Necessary and Sufficient Conditions for Existence and Uniqueness of Recursive Utilities

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ABSTRACT. We study existence, uniqueness and stability of solutions for a class of discrete time recursive utilities models. By combining two streams of the recent literature on recursive preferences—one that analyzes principal eigenvalues of valuation operators and another that exploits the theory of monotone concave operators—we obtain conditions that are both necessary and sufficient for existence and uniqueness. We also show that the natural iterative algorithm is convergent if and only if a solution exists. Consumption processes are allowed to be nonstationary.

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1. INTRODUCTION

Recursive preference models such as those discussed in Koopmans (1960), Epstein and Zin (1989) and Weil (1990) play an important role in macroeconomic and financial modeling. For example, the long-run risk models analyzed in Bansal and Yaron (2004), Hansen et al. (2008), Bansal et al. (2012) and Schorfheide et al. (2017) have employed such preferences in discrete time infinite horizon settings with a

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variety of consumption path specifications to help resolve long-standing empirical puzzles identified in the literature.

In recursive utilities models, the lifetime value of a consumption stream from a given point in time is expressed as the solution to a nonlinear forward-looking equation. While this representation is convenient and intuitive, it can also be vacuous, in the sense that no finite solution to the forward looking recursion exists. Moreover, even when a solution does exist, this solution lacks predictive content unless some form of uniqueness can also be established. In general, identifying restrictions that imply existence and uniqueness of a solution for an empirically relevant class of consumption streams is a nontrivial problem.

The aim of the present paper is to obtain existence and uniqueness results that are as tight as possible in a range of empirically plausible settings, while restricting attention to practical conditions that can be tested in applied work. More specifically, we provide conditions for existence and uniqueness of solutions to the class of preferences studied in Epstein and Zin (1989), while admitting consumption paths that follow a general multiplicative functional specification (see, e.g., Hansen and Scheinkman, 2009, 2012). These conditions are both necessary and sufficient, and hence as tight as possible in the setting we consider. In particular, if the conditions hold then a unique, globally attracting solution exists, while if not then no finite solution exists.

To give more detail on that setting, let preferences be defined recursively by the CES aggregator

$$V_t = \left[(1-\beta)C_t^{1-1/\psi} + \beta \left\{\mathcal{R}_t\left(V_{t+1}\right)\right\}^{1-1/\psi}\right]^{1/(1-1/\psi)}, \tag{1}$$

where $\{C_t\}$ is a consumption path, V_t is the utility value of the path extending on from time t and $\mathcal{R}_t$ is the Kreps–Porteus certainty equivalent operator

$$\mathcal{R}_t(V_{t+1}) := (\mathbb{E}_t V_{t+1}^{1-\gamma})^{1/(1-\gamma)}. \tag{2}$$

The parameter $\beta \in (0,1)$ is a time discount factor, while γ governs risk aversion and ψ is the elasticity of intertemporal substitution (EIS). To ensure that (1) and (2) are well defined, γ and ψ are required to be distinct from 1.

We assume that consumption growth can be expressed as

$$\ln(C_{t+1}/C_t) = \kappa(X_{t+1}, Y_{t+1}, X_t), \tag{3}$$

where κ is a continuous real-valued function, $\{X_t\}$ is a time homogeneous Markov process and $\{Y_t\}$ is an IID innovation process. The persistent component $\{X_t\}$ is required to be compact-valued, while the innovation $\{Y_t\}$ is allowed to be unbounded.

We seek a solution to normalized utility V_t/C_t expressed as a function of the state X_t. Our conditions for existence and uniqueness feature two components. The first is the sign of the composite parameter

$$\theta := \frac{1-\gamma}{1-1/\psi}. \tag{4}$$

The other is $r(K)$, the spectral radius of a valuation operator K determined by the primitives in (1)–(3) and specified below. We focus on the expression $r(K)^{1/\theta}$, which, as shown below, represents the discounted long-term growth rate of the risk-adjusted value of the consumption process that enters recursion (1), adjusted by the intertemporal substitutability of consumption. We show that a unique solution exists if and only if $r(K)^{1/\theta} < 1$. We also prove that the same condition is necessary and sufficient for global convergence of fixed point iteration.

With a view to applications, we then study the test value $r(K)^{1/\theta}$ and show that it can be calculated via a local spectral radius result for positive operators acting on a solid cone. We first apply this idea to a model with deterministic time trend studied in Alvarez and Jermann (2005), where the test value $r(K)^{1/\theta}$ can be obtained analytically. We show that the sufficient condition for existence and uniqueness of a solution in Alvarez and Jermann (2005) can be sharpened appreciably, since the short term fluctuations in consumption accommodated in their condition do not matter over the long horizons that determine lifetime utility. Second, we apply these local spectral radius results to calculate $r(K)^{1/\theta}$ for the long-run risk model specifications from Bansal and Yaron (2004) and Schorfheide et al. (2017), in order to infer regions of the parameter space where a unique continuation value exists.

On the level of techniques, the main innovation in this paper is to combine the idea in Hansen and Scheinkman (2012) of extracting information on long run dynamics from the spectral radius of K with the theory of monotone concave operators and their connections to recursive utility models, as developed in a set of papers referenced below. In particular, we show that a continuous transformation of lifetime utility per unit of consumption can be expressed as the fixed point of an operator that combines the linear valuation operator K and a nonlinear component. When

$\theta < 0$ or $\theta \geqslant 1$, the nonlinear component is monotone and concave, with either zero or one nontrivial fixed point. We show that the latter case arises if and only if $r(K)^{1/\theta} < 1$. For the remaining case $0 < \theta < 1$, we combine spectral theory with an extension of Banach's contraction mapping theorem to establish that a unique solution exists if and only if the same condition holds.[1]

Our work builds on a significant literature on solutions to recursive preference models. The groundwork for studying Epstein–Zin preferences over infinite horizon consumption streams was provided by Epstein and Zin (1989), who in turn built on the finite-horizon framework of temporal lotteries found in Kreps and Porteus (1978). Epstein and Zin (1989) obtained sufficient conditions for existence across a broad set of parameters, while allowing geometric consumption growth and eschewing a Markov assumption. These findings were strengthened by Marinacci and Montrucchio (2010), who provided sufficient conditions for both existence and uniqueness of solutions, as well as convergence of successive approximations. Their results were obtained via an innovative fixed point approach that exploits concavity and monotonicity properties possessed by Epstein–Zin preferences with empirically plausible parameterizations.

One issue with the conditions of Epstein and Zin (1989) and Marinacci and Montrucchio (2010) is that they require, at least asymptotically, a finite bound M_c on consumption growth C_{t+1}/C_t that holds with probability one. This fails for many standard consumption processes. One example is Bansal and Yaron (2004), where

$$\ln(C_{t+1}/C_t) = \mu_c + z_t + \sigma_t \, \eta_{t+1}. \tag{5}$$

Here $\{z_t\}$ and $\{\sigma_t\}$ are stationary processes and $\{\eta_{t+1}\}$ is IID and $N(0,1)$. Evidently consumption growth is unbounded above.

[1] The Krein–Rutman theorem applies to the valuation operator K in our setting, and its spectral radius equals its dominant eigenvalue. Action of K on the corresponding eigenfunction of the valuation operator is then representative of the long run dynamics induced by K on positive functions. These objects—the principal eigenpairs of valuation operators associated with future cash and utility payoffs—have increasingly been used to understand long run risks and long run values in macroeconomic and financial applications by inducing a decomposition of the stochastic discount factor (see, e.g., Alvarez and Jermann (2005); Hansen and Scheinkman (2009); Qin and Linetsky (2017); Christensen (2017b)). In connecting the role of principal eigenpairs of the valuation operator to the theory of monotone concave operators, we link two active strands of research on present values associated with cash and utility flows.

In fact the problem is not so much that the bounded growth condition fails, since the shocks in (5) can be truncated at suitably large values with minimal impact on the consumption growth process. Rather, the issue is that the resulting restrictions on parameters, which are used to ensure finiteness of the solution to the recursive utility model, are excessively conservative. The intuition behind this is that the probability one bound uniformly restricts utility along every future consumption trajectory. We obtain sharper results by considering what happens "on average" across all paths. This strategy is successful because recursive utility specifications, while nonlinear, are still defined using integration over future continuation values. Hence the whole distribution of the consumption process matters, not just extreme tail realizations. By applying our results to two empirical set ups from the recent literature (Bansal and Yaron (2004), Schorfheide et al. (2017)), we demonstrate that the gap between the necessary and sufficient conditions developed here and the sufficient conditions arising from probability one bounds is both large and significant for modern quantitative applications.[2]

In addition to ourselves, a number of other researchers have sought to extend the work of Epstein and Zin (1989) and Marinacci and Montrucchio (2010). The idea of exploiting monotonicity and concavity of the Koopmans operator has been adapted and extended by Balbus (2016), Becker and Rincón-Zapatero (2017), Bloise and Vailakis (2018) and Marinacci and Montrucchio (2017). While these contributions do not resolve the issues associated with using probability one bounds on consumption growth discussed above, they do elucidate the links between the monotonicity and concavity properties of certain aggregators and fixed point results in partially ordered vector spaces. Such results also lie at the heart of this paper and our proofs draw extensively on their ideas.[3]

Another paper upon which we draw heavily is the study of Epstein–Zin utility models with unbounded consumption growth specifications in Hansen and

[2]The preceding discussion should be qualified by the fact that Epstein and Zin (1989) and Marinacci and Montrucchio (2010) treat a much larger range of state processes than we consider here. Thus, while our results are sharper for the problems we consider, their studies are more comprehensive.

[3]Prior to Marinacci and Montrucchio (2010), contributions to the literature on existence and uniqueness of solutions to recursively defined utility specifications (as well as the closely related problem of optimality of dynamic programs with general aggregators), were made by Koopmans (1960), Lucas and Stokey (1984), Becker et al. (1989), Streufert (1990), Boyd (1990), Ozaki and Streufert (1996), Le Van and Vailakis (2005) and Rincón-Zapatero and Rodríguez-Palmero (2007).

Scheinkman (2012), already mentioned above. Their approach is to connect the solution to the Epstein–Zin utility recursion and the Perron–Frobenius eigenvalue problem associated with a linear operator, denoted in their paper by $\mathbb{T}$, that is proportional to the operator K discussed above. Consumption growth obeys (3) and, unlike this paper, $\{X_t\}$ is not required to be compact-valued. In this very general setting they show that a solution exists when a joint restriction holds on the spectral radius of $\mathbb{T}$ and the preference parameters, in addition to certain auxiliary restrictions. They also obtain a uniqueness result for the case $\theta \geqslant 1$.[4]

In terms of results, the advantage of the conditions in Hansen and Scheinkman (2012) is the lack of a compactness restriction on X_t. The advantages of our approach are as follows: First, we obtain uniqueness of the solution for all θ, not just for $\theta \geqslant 1$. (One reason this matters is that empirical studies typically find that $\theta < 0$.) Second, we obtain conditions that are necessary as well as sufficient, both for existence and for uniqueness. Third, we obtain a globally convergent method of computation, and show that it converges if and only if a solution exists. Fourth, the auxiliary conditions in Hansen and Scheinkman (2012), which generalize our compactness assumption, involve testing integrability restrictions on the eigenfunctions of the operator $\mathbb{T}$. In general these kinds of conditions are difficult to test, unless, of course, one truncates the state space—which is essentially what we do here.[5]

While compactness must be imposed on persistent components of consumption growth in order to use our conditions, these components are bounded in probability in the applications we consider, and hence one can always choose a compactification of the state space such that the impact on the stochastic process for consumption is arbitrarily small. Further comments on compactness are given at the beginning of section 4.

Earlier authors used monotonicity and concavity to obtain fixed points of forward looking recursive models with capital and savings choices. See, for example, Coleman (1991), Datta et al. (2002) and Mirman et al. (2008).

[4]See proposition 6 of Hansen and Scheinkman (2012). Note that the symbol α in their study corresponds to θ here.

[5]Another recent paper that works in a noncompact state setting is Christensen (2017a). There the focus is on robust decision makers, who can also be viewed as utility maximizers with risk-sensitive preferences.

In one final related study, Guo and He (2017) consider an extension to the Epstein–Zin recursive utility model that includes utility measures for investment gains and losses. As a part of that study they obtain results for existence, uniqueness and convergence of solutions to Epstein–Zin recursive utility models with consumption specifications analogous to those in Hansen and Scheinkman (2012), except that the state space is restricted to be finite. In comparison, we allow for the state space to be countably or uncountably infinite and we establish not just sufficiency but also necessity.

The paper is structured as follows: Section 2 states our main results. Sections 3–4 discuss applications. Section 5 begins the process of providing proofs by developing a set of fixed point results in an abstract setting. Section 6 completes the proofs by connecting these abstract results to the Epstein–Zin recursive utility setting.

2. Main Results

To maintain focus on results and applications, we first state all findings on existence, uniqueness and convergence of successive approximations for Epstein–Zin recursive preference models.

2.1. **Definitions and Assumptions.**

Let consumption $\{C_t\}$ be as defined in (3), where the state process $\{X_t\}$ is time homogeneous and Markovian, taking values in some metric space $\mathbb{X}$. The stochastic kernel for $\{X_t\}$ will be denoted by Q. In particular,

$$Q(x, B) = \mathbb{P}\{X_{t+1} \in B \mid X_t = x\}$$

for all $x \in \mathbb{X}$ and all Borel sets $B \subset \mathbb{X}$. The innovation process $\{Y_t\}$ is IID, independent of $\{X_t\}$, and takes values in some topological space $\mathbb{Y}$. The common distribution of each Y_t is a Borel probability measure on $\mathbb{Y}$ denoted by ν.

Assumption 2.1. $\mathbb{X}$ is compact and the growth function κ is continuous.

Let $\mathscr{C}$ be all continuous real-valued functions on $\mathbb{X}$ and let $\|\cdot\|$ be the supremum norm on $\mathscr{C}$. Let $\mathscr{C}_+$ be all nonnegative functions in $\mathscr{C}$ and let $\mathscr{C}_{++}$ be all strictly positive functions. A linear operator K from $\mathscr{C}$ to itself is called *strongly positive* if it maps nonzero elements of $\mathscr{C}_+$ into $\mathscr{C}_{++}$. K is called *compact* if the image under K

of the unit ball in $\mathscr{C}$ is relatively compact. The *operator norm* and *spectral radius* of K are defined by

$$\|K\| := \sup\{\|Kg\| : g \in \mathscr{C}, \ \|g\| \leqslant 1\} \quad \text{and} \quad r(K) := \lim_{n\to\infty} \|K^n\|^{1/n} \tag{6}$$

respectively.[6]

2.2. **The Fixed Point Problem.** Our interest centers on existence, uniqueness and computability of V_t in (1)–(2). In Hansen and Scheinkman (2012), this is converted into a fixed point problem by manipulating (1) to yield the forward looking restriction

$$W_t = \zeta + \beta \left\{ \mathbb{E}_t W_{t+1}^{\theta} \exp[(1-\gamma)\kappa(X_{t+1}, Y_{t+1}, X_t)] \right\}^{1/\theta} \tag{7}$$

where

$$\zeta := 1 - \beta, \qquad W_t := \left(\frac{V_t}{C_t}\right)^{1-1/\psi},$$

and θ is as defined in (4). They then seek a Markov solution $W_t = w(X_t)$ for some $w \colon \mathbb{X} \to \mathbb{R}$.[7]

One disadvantage of treating (7) directly is that the operation on the right hand side involves the composition of a nonlinear transformation (taking W_t to the power θ), a linear transformation (the integral operation embedded in the expectation) and another nonlinear transformation (taking the result to the power $1/\theta$). In order to generate a simpler decomposition, we rewrite (7) as

$$G_t = \left\{ \zeta + \beta \left\{ \mathbb{E}_t G_{t+1} \exp[(1-\gamma)\kappa(X_{t+1}, Y_{t+1}, X_t)] \right\}^{1/\theta} \right\}^{\theta} \tag{8}$$

where $G_t := W_t^{\theta} = (V_t/C_t)^{1-\gamma}$. In terms of the Markov solution $G_t = g(X_t)$, the restriction in (8) translates to

$$g(x) = \left\{ \zeta + \beta \left\{ \int g(x') \int \exp[(1-\gamma)\kappa(x', y', x)] \nu(\mathrm{d}y') Q(x, \mathrm{d}x') \right\}^{1/\theta} \right\}^{\theta}. \tag{9}$$

If we now let K be the linear operator defined on $\mathscr{C}$ by

$$Kg(x) = \beta^{\theta} \int g(x') \int \exp[(1-\gamma)\kappa(x', y', x)] \nu(\mathrm{d}y') Q(x, \mathrm{d}x') \tag{10}$$

[6] The spectral radius can alternatively be defined as the supremum of the modulus of elements of the spectrum of K. These two definitions are equivalent for linear operators on $\mathscr{C}$. In the case where $\mathbb{X}$ is finite, the spectral radius of K reduces to $\max_{\lambda} |\lambda|$, where λ ranges over the eigenvalues of K. See, for example, Kolmogorov and Fomin (1975).

[7] Although $\zeta = 1 - \beta$ in the present case, the results below are valid for any positive constant ζ.

and let ϕ be the scalar function

$$\phi(t) = \left\{\zeta + t^{1/\theta}\right\}^{\theta} \tag{11}$$

on $\mathbb{R}_+$, then (9) can be written more concisely as $g = Ag$, where A is the operator defined at $g \in \mathscr{C}_+$ by

$$Ag(x) = \phi(Kg(x)). \tag{12}$$

Note that A is the composition of two maps: the linear but infinite dimensional operator K and the nonlinear but one dimensional function ϕ. The fact that nonlinear structure is isolated in ϕ is exploited extensively in the proofs. In doing so we adopt the convention $0^{\alpha} = \infty$ and $\infty^{\alpha} = 0$ whenever $\alpha < 0$. In particular, $\phi(0) = 0$ when $\theta < 0$.

Assumption 2.2. K is a strongly positive compact linear operator from $\mathscr{C}$ to itself.

Assumption 2.2 holds in all applications we consider. The following examples help illustrate. Linearity is not discussed since K is obviously linear on $\mathscr{C}$.

Example 2.1. Suppose that $\mathbb{X} \subset \mathbb{R}^n$ and $Q(x, \mathrm{d}x') = q(x, x')\mathrm{d}x'$ for some continuous positive function q. Then K is an integral operator $Kg(x) = \int g(x')k(x, x')\mathrm{d}x'$ with kernel

$$k(x, x') := \beta^{\theta} \int \exp[(1-\gamma)\kappa(x', y', x)]\nu(\mathrm{d}y')q(x, x').$$

Evidently Kg is strictly positive whenever g is nonnegative and nonzero. Joint continuity of k combined with compactness of $\mathbb{X}$ implies compactness of K as a linear operator on $\mathscr{C}$.[8] Hence assumption 2.2 is valid.

Example 2.2. Consider the setting of example 2.1, except that $\mathbb{X}$ is finite, endowed with the discrete topology, and q is a density with respect to the counting measure instead of Lebesgue measure (i.e., q is a stochastic matrix). In this setting all linear operators are compact, so the conditions of assumption 2.2 are again satisfied. The linear operator K can be represented as a matrix with strictly positive entries acting on a space of vectors, and the Perron–Frobenius theorem for strictly positive matrices implies that $r(K)$ is equal to the largest eigenvalue of K.

In seeking fixed points of A we need only consider elements of $\mathscr{C}_{++}$. While $g = 0$ is a fixed point of A when $\theta < 0$, this corresponds to the solution $G_t \equiv 0$, for all

[8] See, for example, Kolmogorov and Fomin (1975), §24.

t, which, in view of $W_t = G_t^{1/\theta}$ indicates infinite utility—a trivial solution for the recursive utility problem. Moreover, if $g \in \mathscr{C}_+$ and g is not everywhere zero, then Ag is strictly positive by assumption 2.2. Hence only in $\mathscr{C}_{++}$ can nontrivial fixed points exist.

For fixed points of A on $\mathscr{C}_{++}$ we have the following result:

Theorem 2.1. *When assumptions 2.1–2.2 hold, the following statements are equivalent:*

(a) $r(K)^{1/\theta} < 1$.
(b) *A has a fixed point in $\mathscr{C}_{++}$.*
(c) *There exists a $g \in \mathscr{C}_{++}$ such that $\{A^n g\}_{n \geqslant 1}$ is convergent in $\mathscr{C}_{++}$.*
(d) *A has a unique fixed point in $\mathscr{C}_{++}$.*
(e) *A has a unique fixed point g^* in $\mathscr{C}_{++}$ and $A^n g \to g^*$ as $n \to \infty$ for any g in $\mathscr{C}_{++}$.*

In any one of the above cases, $G_t = g^*(X_t)$ solves (8) and $V_t := C_t G_t^{1/(1-\gamma)}$ solves the Epstein–Zin recursion (1).

Regarding the proof of theorem 2.1, observe that A is monotone, being the composition of two monotone increasing maps ϕ and K. Moreover, ϕ is concave whenever $\theta < 0$ or $\theta \geqslant 1$. Since K is linear, the operator A is likewise concave in both of these settings. In general, monotone increasing concave operators have unique positive fixed points when their iterates neither collapse to zero nor diverge to infinity. These asymptotics depend on the spectral radius of K. For the remaining case $0 < \theta < 1$ we use a contraction mapping argument, where contraction is required eventually, rather than in the first step. Again, this low frequency property is controlled by the spectral radius $r(K)$. Sections 5–6 provide further details. An interpretation of the condition $r(K)^{1/\theta} < 1$ in terms of intertemporal preferences is given in section 2.3.[9]

There are two interesting implications of theorem 2.1 not previously discussed. One is that, if a solution exists, then, since (b) implies (d), a unique solution exists. In other words, at most one solution exists for every parameterization, and hence uniqueness is never problematic.

[9]In Hansen and Scheinkman (2012) and Guo and He (2017), the condition $r(K)^{1/\theta} < 1$ is expressed in different notation but the corresponding condition (e.g., Assumption 3 in Hansen and Scheinkman (2012)) is in each case equivalent, modulo the different function spaces and hence the definition of the spectral radius.

Another interesting implication is that, since (c) is equivalent to both (d) and (e), convergence of successive approximations from any starting point in $\mathscr{C}_+$ implies that a unique solution exists and this solution is equal to the limit of the successive approximations from every initial condition. Thus, if computing the solution to the model at a given set of parameters is the primary objective, then convergence of the iterative method itself justifies the claim that the limit is a solution, and no other solution exists in the candidate space $\mathscr{C}_+$.

2.3. **Evaluation and Interpretation.** To test the conditions of theorem 2.1, it is necessary to calculate the test value $r(K)^{1/\theta}$, which involves the spectral radius of the operator K defined in (10). For this purpose we adapt a result concerning the local spectral radius of positive operators originally due to V. Ya. Stet'senko (see the proof of proposition 2.2 below). In doing so we also obtain a more natural interpretation of our key condition. In all of what follows we adopt the notation

$$\mathscr{R}[C] := \lim_{n \to \infty} \sup_{x \in \mathbb{X}} \left\{ \mathcal{R}_x \left[\frac{C_n}{C_0} \right] \right\}^{1/n}, \tag{13}$$

where $\mathcal{R}_x$ is defined by

$$\mathcal{R}_x[Y] := \left\{ \mathbb{E}[Y^{1-\gamma} \mid X_0 = x] \right\}^{1/(1-\gamma)}.$$

The term $\mathscr{R}[C]$ represents the risk-adjusted (certainty equivalent) long-run mean consumption growth rate.

Proposition 2.2. *If assumptions 2.1–2.2 hold, then* $r(K)^{1/\theta} = \beta \mathscr{R}[C]^{1-1/\psi}$.

Condition (a) in theorem 2.1 can now be written as

$$\beta \mathscr{R}[C]^{1-1/\psi} < 1. \tag{14}$$

The spectral radius condition thus separates the contributions of time discounting β, intratemporal risk adjustment of the consumption stream embedded in $\mathscr{R}[C]$, and intertemporal substitutability of consumption ψ to the valuation of the consumption stream.

More impatience (a lower β) always makes it more likely that (14) holds. Higher intratemporal risk adjustment γ decreases $\mathscr{R}[C]$ but its impact on our condition depends on the EIS parameter. When preferences are elastic ($\psi > 1$), the income effect of a higher value of future consumption arising from an increase in $\mathscr{R}[C]$ is stronger than the change in the marginal rate of substitution between current and future consumption. Since W_t is denominated in utils per unit of current consumption, it increases and diverges to $+\infty$ as $\mathscr{R}[C]$ increases. The relative strength of the income and substitution effect switches when $\psi < 1$, and $r(K)^{1/\theta} < 1$ is violated when $\mathscr{R}[C]$ is sufficiently low.

Finally, details of the dynamics of the consumption growth process (for instance, its persistence or higher moments of its innovations) matter for our condition only through the long run distribution of consumption growth encoded in $\mathscr{R}[C]$.

2.4. **Connection to Tests Using Almost Sure Bounds.** The preceding discussion facilitates comparison between theorem 2.1 and the tests for existence and uniqueness of continuation values in Epstein and Zin (1989) and Marinacci and Montrucchio (2010). In particular, theorem 3.1 of Epstein and Zin (1989) shows that a solution to the recursive utility problem exists whenever $\psi > 1$ and

$$\beta M_c^{1-1/\psi} < 1, \tag{15}$$

where M_c is an almost sure (i.e., probability one) upper bound on C_{t+1}/C_t. This condition is directly comparable with (14), since $C_{t+1}/C_t \leqslant M_c$ implies $C_n/C_0 \leqslant M_c^n$, and so

$$\mathscr{R}[C] = \lim_{n\to\infty} \sup_{x\in\mathbb{X}} \left\{ \mathcal{R}_x \left[\frac{C_n}{C_0} \right] \right\}^{1/n} \leqslant M_c. \tag{16}$$

Hence the spectral radius based condition in (14) is always weaker. Moreover, the inequality in (16) is strict whenever shocks are non-degenerate, and the gap between the conditions turns out to be large in the quantitative applications we consider (see section 4).

3. Application I: Consumption With a Deterministic Time Trend

In this section we consider a relatively simple case where consumption obeys the geometric trend specification adopted in Alvarez and Jermann (2005). That is,

$$C_t = \tau^t X_t \quad \text{with} \quad \{X_t\} \overset{\text{IID}}{\sim} \pi \quad \text{and} \quad \tau > 0. \tag{17}$$

Here π is a distribution concentrated on $[a, b] \subset \mathbb{R}$ for some positive scalars $a < b$. We now show that the stability coefficient from theorem 2.1 can be calculated analytically when (17) holds. We then demonstrate how the result we obtain improves upon the stability condition presented in Alvarez and Jermann (2005).

When consumption obeys (17), Alvarez and Jermann (2005) show that a unique solution to the Epstein–Zin recursive utility problem exists whenever

$$\beta\tau^{1-\frac{1}{\psi}} \max_{a \leqslant x \leqslant b} \left\{ \int \left(\frac{x'}{x} \right)^{1-\gamma} \pi(\mathrm{d}x') \right\}^{\frac{1}{\theta}} < 1. \tag{18}$$

See Alvarez and Jermann (2005), proposition 9 and lemma A.1 for details.[10]

By comparison, from the definition in (13), we have

$$\begin{aligned} \beta\,\mathscr{R}[C]^{1-\frac{1}{\psi}} &= \beta \lim_{n\to\infty} \sup_{x\in\mathbb{X}} \left\{ \mathcal{R}_x \left(\frac{\tau^n X_n}{X_0} \right) \right\}^{\frac{1-1/\psi}{n}} \\ &= \beta\,\tau^{1-\frac{1}{\psi}} \lim_{n\to\infty} \left\{ \max_{a \leqslant x \leqslant b} \int \left(\frac{x'}{x} \right)^{1-\gamma} \pi(\mathrm{d}x') \right\}^{\frac{1}{n\theta}} . \end{aligned} \tag{19}$$

Taking the limit in (19) and appealing to proposition 2.2 and theorem 2.1, we find the exact necessary and sufficient condition for a unique solution to exist is

$$\beta\,\tau^{1-\frac{1}{\psi}} < 1.$$

This condition is weaker than (18), since the maximized term in (18) exceeds unity. To understand the difference between the two conditions, observe that (18) and (19) are identical apart from the fact that (19) replaces θ with $n\theta$ and takes the limit. This difference arises because the condition from Alvarez and Jermann (2005) enforces contraction in one step. In contrast, the condition $\beta\,\mathscr{R}[C]^{1-1/\psi} < 1$ used in this paper is an asymptotic condition that ignores short-run fluctuations in consumption. This leads to a weaker condition because short-run fluctuations do not impinge on asymptotic outcomes.

4. Application II: Long-Run Risk

Next we turn to two specifications from recent empirical studies and apply theorem 2.1. Since the spectral radius based condition in theorem 2.1 is necessary and sufficient, the outcome determines exactly which parameterizations are stable and which are unstable in a given application. In the stable case $r(K)^{1/\theta} < 1$, a unique solution exists. In the unstable case $r(K)^{1/\theta} \geqslant 1$, no solution exists. To give some

[10]The fixed point of the functional equation solved by Alvarez and Jermann (2005) is equal to $g^{1/\theta} + 1$ in our notation, where g solves (9) with $\zeta := 1$. The conditions for existence and uniqueness are identical in each case.

basis for comparison, we also investigate when the sufficient condition of Epstein and Zin (1989) holds, as given in equation (15).

The test of Epstein and Zin (1989) assumes that consumption growth is bounded. In order to apply it we truncate the stationary distribution of consumption growth at its 95th percentile. While other percentiles can also be considered, our main conclusion is that the probability one tests are too strict in the applications we consider, in the sense that many stable parameterizations fail to meet their conditions. This conclusion is further strengthened if higher percentiles are used to truncate consumption growth. On the other hand, lower percentiles open an increasingly significant gap between the theoretical model and the truncated version.

For the spectral radius based test, the state space must be compact before theorem 2.1 applies. This requires truncating the innovations to the state process in the specifications below. In fact a weak truncation occurs automatically when we implement the models on a computer, with $\mathscr{R}[C]$ in (13) calculated by simulating over consumption paths and taking the (risk adjusted) average. For paths implemented in this way, truncation is inherent, with bounds equal to the smallest and largest floating point numbers that can be implemented on a machine of given precision.[11] Code that replicates all of our computations can be found at

`https://github.com/jstac/recursive_utility_code`

4.1. **Bansal–Yaron Consumption Dynamics.** Suppose first that consumption growth obeys the Bansal and Yaron (2004) specification

$$\begin{aligned} \ln(C_{t+1}/C_t) &= \mu_c + z_t + \sigma_t \, \eta_{c,t+1}, \\ z_{t+1} &= \rho z_t + \phi_z \, \sigma_t \, \eta_{z,t+1}, \\ \sigma^2_{t+1} &= \max\left\{ v \, \sigma^2_t + d + \phi_\sigma \, \eta_{\sigma,t+1}, \, 0 \right\}. \end{aligned}$$

Here $\{\eta_{i,t}\}$ are IID and standard normal for $i \in \{c, z, \sigma\}$. The Markov state for consumption growth is given by $X_t = (z_t, \sigma_t)$.

[11] With standard 64 bit double precision floating point integers, truncation takes place at $\pm 1.8 \times 10^{308}$. The amount of probability mass outside this range is infinitesimally small. Aside from Monte Carlo methods for evaluating (33), we also experimented with projection methods and discretization methods. These computations were slower to run and implement but generated almost identical results.

We calculate the spectral radius based test value $r(K)^{1/\theta}$ associated with these dynamics, as well as the test of Epstein and Zin (1989) discussed in section 2.4.[12] The parameterization is as given in Bansal and Yaron (2004), where the preference parameters are set to $\gamma = 10.0$, $\beta = 0.998$ and $\psi = 1.5$. Here and in all subsequent cases $\zeta = 1 - \beta$. The parameters in the consumption process are $\mu_c = 0.0015$, $\rho = 0.979$, $\phi_z = 0.044$, $v = 0.987$, d =7.9092e-7 and ϕ_σ =2.3e-6.

Evaluating the Epstein and Zin (1989) test value on the left hand side of (15) at these parameters using the methodology described above yields 1.003. At the same time, $r(K)^{1/\theta}$ evaluates to 0.9983. Hence, by theorem 2.1, a unique solution exists. This tells us (i) that the solution to the Bansal–Yaron specification is well-defined, unique and can be computed from any initial condition in $\mathscr{C}_+$ by successive approximation, and (ii) that the probability one condition of Epstein and Zin (1989) is too strict to effectively treat this model.

Figure 1a further illustrates the same point by showing the value of the expression on the left hand side of (15) at the Bansal–Yaron parameterization and also at neighboring parameterizations obtained by varying ψ and μ_c. Almost all values exceed unity, indicating failure of the test, apart from a small measure of parameterizations to the left of the 1.0 contour line. Nonetheless, most of these models are in fact stable, with a unique, globally attracting solution. This is true because, as shown in figure 1b, only for combinations of a very high average growth rate and high intertemporal elasticity of substitution do we find parameterizations with $r(K)^{1/\theta} \geqslant 1$.

4.2. **Schorfheide–Song–Yaron Consumption Dynamics.** Next consider the consumption specification adopted in Schorfheide et al. (2017), where

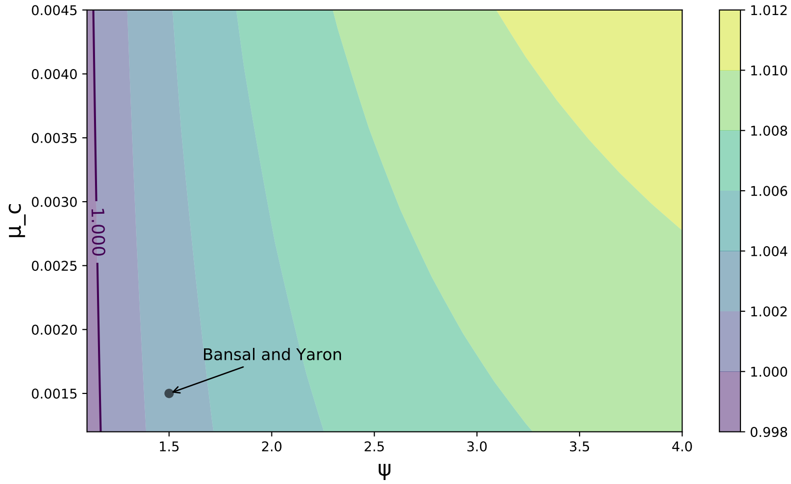
$$\ln(C_{t+1}/C_t) = \mu_c + z_t + \sigma_{c,t}\, \eta_{c,t+1},$$

$$z_{t+1} = \rho\, z_t + \sqrt{1-\rho^2}\, \sigma_{z,t}\, \eta_{z,t+1},$$

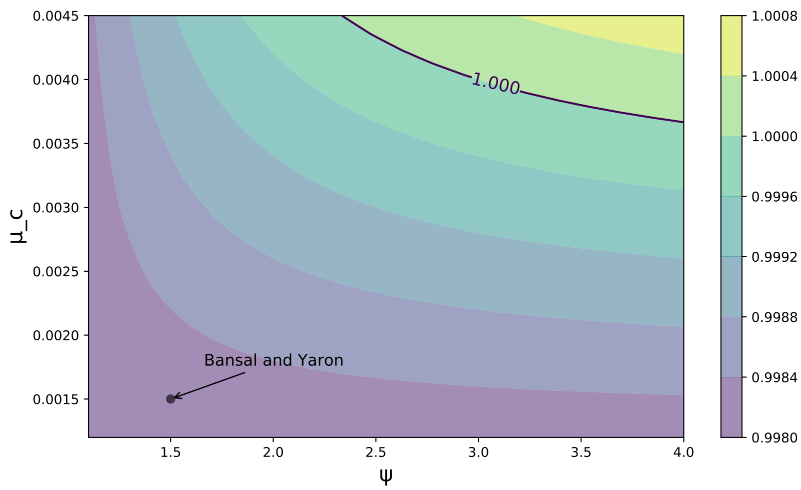
$$\sigma_{i,t} = \phi_i\, \bar\sigma \exp(h_{i,t}) \quad \text{with} \quad h_{i,t+1} = \rho_{h_i} h_i + \sigma_{h_i} \eta_{h_i,t+1}, \quad i \in \{c, z\}.$$

The innovations $\{\eta_{c,t}\}$ and $\{\eta_{h_i,t}\}$ are IID and standard normal for $i \in \{c, z\}$. The state can be represented as $X_t = (\sigma_{c,t}, \sigma_{z,t}, z_t)$.

[12]Since consumption growth is in fact unbounded in this model, to obtain finite M_c in (15) we restrict it by truncating the stationary distribution of consumption growth at the 95th percentile, as discussed above.



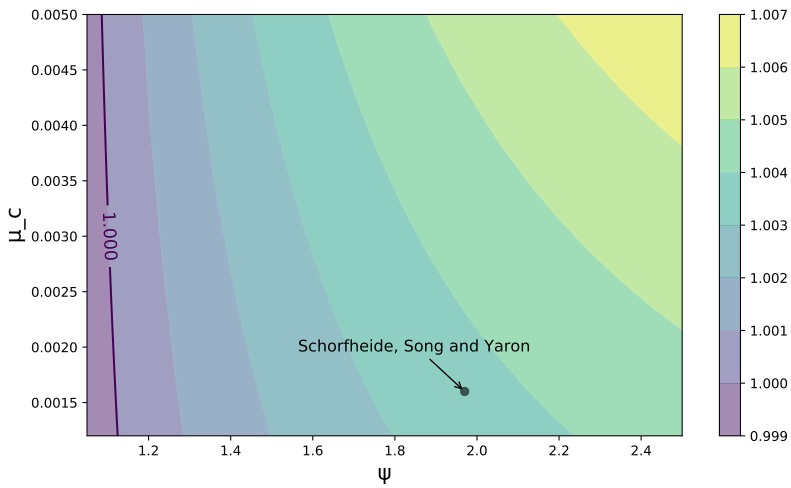
(A) The probability one test value $\beta M_c^{1-1/\psi}$



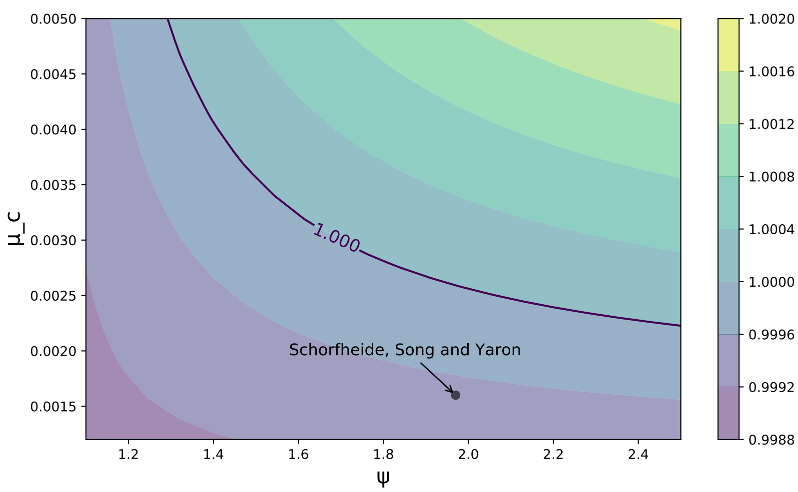
(B) The spectral radius test value $r(K)^{1/\theta}$

FIGURE 1. Stability tests for the Bansal–Yaron model

In Schorfheide et al. (2017), the preference parameters are calibrated to be $\gamma = 8.89$, $\beta = 0.999$ and $\psi = 1.97$, while the parameters in the consumption process are $\mu_c = 0.0016$, $\rho = 0.987$, $\phi_z = 0.215$, $\bar{\sigma} = 0.0032$, $\phi_c = 1.0$, $\rho_{h_z} = 0.992$, $\sigma^2_{h_z} = 0.0039$, $\rho_{h_c} = 0.991$, and $\sigma^2_{h_c} = 0.0096$. Following procedures analogous to those used for the model of Bansal and Yaron (2004), we find that the left hand side of (15) evaluates to 1.003. Thus, the sufficient condition for existence based on the



(A) The probability one test value $\beta M_c^{1-1/\psi}$

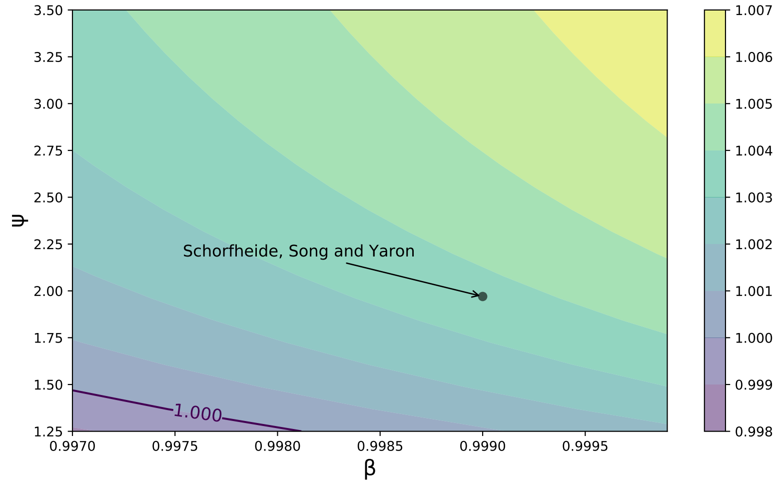


(B) The spectral radius test value $r(K)^{1/\theta}$

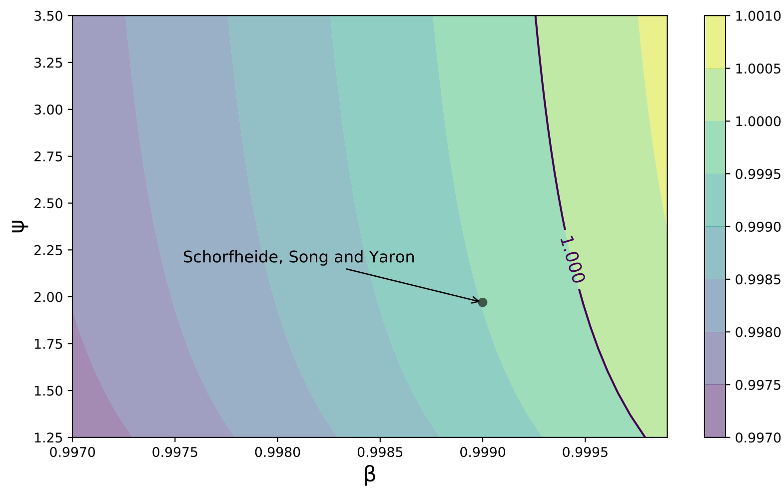
FIGURE 2. Stability tests for the Schorfheide–Song–Yaron model

probability one bound fails to hold. At the same time, by calculations analogous to those used to compute the spectral radius in the Bansal and Yaron (2004) case, we find that $r(K)^{1/\theta} = 0.9995$. It follows that a unique solution does exist, by theorem 2.1.

Figures 2a–2b illustrate further by repeating the exercise of examining the tests at neighboring parameterizations. The results are similar to those obtained for the



(A) The probability one test value $\beta M_c^{1-1/\psi}$



(B) The spectral radius test value $r(K)^{1/\theta}$

FIGURE 3. Stability tests for the Schorfheide–Song–Yaron model

Bansal–Yaron parameterization: The probability one based test is too conservative, excluding many parameterizations that do in fact have unique, well-defined solutions. A similar outcome is observed in figures 3a–3b, when β is varied instead of μ_c.

5. Proofs Part I: General Fixed Point Results

We begin with an abstract fixed point problem for an operator A of the form $Ag(x) = \phi(Kg(x))$, where K is a linear operator and ϕ is a scalar function. We obtain a range of fixed point results depending on the properties of ϕ and K. While our motivation is to solve the formulation of the recursive preference equation given in (12), the abstract view presented here is intended to strip away unnecessary details and facilitate the use of the results below in alternative applications.

5.1. **Preliminaries.** Let $\mathbb{X}$ be any compact metric space. As before, let $\mathscr{C}$ be all continuous real-valued functions on $\mathbb{X}$, let $\leqslant$ be the usual pointwise partial order on $\mathscr{C}$ and let $\| \cdot \|$ be the supremum norm. Note that $\mathscr{C}$ is a Banach lattice with this norm and partial order.[13]

The symbol $\ll$ denotes strict pointwise inequality, so that $f \ll g$ means $f(x) < g(x)$ for all $x \in \mathbb{X}$. The statement $f < g$ means that $f \leqslant g$ and $f(x) < g(x)$ for some $x \in \mathbb{X}$. Given $f \leqslant g$ in $\mathscr{C}$, let

$$[f, g] := \{h \in \mathscr{C} : f \leqslant h \leqslant g\}.$$

The *positive cone* of $\mathscr{C}$, denoted below by $\mathscr{C}_+$, is all $g \in \mathscr{C}$ such that $0 \leqslant g$. As an order cone in $\mathscr{C}$, the set $\mathscr{C}_+$ is solid, normal and reproducing.[14] The interior of $\mathscr{C}_+$ is denoted $\mathscr{C}_{++}$ and contains all $g \in \mathscr{C}$ with $0 \ll g$. An operator A from $\mathscr{C}_0 \subset \mathscr{C}$ into $\mathscr{C}$ is called *increasing* if $Af \leqslant Ag$ whenever $f, g \in \mathscr{C}_0$ and $f \leqslant g$. It is called *concave* if $\mathscr{C}_0$ is convex and, for all $\alpha \in [0, 1]$ and $f, g \in \mathscr{C}_0$,

$$\alpha Af + (1 - \alpha)Ag \leqslant A\{\alpha f + (1 - \alpha)g\}. \tag{20}$$

For a linear operator L from $\mathscr{C}$ to itself, the operator norm, spectral radius, and properties of compactness and strong positivity, are as defined in section 2.1. Note that, for any increasing linear map L on $\mathscr{C}$ we have $|Lf| \leqslant L|f|$ for every $f \in \mathscr{C}$.

A map F from some $\mathscr{C}_0 \subset \mathscr{C}$ to itself is called *globally asymptotically stable* if F has a unique fixed point w^* in $\mathscr{C}_0$ and $F^n w$ converges to w^* from any $w \in \mathscr{C}_0$. We will exploit the following fixed point theorem for monotone concave operators, which

[13] In particular, the metric induced by $\| \cdot \|$ on $\mathscr{C}$ is complete, $\mathscr{C}$ is closed under the taking of pairwise suprema and $|f| \leqslant |g|$ implies $\|f\| \leqslant \|g\|$ for all f, g in $\mathscr{C}$. See Zaanen (1997) for details.

[14] A cone C in $\mathscr{C}$ is called *solid* if it has nonempty interior, *normal* if there exists a constant N with $\|f\| \leqslant N\|g\|$ whenever $f, g \in C$ with $f \leqslant g$, and *reproducing* if every element f of $\mathscr{C}$ can be written as a linear combination of elements of C. See, for example, Du (2006).

is implied by corollary 2.1.1 of Zhang (2013) and the fact that $\mathscr{C}_+$ is both normal and solid in $\mathscr{C}$:

Theorem 5.1 (Du–Zhang)**.** *Let A be increasing and concave on $\mathscr{C}_+$. If, in addition, there exist functions $f_1 \leqslant f_2$ in $\mathscr{C}_+$ with $Af_1 \gg f_1$ and $Af_2 \leqslant f_2$, then A is globally asymptotically stable on $[f_1, f_2]$.*

5.2. **Set Up.** Let $K\colon \mathscr{C} \to \mathscr{C}$ be an increasing linear operator and let $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be continuous. Let Φ be the operator on $\mathscr{C}_+$ defined by $\Phi g = \phi \circ g$. Let A be the operator on $\mathscr{C}_+$ defined by $A = \Phi \circ K$, or, equivalently,

$$(Ag)(x) = \phi(Kg(x)) \qquad (x \in \mathbb{X}). \tag{21}$$

(In what follows, compositions such as $\Phi \circ K$ are written more simply as ΦK.) Note that $A\mathscr{C}_+ \subset \mathscr{C}_+$. Indeed, given that K is linear and increasing, for each fixed $g \in \mathscr{C}_+$ we have $Kg \in \mathscr{C}_+$. Since ϕ is continuous and nonnegative, $\Phi\, Kg$ is also in $\mathscr{C}_+$. Below we consider fixed points of A in $\mathscr{C}_+$ under a range of auxiliary assumptions. Assumptions placed on K and ϕ in this section are always in force.

5.3. **Long Run Contractions.** We begin with a long run contraction result that uses the following Lipschitz bound:

$$|\phi(s) - \phi(t)| \leqslant \ell\, |s - t| \text{ for all } s, t \geqslant 0. \tag{22}$$

Proposition 5.2. *If (22) holds for some $\ell > 0$ and $r(K) < 1/\ell$, then there exists an $n \in \mathbb{N}$ such that A^n is a contraction on $\mathscr{C}_+$. In particular, A is globally asymptotically stable.*

Proof of proposition 5.2. Fix $f, g \in \mathscr{C}_+$. As a first step we claim that

$$|A^n f - A^n g| \leqslant (\ell K)^n |f - g| \tag{23}$$

holds pointwise on $\mathbb{X}$ for all integers $n \geqslant 0$. It holds for $n = 0$, since A^0 and K^0 are by definition identity maps. Now suppose that it holds for some $n \geqslant 0$. We claim it also holds at $n + 1$. Indeed

$$\left| A^{n+1} f - A^{n+1} g \right| = |\Phi K\, A^n f - \Phi K\, A^n g| \leqslant \ell\, |K A^n f - K A^n g|\,,$$

where the inequality is due to (22). Using this bound and the linearity and monotonicity of K leads us to

$$\left| A^{n+1} f - A^{n+1} g \right| \leqslant \ell\, |K(A^n f - A^n g)| \leqslant \ell K\, |A^n f - A^n g| \leqslant (\ell K)^{n+1} |f - g|,$$

where the last inequality uses the induction hypothesis combined with the monotonicity of K. We have now confirmed that (23) holds for all $n \geqslant 0$.

Taking the supremum over (23) yields $\|A^n f - A^n g\| \leqslant \|(\ell K)^n |f - g|\|$. The definition of the operator norm then gives

$$\|A^n f - A^n g\| \leqslant \|(\ell K)^n\| \cdot \|f - g\| = \ell^n \, \|K^n\| \cdot \|f - g\|.$$

From the definition $r(K) = \lim_{n\to\infty} \|K^n\|^{1/n}$, we have

$$(\ell^n \, \|K^n\|)^{1/n} \to \ell r(K) < 1.$$

Hence, for sufficiently large n, the map A^n is a contraction with modulus $\ell^n \|K^n\|$. All claims in proposition 5.2 follow from this property, completeness of $\|\cdot\|$, the fact that $\mathscr{C}_+$ is closed in $\mathscr{C}$, and a well-known extension to the Banach contraction mapping theorem (see, e.g., p. 272 of Wagner (1982)). □

5.4. **Fixed Points under Monotonicity and Concavity.** If $\phi(0) = 0$, then $0 \in \mathscr{C}_+$ is clearly a fixed point of A. In applications this fixed point is often trivial, and interest centers on positive fixed points. In such settings it turns out that notions of concavity and monotonicity are more helpful than contractivity arguments for establishing fixed point results. We begin with a simple lemma.

Lemma 5.3. *For the operator $A = \Phi K$ defined above, the following statements hold.*

(a) If ϕ is increasing on $\mathbb{R}_+$, then A is increasing on $\mathscr{C}_+$.
(b) If ϕ is concave on $\mathbb{R}_+$, then A is concave on $\mathscr{C}_+$.

Proof. Claim (a) is obviously true, given that K is already assumed to be increasing. Regarding claim (b), since K is linear, for any fixed $f, g \in \mathscr{C}_+$, $x \in \mathbb{X}$, and $\alpha \in [0,1]$, we have

$$\begin{aligned}\alpha\phi(Kf(x)) + (1-\alpha)\phi(Kg(x)) &\leqslant \phi(\alpha Kf(x) + (1-\alpha)Kg(x)) \\ &= \phi[K(\alpha f(x) + (1-\alpha)g(x))].\end{aligned}$$

In other words, $\alpha Af(x) + (1-\alpha)Ag(x) \leqslant A[\alpha f(x) + (1-\alpha)g(x)]$ for any $x \in \mathbb{X}$, as was to be shown. □

Lemma 5.4. *If ϕ is strictly concave, K is strongly positive, and, in addition,*

$$\phi(0) = 0, \quad \lim_{t\downarrow 0} \frac{\phi(t)}{t} \leqslant 1 \quad \textit{and} \quad r(K) \leqslant 1, \tag{24}$$

then the only fixed point of A in $\mathscr{C}_+$ is 0.

Proof. Since K is linear we have $K0 = 0$. Hence $\phi(0) = 0$ implies $A0 = 0$. Thus, the zero element is a fixed point. It remains to show that no other fixed point exists in $\mathscr{C}_+$. In doing so we use the fact that ϕ must be strictly positive everywhere on $(0, \infty)$, since the existence of a positive t with $\phi(t) = 0$ violates our assumption that ϕ is both strictly concave and nonnegative.

Seeking a contradiction, suppose that $g \in \mathscr{C}_+$ is a nonzero fixed point of A. Observe that, since $Ag = g$ and g is nonzero, the fact that K is a strongly positive operator and ϕ is positive on $(0, \infty)$ implies that $g \gg 0$. In particular, the constant $m := \min g$ is strictly positive. Let $\eta := \phi(m)/m$. Note that, by strict concavity of ϕ and the assumption that $\frac{\phi(t)}{t} \to 1$ as $t \downarrow 0$, we have

$$\eta < 1 \quad \text{and} \quad \phi(t) \leqslant \eta t \text{ whenever } t \geqslant m. \tag{25}$$

Observe that $Kg \gg g$ must hold. To see why, suppose that $Kg(x) \leqslant g(x)$ for some x. Invoking strict concavity and the limit in (24) again, we have $\phi(t) < t$ for any positive t, and hence $Ag(x) = \phi(Kg(x)) < Kg(x) \leqslant g(x)$. This contradicts the assumption that g is a fixed point of A. Our claim that $Kg \gg g$ is confirmed.

Next we claim that $A^n g \leqslant (\eta K)^n g$ for all n. Evidently this holds at $n = 0$, and, assuming it holds at n, we have

$$A^{n+1} g = \Phi K A^n g \leqslant \eta K A^n g \leqslant \eta K (\eta K)^n g = (\eta K)^{n+1} g.$$

In the first inequality we used the fact that $Kg \gg g$ and g is a fixed point of A, so that $KA^n g = Kg \gg g \geqslant m$ when m is as given in (25). In the second inequality we used the induction hypothesis and the monotonicity of K.

We have now shown by induction that $A^n g \leqslant (\eta K)^n g$ for all $n \in \mathbb{N}$. Hence

$$\|A^n g\| \leqslant \eta^n \|K^n g\| \leqslant \eta^n \|K^n\| \, \|g\| \tag{26}$$

for all n. Since $r(K) \leqslant 1$ and $\eta < 1$, the definition of $r(K)$ implies existence of an $n \in \mathbb{N}$ such that $\|K^n\|^{1/n} < 1/\eta$, or $\|K^n\| < (1/\eta)^n$. Evaluating (26) at this n gives $\|A^n g\| < \|g\|$, contradicting our assumption that g is a fixed point of A. $\square$

Below we will make use of the following version of the Krein–Rutman theorem, the value of which for studying recursive preference models was identified and illustrated in Hansen and Scheinkman (2009, 2012).

Lemma 5.5 (Krein–Rutman)**.** *If, in addition to being linear and increasing, K is also strongly positive and compact, then $r(K) > 0$ and $r(K)$ is an eigenvalue of K. In particular, there exists an $e \in \mathscr{C}_+$ such that $Ke = r(K)e$, and $e \gg 0$.*

Lemma 5.5 follows directly from theorem 1.2 of Du (2006), given that $\mathscr{C}_+$ is both solid and reproducing. The element e in lemma 5.5 is unique up to a scale factor. In what follows we normalize by requiring that $\|e\| = 1$, and call e the *Perron–Frobenius* eigenfunction of K.

Lemma 5.6. *If K is strongly positive and compact, and ϕ and K jointly satisfy*

$$\lim_{t \downarrow 0} \frac{\phi(t)}{t} r(K) > 1 \quad \text{and} \quad \lim_{t \uparrow \infty} \frac{\phi(t)}{t} r(K) < 1, \tag{27}$$

then there exist positive constants $c_1 < c_2$ with the following properties:

(a) If $0 < c \leqslant c_1$ and $f = ce$, then $f \ll Af$.
(b) If $c_2 \leqslant c < \infty$ and $f = ce$, then $Af \ll f$.

Proof. Let $\lambda := r(K)$. Since K is strongly positive and compact, the Perron–Frobenius eigenfunction e discussed above is well defined. Regarding claim (a), observe that, in view of (27), there exists an $\epsilon > 0$ such that

$$\frac{\phi(t)}{t} \lambda > 1 \quad \text{whenever} \quad 0 < t < \epsilon.$$

Choosing c_1 such that $0 < c_1 < \epsilon/\lambda$ and $c \leqslant c_1$, we have $c\lambda e(x) \leqslant c_1 \lambda \|e\| = c_1 \lambda < \epsilon$, and hence

$$Ace(x) = \phi(cKe(x)) = \phi(c\lambda e(x)) = \frac{\phi(c\lambda e(x))}{c\lambda e(x)} c\lambda e(x) > ce(x).$$

Since $x \in \mathbb{X}$ was arbitrary, the first claim in the lemma is verified.

Turning to claim (b) and using again the hypotheses in (27), we can choose a finite M such that

$$\frac{\phi(t)}{t} \lambda < 1 \quad \text{whenever} \quad M < t.$$

Let m be the minimum of e on $\mathbb{X}$. Since $\mathbb{X}$ is compact and $e \gg 0$, we have $m > 0$. Let c_2 be a constant strictly greater than $\max\{M/(\lambda m), c_1\}$ and let c lie in $[c_2, \infty)$. By the definition of m we have $c\lambda e(x) \geqslant c_2 \lambda e(x) > M$ for all $x \in \mathbb{X}$, from which it follows that

$$Ace(x) = \phi(c\lambda e(x)) = \frac{\phi(c\lambda e(x))}{c\lambda e(x)} \lambda ce(x) < ce(x).$$

By construction, $0 < c_1 < c_2$, so all claims are now established. $\square$

Proposition 5.7. *Let K be strongly positive and compact, and let the conditions in (27) hold. If, in addition, ϕ is increasing and concave, then A is globally asymptotically stable on $\mathscr{C}_{++}$.*

Proof. Given that ϕ is increasing and concave on $\mathbb{R}_+$, lemma 5.3 implies that A is increasing and concave on $\mathscr{C}_+$. Since lemma 5.6 implies existence of a pair f_1, f_2 such that $Af_1 \gg f_1$ and $Af_2 \leqslant f_2$, and since the function f_1 can be chosen from $\mathscr{C}_{++}$, theorem 5.1 implies that A has a fixed point g^* in $\mathscr{C}_{++}$.

Next we claim that

$$\forall\, g \in \mathscr{C}_{++}, \;\; \exists\, f_1, f_2 \in \mathscr{C}_{++} \text{ such that } f_1 \leqslant g, g^* \leqslant f_2, \;\; Af_1 \gg f_1 \text{ and } Af_2 \leqslant f_2. \tag{28}$$

To see this, fix $g \in \mathscr{C}_{++}$. Since $g \gg 0$ and $\mathbb{X}$ is compact, g attains a finite maximum and strictly positive minimum on $\mathbb{X}$. The same is true of the existing fixed point g^* and the Perron–Frobenius eigenfunction e. Hence, we can choose constants a_1 and a_2 such that $0 \ll a_1 e \leqslant g^*, g \leqslant a_2 e$. With a_1 chosen sufficiently small and a_2 sufficiently large, lemma 5.6 implies that $a_1 e \ll A(a_1 e)$ and $A(a_2 e) \ll a_2 e$. Thus (28) holds.

Turning to uniqueness, let g^{**} be a second fixed point of A in $\mathscr{C}_{++}$. By (28) there exist $f_1, f_2 \in \mathscr{C}_{++}$ such that $f_1 \leqslant g^{**}, g^* \leqslant f_2$ with $f_1 \ll Af_1$ and $Af_2 \ll f_2$. By theorem 5.1, the interval $[f_1, f_2]$ contains only one fixed point. Thus, $g^* = g^{**}$.

Finally, regarding convergence, let g be an element of $\mathscr{C}_{++}$. Invoking (28) establishes the existence of $f_1, f_2 \gg 0$ such that $f_1 \leqslant g, g^* \leqslant f_2$ with $f_1 \ll Af_1$ and $Af_2 \ll f_2$. By theorem 5.1, every element of $[f_1, f_2]$ converges to g^* under iteration of A. In particular, $A^n g \to g^*$ as $n \to \infty$. $\square$

6. Proofs Part II: The Recursive Utility Problem

Now we turn to the specific operator A given in (12), with ϕ defined as before by

$$\phi(t) = \left\{ \zeta + t^{1/\theta} \right\}^{\theta} \tag{29}$$

and K is as given in (10). Our aim is to establish the claims in theorem 2.1 using the abstract fixed point results from section 5.

To establish theorem 2.1, it suffices to show that $(a) \implies (e) \implies (d) \implies (c) \implies (b) \implies (a)$. Of these, $(e) \implies (d)$ and $(d) \implies (c)$ are obvious (for $(d) \implies (c)$, just take g to be the fixed point). We now prove the remainder.

($(a) \implies (e)$ when $\theta < 0$). Suppose that $r(K)^\theta < 1$, or, equivalently, $r(K) > 1$. We need to show that A is globally asymptotically stable on $\mathscr{C}_{++}$, for which it suffices that the conditions of proposition 5.7 hold. The operator K is strongly positive by assumption. Given that $\theta < 0$, the map ϕ is strictly increasing and strictly concave on $\mathbb{R}_+$. Thus, we need only show that the two inequalities in (27) are valid. Regarding the first inequality, we have

$$\frac{\phi(t)}{t} = \left\{ \frac{\zeta}{t^{1/\theta}} + 1 \right\}^\theta \uparrow 1 \quad \text{as } t \downarrow 0.$$

Since $r(K) > 1$, the first inequality holds. Regarding the second inequality in (27), evidently $\phi(t)/t \to 0$ as $t \to \infty$, so this bound certainly holds. The proof of (e) is therefore complete.

($(a) \implies (e)$ when $\theta > 0$). Suppose first that $\theta \in (0, 1)$. The conditions of proposition 5.2 are then satisfied, since $r(K) < 1$ by assumption and ϕ is Lipschitz of order 1. From this proposition we see that A is globally asymptotically stable on $\mathscr{C}_+$, with a unique fixed point g^*. This fixed point lies in $\mathscr{C}_{++}$, since 0 is not a fixed point of A (because $A0 = \phi(K0) = \phi(0) > 0$), and, moreover, if g^* is nonzero then Kg^* is strictly positive, and hence so is $Ag^* = \Phi K g^*$. Thus, A is also globally asymptotically stable on $\mathscr{C}_{++}$. Hence (e) is valid.

Now consider the remaining case $\theta \geqslant 1$, while continuing to assume that $r(K) < 1$. For such θ the function ϕ is increasing and concave, and, since $r(K) < 1$, the conditions in (27) are both satisfied. Hence proposition 5.7 applies, and A is globally asymptotically stable on $\mathscr{C}_{++}$. Hence (e) holds

($(c) \implies (b)$). Let $g \in \mathscr{C}_{++}$ be such that $\{A^n g\}$ is convergent in $\mathscr{C}_{++}$, with limit denoted by $\bar{g}$. Evidently A is continuous, being the composition of continuous mappings.[15] Thus, $A\bar{g} = A(\lim_{n\to\infty} A^n g) = \lim_{n\to\infty} A^{n+1} g = \bar{g}$. In particular, $\bar{g}$ is a fixed point of A.

($(b) \implies (a)$ when $\theta < 0$). Suppose to the contrary that $r(K) \leqslant 1$. To show that A has no fixed point in $\mathscr{C}_{++}$, recall lemma 5.4. We have $r(K) \leqslant 1$ by assumption and

[15] Here we are using the fact that K is a compact operator and hence continuous.

the other conditions (24) have already been checked. Hence lemma 5.4 applies, and A has no fixed point in $\mathscr{C}_{++}$.

($(b) \implies (a)$ when $\theta > 0$). First we make some observations about $\phi(t) = (\zeta + t^{1/\theta})^\theta$ on $\mathbb{R}_+$ when $\theta > 0$. Evidently ϕ is continuous and increasing with $\phi(t) > t$ for all $t \in \mathbb{R}_+$. It is not difficult to see that, in addition,

$$0 \leqslant s \leqslant t \implies \phi(s) \geqslant \frac{\phi(t)}{t} s \qquad \text{and} \qquad \lim_{n\to\infty} \phi^n(t) = \infty, \; \forall\, t \geqslant 0. \tag{30}$$

Now suppose that A has a fixed point $g \in \mathscr{C}_{++}$ and yet $r(K) \geqslant 1$. Let e be the Perron–Frobenius eigenfunction of K and let c be a positive constant such that $e_c := ce \leqslant g$. Such a c exists because $\min_{x\in\mathbb{X}} g(x)$ is strictly positive and $\max_{x\in\mathbb{X}} e(x)$ is finite. Let t_0 be a positive constant such that $e_c \leqslant t_0$ on $\mathbb{X}$. We claim that

$$\forall n \in \mathbb{N}, \quad A^n e_c \geqslant \frac{\phi^n(t_0)}{t_0} e_c \;\text{ on }\; \mathbb{X}. \tag{31}$$

To see this, observe that (31) holds at $n = 0$. Now suppose that (31) holds at some $n \geqslant 0$. We then have

$$A^{n+1} e_c(x) = \phi(K A^n e_c(x)) \geqslant \phi\left(\frac{\phi^n(t_0)}{t_0} K e_c(x)\right) \geqslant \phi\left(\frac{\phi^n(t_0)}{t_0} e_c(x)\right).$$

Here the first inequality is by the induction hypothesis, the monotonicity of ϕ and K and the linearity of K. The second is from $r(K) \geqslant 1$, which gives $K e_c = cKe = cr(K)e \geqslant ce = e_c$. Using $e_c(x) \leqslant t_0$ and the first property in (30), we have

$$A^{n+1} e_c(x) \geqslant \phi\left(\frac{\phi^n(t_0)}{t_0} e_c(x)\right) \geqslant \frac{\phi(\phi^n(t_0))}{\phi^n(t_0)} \frac{\phi^n(t_0)}{t_0} e_c(x) = \frac{\phi^{n+1}(t_0)}{t_0} e_c(x).$$

Thus, the statement in (31) is valid.

From (31) and the second property in (30) we conclude that $A^n e_c$ diverges to $+\infty$. Moreover, the fixed point g satisfies $g \geqslant e_c$, so $A^n g \geqslant A^n e_c$. Hence $A^n g$ eventually exceeds g, contradicting our assumption that g is a fixed point of A.

The proof of theorem 2.1 is now done.

APPENDIX

It remains to complete the proof of proposition 2.2:

Proof of proposition 2.2. Theorem 9.1 of Krasnosel'skii et al. (2012) shows that if L is a linear operator that leaves a solid normal order cone invariant, then

$$r(L) = \lim_{n\to\infty} \|L^n g\|^{1/n} \tag{32}$$

whenever g lies in the interior of that cone. Since the positive cone $\mathscr{C}_+$ of $\mathscr{C}$ is both solid and normal, and since the operator K is strongly positive, we can apply (32) to the interior element $g = \mathbb{1} \equiv 1$ to obtain $r(K) = \lim_{n\to\infty} \|K^n \mathbb{1}\|^{1/n}$. A straightforward inductive argument shows that

$$\lim_{n\to\infty} \|K^n \mathbb{1}\|^{1/n} = \beta^\theta \lim_{n\to\infty} \left\{ \sup_{x\in\mathbb{X}} \mathbb{E}_x \exp\left[(1-\gamma) \ln\left(\frac{C_n}{C_0}\right)\right]\right\}^{1/n}. \tag{33}$$

We have now shown that $r(K)$ is equal to the right hand side of (33), which is equivalent to the claim in proposition 2.2. □

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