+} + \mathbb{E}_\epsilon\Big\{\epsilon\ u(q^A_\epsilon) + \beta\phi^+[\hat{m}^{A'} + \frac{T^{A+}}{\rho_T} + R(d^A_\epsilon - T^{A+}) - \tau M]\Big\} + \beta V^{A'}$$

so that $\phi B = V^B - V^A$ and

$$\begin{aligned}\phi B &= \phi(m^{A+} - m^+) + \mathbb{E}_\epsilon[\epsilon u(q_\epsilon) - \epsilon u(q^A_\epsilon)] \\ &\quad + \mathbb{E}_\epsilon \beta\phi'\Big\{(\frac{T^+ - T^{A+}}{\rho_T}) - \frac{l_\epsilon}{\rho_l} + R(d_\epsilon - T^+) - R(d^A_\epsilon - T^{A+})\Big\} + \beta\phi' B' \qquad (55)\end{aligned}$$

where we used the result that $\hat{m}' = 0 = \hat{m}^{A'}$. Notice that none of the autarky terms depend on B, while ρ_l and T^+ as well as the allocation do.

Appendix B. Household Decisions

B.1. **Settlement Market with Voluntary Reserve Targets.** Here we close the buyers' choice for settlement period balances. From the settlement period problem, we have the first order condition on money balances which guarantees that $\phi = W^B_M(m|T)$. $W^B_M(m|T)$ is just the expectation of $W^B_M(m|T,\epsilon)$ which is just $\beta\phi^+\lambda_\epsilon$. This final quantity can be derived from the FOC for q: $\epsilon u'(q) - p\beta\phi^+\lambda_\epsilon = 0$.

The envelope condition from the decision problem in the settlement market is:

$$V^i(\hat{m}, l, d|T) = \phi\left(\hat{m} + T/\rho_T + R(d - T) - l/\rho_l - \tau M\right) + \max_{(m^+,T^+)}\left\{-\phi m^+ + W^i(m^+|T^+)\right\}.$$

where the first order condition for m^+ yields $-\phi + \frac{\partial W^i}{\partial m^+} \leq 0$, with $W^B(m^+|T^+) = \mathbb{E}_\epsilon[W^B(m^+, T^+|\epsilon)]$. Consider then the decision problem in money and goods market:

$$W^B(m|T,\epsilon) = \max_{(q_\epsilon, l_\epsilon, d_\epsilon)} \epsilon u(q_\epsilon) + \beta V^B(\hat{m}', l_\epsilon, d_\epsilon|T)$$

$$\text{s.t.} \quad m + l_\epsilon - pq_\epsilon - d_\epsilon - \hat{m}' \geq 0, \quad \text{and} \quad \rho_l B - l_\epsilon \geq 0.$$

and, with $z = (\hat{m}', q_\epsilon, l_\epsilon, d_\epsilon)$, the Lagrangean

$$L(z) = \epsilon u(q_\epsilon) + \beta V^B(\hat{m}', l_\epsilon, d_\epsilon|T) + \beta\phi'\lambda_\epsilon(m + l_\epsilon - d_\epsilon - pq_\epsilon - \hat{m}') + \beta\phi'\lambda_B(\rho_l B - l_\epsilon) + \beta\phi'\lambda_d d_\epsilon + \beta\phi'\lambda_q q_\epsilon$$

we then have $\frac{\partial W^B(m|T,\epsilon)}{\partial m^+} = \beta\phi'\lambda_\epsilon$ and $W^B(m^+|T^+) = \mathbb{E}_\epsilon\beta\phi'\lambda_\epsilon$. Thus the first order condition for m^+ is simply $-\phi + \mathbb{E}_\epsilon[\beta\phi'\lambda_\epsilon] \leq 0$.

Putting this all together gives

$$W^B_M = \frac{\rho_T}{p}\left(\int_0^{\epsilon_B} \frac{1}{\rho_l} + \int_{\epsilon_B}^{\epsilon_T} \frac{1}{\rho_l}\frac{\epsilon}{\epsilon_B} dF(\epsilon) + \int_{\epsilon_T}^{\epsilon_M} \frac{1}{\rho_P} dF(\epsilon) + \int_{\epsilon_M}^{\infty} \frac{1}{\rho_P}\frac{\epsilon}{\epsilon_M} dF(\epsilon)\right).$$

Setting this equal to ϕ and given that $\rho_t/p = \beta\phi^+$ with $\phi/\phi^+ = \gamma$ we get

$$\frac{\gamma}{\beta} = \int_0^{\epsilon_B} \frac{1}{\rho_l} + \int_{\epsilon_B}^{\epsilon_T} \frac{1}{\rho_l}\frac{\epsilon}{\epsilon_B} dF(\epsilon) + \int_{\epsilon_T}^{\epsilon_M} \frac{1}{\rho_P} dF(\epsilon) + \int_{\epsilon_M}^{\infty} \frac{1}{\rho_P}\frac{\epsilon}{\epsilon_M} dF(\epsilon)$$

Similarly, the choice of money holdings for autarkic buyers satisfies $-\phi + \beta\phi'\mathbb{E}_\epsilon\lambda_\epsilon \leq 0$, that, for $m^{A+} > 0$, is

$$F(\epsilon_E^A)\frac{1}{\rho_E} + \int_{\epsilon_E^A}^{\epsilon_P^A} \frac{\epsilon}{\rho_E \epsilon_E^A} dF(\epsilon) + \int_{\epsilon_P^A}^{\epsilon_M^A} \frac{1}{\rho_P} + \int_{\epsilon_M^A}^{\infty} \frac{\epsilon}{\epsilon_E^A \rho_P} \quad = \quad \phi \tag{56}$$

(57)

Appendix C. Flat Rate

C.1. **Floors Happen when $\imath_E = \pi$.** *Notation:* Write $z = \phi m$, $z^A = \phi m^A$, $L = \phi B$ (so, in the current period, the borrowing limit is $l \leq \rho B$), and $\zeta = \beta/\gamma$. Given this, one can derive that

$$\epsilon_B = \frac{\zeta}{\rho_E}(z + \rho_l L) \qquad \text{and} \qquad \epsilon_M = \frac{\zeta}{\rho_E} z^A$$

Also, write

$$\Delta\, \mathbb{E}[u] = \int \epsilon \left(u(q_\epsilon) - u(q_\epsilon^A)\right) dF$$

for the difference in utility from consumption between buyers and autarks.

Given these preliminaries, we derive some alternate forms on the three equilibrium conditions.

Lemma 12. The two money-choice equations can be equivalently expressed as

$$\frac{\rho_l}{\rho_E}(z + \rho_l L) = \frac{\rho_l}{\zeta}\epsilon_B = \mathbb{E}[\epsilon] + \int_0^{\epsilon_B} F\, d\epsilon,$$

and

$$z^A = \frac{\rho_E}{\zeta}\epsilon_M = \mathbb{E}[\epsilon] + \int_0^{\epsilon_M} F(\epsilon)\, d\epsilon.$$

Proof. As the proofs are similar, consider the autark's money choice equation. In the new notation, the equation in the text reads

$$\frac{\rho_E}{\zeta} = \int_0^{\epsilon_M} dF + \int_{\epsilon_M}^{\infty} \frac{\epsilon}{\epsilon_M} dF.$$

First, note that $\mathbb{E}[\epsilon] = \int_0^{\epsilon_M} \epsilon\, dF + \int_{\epsilon_M}^{\infty} \epsilon\, dF$. Also, after integration by parts,

$$\int_0^{\epsilon_M} \epsilon\, dF = [\epsilon F(\epsilon)]|_0^{\epsilon_M} - \int_0^{\epsilon_M} F(\epsilon)\, d\epsilon. \tag{58}$$

Hence, we can write

$$\int_{\epsilon_M}^{\infty} \epsilon\, dF = \mathbb{E}[\epsilon] - \int_0^{\epsilon_M} \epsilon\, dF = \mathbb{E}\left[\epsilon\right] + \int_0^{\epsilon_M} F(\epsilon)\, d\epsilon - \epsilon_M F(\epsilon_M).$$

Combining this with the original equation gives the result. □

Lemma 13. Deposits of autarks can be written as

$$\mathbb{E}[d_\epsilon^A] = \frac{\rho_E}{\zeta}\left[z^A - \mathbb{E}\left[\epsilon\right]\right].$$

Proof. Deposits for autarks consist of all unspent funds. That is

$$
\begin{aligned}
\mathbb{E}[d_\epsilon^A] &= \int_0^{\epsilon_M} z^a - pq_\epsilon^a \, dF, \\
&= \int_0^{\epsilon_M} z^a - \frac{\rho_E}{\zeta}\epsilon \, dF && \text{by definition of } q \text{ and } p, \\
&= \frac{\rho_E}{\zeta}\int_0^{\epsilon_M} \epsilon_M - \epsilon \, dF, \\
&= \frac{\rho_E}{\zeta}\left[\frac{\rho_E}{\zeta}\epsilon_M - \int_{\epsilon_M}^{\infty} \epsilon \, dF - \int_0^{\epsilon_M} \epsilon \, dF\right] && \text{from money choice,} \\
&= \frac{\rho_E}{\zeta}\left(z^A - \mathbb{E}\left[\epsilon\right]\right).
\end{aligned}
$$

□

Lemma 14. The money market equation can be written as $\zeta L = \int_0^{\epsilon_B} F(\epsilon) d\epsilon$.

Proof. Market clearing in the money market demands that $0 = \int l_\epsilon dF$. Substituting in terms, we have

$$
\begin{aligned}
0 &= \int_0^{\epsilon_B} (pq_\epsilon - m) dF + \rho_l B \int_{\epsilon_B}^{\infty} dF && \text{market clearing.} \\
&= \int_0^{\epsilon_B} \left(\frac{\rho_E}{\zeta}\frac{\rho_l}{\rho_E}\epsilon - z\right) dF + \rho_l L[1 - F(\epsilon_B)]. && \text{def'n of } q, p, \text{ mult. by} \phi. \\
&= \int_0^{\epsilon_B} \left[\frac{\rho_E}{\zeta}\epsilon - (z + \rho_l L)\right] dF + \rho_l L && \text{splitting } (1-F) \text{ inside.} \\
&= \frac{\rho_l}{\zeta}\int_0^{\epsilon_B} (\epsilon - \epsilon_B)\, dF + \rho_l L && \text{definition of } \epsilon_B. \\
&= \frac{\rho_l}{\zeta}\left[\int_0^{\epsilon_B} \epsilon \, dF + \int_{\epsilon_B}^{\infty} \epsilon \, dF - \epsilon_B\left(\int_0^{\epsilon_B} dF + \int_{\epsilon_B}^{\infty}\frac{\epsilon}{\epsilon_B} dF\right)\right] + \rho_l L. \\
&= \frac{\rho_l}{\zeta}\left(\mathbb{E}\left[\epsilon\right] - \frac{\rho_l}{\zeta}\epsilon_B\right) \\
\implies \zeta L &= \int_0^{\epsilon_B} F(\epsilon) d\epsilon && \text{substitute m-eqn.}
\end{aligned}
$$

□

Lemma 15. When $\rho_E > \rho_l$, so buyers make no deposits, the limited commitment constraint can be written as

$$\left(\frac{1}{\gamma} - \zeta\right) L = \mathbb{E}\left[\epsilon\right] - z + \Delta\, \mathbb{E}[u],$$

Proof. Assuming $\rho_E > \rho_l$ allows one to ignore the potential for buyers' deposits. Besides that, the result follows directly from the definition of L and the previous lemmas. □

Proposition 3. The difference in expected utility from consumption, $\Delta\, \mathbb{E}[u]$, decreases in ρ_l and is zero at when $\rho_l = \rho_E$.

Proof. That $\Delta\,\mathbb{E}[u] = 0$ when $\rho_l = \rho_E$ follows from two observations. First, consider the two money holding equations. If $\rho_l = \rho_E$ the equations become the same, so $\epsilon_B = \epsilon_M$. Second, if $\rho_l = \rho_E$ and $\epsilon_B = \epsilon_M$, then $q_\epsilon = q_\epsilon^A$. Identical consumption, of course, implying equal utility from such, given common preferences.

For the next part, we need the derivative of ϵ_B with respect to ρ_l. To calculate this, consider the money holding equation from lemma 12:

$$\frac{\rho_l}{\zeta}\epsilon_B = \mathbb{E}[\epsilon] + \int_0^{\epsilon_B} F(\epsilon)\,d\epsilon$$

Moving the ϵ_B/ζ over and differentiating with respect to ϵ_B gives

$$\frac{\partial}{\partial \epsilon_B}\left[\zeta \frac{\mathbb{E}[\epsilon] + \int_0^{\epsilon_B} F(\epsilon)\,d\epsilon}{\epsilon_B}\right] = \zeta \frac{F(\epsilon_B)\epsilon_B - \left(\mathbb{E}[\epsilon] + \int_0^{\epsilon_B} F(\epsilon)\,d\epsilon\right)}{\epsilon_B^2} = -\frac{\zeta}{\epsilon_B^2}\int_{\epsilon_B}^{\infty} \epsilon\, dF < 0$$

where the last line follows from integration by parts, as in equation (58). This, then, implies that

$$\frac{\partial \epsilon_B}{\partial \rho_l} = -\frac{\epsilon_B^2}{\zeta \int_{\epsilon_B}^{\infty} \epsilon\, dF}.$$

To show that $\Delta\,\mathbb{E}[u]$ is decreasing, consider its derivative.

$$\begin{aligned}
\frac{d\Delta\,\mathbb{E}[u]}{d\rho_l} &= \frac{d}{d\rho_l}\left[\int \epsilon\left(u(q_\epsilon) - u(q_\epsilon^A)\right) dF\right]. \\
&= \frac{d}{d\rho_l}\left[\int_0^{\epsilon_B} \epsilon u\left(\frac{\rho_l}{\rho_E}\epsilon\right) dF + \int_{\epsilon_B}^{\infty} \epsilon u(\epsilon_B)\, dF\right]. \\
&= \int_0^{\epsilon_B} \epsilon \frac{\frac{1}{\rho_E}\epsilon}{\frac{\rho_l}{\rho_E}\epsilon} dF + \frac{1}{\epsilon_B}\frac{\partial \epsilon_B}{\partial \rho_l}\int_{\epsilon_B}^{\infty} \epsilon\, dF && u'(x) = \frac{1}{x}. \\
&= \frac{1}{\rho_l}\int_0^{\epsilon_B} \epsilon\, dF - \frac{\epsilon_B}{\zeta} \\
&= \frac{1}{\rho_l}\left[\int_0^{\epsilon_B} \epsilon\, dF - \left(\mathbb{E}[\epsilon] + \int_0^{\epsilon_B} F\, d\epsilon\right)\right] && \text{Lemma 12} \\
&= -\frac{1}{\rho_l}\left(\int_{\epsilon_B}^{\infty} \epsilon\, dF + \int_0^{\epsilon_B} F(\epsilon)\, d\epsilon\right) < 0.
\end{aligned}$$

□

Theorem. If the central bank deposit rate equals or exceeds the rate of inflation, $\gamma\rho_E \leq 1$, then the private money market rate equals the deposit rate, $\rho_l = \rho_E$.

Proof. Note that $\rho_l > \rho_E$ is not possible: no one would lend money at a worse rate than can be had from the central bank. We will require the derivatives of z and L with respect to ρ_l. For L, Lemma 14 imlies that

$$\frac{\partial L}{\partial \rho_l} = \frac{F(\epsilon_B)}{\zeta}\frac{\partial \epsilon_B}{\partial \rho_l}.$$

Given this, rewrite z in terms of ϵ_B and L as

$$z = \frac{\rho_E}{\zeta}\epsilon_B - \rho_l L$$

to calculate

$$\frac{\partial z}{\partial \rho_E} = \left[\frac{\rho_E}{\zeta} - \frac{\rho_l}{\zeta} F(\epsilon_B)\right] \frac{\partial \epsilon_B}{\partial \rho_l} - L.$$

And, as shown in the proof of Proposition 3,

$$\frac{\partial \Delta\, \mathbb{E}[u]}{\partial \rho_l} = \frac{1}{\rho_l} \int_0^{\epsilon_B} \epsilon\, dF + \frac{1}{\epsilon_B} \int_{\epsilon_B}^{\infty} \epsilon\, dF \frac{\partial \epsilon_B}{\partial \rho_l}.$$

Towards a contradiction, suppose $\rho_l < \rho_E$, so that Lemma 15 holds, and rewrite the LC constraint as

$$0 = \mathbb{E}[\epsilon] - z - \left(\frac{1}{\gamma} - \zeta\right) L + \Delta\, \mathbb{E}[u].$$

Taking a derivative with respect to ρ_l yields

$$0 = -\underbrace{\left[\left(\frac{\rho_E}{\zeta} - \frac{\rho_l}{\zeta} F(\epsilon_B)\right) \frac{\partial \epsilon_B}{\partial \rho_l} - \frac{1}{\zeta} \int_0^{\epsilon_B} F(\epsilon)\, d\epsilon\right]}_{\partial z / \partial \rho_l}$$

$$- \underbrace{\left(\frac{1}{\gamma} - \zeta\right) \frac{1}{\zeta} F(\epsilon_B) \frac{\partial \epsilon_B}{\partial \rho_l}}_{\partial L / \partial \rho_l} + \underbrace{\frac{1}{\rho_l} \int_0^{\epsilon_B} \epsilon\, dF + \frac{1}{\epsilon_B} \int_{\epsilon_B}^{\infty} \epsilon\, dF \frac{\partial \epsilon_B}{\partial \rho_l}}_{\partial \Delta\, \mathbb{E}[u] / \partial \rho_l}.$$

Collect the terms involving $\frac{\partial \epsilon_B}{\partial \rho_l}$. Using the money market equation to substitute out the integral give

$$-\left(\frac{1}{\gamma} - \zeta\right) \frac{1}{\zeta} F(\epsilon_B) - \left(\frac{\rho_E}{\zeta} - \frac{\rho_l}{\zeta} F(\epsilon_B)\right) + \frac{\rho_l}{\zeta} - F(\epsilon_B)$$

which can be rearranged to read

$$\left(\frac{\rho_l}{\zeta} - \frac{1}{\gamma\zeta}\right) F(\epsilon_B) - \frac{\rho_E}{\zeta} + \frac{\rho_l}{\zeta} \leq \left(\frac{\rho_l}{\zeta} - \frac{\rho_E}{\zeta}\right) (1 + F(\epsilon_B)) < 0$$

Where the first inequality follows since $1/\gamma \geq \rho_E$ and the second because $\rho_l < \rho_E$. As $\frac{\partial \epsilon_B}{\partial \rho_l} < 0$, the two multiplied together give a positive term. The final terms of the derivative are obviously positive. Thus, the whole right hand side of the equilibrium condition is increasing in ρ_l.

Hence, there can exist at most one solution with $\rho_l > 0$. But, for Hence, as $\rho_l = \rho_E$ is a solution, it must be the only solution. □