Platform Trading with an OTC Market Fringe*

Jérôme Dugast
Luxembourg School of Finance

Semih Üslü
Johns Hopkins Carey Business School

Pierre-Olivier Weill
University of California, Los Angeles

PRELIMINARY AND INCOMPLETE. PLEASE DO NOT CIRCULATE.

February 8, 2018

Abstract

We study the privately and socially optimal participation of investors in a centralized platform or in an over-the-counter (OTC) market. Investors incur costs to trade in the platform, in the OTC market, or in both at the same time. Investors differ from each other in risk-sharing needs and OTC market trading capacities. We show that investors with low risk-sharing needs and large trading capacities endogenously emerge as OTC intermediaries, and have the strongest private incentives to enter the OTC market vs. the trading platform. Investors with strong risk-sharing needs and low trading capacities endogenously emerge as OTC customers, and have the weakest private incentive to enter the OTC market vs. the trading platform. Turning to social welfare, we provide two necessary conditions for customers' private incentives to be excessively large relative to their social contribution. Mandating or subsidizing trade in a centralized venue can be welfare improving only if these conditions are satisfied. First, investors must differ mostly in terms of OTC trading capacities. Second, participation costs must induce exclusive participation decisions. Based on the empirical trading patterns generated by closed-form examples of our model, we argue that the real-world OTC markets might satisfy the conditions under which mandating or subsidizing centralized trade is welfare improving.

*We would like to thank, for fruitful comments and suggestions, Valentin Haddad, Gregor Jarosch, Peter Kondor, and Zhaogang Song, as well as seminar participants at UCLA, University of Luxembourg, the Bank of Canada Workshop on Money, Banking, Payments, and Finance, the 2nd LAEF OTC Markets and Securities Workshop at UCSB, and the 2018 ASSA Meetings in Philadelphia.

1 Introduction

Over-the-counter (OTC) financial markets, for bonds, asset-backed securities, or swaps, have a decentralized market structure: trade is bilateral, opaque, and involves bargaining. In recent years, especially since the 2008 financial crisis, the decentralized structure of OTC markets has been the focus of policy discussions (see, for example, The Squam Lake Report, 2010). Regulators have made proposals and taken measures to make OTC markets more centralized (see, for example, Financial Stability Board, 2016). For example, regulators have mandated that some swaps trade multilaterally, via "swap execution facilities."[1] Does the mandate to centralize OTC markets improve welfare? The answer is not obvious. After all, investors have been content with trading over-the-counter for many years before regulators started to discuss centralization. A commonly held view is indeed that dealers greatly benefit from an OTC market structure because they profit from price dispersion when providing intermediation services to customers. One could argue that customers are satisfied with trading in OTC markets as well: otherwise, private parties could have attracted these customers in more centralized trading venues.

In this paper, we study theoretically whether mandating or subsidizing centralized trade is welfare improving. To do so, we study the privately and socially optimal participation of heterogeneous investors in a centralized platform, modeled as a Walrasian market, or in an OTC market. We assume that investors are heterogeneous in two dimensions. First, they are heterogeneous in terms of their initial endowment of risky assets, which determines the intensity of their risk-sharing needs. Second, they are heterogeneous in their ability to take large positions in the OTC market, which we call "OTC market trading capacity." We assume that investors must incur costs to participate in the Walrasian market, in the OTC market, or both at the same time. We first characterize the equilibrium abstractly and then illustrate its properties via examples.

When making their participation decisions, investors face a trade-off between sharing risk and earning intermediation profits. Specifically, while the centralized structure of the Walrasian market allows investors to change their risk exposures more easily, the decentralized structure of the OTC market creates price dispersion and allows investors to earn intermediation profits. We show that, under natural participation cost structures, investors with low risk-sharing needs and high trading capacity participate exclusively in the OTC market. Amongst the exclusive

[1] In 2009, G20 Leaders agreed that "all standardized OTC derivative contracts should be traded on exchanges or electronic trading platforms." And, as of June 2017, "12 jurisdictions have in force comprehensive assessment standards or criteria for determining when products should be platform traded, and an appropriate authority regularly assesses transactions against these criteria" (Financial Stability Board, 2017).

OTC investors, those with relatively large risk-sharing needs and small trading capacity behave as customers: they engage in small trades, mostly in the same direction, and so make little intermediation profits. On the other hand, investors with relatively small risk-sharing needs and large trading capacity behave as dealers: they engage in large trades in opposing directions and so make large intermediation profits. Depending on the cost structure, investors with high risk-sharing needs and low trading capacity participate exclusively in the Walrasian market or in both markets at the same time.

We show that customers are the marginal OTC market participants: relative to dealers, they have a weaker preference for the OTC over the Walrasian market. Their characteristics are not suitable to earn large intermediation profit. However, under natural participation cost structures, customers marginally prefer cheap but low-quality risk-sharing of the OTC market over expensive but high-quality risk-sharing of the Walrasian market. The finding that customers are marginal OTC participants is in line with the commonly held view that dealers benefit relatively more from OTC market trading, while customers benefit relatively less.

Of course, the finding that customers benefit relatively less from OTC market trading does not necessarily imply that mandating them to trade in the Walrasian market would be welfare improving. It only indicates that, compared to dealers, customers have relatively low private incentives to participate in the OTC vs. the Walrasian market. To study welfare, we need to compare the customers' private incentives to participate in the OTC vs. the Walrasian market with the social contribution of doing so. Moving OTC market customers to the Walrasian market is welfare improving only if these private incentives are larger than the corresponding social contribution. We provide two necessary conditions for customers' private incentives to enter the OTC market to be excessively large relative to their social contribution. Mandating or subsidizing trade in a centralized venue can be welfare improving only if these conditions are satisfied. First, investors must differ mostly in terms of OTC trading capacities. Second, participation costs must induce exclusive participation decisions. The intuition arises from a composition externality typical of random matching environments with *ex post* bargaining. If investors differ mostly in terms of OTC trading capacities, the marginal OTC participant has low trading capacity compared to other OTC participants. If she changes her decision to participating exclusively in the Walrasian market, she destroys her small trades and allows the remaining OTC participants to create larger trades among themselves, and hence, improves welfare. However, she does not have sufficiently high private incentives to do so because she captures a disproportionately high surplus in the OTC market. Alternatively, if investors differ mostly in terms of risk-sharing needs, this normative result is reversed. In this case, the marginal OTC participant has strong risk-sharing needs compared to other OTC participants. Her departure from the OTC market to participate exclusively in the Walrasian market destroys large-surplus trades and generates low-surplus trades, and hence, reduces welfare. Thus, moving marginal OTC market customers to the Walrasian market is not welfare-improving if investors differ mostly in terms of risk-sharing needs. Finally, notice that exclusive participation decisions play a key role in generating these welfare gains or losses. This is because, if an investor does not depart from the OTC market to trade exclusively in the Walrasian market, but depart to trade in both markets, then she neither destroys nor creates any trade.

An implication of this finding is that, in order to evaluate whether mandating or subsidizing trade in a centralized trading venue is welfare improving, it is crucial to empirically distinguish an economy in which investors differ mostly in terms of OTC trading capacity, from an economy in which investors differ mostly in terms of their risk sharing needs. Our examples suggest the following empirical distinctions. When investors differ mostly in terms of risk sharing needs, dealers have lower net trading volume, and dealers and customers have comparable gross trading volume. In contrast, when investors differ mostly in terms of OTC trading capacity, the net trading volume of dealers can be large, and the per-dealer gross trading volume can be much larger than the per-customer gross volume. In light of the empirical evidence, the latter case may fit better OTC trading patterns in practice. As OTC markets typically feature a core-periphery structure, most trades are intermediated by a handful of dealers, which implies that the per-dealer gross trading volume is larger than the per-customer gross volume, even after controlling for size.[2] In addition, Siriwardane (2015) finds that, in the CDS market, the net notionals are concentrated in the hands of dealers. In particular, he shows that, dealers are responsible for 55% of all net buying and 60% of all net selling on average between 2010 and 2014, which implies that the net trading volume of dealers in the CDS market is large.

A Short Literature Review. We build on the model of Atkeson, Eisfeldt, and Weill (2015, henceforth AEW), who have developed a tractable framework, based on insights from both the search- and network-theoretic literature, to study entry and trading patterns in OTC market. We extend AEW in two substantive ways. First, we allow investors to trade in two markets, one centralized and one decentralized, instead of only one decentralized market in Atkeson, Eisfeldt, and Weill (2015). This is obviously essential to analyze our main research question. Second, we

[2]See Bech and Atalay (2010), Li and Schürhoff (2012), Atkeson, Eisfeldt, and Weill (2013), Di Maggio, Kermani, and Song (2017), and Hollifield, Neklyudov, and Spatt (2017).

assume that investors have two dimensions of heterogeneity, endowment and trading capacity. Atkeson, Eisfeldt, and Weill (2015) only considered heterogeneity in endowments. We show that the two dimensions of heterogeneity generate opposite normative results, when participation costs induce exclusive participation decisions.

A branch of the literature compares the costs and benefits associated with centralized and decentralized trading structures without endogenous participation decision. See, for example, Glode and Opp (2015) and Geromichalos and Herrenbrueck (2016). Another branch of the literature has studied the trade-off between exclusive participation in a centralized or a decentralized market. See, for example, Yavaş (1992), Gehrig (1993), Rust and Hall (2003), and Miao (2006). Our contribution relative to these papers is to generate dealer and customer trading patterns endogenously in the OTC market, and relate these patterns to private and social participation incentives. In addition, we also study non-exclusive participation, i.e., the possibility that investors participate simultaneously in two markets. We show that the positive and normative analysis of non-exclusive participation is conceptually different, and sometimes generates opposite normative results.

Many recent papers have studied the manner in which customer and dealer trading patterns emerge endogenously in OTC markets, based on alternative assumptions regarding investor's heterogeneity. See, for example, Babus (2009), Afonso and Lagos (2015), Babus and Kondor (2013), Hugonnier, Lester, and Weill (2014), Üslü (2015), Farboodi, Jarosch, and Shimer (2015), Farboodi, Jarosch, and Menzio (2016), Wang (2016), Neklyudov and Sambalaibat (2017). Our paper builds on their insights with a different modeling framework to study equilibrium and socially optimal participation in OTC vs. centralized market. The advantage of our static modeling framework is that it allows for a rigorous, transparent, and simple characterization of the composition externalities induced by participation decisions.

A few papers have explored the manner in which market fragmentation may emerge as an equilibrium outcome due to information and price-setting frictions, and may dominate a centralized exchange. See Kawakami (2017), Malamud and Rostek (2017), Babus and Parlatore (2017). We do not seek to explain fragmentation per-se. Instead, we study whether investors make socially optimal participation decision in fragmented markets that we take as given.

The rest of the paper is organized as follows. In Section 2 we lay out a general model of participation in a centralized and a decentralized markets and we study its abstract properties. In Section 3 we consider three tractable examples of the general model, under alternative assumptions about investor heterogeneity and participation cost. We derive our main normative results regarding the social optimality of encouraging participation in the centralized market.

2 The model

In this section we generalize the model of Atkeson, Eisfeldt, and Weill (2015) (henceforth AEW) in several dimensions. First, we open two markets: a decentralized OTC market and a centralized Walrasian market. Second, we introduce heterogeneity in both risk endowment and OTC trading capacity. Third, we consider general distributions over risk endowments and trading capacities, instead of discrete distributions.[3] This leads to generality and clarity gains and, as illustrated in Section 3, this makes it easier to characterize equilibrium in some important cases of interest.

2.1 Overview of the economic environment

Agents and Assets. There are four dates $t \in \{0, 1, 2, 3\}$, one good consumed at the terminal date, $t = 3$, and one risky asset with normally distributed payoff denoted by v. There is a measure one of traders with Constant Absolute Risk Aversion (CARA) utility over $t = 3$ consumption. We allow for negative consumption or, equivalently, we allow traders to produce the good at unit marginal cost. Namely, if a trader consumes $C \geq 0$ and produces $H \geq 0$, then her utility is $-\frac{1}{\eta}e^{-\eta(C-H)}$, for some $\eta > 0$. We assume that traders are organized in a measure one of large coalitions, called "banks." Each bank starts with an endowment, $\omega \in [0, \Omega]$, of the risky asset and with some OTC market trading capacity, $k \in [0, K]$. We let Φ denote the joint distribution of endowments and trading capacities, (ω, k), over the set $[0, \Omega] \times [0, K]$ equipped with its Borel σ-algebra. At this stage of the analysis we place no restriction on this distribution: it can be continuous, discrete, or a mixture of both.

Participation. At $t = 0$, banks make a participation decision, denoted by π. They can pay participation costs to trade the risky asset in a decentralized OTC market ("π = o"), in a centralized Walrasian market ("π = w"), or in both markets at the same time ("π = ow"). They can also stay in autarky ("π = a"). We let $c(\pi) \geq 0$ denote the participation cost associated with decision $\pi \in \{\mathrm{a}, \mathrm{o}, \mathrm{w}, \mathrm{ow}\}$, and we normalize $c(\mathrm{a}) = 0$.

The type of a bank at time $t = \{1, 2, 3\}$ is denoted by $x \equiv (\omega, k, \pi)$, for endowment, trading

[3] While the economics of the problem remains essentially the same, this creates important technical difficulties, as all optimization and fixed point arguments must be formulated and solved in infinite dimensional vector spaces.

capacity, and participation decisions. With a slight abuse of notation, we denote the functions mapping the bank type $x \in X$ to its endowment by $\omega(x)$, to its trading capacity by $k(x)$, and to its participation decision by $\pi(x)$. We let $X \equiv [0, \Omega] \times [0, K] \times \{\mathrm{a}, \mathrm{o}, \mathrm{w}, \mathrm{ow}\}$ denote the set of all possible bank types. The set of bank types who trade in the OTC market is denoted by $X_{\mathrm{o,ow}} \equiv \pi^{-1}(\{\mathrm{o}, \mathrm{ow}\})$ and, likewise, the set of bank types who trade in the Walrasian market is denoted by $X_{\mathrm{w,ow}} \equiv \pi^{-1}(\{\mathrm{w}, \mathrm{ow}\})$.

Banks' participation decisions induce a measure N over X, which we will refer to as "the participation path." The participation path must be consistent with the primitive distribution of endowment and trading capacities, Φ. Formally, given any measurable subset $A \subseteq [0, \Omega] \times [0, K]$ of endowments and trading capacities it must be the case that:

$$\Phi(A) = \int \mathbb{I}_{\{(\omega(x), k(x)) \in A\}} \, dN(x). \tag{1}$$

In most of the analysis that follows we assume for simplicity that both markets are open, i.e. the participation path assigns strictly positive measure of banks to both markets, $N(X_{\mathrm{o,ow}}) > 0$ and $N(X_{\mathrm{w,ow}}) > 0$. It is straightforward to extend our arguments to the simpler cases in which one or both markets is empty.

Timing. The timing after participation decisions have been made is the following. At $t = 1$, banks who chose $\pi \in \{\mathrm{o}, \mathrm{ow}\}$ trade in the OTC market. At $t = 2$, banks who chose $\pi \in \{\mathrm{w}, \mathrm{ow}\}$ trade in the Walrasian market. At $t = 3$, every bank consolidates all its traders' positions, and the risky asset pays off. Then, each trader receives a consumption equal to the average per-trader payoff of her bank.

2.2 Trading and Payoff

In this section we describe the trading protocol and final payoffs of banks in the OTC and Walrasian markets, for a given participation path, N.

2.2.1 Bilateral trades in the OTC market

At $t = 1$, banks with type $x \in X_{\mathrm{o,ow}}$ send their traders out to the decentralized OTC market, where they are paired uniformly to bargain over a bilateral trade. When a trader from a type-x bank is paired with a trader from a type-x' bank the trader of type x buys a quantity $\gamma(x, x')$ of assets from the trader of type x', in exchange for the payment $P_{\mathrm{o}}(x, x')\gamma(x, x')$. A positive $\gamma(x, x')$ is an outright purchase, and a negative an outright sale. OTC market trades must satisfy two elementary feasibility constraints:

$$\gamma(x, x') + \gamma(x', x) = 0 \text{ for all } (x, x') \in X^2 \tag{2}$$

$$\gamma(x, x') = 0 \text{ if } (x, x') \notin X^2_{\text{o,ow}}. \tag{3}$$

Condition (2) imposes bilateral feasibility and condition (3) rules out trades between types who do not participate in the OTC market.

We add a crucial constraint which prevents banks from fully sharing their risk by trading in OTC markets. We assume that traders are subject to a bilateral trading capacity constraint: the maximum that a trader of type x can purchase from a trader of type x' is given by a continuous function of their respective endowment and trading capacities, which in this section we denote by $M(x, x')$. Vice versa, the maximum that the trader can sell is given by $M(x', x)$.

$$-M(x', x) \leq \gamma(x, x') \leq M(x, x'). \tag{4}$$

The function M stands in for a broad range of institutional frictions that limit trade size in practice, arising for example from short-selling constraints, collateral requirements, or risk-management concerns. For an example involving short-selling constraints, imagine that a fraction $\lambda(k)$ of the assets endowed to the (ω, k) bank can be traded in the OTC market, and that each bank distributes its tradeable asset endowment uniformly to its traders before they enter the OTC market. Then, the short-selling constraint for a type-x trader can be represented by $M(x, x') = \lambda(k')\omega(x')$, i.e., the maximum that a trader can purchase is determined by the amount of asset brought by its OTC market counterparty. For an example involving collateral requirements, imagine that the asset being traded is a derivative contract, that each derivative contract sold must be backed by one unit of collateral. Let $k(x)$ denote the amount of collateral that a type-x bank endows each traders prior to matching in OTC market. Then, the number of contracts that a type-x trader can purchase is bounded by the amount of collateral brought by its counterparty, $M(x, x') = k(x')$. Finally, a common risk-management practice in banks is to impose limit (notional or risk-based) on the size of the position taken by its trading desk. Such limits can be represented in our framework by assuming that $M(x, x')$ is some increasing function of the trading capacities of both counterparties, (k, k').

Taking stock, a collection of OTC market trade $\gamma : X \to \mathbb{R}^2$ is *feasible* if it is measurable and if it satisfies (2), (3), and (4).

2.2.2 Multilateral trades in the Walrasian market

We assume that, in the centralized market, trading is multilateral and frictionless. That is, a bank of type x purchases $\varphi(x)$ unit of asset, at a price P_{w}. A collection of Walrasian-market trades is described by some measurable function $\varphi : X \to \mathbb{R}$. Walrasian-market trades are *feasible* if:

$$\int \varphi(x)\, dN(x) = 0 \tag{5}$$

$$\varphi(x) = 0 \text{ if } x \notin X_{\mathrm{w,ow}}. \tag{6}$$

Condition (5) is the market-clearing condition in the Walrasian market. Condition (6) simply states that types who do not participate in the Walrasian market cannot trade in the Walrasian market.

2.2.3 Certainty equivalent payoffs

After trading in the OTC and the Walrasian market, a bank consolidates all its trades. Each trader receives a consumption equal to the average per-trader payoff of the bank. That is, for each trader, consumption net of production is:

$$C(x) - H(x) = -c\left[\pi(x)\right] + \omega(x)v + \int \gamma(x, x')\left[v - P_{\mathrm{o}}(x, x')\right]\, dN(x' \mid X_{\mathrm{o,ow}}) + \varphi(x)\left(v - P_{\mathrm{w}}\right).$$

The first term, $-c\left[\pi(x)\right]$, is the participation cost. The second term, $\omega(x)v$, is the payoff of the initial asset endowment. The third term, $\int \gamma(x, x')\left[v - P_{\mathrm{o}}(x, x')\right]\, dN(x' \mid X_{\mathrm{o,ow}})$, is the net payoff from OTC-market trades. Finally, the fourth term, $\varphi(x)\left(v - P_{\mathrm{w}}\right)$, is the net payoff from Walrasian-market trades. Given its initial endowment, and after OTC- and Walrasian-market trade, the *post-trade* exposure of the bank to the risky asset is defined as:

$$g(x) \equiv \omega(x) + \int \gamma(x, x')\, dN(x' \mid X_{\mathrm{o,ow}}) + \varphi(x). \tag{7}$$

The first term is the initial endowment, the second term is the exposure gained via OTC-market trades, and the third term is the exposure gained via Walrasian-market trades. With this notation in mind, we calculate the certainty equivalent payoff of the bank:

$$\begin{aligned} & -\frac{1}{\eta} \log \mathbb{E}\left[e^{-\eta[C(x)-H(x)]}\right] \\ &= -\, c\left[\pi(x)\right] + U\left[g(x)\right] - \int \gamma(x, x') P_{\mathrm{o}}(x, x')\, dN(x' \mid X_{\mathrm{o,ow}}) - \varphi(x) P_{\mathrm{w}}, \end{aligned} \tag{8}$$

where $U(g) \equiv \mathbb{E}[v]\, g - \frac{\eta}{2}\mathbb{V}[v]\, g^2$ is the mean-variance payoff that obtains with CARA utility and normally distributed risky asset.[4]

2.3 Equilibrium and social planning problem given participation

Optimality condition in the OTC market. We assume that, in the OTC market, traders view themselves as small relative to their bank's coalition, and do not coordinate their trades with other traders in the same bank coalition. As a result, we assume that a trader's objective is to maximize the value to the bank of its bilateral trade, taking as given all other bilateral and centralized trades in the bank coalition.[5] Formally, let $U_g(g) \equiv \mathbb{E}[v] - \eta\mathbb{V}[v]\, g$ denote the derivative of $U(g)$. Then, the objective of a type-x trader who meets a type-x' trader in the OTC market is to maximize its marginal impact on the certainty equivalent payoff, (8):

$$\gamma(x, x') \left\{U_g\left[g(x)\right] - P_{\mathrm{o}}(x, x')\right\},$$

taking all other trades, as summarized by the post-trade exposure $g(x)$, as given.

Assuming that bilateral trades are the outcome of symmetric Nash bargaining between the two traders, we obtain the following optimality conditions. Optimal OTC market trades must satisfy:

$$\gamma(x, x') = \begin{cases} M(x, x') & \text{if } g(x) < g(x') \\ \in [-M(x', x), M(x, x')] & \text{if } g(x) = g(x') \\ -M(x', x) & \text{if } g(x) > g(x') \end{cases} \tag{9}$$

for all $(x, x') \in X^2_{\mathrm{o,ow}}$. That is, if the type-$x$ trader expects a lower post-trade exposure than the type-x' trader, then he should purchase some asset. Given that the type-x trader views itself as small relative to its coalition, it is optimal to purchase as much as feasible given the bilateral trading capacity constraint (4). The asset price between x and x' is set to split the bilateral gains from trade in half:

$$P_{\mathrm{o}}(x, x') = \frac{1}{2}\left\{U_g\left[g(x)\right] + U_g\left[g(x')\right]\right\}. \tag{10}$$

[4]In general, if v is not normally distributed, many of our results would go through because $U(g)$ would be a well behaved concave function. While not crucial, the assumption of quadratic payoffs is used in two places. First, when the support of the distribution of (ω, k) is not discrete, we use the assumption of quadratic payoff to guarantee a continuity property and complete the final steps of the proof of Proposition 2. Second, in Section 3, the assumption of quadratic payoffs guarantee that participation incentives are appropriately symmetric in the bank populations, which is useful to produce tractable parametric examples.

[5]This approach allows for perfect risk sharing within coalitions and used in monetary economics literature as well as by AEW. See Lucas (1990) and Shi (1997), Andolfatto (1996) and Shimer (2010), among others.

One sees from (9) that OTC market trades tend to bring banks' post-trade exposures closer together, in that banks with small exposures tend to buy from banks with high exposures. However, in general, banks do not equalize their post-trade exposures, for two reasons. First, the trading capacity constraint (4) limits the size of OTC market trades. Second, the bilateral trading protocol implies that traders in the same bank will trade in opposite direction depending on who they meet. For example, type-x traders purchase from type-x' traders if $g(x) < g(x')$, but they sell if $g(x) > g(x')$. Trades of the same size going in opposite direction will net out to zero, and so do not contribute to the equalization of post-trade exposures.

Optimality condition in the Walrasian market. In the Walrasian market all banks trade at the same price, P_{w}, without any constraint. Taking derivative of (8) with respect to φ, this clearly implies that

$$U_g\left[g(x)\right] = P_{\text{w}} \quad \forall x \in X_{\text{w,ow}} \quad \Longleftrightarrow \quad g(x) = \int_{X_{\text{w,ow}}} g(x')\, dN(x' \,|\, X_{\text{w,ow}})\, \forall x \in X_{\text{w,ow}}. \tag{11}$$

In contrast to the OTC market, the Walrasian market places no capacity constraints on banks, and allows banks to trade in a multilateral fashion. As a result, banks who participate in the Walrasian market are able to fully equalize their post-trade exposures.

Definition of equilibrium given participation. With the derivation above in mind, we define an equilibrium given some participation path, N, satisfying $N(X_{\text{o,ow}}) > 0$ and $N(X_{\text{w,ow}}) > 0$, to be a collection $(\gamma, \varphi, g, P_{\text{o}}, P_{\text{w}})$ of feasible OTC market trades, γ, feasible Walrasian market trades, φ, post-trade exposures, g, OTC market prices, P_{o}, and Walrasian price, P_{w}, such that (7), (9), (10) and (11) hold.

Notice that our definition of equilibrium requires that the optimality conditions (9) and (11) hold *everywhere*, even for sets of types that have measure zero according to N. Economically, this means that we require banks' trading decisions to be optimal both on and off the participation path. This is crucial to evaluate the value of all possible participation decisions and solve for an equilibrium participation path.

To be more specific, consider for example some banks with endowments and trading capacities (ω, k) in some set A. Suppose that, in an equilibrium, these banks only participate in the Walrasian market, i.e., $N(A \times \{\text{w}\}) > 0$ but $N(A \times \{\text{o}, \text{ow}\}) = 0$. To verify whether participating only in the Walrasian market is indeed optimal, these banks with $(\omega, k) \in A$ evidently need to compare the value of all participation decisions, $\pi \in \{\text{o}, \text{w}, \text{ow}\}$. This means that we need to solve for trades, (g, γ, φ), and payoffs for all types $x \in A \times \{\mathrm{o}, \mathrm{w}, \mathrm{ow}\}$, even if some of these types have measure zero on the equilibrium participation path, N.

Definition of the social planning problem given participation. Establishing equilibrium existence can be boiled down to the following fixed-point problem for post-trade exposures. Namely, given some post-trade exposures, g, the two optimality conditions (9) and (11) pin down a collection of OTC and Walrasian trades (γ, φ). These trades, in turn, must aggregate to the postulated post-trade exposure, via equation (7).

Instead of directly solving this fixed-point problem, we take a different route: we show that the equilibrium allocation solves the following social planning problem:

$$W^\star(N) = \sup \int \{U\left[g(x)\right] - c[\pi(x)]\} \, dN(x),$$

with respect to feasible OTC market trades, γ, feasible Walrasian market trades, φ, and post-trade exposures g generated by (γ, φ) according to (7).

Main existence result. Equipped with these definitions, we obtain:

Proposition 1. *There exists an equilibrium given some participation path N, satisfying $N(X_{\mathrm{o,ow}}) > 0$ and $N(X_{\mathrm{w,ow}}) > 0$. All equilibria solve the planning problem given participation. The equilibrium is essentially unique in the sense that all equilibria share the same post-trade risk exposures, g, OTC prices, P_{o}, Walrasian price, P_{w}, and certainty equivalent payoff,* (8)*. Equilibria may only differ in terms of OTC and Walrasian trades, (γ, φ).*

Showing that equilibria solve the planner's problem is straightforward: it follows from direct comparison of the planner's first-order conditions with the equilibrium optimality condition (9)-(11). Showing that the planner's problem has a solution follows from standard results on convex optimization in infinite dimensional vector spaces (see, for example, Proposition 1.2, Chapter II in Eckland and Témam, 1987).

There is, however, a key difficulty. To understand why, consider again banks with endowments and trading capacities (ω, k) in some set A who only participate in the Walrasian market, $N(A \times \{o, \mathrm{ow}\}) = 0$. As argued above, the equilibrium requires to determine their payoff and trades on and off the participation path, that is, for all participation decisions $\pi \in \{\mathrm{w}, \mathrm{o}, \mathrm{ow}\}$. But since these banks only participate in the Walrasian market, they are in zero measure in the OTC market. Formally, banks of type $x \in A \times \{\mathrm{o}, \mathrm{ow}\}$ are in measure zero according to N, which implies that they have zero weight in the planner's objective, and so their socially optimal trades are indeterminate. Put differently, the planner only cares about determining trades on the participation path, while our definition of equilibrium requires to determine these trades both on and off the path.

As a result of this difficulty, our existence proof goes in two steps. In the first step we establish that the planner's problem has a solution and find the post-trade exposures on the participation path. Technically, the planner's problem determines the post-trade exposures, $g(x)$, almost everywhere according to N. Economically, the planner's problem determines "aggregate conditions" in the OTC and the Walrasian market: the post-trade exposures of potential counterparties in the OTC market, $g(x)$, and the price in the Walrasian market. Then, equipped with this characterization, we turn to the determination of the trades off the participation path. For example, the trade that a bank who participates in the Walrasian market only would find it optimal in case it participated instead in the OTC market.

2.4 Equilibrium participation

The value of participation. The certainty equivalent of a type-x bank, before participation cost, can be written:

$$\begin{aligned}
&U\left[g(x)\right] - \int \gamma(x,x')\, P_{\mathrm{o}}(x,x')\, dN(x' \mid X_{\mathrm{o,ow}}) - \varphi(x)\, P_{\mathrm{w}} \\
=&U\left[g(x)\right] - U_g\left[g(x)\right]\left\{g(x) - \omega(x)\right\} + U_g\left[g(x)\right]\left[\varphi(x) + \int \gamma(x,x')\, dN(x' \mid X_{\mathrm{o,ow}})\right] \\
&- \int \gamma(x,x') P_{\mathrm{o}}(x,x')\, dN(x' \mid X_{\mathrm{o,ow}}) - \varphi(x) P_{\mathrm{w}} \\
=&U\left[g(x)\right] - U_g[g(x)]\left[g(x) - \omega(x)\right] + \frac{1}{2}\int \gamma(x,x')\left\{U_g\left[g(x)\right] - U_g\left[g(x')\right]\right\}\, dN(x' \mid X_{\mathrm{o,ow}}).
\end{aligned}$$

The first equality follows by adding and subtracting $U_g\left[g(x)\right]\left\{g(x) - \omega(x)\right\}$, and using (7). The second equality follows from using the formula for OTC market prices, (10). Using the optimality condition (9), we obtain:

Lemma 1. *Consider some participation path N, such that $N(X_{\mathrm{o,ow}}) > 0$ and $N(X_{\mathrm{w,ow}}) > 0$. Then, the certainty equivalent of a bank of type $x \in X$ can be written*

$$U\left[\omega(x)\right] + \mathrm{MPV}(x), \text{ where } \mathrm{MPV}(x) = K(x) + \frac{F(x)}{2},$$

where $\mathrm{MPV}(x)$ *is the marginal private value of the bank,* $K(x)$ *the competitive surplus, and*

$F(x)$ *the frictional surplus, defined as:*

$$K(x) \equiv U\left[g(x)\right] - U\left[\omega(x)\right] - U_g\left[g(x)\right]\left[g(x) - \omega(x)\right] \tag{12}$$

$$F(x) \equiv \mathbb{I}_{\{\pi(x)\in\{\mathrm{o,ow}\}\}} \int_{X_{\mathrm{o,ow}}} \left\{ \left(U_g\left[g(x)\right] - U_g\left[g(x')\right]\right)^+ M(x,x') \right.$$
$$\left. + \left(U_g\left[g(x)\right] - U_g\left[g(x')\right]\right)^- M(x',x) \right\} dN(x' \mid X_{\mathrm{o,ow}}). \tag{13}$$

The marginal private value is the extra payoff relative to autarky. It can be decomposed in two components, which we term the "competitive" and the "frictional" surplus. The competitive surplus is the value of changing exposure, assuming that all assets are bought and sold at marginal value, $U_g\left[g(x)\right]$. This indeed corresponds to the asset price if the bank reaches the same post-trade exposure of $g(x)$ by trading in a Walrasian market. However, in the OTC market, the bank is typically able to bargain a better price than marginal value. Specifically, when a type-x trader expects a lower post-trade exposure than its counterparty, it purchases a quantity $M(x, x')$ below marginal value. Vice versa when it expects a higher post-trade exposure it sells a quantity $M(x', x)$ above marginal value. The sum of all these OTC bargaining gains for a bank of type x is equal to half of the frictional surplus, $\frac{F(x)}{2}$, due to the symmetry in bargaining powers.

Definition of an equilibrium with positive participation. An equilibrium with positive participation in both markets is a positive measure, N, over the set of banks' types, X, satisfying the following three conditions. First, participation is positive in both markets: $N(X_{\mathrm{o,ow}}) > 0$ and $N(X_{\mathrm{w,ow}}) > 0$. Second, the participation path must satisfy (1), that is, it must be consistent with the primitive exogenous distribution of risk endowment and trading capacities, Φ. Third, the participation path must be generated by optimal participation decisions, that is:

$$\int \left(\mathrm{MPV}(x) - c\left[\pi(x)\right] - \max_{\pi\in\{\mathrm{a,o,w,ow}\}} \left\{ \mathrm{MPV}\left(\omega(x), k(x), \pi\right) - c(\pi) \right\} \right) dN(x) = 0.$$

It is conceptually more subtle to define an equilibrium in which participation is zero in one or in both markets, $N(X_{\mathrm{o,ow}}) = 0$ or $N(X_{\mathrm{w,ow}}) = 0$. Indeed, in that case, one must take a stand about a bank's rational belief regarding its payoff if it chooses to enter an "empty" market, that is, a market in which there is no participation. One natural choice of beliefs, as in Atkeson, Eisfeldt, and Weill (2015), is to assume that, if no one else participates in a market, then the payoff of participation is zero. But this creates coordination failures: no participation is always an optimal choice if the market is expected to be empty. Another choice is to use a refinement that is in the spirit of a competitive search equilibrium. That is, if a bank chooses to enter in an empty market, it expects to attract the banks who have most incentives to enter. Because we have not figured out a formal definition, we focus for now on equilibria such that participation is positive in all markets.

2.5 Marginal private vs. social value of a bank

In the next section we will characterize the equilibrium participation path under alternative assumptions regarding bank heterogeneity and participation cost, and we will ask whether the participation path is efficient. To answer this question, we take a first-order approach: we study the social benefit of reallocating a small measure of banks between markets, e.g., by moving some banks from the OTC market to the Walrasian market and vice versa. To objective of the present section is to provide an explicit and general formula for this social benefit.

The main proposition. Consider any participation path N such that $N(X_{\mathrm{o,ow}}) > 0$ and $N(X_{\mathrm{w,ow}}) > 0$. The reallocation of banks increases participation in some markets, and decreases participation in others. Formally, reallocation induces a new participation path of the form:

$$N + \varepsilon\, n.$$

The parameter ε controls the scale of the reallocation. The parameter n, which controls the direction of the reallocation, is a finite signed measure, that is $n \equiv n^+ - n^-$ for some positive and finite measures n^+ and n^-. Increases in participation are captured by the positive part, n^+. Vice versa, decreases in participation are captured by the negative part.

A direction of reallocation, n, is said to be *admissible* if it satisfies the following two conditions. First, it must conserve the distribution of endowments and trading capacities, i.e. $N+\varepsilon\, n$ must satisfy (1). Since N already satisfies (1), this can be written

$$\int \mathbb{I}_{\{(\omega(x),k(x))\in A\}}\, dn(x) = 0$$

for all measurable sets $A \subseteq [0,\Omega] \times [0,K]$ of endowments and trading capacities. Second, the direction of reallocation must keep the participation path $N+\varepsilon\, n$ positive for all ε small enough. Formally, we require that the negative part of the direction of reallocation, n^-, is absolutely continuous with respect to N, with a bounded Radon-Nikodym derivative.

Next we study the directional derivative of equilibrium social welfare in any admissible direction of reallocation. From Proposition 1, we know that equilibrium trades are socially efficient, which implies that equilibrium social welfare given the participation path $N + \varepsilon\, n$

is equal to $W^\star(N + \varepsilon\, n)$, the maximized value of the social planner given the participation path $N + \varepsilon\, n$. This allows us to use Envelope Theorems to calculate the derivative. Precisely, adapting argument from Milgrom and Segal (2002), we obtain:[6]

Proposition 2. *Assume that, $N(X_{\text{o,ow}}) > 0$ and $N(X_{\text{w,ow}}) > 0$ and consider any admissible direction of reallocation n. Then, the function $\varepsilon \mapsto W^\star(N + \varepsilon\, n)$ is right-hand differentiable at $\varepsilon = 0$, with derivative:*

$$\frac{d}{d\varepsilon}\left[W^\star(N + \varepsilon\, n)\right](0+) = \int \left\{\text{MSV}(x) - c[\pi(x)]\right\}\, dn(x)$$

with

$$\text{MSV}(x) \equiv K(x) + F(x) - \mathbb{I}_{\{\pi(x) \in \{\text{o,ow}\}\}} \frac{\bar{F}}{2},$$

where $K(x)$ and $F(x)$ are, respectively, the equilibrium competitive and frictional surplus given participation path N, defined in (12) and (13), and $\bar{F} = \int F(x')\, dN(x' \,|\, X_{\text{o,ow}})$ is the average frictional surplus.

The marginal social value formula. The first term of the marginal social value is the competitive surplus, (12). It reflects the utility gain of changing the exposure of type-x banks, $U\,[g(x)] - U\,[\omega(x)]$, net of the cost of changing exposure evaluated at the marginal value of a type-x bank, $U_g\,[g(x)]\,[g(x) - \omega(x)]$.

The remaining terms arise because participation in the OTC market results in match creation and match destruction: new matches between the participant and the incumbents are created, and old matches between incumbents are destroyed.

The value of match creation is measured by the frictional surplus, $F(x)$. Indeed in the OTC market, a new type-x participant always purchase from incumbents with higher exposure, $g(x') \leq g(x)$. This pushes down the unit social cost of increasing the new participant exposure below its marginal value, $U_g\,[g(x)]$, by an amount equal to $U_g\,[g(x)] - U_g\,[g(x')] > 0$. The opposite is true when the new type-x participant sells.

The cost of match destruction is equal to half the average frictional surplus, $\bar{F}/2$. This formula for the cost of match destruction can be understood as follows. First, one new match involves only one incumbent, but an old match involves two. This implies that the creation of one new match only requires the destruction of half an old match. Second, the matching protocol

[6] Relative to Atkeson, Eisfeldt, and Weill (2015), the main difficulty of the proof is to establish that the continuity properties required by Milgrom and Segal hold when the planner makes choices in an infinite dimensional vector space.

implies that old matches are effectively destroyed at random, so that the value destroyed per match is the average frictional surplus.

Socially optimal trades on vs. off the participation path. Earlier in the paper we noted that, off the participation path, socially optimal trades are indeterminate. Indeed, banks that are off the participation path are given zero weights in the planner's objective. Matters are different when applying the Envelope Theorem. Consider as before banks with endowments and trading capacities (ω, k) in some set A who only participate in the Walrasian market, $N(A \times \{\mathrm{o}, \mathrm{ow}\}) = 0$. If the proposed direction of reallocation, n, moves some $x \in A \times \{\mathrm{o}, \mathrm{ow}\}$ in the OTC market, the planner starts to put some weight on them and their socially optimal trades are no longer indeterminate. This is why, in Proposition 2, the marginal social value is calculated based on the *equilibrium* competitive and frictional surplus. That is, the two surpluses are calculated assuming that banks follow their optimal equilibrium trades, both on and off the participation path.[7]

Social gain/loss from centralization. In our examples in Section 3, we study whether participation decisions are socially efficient, with the following thought experiment. Consider a marginal bank with endowment $\omega^\star$ and capacity $k^\star$, who decides to participate in the OTC market only but is actually indifferent between $\pi = \mathrm{o}$ and another participation decision $\pi' \in \{\mathrm{w}, \mathrm{ow}\}$. What would be the effect on social welfare of requiring this bank to choose π'?

Formally, this amounts to applying Proposition 2, with a direction of reallocation such that $dn(\omega, k, \pi') = -dn(\omega, k, \mathrm{o}) > 0$ in a small neighborhood of $(\omega^\star, k^\star)$ in the direction of (ω, k)s who strictly prefer the OTC market, and $dn(x) = 0$ everywhere else. The change in social welfare for the reallocation of the type $(\omega^\star, k^\star)$ bank is given by:

$$\Delta W(\pi') \equiv \left[\mathrm{MSV}\left(\omega^\star, k^\star, \pi'\right) - c(\pi')\right] - \left[\mathrm{MSV}\left(\omega^\star, k^\star, \mathrm{o}\right) - c(\mathrm{o})\right].$$

But since $(\omega^\star, k^\star)$ is the marginal bank,

$$\left[\mathrm{MPV}(\omega^\star, k^\star, \pi') - c(\pi')\right] - \left[\mathrm{MPV}\left(\omega^\star, k^\star, \mathrm{o}\right) - c(\mathrm{o})\right] = 0.$$

Therefore,

$$\Delta W(\pi') = \left[\mathrm{MSV}\left(\omega^\star, k^\star, \pi'\right) - \mathrm{MPV}\left(\omega^\star, k^\star, \pi'\right)\right] - \left[\mathrm{MSV}\left(\omega^\star, k^\star, \mathrm{o}\right) - \mathrm{MPV}\left(\omega^\star, k^\star, \mathrm{o}\right)\right]. \quad (14)$$

[7]In Milgrom and Segal formalism, the right-hand derivatives are obtained by maximizing the partial derivative of social welfare with respect to all optimal trades. We show in our proof that equilibrium trades maximizes the partial derivative.

Hence, the gain is the difference between the marginal social and private value of choosing π'. Vice versa, the cost is the difference between the social and private value of OTC market participation. Thus, the net welfare impact equals the difference between *marginal value wedges* of choosing π' and choosing $\pi = \mathrm{o}$.

Recalling the formulas for MSV and MPV derived in Lemma 1 and Proposition 2, the marginal value wedge of a participation decision π is

$$\mathrm{MSV}\left(\omega^\star, k^\star, \pi\right) - \mathrm{MPV}\left(\omega^\star, k^\star, \pi\right) = \frac{1}{2}\left[F\left(\omega^\star, k^\star, \pi\right) - \bar{F}\right]. \tag{15}$$

The economic intuition for this formula is as follows. The competitive surplus term is common between marginal social and private values because it reflects the fully internalized part of the entry incentives. If $\pi \in \{\mathrm{o}, \mathrm{ow}\}$, the marginal bank engages in trades in the decentralized OTC market which is subject to composition externalities. The social planner attributes to this bank the difference between the full frictional surplus that the entry of this bank creates and the value of the frictional surplus that the entry of this bank destroys. However, when calculating its private entry incentives, the bank takes into account only the half of the frictional surplus it creates because this is the amount it captures through bargaining. As a result, a marginal value wedge equal to (15) arises if $\pi \in \{\mathrm{o}, \mathrm{ow}\}$.

In order to demonstrate that the welfare implication of centralization might be very different in an equilibrium with exclusivity and in another equilibrium without exclusivity, we calculate (14) for $\pi' = \mathrm{w}$ and $\pi' = \mathrm{ow}$. Since there is no wedge between the MSV and MPV of Walrasian market participation,

$$\Delta W(\mathrm{w}) = -\frac{1}{2}\left[F\left(\omega^\star, k^\star, \mathrm{o}\right) - \bar{F}\right]. \tag{16}$$

Therefore, calculating the social gain or loss of centralization in an equilibrium with exclusivity boils down to comparing the frictional surplus of the marginal bank and the average frictional surplus of OTC banks. If the marginal bank has a higher surplus than the average OTC bank, the centralization will be welfare reducing. If the marginal bank has a lower surplus than the average OTC bank, the centralization will be welfare improving. In the next section, we will demonstrate that this comparison depends heavily on the underlying heterogeneity of banks.

In an equilibrium with full participation in the OTC market and partial participation in the Walrasian market, the social gain formula for centralization becomes

$$\Delta W(\mathrm{ow}) = \frac{1}{2}\left[F\left(\omega^\star, k^\star, \mathrm{ow}\right) - F\left(\omega^\star, k^\star, \mathrm{o}\right)\right]. \tag{17}$$

In the exclusivity case, the result was based on comparing the value of matches created and the value of matches destructed. In this case, the process of match creation and destruction will not matter because the participation decisions that prevail in equilibrium, o and ow, both feature trading in the OTC market, and hence, the set of OTC matches are exactly the same even when the decision of the marginal bank is impacted. However, the composition of match surpluses is different. In particular, if the marginal bank enters the Walrasian market on top of the OTC market, we will show that it will become an intermediary in the OTC market due to the perfect risk sharing it achieves with its Walrasian trades. Therefore, the main question will become whether it is socially beneficial to transform the marginal OTC bank into an OTC intermediary. In the next section, we show that this is never socially beneficial because it destroys some frictional surplus without creating any benefit.

3 Three Examples

In this section we characterize participation equilibrium under specific parametric assumptions about underlying bank heterogeneity and participation costs.

3.1 Heterogeneous endowments and exclusive participation

In this section, we assume that banks have heterogeneous endowments but have homogenous trading capacity. Precisely, we let ω be uniformly distributed over $[0, 1]$. We let the trading capacity constraint be $M(x, x') = k(x)$, with $k < \frac{1}{2}$ being identical for all banks. To create a meaningful trade-off between OTC and Walrasian market participation, we assume that $c(\text{o}) < c(\text{w})$. Finally, we assume that $c(\text{ow})$ is sufficiently large so that participation is exclusive: banks either participate in the OTC market or in the Walrasian market, but not in both at the same time. We guess and verify that, under parameter restrictions to be determined, there exists an equilibrium with exclusive participation, with the following two features.

1. Participation is symmetric around $\omega = 1/2$: banks with endowment ω and $1 - \omega$ make the same participation decision.

2. Extreme-ω banks participate in the Walrasian market, and middle-ω banks in the OTC market. Precisely, there is some $\omega^\star \in [0, 1/2]$ such that banks with $\omega \in [0, \omega^\star) \cup (1 - \omega^\star, 1]$ participate in the Walrasian market, and banks with $\omega \in [\omega^\star, 1 - \omega^\star]$ participate in the OTC market, where $\omega^\star$ satisfies $\omega^\star + k < 1/2$.

We can obtain the first property because the marginal certainty equivalent, U_g, is linear, which ensures that banks ω and $1-\omega$ have identical marginal private values. The second property is similar to the one in Gehrig (1993), Miao (2006), and Lester, Rocheteau, and Weill (2015). The banks with the strongest risk sharing needs find it optimal to use the most efficient trading venue. As will be clear below, the condition $\omega^\star + k < 1/2$ ensures that the marginal bank shares risk imperfectly in the OTC market, and so faces meaningful trade-off between the OTC and the Walrasian market.

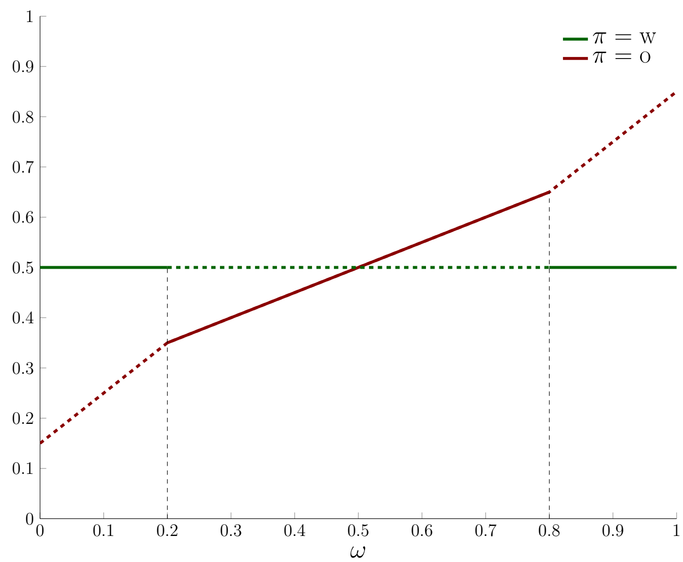

Figure 1: Post-trade exposures as a function of endowment in the example with heterogenous endowment and exclusive participation

3.1.1 Exclusive participation in the OTC market

Let us guess and verify that the post-trade exposures of a bank conditional on participating in the OTC market, $g(\omega, k, \mathrm{o})$, is strictly increasing in ω. In that case, the optimality condition (9) implies that an ω trader always sells k units to $\omega' < \omega$ traders, and always purchases k units from $\omega' > \omega$ traders:

$$g(\omega, k, \mathrm{o}) = \omega - kN(\omega \,|\, X_\mathrm{o}) + k\left[1 - N(\omega \,|\, X_\mathrm{o})\right] = \omega + k\left[1 - 2N(\omega \,|\, X_\mathrm{o})\right], \tag{18}$$

where $N(\omega \,|\, X_\mathrm{o})$ is the conditional distribution of endowment in the OTC market. Given our assumed participation decisions and given uniform distribution,

$$N(\omega \,|\, X_\mathrm{o}) = \begin{cases} 0 & \text{if } \omega \in [0, \omega^\star) \\ \dfrac{\omega - \omega^\star}{1 - 2\omega^\star} & \text{if } \omega \in [\omega^\star, \frac{1}{2}], \end{cases}$$

and, by symmetry, $N(\omega \mid X_o) = N(1-\omega \mid X_o)$ for $\omega \geq \frac{1}{2}$. Plugging the expression for the conditional distribution in (18), and keeping in mind that $\omega^\star + k < \frac{1}{2}$, one easily sees that $g(\omega, k, o)$ is strictly increasing, so our guess is verified.[8]

As in AEW, banks with extreme ω trade like "customers," in the sense that most of their trades go in the same direction. Specifically, low-ω banks mostly purchase assets, and high-ω banks mostly sell assets. Middle-ω banks, on the other hand, trade like "intermediaries." They trade in all directions, buying from high-ω banks and selling to low-ω banks.

After substituting into (12) and (13), we obtain the competitive and frictional surpluses:

$$K(\omega, k, o) = \frac{|U_{gg}|}{2}\left[g(\omega, k, o) - \omega\right]^2$$
$$F(\omega, k, o) = |U_{gg}| k \int \left| g(\omega', k, o) - g(\omega, k, o) \right| dN(\omega' \mid X_o).$$

One sees that the surpluses are U-shaped and symmetric around $\frac{1}{2}$.

Mathematically, the competitive surplus is U-shaped because the net exposure, $g(\omega) - \omega$, is decreasing in ω, and equal to zero at $\omega = \frac{1}{2}$. Indeed, as ω becomes closer and closer to $1/2$, banks transition from being "customers" to becoming "intermediaries": they trade less and less in the same direction, and more and more in opposing directions. As a result, their net exposure is smaller and smaller.

Mathematically, the frictional surplus is also U-shaped because it is proportional to the average distance between the bank's post-trade exposure, and the post-trade exposures of other banks in the OTC market. As is well known, this average distance is minimized at the median bank in the OTC market, that is, bank $\omega = \frac{1}{2}$.

Economically, the U-shaped property simply reflects that extreme-ω banks gains more by trading, for two reasons. First, as measured by the competitive surplus, they gain more by changing their exposures. Second, as measured by the frictional surplus, they are able to bargain better deals. This does not mean, however, that extreme-ω banks have more incentive to enter the OTC market than middle-ω banks. In fact, we show that the opposite is true because extreme-ω banks have even stronger incentives to enter the Walrasian market.

[8] Note that formula (18) applies to all banks: the one who actually decide to participate in the OTC market (in the support of $N(\omega \mid X_o)$) and the one who do not (outside the support). Figure 1 depicts it with solid lines for the banks in the support and with dashed lines for the banks outside the support.

3.1.2 Exclusive participation in the Walrasian market

As shown in Section 2, banks who participate in the Walrasian market equalize their risk exposures. Given symmetric participation, this means that:

$$g(\omega, k, \mathrm{w}) = \frac{1}{2}$$

for all banks in the Walrasian market. Hence the competitive and frictional surpluses are:

$$K(\omega, k, \mathrm{w}) = \frac{|U_{gg}|}{2}\left(\frac{1}{2} - \omega\right)^2$$
$$F(\omega, k, \mathrm{w}) = 0.$$

As for the OTC market, incentives to trade in the Walrasian market are strictly U shaped in ω.

3.1.3 The trade-off between OTC and Walrasian markets

The key result in this section is:

Lemma 2. *Given our conjectured participation patterns, the difference between the* MPV *of participating in the Walrasian market and that of participating in the OTC market* $\mathrm{MPV}(\omega, k, \mathrm{w}) - \mathrm{MPV}(\omega, k, \mathrm{o})$ *is strictly U shaped and symmetric around* $\omega = \frac{1}{2}$.

Therefore, although extreme-ω banks have higher MPV of participating in the OTC market, their MPV of participating in the Walrasian market is even higher. This is a key step in verifying that our guessed participation patterns are indeed optimal.

The marginal bank indifference condition. The marginal bank, $\omega^\star$, must be indifferent between participating in the Walrasian and the OTC market. The indifference condition is easily derived by calculating the competitive and frictional surplus of the marginal bank, $\omega^\star$, as follows.

The post-trade exposure of the marginal bank is (almost everywhere according to N) lower than that of other banks who participate in the OTC market. Hence, its traders always purchase the asset, up to the trading limit, which implies that $g(\omega^\star) = \omega^\star + k$. Hence, the competitive surplus is:

$$K(\omega^\star, k, \mathrm{o}) = \frac{|U_{gg}|}{2}k^2.$$

Using again that the post-trade exposure of the marginal bank is, almost everywhere according to N, lower than that of other banks who participate in the OTC market, we obtain that the frictional surplus is proportional to the average distance between the post-trade exposure of the marginal bank, and the average post-trade exposure of OTC market participant. By symmetry, the average post-trade exposure of OTC market participant is $\frac{1}{2}$. Therefore:

$$F(\omega^\star, k, \mathrm{o}) = k|U_{gg}| \left(\frac{1}{2} - (\omega^\star + k) \right).$$

After some algebra, one then sees that the indifference condition for the marginal bank, $\mathrm{MPV}(\omega^\star, k, \mathrm{w}) - \mathrm{MPV}(\omega^\star, k, \mathrm{o}) = c(\mathrm{w}) - c(\mathrm{o})$, writes:

$$\frac{|U_{gg}|}{2} \left(\frac{1}{2} - \omega^\star \right) \left(\frac{1}{2} - (\omega^\star + k) \right) = c(\mathrm{w}) - c(\mathrm{o}). \tag{19}$$

The left-side is a decreasing function of $\omega^\star \in \left[0, \frac{1}{2} - k\right]$. Hence, as long as $c(\mathrm{w}) - c(\mathrm{o})$ is not too large relative to $|U_{gg}|$, there is a unique solution such that $\omega^\star + k < \frac{1}{2}$. This determines $\omega^\star$.

One obtains three natural comparative statics from the indifference condition. Participation in the OTC market increases (the threshold $\omega^\star$ decreases) when the OTC market trading technology is relatively more efficient, as proxied by an increase in k, a decrease in $c(\mathrm{o})$, or an increase in $c(\mathrm{w})$, and when risk-sharing is less important to banks, as proxied by a decrease in $|U_{gg}|$.

Verifying that our guessed participation decisions are optimal. In order to verify that the exclusive and symmetric participation pattern that we assumed at the beginning arises endogenously, we must show that (i) the value of exclusive participation in the Walrasian market net of its cost dominates the value of exclusive participation in the OTC market net of its cost, for extreme-ω banks and (ii) the autarky and the joint participation in the Walrasian and OTC markets are dominated options for every bank. We go through these verification steps in the appendix and derive sufficient conditions on costs for the guessed participation decision to be optimal.

Proposition 3. *There exists an equilibrium in which banks with $\omega \in [0, \omega^\star) \cup (1 - \omega^\star, 1]$ participate exclusively in the Walrasian market, and banks with $\omega \in [\omega^\star, 1 - \omega^\star]$ participate exclusively in the OTC market, for some $\omega^\star + k < 1/2$ if and only if $\omega^\star$ is the unique solution of the indifference condition (19) belonging to $\left[0, \frac{1}{2} - k\right]$, $3\,c(\mathrm{o}) + \frac{8}{|U_{gg}|k^2} c(\mathrm{o})^2 < c(\mathrm{w}) < c(\mathrm{o}) + \frac{|U_{gg}|}{8}(1 - 2k)$, and $c(\mathrm{ow})$ is sufficiently large relative to $c(\mathrm{o}) + c(\mathrm{w})$.*

The Proposition imposes natural restrictions on the level of participation costs and on their difference. In particular, the level of $c(\text{o})$ and $c(\text{w})$ cannot be too large, otherwise autarky would be preferred to market participation. The difference between $c(w)$ and $c(\text{o})$ cannot be too large either, otherwise there would not be positive participation in both markets at the same time: namely, if $c(\text{w}) \gg c(\text{o})$ then all banks would find it optimal to participate in the OTC market only. Another restriction required by our constructive proof is that participation costs are convex, in the sense that $c(\text{ow}) > c(\text{o}) + c(\text{w})$. Otherwise, we cannot guarantee that participation is exclusive. We consider non-exclusive participation in Section 3.3.

3.1.4 Inefficiently large Walrasian market

Next, we study whether participation decisions are socially efficient, with the following thought experiment. Consider the marginal bank, with endowment $\omega^\star$, who decides to participate in the OTC market only. What would be the effect on social welfare of requiring this bank to participate in the Walrasian market?

Formally, this amounts to applying Equation (16). Namely, we obtain that

$$\Delta W \;=\; -\left[\text{MSV}\left(\omega^\star, k, \text{o}\right) - \text{MPV}\left(\omega^\star, k, \text{o}\right)\right] = -\frac{1}{2}\left[F\left(\omega^\star, k, \text{o}\right) - \bar{F}\right] < 0.$$

To understand why $\Delta W < 0$, recall that the frictional surplus is U-shaped: this implies that, amongst all banks who participate int the OTC market, a $\omega^\star$-bank has the largest frictional surplus. In particular, this frictional surplus is larger than $\bar{F}$, the average surplus of banks who participate in the OTC market. Economically, the result arises because the marginal bank appropriate less surplus than it creates in the OTC market. Hence, from the perspective of the planner, the marginal bank has too little private incentive to stay in the OTC market, and too much private incentive to switch to the Walrasian market.

Therefore, we conclude that, with heterogeneity in endowments, the Walrasian market is inefficiently large. Encouraging further Walrasian market participation would be welfare reducing. In the next section we consider an environment with heterogeneity in OTC trading capacity, and we show that this normative result is reversed.

3.1.5 Trading volume

To empirically distinguish our examples from each other, we look at the customers' and dealers' equilibrium trade patterns in the OTC market. More precisely, we define the net (signed) trading volume and the gross trading volume as

$$NV(\omega, k, o) \equiv \int \gamma\left[(\omega, k, o), (\omega', k', \pi')\right] dN\left[(\omega', k', \pi')|X_{\text{o,ow}}\right] = g(\omega, k, o) - \omega \tag{20}$$

and

$$GV(\omega, k, o) \equiv \int \left|\gamma\left[(\omega, k, o), (\omega', k', \pi')\right]\right| dN\left[(\omega', k', \pi')|X_{\text{o,ow}}\right], \tag{21}$$

respectively.

In this specific example with heterogeneous endowment and exclusive participation, one can calculate these volumes using (9) and (18):

$$\begin{aligned} NV(\omega, k, o) &= k\left[1 - 2N(\omega \mid X_{\text{o}})\right] \\ GV(\omega, k, o) &= k. \end{aligned}$$

The implication of this is that dealers have lower net trading volume, and dealers and customers have comparable gross trading volume.

3.2 Heterogenous capacity and exclusive participation

In this section, we consider our second special case with endogenous exclusivity.

As our focus of attention is on the heterogeneity in trading capacities in this case, we impose minimal heterogeneity in endowments, only to generate gains from trade: we assume that $\omega \in \{0, 1\}$. On the other hand, we consider a continuous uniform distribution over trading capacities, $k \in [0, 1]$. We let the trading capacity constraint be $M(x, x') = \max\{k(x), k(x')\}$. This assumption means that high capacity banks are not restricted by the lower capacity of their counterparties. This may appear as an extreme assumption but it captures the observation that, in OTC markets, investors with low sophistication benefit from their counterparties' higher sophistication. A theoretical motivation is the work of Üslü (2015), who shows that trading with a counterparty with higher search ability provides better risk sharing. An empirical motivation is the observation that, in OTC markets, the customer-to-customer volume is considerably lower than the customer-to-dealer volume. Therefore, customers appear to benefit from dealers' superior trading technology.

To create a meaningful trade-off between OTC and Walrasian market participation, we assume that $c(\text{o}) < c(\text{w})$. To guarantee exclusivity, we assume that $c(\text{ow})$ is large enough. Following similar step as in the previous section, we guess and verify that, under parameter restrictions to be determined, there exists an equilibrium with the following three features.

1. Participation is independent of ω: banks with endowment 0 and 1 make the same participation decision if they have the same trading capacity.

2. Small-k banks participate in the Walrasian market, and large-k banks in the OTC market. Precisely, there is some $k^\star \in [0,1]$ such that banks with $k \in [0, k^\star)$ participate in the Walrasian market, and banks with $k \in [k^\star, 1]$ participate in the OTC market.

3.2.1 Exclusive participation in the OTC market

The atom property. Let us guess and verify that post-trade exposures conditional on participating in the OTC market have the "atom property": all banks with the same endowment ω who participate in the OTC markets have the same post-trade exposures, regardless of their capacity. Formally, for each $\omega \in \{0,1\}$, $g(\omega, k, \mathrm{o})$ is independent of $k \in [0,1]$.[9] In addition, we conjecture that risk-sharing is imperfect, that is $\omega = 0$-banks do not equalize their post-trade exposures with $\omega = 1$-banks, $g(0,k,\mathrm{o}) < 1/2 < g(1,k,\mathrm{o})$.

If the atom property holds, and if there is imperfect risk sharing, the optimality condition (9) implies that, when an $\omega = 0$ trader is paired with an $\omega = 1$ trader, the $\omega = 0$ trader buys $\max\{k, k'\}$ units from $\omega = 1$ trader. When two $\omega = 0$-traders meet, they anticipate the same post-trade exposures and so are indifferent between any quantity in $[-\max\{k,k')\}, \max\{k,k'\}]$. The post-trade exposure of an $\omega = 0$ bank conditional on participating in the OTC market can thus be written:

$$g(0,k,\mathrm{o}) = \frac{1}{2}\int \gamma(k,k')\, dN(k' \mid X_\mathrm{o}) + \frac{1}{2}\int \max\{k,k'\}\, dN(k' \mid X_\mathrm{o}), \tag{22}$$

where, with a slight abuse of notation "$\gamma(k,k')$" stands for "$\gamma\left[(0,k,\mathrm{o}),(0,k',\mathrm{o})\right]$." The first term is the net trade with $\omega = 0$ banks. The second term is the net trade with $\omega = 1$ banks. Now if we aggregate (22) over $k \in [k^\star, 1]$, the bilateral feasibility constraint (2) implies that the aggregate net trade between $\omega = 0$ banks is equal to zero. Therefore, if the atom property holds, we must have:

$$g(0,k,\mathrm{o}) = \int g(0,k',\mathrm{o})\, dN(k' \mid X_\mathrm{o}) = \frac{1}{2}\mathbb{E}\left[\max\{k',k''\} \mid (k',k'') \in X_\mathrm{o}^2\right], \tag{23}$$

where, with a slight abuse of notation "$k' \in X_\mathrm{o}$" stands for "$(0,k',\mathrm{o}) \in X_\mathrm{o}$". In words, the post-trade exposure of $\omega = 0$ bank is equal to $\frac{1}{2}$, which is the probability of a meeting between

[9] Notice that this property is assumed to hold for all k. That is, if an ω-bank makes the possibly suboptimal choice to participate in the OTC market, it will equalize its exposure with that other ω banks in the OTC market.

an $\omega = 0$ and an $\omega = 1$ traders, multiplied by $\mathbb{E}\left[\max\{k', k''\} \mid (k', k'') \in X_{\text{o}}^2\right]$, which is the average trade size between an $\omega = 0$ and an $\omega = 1$ trader.

To verify our guess, we need to find bilateral trades between $\omega = 0$-traders, $\gamma(k, k')$, that have two properties. First, they must satisfy the bilateral trading capacity constraint (4) for all (k, k'). Second, the post-trade exposures resulting from these bilateral trades, (23), must be equalized. A natural candidate for these bilateral trades is:

$$\gamma(k, k') = \mathbb{E}\left[\max\{k', k''\} \mid k'' \in X_{\text{o}}\right] - \mathbb{E}\left[\max\{k, k''\} \mid k'' \in X_{\text{o}}\right]. \tag{24}$$

That is, when two $\omega = 0$-traders meet, they "swap" the exposures their banks acquired from $\omega = 1$ banks. When aggregated across all possible $\omega = 0$ counterparties, these swaps mechanically equalize exposure of all $\omega = 0$ banks who participate in the OTC market. To see that these swaps also satisfy the bilateral trading capacity constraint, note that

$$\max\{k', k''\} - \max\{k, k''\} = \max\{k' - k'', 0\} - \max\{k - k'', 0\} \in [-k, k'],$$

and so satisfies the bilateral trading capacity constraint. We have thus verified our conjecture that the post-trade exposures of OTC market participants are independent of k.

Same post-trade exposure, different trades. Notice that, although banks with different trading capacity arrive at the same post-trade exposure, they have very different trading patterns. Indeed, given some trading capacity k, the aggregate net trade of an $\omega = 0$ bank with other $\omega = 0$ banks is

$$\frac{1}{2}\left\{\mathbb{E}\left[\max\{k', k''\} \mid (k', k'') \in X_{\text{o}}^2\right] - \mathbb{E}\left[\max\{k, k''\} \mid k'' \in X_{\text{o}}\right]\right\}.$$[10]

It is clearly decreasing in k. Since it averages out to zero across $k \in X_{\text{o}}$, it has to be positive for small $k \in X_{\text{o}}$ and negative for large k. Therefore, banks with small k behave like "customers," in the sense that their trade mostly in the same direction: they always buy from $\omega = 1$ banks, and they mostly buy from fellow $\omega = 0$ banks. On the other hand, large-k banks behave like "intermediaries," in the sense that they trade in all directions. They always buy from $\omega = 1$ banks, and they mostly sell to fellow $\omega = 0$ banks. That is, large-k bank mostly engages in simultaneous buying and selling, and appears as intermediary in the OTC market.

[10] Although some equilibrium bilateral trade are indeterminate, this expression for aggregate net trade is not. This is because the aggregate net trade with $\omega = 0$ is equal to the difference between the post-trade exposure, and the aggregate net trade with $\omega = 1$ banks, which are both determinate. First, from our uniqueness result in Proposition 1 it follows that equilibrium post-trade exposures are determinate. Namely, all equilibria satisfy the atom property, and the atom is given by (23). Second since $g(0, k, \text{o}) < g(1, k, \text{o})$, the aggregate net trade with $\omega = 1$ banks is determinate.

After substituting into (12) and (13), we obtain that the competitive and frictional surpluses are:

$$\begin{aligned}
K(\omega, k, \mathrm{o}) &= \frac{|U_{gg}|}{2} \left[g(0, k, \mathrm{o})\right]^2 \\
F(\omega, k, \mathrm{o}) &= \frac{1}{2} \int \left| U_g[g(0, k, \mathrm{o})] - U_g[g(1, k', \mathrm{o})] \right| \max\{k, k'\} \; dN(k' \,|\, X_\mathrm{o}) \\
&= \frac{|U_{gg}|}{2} \left[1 - 2g(0, k, \mathrm{o})\right] \int \max\{k, k'\} \; dN(k' \,|\, X_\mathrm{o}),
\end{aligned} \tag{25}$$

where: on the second line, there is a $\frac{1}{2}$ coefficient because the frictional surplus is strictly positive only for matches between an $\omega = 0$ and an $\omega = 1$ bank; on the third line we used that the post-trade exposures are independent of k and symmetric between $\omega = 0$ and $\omega = 1$. One sees that, as a result of the atom property, the competitive surplus is independent of k. However, the frictional surplus is an increasing function of k.[11] Although the per-unit frictional surplus is a constant due to the atom property, banks with larger k trade larger quantities and generate larger frictional surplus. As a consequence, investors with sufficiently large k find it optimal to enter exclusively the OTC market in equilibrium.

3.2.2 Exclusive participation in the Walrasian market

As shown in Section 2, banks who participate in the Walrasian market equalize their risk exposures. Given symmetric participation, this means that:

$$g(\omega, k, \mathrm{w}) = \frac{1}{2}$$

for all banks in the Walrasian market. Hence the competitive and frictional surpluses are:

$$\begin{aligned}
K(\omega, k, \mathrm{w}) &= \frac{|U_{gg}|}{8} \\
F(\omega, k, \mathrm{w}) &= 0.
\end{aligned}$$

Since we have shut down heterogeneity in endowments and Walrasian trades are not subject to size limits, entering the Walrasian market becomes a fixed outside option in this example.

3.2.3 The trade-off between OTC and Walrasian markets

Re-stating earlier observations about competitive and frictional surpluses, we obtain a a simple but key result in this section:

[11] To be precise, the frictional surplus is strictly increasing function of k for $k \in [k^\star, 1]$ and constant for $k \in [0, k^\star)$, due to the max specification of trade size limit.

Lemma 3. *Given our conjectured participation patterns, the difference between the MPV of participating in the Walrasian market and that of participating in the OTC market,* $\text{MPV}(\omega, k, \text{w}) - \text{MPV}(\omega, k, \text{o})$*, is constant in* $k \in [0, k^\star)$ *and strictly decreasing in* $k \in [k^\star, 1]$*.*

Therefore, banks with larger trading capacities have stronger preferences to participate in the OTC market. In this example, this stronger preference stems from larger trade size, leading to a larger frictional surplus.

The marginal bank indifference condition. We proceed as before to derive the marginal bank's indifference condition. Recall that, in the OTC market, the marginal bank has the smallest trading capacity. Therefore, in the frictional surplus formula, (25), the average trade size with banks from the other atom is equal to $\int \max\{k^\star, k'\}\, dN(k' \,|\, X_\text{o}) = \int k'\, dN(k' \,|\, X_\text{o}) = \mathbb{E}\,[k' \,|\, k' \in X_\text{o}]$. Using formula (23) for the post-trade exposure, we obtain that, when banks $k \in [k^\star, 1]$ participate exclusively in the OTC market, the MPV of the marginal bank is:

$$\mathcal{M}\,[x(k^\star), y(k^\star)]\,, \text{ where } \mathcal{M}(x, y) = \frac{|U_{gg}|}{8} x^2 + \frac{|U_{gg}|}{4}\,(1 - x)\, y,$$

$x(k^\star) \equiv \mathbb{E}\,[\max\{k', k''\} \,|\, (k', k'') \geq k^\star]$ and $y(k^\star) = \mathbb{E}\,[k' \,|\, k' \geq k^\star]$. Since $0 \leq k' \leq \max\{k', k''\} \leq 1$, it immediately follows that $0 \leq y(k^\star) \leq x(k^\star) \leq 1$. It is easy to see that the partial derivative of $\mathcal{M}(x, y)$ are positive, when evaluated at $x(k^\star)$ and $y(k^\star)$. Therefore, the marginal private value of the marginal bank is increasing in the marginal bank threshold, $k^\star$.[12]

Given uniform distribution, we can plug in explicit expressions for $x(k^\star)$ and $y(k^\star)$[13] and we obtain that: $\mathcal{M}\,[x(k^\star), y(k^\star)] = \frac{|U_{gg}|}{8} - \frac{|U_{gg}|}{36}\,(1 - k^\star)^2$. Given that the marginal private value of exclusive participation in the Walrasian market is $\frac{|U_{gg}|}{8}$, this means that the marginal bank indifference condition is:

$$\frac{|U_{gg}|}{36}\,(1 - k^\star)^2 = c(\text{w}) - c(\text{o}). \tag{26}$$

As before this indifference condition implies natural comparative statics. The OTC market is larger, as proxied by a smaller $k^\star$, when risk-sharing considerations matter less to investors (smaller $|U_{gg}|$) and when the participation cost is high in the Walrasian market relative to the OTC market (larger $c(\text{w}) - c(\text{o})$).

[12] Earlier we showed that, holding participation decision constant, $\text{MPV}(\omega, k, \text{o})$ is increasing in k. What we just showed is different: as we change participation decisions by moving the marginal bank threshold, the MPV of the marginal bank is increasing.

[13] Namely, we use that the expectation of the maximum of n IID random variables uniformly distributed over $[a, b]$ is equal to $\frac{a+nb}{1+n}$.

Verifying that our guessed participation decisions are optimal. In order to verify that the assumed exclusive participation pattern with threshold property arises endogenously, we must show that (i) the value of exclusive participation in the Walrasian market net of its cost dominates the value of exclusive participation in the OTC market net of its cost, for small-k banks and (ii) the autarky and the joint participation in the Walrasian and OTC markets are dominated options for every bank. We go through these verification steps in the appendix and derive associated necessary and sufficient conditions on costs.

Proposition 4. *There exists an equilibrium in which banks with $k \in [0, k^\star)$ participate exclusively in the Walrasian market, and banks with $k \in [k^\star, 1]$ participate exclusively in the OTC market, if and only if $k^\star$ solve (26), $0 \leq c(\mathrm{o}) < c(\mathrm{w}) \leq \min\left\{\frac{|U_{gg}|}{8}, c(\mathrm{o}) + \frac{|U_{gg}|}{36}\right\}$, and $c(\mathrm{ow})$ is large enough.*

The restrictions on participation costs have a similar interpretation as before.

3.2.4 Inefficiently large OTC market

Now we conduct the same thought experiment as in the heterogenous ω case. The effect on social welfare of moving the marginal bank from the OTC to the Walrasian market is given by the same formula as before:

$$\Delta W = -\frac{1}{2}\left[F\left(\omega, k^\star, \mathrm{o}\right) - \bar{F}\right] > 0.$$

Differently from before, the frictional surplus is now increasing in $k \in [k^\star, 0]$, which implies that the frictional surplus of the marginal bank is lower than the average frictional surplus in the OTC market. Therefore, ΔW is positive; i.e., moving the marginal bank from the OTC to the Walrasian market is welfare improving. This means that there is too much participation in the OTC market and too little in the Walrasian market.

This stems from the fact that the marginal bank has the lowest OTC trading capacity. By removing it from the OTC market, the social planner destroys small trades by allowing the remaining banks to create larger trades with larger surplus. The marginal bank itself does not have the incentive to do so because it captures a disproportionately high fraction of the individual trade surpluses.

Finally, we note that this result heavily depends on exclusivity. When the marginal bank leaves the OTC market, it results in a match destruction, making the other banks able to generate new matches.

3.2.5 Trading volume

Substituting (9), (23), and (24) into (20) and (21), the net and gross trading volumes are:

$$
\begin{aligned}
NV(0,k,o) &= \frac{1}{2}\mathbb{E}\left[\max\{k',k''\} \,|\, (k',k'') \in X_{\mathrm{o}}^2\right] \\
NV(1,k,o) &= -NV(0,k,o) \\
GV(\omega,k,o) &= \frac{1}{2}\int |\gamma(k,k')| \; dN(k' \,|\, X_{\mathrm{o}}) + |NV(\omega,k,o)| \\
&= \frac{1}{2}k\left\{\mathbb{E}\left[\max\{k,k''\} \,|\, k'' \in X_{\mathrm{o}}\right] - \mathbb{E}\left[\max\{k',k''\} \,|\, k'' \in X_{\mathrm{o}}, k' < k\right]\right\} \\
&+ \frac{1}{2}(1-k)\left\{\mathbb{E}\left[\max\{k',k''\} \,|\, k'' \in X_{\mathrm{o}}, k' > k\right] - \mathbb{E}\left[\max\{k,k''\} \,|\, k'' \in X_{\mathrm{o}}\right]\right\} \\
&+ |NV(\omega,k,o)|.
\end{aligned}
$$

In contrast to the case with heterogeneous endowment, the net trading volume of dealers does not have to be smaller than that of customers. And, the per-dealer gross trading volume can be much larger than the per-customer gross volume.

3.3 Heterogenous endowments and non-exclusive participation

In this section we consider our first special case with endogenous *non*-exclusivity, that is an equilibrium in which some banks find it optimal to trade on both markets. As in section 3.1, we focus on heterogeneity in endowments. Thus, we make the same assumptions. We let ω be uniformly distributed over $[0,1]$. We let the trading capacity constraint be $M(x,x') = k(x)$, with k being identical for all banks. We assume that $c(\mathrm{o}) < c(\mathrm{w})$. The only difference with the setting of section 3.1 is that $c(\mathrm{ow}) = c(\mathrm{o}) + c(\mathrm{w})$. We guess and verify that, under parameter restrictions to be determined, there exists an equilibrium with the following features.

1. Participation is symmetric around $\omega = 1/2$: banks with endowment ω and $1-\omega$ make the same participation decision.

2. Extreme-ω banks participate in the Walrasian and the OTC markets, and middle-ω banks in the OTC market only. Precisely, there is some $\omega^\star \in [0,1/2]$ such that banks with $\omega \in [0,\omega^\star) \cup (1-\omega^\star,1]$ participate in both markets, and banks with $\omega \in [\omega^\star, 1-\omega^\star]$ participate in the OTC market, where $\omega^\star$ satisfies $\omega^\star + k < 1/2$.

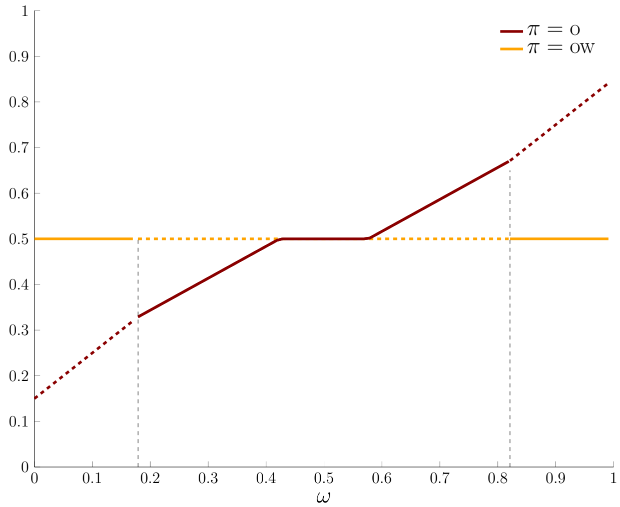

Figure 2: Post-trade exposures as a function of endowment in the example with heterogenous endowment and non-exclusive participation

3.3.1 The value of participation in the OTC and Walrasian markets

The value of exclusive participation in the OTC market. Let us guess and verify that there exists some $\omega_1 \in \left(\omega^\star, \frac{1}{2}\right]$ such that the post-trade exposure conditional on participating in the OTC market, $g(\omega, k, \text{o})$, is continuous and symmetric around $\omega = \frac{1}{2}$, strictly increasing for $\omega \in [0, \omega_1] \cup [1-\omega_1, 1]$, and constant $g(\omega, k, \text{o}) = 1/2$ for $\omega \in (\omega_1, 1-\omega_1)$. Figure 2 illustrates.

Let us focus on some trader with $\omega \in [0, \omega_1]$ (the case $\omega \in [1-\omega_1, 1]$ is symmetric). Then, the optimality condition (9) implies that an ω-trader always sells k units when she meets a counterparty with strictly lower post-trade exposures. From the the figure, one sees that the set of counterparties with strictly lower post-trade exposure is $[\omega^\star, \max\{\omega, \omega^\star\})$. Vice versa, the ω-trader always buy k unit when she meets a counterparty in the complementary set, with strictly larger post-trade exposure. Therefore, for $\omega \in (\omega^\star, \omega_1]$

$$\begin{aligned} g(\omega, k, \text{o}) &= \omega - k\left[N(\max\{\omega, \omega^\star\}) - N(\omega^\star)\right] + k\left[1 - N(\max\{\omega, \omega^\star\}) + N(\omega^\star)\right] \\ &= \omega + k\left(1 - 2\max\{\omega - \omega^\star, 0\}\right), \end{aligned}$$

where the second line follows from using our assumption that endowment are uniformly distributed. The value of the threshold ω_1 is pinned down by the equation,

$$g_o(\omega_1) = \frac{1}{2} \Rightarrow \omega_1 = \frac{1 - 2k(1+2\omega^\star)}{2(1-2k)} < \frac{1}{2}.$$

One can verify that as long as $\omega^\star + k < 1/2$, we have $\omega_1 > \omega^\star$. The general formulas for competitive and frictional surpluses are as follows,

$$K(\omega, k, \mathrm{o}) = \frac{|U_{gg}|}{2}\left(g(\omega, k, \mathrm{o}) - \omega\right)^2$$
$$F(\omega, k, \mathrm{o}) = |U_{gg}|k\left[N(X_{\mathrm{o}})\int \left|g(\omega', k, \mathrm{o}) - g(\omega, k, \mathrm{o})\right| dN(\omega' \,|\, X_{\mathrm{o}}) + N(X_{\mathrm{ow}})\left|\frac{1}{2} - g(\omega, k, \mathrm{o})\right|\right].$$

As before, it is easy to see that both the competitive and the frictional surplus are U-shaped and symmetric in ω.

The value of joint participation in the Walrasian and OTC market. As previously, a bank in both markets has the following post-trade exposure:

$$g(\omega, k, \mathrm{ow}) = \frac{1}{2}$$

and simultaneously trade in the OTC market given its post-trade exposure which is the same as a "middle" bank. Thus, the competitive and frictional surplus are:

$$K(\omega, k, \mathrm{ow}) = \frac{|U_{gg}|}{2}\left(\frac{1}{2} - \omega\right)^2$$
$$F(\omega, k, \mathrm{ow}) = F\left(\frac{1}{2}, k, \mathrm{o}\right).$$

The value of exclusive participation in the Walrasian market. As previously, a bank in the Walrasian market has the following post-trade exposure:

$$g(\omega, k, \mathrm{w}) = \frac{1}{2}$$

Hence the competitive and frictional surpluses are:

$$K(\omega, k, \mathrm{w}) = \frac{|U_{gg}|}{2}\left(\frac{1}{2} - \omega\right)^2$$
$$F(\omega, k, \mathrm{w}) = 0.$$

3.3.2 The trade-off between both markets and the OTC market only

A key result in this section is:

Lemma 4. *Given our conjectured participation patterns, the difference between the MPV of participating in both markets and that of participating in the OTC market,* $\mathrm{MPV}(\omega, k, \mathrm{ow}) - \mathrm{MPV}(\omega, k, \mathrm{o})$*, is U-shaped and symmetric around* $\omega = \frac{1}{2}$*.*

Therefore, although extreme-ω bank have higher MPV of participating in the OTC market, their MPV of participating in both markets is even higher. This is a key step in verifying that our guessed participation patterns are indeed optimal.

The marginal bank indifference condition. For the marginal bank, with endowment ω^*, on the OTC market only, the post-trade exposure is $g(\omega^\star, k, \mathrm{o}) = \omega^\star + k$. Therefore, its competitive surpluses can be computed as

$$K(\omega^\star, k, \mathrm{o}) = \frac{|U_{gg}|}{2} k^2$$

Notice that the marginal bank has lower post-trade exposure than any of its potential OTC market counterparties. Therefore, the average distance between the post-trade exposure of the marginal bank, and the post-trade of its potential counterparties, is simply equal to the distance between the marginal bank and the average counterparty. Given symmetry, the post-trade exposure of the average counterparty is equal to 1/2. Therefore, the frictional surplus is:

$$F(\omega^\star, k, \mathrm{o}) = |U_{gg}| k \left(\frac{1}{2} - g(\omega^\star, k, \mathrm{w}) \right) = |U_{gg}| k \left(\frac{1}{2} - (\omega^\star + k) \right).$$

Consider the same marginal bank, which enters both market. Its post-trade exposure is $\frac{1}{2}$ and the its competitive surplus is

$$K(\omega, k, \mathrm{o}) = \frac{|U_{gg}|}{2} \left(\frac{1}{2} - \omega^\star \right)^2.$$

A trader from a bank with post-trade exposure equal to 1/2 generates strictly positive surplus only with counterparties with post-trade exposure different from 1/2, that is, counterparties with endowment in $[\omega^\star, \omega_1) \cup (1-\omega_1, 1-\omega^\star]$. Moreover, given symmetry, the surplus with some counterparty $\omega' \in [\omega^\star, \omega_1)$ is equal to the surplus with the symmetric counterparty $1-\omega$. This implies that the frictional surplus can be written:

$$F(\omega^\star, k, \mathrm{ow}) = F\left(\frac{1}{2}, k, \mathrm{o}\right) = 2|U_{gg}| k \int_{\omega^\star}^{\omega_1} \left(\frac{1}{2} - g(\omega', k, \mathrm{o}) \right) d\omega' = |U_{gg}| \frac{k}{1-2k} \left(\frac{1}{2} - \omega^\star - k \right)^2.$$

In equilibrium, the marginal bank $\omega^\star$ is such that $MPV(\omega, k, \mathrm{ow}) - c(\mathrm{w}) - c(\mathrm{o}) = MPV(\omega, k, \mathrm{o}) - c(\mathrm{o})$. That is,

$$\frac{|U_{gg}|}{2} \left[\left(\frac{1}{2} - \omega^* \right)^2 + \frac{k}{1-2k} \left(\frac{1}{2} - \omega^\star - k \right)^2 - k^2 - k \left(\frac{1}{2} - \omega^\star - k \right) \right] = c(\mathrm{w}). \tag{27}$$

Verifying that our guessed participation decisions are optimal. In order to verify that the non-exclusive and symmetric participation pattern that we assumed at the beginning arises endogenously, we must show that (i) the value of participation in both markets net of their costs dominates the value of exclusive participation in the OTC market net of its cost, for extreme-ω banks, (ii) the participation in the Walrasian market only is a dominated option for every bank, and (iii) autarky is a dominated option for every bank. We go through these verification steps in the appendix and derive sufficient conditions on costs for the guessed participation decision to be optimal.

Proposition 5. *Suppose* $0 < c(\mathrm{w}) < \frac{|U_{gg}|}{8}(1+k)(1-2k)$. *Then, equation* (27) *has a unique solution* $\omega^\star \in \left(0, \frac{1}{2} - k\right)$, *with expression*

$$\omega^\star = \frac{1}{2} - k - \frac{k(1-2k)}{2(1-k)} \left[\left(\frac{1}{k^2} - \frac{1-k}{k^2(1-2k)} \left((1+k)(1-2k) - \frac{8c(\mathrm{w})}{|U_{gg}|} \right) \right)^{1/2} - 1 \right].$$

The threshold $\omega^\star$ *is decreasing in* $c(\mathrm{w})$. *Under the following condition,*

$$c(\mathrm{o}) < \frac{|U_{gg}|k}{2(1-2k)} \left(\frac{1}{2} - \omega^\star - k \right)^2,$$

there exists an equilibrium without exclusivity in which banks with $\omega \in [0, \omega^\star) \cup (1-\omega^\star, 1]$ *participate in both the Walrasian and the OTC markets, and banks with* $\omega \in [\omega^\star, 1-\omega^\star]$ *participate in the OTC market.*

Intuitively, the necessary condition for nonexclusive market participation to arise in equilibrium is that any bank would prefer to go on the OTC market than remaining in autarky. As Lemma 4 shows, the bank with the least private gain on the OTC market is the $\omega = 1/2$ bank. The only source of private gain for such a bank is the frictional surplus, $F(1/2, k, \mathrm{o})$. Thus, we assume $c(\mathrm{o}) < F(1/2, k, \mathrm{o})/2$.

Interestingly, if we set $c(\mathrm{o}) = 0$ and we take the cost of entering the Walrasian market to the limit $c(\mathrm{w}) \to 0$, the asymptotic equilibrium still exhibit participation in the OTC market. Indeed in this case $\omega^\star$ converges toward $1/2 - k$. Yet, equilibrium post-trade exposures are equal to $1/2$ for all banks. In terms of welfare, the asymptotic equilibrium is as efficient as an equilibrium with full participation to the Walrasian market.

3.3.3 Inefficiently large Walrasian market

As in section 3.1, consider the marginal bank, with endowment $\omega^\star$, who decides to participate in the OTC market only. Similarly, if we change its participation decision to both Walrasian and OTC markets, the effect on social welfare is given by Equation (17). Namely, we obtain that

$$\Delta W = \frac{1}{2}\left[F(\omega^\star, k, \mathrm{ow}) - F(\omega^\star, k, \mathrm{o})\right] = \frac{1}{2}\left[F\left(\frac{1}{2}, k, \mathrm{o}\right) - F(\omega^\star, k, \mathrm{o})\right].$$

Keeping in mind that the frictional surplus is strictly U-shaped and symmetric, we obtain that $\Delta W < 0$, i.e., the Walrasian market is inefficiently large. By moving a marginal bank, with endowment $\omega^\star$, back in the OTC market only, it would generate the highest frictional surplus possible, the one of the most extreme post-trade exposure bank, $g_o(\omega^\star)$, instead of the lowest frictional surplus, the one of banks with middle post-trade exposure.

References

Gara Afonso and Ricardo Lagos. Trade dynamics in the market for federal funds. *Econometrica*, 83:263–313, 2015. 5

David Andolfatto. Business cycles and labor-market search. *American Economic Review*, 86 (1):112–132, 1996. 10

Andrew G. Atkeson, Andrea L. Eisfeldt, and Pierre-Olivier Weill. The market for otc credit derivatives. Working paper, UCLA, 2013. 4

Andrew G. Atkeson, Andrea L. Eisfeldt, and Pierre-Olivier Weill. Entry and exit in over-the-counter markets. *Econometrica*, 83:2231–2292, 2015. 4, 5, 6, 14, 16, 40

Ana Babus. Endogenous intermediation in over-the-counter markets. Working paper, Federal Reserve Bank of Chicago, 2009. 5

Ana Babus and Péter Kondor. Trading and information diffusion in over-the-counter markets. Working paper, Federal Reserve Bank of Chicago Central European University, 2013. 5

Ana Babus and Cecilia Parlatore. Strategic fragmented market. Working paper, 2017. 5

Morten L. Bech and Enghin Atalay. The topology of federal funds market. *Physica A*, 389(22): 5223–5246, 2010. 4

Marco Di Maggio, Amir Kermani, and Zhaogang Song. The value of trading relations in turbulent times. *Journal of Financial Economics*, 124(2):266–284, 2017. 4

Ivan Eckland and Roger Témam. *Convex Analysis and Variational Problems*. Society for Industrial Mathematics, 1987. 12

Maryam Farboodi, Gregor Jarosch, and Robert Shimer. The emergence of market structure. Working paper, 2015. 5

Maryam Farboodi, Gregor Jarosch, and Guido Menzio. Intermediation as rent extraction. Working paper, 2016. 5

Financial Stability Board. Otc derivatives market reform, eleventh progress report on implementation. Technical report, 2016. 2

Thomas Gehrig. Intermediation in search markets. *Journal of Economics and Management Strategy*, 2:97–120, 1993. 5, 20

Athanasios Geromichalos and Lucas Herrenbrueck. Monetary policy, asset prices, and liquidity in over-the-counter markets. *Journal of Money, Credit, and Banking*, 48(1):35–79, 2016. 5

Vincent Glode and Christian C. Opp. Over-the-counter vs. limit-order markets: The role of traders' expertise. Working paper, 2015. 5

Burton Hollifield, Artem Neklyudov, and Chester Spatt. Bid-ask spreads, trading networks, and the pricing of securitizations. *The Review of Financial Studies*, 30(9):3048–3085, 2017. 4

Julien Hugonnier, Benjamin Lester, and Pierre-Olivier Weill. Heterogeneity in decentralized asset market. Working paper, 2014. 5

Kei Kawakami. Welfare consequences of information aggregation and optimal market size. *American Economic Journal: Microeconomics*, Forthcoming, 2017. 5

Benjamin Lester, Guillaume Rocheteau, and Pierre-Olivier Weill. Competing for Order Flow in OTC Markets. *Journal of Money, Credit, and Banking*, 47:77–126, 2015. 20

Dan Li and Norman Schürhoff. Dealer networks. Working paper, 2012. 4

Robert E. Jr. Lucas. Liquidity and interest rates. *Journal of Economic Theory*, 50:237–264, 1990. 10

Semyon Malamud and Marzena Rostek. Decentralized exchange. *American Economic Review*, Forthcoming, 2017. 5

Jianjun Miao. A search model of centralized and decentralized trade. *Review of Economic Dynamics*, 9(1):68–92, 2006. 5, 20

Paul Milgrom and Ilya Segal. Envelope theorem for arbitrary choice sets. *Econometrica*, 2: 583–601, 2002. 16, 17

Artem Neklyudov and Batchimeg Sambalaibat. Endogenous specialization and dealer networks. Working paper, 2017. 5

John Rust and George Hall. Middlemen versus market makers: A theory of competitive exchange. *Journal of Political Economy*, 111(2):pp. 353–403, 2003. 5

Shouyang Shi. A divisible search model of fiat money. *Econometrica*, 65:75–102, 1997. 10

Robert Shimer. *Labor Markets and Business Cycles (CREI Lectures in Macroeconomics)*. Princeton University Press, 2010. 10

Emil Siriwardane. Concentrated capital losses and the pricing of corporate credit risk. Working paper, 2015. 4

The Squam Lake Report. *Fixing the Financial System*. Princeton University Press, 2010. 2

Semih Üslü. Pricing and liquidity in decentralized asset markets. Working paper, University of California at Los Angeles, 2015. 5, 25

Chaojun Wang. Core-periphery trading networks. Working paper, 2016. 5

Abdullah Yavaş. Marketmakers versus matchmakers. *Journal of Financial Intermediation*, 2: 33–58, 1992. 5

A Some omitted proofs

A.1 Proof of Proposition 1 and 2

The proof are available from the authors upon request. They extend arguments of Atkeson, Eisfeldt, and Weill (2015) to multiple trading venues, heterogeneity in both endowment and trading capacities, and general distribution over endowment and trading capacities.

A.2 Proof of Lemma 2

Direct calculations show that:

$$\frac{d}{d\omega}K(\omega,k,\mathrm{o}) = |U_{gg}|\,(g-\omega)\left(\frac{dg}{d\omega}-1\right)$$
$$\frac{d}{d\omega}F(\omega,k,\mathrm{o}) = |U_{gg}|k\left[2N(\omega\,|\,X_{\mathrm{o}})-1\right]\frac{dg}{d\omega},$$

where g and $dg/d\omega$ denote, respectively, the post-trade exposure and the right-derivative of a ω-bank who participate the OTC market. Using that $g-\omega = k\left[1-2N(\omega\,|\,X_{\mathrm{o}})\right]$, we obtain:

$$\frac{d}{d\omega}\mathrm{MPV}(\omega,k,\mathrm{o}) = |U_{gg}|k\left[1-2N(\omega\,|\,X_{\mathrm{o}})\right]\left[\frac{1}{2}\frac{dg}{d\omega}-1\right].$$

Now use that $\frac{d}{d\omega}\mathrm{MPV}(\omega,k,\mathrm{w}) = -|U_{gg}|\left(\frac{1}{2}-\omega\right)$, and obtain:

$$\frac{d}{d\omega}\left[\mathrm{MPV}(\omega,k,\mathrm{w})-\mathrm{MPV}(\omega,k,\mathrm{o})\right] = -|U_{gg}|\left\{\frac{1}{2}-\omega+k\left[1-2(\omega\,|\,X_{\mathrm{o}})\right]\left[\frac{1}{2}\frac{dg}{d\omega}-1\right]\right\}$$

For $\omega<\omega^\star$, then $dg/d\omega=1$, $N(\omega\,|\,X_{\mathrm{o}})=0$, so that:

$$\frac{d}{d\omega}\left[\mathrm{MPV}(\omega,k,\mathrm{w})-\mathrm{MPV}(\omega,k,\mathrm{o})\right] = -|U_{gg}|\left\{\frac{1}{2}-\omega-\frac{k}{2}\right\}<0,$$

given that $\omega<\omega^\star$ and $\omega^\star+k<\frac{1}{2}$. Next, for $\omega\in[\omega^\star,1]$, $N(\omega\,|\,X_{\mathrm{o}}) = \frac{\omega-\omega^\star}{1-2\omega^\star}$ and $\frac{dg}{d\omega} = 1-\frac{2k}{1-2\omega^\star}$. Substituting and rearranging, we obtain after a few lines of algebra:

$$\frac{d}{d\omega}\left[\mathrm{MPV}(\omega,k,\mathrm{w})-\mathrm{MPV}(\omega,k,\mathrm{o})\right] = -|U_{gg}|\frac{1-2\omega}{1-2\omega^\star}\left(\frac{1}{2}-\omega^\star-k\right)\left(\frac{1}{2}+\frac{k}{1-2\omega^\star}\right)<0,$$

as well.

A.3 Proof of Proposition 3

Two missing conditions. The left side of equation (19) is zero at $\omega=\frac{1}{2}-k$ and it is strictly decreasing for $\omega\in\left[0,\frac{1}{2}-k\right]$. Hence, a necessary and sufficient condition for (19) to have a solution in $\left[0,\frac{1}{2}-k\right]$ is that the left-side evaluated at $\omega=0$ is greater than the right side:

$$\frac{|U_{gg}|}{8}(1-2k)>c(\mathrm{w})-c(\mathrm{o}). \tag{28}$$

Once $\omega^\star$ is found, Lemma 2 ensures that banks in $\omega\in[0,\omega^\star]$ prefer the Walrasian market to the OTC market, and banks in $\omega\left[\omega^\star,\frac{1}{2}\right]$ prefer the OTC market to the Walrasian market (incentives are symmetric for $\omega\geq\frac{1}{2}$.

We next need to make sure that banks prefer their choices of market to autarky.

$$\max\left\{\text{MPV}(\omega,k,\text{w})-c(\text{w}),\text{MPV}(\omega,k,\text{o})-c(\text{o})\right\}>0.$$

Clearly, this function is decreasing since both MPV's are decreasing. Therefore, the condition holds if and only if it holds for $\omega=\frac{1}{2}$.

$$\begin{aligned}\text{MPV}\left(\frac{1}{2},k,\text{o}\right)-c(\text{o})>0 &\Leftrightarrow \frac{1}{2}F\left(\frac{1}{2},k,\text{o}\right)-c(\text{o})>0\\ &\Leftrightarrow \frac{|U_{gg}|k}{4}\left(\frac{1}{2}-\omega^\star-k\right)-c(\text{o})>0. \qquad (29)\end{aligned}$$

The first equivalence follows because, for a $\omega=\frac{1}{2}$-bank, the competitive surplus is zero. The second equivalence because the frictional surplus is proportional to the average distance between the post-trade exposure of the $\omega=\frac{1}{2}$ bank, and the post-trade exposures of other banks. Given uniform distribution and symmetry, this average distance is equal to half the distance between the post trade exposure of the $\omega=\frac{1}{2}$ bank, $g\left(\frac{1}{2},k,\right)=\frac{1}{2}$, and the post-trade exposure of the $\omega^\star$-bank, $g(\omega^\star,k,\text{o})=\omega^\star+k$.

Conditions (28) and (29) in the $c(\mathbf{o})$, $c(\mathbf{w})$ plane. Condition (28) can be evidently written:

$$c(\text{w})<c(\text{o})+\frac{|U_{gg}|}{8}(1-2k),$$

i.e $c(\text{w})$ must be below a line with slope one and intercept, $\frac{|U_{gg}|}{8}(1-2k)$. Condition (29) requires some work because it depends on $\omega^\star$, which is itself a function of $c(\text{o})$ and $c(\text{w})$. To obtain a workable representation, note first that (29) can be equivalently written as

$$\omega^\star\leq\frac{1}{2}-k-\frac{4c(\text{o})}{|U_{gg}|k},$$

where $\omega^\star$ solves (19). Since the left-hand side of (19) is strictly decreasing in $\omega<\frac{1}{2}-k$, this is equivalent to requiring that, when evaluated at $\frac{1}{2}-k-\frac{4c(\text{o})}{|U_{gg}|k}$ the left-hand side of (19) is less than the right-hand side:

$$\frac{|U_{gg}|}{2}\left(k+\frac{4\,c(\text{o})}{|U_{gg}|k}\right)\frac{4\,c(\text{o})}{|U_{gg}|k}<c(\text{w})-c(\text{o})\Leftrightarrow 3\,c(\text{o})+\frac{8\,c(\text{o})^2}{|U_{gg}|k^2}<c(\text{w}),$$

and we are done.

A.4 Proof of Proposition 4

First, for equation (26) to have a solution, we need that the left-hand side is larger than the right-hand side at $k^\star=0$, and smaller at $k^\star=1$, which can be written $0\leq c(\text{w})-c(\text{o})\leq\frac{U_{gg}}{36}$.

Now given a solution $k^\star$ to the (26), Lemma 3 implies that banks with $k\leq k^\star$ prefer the Walrasian market to the OTC market, while banks with $k\geq k^\star$ prefer the OTC market to the Walrasian market. To ensure that they participating in some market to stay in autarky, we need that:

$$\max\{\text{MPV}(\omega,k,\text{w})-c(\text{w}),\text{MPV}(\omega,k,\text{o})-c(\text{o})\}\geq 0$$

for all $k\in[0,1]$. Lemma 3 implies that this function is increasing in k, hence it is positive if and only if it is positive at $k=0$. But when $k=0$, the bank prefers the Walrasian market to the OTC market. We thus conclude that banks prefer to participate in some market than to stay in autarky if and only if $c(\text{w})\leq\frac{|U_{gg}|}{8}$. Finally, we can rule out participation in two market by setting $c(\text{ow})$ large enough, and we are done.

A.5 Proof of Lemma 4

Lemma 4 assumes that the following condition (a) is verified:

$$(a)\ \exists \omega^\star \in \left(0, \frac{1}{2} - k\right),\ \mathrm{MPV}(\omega^\star, k, \mathrm{ow}) - \mathrm{MPV}(\omega^\star, k, \mathrm{o}) = c(\mathrm{w}) > 0$$

The marginal private value of a pure OTC bank. First, let us derive the formula for surpluses of bank in the OTC market only. The competitive surplus is as follows,

$$K(\omega, k, \mathrm{o}) = \frac{|U_{gg}|}{2}\left(g(\omega, k, \mathrm{o}) - \omega\right)^2 = \begin{cases} \frac{|U_{gg}|}{2}\left(g_o(\omega) - \omega\right)^2, \text{ if } \omega \in (\omega^\star, \omega_1] \\ \\ \frac{|U_{gg}|}{2}\left(\frac{1}{2} - \omega\right)^2, \text{ if } \omega \in (\omega_1, 1-\omega_1) \\ \\ \frac{|U_{gg}|}{2}\left(1 - g_o(1-\omega) - \omega\right)^2, \text{ if } \omega \in [1-\omega_1, 1-\omega^\star) \end{cases}$$

$K(\omega, k, \mathrm{o})$ is a U-shape function of ω with a minimum in $\omega = 1/2$. Take the general formula for the frictional surplus,

$$F(\omega, k, \mathrm{o}) = |U_{gg}|k\left[N(X_\mathrm{o})\int \left|g(\omega', k, \mathrm{o}) - g(\omega, k, \mathrm{o})\right| dN(\omega' \mid X_\mathrm{o}) + N(X_\mathrm{ow})\left|\frac{1}{2} - g(\omega, k, \mathrm{o})\right|\right].$$

It is natural to separate two types of banks that participate to the OTC market only: (i) "middle" banks, that is with in initial endowment $\omega \in (\omega_1, 1-\omega_1)$, and (ii) "external" banks. that is with in initial endowment $\omega \in (\omega^\star, \omega_1] \cup [1-\omega_1, 1-\omega^\star)$. For a "middle" bank the frictional surplus is the same as for a bank with initial endowment $\omega = 1/2$, that is

$$F\left(\frac{1}{2}, k, \mathrm{o}\right) = |U_{gg}|k\left[\int_{\omega^\star}^{\omega_1}\left(\frac{1}{2} - g(\omega', k, \mathrm{o})\right) d\omega' + \int_{1-\omega_1}^{1-\omega^\star}\left(g(\omega', k, \mathrm{o}) - \frac{1}{2}\right) d\omega'\right] = 2|U_{gg}|k\int_{\omega^\star}^{\omega_1}\left(\frac{1}{2} - g_o(\omega')\right) d\omega'$$

Now consider the frictional surplus of an "external" bank. Because of the symmetry post-trade exposure, the frictional surplus is also symmetric, that is $F(\omega, k, \mathrm{o}) = F(1-\omega, k, \mathrm{o})$. Therefore, we can focus on banks with initial endowment $\omega \in (\omega^\star, \omega_1]$. The frictional surplus is as follows:

$$F(\omega, k, \mathrm{o}) = |U_{gg}|k\left[\int_{\omega^\star}^{\omega_1}\left|g_o(\omega') - g_o(\omega)\right| d\omega' + \int_{\omega_1}^{\omega^\star}\left(1 - g_o(\omega') - g_o(\omega)\right) d\omega' + \left(1 - 2(\omega_1 - \omega^\star)\right)\left(\frac{1}{2} - g_o(\omega)\right)\right]$$

For a "middle" bank, that is for $\omega \in [\omega_1, 1-\omega_1]$, the marginal private value is equal to,

$$\mathrm{MPV}(\omega, k, \mathrm{o}) = \frac{|U_{gg}|}{2}\left(\frac{1}{2} - \omega\right)^2 + \frac{1}{2}F\left(\frac{1}{2}, k, \mathrm{o}\right).$$

It is U-shape with a minimum in $\omega = 1/2$. For an "external" bank with endowment, $\omega \in (\omega^\star, \omega_1]$, it is equal to

$$\mathrm{MPV}(\omega, k, \mathrm{o}) = \frac{|U_{gg}|}{2}\left(g_o(\omega) - \omega\right)^2 + \frac{1}{2}F\left(\omega, k, \mathrm{o}\right),$$

with $F(\omega, k, \text{o})$ that can be rewritten as,

$$F(\omega, k, \text{o}) = |U_{gg}|k \left[\int_{\omega^\star}^{\omega_1} \left| g_o(\omega') - g_o(\omega) \right| d\omega' + \int_{\omega_1}^{\omega^\star} \left(1 - g_o(\omega') - g_o(\omega)\right) d\omega' + (1 - 2(\omega_1 - \omega^\star)) \left(\frac{1}{2} - g_o(\omega) \right) \right]$$

$$\begin{aligned}
&= |U_{gg}|k \left[\int_{\omega^\star}^{\omega} \left(g_o(\omega) - g_o(\omega')\right) d\omega' + \int_{\omega}^{\omega_1} \left(g_o(\omega') - g_o(\omega)\right) d\omega' \right] \\
&+ |U_{gg}|k \left[\int_{\omega_1}^{\omega^\star} \left(1 - g_o(\omega') - g_o(\omega)\right) d\omega' + [1 - 2(\omega_1 - \omega^\star)] \left(\frac{1}{2} - g_o(\omega) \right) \right] \\
&= |U_{gg}|k \left[\int_{\omega^\star}^{\omega} \left(1 - 2g_o(\omega')\right) d\omega' + \int_{\omega}^{\omega_1} (1 - 2g_o(\omega)) \, d\omega' \right] \\
&+ |U_{gg}|k [1 - 2(\omega_1 - \omega^\star)] \left(\frac{1}{2} - g_o(\omega) \right)
\end{aligned}$$

For $\omega \in (\omega^\star, \omega_1]$, the derivative of $F(\omega, k, \text{o})$ is equal to

$$\frac{\partial F(\omega, k, \text{o})}{\partial \omega} = |U_{gg}|k \left[\underbrace{(1 - 2g_o(\omega)) - (1 - 2g_o(\omega))}_{=0} - 2 \int_{\omega}^{\omega_1} \frac{\partial g_o}{\partial \omega} d\omega' - [1 - 2(\omega_1 - \omega^\star)] \frac{\partial g_o}{\partial \omega} \right] = -\frac{\partial g_o}{\partial \omega} < 0$$

Since $(g_o(\omega) - \omega)^2$ is also decreasing, then $\text{MPV}(\omega, k, \text{o})$ is decreasing on $(\omega^\star, \omega_1]$ and increasing on $[1 - \omega_1, 1 - \omega^\star)$, by symmetry.

We can also define the competitive and frictional surplus of type $\omega \in [0, \omega^\star]$, who does *not* participate in the OTC market. Such an investor would find optimal to buy from all OTC market participants. Indeed, his post trade exposure would be $\omega + k$, that is lower than $\omega^\star + k$ and thus than any OTC market participant post-trade exposure. Therefore his competitive and frictional surplus would be,

$$\begin{aligned}
K(\omega, k, \text{o}) &= \frac{|U_{gg}|}{2} k^2 \\
F(\omega, k, \text{o}) &= |U_{gg}|k \left(\frac{1}{2} - \omega - k \right)
\end{aligned}$$

Therefore, $\text{MPV}(\omega, k, \text{o})$ is decreasing on $[0, \omega^\star]$ and increasing on $[1 - \omega^\star, 1]$, by symmetry. Overall, the marginal private value $\text{MPV}(\omega, k, \text{o})$ is a U-shape function with a minimum in $\omega = 1/2$.

Walrasian + OTC versus OTC. In order to obtain in equilibrium that participation in both markets dominated for extreme-ω banks, sufficient conditions are:

$$\begin{aligned}
&(b)\ \forall \omega \in [0, \omega^\star),\ \text{MPV}(\omega, k, \text{ow}) - \text{MPV}(\omega, k, \text{o}) > \text{MPV}(\omega^\star, k, \text{ow}) - \text{MPV}(\omega^\star, k, \text{o}), \\
&(b')\ \forall \omega \in \left(\omega^\star, \frac{1}{2}\right],\ \text{MPV}(\omega, k, \text{ow}) - \text{MPV}(\omega, k, \text{o}) < \text{MPV}(\omega^\star, k, \text{ow}) - \text{MPV}(\omega^\star, k, \text{o}).
\end{aligned}$$

Proving that $\text{MPV}(\omega, k, \text{ow}) - \text{MPV}(\omega, k, \text{o})$ is (weakly) decreasing over $[0, 1/2]$ would imply (b) and (b'). Because of the problem's symmetry, as long as $\text{MPV}(\omega, k, \text{ow}) - \text{MPV}(\omega, k, \text{o})$ is (weakly) decreasing over $[0, 1/2]$, then it is U-shape on $[0, 1]$. To show this result, we must compute the difference in marginal private values for $\omega \in \left[0, \frac{1}{2}\right]$. Several cases must be considered.

Consider a bank with $\omega \in [0, \omega^\star)$, in the equilibrium guess, it is on both markets. If instead it goes on the OTC market only, as shown before, its post trade exposure would be $\omega + k$, and thus the difference in private valuations would be

$$\begin{aligned}
&MPV(\omega, k, \text{ow}) - MPV(\omega, k, \text{o}) \\
&= \frac{|U_{gg}|}{2}\left[\left(\frac{1}{2}-\omega\right)^2 - k^2 - k\left(\frac{1}{2}-\omega-k\right)\right] + \frac{1}{2}F\left(\frac{1}{2}, k, \text{o}\right)
\end{aligned}$$

The derivative of the difference in marginal private values is equal to

$$\frac{\partial}{\partial \omega}\left(MPV(\omega, k, \text{ow}) - MPV(\omega, k, \text{o})\right) = \frac{|U_{gg}|}{2}\left[-1 + 2\omega + k\right] < 0$$

Therefore, $\text{MPV}(\omega, k, \text{ow}) - \text{MPV}(\omega, k, \text{o})$ is strictly decreasing on $[0, \omega^\star)$

$$\forall \omega \in [0, \omega^\star), MPV(\omega, k, \text{ow}) - MPV(\omega, k, \text{o}) > MPV(\omega^\star, k, \text{ow}) - MPV(\omega^\star, k, \text{o})$$

Consider a bank with $\omega \in (\omega^\star, \omega_1]$, in the guess equilibrium it goes on the OTC market only. If instead, it goes on both market, it obtain a $1/2$ post trade exposure and a frictional surplus equal to $\frac{1}{2}F\left(\frac{1}{2}, k, \text{o}\right)$. Therefore the difference in MPV is

$$\begin{aligned}
&MPV(\omega, k, \text{ow}) - MPV(\omega, k, \text{o}) \\
&= \frac{|U_{gg}|}{2}\left[\left(\frac{1}{2}-\omega\right)^2 - (g_o(\omega)-\omega)^2 + 2k\int_{\omega^\star}^{\omega_1}\left(\frac{1}{2} - g_o(\omega')\right)d\omega'\right] \\
&- \frac{|U_{gg}|}{2}\left[k\int_{\omega^\star}^{\omega}\left(1 - 2g_o(\omega')\right)d\omega' + k\int_{\omega}^{\omega_1}\left(1-2g_o(\omega)\right)d\omega' + k\left(1 - 2(\omega_1 - \omega^\star)\right)\left(\frac{1}{2} - g_o(\omega)\right)\right]
\end{aligned}$$

with,

$$g_o(\omega) = \omega(1-2k) + k(1+2\omega^\star) \quad \left(= \frac{1}{2} - \left(\frac{1}{2} - \omega^\star - k\right)\frac{\omega_1 - \omega}{\omega_1 - \omega^\star}\right)$$

Since $g_o(\omega)$ is an affine, $\text{MPV}(\omega, k, \text{ow}) - \text{MPV}(\omega, k, \text{o})$ is quadratic function of ω on $(\omega^\star, \omega_1]$. It is equal to 0 in $\omega = \omega_1$. Therefore the following equivalence holds,

$$\begin{aligned}
&\forall \omega \in (\omega^\star, \omega_1], MPV(\omega, k, \text{ow}) - MPV(\omega, k, \text{o}) < MPV(\omega^\star, k, \text{ow}) - MPV(\omega^\star, k, \text{o}) \\
\Leftrightarrow\ &\frac{\partial}{\partial \omega}\left(MPV(\omega, k, \text{ow}) - MPV(\omega, k, \text{o})\right)\Big|_{\omega = (\omega^\star)^+} < 0
\end{aligned}$$

We must verify under which condition the right derivative, in $\omega^\star$, of the difference in MPVs is strictly negative.

$$\begin{aligned}
&\frac{\partial}{\partial \omega}\left(MPV(\omega, k, \text{ow}) - MPV(\omega, k, \text{o})\right) \\
&= \frac{|U_{gg}|}{2}\left[-(1-2\omega) + 4k\left(k(1+2\omega^\star) - 2k\omega\right) + 2k\int_{\omega}^{\omega_1}\frac{\partial g_o}{\partial \omega}d\omega' + k\left(1 - 2(\omega_1-\omega^\star)\right)\frac{\partial g_o}{\partial \omega}\right] \\
&= \frac{|U_{gg}|}{2}\left[-(1-2\omega) + 4k\left(k(1+2\omega^\star) - 2k\omega\right) + 2k\int_{\omega}^{\omega_1}\frac{\partial g_o}{\partial \omega}d\omega' + k\left(1 - 2(\omega_1-\omega^\star)\right)\frac{\partial g_o}{\partial \omega}\right] \\
&= \frac{|U_{gg}|}{2}\left[-(1-2\omega) + 4k\left(k(1+2\omega^\star) - 2k\omega\right) + k\frac{\partial g_o}{\partial \omega}\right] \\
&= \frac{|U_{gg}|}{2}\left[-(1-2\omega) + 4k\left(k(1+2\omega^\star) - 2k\omega\right) + k(1-2k)\right]
\end{aligned}$$

In $\omega = \omega^\star$, this derivative is equal to

$$\frac{|U_{gg}|}{2}\left[-\left(1-2\omega^\star\right)+4k^2+k(1-2k)\right] = \frac{|U_{gg}|}{2}\left[-1+2\omega^\star+4k^2+k(1-2k)\right]$$

This is an increasing function of $\omega^\star$. The largest value that $\omega^\star$ can take is $\frac{1}{2}-k$. Therefore in $\omega^\star = \frac{1}{2}-k$, the derivative is equal to

$$\frac{|U_{gg}|}{2}\left[-2k+4k^2+k(1-2k)\right] = \frac{|U_{gg}|}{2}\left[-k+2k^2\right] = -\frac{|U_{gg}|}{2}k(1-2k) < 0$$

Consequently, $\mathrm{MPV}(\omega,k,\mathrm{ow}) - \mathrm{MPV}(\omega,k,\mathrm{o})$ is strictly decreasing on $(\omega^\star,\omega_1]$, and

$$\forall\omega\in(\omega^\star,\omega_1],\ MPV(\omega,k,\mathrm{ow}) - MPV(\omega,k,\mathrm{o}) < MPV(\omega^\star,k,\mathrm{ow}) - MPV(\omega^\star,k,\mathrm{o})$$

Finally, consider a bank with $\omega\in(\omega_1,1/2]$, in the equilibrium guess it is in the OTC market only. Yet, its post trade exposure is 1/2, the same as a bank who would also be on the Walrasian market. Hence, frictional and competitive surplus would remain unchanged if the bank would also join the Walrasian market, that is

$$MPV(\omega,k,\mathrm{ow}) - MPV(\omega,k,\mathrm{o}) = 0 < MPV(\omega^\star,k,\mathrm{ow}) - MPV(\omega^\star,k,\mathrm{o}) = c(\mathrm{w})$$

A.6 Proof of Proposition 5

Existence and uniqueness of $\omega^\star$. In order to obtain the existence result for $\omega^\star$, compute the following function,

$$\begin{aligned}
G(\omega^\star) &= MPV(\omega^\star,k,\mathrm{ow}) - MPV(\omega^\star,k,\mathrm{o})\\
&= \frac{|U_{gg}|}{2}\left[\left(\frac{1}{2}-\omega^\star\right)^2 - k^2 + 2k\int_{\omega^\star}^{\omega_1}\left(\frac{1}{2}-g_o(\omega')\right)d\omega' - k\left(\frac{1}{2}-g_o(\omega^\star)\right)\right]\\
&= \frac{|U_{gg}|}{2}\left[\left(\frac{1}{2}-\omega^\star\right)^2 - k^2 + 2k\left(\frac{1}{2}-\omega^\star-k\right)\int_{\omega^\star}^{\omega_1}\frac{\omega_1-\omega}{\omega_1-\omega^\star}d\omega' - k\left(\frac{1}{2}-\omega^\star-k\right)\right]\\
&= \frac{|U_{gg}|}{2}\left[\left(\frac{1}{2}-\omega^\star\right)^2 - k^2 + 2k\left(\frac{1}{2}-\omega^\star-k\right)\frac{\omega_1-\omega^\star}{2} - k\left(\frac{1}{2}-\omega^\star-k\right)\right]\\
&= \frac{|U_{gg}|}{2}\left[\left(\frac{1}{2}-\omega^\star\right)^2 - k^2 + \frac{k}{1-2k}\left(\frac{1}{2}-\omega^\star-k\right)^2 - k\left(\frac{1}{2}-\omega^\star-k\right)\right]\\
&= \frac{|U_{gg}|}{2}\left[\left(\frac{1}{2}-\omega^\star\right)^2 - k^2 + 2k\left(\frac{1}{2}-\omega^\star-k\right)\frac{\omega_1-\omega^\star}{2} - k\left(\frac{1}{2}-\omega^\star-k\right)\right]\\
&= \frac{|U_{gg}|}{2}\left[\left(\frac{1}{2}-\omega^\star-k\right)^2 + 2k\left(\frac{1}{2}-\omega^\star-k\right) + \frac{k}{1-2k}\left(\frac{1}{2}-\omega^\star-k\right)^2 - k\left(\frac{1}{2}-\omega^\star-k\right)\right]\\
&= \frac{|U_{gg}|}{2}\left[\frac{1-k}{1-2k}\left(\frac{1}{2}-\omega^\star-k\right)^2 + k\left(\frac{1}{2}-\omega^\star-k\right)\right]
\end{aligned}$$

G is decreasing with $\omega^\star$ on the interval $[0, 1/2-k]$. Therefore, there exists a unique $\omega^\star$ if and only if $0 < c(\mathrm{w}) < G(0)$. Let us compute $G(0)$,

$$\begin{aligned}
G(0) &= \frac{|U_{gg}|}{2}\left[\frac{1-k}{1-2k}\left(\frac{1}{2}-k\right)^2 + k\left(\frac{1}{2}-k\right)\right] \\
&= \frac{|U_{gg}|}{8}\left[(1-k)(1-2k) + 2k(1-2k)\right] \\
&= \frac{|U_{gg}|}{8}(1+k)(1-2k)
\end{aligned}$$

Define

$$X(\omega^\star) = \frac{1}{2} - \omega^\star - k$$

It is the solution to

$$X^2 + \frac{k(1-2k)}{1-k}X = \frac{2(1-2k)c(\mathrm{w})}{(1-k)|U_{gg}|}$$

The 2 roots to this equation are,

$$-\frac{k(1-2k)}{2(1-k)} \pm \left[\left(\frac{k(1-2k)}{2(1-k)}\right)^2 + \frac{2(1-2k)c(\mathrm{w})}{(1-k)|U_{gg}|}\right]^{1/2}$$

When $c(\mathrm{w})$ increases, ω^* decreases, therefore $X(\omega^\star)$ increases. Then

$$\begin{aligned}
X(\omega^\star) &= \frac{k(1-2k)}{2(1-k)}\left[\left(1 + \frac{8(1-k)c(\mathrm{w})}{k^2(1-2k)|U_{gg}|}\right)^{1/2} - 1\right] \\
&= \frac{k(1-2k)}{2(1-k)}\left[\left(\frac{1}{k^2} - \frac{1-k}{k^2(1-2k)}\left((1+k)(1-2k) - \frac{8c(\mathrm{w})}{|U_{gg}|}\right)\right)^{1/2} - 1\right]
\end{aligned}$$

Therefore, we have

$$\omega^\star = \frac{1}{2} - k - \frac{k(1-2k)}{2(1-k)}\left[\left(\frac{1}{k^2} - \frac{1-k}{k^2(1-2k)}\left((1+k)(1-2k) - \frac{8c(\mathrm{w})}{|U_{gg}|}\right)\right)^{1/2} - 1\right]$$

Autarky dominated. First, a sufficient condition for autarky being a dominated option for every bank is that for any bank, entering the OTC market yields a positive net MPV, that is

$$\forall \omega, MPV(\omega, k, \mathrm{o}) - c(\mathrm{o}) > 0$$

Given that $MPV(\omega, k, \mathrm{o})$ has a minimum in 1/2, this is equivalent to

$$MPV\left(\frac{1}{2}, k, \mathrm{o}\right) = \frac{1}{2}F\left(\frac{1}{2}, k, \mathrm{o}\right) = |U_{gg}|k\int_{\omega^\star}^{\omega_1}\left(\frac{1}{2} - g_o(\omega')\right)d\omega' > c(\mathrm{o}).$$

As long as there is frictional surplus to be generated, $F\left(\frac{1}{2}, k, \mathrm{o}\right) > 0$, that is if and only $\omega_1 > \omega^\star \Leftrightarrow \omega^\star + k < \frac{1}{2}$, we can find a $c(\mathrm{o})$ small enough so that this condition is verified.

To obtain the condition on $c(\mathrm{o})$, notice that we can rewrite the function $g_o(\omega)$ as,

$$g_o(\omega) = \frac{1}{2} - \left(\frac{1}{2} - \omega^\star - k\right)\frac{\omega_1 - \omega}{\omega_1 - \omega^\star}$$

Since the slope of $g_o(\omega) = (1 - 2k)$, we also have the equality,

$$\left(\frac{1}{2} - \omega^\star - k\right)\frac{1}{\omega_1 - \omega^\star} = 1 - 2k.$$

Overall, it yields

$$\int_{\omega^\star}^{\omega_1}\left(\frac{1}{2} - g_o(\omega')\right)\,d\omega' = \left(\frac{1}{2} - \omega^\star - k\right)\frac{\omega_1 - \omega^\star}{2} = \frac{1}{2(1-2k)}\left(\frac{1}{2} - \omega^\star - k\right)^2$$

Therefore, the condition on $c(\mathrm{o})$ is

$$c(\mathrm{o}) < \frac{|U_{gg}|k}{2(1-2k)}\left(\frac{1}{2} - \omega^\star - k\right)^2$$

Walrasian only dominated. Second, a sufficient condition for the Walrasian market only being a dominated option for every bank is that for any bank, entering both markets is more profitable than entering the Walrasian market only, that is

$$\forall\omega, MPV(\omega, k, \mathrm{ow}) - c(\mathrm{o}) - c(\mathrm{w}) - (MPV(\omega, k, \mathrm{w}) - c(\mathrm{w})) > 0$$

This is equivalent to

$$\frac{1}{2}F\left(\frac{1}{2}, k, \mathrm{o}\right) > c(\mathrm{o}).$$

Hence, whenever autarky is dominated because OTC markets is profitable for any bank, then it is also the case that any bank prefers going to both markets than in the Walrasian market only.

Inefficiently large Walrasian market. Since $g_o(\omega)$ is increasing for $\omega \in (\omega^\star, \omega_1]$, we have

$$F(\omega^\star, k, \mathrm{o}) = |U_{gg}|k\left(\frac{1}{2} - g_o(\omega^\star)\right) > 2|U_{gg}|k\int_{\omega^\star}^{\omega_1}\left(\frac{1}{2} - g_o(\omega^\star)\right) > 2|U_{gg}|k\int_{\omega^\star}^{\omega_1}\left(\frac{1}{2} - g_o(\omega')\right) = F\left(\frac{1}{2}, k, \mathrm{o}\right)$$