Partnership with Persistence

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VERY PRELIMINARY AND INCOMPLETE—PLEASE DO NOT CIRCULATE

Abstract

In this paper we analyze a continuous-time model of partnership with persistence. In the model, agents exert private efforts affecting persistent internal capital, which drives the profitability of the partnership. We characterize the optimal equilibrium with a novel Hamilton-Jacobi-Bellmann equation. It describes the maximal incentives for the partners, as a function of continuation values net of the internal capital. We show that imperfect monitoring of the internal capital discontinuously helps the agents. Even a partnership with high level of internal capital may unravel as a consequence of a short spell of bad outcomes. Good profit outcomes increase effort when partnership is doing badly, but decrease effort when partnership is doing well.

1 Introduction

Partnerships are one of the dominant forms of organizing a joint enterprise. Characterized by a fixed rule for sharing the joint benefits, partnerships are common not only as informal arrangements, but are also one of the key ways to organize firms. They are particularly prevalent in the professional sector, such as with law firms, financial

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consulting or tech start-ups. In the absence of strict monitoring of partners' actions, they also give rise to an obvious free-riding problem: a partner exerts private effort to contribute to a common good. Economists have long recognized the problem and tried to understand how partnerships may overcome this inherent incentive issue.

In this paper we present a partnership model with persistent effect of effort, and develop a novel method to characterize the optimal incentive provision. In our model each agent exerts a costly effort to contribute to the persistent "internal capital" of the partnership. This is an umbrella term that captures all the partnership's intangible assets, such as quality of the products, breadth and loyalty of the customer base, or quality of the internal organization. Neither private efforts nor the internal capital are publicly observed, but the internal capital affects the observable profits which follow a Brownian diffusion process. The model thus gives rise to a novel Brownian stochastic game with imperfectly observable state.

Despite the well known difficulties of solving dynamic games with persistence, compounded by imperfect monitoring, we derive a tractable characterization of the optimal perfect public equilibrium. We characterize the optimal equilibrium with a novel Hamilton-Jacobi-Bellmann (HJB) ordinary differential equation. It uses a novel state variable—the continuation value net of the effects of inherited internal capital—and characterizes not the maximal continuation values, as is standard in the literature, but the maximal stock of incentives for the partners. We show that imperfect monitoring of internal capital provides *fine sand in the wheels* of the partnership, increasing its efficiency. While with perfectly monitored internal capital any dynamic incentive provision is impossible, imperfect monitoring discontinuously restores some of the efficiency.

Furthermore, the persistent effects of actions gives rise to a rich dynamics of efforts, internal capital levels, and profits. In particular, we show that a high level of internal capital does not guarantee a long-lived partnership. We show that a *Beatles' break-up* is possible—even a partnership with high level of internal capital may unravel as a consequence of a short spell of bad outcomes. Regarding the provision of effort, when the future of the partnership is looking badly, good profit outcomes increase efforts of the partners through a standard ratchet effect. However, partners may *coast it* and decrease their effort after a string of good outcomes, if the future of the partnership is looking good.

The two crucial features of our model are a persistent state, which is imperfectly observed by agents, and a parsimonious signal structure. Persistence and imperfect monitoring are well-understood to make models intractable, yet they are an important feature of many dynamic relationships. The persistent state in our model is the internal capital of the company. It is a reduced form of all the slowly changing intangible assets of the company. In our continuous-time model, the profits at any point of time depend not only on this moment's efforts, but on the internal capital, slowly shaped by both partners' effort in the past.

Beside the usual moral hazard assumption of unobservable efforts, we assume that the only public signal is the profit of the partnership. This assumption has two main implications. First, the internal capital is not observed by either player. Second, each partner is not separately monitoring the other. Allowing unobservable internal capital will end up playing a crucial role in our model. We take this relaxing of the perfect observability as a step to make the model more realistic. On the other hand, not allowing the signals to separately identify each partner's actions is restrictive. In the solution, it allows us to focus on the strongly symmetric perfect public equilibria (SSPPE), without asymmetric punishments of potential deviators. Clearly, each player investing in a separate monitoring technology of the other agent's actions would be an alternative way to address the moral hazard problem. We are interested in how the incentive problem may be alleviated with only the minimal, and readily available, information of the firm's profits.

The first contribution of the paper is a novel method to characterize the solution of this model with persistence. The first step is the choice of a state space and a maximized variable. The standard approach is to consider *continuation value*, and/or perhaps an exogenous state variable, and maximize the continuation utility of the other agent. In our case, we show that the state space can be reduced to a single variable: the continuation value of the partners net of the expected discounted profits from the inherited internal capital. It is an analogue of the continuation value in models without persistence, and can be understood as partners' goodwill towards the future of the relationship. Our characterization is based on a novel HJB ordinary differential equation, which parametrizes the maximal level of marginal benefits of effort, or simply *incentives* deliverable in a SSPPE, for any level of net continuation value.

Maximizing incentives might seem like a strange choice. After all, the optimal level of effort in our model is interior, and so it is well possible to overincentivize the partners. Moreover, we are ultimately interested in the maximal level of net continuation values deliverable in equilibrium. Our approach is based on an idea of tracing out the upper boundary of the set of incentive-net continuation value pairs that are achievable in a SSPPE, as a way of getting to the righter-most point of maximal net continuation value. In the proof we establish that this boundary is self-generating and smooth, and so satisfies the HJB equation. More precisely, the HJB equation is based on the First Order Approach of maximizing incentives only in local SSPPE, which prevent only local deviations. In a separate result we verify analytically that as long as the additively separable cost of effort is convex enough, the constructed strategies are fully incentive compatible.

With perfectly observable internal capital, Sannikov and Skrzypacz (2007) and Bohren (2016) established that the only equilibria are Markovian. In our case, the only Markovian equilibrium is the inefficient repeated static Nash (Proposition 2). Somewhat paradoxically, imperfectly observable state allows for nontrivial equilibria and improves efficiency. Moreover, the optimal setting is one with "fine sand in the wheels": as long as the observational noise is positive, efficiency increases when noise goes down . This means that the trivial equilibrium is an anomaly, or, formally, it lies at a point where upper hemi-continuity fails. The intuition why the triviality is avoided is as follows. With persistence and imperfectly observable state, signals about high effort today will be leaking throughout the future. When the state is perfectly observed, it is not possible to deliver incentives at the bliss point of the highest (net) continuation value (Sannikov and Skrzypacz (2007)). With noise, agents' effort will be rewarded later on, once the continuation value drifts below the bliss point.

The optimal SSPPE exhibits a rich dynamics of mean internal capital, partners' efforts and profits. Its non-Markovian nature implies that the internal capital does not determine the state or future of the partnerhip. In particular, the model can explain why even the apparently successful and profitable partnerships unravel ("Beatles' breakup"). In the quilibrium, a brief spell of few bad signals can bring partnership to the inefficient repeated static Nash equilibrium, while at the same time barely affect a sound level of internal capital, and so profitability.

Moreover, in a model without persistence, rewards and punishments are meted out period by period, and so high signal may never lead to low effort. With persistence, when effort will be rewarded throughout the future, equilibrium exhibits both "ratcheting" and "coasting". When partnership is close to unravelling, effort will likely not matter for the partnership beyond the direct noncooperative effect on the internal capital. As a consequence, a good profit outcome brings the partnership further away from the brink and encourages higher effort. In contrast, when partnership is close to the optimal bliss-point, the effort of the partners decreases after good outcomes.

1.1 Relationship to the Literature (soon)

2 Model

We consider a game that unfolds in continuous time as follows. At every moment of time $t \geq 0$ each of two partners privately chooses effort $a_t^i \in \mathbb{R}_+$, $i = 1, 2$. Total effort contributes to the *internal capital* μ_t of the partnership, which deteriorates at rate $\alpha > 0$, and is unobservable to the partners. Internal capital in turn determines the publicly observable flow of partnership profits dY_t,

$$d\mu_t = (a_t^1 + a_t^2 - \alpha\mu_t)dt + \sigma_\mu dB_t^\mu, \tag{1}$$
$$dY_t = \mu_t dt + \sigma_Y dB_t^Y,$$

where $\{B_t^\mu\}_{t\geq 0}$ and $\{B_t^Y\}_{t\geq 0}$ are two independent Brownian Motions.

Exerting effort a involves private flow cost $c(a)$, for a twice differentiable, increasing, and strictly concave cost of effort function c. Partners split the profits evenly, are risk neutral, and discount the future at rate $r > 0$. For fixed effort strategies of both partners this results in $i's$ expected discounted continuation payoffs

$$W_\tau^i = \mathbb{E}^{\{a_t^1, a_t^2\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\mu_t}{2} - c(a_t^i)\right)dt\right].$$

The dynamic game has one persistent state variable, the internal capital μ_t. It changes stochastically, driven by the efforts of the partners and subject to *production noise* $\sigma_\mu dB_t^\mu$. The only publicly observable signal is the profit flow dY_t, which is an imperfect signal of the state, subject to *observation noise* $\sigma_Y dB_t^Y$. The game thus has a novel structure of a continuous time *stochastic game with imperfectly observable state.* In the case of $\sigma_Y = 0$ it reduces to a stochastic game with perfect monitoring (see Bohren (2016)).

Exploiting the exponential decay of the internal capital, we may rewrite the continuation payoffs as

$$\begin{aligned} W_\tau^i &= \mathbb{E}^{\{a_t^1, a_t^2\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{u_\tau}{2}e^{-\alpha(t-\tau)} + \frac{1}{2}\int_\tau^t (a_s^1 + a_s^2)e^{-a(t-s)}ds - c(a_t^i)\right)dt\right] \qquad (2) \\ &= \frac{\overline{\mu}_\tau}{2(r+\alpha)} + \mathbb{E}^{\{a_t, a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{a_t^1 + a_t^2}{2(r+\alpha)} - c(a_t^i)\right)dt\right], \end{aligned}$$

where

$$\overline{\mu}_\tau = \mathbb{E}^{\{a_t^1, a_t^2\}}[\mu_\tau]$$

is the expected internal capital, given the pair of strategies $\{a_t^1, a_t^2\}_{t\geq 0}$ and the history of public signals $\{Y_t\}_{t\in[0,\tau)}$.

In the following we characterize the optimal Perfect Public Equilibria of this partnership game. Formally, we say that a strategy $\{a_t^i\}_{t\geq 0}$ is public if at any time $\tau > 0$ it depends only on the public history of cumulative profits $\{Y_t\}_{t\in[0,\tau)}$. A pair of public strategies $\{a_t^1, a_t^2\}_{t\geq 0}$ is a *Perfect Public Equilibrium (PPE)* if for each partner i, at any time $\tau \geq 0$ and after any public history $\{Y_t\}_{t\in[0,\tau)}$

$$\begin{aligned} &\mathbb{E}^{\{a_t^i, a_t^{-i}\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\mu_t}{2} - c(a_t^i)\right) dt \middle| \{Y_t\}_{t\in[0,\tau)}\right] \\ &\geq \mathbb{E}^{\{\widetilde{a}_t^i, a_t^{-i}\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\mu_t}{2} - c(\widetilde{a}_t^i)\right) dt \middle| \{Y_t\}_{t\in[0,\tau)}\right]. \end{aligned} \tag{3}$$

The defining feature of a partnership is a fixed rule for splitting the profits. This gives rise to an obvious freeriding problem. At any point of time, a partner incurs private cost of effort, whereas the (expected discounted) benefits of the effort will be split evenly among the two partners. It follows from (2) that the efficient level effort level a^{EF} is constant over time and satisfies

$$\frac{a_{EF}}{r+\alpha} = c'\left(a_{EF}\right), \tag{4}$$

whereas the unique stationary equilibrium, in which each partner chooses the same action a^{NE}, satisfies

$$\frac{a_{NE}}{2\left(r+\alpha\right)} = c'\left(a_{NE}\right).$$

With some abuse of terminology, we call a pair of strategies that play a_{NE} after every hstory a *repeated static Nash equilibrium.* A partnership *unravels* if from that point on partners play repeated static Nash equilibrium.

One partnership game is characterized by a parsimonious information structure. The only information about the effort of the partners comes from the stream of profits, and the efforts enter the profits additively. Consequently, it is not possible to idetify which of the partners did, and which one did not contribute to the common good based solely on the public signals. This implies that it is not possible to provide incentives by continuation value "transfers" between the agents, shifting resources from likely deviators.

More formally, we say that a pair of public strategies $\{a_t^1, a_t^2\}_{t\geq 0}$ is *strongly symmetric* if after every public history $\{Y_t\}_{t\in[0,\tau)}$

$$a_\tau^1(\{Y_t\}_{t\in[0,\tau)}) = a_\tau^2(\{Y_t\}_{t\in[0,\tau)}).$$

A *Strongly Symmetric Perfect Public Equilibrium (SSPPE)* is a Perfect public equilibrium in strongly symmetric strategies. In the rest of the paper we focus solely on SSPPE strategies, which is justified by the following:

Proposition 1 *Any sum of payoffs $W_0^1 + W_0^2$ achievable in some PPE is also achievable in a SSPPE.*

3 Solution

In this section we characterize the optimal SSPPE.

While partners do not observe the internal capital, which is the state that their efforts control, in equilibrium they share public beliefs about it. A simple application of Kalman-Bucy filter yields that for a fixed pair of strategies $\{a_t^1, a_t^2\}_{t\geq 0}$ the publicly expected internal capital $\overline{\mu}_t$ follows

$$d\overline{\mu}_t = (a_t^1 + a_t^2 - \alpha\overline{\mu}_t)dt + \gamma_t[dY_t - \overline{\mu}_t dt], \tag{5}$$

for an appropriate gain parameter γ_t. For simplicity, we will assume that at time zero partners believe that μ_0 is Normally distributed with steady state variance σ^2. This implies that both the posterior variance σ_t^2 of their estimate as well as the gain parameter γ_t remain constant throughout the game, $\sigma_t^2 = \sigma^2$ and $\gamma_t = \gamma$, and equal

$$\gamma = \sqrt{\alpha^2 + \left(\frac{\sigma_\mu}{\sigma_Y}\right)^2} - \alpha,$$
$$\sigma^2 = \gamma \times \sigma_Y^2.$$

Let us first establish the following two benchmark results. First, with no observational noise, $\sigma_Y = 0$, partners observe the process of internal capital, and so play a stochastic game with perfect monitoring. While on the first blush no noise should benefit the partners, we have the following result (see also Sannikov and Skrzypacz (2007), Bohren (2016)).[1]

Proposition 2 *With no observational noise, $\sigma_Y = 0$, the essentially unique SSPPE is the repeated static Nash equilibrium.*

Second, both the continuation values in (2) and the law of motion (5) of the mean belief $\overline{\mu}_t$ depend only on the strategies of the agents and $\overline{\mu}_t$. This means that our imperfect monitoring game can be analyzed as a stochastic game with a state variable $\overline{\mu}_t$. We define a *Markov SSPPE* as a SSPPE in which the equilibrium strategies depend on the past only via the state variable $\overline{\mu}_t$,

$$a_\tau^1(\{Y_t\}_{t\in[0,\tau)}) = a_\tau^2(\{Y_t\}_{t\in[0,\tau)}) = a\left(\overline{\mu}_\tau\right),$$

for some measurable function $a : \mathbb{R} \to \mathbb{R}_+$.

Proposition 3 *The only Markov SSPPE is the repeated static Nash equilibrium.*

In the partnership we consider the benefits and costs of effort are constant and, in particular, do not depend on the past. It follows that the only channel through which the state could help partners is informational, as a statistic of past profit flows. Intuitively, changes in the mean beliefs $d\overline{\mu}_t$ and profit flows dY_t are colinear with

[1] As is standard, by essential uniqueness we mean that any other SSPPE may not be different with nonzero probability.

positive slope (see (5)), and so high current beliefs $\overline{\mu}_\tau$ indicate high profit realizations. The proposition shows that in our Brownian setting coordinating on the state does not provide any coordination benefits.

In what follows we use the following endogenous state. For a fixed pair of strategies define *net continuation value* as

$$w_\tau = W_\tau - \frac{\overline{\mu}_\tau}{2\,(r+\alpha)} = \mathbb{E}^{\{a_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{a_t^1+a_t^2}{2\,(r+\alpha)} - c(a_t^i)\right)dt\right]. \qquad (6)$$

The following result is standard in the continuous time literature. It applies Martingale Representation Theorem to show that locally net continuation values responds linearly to the profit realizations.

Lemma 1 *Consider a symmetric strategy profile* $\{a_t, a_t\}_{t\geq 0}$ *that gives rise to a bounded process of continuation values* $\{W_t\}_{t\geq 0}$*. A process* $\{w_t\}_{t\geq 0}$ *is the associated process of net continuation values as in (6) if and ony if there is a process* $\{I_t\}_{t\geq 0}$ *in* L^2 *such that*

$$dw_t = \left(rw_t - \left(\frac{a_t}{\alpha+r} - c(a_t)\right)\right)dt + I_t \times (dY_t - \overline{\mu}_t dt)\,, \qquad (7)$$

and the transversality condition $\mathbb{E}\left[e^{-rt}w_t\right] \rightarrow_{t\rightarrow\infty} 0$ *holds.*

The process $\{I_t\}_{t\geq 0}$ measures the sensitivities of the net continuation values to public signals. As partners' efforts translate into higher internal capital, which in turn pushes the profits up, the sensitivities seem the right tool to incentivise them to exert effort.

Before we make this statement precise, let us emphasize three features of our setup. First, at any point of time the *level* of internal capital, or the beliefs about it plays no role for the incentives of the agents. As the "inherited" level of capital does not interact with the current effort to affect the public profitability (1), the incentives depend only on how the continuation value *net* of the believed effects of the inherited capital responds to the public signals. This is the reason justifying net continuation value as the state variable. For example, in the repeated static Nash equilibrium

$\{I_t\}_{t\geq 0}$ is the null process and the net continuation values are constant, despite the fact that both the beliefs $\overline{\mu}_t$ and so the continuation value W_t change over time.

Second, partners' efforts at any point of time have a persistent effect on profitability at any future time. Finally, imperfect monitoring of the internal capital implies that the whole future stream of profits is relevant for estimating the level of effort at any point of time. We stress that even with persistent effect of effort, when the internal capital is perfectly monitored, the immediate change of capital is a sufficient statistic for an affort at any time.

Formally, we have the following:

Lemma 2 *A necessary condition for a symmetric strategy profile $\{a_t, a_t\}_{t\geq 0}$ to be a SSPPE is that for the associated process $\{I_t\}_{t\geq 0}$ as in Lemma 1 we have*

$$c'(a_\tau) = \mathbb{E}^{\{a_t,a_t\}}\left[\int_\tau^\infty e^{-(r+\alpha)(t-\tau)}\left(\frac{1}{2}+I_t\right)dt\,\middle|\,\{Y_t\}_{t\in[0,\tau)}\right] =: F_\tau. \tag{8}$$

In the rest of the paper we will call $\{I_t\}_{t\geq 0}$ the *flow of incentives* and $\{F_t\}_{t\geq 0}$ the *incentives* processes. The proposition shows that F_τ, which is an expected integral of shifted incentive flows in the future, is the marginal benefit of exerting effort at time τ, after any history. To understand the accounting of the incentives in the lemma, consider the effect of increasing effort at time τ. The increase translates into a higher internal capital and so profit realization today one-to-one, half of which is captured by a partner, and an increase in the net continuation value by I_τ. Moreover, due to persistence, the increased internal capital today will have a knock-on effect of increased capital in the next instant $\tau+\Delta$, scaled down by $e^{-\alpha\Delta}$ and appropriately discounted. The integral captures thus the total marginal benefit of the extra effort, calculating its expected knock-on effects in the whole future.

The lemma characterizes only the necessary local incentive constraints. In the rest of the section we deal with the relaxed problem of characterizing the optimal SSPPE under only the local incentives constraints (8), which we call *local SSPPE*. In Theorem 2 we verify analytically that under an appropriate assumption the solution is truly an equilibrium, i.e., satisfies the global incentive compatibility constraints (3).

With one more application of the Martingale Representation Theorem, we summarize in the following Proposition the strategy of characterizing local SSPPE.

Proposition 4 *A symmetric strategy profile $\{a_t, a_t\}_{t\geq 0}$ is a local SSPPE if and only if there are L^2 processes $\{I_t\}_{t\geq 0}$ and $\{J_t\}_{t\geq 0}$ such that*

$$
\begin{aligned}
dw_t &= \left(r w_t - \left(\frac{a_t}{\alpha + r} - c(a_t) \right) \right) dt + I_t \times \left(dY_t - \overline{\mu}_t dt \right), \\
dF_t &= \left(\left(r + \alpha \right) F_t - \left(\frac{1}{2} + I_t \right) \right) dt + J_t \times \left(dY_t - \overline{\mu}_t dt \right), \\
a_t &= c'^{-1}(F_t) =: a(F_t),
\end{aligned}
$$

and the transversality conditions $\mathbb{E}\left[e^{-rt} w_t\right], \mathbb{E}\left[e^{-(r+\alpha)t} F_t\right] \to_{t\to\infty} 0$ hold.

In other words, constructing local SSPPE is equivalent to constructing two L^2 processes $\{I_t\}_{t\geq 0}$ and $\{J_t\}_{t\geq 0}$, so that the derived net continuation and incentive processes $\{w_t\}_{t\geq 0}$ and $\{F_t\}_{t\geq 0}$ satisfy transversality conditions. Given strict concavity and continuity of the cost of effort function, local equilibrium efforts are fully pinned down by the (local) incentives.

Our solution strategy relies on the following novel parametrization. In order to construct local SSPPE we will 1) use the net continuation value as a state variable, and 2) parametrize maximal incentives as a function of it. Unlike in the Principal-Agent problems, or games with signals that identify the deviators, our problem does not yield a natural choice of "objective" function to be maximized, such as one agent's continuation value as a function of the other agent's continuation value. Maximizing incentives, as opposed to continuation values might also seem counterintuitive: in our problem the first-best efficient level of effort is interior (see (4)), and so it is well possible to overincentivize the partners.

To highlight the role of the new parametrization, we split the problem of characterizing the optimal local SSPPE in two steps. The following proposition suggests a strategy of constructing any, not necessarily optimal equilibria. Let

$$w_{NE} = \int_0^\infty e^{-rt}\left(\frac{a_{NE}}{r+\alpha} - c(a_{NE})\right) = \frac{1}{r}\left(\frac{a_{NE}}{r+\alpha} - c(a_{NE})\right)$$

be the net continuation value from playing the repeated static Nash equilibrium.[2]

Proposition 5 *Consider a bounded measurable $I : [w_{NE}, \overline{w}] \to R_+$ and a C^2 function $F : [w_{NE}, \overline{w}] \to R$ that satisfy the differential equation*

$$(r+\alpha)F(w) = \left(\frac{1}{2} + I(w)\right) + F'(w) \times \left(rw - \left(\frac{a(F(w))}{\alpha + r} - c(a(F(w)))\right)\right) + \frac{F''(w)}{2}\sigma_Y^2 I^2(w), \tag{9}$$

with boundary conditions

$$\begin{aligned} F(w_{NE}) &= \frac{1}{2(r+\alpha)}, \\ (r+\alpha)F(\overline{w}) &= \frac{1}{2} + F'(\overline{w}) \times \left(r\overline{w} - \left(\frac{a(F(\overline{w}))}{\alpha + r} - c(a(F(\overline{w})))\right)\right), \\ &\left(r\overline{w} - \left(\frac{a(F(\overline{w}))}{\alpha + r} - c(a(F(\overline{w})))\right)\right) \leq 0. \end{aligned} \tag{10}$$

Then $\{a(F(w_t))\}_{t\geq 0}$ is a local SSPPE with the net continuation value process $\{w_t\}_{t\geq 0}$, as in (7), with $a_t = a(F(w_t))$.

The proposition reduces the problem of finding local SSPPE to that of solving the Ordinary Differential Equation (9). Function I parametrizes the flow of incentives, $I_t = I(w)$ for w in $[w_{NE}, \overline{w}]$. The differential equation then characterizes the incentive function $F_t = F(w)$, defined by (8), when the net continuation values follow the process (7). It follows from the Ito formula that the process $\{F(w_t)\}_{t\geq 0}$ satisfies the conditions of Proposition 4, for the sensitivities $J_t = F'(w_t) \times I(w_t)$.

[2] One may verify that for the continuation value $W_{NE} = \frac{1}{r}\left(\frac{\mu_{NE}}{2} - c(a_{NE})\right)$, we have $w_{NE} = W_{NE} - \frac{\mu_{NE}}{2(r+\alpha)}$, where $\mu_{NE} = \frac{2a_{NE}}{\alpha}$ is the constant level of internal capital in the repeted static Nash equilibrium.

The first boundary condition simply states that when partners play the least efficient repeated static Nash equilibrium, their incentives derive solely from the direct effect that the effort has on the profitability of the firm. There is no strategic "ratchet" effect, by which higher profit flows triger partners to work harder.

The second equation is only slightly more complicated. At the highest net continuation value $\overline{w}$ the flow of incentives $I(\overline{w})$ must be zero. If instead the sensitivity of net continuation value with respect to the flow profits was strictly positive, then with positive probability it would "escape" past the maximal value, leading to a contradiction. Precisely this feature results in nonexistence of nontrivial equilibria in the partnership model without persistence (Sannikov and Skrzypacz (2007)). In our case, the second boundary condition says that the (stock of) incentives $F(\overline{w})$ is made up only of the direct effect on the profitability, as well as the "discounted" stock of incentives in the next instant, due to persistence. The last clause requires that at the maximal value, the net continuation value drifts leftwards, to satisfy the transversality condition.

The following is one of the main results of this paper.

Theorem 1 *Let w^* be the highest net continuation value achievable in a local SSPPE. Then there exists a C^2 strictly convex function F on $[w_{NE}, w^*]$ that satisfies*

$$\begin{aligned}(r+\alpha)F(w) &= \max_I \left\{ \frac{1}{2} + I + F'(w)\left(rw - \left(\frac{a(F(w))}{\alpha+r} - c(a(F(w))) \right) \right) + \frac{F''(w)\sigma_Y^2}{2} I^2 \right\} \\ &= \frac{1}{2} + F'(w)\left(rw - \left(\frac{a(F(w))}{\alpha+r} - c(a(F(w))) \right) \right) - \frac{1}{2\sigma_Y^2 F''(w)},\end{aligned} \tag{11}$$

with boundary conditions:

$$\begin{aligned}F(w_{NE}) &= \frac{1}{2(r+\alpha)}, \\ (r+\alpha)F(w^*) &= \frac{1}{2} + F'(w^*) \times \left(rw^* - \left(\frac{a(F(w^*))}{\alpha+r} - c(a(F(w^*))) \right) \right).\end{aligned}$$

The novel Hamilton-Jacobi-Belman (HJB) equation represents the problem of maximizing the stock of incentives F, over all possible incentive flows I, given the "promise keeping" law of motion of the net continuation value (7). The first term in the maximization problem is the total flow of incentives, the second term is the change of the stock due to the change in the net continuation value; and the last term is the change in the incentives due to the second order variation. The maximization problem is easily solved, with the incentive flow inversely proportional to the curvature of F, $I^*(w) = -\frac{1}{\sigma_Y^2 F''(w)}$.

The idea behind the maximization problem is to trace out the upper boundary of the set of the net continuation value and stock of incentives pairs achievable across all the local SSPPE. The proof of the theorem establishes, among others, that the boundary is self generating and smooth, and so must satisfy the HJB equation. While we are primarily interested in the point w^* of the highest net continuation value achievable in a local SSPPE, the upper boundary of the set satisfying the HJB equation helps us to get to this point. Intuitively, we can trace out solutions of the HJB equation that pass through the starting point $\left(w_{NE}, \frac{1}{2(r+\alpha)}\right)$ and see which one reaches furthest (see Section XXX). In particular, recall from Proposition 5 that each such curve, and in particular the optimal one, gives rise to a local SSPPE.

So far we have characterized only local equilibria. The following result shows that with sufficiently convex cost function, the local equilibria satisfy full incentive compatibility constraints (see Williams (2011), Edmans et al. (2012), Sannikov (2014)).

Theorem 2 *If cost of effort is sufficiently convex ($c''(a) > C$ for sufficiently large C), then any local SSPPE is a SSPPE.*

4 Value of Partnership and Equilibrium Dynamics

The results in the previous section provide a convenient tool to explore the comparative statics of the value of the partnership. They also provide a complete characterization of the dynamics of the internal capital, or profitability of the partnership, together with the level of effort that the partnership sustains. In this section we explore some of its properties.

Most importantly, it is still incumbent on us to show that the ODEs in Proposition

5 or in Theorem 1 have nontrivial solutions, and so the nontrivial SSPPE exist. Such equilibria in our partnership game provide incentives based purely on the self-fulfilling prophecy: the value of the partnership net of the inherited internal capital responds positively to the strings of unexpectedly good outcomes, and decreases after unexpectedly bad outcomes. No signal may statistically identify which of the partners did and which did not work hard enough, and so both are rewarded and punished simultaneously. As the level of incentives, and so the level of equilibrium punishments may not disappear even as the partners are very patient, this suggests that the Folk Theorem fails in our result.

Let

$$w_{EF} = \int_0^\infty e^{-rt} \left(\frac{a_{EF}}{r+\alpha} - c(a_{EF}) \right) = \frac{1}{r} \left(\frac{a_{EF}}{r+\alpha} - c(a_{EF}) \right)$$

be the first-best net continuation value achievable in the partnership.

Proposition 6 *The highest value w^* achievable in a local SSPPE equals w_{NE}, when r is high enough. If partners are sufficiently patient (r low), then $w^* > w_{NE}$. As r approaches zero, w^* is bounded away from w_{EF}.*

Regarding the quality of the monitoring technology, parametrized by the variance of the observational noise σ_Y^2, recall from Proposition 2 that perfect monitoring prevents any nontrivial equilibria. However, we have the following:

Proposition 7 ("Fine Sand in the Wheels") *The value of partnership w^* increases discontinuously at $\sigma_Y = 0$, and is decreasing in the monitoring technology σ_Y, for $\sigma_Y > 0$.*

It is intuitive that the equilibrium becomes more efficient as monitoring technology improves. With better monitoring, for the fixed sensitivity of net continuation value with respect to the signals, and so for fixed level of incentives, the volatility of continuation value goes down. This allows the partnership to operate longer, before it reaches the absorbing inefficient state w_{NE}. Eliminating the monitoring noise, however, eliminates the possibility of providing any incentives at the most efficient state $w^* > w_{NE}$, and so the equilibrium unravels. The result highlights that the impossibility of collusion in Sannikov and Skrzypacz (2007) is very sensitive to adding the monitoring noise.

Regarding the dynamics of the optimal equilibrium, recall first the benchmark of discrete-time model without persistence (Abreu (1986), Abreu, Pearce, and Stacchetti (1986)). There are typically many optimal equilibria, the simplest one of which, the "bang-bang" equilibrium, has the game transitioning between two states only. The partners play a stationary strategy profile, until the partnership unravels as a consequence of a bad signal. In particular, good signals never lead to lower effort.

Proposition 8 ("Ratcheting and Coasting") *The incentive function F in the optimal local SSPPE is strictly concave, with $F'(w^\#) = 0$ for some $w^\# \in (w_{NE}, w^*)$. In the optimal local SSPPE equilibrium effort $a(w)$ is increasing to the left, and decreasing to the right of $w^\#$.*

When the partnership has done well in the past, the net continuation value increases, since $I(w)$ is positive. The result says that after a string of good outcomes, when the net continuation value is sufficiently high, the effort of the partners decreases after good outcomes. In other words, the partners "coast" on their past good performance. As the net continuation value approaches the maximum w^*, the flow incentives must diminish (and eventually die out). The incentives are provided mostly by discounting back the future incentives, from the moment when the partnership has "drifted back to the mediocrity".

In contrast, if the partnership has done poorly in the past and the net continuation value is low, the possibility of the partnership unraveling looms large. In this case effort will likely not matter for the partnership beyond the direct effect on the internal capital. As a consequence, a good profit outcome brings the partnership further away from the brink and encourages higher effort. Thus only for low net continuation values the equilibrium exhibits "ratcheting", as better profit outcomes induce partners to work harder.

One of the crucial features of the optimal equilibrium is its "non-Markovian" nature. The internal capital, or the public beliefs about the internal capital on the one hand, and the net continuation value, or the effort level respond differently to the past outcomes, and are not functionally related in any way.

The following is one of the implications.

Proposition 9 ("Beatles' break-up") *Partnership unravels with probability one. Conditional on unraveling, the distribution of mean internal capital $\overline{\mu}_t$ is nondegenerate. As production noise σ_μ vanishes, partnership can unravel at any level of mean internal capital $\overline{\mu}_t$ that is possible on the equilibrium path.*

The second part of the result follows because the sensitivity of mean internal capital with respect to profit flow is

$$\frac{d\overline{\mu}_t}{dY_t} = \gamma = \sqrt{\alpha^2 + \left(\frac{\sigma_\mu}{\sigma_Y}\right)^2} - \alpha,$$

whereas it is $I(w)$ in the case of net continuation value. With low σ_μ, and so low $d\overline{\mu}_t/dY_t$, a short string of very low profit realizations will unravel the partnership, with hardly any effect on the internal capital. Even very profitable partnerships unravel, when the net continuation value, or the goodwill is tested by a series of adverse outcomes.

5 Proofs (soon)

Proof of Proposition 6.

Part 1) $w^* = w_{NE}$ for r sufficiently high.

Fix $\alpha, r > 0$ and consider the solution of the HJB equation (11), for fixed initial condition $(w_{NE}, F(w_{NE}), F'(w_{NE}))$.Let w_0 be the point for which $F'(w_0) = 0$. Let w_0' be defined by $G'(w_0') = 0$, where G solves

$$(r + \alpha)G(w) = \frac{1}{2} - \frac{1}{2G''(w)},$$

and has the same initial condition, $(w, G(w_{NE}), G'(w_{NE})) = (w_{NE}, F(w_{NE}), F'(w_{NE}))$. We claim that $w_0 > w_0^{'}$. This follows from $G''(w) \geq F''(w)$ and $G'(w) \geq F'(w)$ on $[w_{NE}, w_0]$.

Second, for fixed $\alpha > 0$ looking at functions G satisfying the above differential equation with $r > 0$ and initial condition $G'(w_{NE}) = d \geq \frac{1}{2(r+\alpha)}$ on $[w_{NE}, w_0'(r,d)]$, we see that $w_0'(r,d)$ is increasing in r and d. The argument is the same as above.

Finally, let w_{EF} be the payoff from the perfectly collusive outcome, and note that $\lim_{r\to\infty} |w_{EF} - w_{NE}(r)| = 0$. We have $w_{EF} - w_{NE}(r) < w_0'(r,d) - w_{NE}(r) \leq w_0(r,d) - w_{NE}(r)$ for all r sufficiently large. This however contradicts $w_0(r,d) < w_{EF}$.

Part 2) $w^* > w_{NE}$ when r is small enough.

The proof strategy is to construct a a C^2 function $F : [w_{NE}, \overline{w}] \to R$ that satisfies the differential inequality

$$(r+\alpha)F(w) \leq \frac{1}{2} + F'(w) \times \left(rw - \left(\frac{a(F(w))}{\alpha + r} - c(a(F(w))) \right) \right) - \frac{1}{2\sigma_Y^2 F''(w)}, \quad (12)$$

together with the boundary conditions (10). The right hand side in the inequality above is the maximal value of the corresponding expression in Proposition 5, over all positive $I(w)$, with the maximizer $I^*(w) = -\frac{1}{\sigma_Y^2 F''(w)}$. Given such an F, it is always possible to find an $I(w) > 0$ for which the equation (9) in Proposition 5 holds at every $w \in (w_{NE}, \overline{w})$.

XXThe proof is for $c(a) = \frac{C}{2}a^2$, with $C > 0$.

For any r and α let $(w, F^*(w))$ be the "inverse parabola" set of achievable pairs of (w, F) ignoring any incentive issues, defined implicitly by

$$rw = \frac{F^*(w)}{(\alpha + r)\, C} - \frac{F^*(w)^2}{2C}.$$

Note that the static Nash has

$$\begin{aligned}
F &= \frac{1}{2(\alpha+r)} \\
w_{NE} &= \frac{1}{r}\left(\frac{F}{(\alpha+r)\,C} - \frac{F^2}{2C}\right) \\
&= \frac{1}{r}\left(\frac{1}{2C\,(\alpha+r)^2} - \frac{1}{8C\,(\alpha+r)^2}\right) \\
&= \frac{3}{8r}\frac{1}{C\,(\alpha+r)^2}.
\end{aligned}$$

and lies on the curve.

Fix $\alpha, r, \sigma_Y > 0$. First, note that at $(w_{NE}, F^*(w_{NE}))$ we have

$$\frac{1}{r}F^{*\prime}(w_{NE}) = \left(\frac{1}{(\alpha+r)\,C} - \frac{F}{C}\right)^{-1} = \left(\frac{1}{(\alpha+r)\,C} - \frac{1}{2C(\alpha+r)}\right)^{-1} = 2\alpha C,$$

We will construct a curve F over $[w_{NE}, \overline{w}]$ for $\overline{w}$ with $r\overline{w} = rw_{NE} + \varepsilon$, with ε small. We can approximate

$$F^*(\overline{w}) \approx F^*(w_{NE}) + \varepsilon 2\alpha C,$$

and we will set the right boundary condition

$$F(\overline{w}) = F^*(w_{NE}) + 2 \times (F^*(\overline{w}) - F^*(w_{NE})) \approx F^*(w_{NE}) + 4\varepsilon\alpha C,$$

which implies

$$
\begin{aligned}
F'\left(\overline{w}\right) &= \frac{(r+\alpha)F(\overline{w})-\frac{1}{2}}{r\overline{w}-\left(\frac{F(\overline{w})}{(\alpha+r)C}-\frac{F(\overline{w})^2}{2C}\right)} \\
&= \frac{(r+\alpha)\left(F^*(w_{NE})+4\varepsilon\alpha C+o(\varepsilon)\right)-\frac{1}{2}}{rw_{NE}+\varepsilon-\left(\frac{F^*(\overline{w})}{(\alpha+r)C}-\frac{F^*(\overline{w})^2}{2C}+\left(\frac{1}{(\alpha+r)C}-\frac{F^*(\overline{w})}{C}\right)\left(F(\overline{w})-F^*(\overline{w})\right)+o(\varepsilon)\right)} \\
&= \frac{(r+\alpha)\left(4\varepsilon\alpha C+o(\varepsilon)\right)}{-\left(\frac{1}{(\alpha+r)C}-\frac{F^*(\overline{w})}{C}\right)\left(F(\overline{w})-F^*(\overline{w})+o(\varepsilon)\right)} = \frac{(r+\alpha)\left(4\varepsilon\alpha C+o(\varepsilon)\right)}{-\left(\frac{1}{2(\alpha+r)C}-O(\varepsilon)\right)\left(2\varepsilon\alpha C+o(\varepsilon)\right)} \\
&\approx -\frac{(r+\alpha)\,4\varepsilon\alpha C}{-\frac{1}{2C(\alpha+r)}2\varepsilon\alpha C} = -\frac{(r+\alpha)\,2}{-\frac{1}{2C(\alpha+r)}} = -4C\left(\alpha+r\right)^2,
\end{aligned}
$$

where we first carried through all the $o(\varepsilon)$ terms, and the approximation is for any r and α, when ε is small.

OK, so now we want to find the constant second derivative $D<0$ of F that will fit. We need to pick D so that

$$
\begin{aligned}
F(\overline{w})-F(w_{NE}) &= 4\varepsilon C\alpha, \\
\overline{w}-w_{NE} &= \frac{\varepsilon}{r}, \\
F'(\overline{w}) &= -4C\left(\alpha+r\right)^2, \\
F(w_{NE}) &= \frac{1}{2(r+\alpha)}
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
4\varepsilon C\alpha = F(\overline{w})-F(w_{NE}) &= \int_{w_{NE}}^{\overline{w}} F'(x)dx = \int_{w_{NE}}^{\overline{w}} \left[F'(\overline{w})-D\left(\overline{w}-x\right)\right]dx \\
&= F'(\overline{w})\times\frac{\varepsilon}{r}-D\frac{1}{2}\left(\frac{\varepsilon}{r}\right)^2, \\
D &= -\left(4\varepsilon C\alpha+4C\left(\alpha+r\right)^2\times\frac{\varepsilon}{r}\right)\left(2\frac{r^2}{\varepsilon^2}\right)\approx -8C\left(\alpha+r\right)^2\frac{r}{\varepsilon}.
\end{aligned}
$$

Notice that:

$$
\begin{aligned}
(r+\alpha)F(w) &= O(\max\{\alpha, r\}), && (13)\\
|F'(w)| &= O(1),\\
\left|\left(rw - \left(\frac{F(w)}{\alpha C} - \frac{F(w)^2}{2C}\right)\right)\right| &= O\left(\varepsilon(\alpha+r)^2\right),\\
-\frac{1}{2\sigma_Y^2 F''(w)} &= O\left(\frac{\varepsilon}{\sigma_Y^2 r(\alpha+r)^2}\right).
\end{aligned}
$$

When r is sufficiently small, it suffices to pick ε such that $\alpha \gg \varepsilon \gg r$. This establishes the proof of part 2.

Part 3) $w^* \ll w_{EF}$ for every r.

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Proof of Proposition 7

In order to show that there exist nontrivial local SSPPE when σ_Y is sufficiently small, it is sufficient to establish that the function F constructed in the proof of Proposition 6 satisfies inequality (12). This in turn follows since the last term on the right can be made arbitrarily large when σ_Y approaches zero, and all the other terms are bounded (see (13)).

In order to establish monotonicity, note that decreasing σ_Y changes the HJB equation (11) only by increasing the last term. This means that the function F characterizing the optimal local SSPPE for σ_Y, which satisfies the HJB equation and the boundary conditions, will satisfy the same boundary conditions and the left inequality version of the HJB equation for any $\sigma_Y' < \sigma_Y$. As in the proof of Proposition 6, with appropriately increased the policy function $I(w)$, the pair (F, I) satisfies the conditions of Proposition 5 for σ_Y', and so gives rise to a local SSPPE. This establishes the proof.

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