A three mutual fund separation theorem

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Abstract

We analyze a one period economy with CARA preferences, and normally distributed aggregate risk. We allow an arbitrary distribution of (dogmatic) beliefs on the expected value and variance of the aggregate risk, as well as on the agent's risk tolerance parameter. We show that there is a representative agent whose risk tolerance, and beliefs over expected value and variance are appropriately defined weighted averages of the same objects across all agents. We show a type of 3 fund separation theorem: any efficient allocation can be decentralized with three asset: an uncontingent bond, a Lucas's tree, and a simple variance swap –whose price is given by the SVIX index, an index closely related to VIX, given by the the price of a portfolio of out of the money puts and calls. We give simple expressions for the relative prices of these securities in terms of the representative agent's parameters. We also give simple expressions for each agent's holding on the Lucas' tree and the simple variance swap as function of the agent's differences with the representative agent's values. The main novelty of the setup is the explicit role for a simple variance swap –or the portfolio of puts and calls– in the decentralization of efficient allocations. In equilibrium there are different positions in the the variance swap –or to the options portfolio– if and only if there is heterogeneity in the beliefs of the aggregate risk of the economy, and hence these heterogeneity maps into trade volume. To understand the role of different assumption we also study the case of arbitrary belief and utility function for a small noise. Finally, we extend the results to a multiperiod setting, for which they hold conditionally on the history for each period.

1 Introduction (To be completed)

1. Start with an extended abstract with summary of results.
2. Summary of the literature review on $K-$mutual fund separation results, to put our result in perspective.
3. Summary of what a simple variance swap, portfolio of options and SVIX. Relationship with variance swap, and VIX, to put result in perspective.
4. Discussion of difference of the results in the literature: difference on beliefs, and role of a variance swap. In particular, comparison with other 3 fund separation results.
5. Discuss small noise approximation.
6. Discuss extension to dynamic set up.
7. Discuss limitations of the results: dogmatic priors, CARA-Normal set up, dogmatic priors as opposed to rational expectations, fast learning of variance in dynamic case.

2 Literature Review on $K-$Mutual Fund Separation Results

The original contributions is by Tobin (1958), who show that mean-variance investors, which have access to a risk free return, can be restricted to chose between two portfolios.

The literature has, roughly speaking, developed in two main branches to find sufficient conditions –or to produce characterizations for $K-$ mutual fund separation. One branch studies the problem of finding class of utility functions which produce separation for arbitrary returns and levels of investors wealth, and the other branch studies the class of returns distribution so that separation occurs for arbitrary concave utility functions.

Among the original contribution on sufficient conditions on the set of returns so that any risk averse investor are indifferent to select $K-$ mutual fund is Samuelson (1967), and on the necessary and sufficient conditions is Ross (1978). On the characterization of the class of utility function so that for any distribution of returns and any level of wealth investors are indifferent to select among $K-$ mutual funds there is the seminal paper by Cass and Stiglitz (1970) and subsequent work, such as results on Chen, Pelsser, and Vellekoop (2011), and most notably the recent contributions by Dybvig and Liu (2016).

Other distinctions are also present in the literature. One is whether the set of asset for which the investor can choose is complete or not. For instance, Black (1972) shows that with mean variance preferences, two fund separation obtains even in incomplete markets. Yet, most of the literature studies the problem with complete markets. Another distinction made in the literature is whether a risk free asset is either available among the set of assets (in the case of incomplete market), or whether one of the $K-$mutual funds is a risk free assets.

There is a handful of 3 mutual fund separation results. One such result is due to Merton (1973) in his intertemporal CAPM model, where the changing nature of the investment set are at the heart of the result. Two important cases of 3 mutual fund which do not rely on a changing investment set, are agents with the SAHARA and GOBI utility function as proposed Chen, Pelsser, and Vellekoop (2011) and Dybvig and Liu (2016) respectively. Finally, and more closely related to our paper, are the results by Chabi-Yo, Dietmar, and Renault (2014) and especially by Chabi-Yo, Ghysels, and Eric (2008). In both paper the authors consider general utility functions, in the sense that they use analytics expansion around small noise, and in both papers they consider heterogeneity among investors. They give a characterization of the equilibrium risk-premium and a mutual fund separation result, where in a special case there are three funds. In particular, Chabi-Yo, Ghysels, and Eric (2008) is even closer to ours in that the authors consider heterogeneity in investor's beliefs. Yet the type of heterogeneity, and the features of distribution of returns highlighted (its skewness), are different from the ones we emphasize in this paper, as discuss below. Also the mutual fund they consider do not include a variance swap. Thus, we view the results as complementary to ours.

Merton's (1973) analyzes the dynamic consumption and portfolio problem of an investor who faces assets whose prices follow diffusions, and whose instantaneous expected return and variance also follow diffusions. He shows that in the case that the interest rate follows a diffusion a 3 mutual fund separation result for any strictly concave and increasing period utility function. The mutual funds are the instantaneous risk-free asset, a mean variance efficient portfolio, and a hedging portfolio –in this case one with innovations perfectly negatively correlated to the ones in short term interest rates. The continuos time set up and the assumption of diffusions are both central to the result. This is in contrast with our result which can be set up in a one period model, and hence cannot be driven by hedging demands. Dybvig and Liu (2016) characterize the three mutual fund separation that occurs in the one period case with SAHARA and GOBI utility functions. In the SAHARA utility function, the three mutual funds are: a risk-less bond, and two risky assets, one equal to the discount factor raise to a power, and the other equal to the reciprocal of the first. In the GOBI utility function, the three mutual funds are: a riskless bond, and two risky assets, one is a power of the discount factor, and the other is the square of the first. The cases of SAHARA and GOBI preferences are different from the preference we consider, and also, as explained, do not produce a separating mutual fund with payoffs similar to a variance swap.

The set-up in both Chabi-Yo, Ghysels, and Eric (2008) and Chabi-Yo, Dietmar, and Renault (2014) allows agent to have difference utility functions, and the ones in Chabi-Yo, Ghysels, and Eric (2008) also allow for different beliefs on the moments of the return to each asset. The authors find expressions for expansions as the function of the "noise", which are carried out up to 3rd or 5th order.[1] By the nature of the parameterization that the authors use on the "noise", investor can only have different beliefs on the risk premium, i.e. the expected return of each asset relative to the risk-free asset. In this set-up, skewness and hyper-skewness of the returns (i.e. lack of symmetry of returns) is allowed, as well as different fatness of tails (i.e. different degree of excess kurtosis). The authors find that separating portfolios are of three types: first, as in the traditional theory, a market portfolio and a risk-free bond. Second, a portfolio that weights the co-skewness of the individual portfolios (so it depends on the lack of symmetry of the returns, and its covariation with the market

[1] The "noise" is a parameter that multiplies the random component of the return, so when it is set to zero the model is deterministic. The authors work out expansions of equilibrium asset prices and portfolios as functions of the noise. In particular the authors work out Taylor expansion of the noise, up to fifth order, for the excess expected return and for the separating portfolios.

portfolio). Third a set of portfolios that span the difference in beliefs on the expected returns of the assets. The authors explain that in the gaussian case (such as ours), due to the symmetry and proportionality of its even moments, many of the notable effects highlighted in their paper disappear. In particular, a variance swap is not one of the mutual funds that are required in their set up. Furthermore, heterogeneity in variances is not allowed in their set-up, and it is the central one in ours.[2] Finally, they work out the case of incomplete market (i.e. they take as given a finite number of assets), while we consider a case where with three funds one can decentralize the complete market allocation. On the other hand, as explained, their set-up is otherwise more general in preferences and in the distribution of assets. Thus, this is the sense in which we view the result of our paper as complementary to theirs –since we have normal aggregate uncertainty and allow for heterogeneity on the beliefs on variances.

3 Variance Swaps, VIX and SVIX

In our one period model, one the 3 mutual funds that suffice to support the complete market allocation are a risk-free bond, a market portfolio, and a simple variance swap on the market portfolio. A one period simple variance swap in a security is a contract where one part agrees to pay a price before the realization of uncertainty and in exchange receives the square of the realized simple return in that security. The cash flow (and the price) of the simple variance swap is equal to the cash flow (and the price) of an equally weighted average of out of the money of calls and puts. Indeed the SVIX index is computed using the price of all available out of the money calls and puts. The SVIX index is closely related, but different, from the VIX index, see Martin (2011,2013). While there is no trade in the VIX index per se, there are actively traded futures markets on the VIX. Furthermore, the VIX index is the most commonly used market based measure of volatility.

We briefly outline the difference between the VIX, SVIX, as well as between their associated variance and simple variance swaps. Under the conditions stated below, the value of the VIX index is equal to the arbitrage free value of a variance swap. A variance swap of is contract where a seller at its inception date promises to pay the square of the log (i.e. continuously compounded) daily returns of an underlying asset from inception until maturity, and in exchange the seller receives at maturity a fixed amount, which we also refer to as the value or price of the variance swap. Yet variance swaps, those whose price is closely related to the VIX, differ from simple variance swaps, those used in the model as a mutual fund and whose price is closely related to the SVIX. In particular, a variance swap uses the square of the daily log returns, while the simple variance swap uses the square of the daily simple returns, where simple returns are defined as the difference in price of the underlying security over consecutive days, scaled, essentially, by the price of the underlying asset at inception. [3] The simple variance swap was proposed, its properties were characterized, and its synthetic

[2]We also allow heterogeneity on risk-tolerance and beliefs on expected values, but they are not relevant for the demand of a variance swap, unless there is heterogeneity on the beliefs of the variance of the aggregate shocks.

[3]It is actually divided by the forward price at inception with maturity at the day returns are measured. For instance, if interest are zero and there are no dividends the forward price at inception is simply the price of the underlying asset at inception. With constant positive interest rates it will be proportional

historial prices analyzed and compared with the price of standard variance swap in Martin (2011,2013).

The price of a variance swap is equal to the value of the VIX index under additional conditions, detailed below. The current value of VIX index is defined as, essentially, a weighted average of the current prices of the out of the money calls and puts, with the SP500 as underlying, all of them with the same maturity, and weighted by the reciprocal of the square of the strike prices of the options. Taking the underlying asset to be the SP500, and the current date as the inception date of the variance swap, the value of the VIX index should equal the arbitrage free price of the variance swap under the following assumptions: i) there are no arbitrage opportunities and no transaction cost, ii) the returns are measured at infinitesimal intervals of time (as opposed to daily), iii) there is a continuum of options with all positive values for the strikes (as opposed to a finite number of strikes), iv) interest rates and dividend yields are constant, and v) the process for the underlying follows a diffusion, perhaps with random expected returns and volatility, but with no jumps.

Likewise, the simple variance swap can also be priced using, essentially, an equally weighted average of the out of the money puts and calls for all positive strikes with the same maturity as the variance swap and over the the same underlying –a quantity that Martin named and SVIX. Martin (2011,2013) uses prices of traded options across strikes to approximate the SVIX, in the same way that the VIX approximated by such finite sum. The conditions for this equality hold are weaker than the ones for the equality between the VIX and the price of the variance swap, in particular it does not require that prices has no jumps, see Martin (2011,2013).

Finally, the one period model uses, naturally, the price of a market portfolio with dividend on a single date, and also a one period simple variance swap, while in both traded variance swaps, and the VIX and SVIX indices are defined using multi-period daily returns. In the general multiperiod version of the model our 3 fund separation theorem uses a one period bond, a one period dividend strip on the market portfolio and a one period variance swap in the one period dividend strip of the market. Yet, under certain circumstances (for instance, when aggregate consumption is a random walk), we extend our 3 fund separation theorem to use a multiperiod setup were we use a one period bond, the market return, and a multiperiod simple variance swap on the market return as separating mutual funds.

4 Static Model

4.1 Set up

The economy has two periods, $t = 0$ and $t = 1$. In period $t = 0$ agents trade financial assets whose payoffs are realized in period $t = 1$. There are I types of agents, each type denoted by i, with measure $\mu_i \geq 0$. Total output at time $t = 1$ takes values y in the set $Y \subset \mathbb{R}$. A type i agent has beliefs $\pi_i : Y \to \mathbb{R}$, where $\pi_i(\cdot)$ is the (density of the) distribution over values of $y \in Y$. A type i agent has utility function $u_i : X \to \mathbb{R}$, where $X \subset \mathbb{R}$. We assume that all u_i are strictly increasing and concave. Thus, agent's use expected utility given by:

$$\int_{y\in Y} u_i\left(c_i\left(y\right)\right)\pi_i\left(y\right)dy \tag{1}$$

where $c_i : X \to \mathbb{R}$ is the consumption of agent type i, contingent on aggregate output to be y. We are not going to work with this level of generality on utility functions and beliefs, but it is useful to defined object at this level of generality to compare with other results in the literature. We will specialize into the case define as CARA-Normal-Heterogenous economies.

CARA-Normal-Heterogeneous Economies. Our results are for the special case of economies which have CARA utility function with risk tolerance parameter τ_i, so that $u_i(x) = -\exp(x/\tau_i)$, and beliefs π_i about $y \in Y = \mathbb{R}$ to be normally distributed with expected value $\bar{y}_i$ and variance $\sigma_i^2 > 0$. Thus a CARA-Normal-Heterogeneous economy is parametrized by $\theta \in \mathbb{R}^{4I}$ containing $\theta_i = (\tau_i, \bar{y}_i, \sigma_i^2\, \mu_i) \in \mathbb{R}^4_+$ for all $i = 1, \ldots, I$ and with $\sum_i \mu_i = 1$.

Feasible allocations. An allocation $\{c_i(\cdot)\}_{i=1}^I$ is feasibility if the implied aggregate consumption at time $t = 1$ equals total output for each realization of aggregate output:

$$y = \sum_{i=1}^{I} c_i\,(y)\,\mu_i \text{ for all } y \in Y\,. \tag{2}$$

Complete Market Equilibrium. We consider a complete set of time $t = 0$ Arrow securities, i.e. agents can buy a unit of consumption $c_i(y)$, contingent to the realization of y. Arrow prices are given by $p : Y \to \mathbb{R}$, so that $p(y)$ has the interpretation of the time $t = 0$ price of one unit of consumption at time $t = 1$ contingent on y. We normalize prices so that the price of a unit in all states of nature is 1, i.e.:

$$\int_{y \in Y} p(y)dy = 1. \tag{3}$$

We denote by the endowment of agent i of a security that pays $e_i(y)$ when y is realized. Thus agent's i problem is:

$$\max_{c_i(\cdot)} \int_{y\in Y} u_i\,(c_i\,(y))\,\pi_i\,(y)\,dy \;\text{ s.t }\; \int_{y \in Y} p(y)\,[c_i\,(y) - e_i\,(y)]\,dy = 0\,. \tag{4}$$

We require agent's endowments $\{e_i(\cdot)\}$ to be feasible for this economy, i.e.:

$$\sum_{i=1}^{I} e_i(y)\mu_i = y \text{ for all } y \in Y\,. \tag{5}$$

K-fund separation. Fix an economy $\{u_i, \pi_i, \mu_i\}$ with agent's endowment of securities $e \equiv \{e_i\}$, and consider the allocation of a competitive equilibrium $\bar{c} \equiv \{\bar{c}_i\}$. We say that there is a $K-$mutual separation if there are K funds: $f_k : Y \to \mathbb{R}$ and K holdings for each agent type α_k^i such that

$$\sum_{k=0}^{K-1} \alpha_k^i\, f_k(y) = \bar{c}_i(y) \text{ for all } y \in Y \text{ and all } i = 1, \ldots, I\,.$$

Moreover, K should be the smallest integer for which this property holds.

In words, the equilibrium allocations can be obtained by a linear combination of K random variables, or portfolios. The holdings α's can depend on characteristics of the agent i, but the random variables or portfolios f's can't. This property can also be defined for a set of economies $\{u_i, \pi_i, \mu_i\}$ or for a set of security endowments e, or both.

4.2 Results

We work out the conditionally efficient allocations, shows that equilibrium prices can be obtained by reading the marginal utility of a representative agent, and that the efficient allocation can be decentralized with three assets: a risk-free bond, a Lucas tree and a one period variance swap –which we refer to as the "VIX". In this sense there is a three fund separation theorem.

Proposition 1. Take a CARA-Normal-Heterogeneous economy with arbitrary endowments $e = \{e_i\}$. Let $p(\cdot)$ be the Arrow-Debreu price of contingent claim. This price coincides with one for an economy with only one agent with the average risk tolerance $\bar{\tau}$, and beliefs about the expected value and variance $(\bar{y}, \bar{\sigma}^2)$ given by the following weighted averages of the parameter for each agent:

$$\bar{\tau} \equiv \sum_i \tau_i \mu_i \tag{6}$$

$$\tilde{\mu}_i \equiv \frac{\tau_i \mu_i}{\sum_j \tau_j \mu_j} \text{ for all } i = 1, \ldots, I \tag{7}$$

$$\frac{1}{\bar{\sigma}^2} \equiv \sum_i \frac{1}{\sigma_i^2} \tilde{\mu}_i = \sum_i \frac{\tau_i}{\bar{\tau}\, \sigma_i^2} \mu_i \text{ and} \tag{8}$$

$$\bar{y} \equiv \sum_i \frac{\frac{\tilde{\tau}_i}{\sigma_i^2}}{\sum_j \frac{\tilde{\tau}_j}{\sigma_j^2}} \bar{y}_i = \frac{1}{\sum_j \frac{\tau_j \mu_j}{\sigma_j^2}} \sum_i \bar{y}_i \frac{\tau_i \mu_i}{\sigma_i^2} \tag{9}$$

The Arrow prices are given by:

$$p(y) = e^{-\frac{1}{2}\left(y - [\bar{y} - \bar{\sigma}^2/\bar{\tau}]\right)^2 \frac{1}{\bar{\sigma}^2}} \frac{1}{\sqrt{2\pi\bar{\sigma}^2}} \text{ for all } y \in \mathbb{R}\,, \tag{10}$$

and the equilibrium prices coincide with the ones of an economy where there is only one agent with utility function $\bar{u}(\cdot)$ and beliefs $\bar{\pi}(\cdot)$:

$$\bar{u}(c) = -\bar{\tau} \exp\left(c/\bar{\tau}\right) \text{ and } \bar{\pi}(y) \sim \mathcal{N}(\bar{y}, \bar{\sigma}^2)\,. \tag{11}$$

Comments. The triplet $(\bar{\tau}, \bar{y}, \bar{\sigma}^2)$ are different type of weighted averages.

1. The representative agent risk tolerance $\bar{\tau}$ is the population average of the agent's risk tolerance $\{\tau_i\}$.

2. The semi-elasticity of the pricing function $p'(y)/p(y)$ is equal to:

$$\frac{p'(y)}{p(y)} = -\frac{1}{\bar{\tau}} - \frac{[y-\bar{y}]}{\bar{\sigma}^2}.$$

3. The representative agent variance $\bar{\sigma}^2$ is the weighed harmonic mean of the agents' variances $\{\sigma_i^2\}$. The weights of each precision are directly proportional to population and risk tolerant of each type.

4. The representative agent variance $\bar{\sigma}^2$ is independent of a proportional change in the risk tolerance for *all* agents.

5. The representative agent expected value $\bar{y}$ is the weighed average of the agents' expected values $\bar{y}_i$. The weights are directly proportional to population divided by the product of agent's type risk aversion and agent's variance.

6. The representative agent expected value $\bar{y}$ is independent of a proportional change to risk tolerance for *all* agents. The representative agent expected value $\bar{y}$ is also independent of a proportional change to variance of all *all* agents.

Market Portfolio (Lucas's tree), Simple Variance Swap, and an Option's portfolio. We define as the market portfolio (or Lucas' tree) an asset that pays y at $t=1$ when the aggregate output is y, with price F at time $t=0$. With our normalization, F has the units of a forward price, since it is relative to the price of a risk-less bond. We define a simple variance swap as an asset that pays $(y-F)^2$ at $t=1$ when the aggregate output is y. We denote the time $t=0$ price of this asset as V, again in terms of an asset that pays 1 in each state of nature. In terms of the definition of the notation for $K-$mutual fund theorem, we can set:

$$f_0(y) = 1,\ f_1(y) = y,\ \text{and}\ f_2(y) = (y-F)^2\ . \tag{12}$$

Note that the third fund, is defined in terms of the market price of the second fund. This makes is closer to a simple variance swap, an asset traded in actual markets, but different from typical definitions of the mutual fund separation theorem. Clearly, the funds are not uniquely defined. We can take any other quadratic function of y.

We return now the pricing of the two key assets in our economy.

Corollary 1. Using the expression from p in equation (10) from Proposition 1 we obtain that the price of the Market and Variance Swap are:

$$F \equiv \frac{\int_{-\infty}^{\infty} y\, p(y)dy}{\int_{-\infty}^{\infty} p(y)dy} = \bar{y} - \frac{\bar{\sigma}^2}{\bar{\tau}} \text{ and}$$
$$V \equiv \frac{\int_{-\infty}^{\infty} (y-F)^2\ p(y)dy}{\int_{-\infty}^{\infty} p(y)dy} = \bar{\sigma}^2\ .$$

1. F, the forward price of the Lucas' tree, is given by the representative agent expected value $\bar{\mu}$ by the risk premium $\bar{\sigma}^2/\bar{\tau}$.

2. V, the price of the variance swap, is given by the representative agent's variance $\bar{\sigma}^2$, so it there is no direct risk adjustment, except that the definition of its payoffs uses F, which itself is impacted by the risk premium.

3. Any economy with different distribution of $(\tau_i, \bar{y}_i, \sigma_i^2, \mu_i)$ but with the same $(\bar{\tau}, \bar{y}, \bar{\sigma}^2)$ has the same prices F, V.

We now relate the one period variance swap to the cash-flow of a portfolio of a portfolio of out of the money puts and calls, and its price, the SVIX. Let call(K) and put(K) be the price at time $t = 0$, in terms of one unit of consumption at date $t = 1$ of a call and put with strike price K, i.e.:

$$\text{call}(K) = \frac{\int_{-\infty}^{\infty} \max\{y - K, 0\}\, p(y)dy}{\int_{-\infty}^{\infty} p(y)dy} \text{ and } \text{put}(K) = \frac{\int_{-\infty}^{\infty} \max\{K - y, 0\}\, p(y)dy}{\int_{-\infty}^{\infty} p(y)dy} \tag{13}$$

We have:

Proposition 2. An equally weighted portfolio with 2 of each of the out of the money puts and calls (i.e. puts with all the strikes $K < F$ and calls with all the strikes $K > F$), has the same payoff at $t = 1$ than the simple variance swap $(y - F)^2$, and thus has a price at $t = 0$ equal to V:

$$V = 2\left[\int_{-\infty}^{F} \text{put}(K)dK + \int_{F}^{\infty} \text{call}(K)dK\right]. \tag{14}$$

We have few comments on this proposition.

1. This result implies that agents can either have access to a one period simple variance swap, or equivalently to a portfolio of puts and calls on the market portfolio.

2. The Proposition 2 is very closely related to Result 4 in Martin (2011,2013). There are minor differences in the set-ups, such as that we have a only period model and that y is unbounded from below, which requires some modifications on his proof.

3. Martin (2011,2013) uses an approximation of the expression in Proposition 2 using finite many strike prices and observed options prices on the SP500 to compute empirical counterparts of V, which he named SVIX. He also compares it with VIX.

The next result gives simple expression for the equilibrium portfolios.

Proposition 3. Three Fund Separation. The complete market allocation can be decentralized using three assets: an uncontingent bond, the market portfolio in terms of risk-less bonds (a Lucas's tree), and a one-period simple variance swap on the Lucas's tree (or equivalently access to a portfolio of options with all strikes). We denote agent's i holdings of the Lucas's tree by α_1^i, and her holdings of the simple variance swap by α_2^i. They are given by:

$$\alpha_1^i = \frac{\tau_i}{\bar{\tau}} + [\bar{y}_i - \bar{y}]\,\frac{\tau_i}{\sigma_i^2} + \left[\frac{\bar{\sigma}^2 - \sigma_i^2}{\sigma_i^2}\right]\frac{\tau_i}{\bar{\tau}}, \quad \text{for all } i = 1, \dots, I, \tag{15}$$

$$\alpha_2^i = \frac{1}{2}\left[\frac{\sigma_i^2 - \bar{\sigma}^2}{\bar{\sigma}^2}\right]\frac{\tau_i}{\sigma_i^2}, \quad \text{for all } i = 1, \dots, I. \tag{16}$$

The holding of the uncontingent bond depend on the initial α_1^i depend on the parameters as well as individual endowments.

Few comments are in order.

1. The representative agent holds one unit of the Lucas's tree, i.e. $\alpha_1^i = 1$ and no exposure to the simple variance swap of, i.e. $\alpha_2^i = 0$.

2. For all distributions of the triplets $(\tau_i, y_i, \sigma_i^2)$ we have that averaging across agent's types: $\sum_i \alpha_1^i \, \mu_i = 1$ and $\sum_i \alpha_2^i \, \mu_i = 0$.

3. If an agent believes that y has higher expected value than the representative agent, i.e. if $\bar{y}_i > \bar{y}$, she will buy more units of the Lucas's tree, i.e. α_2^i will be higher. How much higher will depend on this agent risk tolerance relative to perceived variance, i.e. relative to τ_i/σ_i^2.

4. If the agent is more risk tolerant than the representative agent, i.e. if $\tau_i > \bar{\tau}$, then she will buy more units of the Lucas's tree, i.e. α_1^i will be higher.

5. If the agent believes the economy is more volatile than the representative agent, i.e. if $\sigma_i^2 > \bar{\sigma}^2$, then this agent will adjust both her holdings of the Lucas's tree and of the VIX. She will hold fewer units of the Lucas' tree, i.e. α_1^i will be lower, but she will buy the variance swap, i.e. $\alpha_2^i > 0$. The sensitivity of these adjustment depends on the relative variance, as well as on the agent's risk tolerance.

6. If there is no heterogeneity in the variances σ_i^2, the first two funds suffice, i.e. there is no need to different positions on the variance swap, even if heterogeneity in risk-tolerance.

7. We have not written an expression for the portfolio of uncontingent bonds α_1^i because it depends on the initial wealth of each agent. Agents will higher wealth will hold more uncontingent bonds. On the other hand, the wealth of each agent does not affect the demand for Lucas' trees nor the demands for variance swap.

Trade volume. Under certain additional conditions Proposition 3 has sharp implications on trade volume. Essentially, trade volume is given by the mean absolute deviation on the cross sectional heterogeneity. In particular, suppose that the initial portfolios of agents are all the same, and hence they must be equal to the ones of the representative agent. To be concrete, every agent start with one unit of the Lucas tree, no risk-free bonds, and no position on the variance swaps. In this case, the difference between the optimal portfolio in Proposition 3 and the representative agent's portfolio gives each agent's trade. Hence trade volume for the Lucas's tree (denoted by TV_2) and the trade volume in the variance swap (denoted by TV_3) can be readily computed by taking the difference in the absolute value between the expression in equation (15) and equation (16) respectively:

$$TV_2 = \sum_{i=1}^{I} \left| \left[\frac{\tau_i - \bar{\tau}}{\tau_i} \right] \frac{\sigma_i^2}{\bar{\sigma}^2} + [\bar{y}_i - \bar{y}] \frac{\bar{\tau}}{\bar{\sigma}^2} + \left[\frac{\bar{\sigma}^2 - \sigma_i^2}{\bar{\sigma}^2} \right] \right| \hat{\mu}_i \tag{17}$$

$$TV_3 = \frac{1}{2} \frac{\bar{\tau}}{\bar{\sigma}^2} \sum_{i=1}^{I} \left| \frac{\sigma_i^2 - \bar{\sigma}^2}{\bar{\sigma}^2} \right| \hat{\mu}_i \text{ where} \tag{18}$$

$$\hat{\mu}_i = \frac{\bar{\sigma}^2}{\bar{\tau}} \frac{\tau_i}{\sigma_i^2} \mu_i \text{ so that } \sum_{i=1}^{I} \hat{\mu}_i = 1\,. \tag{19}$$

The expressions for both trade values use $\hat{\mu}_i$ as weights on the different types, which are fractions of agents μ_i, reweighted by the ratio τ_i/σ_i^2. Trade volume of the Lucas's tree (TV_2) has three components, corresponding to differences in risk-tolerance, beliefs on the expected value of the market portfolio, and beliefs on the variance of the market portfolio. In the case in which there is heterogeneity in only one of these three dimensions, trade volume is proportional to the mean absolute value in each of the parameters. The trade volume on the variance swap (TV_3) depends only on the heterogeneity on the beliefs on the variance of the aggregate shocks, and it is proportional to the mean absolute deviation of the variance beliefs. The constant of proportionality is the ratio of the representative agent's risk-tolerance to the variance, so if agents are, in average, more risk tolerant, the same dispersion on variance beliefs generates more trade. Likewise, an increase in the average variance belief, while keeping the same distribution (say with a multiplicative shift the variance beliefs of each agent) proportionally decreases trade volume in the variance swap.

5 Small Noise Approximations

In this section instead of using a parametric version for preferences (i.e. CARA u_i) and belief distribution (i.e. normal y) we develop a small noise approximation, or as termed by Samuelson (1970) as "compact distributions". A similar small noise approximation is also used, in a related but different context, by Chabi-Yo, Ghysels, and Eric (2008) and Chabi-Yo, Dietmar, and Renault (2014). We parameterize the belief distributions by a common value $\sigma > 0$, and compute the limit of the efficient allocations as $\sigma \downarrow 0$. The beliefs are thus defined as:

$$\pi_i(y) = \frac{1}{v_i \sigma} \, \pi \left(\frac{y - \hat{y} - a_i \, \sigma^2}{v_i \sigma} \right) \quad \text{where } v_i > 0, \sigma > 0. \tag{20}$$

and where $\pi(\cdot)$ is density common to all agents. The interpretation of this parametrization is that agent i believes that:

$$y = \hat{y} + a_i \sigma^2 + \sigma \, v_i \, \epsilon \text{ with } \epsilon \sim \pi(\cdot) \tag{21}$$

for the same random variable ϵ. We assume that $\int \epsilon \pi(\epsilon) d\epsilon = 0$ and $\int \pi(\epsilon) d\epsilon = \int \epsilon^2 \pi(\epsilon) d\epsilon = 1$. Thus agent's i belief for the expected value of y is

$$\bar{y}_i = \hat{y} + a_i \, \sigma^2,$$

which contains a common component $\hat{y}$, and an agent specific component, a_i, which scales with σ^2. The variance of agent's i belief for y is given by

$$\sigma_i^2 = (v_i\, \sigma)^2 \,,$$

where v_i is the agent specific component of this variance. The parameterization of $\bar{y}_i$, as first suggested by Samuelson's (1970), ensures that as $\sigma \downarrow 0$, the trade off of risk and expected return for each agent are commensurate, so that it avoids corner solutions.

We study the shape of the efficient allocation $c_i(y;\sigma)$ as $\sigma \downarrow 0$. In particular, we are interested in a second order approximation to c_i with respect to y, evaluated at $\hat{y}$, as σ vanishes. For small σ we have:

$$c_i(y;\sigma) \approx c_i(\hat{y};\sigma) + c_i'(\hat{y};\sigma)\,(y-\hat{y}) + \frac{1}{2}c_i''(\hat{y};\sigma)\,(y-\hat{y})^2 + o\left((y-\hat{y})^2\right) \tag{22}$$

We will compare the values of $c_i'(\hat{y};\sigma)$ and $(1/2)c_i''(\hat{y};\sigma)$ for small σ with those of α_1^i and α_2^i in the CARA-Normal case. We will see that, if either $\pi'(0) = 0$ and $\pi''(0)/\pi(0) = -1$, as it is for Normal variable, *or* if the product of the absolute risk tolerance coefficient to the absolute prudence coefficient is one, as it is for a CARA utility function, then the expression for α_2^i and $(1/2)c_i''(\hat{y};\sigma)$ are the same. Also we will see that, as in the case of CARA-Normal, differences in variances have a large impact in $c_i''(\hat{y};\sigma)$, in the sense that $c_i''(\hat{y};\sigma) \propto 1/\sigma^2$ for small σ.

For the next proposition we define, abusing notation, the absolute risk tolerance and absolute prudence coefficients:

$$\tau_i \equiv -\frac{u_i'\left(c_i\left(\hat{y};0\right)\right)}{u_i''\left(c_i\left(\hat{y};0\right)\right)} \text{ and } \kappa_i \equiv -\frac{u_i'''\left(c_i\left(\hat{y};0\right)\right)}{u_i''\left(c_i\left(\hat{y};0\right)\right)}\,. \tag{23}$$

Using the parameterizations for $\sigma_i = \sigma v_i$ and $\bar{y}_i = \hat{y} + \sigma^2 a_i$ in the definitions of $\bar{\tau}$, $\bar{y}$ and $\bar{\sigma}$ in equation (6), equation (9) and equation (8) we have

$$\bar{y} = \hat{y} + \sigma^2 \frac{1}{\sum_j \frac{\tau_j \mu_j}{v_j^2}} \sum_i a_i \frac{\tau_i \mu_i}{v_i^2} \tag{24}$$

$$\bar{y}_i = \bar{y} + \sigma^2 \frac{1}{\sum_j \frac{\tau_j \mu_j}{v_j^2}} \sum_i a_i \frac{\tau_i}{v_i^2}\mu_i \tag{25}$$

$$\frac{1}{\bar{v}^2} = \frac{1}{\sum_j \tau_j \mu_j} \sum_i \frac{1}{v_i^2}\tau_i \mu_i \,, \tag{26}$$

We define $\bar{v}^2$, $\hat{v}$ and $\bar{a}$ as the harmonic mean of the normalized variance and standard deviation, and the mean of the normalized mean belief, which are independent of σ, as follows:

$$\frac{1}{\bar{v}^2} \equiv \frac{1}{\sum_k \tau_k \mu_k} \sum_i \frac{1}{v_i^2}\,\tau_i \mu_i \tag{27}$$

$$\frac{1}{\hat{v}} \equiv \frac{1}{\sum_k \tau_k \mu_k} \sum_i \frac{1}{v_i}\,\tau_i \mu_i \text{ and} \tag{28}$$

$$\bar{a} \equiv \frac{1}{\sum_k \frac{\tau_k \mu_k}{v_k^2}} \sum_i a_i\,\frac{\tau_i \mu_i}{v_i^2} \tag{29}$$

Thus, the following expressions are independent of σ:

$$\frac{\bar{y}-\hat{y}}{\bar{\sigma}^2} = \frac{1}{\bar{\tau}}\sum_i \frac{a_i\,\tau_i}{v_i^2}\mu_i \quad \text{and} \quad \frac{\bar{y}_i-\bar{y}}{\sigma_i^2} = \frac{a_i-\bar{a}}{v_i^2} \tag{30}$$

With these definitions at hand, we show that the result of the CARA-Normal case extend to symmetric distribution with small noise. There are four cases of interest, depending on whether there is heterogeneity on v_i's and a_i's, and whether $\pi'(0) = 0$ or not. We break it into two propositions since the most interesting case for us is the one with heterogeneity on the beliefs on variances.

Proposition 4. *Assume that $v_i \neq \bar{v}$ for some i.*

If $\pi'(0)/\pi(0) = 0$:

$$\lim_{\sigma\to 0} c_i'(\hat{y};\sigma) = \frac{\tau_i}{\bar{\tau}} + \frac{\tau_i}{v_i^2}\left[a_i-\bar{a}\right]\left[\frac{-\pi''(0)}{\pi(0)}\right] \tag{31}$$

$$\lim_{\sigma\to 0} \sigma^2 c_i''(\hat{y};\sigma) = \frac{\tau_i}{v_i^2}\left[\frac{v_i^2-\bar{v}^2}{\bar{v}^2}\right]\left[\frac{-\pi''(0)}{\pi(0)}\right] \tag{32}$$

If $\pi'(0)/\pi(0) \neq 0$:

$$\lim_{\sigma\to 0} \sigma\, c_i'(\hat{y};\sigma) = \left[\frac{\pi'(0)}{\pi(0)}\right]\frac{\tau_i}{v_i}\left[\frac{\hat{v}-v_i}{\hat{v}}\right] \tag{33}$$

$$\begin{aligned}\lim_{\sigma\to 0} \sigma^2 c_i''(\hat{y};\sigma) = {} & \left[\frac{\pi'(0)}{\pi(0)}\right]^2 \frac{\tau_i}{\hat{v}^2}\left(\sum_k \left\{\left[\frac{\hat{v}-v_k}{v_k}\right]^2[\kappa_k\tau_k-1] - \left[\frac{\hat{v}-v_i}{v_i}\right]^2[\kappa_i\tau_i-1]\right\}\mu_k\right) \\ & - \left[\frac{\pi''(0)}{\pi(0)} - \left(\frac{\pi'(0)}{\pi(0)}\right)^2\right]\frac{\tau_i}{v_i^2}\left[\frac{v_i^2-\bar{v}^2}{\bar{v}^2}\right]\end{aligned} \tag{34}$$

We have several comments on the this proposition. We first comment on c_i''.

1. For small σ, the expressions for $c_i'' \propto 1/\sigma^2$. This is the same proportionality factor than in the CARA-Normal case for α_2^i.

2. If the beliefs have a symmetric distribution, then the density evaluated a the mean value must be zero, $\pi'(0) = 0$ and $\pi'''(0) = 0$.

3. If $\pi'(0) = 0$, the expression for c_i'' are the same regardless of the value of the ratios $\kappa_i\tau_i$.

4. If the utility function is CARA then the product of prudence and risk tolerance is one, i.e. $\kappa_i\tau_i = 1$. In this case the expressions for c_i'' are the same regardless of the value of $\pi'(0)$.

5. If $\pi(\cdot)$ is the density of a Normal distribution, then $\pi'(0)/\pi(0) = 0$, and $\pi''(0)/\pi(0) = -1$, so that the expressions for $(1/2)c_i''$ are identical to the values for α_2^i for the CARA-Normal case.

For c_i' we note that:

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Now we state the results in the case of no heterogeneity in the beliefs about variances.

PROPOSITION 5. Assume that $v_i = \bar{v}$ for all i.

If $a_i \neq \bar{a}$ for some i :

$$\lim_{\sigma \to 0} c_i'(\hat{y};\sigma) = \frac{\tau_i}{\bar{\tau}} + \frac{\tau_i}{\bar{v}^2}\left[a_i - \bar{a}\right]\left(\frac{\pi''(0)}{\pi(0)} - \left(\frac{\pi'(0)}{\pi(0)}\right)^2\right) \tag{35}$$

$$\lim_{\sigma \to 0} \sigma\, c_i''(\hat{y};\sigma) = \left(-\frac{1}{\bar{v}^3}\right)\left[a_i - \bar{a}\right] \times$$
$$\left[\frac{\pi'''(0)}{\pi(0)} - \left(\frac{\pi'(0)}{\pi(0)}\right)^2 - 2\frac{\pi'(0)}{\pi(0)}\left(\frac{\pi''(0)}{\pi(0)} - \left(\frac{\pi'(0)}{\pi(0)}\right)^2\right)\right] \tag{36}$$

If $a_i = \bar{a}$ for all i and σ :

$$c_i'(\hat{y};\sigma) = \frac{\tau_i}{\bar{\tau}} \tag{37}$$

$$c_i''(\hat{y};\sigma) = \left[\frac{\kappa_i \tau_i - 1}{\tau_i}\right]\left(\frac{\tau_i}{\bar{\tau}}\right)^2 - \frac{\tau_i}{\bar{\tau}}\sum_k \left[\frac{\kappa_k \tau_k - 1}{\tau_k}\right]\left(\frac{\tau_k}{\bar{\tau}}\right)^2 \mu_k \tag{38}$$

6 A Dynamic Version

In this section we extend the result to a multiperiod set-up. The result on the one period model are clear, the asset used on it differ from the actual market portfolio and variance swap in that in while in the model they are based on one period payoffs, the actual assets involve cash-flows on several periods.

6.1 Set up

In this section we extend the results to a multi period model. Time is discrete and runs $t = 0, 1, \ldots, T$, where T can take the value $+\infty$. Agents use discounted expected utility, with possibly time varying discount factor β_t between period $t-1$ and t, common across agents, with $\beta_0 = 1$ and $\beta_t < .1$ all $t \geq 1$. The period utility for is state independent, but possibly heterogeneous, in particular agents have specific risk tolerance τ_i.[4]. As in the static case, we consider a pure endowment case. We denote by y_t the aggregate endowment a time t. Histories of aggregate endowment $y^t \in \mathbb{R}^{t+1}$ are denoted by $y^t = (y_t, y_{t-1}, \ldots, y_1, y_0)$, which solve the recursion $y^t = (y_t, y^{t-1})$. The economy starts at $t = 0$ where y_0 has been realized. Each agent i uses a density $\pi_{i,t+1}(y^t)$ over histories, with the property that:

$$\pi_{i,t+1}\left(\cdot|y^t\right) : Y \to \mathbb{R}, \text{ with } \pi_{i,t+1}(\cdot|y^t) \sim \mathcal{N}\left(\bar{y}_{i,t+1}(y^t), \bar{\sigma}^2_{i,t+1}(y^t)\right) \text{ for all } y^t, \text{ and all } t \geq 0. \tag{39}$$

[4]We can consider random risk aversion, whereda shocks to risk aversion have both aggregate and idiosyncratic components, as done in Alvarez and Atkeson (2017). In that paper we argue that in the presence of shocks to risk aversion it is more appropriate to use non-expected utility.

so the distribution of y_{t+1}, conditional on the history y^t, is *normal* with expected value and variance $\left(\bar{y}_{i,t+1}(y^t), \bar{\sigma}^2_{i,t+1}(y^t)\right)$. Agent i chose consumption plans: $c_i = \{c_{i,t}(y^t)\}_{t=0}^T$, for all with $y^t \in \mathbb{R}^t$, and for all $t \geq 0$. Thus agent's i objective function is:

$$\sum_{t=0}^{T} \left(\prod_{s=0}^{t} \beta_s \right) \int_{y^t \in Y^t} u_i \left(c_{i,t} \left(y^t \right) \right) \pi_{i,t} \left(dy^t \right) \tag{40}$$

Feasibility is defined history by history as follows:

$$\sum_{i=1}^{I} c_{i,t} \left(y^t \right) \mu_i = y_t, \text{ for all } y^t \in \mathbb{R}^t \text{ and all } t \geq 0 \,. \tag{41}$$

To define an Arrow-Debreu Competitive Equilibrium we introduce a pricing function $p = \{p_t(y^t)\}_{t=0}^T$, for all with $y^t \in \mathbb{R}^t$, and for all $t \geq 0$, with the interpretation that $p_t(y^t)$ is the price of one unit of consumption at date t, after history y^t in terms of date $t = 0$ consumption, i.e. we have normalized prices with $p_0 = 1$.

We will use to three different related prices. First we define the price of one unit of consumption at date $t+1$ contingent on history (y_{t+1}, y^t) relative to the one for the history y^t.

$$p_{t+1} \left(y_{t+1} | y^t \right) \equiv \frac{p_{t+1} \left(y_t, y^t \right)}{p_t \left(y^t \right)} \text{ for all } y^{t+1} \in \mathbb{R}^{t+1} \text{ and all } t \geq 1 \,, \text{ so} \tag{42}$$

$$p_{t+1}(y^{t+1}) = \prod_{s=0}^{t} p_s \left(y_{s+1} | y^s \right) \text{ for all } y^{t+1} \in \mathbb{R}^{t+1} \text{ and all } t \geq 1 \tag{43}$$

We also define $\bar{p}_{t+1} \left(y_{t+1} | y^t \right)$ as the average conditional price, i.e. the price of one unit of consumption at date $t+1$ when y_{t+1} is realized, relative to the price of one unit at time $t+1$ for sure, both contingent on history y^t, so that

$$\bar{p}_{t+1} \left(y_{t+1} | y^t \right) \equiv \frac{p_{t+1} \left(y_t, y^t \right)}{\int_y p_{t+1} \left(y, y^t \right) dy} \text{ for all } y^{t+1} \in \mathbb{R}^{t+1} \text{ and all } t \geq 1 \,, \tag{44}$$

Finally, short term risk-free bonds are defined as the price of good at $t+1$ across all realization of of y_{t+1}, in terms of units of y_t consumption goods, both condition on history y^t.

$$b_t(y^t) = \frac{\int_y p_{t+1} \left(y, y^t \right) dy}{p_t(y^t)} \text{ for all } y^t, \text{ and } t \geq 0 \tag{45}$$

so that short term rates are $R_t^f(y^t) = 1/b_t(y^t)$. Using this definition we can relate the two conditional prices as follows

$$p_{t+1} \left(y_{t+1} | y^t \right) = b_t(y^t) \, \bar{p}_{t+1}(y_{t+1} | y^t) \tag{46}$$

which implies that:

$$p_{t+1} \left(y_{t+1} | y^t \right) = \prod_{s=0}^{t} \frac{\bar{p}_{s+1}(y_{s+1} | y^s)}{R_s^f(y^s)} \tag{47}$$

Another important price is the one for an asset is a one period market dividend strip, i.e. the price of a claim to y_{t+1} in terms of time t consumption, conditional on history y^t. We denote this price by $S_t(y^t)$ and it is given by:

$$S_t(y^t) = \int_y y\, p_{t+1}\left(y|y^t\right) dy \tag{48}$$

and $F_t(y^t)$ is the one-period forward price of the one-period market strip, so that

$$F_t(y^t) = R_t^f(y^t) S_t(y^t) = \int_y y\, \bar{p}_{t+1}\left(y|y^t\right) dy$$

We also let the market portfolio be an asset that at t promise payoffs $\{y_s\}_{s=t+1}^T$. We let the price of this asset be $Q_t(y^t)$, satisfying:

$$Q_t(y^t) = \sum_{s=t+1}^{T} \int_{y_s} \cdots \int_{y_{t+1}} p_s\left(y_s|y^{s-1}\right) \ldots p_{t+1}\left(y_{t+1}|y^t\right) dy_s \ldots dy_{t+1} \tag{49}$$

The price of a one period variance swap, on a one-period market strip is denoted by $V_t(y^t)$, and is given by:

$$0 = \int_{-\infty}^{\infty} \left[\left(y - F_t(y^t)\right)^2 - V_t(y^t)\right] p_{t+1}(y|y^t) dy$$

6.2 Results

The results for the one period economy extends immediately conditionally to the dynamic case, appropriately redefined.

Proposition 6. Take a CARA-Normal-Heterogeneous economy with arbitrary endowments $e = \{e_i\}$. Let $\bar{p}_{t+1}(\cdot|y^t)$ be the average Arrow-Debreu conditional price of contingent claim at $t+1$ conditional on history y^t. This price coincides with one for an economy with only one agent with the average risk tolerance $\bar{\tau}$, and beliefs about the expected value and variance $(\bar{y}_{t+1}(y^t), \bar{\sigma}^2_{t+1}(y^t))$ given by the following weighted averages of the parameter for each agent:

$$\bar{\tau} \equiv \sum_i \tau_i \mu_i \tag{50}$$

$$\tilde{\mu}_i \equiv \frac{\tau_i \mu_i}{\sum_j \tau_j \mu_j} \text{ for all } i = 1, \ldots, I \tag{51}$$

$$\frac{1}{\bar{\sigma}^2_{t+1}(y^t)} \equiv \sum_i \frac{1}{\sigma^2_{i,t+1}(y^t)} \tilde{\mu}_i = \sum_i \frac{\tau_i}{\bar{\tau}\, \sigma^2_{i,t+1}(y^t)} \mu_i \text{ and} \tag{52}$$

$$\bar{y}_{t+1}(y^t) \equiv \sum_i \frac{\frac{\tilde{\tau}_i}{\sigma^2_{i,t+1}(y^t)}}{\sum_j \frac{\tilde{\tau}_j}{\sigma^2_{j,t+1}(y^t)}} \bar{y}_{i,t+1}(y^t) = \frac{1}{\sum_j \frac{\tau_j \mu_j}{\sigma^2_{j,t+1}(y^t)}} \sum_i \bar{y}_{i,t+1}(y^t) \frac{\tau_i \mu_i}{\sigma^2_{i,t+1}(y^t)} \tag{53}$$

The Arrow average conditional prices are given by:

$$\bar{p}_{t+1}(y|y^t) = e^{-\frac{1}{2}\left(y-[\bar{y}_{t+1}(y^t)-\bar{\sigma}^2_{t+1}(y^t)/\bar{\tau}]\right)^2 \frac{1}{\bar{\sigma}^2_{t+1}(y^t)}} \frac{1}{\sqrt{2\pi\bar{\sigma}^2_{t+1}(y^t)}} \text{ for all } y \in \mathbb{R}\,, \tag{54}$$

and the equilibrium prices coincide with the ones of an economy where there is only one agent with utility function $\bar{u}(\cdot)$ and beliefs $\bar{\pi}_{t+1}(\cdot|y^t)$:

$$\bar{u}(c) = -\bar{\tau}\exp\left(c/\bar{\tau}\right) \text{ and } \bar{\pi}_{t+1}(y|y^t) \sim \mathcal{N}\left(\bar{y}_{t+1}(y^t), \bar{\sigma}^2_{t+1}(y^t)\right)\,. \tag{55}$$

COROLLARY 2. Using the expression from $\bar{p}_{t+1}(\cdot|y^t)$ in equation (54) from Proposition 6 we obtain that the forward price of one-period dividend strip on the market portfolio and the one-period Variance Swap are:

$$F_t(y^t) = \frac{\left(\bar{y}_{t+1}(y^t) - \frac{\bar{\sigma}^2_{t+1}(y^t)}{\bar{\tau}}\right)}{R^f_t(y^t)} \text{ and}$$
$$V_t(y^t) \equiv \int_{-\infty}^{\infty} \left(y - F_t(y^t)\right)^2 \bar{p}_{t+1}(y|y^t)dy = \bar{\sigma}^2_{t+1}(y^t)\,.$$

In this case 3-mutual funds that suffice are also, conditionally, a risk free bond, a dividend strip of the market portfolio, and a one period variance swap on the dividend strip of the market portfolio.

$$f_{0,t+1}\left(y_{t+1}|y^t\right) = 1,\ f_{1,t+1}\left(y_{t+1}|y^t\right) = y_{t+1},\ \text{and } f_{2,t+1}\left(y_{t+1}|y^t\right) = \left(y_{t+1} - F_t(y^t)\right)^2\,. \tag{56}$$

The 3 mutual funds are not unique. As we have seen in the static model, and we will repeat below, the variance swap can be replace by a portfolio of 2 out of the money put and calls.

PROPOSITION 7. Conditional three Fund Separation. The complete market allocation can be decentralized using three assets: an uncontingent risk-less bond, a one forward contract on the (a Lucas's tree), and a one-period simple variance swap on the Lucas's tree. We denote agent's i holdings of the Lucas's tree by α^i_1, and her holdings of the simple variance swap by α^i_2. They are given by:

$$\alpha^i_{1,t+1}(y^t) = \frac{\tau_i}{\bar{\tau}} + \left[\bar{y}_{i,t} - \bar{y}_{t+1}(y^t)\right]\frac{\tau_i}{\sigma^2_{i,t+1}(y^t)} + \left[\frac{\bar{\sigma}^2_{t+1}(y^t) - \sigma^2_{i,t+1}(y^t)}{\sigma^2_{i,t+1}(y^t)}\right]\frac{\tau_i}{\bar{\tau}}\,, \quad \text{for all } i = 1, \ldots, I\,, \tag{57}$$

$$\alpha^i_{2,t+1}(y^t) = \frac{1}{2}\left[\frac{\sigma^2_{i,t+1}(y^t) - \bar{\sigma}^2_{t+1}(y^t)}{\bar{\sigma}^2_{t+1}(y^t)}\right]\frac{\tau_i}{\sigma^2_{i,t+1}(y^t)}\,, \quad \text{for all } i = 1, \ldots, I\,. \tag{58}$$

Among the main differences between the one period economy and the multiperiod economy are that with at least two periods we can define and analyze returns, and also that we can consider securities with payoffs in multiple periods. On the first issue, we can define returns by combining the prices describe in the one period economy, which are relative to risk-free securities, and interest rates. For this purpose, we next define and characterize the price of one period bonds $b_t(y^t)$ at t that pays one unit of consumption at $t+1$ as follows

$$b_t(y^t) = \beta_t \int_y \frac{\bar{u}'(y)}{\bar{u}'(y_t)} \, \bar{\pi}_{t+1}\left(y|y^t\right) dy = \beta_t \, e^{\frac{y_t - \bar{y}_{t+1}(y^t)}{\bar{\tau}}} \, e^{\frac{1}{2}\frac{\bar{\sigma}^2_{t+1}(y^t)}{\bar{\tau}^2}}$$

Thus the log of gross interest rates $(R_t(y^t))$ are given by:

$$\log R^f_t(y^t) = -\log b_t(y^t) = -\log \beta_t + \frac{\bar{y}_{t+1}(y^t) - y_t}{\bar{\tau}} - \frac{1}{2}\frac{\bar{\sigma}^2_{t+1}(y^t)}{\bar{\tau}^2}$$

Let's denote by $Put_{t,t+1}(K;y^t)$ and $Call_{t,t+1}(K;y^t)$ the one period call and put on the one-period dividend strip. These are prices time t prices of puts and calls that have $t+1$ payoffs.

$$Put_{t,t+1}(K;y^t) = \int_{-\infty}^{\infty} \max\{K-y,0\} p_{t+1}(y|y^t)dy$$
$$Call_{t,t+1}(K;y^t) = \int_{-\infty}^{\infty} \max\{y-K,0\} p_{t+1}(y|y^t)dy$$

As in the static case, we will obtain that the cash-flow of one period variance swap and a portfolio of puts and calls.

Proposition 8. An equally weighted portfolio at time t with 2 of each of the one period out of the money puts and calls (i.e. puts with all the strikes $K < F_t$ and calls with all the strikes $K > F_t$) on the dividend strip, has the same payoff at $t+1$ than the simple variance swap with payoffs $(y_{t+1} - F_t)^2$, and thus has a price at t equal to:

$$V_t(y^t) = 2\, R^f_t(y^t) \left[\int_{-\infty}^{F_t(y^t)} Put_{t,t+1}(K;y^t)dK + \int_{F_t(y^t)}^{\infty} Call_{t,t+1}(K;y^t)dK \right]$$

Note the small difference with the one period model in that V_t has an gross interest rate. This is because for the dynamic version we define the puts and calls as traded, or with price at, time t, and payoffs at time $t+1$.

Random Walk Example. Let $T=\infty$, and assume that all agents beliefs $\pi_i(\cdot|y^t)$ given by

$$y_{t+1} = y_t + m_{i,t} + \sigma^2_{i,t}\epsilon_{t+1}$$

where ϵ_t is i.i.d normal $(0,1)$, and where $m_{i,t}$ and $\sigma^2_{i,t}$ are determinisitc. Thus the representative agent has beliefs $\bar{y}_{t+1} = y_t + \bar{m}_t$ and $\bar{\sigma}^2_{t+1}$. The price of the market portfolio is:

$$Q_t(y^t) = \phi_t y_t + \kappa_t$$

for two functions of time $\{\phi_t, \kappa_t\}$. We obtain:

$$\phi_t = \prod_{s=t}^{\infty} \frac{1}{1+r_s} \text{ and } \kappa_t = \phi_t(\bar{m}_{t+1} - \bar{\sigma}^2_{t+1}/\bar{\tau}) + \sum_{s=t+1}^{\infty} \frac{\phi_s(\bar{m}_{s+1} - \bar{\sigma}^2_{s+1}/\bar{\tau})}{\prod_{u=t}^{s-1}(1+r_u)}$$

In the case where $\bar{m}_t = \bar{m}$, and $\bar{\sigma}_t^2 = \bar{\sigma}^2$ are constant through time, so aggregate consumption is a random walk, then the net interest $\bar{r} \equiv R_t^f - 1 = (1/\beta)e^{\frac{\bar{m}}{\bar{\tau}} - \frac{1}{2}\frac{\bar{\sigma}^2}{\bar{\tau}^2}} - 1$ as well as ϕ and κ are constant throug time:

$$\phi = \frac{1}{\bar{r}} \text{ and } \kappa = \left(\frac{1+\bar{r}}{\bar{r}^2}\right)(\bar{m} - \bar{\sigma}^2/\bar{\tau}) \text{ so that}$$
$$Q_t(y^t) = \frac{y_t}{\bar{r}} + \left(\frac{1+\bar{r}}{\bar{r}^2}\right)(\bar{m} - \bar{\sigma}^2/\bar{\tau})$$

Note that in the case of a random walk, Q_t is a linear function of y_t, so a portfolio with $\bar{r}$ number of share of the market portfolio and short in consol with coupon $\bar{r}\bar{\kappa}$ give a payoff equal to $\{y_t\}$. Thus, in this dynamic economy, a multiperiod simple variance swap, and a simple portfolio on the aggregate portfolio will give the separating mutual funds.

Learning Example. Suppose that each agent believes that y_t evolves according to:

$$y_{t+1} = \bar{m} + y_t + \omega_{t+1}$$

where ω_t as a time invariant but unknown variance σ^2. Suppose additionally that each agent learning is characterized by two parameters $\sigma_{i,0}^2$ and k_i. The first has the interpretation of agent's i initial expected value of her estimate of σ^2. The second has the interpretation of the precision of her initial estimate. Then we assume that the agent update her estimate with the observation of y_t as follows:

$$\sigma_{i,t+1}^2(y^t) = \frac{1}{t+k_i}(y_t - \bar{m} - y_{t-1})^2 + \frac{t+k_i-1}{t+k_i}\sigma_{i,t}^2(y^{t-1}) \text{ for all } y^t \text{ and } t \geq 1.$$

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7 Appendix

Lemma 1. Let $\{\bar{c}_i(\cdot)\}$ and $p(\cdot)$ be a complete market competitive equilibrium given endowments $\{e_i(\cdot)\}$, where prices are normalized as in equation (3). Then there is vector of weights $\lambda \in \mathbb{R}^I$ so that for each $y \in Y$:

$$\bar{c}(y) \equiv \{\bar{c}_i\,(y)\}_{i=1}^I \in \arg\max_{x_1,\ldots,x_I} \sum_{i=1}^I \lambda_i\, \mu_i\, u_i\,(x_i)\,\pi_i(y) \text{ s.t.: } \sum_{i=1}^I x_i\,\mu_i = y\,, \tag{59}$$

and $p(\cdot)$ is the Lagrange multiplier of the feasibility constraint for y. Likewise, a solution of equation (59) for a vector of weights, corresponds to an equilibrium for some endowments $\{e_i(\cdot)\}$,

Proof. (of Lemma 1)
The economy satisfies local non-satiation, and hence the first welfare theorem holds, and thus the equilibrium allocation is Pareto Optimal. On the other hand, since all $u_i(\cdot)$ are strictly concave, there is a vector $\lambda \in \mathbb{R}_+^I$ so that the equilibrium allocation solves"

$$\sum_{i=1}^I \lambda_i\,\mu_i \int_{y\in Y} u_i\,(c_i\,(y))\,\pi_i\,(y)\,dy + \int_{y\in Y}\left[y - \sum_{i=1}^I \mu_i c_i(y)\right] p\,(y)\,dy \tag{60}$$

On the other hand, given the additive separability of expected utility, the solution must maximize equation (59). For the second part, we use that the solution of the problem equation (59) for each y gives the sufficient conditions for the solution of the problem equation (60). Then, since the preferences are strictly convex, by virtue of the second welfare theorem we can find prices to decentralize the allocation. □

Proof. (of Proposition 1)
Consider the problem (59). The first order condition for the planner's choice of $c_i(y)$ is given by:

$$\lambda_i\mu_i \exp\left(-c_i(y)/\tau_i\right) \exp\left(-\frac{1}{2}\,(y-\bar{y}_i)^2\,\frac{1}{\sigma_i^2}\right)\frac{1}{\sqrt{2\pi\sigma_i^2}} = p(y)$$

where $p(y)$ is the multiplier of the feasibility constraint corresponding to that value of y. Taking logs:

$$\log\lambda_i + \log\mu_i - c_i(y)/\tau_i - \frac{1}{2}\,(y-\bar{y})^2\,\frac{1}{\sigma_i^2} - \frac{1}{2}\log(2\pi\sigma_i^2) = \log p(y)\,.$$

Multiplying by $\tau_i\mu_i$, adding up across i, and using feasibility we have:

$$\sum_i \left[\log(\lambda_i\mu_i) - \frac{1}{2}\log(2\pi\sigma_i^2)\right]\tau_i\mu_i - y\sum_i \mu_i - \frac{1}{2}\sum_i\,(y-\bar{y}_i)^2\,\frac{1}{\sigma_i^2}\tau_i\mu_i = \log p(y)\sum_i \tau_i\mu_i$$

Defining $\bar{\tau}$ as

$$\bar{\tau} \equiv \sum_i \tau_i\mu_i$$

then $p(y)$ is:

$$\log p(y) = \sum_i \left[\log(\lambda_i \mu_i) - \frac{1}{2}\log(2\pi\sigma_i^2)\right] \frac{\tau_i \mu_i}{\bar{\tau}} - y/\bar{\tau} - \frac{1}{2}\sum_i (y - \bar{y}_i)^2 \, \frac{1}{\sigma_i^2}\frac{\tau_i\mu_i}{\bar{\tau}}$$

Note

$$\begin{aligned}
\sum_i (y-\bar{y}_i)^2 \, \frac{1}{\sigma_i^2}\frac{\tau_i\mu_i}{\bar{\tau}} &= \sum_i \frac{1}{\sigma_i^2}\frac{\tau_i\mu_i}{\bar{\tau}} y^2 - 2y\sum_i \frac{1}{\sigma_i^2}\frac{\tau_i\mu_i}{\bar{\tau}}\bar{y}_i + \sum_i \frac{1}{\sigma_i^2}\frac{\tau_i\mu_i}{\bar{\tau}}(\bar{y}_i)^2 \\
&= \sum_j \frac{1}{\sigma_j^2}\frac{\tau_j\mu_j}{\bar{\tau}}\left[y^2 - 2y\sum_i \frac{\frac{1}{\sigma_i^2}\frac{\tau_i\mu_i}{\bar{\tau}}}{\sum_j \frac{1}{\sigma_j^2}\frac{\tau_j\mu_j}{\bar{\tau}}}\bar{y}_i\right] + \sum_i \frac{1}{\sigma_i^2}\frac{\tau_i\mu_i}{\bar{\tau}}(\bar{y}_i)^2 \\
&= (y-\bar{y})^2 \, \frac{1}{\bar{\sigma}^2} - \frac{(\bar{y})^2}{\bar{\sigma}^2} + \sum_i \frac{1}{\sigma_i^2}\frac{\tau_i\mu_i}{\bar{\tau}}(\bar{y}_i)^2
\end{aligned}$$

where we define

$$\begin{aligned}
\tilde{\mu}_i &\equiv \frac{\mu_i\tau_i}{\sum_j \tau_j\mu_j} \\
\bar{y} &\equiv \sum_i \frac{\frac{\tau_i}{\sigma_i^2}\mu_i}{\sum_j \frac{\tau_j}{\sigma_j^2}\mu_j} \; \bar{y}_i \\
\frac{1}{\bar{\sigma}^2} &\equiv \sum_i \tilde{\mu}_i \; \frac{1}{\sigma_i^2}
\end{aligned}$$

Thus we can write:

$$\begin{aligned}
\log p(y) = &\sum_i \left[\log\lambda_i - \frac{1}{2}\log(2\pi\sigma_i^2)\right]\tilde{\mu}_i - \frac{1}{2}\left[\sum_i \frac{1}{\sigma_i^2}\frac{\tau_i\mu_i}{\bar{\tau}}(\bar{y}_i)^2 - \frac{(\bar{y})^2}{\bar{\sigma}^2}\right] \\
&- y/\bar{\tau} - \frac{1}{2}(y-\bar{y})^2 \, \frac{1}{\bar{\sigma}^2}
\end{aligned}$$

Thus

$$e^{-y/\bar{\tau}} e^{-\frac{1}{2}(y-\bar{y})^2 \frac{1}{\bar{\sigma}^2}} \frac{1}{\sqrt{2\pi\bar{\sigma}^2}} \propto p(y) \text{ for all } y \in \mathbb{R}$$

where use $\propto$ to mean that its constant of proportionality doe not depend on y. Note that this is the marginal utility of an agent with $(\bar{\tau}, \bar{y}, \bar{\sigma}^2)$ that consumes the entire endowment, i.e. it is the utility of the representative agent. We can rewrite this expression as:

$$\begin{aligned}
p(y) &\propto e^{-y/\bar{\tau}} e^{-\frac{1}{2}(y-\bar{y})^2 \frac{1}{\bar{\sigma}^2}} \frac{1}{\sqrt{2\pi\bar{\sigma}^2}} \\
&= e^{-\frac{1}{2}\left(y^2 - 2y[\bar{y} - (\bar{\sigma}^2/\bar{\tau})] + \bar{y}^2\right)\frac{1}{\bar{\sigma}^2}} \frac{1}{\sqrt{2\pi\bar{\sigma}^2}} \\
&= e^{[(\bar{\sigma}^2/\bar{\tau})^2 - 2(\bar{y}\bar{\sigma}^2/\bar{\tau})]\frac{1}{2\bar{\sigma}^2}} e^{-\frac{1}{2}\left(y^2 - 2y[\bar{y} - (\bar{\sigma}^2/\bar{\tau})] + \bar{y}^2 + (\bar{\sigma}^2/\bar{\tau})^2 - 2(\bar{y}\bar{\sigma}^2/\bar{\tau})\right)\frac{1}{\bar{\sigma}^2}} \frac{1}{\sqrt{2\pi\bar{\sigma}^2}} \\
&= e^{[(\bar{\sigma}^2/\bar{\tau})^2 - 2(\bar{y}\bar{\sigma}^2/\bar{\tau})]\frac{1}{2\bar{\sigma}^2}} e^{-\frac{1}{2}\left(y - [\bar{y} - \bar{\sigma}^2/\bar{\tau}]\right)^2 \frac{1}{\bar{\sigma}^2}} \frac{1}{\sqrt{2\pi\bar{\sigma}^2}} \\
&\propto e^{-\frac{1}{2}\left(y - [\bar{y} - \bar{\sigma}^2/\bar{\tau}]\right)^2 \frac{1}{\bar{\sigma}^2}} \frac{1}{\sqrt{2\pi\bar{\sigma}^2}}
\end{aligned}$$

so prices are proportional to the pdf of a normal random variable with expected value $F = \bar{y} - \bar{\sigma}^2/\bar{\tau}$, and with variance $\bar{\sigma}^2$.

□

Proof. (of Proposition 2) Fix F and an arbitrary y. We first show that an equally weighted portfolio of 2 out of the money puts and calls as the same cash flow at $t = 1$ that $(y - F)^2$. We want to show that:

$$(y - F)^2 = 2\left[\int_{-\infty}^{F} \max\{K - y, 0\}dK + \int_{F}^{\infty} \max\{y - K, 0\}dK\right] \tag{61}$$

To show this we note that the value that the right hand side takes for two cases.

If $y \leq F$, then

$$\int_{-\infty}^{F} \max\{K - y, 0\}dK = \int_{y}^{F} (K - y)\, dK = \frac{1}{2}(y - F)^2 \text{ and } \int_{F}^{\infty} \max\{y - K, 0\}dK = 0$$

If $y \geq F$, then

$$\int_{F}^{\infty} \max\{y - K, 0\}dK = \int_{F}^{y} (y - K)\, dK = \frac{1}{2}(y - F)^2 \text{ and } \int_{-\infty}^{F} \max\{K - y, 0\}dK = 0$$

Thus the cash-flows in both sides of equation (61) are indeed the same, and hence the market value $(y - F)^2$ at $t = 0$ is the same as an equally weighted portfolio of 2 out of the money puts and calls, i.e. the market value of the right hand side of equation (61) is equal to $\int_{-\infty}^{F} \text{put}(K)dK + \int_{F}^{\infty} \text{call}(K)dK$.

□

Proof. (of Proposition 3)

Note that, by equating the log of the first order condition with the expression for the log of $p(y)$, and multiplying it by τ_i, the optimal allocation for $c_i(y)$ can be written as follows:

$$[\log(\lambda_i\mu_i) - \log(\pi\sigma_i^2/2)]\tau_i - c_i(y) - \frac{1}{2}(y - \bar{y}_i)^2 \frac{\tau_i}{\sigma_i^2} = a_i\tau_i - y\frac{\tau_i}{\bar{\tau}} - \frac{1}{2}(y - \bar{y})^2 \frac{\tau_i}{\bar{\sigma}^2}$$

where a_i is a constant that does not depend on y. Solving for $c_i(y)$:

$$\begin{aligned} c_i(y) = &\left[\log(\lambda_i\mu_i) - \log(\pi\sigma_i^2/2) - a_i\right]\tau_i + y\frac{\tau_i}{\bar{\tau}} \\ &+ \frac{\tau_i}{2}\left[(y - \bar{y})^2 \frac{1}{\bar{\sigma}^2} - (y - \bar{y}_i)^2 \frac{1}{\sigma_i^2}\right] \end{aligned}$$

and

$$
\begin{aligned}
&(y-\bar{y})^2 \frac{1}{\bar{\sigma}^2}-(y-\bar{y}_i)^2 \frac{1}{\sigma_i^2}=\frac{y^2-2y\bar{y}+\bar{y}^2}{\bar{\sigma}^2}-\frac{y^2-2y\bar{y}_i+\bar{y}_i^2}{\sigma_i^2} \\
&=y^2\left[\frac{1}{\bar{\sigma}^2}-\frac{1}{\sigma_i^2}\right]-2y\left[\frac{\bar{y}}{\bar{\sigma}^2}-\frac{\bar{y}_i}{\sigma_i^2}\right]+\left[\frac{\bar{y}^2}{\bar{\sigma}^2}-\frac{\bar{y}_i^2}{\sigma_i^2}\right] \\
&=\left[\frac{1}{\bar{\sigma}^2}-\frac{1}{\sigma_i^2}\right](y-F)^2 \\
&-\left[\frac{1}{\bar{\sigma}^2}-\frac{1}{\sigma_i^2}\right]\left(-2yF+F^2\right)-2y\left[\frac{\bar{y}}{\bar{\sigma}^2}-\frac{\bar{y}_i}{\sigma_i^2}\right]+\left[\frac{\bar{y}^2}{\bar{\sigma}^2}-\frac{\bar{y}_i^2}{\sigma_i^2}\right] \\
&=\left[\frac{1}{\bar{\sigma}^2}-\frac{1}{\sigma_i^2}\right](y-F)^2-2y\left(\left[\frac{\bar{y}}{\bar{\sigma}^2}-\frac{\bar{y}_i}{\sigma_i^2}\right]-\left[\frac{F}{\bar{\sigma}^2}-\frac{F}{\sigma_i^2}\right]\right) \\
&-\left[\frac{1}{\bar{\sigma}^2}-\frac{1}{\sigma_i^2}\right]F^2+\left[\frac{\bar{y}^2}{\bar{\sigma}^2}-\frac{\bar{y}_i^2}{\sigma_i^2}\right]
\end{aligned}
$$

From the previous expression we can write the efficient consumption as:

$$
c_i(y)=\alpha_1^i+\alpha_2^i\, y+\alpha_3^i\,(y-F)^2 .
$$

The direct exposure to y from the linear term depends on the risk tolerance, relative to the average, just as in the case of common variance. To better match it with the variance swap consider the one where the quadratic deviation is with respect with the *market value* of y, so that it reads $(y-F)^2$, where $F=\bar{y}-\bar{\sigma}^2/\bar{\tau}$, i.e., where the coefficients satisfy:

$$
\begin{aligned}
\alpha_2^i &= \frac{\tau_i}{\bar{\tau}}-\left(\left[\frac{\bar{y}}{\bar{\sigma}^2}-\frac{\bar{y}_i}{\sigma_i^2}\right]-\left[\frac{F}{\bar{\sigma}^2}-\frac{F}{\sigma_i^2}\right]\right)\tau_i \\
&= \frac{\tau_i}{\bar{\tau}}-\left(\left[\frac{\bar{y}-F}{\bar{\sigma}^2}-\frac{\bar{y}_i-F}{\sigma_i^2}\right]\right)\tau_i \\
&= \frac{\tau_i}{\bar{\tau}}+\left[\bar{y}_i-\bar{y}\right]\frac{\tau_i}{\sigma_i^2}+\left[\frac{\bar{\sigma}^2-\sigma_i^2}{\sigma_i^2}\right]\frac{\tau_i}{\bar{\tau}} \\
\alpha_3^i &= \frac{1}{2}\left[\frac{1}{\bar{\sigma}^2}-\frac{1}{\sigma_i^2}\right]\tau_i=\frac{1}{2}\left[\frac{\sigma_i^2-\bar{\sigma}^2}{\bar{\sigma}^2}\right]\frac{\tau_i}{\sigma_i^2}
\end{aligned}
$$

□

Proof. (of Proposition 4 and Proposition 5) The first order condition for the planner problem is:

$$
\lambda_i u_i'\left(c_i\left(y;\sigma\right)\right)\frac{1}{v_i\sigma}\,\pi\left(\frac{y-\hat{y}-a_i\,\sigma^2}{v_i\sigma}\right)=p(y;\sigma) \tag{62}
$$

where λ_i is the planner weight on agent type i. Evaluating at $\hat{y}$,

$$
\lambda_i u_i'\left(c_i\left(\hat{y};\sigma\right)\right)\frac{1}{v_i}\,\pi\left(\frac{-a_i\,\sigma}{v_i}\right)=\sigma p(\hat{y};\sigma)
$$

Taking $\sigma\to 0$:

$$
\frac{\lambda_i}{v_i}u_i'\left(c_i\left(\hat{y};0\right)\right)\,\pi\left(0\right)=\lim_{\sigma\to 0}\sigma p(\hat{y};\sigma)
$$

Takin logs on equation (62)

$$\log\frac{\lambda_i}{v_i}+\log u_i'\left(c_i\left(y;\sigma\right)\right)+\log\pi\left(\frac{y-\hat{y}-a_i\,\sigma^2}{v_i\sigma}\right)=\log\sigma p(y;\sigma)$$

and differentiating with respect to y:

$$\frac{u_i''\left(c_i\left(y;\sigma\right)\right)}{u_i'\left(c_i\left(y;\sigma\right)\right)}c_i'\left(y;\sigma\right)+\frac{1}{v_i\sigma}\frac{\pi'\left(\frac{y-\hat{y}-a_i\,\sigma^2}{v_i\sigma}\right)}{\pi\left(\frac{y-\hat{y}-a_i\,\sigma^2}{v_i\sigma}\right)}=\frac{p'(y;\sigma)}{p(y;\sigma)}$$

Evaluating the equation for the first derivative at $y=\hat{y}$:

$$\frac{u_i''\left(c_i\left(\hat{y};\sigma\right)\right)}{u_i'\left(c_i\left(\hat{y};\sigma\right)\right)}c_i'\left(\hat{y};\sigma\right)+\frac{1}{v_i\sigma}\frac{\pi'\left(\frac{-a_i\,\sigma}{v_i}\right)}{\pi\left(\frac{-a_i\,\sigma}{v_i}\right)}=\frac{p'(\hat{y};\sigma)}{p(\hat{y};\sigma)} \tag{63}$$

For future reference we also compute the second derivative of $c_i(y;\sigma)$. We have:

$$\begin{aligned}&\left[\frac{u_i''\left(c_i\left(y;\sigma\right)\right)}{u_i'\left(c_i\left(y;\sigma\right)\right)}\right]c_i''\left(y;\sigma\right)+c_i'\left(y;\sigma\right)\frac{d}{dy}\left[\frac{u_i''\left(c_i\left(y;\sigma\right)\right)}{u_i'\left(c_i\left(y;\sigma\right)\right)}\right]\\&+\frac{1}{(v_i\sigma)^2}\left[\frac{\pi''\left(\frac{y-\hat{y}-a_i\,\sigma^2}{v_i\sigma}\right)}{\pi\left(\frac{y-\hat{y}-a_i\,\sigma^2}{v_i\sigma}\right)}-\left(\frac{\pi'\left(\frac{y-\hat{y}-a_i\,\sigma^2}{v_i\sigma}\right)}{\pi\left(\frac{y-\hat{y}-a_i\,\sigma^2}{v_i\sigma}\right)}\right)^2\right]\\&=\frac{p''(y;\sigma)}{p(y;\sigma)}-\left(\frac{p'(y;\sigma)}{p(y;\sigma)}\right)^2\end{aligned}$$

Evaluating the equation for the second derivative at $y=\hat{y}$:

$$\begin{aligned}&\left[\frac{u_i''\left(c_i\left(\hat{y};\sigma\right)\right)}{u_i'\left(c_i\left(\hat{y};\sigma\right)\right)}\right]c_i''\left(\hat{y};\sigma\right)+c_i'\left(\hat{y};\sigma\right)\frac{d}{dy}\left[\frac{u_i''\left(c_i\left(\hat{y};\sigma\right)\right)}{u_i'\left(c_i\left(y;\sigma\right)\right)}\right]\\&+\frac{1}{(v_i\sigma)^2}\left[\frac{\pi''\left(\frac{-a_i\,\sigma}{v_i}\right)}{\pi\left(\frac{-a_i\,\sigma}{v_i}\right)}-\left(\frac{\pi'\left(\frac{-a_i\,\sigma}{v_i}\right)}{\pi\left(\frac{-a_i\,\sigma}{v_i}\right)}\right)^2\right]\\&=\frac{p''(\hat{y};\sigma)}{p(\hat{y};\sigma)}-\left(\frac{p'(\hat{y};\sigma)}{p(\hat{y};\sigma)}\right)^2\end{aligned} \tag{64}$$

Finally we use feasibility to conclude that for all y and σ we have

$$\sum_i c_i'(y;\sigma)\mu_i=1 \text{ and } \sum_i c_i''(y;\sigma)\mu_i=0$$

Case 1. First derivative when $\pi'(0)=0$:

Taking the limit as $\sigma \to 0$ in equation (63) and using L'Hopital, since $\pi'(0) = 0$ and $\pi(0) > 0$, we get

$$\frac{u_i''\left(c_i\left(\hat{y};0\right)\right)}{u_i'\left(c_i\left(\hat{y};0\right)\right)} c_i'\left(\hat{y};0\right) - \frac{a_i}{(v_i)^2}\frac{\pi''\left(0\right)}{\pi\left(0\right)} = \lim_{\sigma\to 0}\frac{p'(\hat{y};\sigma)}{p(\hat{y};\sigma)}$$

Using the definition of τ_i then

$$c_i'\left(\hat{y};0\right) + \tau_i\frac{a_i}{(v_i)^2}\frac{\pi''\left(0\right)}{\pi\left(0\right)} = -\tau_i\lim_{\sigma\to 0}\frac{p'(\hat{y};\sigma)}{p(\hat{y};\sigma)}$$

Multiplying the previous expression by μ_i, adding up across i and using feasibilty:

$$1 + \frac{\pi''\left(0\right)}{\pi\left(0\right)}\sum_i\frac{a_i}{(v_i)^2}\tau_i\mu_i = -\bar{\tau}\lim_{\sigma\to 0}\frac{p'(\hat{y};\sigma)}{p(\hat{y};\sigma)}$$

or

$$\begin{aligned}
\lim_{\sigma\to 0}\frac{p'(\hat{y};\sigma)}{p(\hat{y};\sigma)} &= -\frac{1}{\bar{\tau}} - \frac{\pi''\left(0\right)}{\pi\left(0\right)}\sum_i\frac{a_i}{(v_i)^2}\frac{\tau_i\mu_i}{\bar{\tau}} = -\frac{1}{\bar{\tau}} - \frac{\pi''\left(0\right)}{\pi\left(0\right)}\sum_i\frac{a_i\sigma^2}{\sigma_i^2}\frac{\tau_i\mu_i}{\bar{\tau}} \\
&= -\frac{1}{\bar{\tau}} - \frac{\pi''\left(0\right)}{\pi\left(0\right)}\frac{1}{\bar{\tau}}\sum_i\left[\bar{y}_i - \hat{y}\right]\frac{\tau_i\mu_i}{\sigma_i^2}
\end{aligned}$$

Using the definition of $\bar{\sigma}^2$ and $\bar{y}$ in equation (8) and equation (9):

$$\lim_{\sigma\to 0}\frac{p'(\hat{y};\sigma)}{p(\hat{y};\sigma)} = -\frac{1}{\bar{\tau}} - \frac{\left[\hat{y} - \bar{y}\right]}{\bar{\sigma}^2}\left[\frac{-\pi''\left(0\right)}{\pi\left(0\right)}\right]$$

Now, replacing on the expression for $c_i'(\hat{y};0)$ we get:

$$\begin{aligned}
c_i'\left(\hat{y};0\right) &= \frac{\tau_i}{\bar{\tau}} + \left[\hat{y} - \bar{y}\right]\frac{\tau_i}{\bar{\sigma}^2}\left[\frac{-\pi''\left(0\right)}{\pi\left(0\right)}\right] - \tau_i\frac{a_i}{(v_i)^2}\frac{\pi''\left(0\right)}{\pi\left(0\right)} \\
&= \frac{\tau_i}{\bar{\tau}} + \left[\hat{y} - \bar{y}\right]\frac{\tau_i}{\bar{\sigma}^2}\left[\frac{-\pi''\left(0\right)}{\pi\left(0\right)}\right] + \frac{\tau_i}{\sigma_i^2}a_i\sigma^2\left[\frac{-\pi''\left(0\right)}{\pi\left(0\right)}\right] \\
&= \frac{\tau_i}{\bar{\tau}} + \left[\hat{y} - \bar{y}\right]\frac{\tau_i}{\bar{\sigma}^2}\left[\frac{-\pi''\left(0\right)}{\pi\left(0\right)}\right] + \frac{\tau_i}{\sigma_i^2}\left[\bar{y}_i - \hat{y}\right]\left[\frac{-\pi''\left(0\right)}{\pi\left(0\right)}\right] \\
&= \frac{\tau_i}{\bar{\tau}} + \frac{\tau_i}{\sigma_i^2}\left[\bar{y}_i - \bar{y}\right]\left[\frac{-\pi''\left(0\right)}{\pi\left(0\right)}\right]
\end{aligned}$$

Case 2. First derivative when $\pi'(0) \neq 0$, and $v_i \neq \bar{v}$ for some i.

Using the definition of τ_i into first derivative, multiplying by μ_i, adding across i, and multiplying both sides by σ:

$$-\sum_i c_i'\left(\hat{y};\sigma\right)\mu_i\sigma + \sum_i\frac{\tau_i(\sigma)\mu_i}{v_i}\frac{\pi'\left(\frac{-a_i\sigma}{v_i}\right)}{\pi\left(\frac{-a_i\sigma}{v_i}\right)} = \sigma\frac{p'(\hat{y};\sigma)}{p(\hat{y};\sigma)}\sum_i\tau_i(\sigma)\mu_i$$

Using feasibility:

$$-\sigma+\sum_i \frac{\tau_i(\sigma)\mu_i}{v_i}\frac{\pi'\left(\frac{-a_i\,\sigma}{v_i}\right)}{\pi\left(\frac{-a_i\,\sigma}{v_i}\right)}=\sigma\frac{p'(\hat{y};\sigma)}{p(\hat{y};\sigma)}\sum_i \tau_i(\sigma)\mu_i$$

Taking the limit as $\sigma\to 0$:

$$\lim_{\sigma\to 0}\frac{p'(\hat{y};\sigma)}{p(\hat{y};\sigma)}\sigma=\left[\frac{\pi'(0)}{\pi(0)}\right]\frac{1}{\bar{\tau}}\sum_i \frac{\tau_i\mu_i}{v_i}$$

Replacing the expression for $\lim p'/p$ back into the equation for c_i':

$$-\lim_{\sigma\to 0}c_i'(\hat{y};\sigma)\,\sigma+\frac{\tau_i}{v_i}\left[\frac{\pi'(0)}{\pi(0)}\right]=\frac{\tau_i}{\bar{\tau}}\sum_i \frac{\tau_i\mu_i}{v_i}\left[\frac{\pi'(0)}{\pi(0)}\right]$$

or

$$\lim_{\sigma\to 0}c_i'(\hat{y};\sigma)\,\sigma=\left[\frac{\pi'(0)}{\pi(0)}\right]\tau_i\left[\frac{1}{v_i}-\frac{1}{\bar{\tau}}\sum_j \frac{\tau_j\mu_j}{v_j}\right]$$

or

$$\lim_{\sigma\to 0}c_i'(\hat{y};\sigma)\,\sigma=\left[\frac{\pi'(0)}{\pi(0)}\right]\tau_i\left[\frac{1}{v_i}-\frac{1}{\hat{v}}\right]=\left[\frac{\pi'(0)}{\pi(0)}\right]\frac{\tau_i}{v_i}\left[\frac{\hat{v}-v_i}{\hat{v}}\right]$$

Case 3. First derivative when $\pi'(0)\neq 0$, and $v_i=\bar{v}$ for all i.

If $v_i=\bar{v}$ for all i then $\lim_{\sigma\to 0}c_i'(\hat{y};\sigma)\,\sigma=0$, so it is consistent with $\lim_{\sigma\to 0}c_i'(\hat{y};\sigma)$ being finite. Indeed if $v_i=\bar{v}$ we have:

$$-\frac{1}{\bar{\tau}(\sigma)}+\frac{1}{\bar{v}\sigma}\sum_i \frac{\tau_i(\sigma)\mu_i}{\bar{\tau}(\sigma)}\frac{\pi'\left(\frac{-a_i\,\sigma}{v_i}\right)}{\pi\left(\frac{-a_i\,\sigma}{v_i}\right)}=\frac{p'(\hat{y};\sigma)}{p(\hat{y};\sigma)}$$

Inserting this into the equation for c_i' we have:

$$-c_i'(\hat{y};\sigma)+\frac{\tau_i(\sigma)}{\bar{v}\sigma}\frac{\pi'\left(\frac{-a_i\,\sigma}{v_i}\right)}{\pi\left(\frac{-a_i\,\sigma}{\bar{v}}\right)}=\tau_i(\sigma)\frac{p'(\hat{y};\sigma)}{p(\hat{y};\sigma)}=-\frac{\tau_i(\sigma)}{\bar{\tau}(\sigma)}+\frac{\tau_i(\sigma)}{\bar{v}\sigma}\sum_j \frac{\tau_j(\sigma)\mu_j}{\bar{\tau}(\sigma)}\frac{\pi'\left(\frac{-a_j\,\sigma}{\bar{v}}\right)}{\pi\left(\frac{-a_j\,\sigma}{\bar{v}}\right)}$$

or

$$c_i'(\hat{y};\sigma)=\frac{\tau_i(\sigma)}{\bar{\tau}(\sigma)}+\frac{\tau_i(\sigma)}{\bar{v}\sigma}\left[\frac{\pi'\left(\frac{-a_i\,\sigma}{v_i}\right)}{\pi\left(\frac{-a_i\,\sigma}{\bar{v}}\right)}-\sum_j \frac{\tau_j(\sigma)\mu_j}{\bar{\tau}(\sigma)}\frac{\pi'\left(\frac{-a_j\,\sigma}{\bar{v}}\right)}{\pi\left(\frac{-a_j\,\sigma}{\bar{v}}\right)}\right]$$

Using L'Hopital we get:

$$\lim_{\sigma\to 0}c_i'(\hat{y};\sigma)=\frac{\tau_i}{\bar{\tau}}+\frac{\tau_i}{\bar{v}^2}\left[a_i-\sum_j \frac{\tau_j\mu_j}{\bar{\tau}}a_j\right]\left(\frac{\pi''(0)}{\pi(0)}-\left(\frac{\pi'(0)}{\pi(0)}\right)^2\right)$$

Hence, in the case where $\pi'(0) \neq 0$ and $\bar{v} = v_i$ for all i the limit for $c_i'(\hat{y};\sigma)$ is finite, and it depends on the differences on expected values.

<u>Case 1. Second derivative when $\pi'(0) = 0$</u>

Multiplying both sides of equation (64) by σ^2 we have:

$$\begin{aligned}
&\left[\frac{u_i''(c_i(y;\sigma))}{u_i'(c_i(y;\sigma))}\right]\sigma^2 c_i''(y;\sigma) + \sigma^2 c_i'(y;\sigma)\frac{d}{dy}\left[\frac{u_i''(c_i(y;\sigma))}{u_i'(c_i(y;\sigma))}\right] \\
&+ \frac{1}{(v_i)^2}\left[\frac{\pi''\left(\frac{y-\hat{y}-a_i\sigma^2}{v_i\sigma}\right)}{\pi\left(\frac{y-\hat{y}-a_i\sigma^2}{v_i\sigma}\right)} - \left(\frac{\pi'\left(\frac{y-\hat{y}-a_i\sigma^2}{v_i\sigma}\right)}{\pi\left(\frac{y-\hat{y}-a_i\sigma^2}{v_i\sigma}\right)}\right)^2\right] \\
&= \sigma^2\left(\frac{p''(y;\sigma)}{p(y;\sigma)} - \left(\frac{p'(y;\sigma)}{p(y;\sigma)}\right)^2\right)
\end{aligned}$$

Evaluating at $y = \hat{y}$:

$$\begin{aligned}
&\left[\frac{u_i''(c_i(\hat{y};\sigma))}{u_i'(c_i(\hat{y};\sigma))}\right]\sigma^2 c_i''(\hat{y};\sigma) + \sigma^2 c_i'(\hat{y};\sigma)\frac{d}{dy}\left[\frac{u_i''(c_i(\hat{y};\sigma))}{u_i'(c_i(\hat{y};\sigma))}\right] \\
&+ \frac{1}{(v_i)^2}\left[\frac{\pi''\left(\frac{-a_i\sigma}{v_i}\right)}{\pi\left(\frac{-a_i\sigma}{v_i}\right)} - \left(\frac{\pi'\left(\frac{-a_i\sigma}{v_i}\right)}{\pi\left(\frac{-a_i\sigma}{v_i}\right)}\right)^2\right] \\
&= \sigma^2\left(\frac{p''(\hat{y};\sigma)}{p(y;\sigma)} - \left(\frac{p'(\hat{y};\sigma)}{p(\hat{y};\sigma)}\right)^2\right)
\end{aligned}$$

Since $\pi'(0) = 0$ implies that $\lim_{\sigma\to 0} c_i'(\hat{y};\sigma)$ is finite, then taking limits we have:

$$\lim_{\sigma\to 0}\sigma^2 c_i''(\hat{y};\sigma) = \frac{\tau_i}{(v_i)^2}\left[\frac{\pi''(0)}{\pi(0)}\right] - \tau_i\lim_{\sigma\to 0}\sigma^2\left(\frac{p''(\hat{y};\sigma)}{p(y;\sigma)} - \left(\frac{p'(\hat{y};\sigma)}{p(\hat{y};\sigma)}\right)^2\right)$$

Multiplying by μ_i adding up and using feasibility:

$$\sum_i \frac{\tau_i}{(v_i)^2}\mu_i\left[\frac{\pi''(0)}{\pi(0)}\right] = \bar{\tau}\lim_{\sigma\to 0}\sigma^2\left(\frac{p''(\hat{y};\sigma)}{p(y;\sigma)} - \left(\frac{p'(\hat{y};\sigma)}{p(\hat{y};\sigma)}\right)^2\right)$$

or

$$\lim_{\sigma\to 0}\sigma^2\left(\frac{p''(\hat{y};\sigma)}{p(y;\sigma)} - \left(\frac{p'(\hat{y};\sigma)}{p(\hat{y};\sigma)}\right)^2\right) = \left[\frac{\pi''(0)}{\pi(0)}\right]\frac{1}{\bar{\tau}}\sum_i\frac{\tau_i}{(v_i)^2}\mu_i = \left[\frac{\pi''(0)}{\pi(0)}\right]\frac{1}{\bar{v}^2}$$

Replacing this expression into the second derivative of agent's i we get:

$$\begin{aligned}
\lim_{\sigma\to 0}\sigma^2 c_i''(\hat{y};\sigma) &= \frac{\tau_i}{(v_i)^2}\left[\frac{\pi''(0)}{\pi(0)}\right] - \tau_i\left[\frac{\pi''(0)}{\pi(0)}\right]\frac{1}{\bar{v}^2} \\
&= \left[\frac{\pi''(0)}{\pi(0)}\right]\tau_i\left[\frac{1}{v_i^2} - \frac{1}{\bar{v}^2}\right] \\
&= \left[\frac{-\pi''(0)}{\pi(0)}\right]\frac{\tau_i}{v_i^2}\left[\frac{v_i^2 - \bar{v}^2}{\bar{v}^2}\right]
\end{aligned}$$

Case 2. Second derivative when $\pi'(0) \neq 0$ and $v_i \neq v$ for some i.

Multiplying equation (64) by μ_i, adding up:

$$-\sum_i c_i''(\hat{y};\sigma)\,\mu_i - \sum_i \tau_i(\sigma)\mu_i c_i'(\hat{y};\sigma)\frac{d}{dy}\left[\frac{u_i''(c_i(\hat{y};\sigma))}{u_i'(c_i(y;\sigma))}\right]$$
$$+\sum_i \frac{\tau_i(\sigma)}{(v_i\sigma)^2}\mu_i\left[\frac{\pi''\left(\frac{-a_i\sigma}{v_i}\right)}{\pi\left(\frac{-a_i\sigma}{v_i}\right)} - \left(\frac{\pi'\left(\frac{-a_i\sigma}{v_i}\right)}{\pi\left(\frac{-a_i\sigma}{v_i}\right)}\right)^2\right]$$
$$=\left[\frac{p''(\hat{y};\sigma)}{p(\hat{y};\sigma)} - \left(\frac{p'(\hat{y};\sigma)}{p(\hat{y};\sigma)}\right)^2\right]\sum_i \tau_i(\sigma)$$

Using feasibility and multiplying both sides by σ^2:

$$-\sum_i c_i'(\hat{y};\sigma)\,\tau_i(\sigma)\mu_i\frac{d}{dy}\left[\frac{u_i''(c_i(\hat{y};\sigma))}{u_i'(c_i(y;\sigma))}\right]\sigma^2$$
$$+\sum_i \frac{\tau_i(\sigma)}{(v_i)^2}\mu_i\left[\frac{\pi''\left(\frac{-a_i\sigma}{v_i}\right)}{\pi\left(\frac{-a_i\sigma}{v_i}\right)} - \left(\frac{\pi'\left(\frac{-a_i\sigma}{v_i}\right)}{\pi\left(\frac{-a_i\sigma}{v_i}\right)}\right)^2\right]$$
$$=\left[\frac{p''(\hat{y};\sigma)}{p(\hat{y};\sigma)} - \left(\frac{p'(\hat{y};\sigma)}{p(\hat{y};\sigma)}\right)^2\right]\bar{\tau}(\sigma)\,\sigma^2$$

or

$$-\sum_i \left(c_i'(\hat{y};\sigma)\right)^2\,\sigma^2\tau_i(\sigma)\mu_i\frac{d}{dc}\left[\frac{u_i''(c_i(\hat{y};\sigma))}{u_i'(c_i(y;\sigma))}\right]$$
$$+\sum_i \frac{\tau_i(\sigma)}{(v_i)^2}\mu_i\left[\frac{\pi''\left(\frac{-a_i\sigma}{v_i}\right)}{\pi\left(\frac{-a_i\sigma}{v_i}\right)} - \left(\frac{\pi'\left(\frac{-a_i\sigma}{v_i}\right)}{\pi\left(\frac{-a_i\sigma}{v_i}\right)}\right)^2\right]$$
$$=\left[\frac{p''(\hat{y};\sigma)}{p(\hat{y};\sigma)} - \left(\frac{p'(\hat{y};\sigma)}{p(\hat{y};\sigma)}\right)^2\right]\bar{\tau}(\sigma)\,\sigma^2$$

Note that

$$\tau_i(\sigma)\frac{d}{dc}\left[\frac{u_i''(c_i(\hat{y};\sigma))}{u_i'(c_i(y;\sigma))}\right] = \frac{\kappa_i(\sigma)\tau_i(\sigma)-1}{\tau_i(\sigma)}$$

where $\kappa_i(\sigma)$ is the absolute prudence coefficient, i.e.:

$$\kappa_i(\sigma) = -\frac{u_i'''(c_i(\hat{y};\sigma))}{u_i''(c_i(\hat{y};\sigma))}\,.$$

so we can write:

$$
\begin{aligned}
&-\frac{1}{\bar{\tau}(\sigma)}\sum_j \left(c_j'\left(\hat{y};\sigma\right)\right)^2\ \sigma^2\left[\frac{\kappa_j(\sigma)\tau_j(\sigma)-1}{\tau_j(\sigma)}\right]\mu_j \\
&+\frac{1}{\bar{\tau}(\sigma)}\sum_j \frac{\tau_j(\sigma)}{(v_j)^2}\left[\frac{\pi''\left(\frac{-a_j\,\sigma}{v_j}\right)}{\pi\left(\frac{-a_j\,\sigma}{v_j}\right)}-\left(\frac{\pi'\left(\frac{-a_j\,\sigma}{v_j}\right)}{\pi\left(\frac{-a_j\,\sigma}{v_j}\right)}\right)^2\right]\mu_j \\
&=\left[\frac{p''(\hat{y};\sigma)}{p(\hat{y};\sigma)}-\left(\frac{p'(\hat{y};\sigma)}{p(\hat{y};\sigma)}\right)^2\right]\sigma^2
\end{aligned}
$$

and from each i:

$$
\begin{aligned}
&-\frac{1}{\tau_i(\sigma)}c_i''\left(\hat{y};\sigma\right)\sigma^2-\left(c_i'\left(\hat{y};\sigma\right)\right)^2\ \sigma^2\left[\frac{\kappa_i(\sigma)\tau_i(\sigma)-1}{\tau_i(\sigma)^2}\right] \\
&+\frac{1}{(v_i)^2}\left[\frac{\pi''\left(\frac{-a_i\,\sigma}{v_i}\right)}{\pi\left(\frac{-a_i\,\sigma}{v_i}\right)}-\left(\frac{\pi'\left(\frac{-a_i\,\sigma}{v_i}\right)}{\pi\left(\frac{-a_i\,\sigma}{v_i}\right)}\right)^2\right] \\
&=\left[\frac{p''(\hat{y};\sigma)}{p(\hat{y};\sigma)}-\left(\frac{p'(\hat{y};\sigma)}{p(\hat{y};\sigma)}\right)^2\right]\sigma^2
\end{aligned}
$$

Equating both expressions we obtain:

$$
\begin{aligned}
&\frac{1}{\tau_i(\sigma)}c_i''\left(\hat{y};\sigma\right)\sigma^2= \qquad (65)\\
&-\left(c_i'\left(\hat{y};\sigma\right)\right)^2\ \sigma^2\left[\frac{\kappa_i(\sigma)\tau_i(\sigma)-1}{\tau_i^2(\sigma)}\right]+\sum_j \left(c_j'\left(\hat{y};\sigma\right)\right)^2\ \sigma^2\left[\frac{\kappa_j(\sigma)\tau_j(\sigma)-1}{\tau_j^2(\sigma)}\right]\mu_j \\
&+\frac{1}{(v_i)^2}\left[\frac{\pi''\left(\frac{-a_i\,\sigma}{v_i}\right)}{\pi\left(\frac{-a_i\,\sigma}{v_i}\right)}-\left(\frac{\pi'\left(\frac{-a_i\,\sigma}{v_i}\right)}{\pi\left(\frac{-a_i\,\sigma}{v_i}\right)}\right)^2\right]-\sum_j\frac{1}{\bar{\tau}(\sigma)}\frac{\tau_j(\sigma)}{(v_j)^2}\left[\frac{\pi''\left(\frac{-a_j\,\sigma}{v_j}\right)}{\pi\left(\frac{-a_j\,\sigma}{v_j}\right)}-\left(\frac{\pi'\left(\frac{-a_j\,\sigma}{v_j}\right)}{\pi\left(\frac{-a_j\,\sigma}{v_j}\right)}\right)^2\right]\mu_j
\end{aligned}
$$

Taking limits as $\sigma\to 0$:

$$
\begin{aligned}
&\lim_{\sigma\to 0} c_i''\left(\hat{y};\sigma\right)\sigma^2= \\
&\sum_j\left(\lim_{\sigma\to 0}\left(c_j'\left(\hat{y};\sigma\right)\sigma\right)^2\ \frac{\tau_i}{\tau_j}\left[\frac{\kappa_j\tau_j-1}{\tau_j}\right]-\lim_{\sigma\to 0}\left(c_i'\left(\hat{y};\sigma\right)\sigma\right)^2\ \left[\frac{\kappa_i\tau_i-1}{\tau_i}\right]\right)\mu_j \\
&-\left[\frac{\pi''\left(0\right)}{\pi\left(0\right)}-\left(\frac{\pi'\left(0\right)}{\pi\left(0\right)}\right)^2\right]\frac{\tau_i}{v_i^2}\left[\frac{v_i^2-\bar{v}^2}{\bar{v}^2}\right]
\end{aligned}
$$

Rearranging

$$\lim_{\sigma\to 0} c_i''\left(\hat{y};\sigma\right)\sigma^2 = \left[\frac{\pi'\left(0\right)}{\pi\left(0\right)}\right]^2 \times$$
$$\sum_j \left\{ \tau_i \left(\tau_j \left[\frac{1}{v_j} - \frac{1}{\bar{\tau}} \sum_k \frac{\tau_k \mu_k}{v_k} \right] \right)^2 \left[\frac{\kappa_j \tau_j - 1}{\tau_j^2} \right] - \left(\tau_i \left[\frac{1}{v_i} - \frac{1}{\bar{\tau}} \sum_k \frac{\tau_k \mu_k}{v_k} \right] \right)^2 \left[\frac{\kappa_i \tau_i - 1}{\tau_i} \right] \right\} \mu_j$$
$$- \left[\frac{\pi''\left(0\right)}{\pi\left(0\right)} - \left(\frac{\pi'\left(0\right)}{\pi\left(0\right)} \right)^2 \right] \frac{\tau_i}{v_i^2} \left[\frac{v_i^2 - \bar{v}^2}{\bar{v}^2} \right]$$

Using $\hat{v}$ we can write:

$$\lim_{\sigma\to 0} c_i''\left(\hat{y};\sigma\right)\sigma^2 = \left[\frac{\pi'\left(0\right)}{\pi\left(0\right)}\right]^2 \tau_i \times$$
$$\sum_j \left\{ \left(\tau_j \left[\frac{1}{v_j} - \frac{1}{\hat{v}} \right] \right)^2 \left[\frac{\kappa_j \tau_j - 1}{\tau_j^2} \right] - \left(\tau_i \left[\frac{1}{v_i} - \frac{1}{\hat{v}} \right] \right)^2 \left[\frac{\kappa_i \tau_i - 1}{\tau_i^2} \right] \right\} \mu_j$$
$$- \left[\frac{\pi''\left(0\right)}{\pi\left(0\right)} - \left(\frac{\pi'\left(0\right)}{\pi\left(0\right)} \right)^2 \right] \frac{\tau_i}{v_i^2} \left[\frac{v_i^2 - \bar{v}^2}{\bar{v}^2} \right]$$

or

$$\lim_{\sigma\to 0} c_i''\left(\hat{y};\sigma\right)\sigma^2 = \left[\frac{\pi'\left(0\right)}{\pi\left(0\right)}\right]^2 \tau_i \times$$
$$\sum_j \left\{ \left(\tau_j \left[\frac{\hat{v} - v_j}{v_j \hat{v}} \right] \right)^2 \left[\frac{\kappa_j \tau_j - 1}{\tau_j^2} \right] - \left(\tau_i \left[\frac{\hat{v} - v_i}{v_i \hat{v}} \right] \right)^2 \left[\frac{\kappa_i \tau_i - 1}{\tau_i^2} \right] \right\} \mu_j$$
$$- \left[\frac{\pi''\left(0\right)}{\pi\left(0\right)} - \left(\frac{\pi'\left(0\right)}{\pi\left(0\right)} \right)^2 \right] \frac{\tau_i}{v_i^2} \left[\frac{v_i^2 - \bar{v}^2}{\bar{v}^2} \right]$$

or

$$\lim_{\sigma\to 0} c_i''\left(\hat{y};\sigma\right)\sigma^2$$
$$= \left[\frac{\pi'\left(0\right)}{\pi\left(0\right)}\right]^2 \frac{\tau_i}{\hat{v}^2} \left(\sum_k \left\{ \left[\frac{\hat{v} - v_k}{v_k} \right]^2 \left[\kappa_k \tau_k - 1\right] - \left[\frac{\hat{v} - v_i}{v_i} \right]^2 \left[\kappa_i \tau_i - 1\right] \right\} \mu_k \right)$$
$$- \left[\frac{\pi''\left(0\right)}{\pi\left(0\right)} - \left(\frac{\pi'\left(0\right)}{\pi\left(0\right)} \right)^2 \right] \frac{\tau_i}{v_i^2} \left[\frac{v_i^2 - \bar{v}^2}{\bar{v}^2} \right]$$

Case 3. Second derivative when $\pi'(0) \neq 0$ and $v_i = \bar{v}$ for all i.

We multiply equation (65) by σ obtaining:

$$c_i''(\hat{y};\sigma)\,\sigma =$$
$$-\left(c_i'(\hat{y};\sigma)\right)^2\,\sigma\left[\frac{\kappa_i(\sigma)\tau_i(\sigma)-1}{\tau_i^2(\sigma)}\right]+\sum_j\left(c_j'(\hat{y};\sigma)\right)^2\,\sigma\left[\frac{\kappa_j(\sigma)\tau_j(\sigma)-1}{\tau_j^2(\sigma)}\right]\mu_j$$
$$+\frac{1}{\sigma\bar{v}^2}\left[\frac{\pi''\left(\frac{-a_i\,\sigma}{\bar{v}}\right)}{\pi\left(\frac{-a_i\,\sigma}{\bar{v}}\right)}-\left(\frac{\pi'\left(\frac{-a_i\,\sigma}{\bar{v}}\right)}{\pi\left(\frac{-a_i\,\sigma}{\bar{v}}\right)}\right)^2\right]-\frac{1}{\sigma\bar{v}^2}\sum_j\frac{\tau_j(\sigma)}{\bar{\tau}(\sigma)}\left[\frac{\pi''\left(\frac{-a_j\,\sigma}{\bar{v}}\right)}{\pi\left(\frac{-a_j\,\sigma}{\bar{v}}\right)}-\left(\frac{\pi'\left(\frac{-a_j\,\sigma}{\bar{v}}\right)}{\pi\left(\frac{-a_j\,\sigma}{\bar{v}}\right)}\right)^2\right]\mu_j$$

We take limits as $\sigma\to 0$, using that when $\pi'(0)\neq 0$ and $v_i=\bar{v}$ for all i, the limit $c_i'(\hat{y};\sigma)$ is finite as $\sigma\to 0$. Thus

$$\lim_{\sigma\to 0}c_i''(\hat{y};\sigma)\,\sigma =$$
$$\frac{1}{\sigma\bar{v}^2}\left(\left[\frac{\pi''\left(\frac{-a_i\,\sigma}{\bar{v}}\right)}{\pi\left(\frac{-a_i\,\sigma}{\bar{v}}\right)}-\left(\frac{\pi'\left(\frac{-a_i\,\sigma}{\bar{v}}\right)}{\pi\left(\frac{-a_i\,\sigma}{\bar{v}}\right)}\right)^2\right]-\sum_j\frac{\tau_j(\sigma)}{\bar{\tau}(\sigma)}\left[\frac{\pi''\left(\frac{-a_j\,\sigma}{\bar{v}}\right)}{\pi\left(\frac{-a_j\,\sigma}{\bar{v}}\right)}-\left(\frac{\pi'\left(\frac{-a_j\,\sigma}{\bar{v}}\right)}{\pi\left(\frac{-a_j\,\sigma}{\bar{v}}\right)}\right)^2\right]\mu_j\right)$$

which using L'Hopital rule gives:

$$\lim_{\sigma\to 0}c_i''(\hat{y};\sigma)\,\sigma =$$
$$\left(-\frac{1}{\bar{v}^2}\right)\left[\frac{\pi'''(0)}{\pi(0)}-\left(\frac{\pi'(0)}{\pi(0)}\right)^2-2\frac{\pi'(0)}{\pi(0)}\left(\frac{\pi''(0)}{\pi(0)}-\left(\frac{\pi'(0)}{\pi(0)}\right)^2\right)\right]\left(\frac{a_i}{\bar{v}}-\frac{1}{\bar{\tau}}\sum_k\frac{a_k}{\bar{v}}\tau_k\mu_k\right)$$
$$=\left(-\frac{1}{\bar{v}^3}\right)\left[\frac{\pi'''(0)}{\pi(0)}-\left(\frac{\pi'(0)}{\pi(0)}\right)^2-2\frac{\pi'(0)}{\pi(0)}\left(\frac{\pi''(0)}{\pi(0)}-\left(\frac{\pi'(0)}{\pi(0)}\right)^2\right)\right]\left(a_i-\frac{1}{\bar{\tau}}\sum_k a_k\tau_k\mu_k\right)$$

or

$$\lim_{\sigma\to 0}c_i''(\hat{y};\sigma)\,\sigma =$$
$$\left(-\frac{1}{\bar{v}^3}\right)\left[\frac{\pi'''(0)}{\pi(0)}-\left(\frac{\pi'(0)}{\pi(0)}\right)^2-2\frac{\pi'(0)}{\pi(0)}\left(\frac{\pi''(0)}{\pi(0)}-\left(\frac{\pi'(0)}{\pi(0)}\right)^2\right)\right]\left(a_i-\frac{1}{\bar{\tau}}\sum_k a_k\tau_k\mu_k\right)$$
$$=\left(-\frac{1}{\bar{v}^3}\right)\left(a_i-\frac{1}{\bar{\tau}}\sum_k a_k\tau_k\mu_k\right)\left[\frac{d^3}{dx^3}\log\pi(x)|_{x=0}\right]$$

Case 4. First and Second derivative when $a_i=\bar{a}=0$, and $v_i=\bar{a}$ for all i .

In the case of homogenous beliefs it is well known that $c(y)$ is independent of the distribution of beliefs. Thus we can define $s(y)$ as

$$s(y;\sigma)\equiv\sigma\frac{p(y;\sigma)}{\pi\left((y-\hat{y})/\sigma\right)}$$

We have:

$$\lambda_i u_i'(c(y;\sigma)=s(y;\sigma)$$

which implies:

$$\frac{u_i''(c(y;\sigma)}{u_i'(c(y;\sigma)}c_i'(y;\sigma) = \frac{s'(y;\sigma)}{s(y;\sigma)}$$

or evaluating at $y = \hat{y}$:

$$-c_i'(\hat{y};\sigma) = \frac{-u_i'(c(\hat{y};\sigma)}{u_i''(c(\hat{y};\sigma)}\frac{s'(\hat{y};\sigma)}{s(y;\sigma)}$$

multiplying by μ_i and adding across i, we get:

$$-\frac{1}{\bar{\tau}(\sigma)} = \frac{s'(\hat{y};\sigma)}{s(y;\sigma)} \text{ and } c_i'(\hat{y};\sigma) = \frac{\tau_i(\sigma)}{\bar{\tau}(\sigma)}$$

Differentiating again:

$$\frac{u_i''(c(\hat{y};\sigma)}{u_i'(c(\hat{y};\sigma)}c_i''(y;\sigma) + \left[\frac{u_i'''(c(\hat{y};\sigma)}{u_i'(c(\hat{y};\sigma)} - \left(\frac{u_i''(c(\hat{y};\sigma)}{u_i(c(\hat{y};\sigma)}\right)^2\right](c_i'(y;\sigma))^2 = \left[\frac{s''(\hat{y};\sigma)}{s(\hat{y};\sigma)} - \left(\frac{s'(\hat{y};\sigma)}{s(\hat{y};\sigma)}\right)^2\right]$$

or

$$\begin{aligned} & -c_i''(y;\sigma) + \frac{-u_i'(c(\hat{y};\sigma)}{u_i''(c(\hat{y};\sigma)}\left[\frac{u_i'''(c(\hat{y};\sigma)}{u_i'(c(\hat{y};\sigma)} - \left(\frac{u_i''(c(\hat{y};\sigma)}{u_i(c(\hat{y};\sigma)}\right)^2\right](c_i'(y;\sigma))^2 \\ & = \frac{-u_i'(c(\hat{y};\sigma)}{u_i''(c(\hat{y};\sigma)}\left[\frac{s''(\hat{y};\sigma)}{s(\hat{y};\sigma)} - \left(\frac{s'(\hat{y};\sigma)}{s(\hat{y};\sigma)}\right)^2\right] \end{aligned}$$

or

$$-\frac{1}{\tau_i(\sigma)}c_i''(y;\sigma) + \left[\frac{\kappa_i(\sigma)\tau_i(\sigma) - 1}{\tau_i^2(\sigma)}\right]\left(\frac{\tau_i(\sigma)}{\bar{\tau}(\sigma)}\right)^2 = \left[\frac{s''(\hat{y};\sigma)}{s(\hat{y};\sigma)} - \left(\frac{s'(\hat{y};\sigma)}{s(\hat{y};\sigma)}\right)^2\right]$$

or

Multiplying by $\mu_i\tau_i(\sigma)$ and adding up across i, and using feasibility:

$$\frac{1}{\bar{\tau}(\sigma)}\sum_k\left[\frac{\kappa_k(\sigma)\tau_k(\sigma) - 1}{\tau_k(\sigma)}\right]\left(\frac{\tau_k(\sigma)}{\bar{\tau}(\sigma)}\right)^2\mu_k = \left[\frac{s''(\hat{y};\sigma)}{s(\hat{y};\sigma)} - \left(\frac{s'(\hat{y};\sigma)}{s(\hat{y};\sigma)}\right)^2\right]$$

Substituting this expression into the one for c_i''"

$$c_i''(y;\sigma) = \left[\frac{\kappa_i(\sigma)\tau_i(\sigma) - 1}{\tau_i(\sigma)}\right]\left(\frac{\tau_i(\sigma)}{\bar{\tau}(\sigma)}\right)^2 - \frac{\tau_i(\sigma)}{\bar{\tau}(\sigma)}\sum_k\left[\frac{\kappa_k(\sigma)\tau_k(\sigma) - 1}{\tau_k(\sigma)}\right]\left(\frac{\tau_k(\sigma)}{\bar{\tau}(\sigma)}\right)^2\mu_k$$

Thus we have:

$$c_i''(\hat{y};\sigma) = \left[\frac{\kappa_i\tau_i - 1}{\tau_i}\right]\left(\frac{\tau_i}{\bar{\tau}}\right)^2 - \frac{\tau_i}{\bar{\tau}}\sum_k\left[\frac{\kappa_k\tau_k - 1}{\tau_k}\right]\left(\frac{\tau_k}{\bar{\tau}}\right)^2\mu_k$$

□

Table 1: Small Noise Approximation

	$\pi'(0) = 0$	$\pi'(0) \neq 0$
$v_i \neq \bar{v}$ some i	$c_i' \approx \frac{\tau_i}{\bar{\tau}} + \frac{\tau_i}{v_i^2} a_i \left[\frac{-\pi''(0)}{\pi(0)}\right]$ $c_i'' \approx \frac{\tau_i}{\sigma^2 v_i^2}\left[\frac{v_i^2 - \bar{v}^2}{\bar{v}^2}\right]\left[\frac{-\pi''(0)}{\pi(0)}\right]$	$c_i' \approx \frac{\tau_i}{\sigma}\left[\frac{1}{v_i} - \frac{1}{\bar{\tau}}\sum_k \frac{\tau_k \mu_k}{v_k}\right]\left[\frac{\pi'(0)}{\pi(0)}\right]$ $c_i'' \approx \frac{\tau_i}{\sigma^2 v_i^2}\left[\frac{v_i^2 - \bar{v}^2}{\bar{v}^2}\right]\left[\left(\frac{\pi'(0)}{\pi(0)}\right)^2 - \frac{\pi''(0)}{\pi(0)}\right] + \frac{\tau_i}{\sigma^2 \hat{v}^2}\left[\frac{\pi'(0)}{\pi(0)}\right]^2 \times$ $\left(\sum_j \left\{\left[\frac{\hat{v} - v_k}{v_k}\right]^2 [\kappa_k \tau_k - 1] - \left[\frac{\hat{v} - v_i}{v_i}\right]^2 [\kappa_i \tau_i - 1]\right\} \mu_j\right)$
$v_i = \bar{v}$ all i	$c_i' \approx \frac{\tau_i}{\bar{\tau}} + \frac{\tau_i}{\bar{v}^2}\left[a_i - \frac{1}{\bar{\tau}}\sum_k \tau_k \mu_k a_k\right]\left[\frac{-\pi''(0)}{\pi(0)}\right]$ $c_i'' \approx -\frac{1}{\sigma\,\bar{v}^3}\left[a_i - \frac{1}{\bar{\tau}}\sum_k a_k \tau_k \mu_k\right]\frac{\pi'''(0)}{\pi(0)}$	$c_i' \approx \frac{\tau_i}{\bar{\tau}} + \frac{\tau_i}{\bar{v}^2}\left[a_i - \frac{1}{\bar{\tau}}\sum_k \tau_k \mu_k a_k\right]\left(\frac{\pi''(0)}{\pi(0)} - \left(\frac{\pi'(0)}{\pi(0)}\right)^2\right)$ $c_i'' \approx -\frac{1}{\sigma\,\bar{v}^3}\left[a_i - \frac{1}{\bar{\tau}}\sum_k a_k \tau_k \mu_k\right] \times$ $\left[\frac{\pi'''(0)}{\pi(0)} - \left(\frac{\pi'(0)}{\pi(0)}\right)^2 - 2\frac{\pi'(0)}{\pi(0)}\left(\frac{\pi''(0)}{\pi(0)} - \left(\frac{\pi'(0)}{\pi(0)}\right)^2\right)\right]$
$v_i = \bar{v}$ $a_i = \bar{a}$ all i	$c_i' \approx \frac{\tau_i}{\bar{\tau}}$ $c_i'' \approx \left[\frac{\kappa_i \tau_i - 1}{\tau_i}\right]\left(\frac{\tau_i}{\bar{\tau}}\right)^2$ $-\frac{\tau_i}{\bar{\tau}}\sum_k \left[\frac{\kappa_k \tau_k - 1}{\tau_k}\right]\left(\frac{\tau_k}{\bar{\tau}}\right)^2 \mu_k$	$c_i' \approx \frac{\tau_i}{\bar{\tau}}$ $c_i'' \approx \left[\frac{\kappa_i \tau_i - 1}{\tau_i}\right]\left(\frac{\tau_i}{\bar{\tau}}\right)^2$ $-\frac{\tau_i}{\bar{\tau}}\sum_k \left[\frac{\kappa_k \tau_k - 1}{\tau_k}\right]\left(\frac{\tau_k}{\bar{\tau}}\right)^2 \mu_k$