Deflation, Sticky Leverage and Asset Prices

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February 13, 2017

Abstract

We develop an asset pricing model with endogenous corporate policies to understand how deflation risk impacts real asset prices. Our key assumption is that nominal coupons paid to long-term corporate debt are fixed, i.e. leverage is sticky creating a nominal rigidity. Our model shows that deflation risk reduces real equity prices and increases equity return volatility. For the US economy, our model shows that deflationary episodes have a strong negative impact on real equity values, whereas inflationary episodes have a relatively mild positive impact. The overall impact of nominal risk on real equity values is therefore negative. In the cross-section, the model predicts that the negative impact of deflation risk on real equity values is stronger for high leverage firms. We find empirical support for our theoretical predictions.

JEL Classification Numbers: E44, G12, G32, G33
Keywords: deflation, leverage, default risk, equity

Deflation is viewed as a major risk for asset prices during times of financial turmoil. Recently, economists have cited deflation as being one of the most-feared risks facing participants in financial markets.[1]

But why should we fear deflation? Fear of deflation hinges on multiple concerns. First, there is the issue of sticky leverage – debt is issued in nominal terms, but is not continuously readjusted. As such, leverage is sticky and acts as a nominal rigidity, ensuring that changes in inflation impact real asset prices. More severe deflation raises the real value of debt, changing firms' default and leverage decisions, which alter the risk profile of real asset returns. Second, there is the well known debt-deflation theory. A fall in producer prices encourages firms to cut wages, thereby weakening consumption demand. Further, the real value of debt increases, which forces borrowers to cut spending to pay down their debts, thereby aggravating the consumption slowdown. Third, there is the issue of the zero lower bound – with greater deflation, nominal interest rates are more likely to hit the zero lower bound, rendering standard monetary policy less effective. Debt-deflation type events seem to be rare. Effective unconventional monetary policies may well diminish the dangers of being at the zero lower bound. However, sticky leverage is omnipresent. Despite this, the sticky leverage channel has received comparatively little attention.[2]

In this paper we explore the impact of deflation risk on real asset returns in the presence of sticky leverage. We want to understand whether or not we should fear the impact of deflation risk on asset prices and returns more than inflation risk – is deflation risk merely the mirror image of inflation risk or is there a fundamental asymmetry at play? We also examine the differential impact of deflation risk on equity versus debt and, for each case, study the firm characteristics and the economic conditions that make assets more sensitive to deflation risk. We are thereby able to analyze aspects of both the cross-sectional and time-series asset-pricing implications of deflation risk. This is important as the U.S. economy has experienced regular and severe deflationary times in the past century (see Figure 1).

Figure 1 [about here]

While there is a growing literature on the measurement of deflation risk, as implied from derivative prices (Fleckenstein, Longstaff, and Lustig, 2013) and the impact of sticky leverage on macroeconomic

[1]See, for example, the articles titled "Risk of Deflation Feeds Global Fears" in the Wall Street Journal (October 16, 2014) and "Eurozone Inflation Rate Dips, Stirring Deflation Fears" in the New York Times (November 28, 2014). Following an article in the Economist (October 25, 2014), Paul Krugman writes that "deflation, not inflation, is now the greatest concern for the world economy." The weak consumption data resulting from the European debt crisis accompanied by the recent fall in energy prices have largely contributed to the risk of deflation globally, that is, in Europe, Japan, and the U.S.

[2]Relatively few papers study the implications of sticky leverage. For example, Dopke and Schneider (2006), Christiano, Motto, and Rostagno (2010), Fernández-Villaverde (2010), Bhamra, Fisher, and Kuehn (2010), De Fiore, Teles, and Tristani (2011), Kang and Pflueger (2015), and Gomes, Jermann, and Schmid (2016). In contrast, the 'debt deflation' mechanism is the subject of a large literature going back to Fisher (1933) and the literature on the implications of the zero lower bound for monetary policy has exploded in recent years.

variables, there is little work on the asset pricing implications. Intuitively, entering a deflation period will reduce firms' nominal earnings, decrease the nominal interest rate, and increase the real value of nominal bond coupons. This would lead to a rise in real financial leverage and eventually to higher default risk, consistent with the evidence of Calomiris and Hubbard (1989) and Giesecke, Longstaff, Schaefer, and Strebulaev (2014). However, in standard asset pricing models, deflation is typically ignored. When it comes to furthering our understanding of how deflation risk impacts asset prices, the power of the modern dynamic asset pricing framework has yet to be fully harnessed.

Our paper makes four contributions. First, we build an asset pricing model of multiple firms which can issue debt and equity with the option to default, where deflation risk impacts firms' asset prices. Second, we show that deflation risk has a more deleterious impact on equity prices than the risk of high inflation and we can understand why by decomposing the effect of deflation risk on asset prices into default risk and discounting rate components. Third, we find that this deflation risk effect is stronger for more highly levered firms. Fourth, we find empirical support for our theoretical predictions.

We model the economy via a dynamic cross-section of firms that issue debt and equity. The theoretical framework endogenizes firms' financing and default policies and provides asset pricing predictions from a corporate finance perspective. We consider a representative agent with standard Epstein-Zin-Weil utility, i.e. with a preference for the early resolution of intertemporal risk associated with a change in real economic conditions.[3] The novelty of our model is that firms face not only real macroeconomic risk but also the risk that the economy switches between high inflation and deflation via a Markov chain. We refer to the fluctuations in the inflation rate as *nominal risk*. In our calibration, the real states of the economy are characterized by the conditional moments of consumption growth using a two-state Markov-regime switching model on quarterly U.S. consumption data over the period 1952Q1-2015Q4. The dynamics of firm real cash flows are given by the moments of quarterly U.S. aggregate earnings from S&P. We consider the Producer Price Index (PPI) from the U.S. Bureau of Labor Statistics to determine the nominal states and identify the deflation regime as when year-on-year PPI growth is negative.

The model allows us to shed new light on the role of fluctuating inflation rate for financial assets. The key to why nominal risk matters for real asset prices in our model is that firms do not adjust their nominal coupons as inflation changes. This stickiness of leverage means real coupons are affected by nominal conditions, and so changes in inflation will impact real asset prices. We show that the impact of nominal risk on real asset values via the sticky leverage is substantial.[4] For a market capitalization of listed companies representing US$19.3 trillion at the NYSE (as of June 2016), modelling the effects

[3]Our approach relates to the models of Chen (2010) and Bhamra, Kuehn, and Strebulaev (2010a,b), who analyze firms' capital and default decisions, as well as levered asset prices, in a consumption-based environment with macroeconomic conditions.

[4]To do so, we compare the model's prediction with that of an hypothetical economy with an expected growth rate of prices set at its unconditional mean.

of sticky leverage shows that nominal risk has a negative impact on equity values, representing a total loss of US$145 billion. Our model thus demonstrates that the impact of fluctuating inflation rate for equity investors is economically important.

A change from high inflation to deflation affects asset prices through a *default risk* and a *discounting* channel. In the first channel, a firm's default probability increases in deflation due to both a lower nominal cash flow growth rate and a higher default boundary optimally selected by shareholders. The second channel implies that the nominal risk-free rate decreases, which increases the present value of debt payments, thereby decreasing the present value of the net cash-flows that shareholders are expected to receive. We show that deflation risk decreases equity value through both channels, which complement one another.

We also show that it is highly misleading to frame our understanding of deflation risk as the simple mirror image of inflation risk. It is tempting to reason that because moving into deflation increases the expected future value of fixed nominal coupons, thereby increasing the real value of debt, while an increase in inflation of the same size will decrease real debt values by the same amount. But, such an analysis is incomplete – it ignores how shifts in real debt values impact levered equity values non-linearly via default risk. The intuition is as follows. A shift into deflation increases real debt values and shareholders decide to default earlier because of this, which increases default risk. Overall, this will depress equity values. Of course, a shift into inflation will have the opposite effect, but it will not be equal. The reason for the asymmetry between the effects of deflation versus inflation stems from the fact that default probabilities are convex in the distance-to-default. This means an increase in default risk depresses the value of equity more than a decrease in default risk of the same size would increase it. We thus find that nominal risk generates novel asset pricing implications.

The main prediction is that the amplified default risk associated with fluctuations in nominal prices increases equity return volatilities and drives up the real equity risk premium. As a shift towards deflation exacerbate firm financial distress by a greater amount than a shift towards inflation of the same size reduces it, firm financial leverage increases with nominal risk, on average. As a consequence, the equity value becomes more sensitive to a potential change in real economic conditions, thereby driving its volatility ex ante. Further, the Epstein-Zin-Weil agent dislikes the uncertainty associated with such unexpected fluctuations and thus demands a greater risk premium. We thus predict that nominal risk fosters the equity risk premium through the default risk channel. This means that the representative agent does not need to be averse to deflation periods to obtain predictions on risk premia. In fact, we model preferences such that they are completely independent of the nominal state, in line with the view that periods of inflation/deflation can be associated with either good or bad economic conditions.[5]

[5] There is no consensus that agents should like inflation and dislike deflation, or vice versa. For example, Piazzesi and Schneider (2006) show that inflation predicts consumption growth negatively, while Boons, Duarte, de Roon, and

In the cross-section, firms with higher leverage are impacted more by the effects of deflation. This effect is driven by the sticky leverage channel, which is naturally stronger for high leverage firms. Similarly, the role of deflation risk for equity prices is countercyclical with respect to real economic conditions, as recessions correspond to times of greater financial stress. In the time-series, the impact of nominal risk implies that equity values will decline not only because of economic contractions, but also because of episodes of deflation. Consequently, our model predicts that real leverage values should decline more strongly for highly levered firms during deflationary episodes. Furthermore, equity return volatilities and risk premia should increase by larger amounts for firms with higher leverage.

Our model also has implications for how inflation risk impact corporate bond prices via the sticky leverage channel. A shift towards deflation raises expected real coupons, which is positive for real corporate bond prices, while the concurrent increase in default risk has a negative impact on bond prices. Overall, we find that the initial effect dominates, implying that a shift towards deflation raises real corporate bond prices. Moreover, we find that debt values are more sensitive to deflation risk for firms with lower leverage and that are financially healthier. Deflation risk also affects bondholders asymmetrically over time but in a procyclical way, as debt value is more sensitive to deflation risk in expansions than in contractions.

We find empirical support that deflation risk decreases equity prices, whereas inflation yields an increases. We use monthly stock returns for all common stocks traded on Nyse, Nadex, and Amex from CRSP. Inflation is the annualized month-on-month change in the seasonally-adjusted producer price index (PPI) from the FRED database at the Federal Reserve Bank of St. Louis. The sample period is from July 1926 till December 2015. We follow Lettau, Maggiori, and Weber (2014) and define dummy variables which equal one if inflation is positive and zero otherwise. At the end of each month, we then estimate inflation and deflation betas regressing monthly excess returns on inflation innovations. We run separate regressions for both inflation and deflation months and sort betas into portfolios. Our results show that firms with a higher sensitivity of equity returns to deflation (inflation) decreases (increases) equity returns. Our contribution is therefore to identify a new source of risk that is priced in the cross-section of individual stock returns. While there exists a large literature that empirically analyzes the effect of inflation on equity returns, the role of deflation risk has remained unexplored to date.

This paper contributes to the literature exploring the asset pricing implications of nominal prices. There is abundant evidence that expected stock returns decrease with inflation.[6] Based on this

Szymanowska (2016) suggest that the relation is strongly time-varying. This finding is consistent with the evidence that inflation periods do not always reflect a bad state of the economy Bekaert and Wang (2010), Campbell, Sunderam, and Viceira (2016), and David and Veronesi (2013).

[6]Studies exploring the relation between stock returns and inflation include Fama (1981), Fama and Schwert (1977), Schwert (1981), Geske and Roll (1983), Gultekin (1983), Solnik (1983), Kaul (1987), Pearce and Roley (1988), Kaul and Seyhun (1990), Boudoukh and Richardson (1993), Duarte (2013), among others.

evidence, there is a stream of the literature which explores theoretically the interaction between inflation and stock returns. These studies include Day (1984), Stulz (1986), Wachter (2006), Gabaix (2008), Hess and Lee (1999), Chen (2010), Bansal and Shaliastovich (2013), and Gomes et al. (2016). Chen, Roll, and Ross (1986) and Ang, Briere, and Signori (2012) find that inflation risk is priced in the cross section of U.S. stock returns, while Boons et al. (2016) show that the inflation risk premium varies over time conditional on the relation between inflation and the real economy. The above papers focus on the impact of an abrupt increase in inflation, whereas our paper focuses on deflation risk. Our work is closely related to work by Weber (2015), showing how inflation risk impacts equity returns via the sticky prices channel. In contrast, our focus is on deflation and the sticky leverage channel. Finally, another closely related paper is Kang and Pflueger (2015), which studies how inflation risk impact corporate bond prices, whereas the focus of our paper is on equity prices and deflation risk.

Section 1 describes a consumption-based asset pricing model with inflation and deflation risk, while Section 2 derives asset prices together with optimal default and capital structure decisions. Section 3 shows how we calibrate the model. Section 4 discusses the model's theoretical predictions. Section 5 tests the theoretical predictions in the data. Section 6 concludes.

1 Model

This section presents a consumption-based corporate finance model with real and nominal risks. We here define aggregate consumption, inflation and derive the real and nominal stochastic discount factors, using an Epstein-Zin-Weil representative agent. We then derive the asset values of firms, which issue nominal debt and equity, and describe their optimal policies.

1.1 Aggregate economic variables

Aggregate consumption at date-t is denoted by C_t and its dynamics are given exogenously by

$$\frac{dC_t}{C_t} = g_t dt + \sigma_{C,t} dZ_t, \tag{1}$$

where Z_t is a standard Brownian motion under the physical measure $\mathbb{P}$. The conditional first and second moments of aggregate consumption growth, g_t and $\sigma_{C,t}$, respectively, can take different values, depending on the current state of the real economy, denoted by ν_t. The real economy is risky and transitions between a recession state, $\nu_t = L$ (for low consumption growth), and a expansion state, $\nu_t = H$ (for high), vary according to a 2-state Markov chain. The probability under the physical measure of moving from the expansion state to the recession state within an instant dt is $\lambda^{\text{real}}_{HL} dt$, where the intensity $\lambda^{\text{real}}_{HL}$ is constant. Similarly, the probability under the physical measure of moving from the recession state to the expansion state within an instant dt is $\lambda^{\text{real}}_{LH} dt$.

Inflation dynamics are specified exogenously. The date-t level of the price index is denoted by P_t and satisfies

$$\frac{dP_t}{P_t} = \mu_{P,t} dt, \tag{2}$$

where we neglect inflation volatility stemming from small Brownian shocks, and assume that date-t conditional expected inflation, $\mu_{P,t}$, depends on the nominal state ϵ_t. We assume 2 nominal states: a positive inflation state, $\epsilon_t = I$, and a state of negative inflation – also referred to as deflation – characterized by $\epsilon_t = D$. The physical probability of moving from the deflationary state to the inflationary state, within the instant dt, is $\lambda^{\$}_{DI} dt$ and the probability of moving back within a subsequent instant is $\lambda^{\$}_{ID} dt$.

For ease of notation we combine the real and nominal states into 4 distinct states, where the current combined state is denoted by $s_t = (\nu_t, \epsilon_t)$. The economy is thus characterized by state 1 (recession with deflation, LD), state 2 (recession with positive inflation, LI), state 3 (expansion with deflation, HD), and state 4 (expansion with positive inflation, HI). In summary, the different states are

	s_t	g_t	$\sigma_{C,t}$	$\mu_{P,t}$
Recession & deflation (LD)	1	g_L	$\sigma_{C,L}$	$\mu_{P,D}$
Recession & positive inflation (LI)	2	g_L	$\sigma_{C,L}$	$\mu_{P,I}$
Expansion & deflation (HD)	3	g_H	$\sigma_{C,H}$	$\mu_{P,D}$
Expansion & positive inflation (HI)	4	g_H	$\sigma_{C,H}$	$\mu_{P,I}$

(3)

where $g_L < g_H$ and $\sigma_{C,L} > \sigma_{C,H}$ to ensure that the mean and volatility of consumption growth are cyclical and countercyclical, respectively. From the definition of the nominal state, $\mu_{P,D} < 0 < \mu_{P,I}$. The transitions between combined real and nominal states are given exogenously by a 4-state Markov chain. The probability under the physical measure of moving from state i to state $j \neq i$ within an instant dt is $\lambda_{ij} dt$, where the intensity λ_{ij} is constant. Real and nominal regimes are assumed to switch independently over period dt, such that the transition intensities for the aggregate state of the economy, Λ, solves

$$e^{\Lambda dt} = \left(S'_{\text{real}}\; e^{\Lambda^{\text{real}} dt}\; S_{\text{real}}\right) \circ \left(S'_{\$}\; e^{\Lambda^{\$} dt}\; S_{\$}\right), \tag{4}$$

where $\circ$ is the Hadamard product and

$$\Lambda^{\text{real}} = \begin{pmatrix} -\lambda^{\text{real}}_{LH} & \lambda^{\text{real}}_{LH} \\ \lambda^{\text{real}}_{HL} & -\lambda^{\text{real}}_{HL} \end{pmatrix}, \quad \Lambda^{\$} = \begin{pmatrix} -\lambda^{\$}_{DI} & \lambda^{\$}_{DI} \\ \lambda^{\$}_{ID} & -\lambda^{\$}_{ID} \end{pmatrix}, \quad S_{\text{real}} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad S_{\$} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

1.2 Representative agent and stochastic discount factors

The representative agent has the continuous-time analog of Epstein-Zin-Weil preferences.[7] The representative agent's real stochastic discount factor (SDF) at time-t, π_t, depends on the state of the real

[7]The continuous-time version of the recursive preferences introduced by Epstein and Zin (1989) and Weil (1989) is known as stochastic differential utility, and is derived in Duffie and Epstein (1992). Schroder and Skiadas (1999) provide a proof of existence and uniqueness.

economy and is given by (see Appendix A.1 for the derivation)

$$\pi_t = \left(\beta e^{-\beta t}\right)^{\frac{1-\gamma}{1-\frac{1}{\psi}}} C_t^{-\gamma} \left(p_{C,t} e^{\int_0^t p_{C,u}^{-1} du}\right)^{-\frac{\gamma-\frac{1}{\psi}}{1-\frac{1}{\psi}}}, \tag{5}$$

where β is the rate of time preference, γ is the coefficient of relative risk aversion (RRA), and ψ is the elasticity of intertemporal substitution under certainty (EIS).

The real stochastic discount factor at date-t, π_t, evolves as follows

$$\frac{d\pi_t}{\pi_t}|_{s_{t-}=i,s_t=j} = -r_i dt - \gamma\sigma_{C,i} dZ_t + (\omega_{ij} - 1) dN^P_{ij,t}, \tag{6}$$

where r_i is the equilibrium real risk-free interest rate in state i, given by

$$r_i = \begin{cases} r_L, & i \in \{1, 2\} \\ r_H, & i \in \{3, 4\} \end{cases} \tag{7}$$

with $r_H > r_L$ such that the real interest rate evolves in a cyclical fashion with the real economy.

The pricing of risk is determined by the martingales Z_t and $N^P_{ij,t}$ introduced in Equation (6). The increment in the standard Brownian motion dZ_t is the relatively small risk from unexpected consumption growth and $\gamma\sigma_{C,i}$ is the associated price of risk, which is higher in recessions. The compensated Poisson process $N^P_{ij,t}$ is given by

$$N^P_{ij,t} = dN_{ij,t} - \lambda_{ij} dt, \tag{8}$$

where $N_{ij,t}$ is a Poisson process which jumps up by one when the (combined) state switches from i to $j \neq i$. The increment in the compensated Poisson process, $dN^P_{ij,t}$, represents the risk stemming from a change in the state of the economy. The associated price of risk is $\omega_{ij} - 1$.

The pricing of securities is based on the risk-neutral switching probabilities per unit of time, $\hat{\lambda}_{ij}$, which are related to the actual switching probabilities per unit of time, λ_{ij}, such that $\hat{\lambda}_{ij} = \lambda_{ij}\omega_{ij}, i \neq j$. The diagonal elements of $\hat{\Lambda}$ are $\hat{\lambda}_{ii} = -\sum_{j\neq i} \hat{\lambda}_{ij}$, ensuring that $\hat{\Lambda}$ is a proper generator matrix under the risk-neutral measure. Under Epstein-Zin-Weil preferences, the distortion factor is defined by $\omega_{ij} = \frac{p_{C,j}}{p_{C,i}}$, where $p_{C,t}$ is the value of the claim to aggregate consumption per unit consumption, i.e. the price-consumption ratio. The price-consumption ratio depends on the current state of the real economy, but not on the nominal state, that is:

$$p_{C,t} = \begin{cases} p_{C,L}, & \text{if } s_t \in \{1, 2\} \\ p_{C,H}, & \text{if } s_t \in \{3, 4\}. \end{cases} \tag{9}$$

Hence, $\omega_{ij} = 1$ if $\nu_i = \nu_j$.

The representative agent cares about future consumption growth and prefers intertemporal risk to be resolved sooner rather than later when $\gamma > 1/\psi$, which implies that $\omega > 1$. Therefore, the risk-neutral probability per unit of time of switching from expansion to recession is higher than the actual probability, as $\hat{\lambda}_{(H,\cdot)(L,\cdot)} > \lambda_{(H,\cdot)(L,\cdot)}$, which means that the agent values securities as if recessions (expansions) were more (less) likely than in reality. However, because only real risks are priced by the agent, the risk-neutral and the actual probabilities related to a change in the nominal state are identical, that is $\hat{\lambda}_{ij} = \lambda_{ij}$ if $\nu_i = \nu_j$.

Financial securities have nominal prices, which requires us to consider a nominal stochastic discount factor for asset pricing. The date-t nominal SDF, denoted by $\pi_t^{\$}$, is defined as

$$\pi_t^{\$} = \frac{\pi_t}{P_t}, \tag{10}$$

whose dynamics satisfy

$$\frac{d\pi_t^{\$}}{\pi_t^{\$}}|_{s_{t-}=i, s_t=j} = -r_i^{\$} dt - \gamma \sigma_{C,i} dZ_t + \sum_{j \neq i} (\omega_{ij} - 1) dN_{ij,t}^P, \tag{11}$$

where $r_i^{\$}$ is the nominal interest rate in state i, given by

$$r_i^{\$} = r_i + \mu_{P,i}. \tag{12}$$

The nominal interest rate depends on both real and nominal states and can thus takes 4 different values – it changes when the conditional moments of consumption growth change and also when expected inflation changes. The nominal risk-free rate is lowest during the recession/deflation state and highest during the expansion/high inflation state.

1.3 Firm cash flows

The date-t level of the real cash-flow of firm n is denoted by $Y_{n,t}$, which evolves according to the process

$$\frac{dY_{n,t}}{Y_{n,t}} = \mu_{Y,t} dt + \sigma_{Y,t}^{id} dZ_{n,t}^{id}, \tag{13}$$

where the real cash flows have a conditional expected growth rate $\mu_{Y,t}$ and a conditional volatility $\sigma_{Y,t}^{id}$. Both moments are identical across firms. The standard Brownian motion $Z_{n,t}^{id}$ represents firm n's idiosyncratic risk factor and its changes are independent across firms and unrelated to shocks to consumption growth.[8]

[8] We ignore a common diffusive component in firm real cash flows, because the associated risk premium is negligible as evidenced by the earlier asset pricing literature.

Systematic risk in real cash flows is exclusively associated with low-frequency changes in economic conditions. The expected growth rate is higher in expansions than in recessions, whereas the conditional idiosyncratic volatility is lower in expansions than in recessions. In sum, for all firms, we have $\mu_{Y,t} = \mu_{Y,L}$, $\sigma^{id}_{Y,t} = \sigma^{id}_{Y,L}$ in recessions and $\mu_{Y,t} = \mu_{Y,H}$, $\sigma^{id}_{Y,t} = \sigma^{id}_{Y,H}$ in expansions, where $\mu_{Y,L} < \mu_{Y,H}$ and $\sigma^{id}_{Y,L} > \sigma^{id}_{Y,H}$.

Because firms issue nominal securities and pay nominal taxes, investors care about the dynamics of the nominal cash flows. Firm n's nominal date-t cash flow level is then given by $X_{n,t}$, where

$$X_{n,t} \equiv P_t Y_{n,t}, \tag{14}$$

which satisfies

$$\frac{dX_{n,t}}{X_{n,t}} = \mu_{X,t} dt + \sigma^{id}_{X,t} dZ^{id}_{n,t}, \tag{15}$$

with $\mu_{X,t} = \mu_{Y,t} + \mu_{P,t}$. Because we ignore instantaneous Brownian shocks on the price index, the volatility of the nominal cash flows is given by $\sigma^{id}_{X,t} = \sigma^{id}_{Y,t}$.

Overall, firms exhibit heterogeneity in their cash flows due to idiosyncratic shocks but, at the same time, all firms have moments in their expected growth rate that similarly vary over a combined nominal and real cycle.

2 Asset Prices and Corporate Financing Decisions

In this section, we derive asset prices together with optimal default and capital structure decisions.

Firms pay taxes on nominal cash flows X_t (for parsimony, we omit the subscript n) and issue debt to shield profits from such taxes. Each firm has a debt contract that is characterized by a constant and perpetual nominal debt coupon. Leverage is sticky because the coupon is nominal and kept fixed. Hence, when the nominal state changes, the real coupon changes, which affects valuations. Consequently, sticky leverage acts as a nominal rigidity. In other words, firms cannot adjust the nominal quantity of debt to news about the inflation/deflation state.[9] When a firm's nominal cash-flows reach a state-dependent boundary $X_{D,i}$, which is selected by equityholders to maximize equity value, the firm is liquidated. Debtholders recover a fraction of the after-tax unlevered asset value of the firm, while the remaining fraction is lost in bankruptcy costs.

[9] A debt structure with constant real coupons would make firm leverage dependent the real state of the economy but not on nominal prices.

2.1 Liquidation value

The nominal asset value at time of liquidation, denoted by $A^{\$}_{i,t}$ in state $i \in \{1, 2, 3, 4\}$, corresponds to the present value of the after-tax nominal unlevered cash flows, which equals

$$A^{\$}_{i,t} = (1-\eta) X_t E_t \left[\int_t^{\infty} \frac{\pi^{\$}_u}{\pi^{\$}_t} \frac{X_u}{X_t} du | s_t = i \right], \tag{16}$$

where η is the corporate tax rate. We now give nominal asset value at time of liquidation in terms of macroeconomic fundamentals.

The liquidation value in state $i \in \{1, 2, 3, 4\}$ is given by

$$A^{\$}_{i,t} = (1-\eta) X_t \frac{1}{r_{A,i}}, \tag{17}$$

where $\frac{1}{r_{A,i}}$ is defined by

$$\frac{1}{r_{A,i}} = E_t \left[\int_t^{\infty} \frac{\Lambda^{\$}_u}{\Lambda^{\$}_t} \frac{X_u}{X_t} du | s_t = i \right]. \tag{18}$$

The value of $r_{A,i} = v_{A,i}^{-1}$ is given by the i'th element of the vector $V_A = [v_{A,1}, \ldots, v_{A,4}]^{\top}$, where

$$V_A = (R_A - \widehat{\Lambda})^{-1} 1_{4\times1}, \tag{19}$$

$1_{4\times1}$ is the 4 by 1 vector of ones and R_A is the following 4 by 4 diagonal matrix

$$R_A = \mathrm{diag}(r_1 - \mu_{Y,1}, \ldots, r_4 - \mu_{Y,4}) = \mathrm{diag}(r^{\$}_1 - \mu^{\$}_{Y,1}, \ldots, r^{\$}_4 - \mu^{\$}_{Y,4}) \tag{20}$$

and $\widehat{\Lambda}$ is the 4 by 4 risk-neutral generator matrix of the Markov chain characterizing the real and nominal states of the economy, as defined in Section 1.2.

Our assumption that the rate of change of the price level is locally risk-free means that there is no inflation risk premium. Inflation will have no impact on the real values of assets, which are unlevered. Inflation risk will however, impact the real values of levered claims, because leverage is sticky.

The matrix R_A, which shows how macroeconomic growth rates impact the discounting of future cashflows, is independent of inflation. We can intrepret $r_{A,i}$ as the discount rate for a perpetuity with stochastic real expected growth rate $\mu_{Y,t}$, which is currently equal to $\mu_{Y,i}$. This discount rate is independent of inflation as explained above. Note that if the economy stays in state i forever, the discount rate reduces to the standard expression $r_{A,i} = r_i - \mu_{Y,i}$. In general, however, the economy can change state, and so the discount rate depends on the risk-neutral generator matrix of the Markov chain governing the economy's transitions. The presence of the risk-neutral generator matrix as opposed to the physical risk-neutral generator matrix incorporates the pricing of risk.

2.2 Arrow-Debreu default claims

Default risk is key to firm valuation. We thus express the value of a firm's asset as a function of a set of Arrow-Debreu default claims.

We define an Arrow-Debreu default claim as an asset, which pays out one dollar if default occurs in state j and the current state is i. We denote the nominal price of such a security by $q^{\$}_{D,ij,t}$, which satisfies (see Appendix A.4)

$$q^{\$}_{D,ij,t} = E_t \left[\frac{\Lambda^{\$}_{\tau_D}}{\Lambda^{\$}_t} I_{\{s_{\tau_D}=j\}} \middle| s_t = i \right], \tag{21}$$

where τ_D is the date at which default occurs, making $I_{\{s_{\tau_D}=j\}}$ the indicator function which equals one, if default occurs in state j, and zero elsewhere.

When valuing assets that depend on the level of cash-flows at time of default, X_{τ_D}, we have to consider additional Arrow-Debreu securities. The reason is that our economy features what we call "deep defaults", which can occur when the state of the economy jumps from its current state to a worse state. Default boundaries are countercyclical and can suddenly move upward when the economy deteriorates, for example when a deflation period starts. In such a situation, a fraction of firms may immediately default upon a change in state. Consider a firm which has a nominal cash-flow level of say \$10 while the default boundary is \$8. If the economy suddenly deteriorates by moving into a new state where the default boundary is \$11, the firm will immediately default. Clearly all firms with nominal cash flow level below \$11 would immediately default, thereby creating a default cluster. More formally we can consider a firm with nominal cash flow level X_{τ_D-}, at time τ_D-, which is the time just before default, where X_{τ_D-} is below the new state's default boundary, $X_{D,j}$. This firm will default as soon as the economy enters the new state, and so $X_{\tau_D-} = X_{\tau_D} < X_{D,j}$ ($X_{\tau_D-} = X_{\tau_D}$ because X is a continuous process). Hence, it not necessarily the case that at default, a firm's cash flow level is at the default boundary. Consequently, to value securities which depend on firm's cash flows, we need a modified set of Arrow-Debreu default claims. We derive them in Appendix A.5.

This second type of Arrow-Debreu default claim pays out $\frac{X_{\tau_D}}{X_{D,j}}$ at default if default occurs in state j and the current state is i. The date-t nominal price of this security is denoted by $\tilde{q}^{\$}_{D,ij,t}$, where

$$\tilde{q}^{\$}_{D,ij,t} = E_t \left[\frac{\Lambda^{\$}_{\tau_D}}{\Lambda^{\$}_t} \frac{X_{\tau_D}}{X_{D,j}} I_{\{s_{\tau_D}=j\}} \middle| s_t = i \right]. \tag{22}$$

Overall, there are 16 Arrow-Debreu default prices for each type, because 4 states characterize the aggregate economy.

2.3 Corporate bond value

A firm which issues debt promises to pay a nominal coupon c per unit time. If the firm defaults, the coupon is no longer paid and instead debt holders receive a fraction of the firm's liquidation value. This fraction is known as the recovery rate and the state-t recovery rate is denoted by α_t. Hence, the date-t nominal value of corporate debt, conditional on the current state being i, is given by

$$B_{i,t}^{\$} = cE_t\left[\int_t^{\tau_D} \frac{\Lambda_u^{\$}}{\Lambda_t^{\$}} du\right] + E_t\left[\frac{\Lambda_{\tau_D}^{\$}}{\Lambda_t^{\$}} \alpha_{\tau_D} A_{\nu_D}^{\$}(X_{\tau_D}) du\right]. \tag{23}$$

The above expression is simply the present value of future coupon flows up until some random default time, τ_D, plus the present value of the unlevered firm assets net of bankruptcy costs. We assume that the recovery rate depends solely on the state of the economy, and so α_j denotes its value in state j. We can rewrite the above expression as

$$B_{i,t}^{\$} = c\left(\frac{1}{r_{P,i}^{\$}} - \sum_{j=1}^{4} q_{D,1,ij}^{\$} \frac{1}{r_{P,j}^{\$}}\right) + \sum_{j=1}^{4} \alpha_j A_j^{\$}(X_{D,j}) \tilde{q}_{D,ij,t}^{\$}, \tag{24}$$

where $r_{P,i}^{\$}$ is the nominal discount rate for perpetuity paying a flow of 1 USD, conditional on the current state being i. Observe that

$$\frac{1}{r_{P,i}^{\$}} = E_t\left[\int_t^{\infty} \frac{\Lambda_u^{\$}}{\Lambda_t^{\$}} du | s_t = i\right]. \tag{25}$$

To understand the intuition underlying the expression the the corporate bond price given in (24), observe that $c\frac{1}{r_{P,i}^{\$}}$ is the present value in nominal terms of a bond paying a coupon flow of c USD in perpetuity with no default risk. The expression $c\sum_{j=1}^{4} q_{D,1,ij}^{\$}\frac{1}{r_{P,j}^{\$}}$ is the present value of coupons lost because of the possibility of default and $\sum_{j=1}^{4} \alpha_j A_j^{\$}(X_{D,j})\tilde{q}_{D,ij,t}^{\$}$ is the present value of the assets recovered.

The nominal discount rate for a constant nominal perpetuity, $r_{P,i}^{\$}$, is given by $r_{P,i}^{\$} = v_{B,i}^{-1}$, where $v_{B,i}$ is the i'th element of the vector $V_B = [v_{B,1}, ..., v_{B,4}]'$, given by

$$V_B = (R^{\$} - \widehat{\Lambda})^{-1} 1_{4\times 1}, \tag{26}$$

where $R^{\$}$ represents the 4 by 4 diagonal matrix such that $R_{ii}^{\$} = r_i^{\$}$. Therefore, $r_{P,i}^{\$}$ accounts for the possibility that the nominal risk-free rate takes different future values as macroeconomic fundamentals and inflation fluctuate over time.

2.4 Equity value

Shareholders are entitled to the firm's cash flows net of taxes and debt servicing as long as the firm does not default. When the firm is in default, which occurs at some random time τ_D, shareholders recover nothing and lose their rights to any future cash flows. The nominal value of equity at date-t, conditional on the current state i, is then given by

$$S^{\$}_{i,t} = (1-\eta)E_t\left[\int_t^{\tau_D} \frac{\Lambda^{\$}_u}{\Lambda^{\$}_t}(X_u - c)du \middle| s_t = i\right], \tag{27}$$

which can be rewritten as

$$S^{\$}_{i,t} = A^{\$}_i(X_t) - (1-\eta)\frac{c}{r^{\$}_{P,i}} - \sum_{j=1}^{4}\left(A^{\$}_j(X_{D,j})\tilde{q}^{\$}_{D,ij,t} - (1-\eta)q^{\$}_{D,ij,t}\frac{c}{r^{\$}_{P,j}}\right). \tag{28}$$

The first two terms represent the present value of cash flows net of coupon payments when there is no default, whereas the summation term captures the present value of the net cash flows that shareholders lose in the case of default.

2.5 Default and capital structure decisions

Shareholders maximize the value of their default option by choosing when to default. There exists a state-contingent endogenous default boundary X_{D,s_t} that depends on the current real and nominal state of the economy, i.e. $s_t \in \{1,2,3,4\}$. Expected inflation matters for default decisions, because a change in the nominal cash flow growth is not offset by a change in the nominal coupon rate, i.e. leverage is sticky. Hence, equityholders are entitled to smaller expected future cash flows in deflation than under high inflation.

The default boundaries satisfy the following four standard smooth-pasting conditions:

$$\left.\frac{\partial S^{\$}_{s_t}(X)}{\partial X}\right|_{X=X_{D,s_t}} = 0,\ s_t \in \{1,2,3,4\}. \tag{29}$$

Shareholders also choose the optimal nominal coupon to maximize firm value at date 0 by balancing marginal tax benefits from debt against marginal expected distress costs. There are two important features to note. First, as is standard in the capital structure literature (Leland, 1994), by maximizing firm value shareholders internalize debtholders' value at date 0. However, in choosing default times they ignore the considerations of debtholders. This feature creates the usual conflict of interest between equity and debtholders. Second, the optimal coupon depends on the state of the economy at date 0. To make this clear, we denote the date-0 coupon by $c_{s_0,0}$, where s_0 is the date-0 state of the economy. Shareholders choose the coupon to maximize date-0 firm value, $F^{\$}_{s_0,0} = B^{\$}_{s_0,0} + S^{\$}_{s_0,0}$, i.e.

$$c_{s_0,0} = \text{argmax } F^{\$}_{s_0,0}(c). \tag{30}$$

The optimal default and capital structure decisions are numerically obtained by maximizing Equation (30) subject to the conditions stated in Equation (29). As a result, the optimal default boundaries depend on the debt policy, which is determined by the initial financing state. Hence, if the economy is in state i, the default boundary for nominal earnings is given by $X_i(c_{s_0,0})$, where i denotes the dependence on the current state, the presence of $c_{s_0,0}$ denotes dependence on the coupon and hence on the state of the economy at date 0, s_0.

Numerically, we verify that the default boundaries are ordered such that: $X_1(c_{s_0,0}) > X_2(c_{s_0,0}) > X_3(c_{s_0,0}) > X_4(c_{s_0,0})$. So, we can see that the onset of deflation, without any change in macroeconomic conditions, i.e. a shift from state 2 to state 1 or change from state 4 to change 3, increases the default boundary, which leads to a cluster of defaults – all firms with earnings below the newly increased default boundary will immediately default. Naturally, default clustering also occurs when there is deterioration in macroeconomic conditions without any change in inflation, as in Bhamra et al. (2010a,b) and Chen (2010).

2.6 Equity risk premium and equity volatility

The model allows us to compute the state-i equity risk premium, which is equal to (see Appendix A.7)

$$\mu_{R,i} - r_i^{\$} = \sum_{j \neq i} (1 - \omega_{ij}) \sigma_{R,ij}^P \lambda_{ij}, \tag{31}$$

where $\sigma_{R,ij}^P = \frac{S_j^{\$}}{S_i^{\$}} - 1$ is the conditional volatility of equity associated with a change in real states. Recall that only real risk is priced by the representative agent.

The conditional level of equity volatility in state i is given by (see Appendix A.7)

$$\sigma_{R,i} = \sqrt{\left(\frac{\partial \ln S_{i,t}}{\partial \ln X_t} \sigma_{X,i}^{id} \right)^2 + \sum_{j \neq i} \lambda_{ij} \left(\sigma_{R,ij}^P \right)^2} \tag{32}$$

and thus accounts for the firm's idiosyncratic Brownian shocks multiplied by the sensitivity of equity to cash flow, which is a positive function of financial leverage, and the Poisson shocks stemming from a change in states.

3 Calibration

We calibrate the model to investigate how nominal risk affects firm asset prices. The real states of the economy are characterized by the conditional moments of consumption growth. We obtain the probability of transition from one real state to another, $\lambda_{\nu_{t-}\nu_t}^{\text{real}}$, using a two-state Markov-regime switching model estimated using quarterly U.S. real consumption jointly with real aggregate earnings data over the period 1952Q1-2015Q4.[10] This estimation strategy follows Bhamra et al. (2010a,b).

We proxy for global consumption using data on real non-durable goods plus service consumption expenditures from the Bureau of Economic Analysis, and proxy for real cash-flows using quarterly US aggregate earnings from S&P provided by Robert J. Shiller's website. The personal consumption expenditure chain-type price index is used to deflate nominal earnings. The conditional moments of real consumption and cash flow growth rates are reported in Table 1.[11] In particular, the estimates of the actual probabilities of being in expansion and recession are respectively $f_H = f_1 + f_2 = 76.7\%$ and $f_L = f_3 + f_4 = 23.3\%$.

We consider the Producer Price Index (PPI) from the US Bureau of Labor Statistics to determine the nominal states. We define the deflation regime as when the year-on-year PPI growth is negative, while the high inflation regime prevails otherwise. We use this condition to compute the frequencies and intensities of leaving a given nominal regime, $\lambda_{\epsilon_{t^-}\epsilon_t}$, using a two-state Markov-regime switching model. Our analysis is based on monthly data over the period 1914Q1-2015Q4. The estimates of the actual probabilities of being in deflation and inflation over the sample are respectively $f_D = f_1 + f_3 = 28.4\%$ and $f_I = f_2 + f_4 = 71.6\%$.

For the firm parameters, the corporate tax rate is set to $\eta = 15\%$ and the liquidation value in default is $\alpha = 50\%$. We normalize the initial value of the cash flow to $X_0 = 1$. Regarding the preference parameters, we fix the risk aversion to $\gamma = 10$, the elasticity of intertemporal substitution (EIS) to $\psi = 2$, and the subjective discount factor to $\beta = 0.03$.

Based on this calibration, the characteristics of the economy are as follows. The nominal growth rate of cash-flows varies between -15.9% in the recession-deflation state to 11.4% in the expansion-inflation state. The real-risk free rate is 2.92% in recession and 4.17% in expansion. The inflation rate equals -6.12% in deflation and 5.92% during times of high inflation. The long-run probability of being simultaneously in recession and deflation, which corresponds to the worst state, is 6.63%. The model predicts that equity prices are not only lower during recessions than during expansions, but also lower during deflationary times compared to periods of high inflation (see Table 2).

Tables 1 and 2 [about here]

[10] Following Yogo (2006), Lettau, Ludvigson, and Wachter (2008), and Boguth and Kuehn (2013), we discard the first years after World War II, as consumption appears to be systematically different from later years.

[11] Following Bhamra et al. (2010a,b), we account for an additional 22.58% of idiosyncratic volatility. The total cash flow volatility is thus approximately 25% for our benchmark firm, which is the average volatility of firms with outstanding rated corporate debt.

4 Theoretical predictions

This section discusses the model predictions and sheds light on the role of nominal risk for corporate asset prices, with a particular emphasis on the implications for equity.

4.1 Unconditional effect of nominal risk on equity

We first investigate how the value of equity varies with nominal risk, which relates to the risk of future deflation and high inflation. To do so, we compare the results of the full model with the case in which a firm faces perfect price stability. In the latter case, the expected inflation rate is set at its unconditional mean, which corresponds to a weighted-average of the deflation and the high inflation regimes. To focus specifically on the asset-pricing implications of nominal risk, we consider that firms that do not face variations in the nominal state have the same debt and default policies as the firms in the full model. We discuss later the effect of nominal risk on capital structure and default choices.

Table 3 indicates that the risk of fluctuating nominal prices decreases the value of equity. Based on our calibration of the U.S. economy, the impact associated with uncertain variations in producer prices amounts to 0.75% of shareholder wealth, on average. For a market capitalization of listed companies representing US$19.3 trillion at the NYSE (as of June 2016), this would represent a loss of US$145 billion. Our model thus suggests that the impact of nominal risk for equity investors is economically important.

Table 3 [about here]

This finding implies that nominal risk is expected to negatively affect equity prices, although such risk is not priced by the representative agent in our environment. Indeed, nominal risk has no effect on our pricing kernel, but nonetheless has a non-trivial influence on asset prices. Accounting for a preference for the inflation state or an aversion to nominal risk would clearly amplify this result.[12]

We discuss in the next section the channels through which nominal risk can have a negative impact on equity prices. To best understand the mechanism at play, it will be useful to analyze the equity sensitivity to the risk of deflation and to the risk of high inflation separately.

4.2 Deflation risk vs. high inflation risk

The model predicts that both deflation risk and high inflation risk have sizeable impacts on shareholder wealth. To show that, we compare the full model with the special case of deflation (high inflation) risk only, which means that the inflation rate (i.e. growth rate of producer prices) can switch between

[12]Indeed, recall from Equation 5 that investors have a preference for early resolution of uncertainty regarding the future (real) consumption regimes but not for the nominal regimes. Hence, we do not account for the premium associated with uncertain variations of the inflation/deflation states, which means that only the quantity of risk drives our predictions.

the unconditional mean and the level of the deflation (high inflation) state. Hence, we ignore the possibility to be in the high inflation regime in the first case and the possibility to be in the deflation regime in the other case. As presented in the Panels B and C of Table 3, the value of equity is reduced by 23.2% in the presence of deflation risk but increases by 15.6% with inflation risk. Deflation is thus negative news for shareholders, while a high inflation rate is good news. Notably, the risk of facing deflation in the future more severely affects asset prices, although it is less likely (the long-term probability is 28.4%), than the risk of high inflation.

Deflation risk decreases equity value through two complementary effects.[13] First, the lower nominal cash-flow growth rate in deflation increases a firm's default probability. Greater default risk then reduces the present value of the *unlevered* cash-flows accrued to shareholders because shareholders recover nothing in default.[14] Second, the present value of debt payments increases in deflation, due to the lower nominal risk free rate, which further reduces the discounting value of the *levered* cash-flows entitled to shareholders. This results into an increase in financial leverage. Hence, the risk associated with future deflationary periods has a deleterious impact for shareholders. The possibility of being in the high inflation regime affects shareholders through the same channels, but in the reverse direction.

Importantly, the decomposition of the variations in nominal prices into deflation risk and inflation risk suggests the presence of an asymmetric effect. That is, deflation periods are relatively more negative for shareholders than periods of high inflation are positive, which generates an overall negative effect of nominal risk on equity valuation. In presence of deflation (inflation) risk, default risk is high (low) and equity becomes strongly (weakly) sensitive to a change in state. Hence, the risk of facing deflation reduces the value of equity more than the possibility of high inflation increases it. The main contribution of our model is thus to demonstrate that the presence of nominal risk has strong real effects on financial asset prices through a deterioration of firm creditworthiness. We now discuss how nominal risk affects equity prices differently across firms and times.

4.3 Conditional asset-pricing implications

Table 3 reports theoretical predictions on the asset pricing implications of nominal risk across different states of the economy. The effect of nominal risk on equity is predicted to be strongest during the recession/deflation state, which corresponds to times of higher economic uncertainty and lower nominal cash flow growth. Two effects induce default risk to be highest during such state, thereby enhancing the sensitivity of shareholder wealth to deflation risk. First, the negative economic outlook increases the likelihood to attain a given default boundary. Second, shareholders have more incentives to liquidate

[13]The impact of inflation risk on equity consists arises from the mirror mechanism.

[14]As the nominal risk-free rate and expected cash-flow growth rate decrease by the same amount (i.e, by the expected growth rate in prices), a change in nominal state has essentially no effect on the discounting of the unlevered cash-flows for a *given default risk level.*

the firm and thus to select a higher default boundary during deflationary recessions compared to inflationary recessions (see Table 2).[15] In contrast, nominal risk does not appear to be particularly more important in recession or expansion periods. As a result, nominal risk affects shareholders in a *countercyclical* way with respect to the nominal cycle only. Hence, uncertainty about the future inflation rate affects shareholders differently over time, depending on the current nominal conditions. These predictions are clearly illustrated in Figure 2.

Figure 2 [about here]

We also report conditional predictions regarding the level of the equity risk premium and the volatility of equity returns. Poor economic conditions during recessions typically decrease the value of equity, thereby inducing firms to become financially more levered. The consequence is a greater levered equity risk premium and a higher level of volatility, through the so-called leverage effect (Black, 1976), as displayed in Table 2. The model is thus consistent with the evidence that these asset-pricing moments are typically higher in bad economic times, thus complementing existing asset-pricing theory (Campbell and Cochrane, 1999; Bansal and Yaron, 2004; Gabaix, 2012; Wachter, 2013).

A novel prediction of our paper is that these moments are also strongly countercyclical with respect to the nominal state. That is, the model generates a level of equity risk premium and equity volatility that is greater (lower) during periods of deflation (high inflation) compared to what an economy without nominal risk would imply (see Table 2). Our theory thus contributes to understanding why expected stock returns tend to decrease with the inflation level in the data.[16] Hence, although the uncertainty associated with fluctuations in the inflation rate does not appear to materially affect the *unconditional* levels of stock market volatility and risk premium, it leads to strong predictions regarding their conditional properties.

4.4 Cross-sectional predictions

The model suggests strong heterogeneity across firms in the equity sensitivity to nominal risk. Firms with higher financial leverage display a greater loss in shareholder wealth due to uncertainty of nominal prices. To see that, Table 4 reports the model predictions for firms having different leverage levels. We find that highly levered firms have a sensitivity to nominal risk that is about twice as that of low levered firms. The main reason is that more debt translates into stronger exposure to a change in the nominal risk-free rate, as well as into greater default risk. Similar results are obtained when we decompose nominal risk into deflation risk and high inflation risk, as displayed in Figure 3.

[15] This prediction complements the conclusion of Chen (2010) and Bhamra et al. (2010a,b) that default policies are countercyclical over the real business cycle. Noteworthy, we obtain this result even though the nominal and real states are independent in our economic environment.

[16] See, for example, Fama (1981), Fama and Schwert (1977), Schwert (1981), Geske and Roll (1983), Gultekin (1983), Solnik (1983), Kaul (1987), Pearce and Roley (1988), Kaul and Seyhun (1990), Boudoukh and Richardson (1993), Duarte (2013), among others.

Table 4 and Figure 3 [about here]

Firms vary in their optimal capital choices for two reasons in our analysis. On the one hand, a firm facing greater bankruptcy cost will exhibit a lower expected liquidation value. In default, debtholders are entitled to the unlevered firm value net of the liquidation costs. As debtholders recover less in default, debt is priced at a higher credit spread, which decreases the incentives for high debt issuance. Hence, the model predicts that shareholders of firms with greater bankruptcy costs should be less sensitive to nominal risk. On the other hand, firms choose to issue less debt when the capital structure is set in a more uncertain and slowing economy, as characterized by the recession state. Hence, the model predicts that firms financed during an economic expansion should display higher sensitivity to nominal risk.

Overall, nominal risk affects shareholders differently across firms. Several testable implications for the equity market directly follow from our model. Equity prices are expected to most sensitive to nominal (in particular deflation) risk for firms with high leverage (e.g., with lower expected bankruptcy costs and/or financed during a good state), for financially healthier firms, and with higher default risk (e.g., BBB vs AAA ratings).

4.5 Nominal risk implications across asset classes

We now examine how the impact of nominal risk varies across asset classes by comparing the effect on equity and debt. Table 3 shows that the risk associated with uncertain price changes increases debt value. In contrast to the impact of nominal risk on equity, there are two opposing effects of such a risk on the value of debt. On the one hand, the lower risk-free rate in deflation increases the present value of the coupons accrued to debtholders, thereby increasing the market debt value for a given capital structure. On the other hand, the firm faces a higher default probability, which reduces the value of the debt claim given the presence of bankruptcy costs. Overall, our calibration indicates that the first effect dominates, as nominal risk risk always appears to increase debt value.

Fluctuations in nominal prices thus have opposite effects on equity vs debt. Moreover, the cross-sectional predictions also vary considerably with respect to the equity case. Table 4 shows that nominal risk has a stronger impact on debt value when a firm has lower default risk, that is when it has lower financial leverage and/or during expansion periods. Furthermore, nominal risk affects bondholders asymmetrically over time but in a *procyclical* way with respect to real economic conditions. Debt value is indeed more sensitive to nominal risk in expansions than in recessions. That is, based on these cross-sectional and conditional predictions, the role of nominal risk is greatest for debtholders when it is weakest for shareholders, and vice versa. Our analysis thus highlights two central mechanisms when the economy faces nominal risk: an increase in default risk and a decrease in the nominal discount rate during deflationary times. Both effects go in the same direction for equity valuation, but go in the opposite direction for debt valuation. Hence, the results complement the analysis of Kang and Pflueger (2015), who focus on debt valuation exclusively.

4.6 Capital structure choices

The analysis has thus far discussed how the presence of nominal risk affects corporate asset prices. We considered an identical debt level across cases, as comparing firms with different debt policies is not particularly meaningful. However, it is also insightful to understand how the presence of nominal risk in the economy affects a firm's optimal capital structure. Table 5 reports the optimal coupon levels, and the associated financial leverage ratios, in the case of nominal risk (full model) and in the absence of such a risk.

Table 5 [about here]

The results suggest that firms choose to issue slightly less debt in the presence of nominal risk. Managers determine an optimal leverage level by selecting a lower (higher) coupon when the expected nominal risk-free rate (i.e., bond discount rate) is lower (higher) due to the possible deflation (high inflation) state. Overall, nominal risk increases the market debt value and this increase in financial leverage provides the managers incentive to issue less debt (by selecting a lower coupon) with the objective of reducing default risk. The difference is however not economically substantial, which suggests that the risk of fluctuating inflation rates only weakly affects a firm's capital structure choice. Hence, the asset-pricing implications of nominal risk discussed in this paper thus seem reasonable.

5 Empirical Results

5.1 Data

We use monthly stock returns for all common stocks traded on Nyse, Nadex, and Amex from CRSP. The one-month T-bill rate is from Ken French's website. Inflation, π_t, is the month-on-month change in the seasonally-unadjusted producer price index (PPI) from the FRED database at the Federal Reserve Bank of St. Louis. The sample period is from July 1926 till December 2015.

5.2 Inflation and Deflation Beta

We follow Lettau et al. (2014) and define dummy variables which equal one if inflation is positive and zero otherwise. At the end of each month t, we then estimate inflation and deflation βs regressing monthly excess returns of firm i, R_{it}, on PPI inflation. We run separate regressions for both inflation and deflation months. We obtain similar results if we model inflation as a random walk and use changes in inflation.

We estimate inflation and deflation βs using rolling regressions with a constant window size 120 months. We start our rolling estimation in June 1936. We then sort all stocks with at least 20 non-missing inflation and deflation observations into decile with increasing exposures to inflation and deflation and weight stock returns equally within each portfolio. We then move our window by one month and repeat the procedure.

Table 6 reports in Panel A mean monthly excess returns for the 10 portfolios sorted on inflation betas, whereas Panel B reports portfolio returns for deflation-beta sorted portfolios. We see in Panel A firms with low exposure to inflation have an annualized excess return of 13.84% on average during our sample period. Return increase in exposure to inflation with high inflation beta firms earning an annualized excess return of 21.38% per month. The inflation risk premium of going long high inflation firm and shorting low inflation exposure firms is 7.54% per year.

In Panel B we see firms with high deflation betas earn an annualized excess return of 20.56%. Returns decrease in deflation expsosure, resulting in a large negative deflation risk premium of -4.82% per year.

Consistent with the predictions of the model, we see a positive inflation risk premium and a negative deflation risk premium.

6 Conclusion

We have built a theoretical model showing how nominal risk impacts real asset prices via the sticky leverage channel. The key mechanism at play is that long-term nominal debt coupons are fixed, but inflation is risky. This assumption makes expected future real debt coupons dependent on future inflation, ensuring that nominal risk impacts real corporate bond values and hence real levered equity values. Our model implies that deflation risk lowers real equity values, increases real equity return volatilities and real equity risk premia. This effect is stronger for firms with higher leverage. Importantly, a shift towards deflation has a greater negative impact on real equity values than an equal shift towards to inflation. This intuition is that deflation decreases the distance-to-default and risk-neutral default probabilities are convex in the distance-to-default. The theoretical predictions of our model are supported in the data, lending credence to the idea that the sticky leverage channel is important for understanding the economics behind how deflation risk impacts real asset values.

In our modelling framework the stochastic discount factor is exogenous. As such, understanding how default clusters caused by increased deflation risk impact aggregate consumption and hence the stochastic discount factor, which will in turn impact default decisions is beyond the scope of this paper. Understanding the general equilibrium implications for how deflation risk can amplify default risk and its impact on real asset values would be an interesting topic for future research.

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APPENDIX

A.1 The Economy

First, we introduce some notation related to jumps in the state of the economy. Suppose that during the small time-interval $[t-\Delta t, t)$ the economy is in state i and that at time t the state changes, so that during the next small time interval $[t, t+\Delta t)$ the economy is in state $j \neq i$. We then define the left-limit of ν at time t as

$$s_{t-} = \lim_{\Delta t \to 0} s_{t-\Delta t}, \tag{A.1}$$

and the right-limit as

$$s_t = \lim_{\Delta t \to 0} s_{t+\Delta t}. \tag{A.2}$$

Therefore $s_{t-} = i$, whereas $s_t = j$, so the left- and right limits are not equal. If some function E depends on the current state of the economy i.e. $E_t = E(s_t)$, then E is a jump process which is right continuous with left limits, i.e. RCLL. If a jump from state i to $j \neq i$ occurs at date t, then we abuse notation slightly and denote the left limit of E at time t by E_i, where i is the index for the state. i.e. $E_{t-} = \lim_{s\uparrow t} E_s = E_i$. Similarly $E_t = \lim_{s\downarrow t} E_s = E_j$. We shall use the same notation for all processes that jump, because of their dependence on the state of the economy.

Using simple algebra we can write the normalized Kreps-Porteus aggregator in the following compact form:

$$f(c,v) = \beta \left(h^{-1}(v)\right)^{1-\gamma} u\left(c/h^{-1}(v)\right), \tag{A.3}$$

where

$$\begin{aligned} u(x) &= \frac{x^{1-\frac{1}{\psi}}-1}{1-\frac{1}{\psi}}, \psi > 0, \\ h(x) &= \begin{cases} \frac{x^{1-\gamma}}{1-\gamma}, & \gamma \geq 0, \gamma \neq 1. \\ \ln x, & \gamma = 1. \end{cases} \end{aligned}$$

The representative agent's value function is given by

$$J_t = E_t \int_t^\infty f(C_t, J_t)\, dt. \tag{A.4}$$

Proposition A.1 *The SDF of a representative agent with the continuous-time version of Epstein-Zin-Weil preferences is given by*

$$\pi_t = \begin{cases} \left(\beta e^{-\beta t}\right)^{\frac{1-\gamma}{1-\frac{1}{\psi}}} C_t^{-\gamma} \left(p_{C,t} e^{\int_0^t p_{C,s}^{-1} ds}\right)^{-\frac{\gamma-\frac{1}{\psi}}{1-\frac{1}{\psi}}}, & \psi \neq 1 \\ \beta e^{-\beta \int_0^t \left[1+(\gamma-1)\ln\left(V_s^{-1}\right)\right] ds} C_t^{-\gamma} V_t^{-(\gamma-1)}, & \psi = 1 \end{cases}. \tag{A.5}$$

When $\psi \neq 1$, the price-consumption ratio in state i, $p_{C,i}$, satisfies the nonlinear equation system:

$$p_{C,i}^{-1} = \overline{r}_i + \gamma \sigma_{C,i}^2 - g_i - \left(1 - \frac{1}{\psi}\right) \sum_{j \neq i} \lambda_{ij} \left(\frac{\left(p_{C,j}/p_{C,i}\right)^{\frac{1-\gamma}{1-\frac{1}{\psi}}} - 1}{1-\gamma} \right), \; i \in \{1, \ldots, N\}. \tag{A.6}$$

where

$$\overline{r}_i = \beta + \frac{1}{\psi} g_i - \frac{1}{2} \gamma \left(1 + \frac{1}{\psi}\right) \sigma_{C,i}^2, \; i \in \{1, \ldots, N\}. \tag{A.7}$$

When $\psi = 1$, define V_i via

$$J = \ln(CV). \tag{A.8}$$

Then V_i satisfies the nonlinear equation system:

$$\beta \ln V_i = g_i - \frac{\gamma}{2} \sigma_{C,i}^2 + \lambda_i \frac{\left(V_j/V_i\right)^{1-\gamma} - 1}{1-\gamma} \; i \in \{1, 2\}, \; j \neq i. \tag{A.9}$$

A.2 Derivation of the real SDF

In this section, we derive the real SDF shown in Proposition A.1. When we refer to states of the economy within this proof we mean only the real states, L, H.

Duffie and Skiadas (1994) show that the SDF for a *general* normalized aggregator f is given by

$$\pi_t = e^{\int_0^t f_v(C_s, J_s) dt} f_c\left(C_t, J_t\right), \tag{A.10}$$

where $f_c(\cdot, \cdot)$ and $f_v(\cdot, \cdot)$ are the partial derivatives of f with respect to its first and second arguments, respectively, and J is the value function given in (A.4). The Feynman-Kac Theorem implies

$$f\left(C_t, J_{t-}\right)\big|_{\nu_{t-} = i} dt + E_t\left[dJ_t \,|\, \nu_{t-} = i\right] = 0, \;\; i \in \{L, H\}.$$

Using Ito's Lemma we rewrite the above equation as

$$0 = f\left(C, J_i\right) + C J_{i,C} g_i + \frac{1}{2} C^2 J_{i,CC} \sigma_{C,i}^2 + \lambda_i \left(J_j - J_i\right), \tag{A.11}$$

for $i, j \in \{1, 2\}$, $j \neq i$. We guess and verify that $J = h(CV)$, where V_i satisfies the nonlinear equation system

$$0 = \beta u\left(V_i^{-1}\right) + g_i - \frac{1}{2} \gamma \sigma_{C,i}^2 + \lambda_i \left(\frac{\left(V_j/V_i\right)^{1-\gamma} - 1}{1-\gamma} \right), \; i, j \in \{L, H\}, \; j \neq i. \tag{A.12}$$

Substituting (A.3) into (A.10) and simplifying gives

$$\pi_t = \beta e^{-\beta \int_0^t \left[1 + \left(\gamma - \frac{1}{\psi}\right) u\left(V_s^{-1}\right)\right] dt} C_t^{-\gamma} V_t^{-\left(\gamma - \frac{1}{\psi}\right)}. \tag{A.13}$$

When $\psi = 1$, the above equation gives the second expression in (A.5). We rewrite (A.12) as

$$\beta\left[1+\left(\gamma-\frac{1}{\psi}\right)u\left(V_i^{-1}\right)\right]=\overline{r}_i-\left(\gamma-\frac{1}{\psi}\right)\lambda_i\left(\frac{\left(V_j/V_i\right)^{1-\gamma}-1}{1-\gamma}\right)-\left[\gamma g_i-\frac{1}{2}\gamma\left(1+\gamma\right)\sigma_{C,i}^2\right],\ i,j\in\{L,H\},\ j\neq i, \tag{A.14}$$

where $\overline{r}_i$ is given in (A.7). Setting $\psi = 1$ in (A.14) gives (A.9). To derive the first expression in (A.5) from (A.13) we prove that

$$V_i = \left(\beta p_{C,i}\right)^{\frac{1}{1-\frac{1}{\psi}}},\ \psi \neq 1. \tag{A.15}$$

We proceed by considering the optimization problem for the representative agent. She chooses her optimal consumption, C^*, and risky asset portfolio, φ, to maximize her expected utility

$$J_t^* = \sup_{C^*,\varphi} E_t \int_t^\infty f\left(C_t^*, J_t^*\right) dt.$$

Observe that J^* depends on optimal consumption-portfolio choice, whereas the J defined previously in (A.8) depends on exogenous aggregate consumption. The optimization is carried out subject to the dynamic budget constraint, which we now describe. If the agent consumes at the rate, C^*, invests a proportion, φ, of her remaining financial wealth in the claim on aggregate consumption (the risky asset), and puts the remainder in the locally risk-free asset, then her financial wealth, W, evolves according to the dynamic budget constraint:

$$\frac{dW_t}{W_{t-}} = \varphi_{t-}\left(dR_{C,t} - r_{t-}dt\right) + r_{t-}dt - \frac{C_{t-}^*}{W_{t-}}dt,$$

where $dR_{C,t}$ is the cumulative return on the claim to aggregate consumption. We define $N_{i,t}$ as the Poisson process which jumps upward by one whenever the real state of the economy switches from i to $j \neq i$. The compensated version of this process is the Poisson martingale

$$N_{i,t}^P = N_{i,t} - \lambda_i t, i \in \{L, H\}$$

It follows from applying Ito's Lemma to $P = p_C C$ that the cumulative return on the claim to aggregate consumption is

$$dR_{C,t} = \frac{dP_t + C_t dt}{P_{t-}} = \mu_{R_C,t-}dt + \sigma_{C,t-}dB_{C,t} + \sigma_{R_C,t-}^P dN_{\nu_{t-},t}^P,$$

where

$$\begin{aligned}
\mu_{R_C,t-}\big|_{\nu_{t-}=i} &= \mu_{R_C,i} = g_i + \frac{1}{2}\sigma_{C,i}^2 + \lambda_i\left(\frac{p_{C,j}}{p_{C,i}} - 1\right) + \frac{1}{p_{C,i}}, \\
\sigma_{C,t-}\big|_{\nu_{t-}=i} &= \sigma_{C,i}, \\
\sigma_{R_C,t-}^P\big|_{\nu_{t-}=i} &= \sigma_{R_C,i}^P = \frac{p_{C,j}}{p_{C,i}} - 1,
\end{aligned}$$

for $i \in \{L, H\}$, $j \neq i$. The total volatility of the return to holding the consumption claim, when the current state is i, is given by

$$\sigma_{R_C,i} = \sqrt{\sigma^2_{C,i} + \lambda_i \left(\sigma^P_{R_C,i}\right)^2}.$$

Note that C^* is the consumption to be chosen by the agent, i.e. it is a control, and at this stage we cannot rule out the possibility that it jumps with the state of the economy. In contrast, C is aggregate consumption, and since it is continuous, its left and right limits are equal, i.e. $C_{t-} = C_t$.

The system of Hamilton-Jacobi-Bellman partial differential equations for the agent's optimization problem is

$$\sup_{C^*,\varphi} f\left(C^*_{t-}, J^*_{t-}\right)\big|_{\nu_{t-}=i} dt + E_t\left[dJ^*_t \,|\, \nu_{t-} = i\right] = 0, \;\; i \in \{L, H\}.$$

Applying Ito's Lemma to $J^*_t = J^*\left(W_t, \nu_t\right)$ allows us to write the above equation as

$$\begin{aligned} 0 \;\; = \;\; & \sup_{C^*_i,\varphi_i} f\left(C^*_i, J^*_i\right) + W_i J^*_{i,W}\left(\varphi_i\left(\mu_{R_C,i} - r_i\right) + r_i - \frac{C^*_i}{W_i}\right) + \frac{1}{2} W^2_i J^*_{i,WW}\varphi^2_i \sigma^2_{R_C,i} + \\ & + \lambda_i\left(J^*_j - J^*_i\right), \; i \in \{L, H\}, \; j \neq i. \end{aligned}$$

We guess and verify that $J^*_t = h\left(W_t F_t\right)$, where F_i satisfies the nonlinear equation system

$$0 = \sup_{C^*_i,\varphi_i} \beta u\left(\frac{C^*_i}{W_i F_i}\right) + \left(\varphi_i\left(\mu_{R_C,i} - r_i\right) + r_i - \frac{C^*_i}{W_i}\right) - \frac{1}{2}\gamma\varphi^2_i\sigma^2_{R_C,i} + \lambda_i\left(\frac{\left(F_j/F_i\right)^{1-\gamma} - 1}{1-\gamma}\right), \; i \in \{L, H\}, \; j \neq i.$$

From the first order conditions of the above equations, we obtain the optimal consumption and portfolio policies:

$$\begin{aligned} C^*_i &= \beta^\psi F_i^{-(\psi-1)} W_i, \; i \in \{L, H\}, \\ \varphi_i &= \frac{1}{\gamma}\frac{\mu_{R_C,i} - r_i}{\sigma^2_{R_C,i}}, \; i \in \{L, H\}. \end{aligned}$$

The market for the consumption good must clear, so $\varphi_i = 1$, $W_i = P_i$, $C^*_i = C$ (and thus $J = J^*$). Note that this forces the optimal portfolio proportion to be one and the optimal consumption policy to be continuous. Hence

$$\mu_{R_C,i} - r_i = \gamma\sigma^2_{R_C,i},$$

and

$$p_{C,i} = \beta^{-\psi} F_i^{1-\psi}. \tag{A.16}$$

The above equation implies that for $\psi = 1$, $p_{C,i} = 1/\beta$. The equality, $J = J^*$, implies that $CV_i = WF_i$. Hence, $F_i = p^{-1}_{C,i} V_i$. Using this equation to eliminate F_i from (A.16) gives (A.15). Substituting (A.15) into (A.13) and (A.14) gives the expression in (A.5) for $\psi \neq 1$ and (A.6).

A.3 Liquidation Value

The abandonment or liquidation value of a firm is just its unlevered value, i.e. the present value of future cashflows, ignoring coupon payments to debtholders and default risk. Small, but frequent shocks to a firm's real cashflow growth are modelled by changes in the standard Brownian motion Z_t^{id}, where we omit the subscript n for ease of notation. Small, but frequent shocks to the real SDF are modelled by changes in the standard Brownian motion Z_t. The assumption that $dZ_t dZ_t^{id} = 0$ means that small, but frequent shocks to cashflow growth are not priced. However, changes in the expected real cashflow growth rate are driven by the same Markov chain as those driving jumps in the SDF. Hence, changes in unlevered firm value driven by changes in the expected real cashflow growth rate will be priced.

Suppose the economy is currently in state i. Then, the risk-neutral probability of the economy switching into state $j \neq i$ different state during a small time interval Δt is $\widehat{\lambda}_{ij}\Delta t$ and the risk-neutral probability of not switching is $1 - \widehat{\lambda}_{ij}\Delta t$. We can therefore write the unlevered real firm value in state i as

$$A_i = (1-\eta)X\Delta t + e^{-(r_i - \mu_{Y,i})\Delta t}\left[\left(1 - \widehat{\lambda}_{ij}\Delta t\right)A_i + \sum_{j\neq i}\widehat{\lambda}_{ij}\Delta t A_j\right],\ i, j \in \{1,2,3,4\},\ j \neq i. \quad \text{(A.17)}$$

The first term in (A.17) is the after-tax cash flow received in the next instant and the second term is the discounted continuation value. The discount rate is just the standard discount rate for a perpetuity. Observe that the volatility of real cashflow growth does not appear in the discount rate, because $dZ_t^{id} dZ_t = 0$. The continuation value is the average of A_i and A_j, weighted by the risk-neutral probabilities of being in states i and $j \neq i$ a small instant Δt from now. For example, with risk-neutral probability $\widehat{\lambda}_{ij}\Delta t$ the economy will be in state $j \neq i$ and the unlevered real firm value will be value will be A_j. The continuation value is discounted back at a rate reflecting the discount rate $\overline{\mu}_i$ and the expected earnings growth rate over that instant which is θ_i.

We take the limit of (A.17) as $\Delta t \to 0$, to obtain

$$0 = (1-\eta)\,X - (r_i - \mu_{Y,i})A_i + \sum_{j\neq i}\widehat{\lambda}_{ij}\left(A_j - A_i\right),\ i \in \{1,2,3,4\}, j \neq i.$$

To obtain the solution of the above linear equation system, we define

$$v_{A,i} = \frac{1}{(1-\eta)}\frac{A_i}{X},$$

the before-tax price-earnings ratio in state i. Therefore

$$\left(\text{diag}\left(r_1 - \mu_{Y,1}, \ldots, r_4 - \mu_{Y,4}\right) - \widehat{\Lambda}\right) \begin{pmatrix} v_{A,1} \\ v_{A,2} \\ v_{A,3} \\ v_{A,4} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \tag{A.18}$$

where $\text{diag}\left(r_1 - \mu_{Y,1}, \ldots, r_4 - \mu_{Y,4}\right)$ is a 4×4 diagonal matrix, with the quantities $r_1 - \mu_{Y,1}, \ldots, r_4 - \mu_{Y,4}$ along the diagonal and $\widehat{\Lambda}$, defined by $[\widehat{\Lambda}]_{ij} = \widehat{\lambda}_{ij}$, $i, j \in \{1, 2, 3, 4\}$, where

$$\widehat{\lambda}_{ij} = \omega_{ij} \lambda_{ij}, j \neq i \tag{A.19}$$

$$\widehat{\lambda}_{ii} = -\sum_{j \neq i} \omega_{ij} \lambda_{ij}, j \neq i \tag{A.20}$$

is the generator matrix of the Markov chain under the risk-neutral measure. Solving (A.18) gives (19), if $\det\left(\text{diag}\left(r_1 - \mu_{Y,1}, \ldots, r_4 - \mu_{Y,4}\right) - \widehat{\Lambda}\right) \neq 0$. We now define $P_i^X = p_i X$, the before-tax value of the claim to the earnings stream X in state i. Hence, from the basic asset pricing equation

$$E_t\left[\frac{dP^X + X dt}{P^X} - r dt \middle| \nu_{t-} = i\right] = -E_t\left[\frac{d\pi}{\pi} \frac{dP^X}{P^X} \middle| \nu_{t-} = i\right],$$

we obtain the unlevered risk premium:

$$E_t\left[\frac{dP^X + X dt}{P^X} - r dt \middle| \nu_{t-} = i\right] = \gamma \rho_{XC} \sigma^s_{X,i} \sigma_{C,i} dt - \left(\widehat{\lambda}_i - \lambda_i\right)\left(\frac{p_j}{p_i} - 1\right) dt, i \in \{1, 2\}, j \neq i.$$

Applying Ito's Lemma,

$$dP^X = p_i dX_t + \lambda_i \left(p_j - p_i\right) dt + \left(p_j - p_i\right) dN^P_{i,t}, \ i \in \{1, 2\}, \ j \neq i.$$

Thus, the unlevered volatility of returns on equity in state i is given by

$$\sigma_{R,i} = \sqrt{\sigma^2_{X,i} + \lambda_i \left(\frac{p_j}{p_i} - 1\right)^2}, \ j \neq i,$$

where $\sigma_{X,i} = \sqrt{\left(\sigma^s_{X,i}\right)^2 + \left(\sigma^{id}_{X,i}\right)^2}$.

A.4 Arrow-Debreu Securities

The Arrow-Debreu default claim denoted by $q_{D,ij,t}$ is the value of a unit of consumption paid if default occurs in state j and the current state is i. There are N^2 such claims in our economy: $\{q_{D,ij}\}_{i,j \in \{1,\ldots,N\}}$.

We assume, without loss of generality, that the regimes are labelled so that the default boundaries respect a monotonic ordering $X_{D,1} > X_{D,2} > X_{D,3} > X_{D,4}$.

We say that a firm's earnings are in region k, $k = 0, \ldots, N-1$, when they fall in the interval $[X_{D,k+1}, X_{D,k})$, assuming that $X_{D,0} \to \infty$. Region N is $(\infty, X_{D,N})$.

Proposition A.2 *Let A_k be*

$$A_k = \begin{pmatrix} 0_{N-k \times N-k} & -I_{N-k \times N-k} \\ 2S_{x,k}^{-1}(\widehat{\Lambda}_k - R_k) & 2S_{x,k}^{-1} M_{x,k} \end{pmatrix},$$

where $0_{n\times m} \in \mathbb{R}^{n\times m}$ denotes a matrix of zeros, $I_{n\times n} \in \mathbb{R}^{n\times n}$ denotes the n-dimensional identity matrix, $\widehat{\Lambda}_k$ is the $N-k$ by $N-k$ matrix obtained by removing the first k rows and columns of $\widehat{\Lambda}$, and

$$S_{x,k} = \mathrm{diag}(\sigma_{x,k+1}^2, \ldots, \sigma_{x,N}^2), \qquad R_k = \mathrm{diag}(r_{k+1}, \ldots, r_N), \qquad M_{k,y} = \mathrm{diag}(\mu_{y,k+1}, \ldots, \mu_{y,N}),$$

are $N-k$ by $N-k$ diagonal matrices, with $\widehat{\mu}_{x,i} = \widehat{\mu}_{X,i} - \frac{1}{2}\sigma_{X,i}^2$ and $\sigma_{x,i} = \sigma_{X,i}$ the drift and diffusion coefficient of $x = \log X$ under the risk-adjusted measure.

Given the integration constants $h_{i,j}(\omega)$, the Arrow-Debreu in region k are given by

Region 0:

$$q_{D,0,ij}(x) = \sum_{l=1}^{N} h_{ij}(\omega_{0,l}) e^{-\omega_{0,l} x}, \tag{A.21}$$

where $\omega_{0,1} > \ldots > \omega_{0,N} > 0$ are the N positive eigenvalues of A_0.

Region $k \in \{1, \ldots, N-1\}$:

$$\begin{aligned} q_{D,k,ij}(x) &= \delta_{ij}, & i \in \{1,\ldots,k\}, j \in \{1,\ldots,N\}, \\ q_{D,k,i,j}(x) &= \sum_{l=1}^{2(N-k)} h_{ij}(\omega_l) e^{-\omega_l x} - [A_k^{-1} B_k]_{i-k,j}, & i \in \{k+1,\ldots,N\}, j \in \{1,\ldots,N\}. \end{aligned} \tag{A.22}$$

where $\omega_{k,l}$ are the $2(N-k)$ positive eigenvalues of A_k and

$$B_k = \begin{pmatrix} 0_{N-k \times k} & 0_{N-k \times N-k} \\ B_k^N & 0_{N-k \times N-k} \end{pmatrix}, \qquad B_k^N = \begin{pmatrix} 2\frac{\widehat{\lambda}_{k+1,1}}{\sigma_{k+1}^2} & 2\frac{\widehat{\lambda}_{k+1,2}}{\sigma_{k+1}^2} & \cdots & 2\frac{\widehat{\lambda}_{k+1,k}}{\sigma_{k+1}^2} \\ 2\frac{\widehat{\lambda}_{k+2,1}}{\sigma_{k+2}^2} & 2\frac{\widehat{\lambda}_{k+2,2}}{\sigma_{k+2}^2} & \cdots & 2\frac{\widehat{\lambda}_{k+2,k}}{\sigma_{k+2}^2} \\ \vdots & \vdots & \cdots & \vdots \\ 2\frac{\widehat{\lambda}_{N,1}}{\sigma_N^2} & 2\frac{\widehat{\lambda}_{N,2}}{\sigma_N^2} & \cdots & 2\frac{\widehat{\lambda}_{N,k}}{\sigma_N^2} \end{pmatrix}.$$

Region N:

$$q_{D,k,ij}(x) = \delta_{ij}, \quad \forall i, j. \tag{A.23}$$

In each region, for each ω, the integration constants $h_{k+1,\bullet}(\omega) \equiv [h_{k+1,j}(\omega)]_{j=1,\ldots,N} \in \mathbb{R}^{1\times N}$, are identified by the boundary conditions (Section A.4.3), and the remaining integration constants

$$\overline{H}_k(\omega) = \begin{pmatrix} h_{k+2,1}(\omega) & \cdots & h_{k+2,N}(\omega) \\ \vdots & \cdots & \vdots \\ h_{N,1}(\omega) & \cdots & h_{N,N}(\omega) \end{pmatrix} \tag{A.24}$$

are given by

$$\overline{H}_k(\omega) = \overline{G}_k^{-1}(\omega)\ g_{k+\bullet,1}(\omega)\ h_{k+1,\bullet}(\omega) \tag{A.25}$$

where $g_{k+\bullet,k+1}(\omega) \equiv [g_{i,k+1}(\omega)]_{i=k+2,\ldots,N} \in \mathbb{R}^{(N-k-1)\times 1}$ *comprises the last* $N-k-1$ *elements of the first column of*

$$G(\omega) = 2S_x^{-1}(\widehat{\Lambda} - R) - \omega(2S_x^{-1}M_x - \omega I_{N\times N}). \tag{A.26}$$

and $\overline{G}_k(\omega)$ *is the* $N-k-1$ *by* $N-k-1$ *matrix obtained by removing the first* $k+1$ *rows and columns of* $G(\omega)$*,*

The next two subsections outline the proof of Proposition A.2.

A.4.1 Region 0: $X_t \geq X_{D,1}$

We start by analyzing the case where earnings at the current date t are above the highest default boundary, i.e. $X_t > X_{D,1}$. Hence, if earnings hit the boundary $X_{D,j}$ from above for the first time in state j, $\{q_{D,ij}\}_{i,j\in\{1,\ldots,N\}}$ will pay one unit of consumption; otherwise, the security expires worthless. Since each Arrow-Debreu default claim is effectively a perpetual digital put, their values can be derived by solving a system of ordinary differential equations, derived from the standard equations

$$E_t^{\mathbb{Q}}[dq_{D,ij} - r_i q_{D,ij}dt] = 0,\ i,j \in \{1,\ldots,N\}. \tag{A.27}$$

Using Ito's Lemma, the above equation can be rewritten as the following second-order ordinary differential-equation system:[17]

$$\frac{1}{2}\sigma_{x,i}^2 \frac{d^2 q_{D,ij}}{dx^2} + \widehat{\mu}_{x,i}\frac{dq_{D,ij}}{dx} + \sum_{k\neq i}\widehat{\lambda}_{ik}\left(q_{D,kj} - q_{D,ij}\right) = r_i q_{D,ij},\ i,j \in \{1,\ldots,N\}, \tag{A.28}$$

where $\widehat{\mu}_{x,i} = \widehat{\mu}_{X,i} - \frac{1}{2}\sigma_{X,i}^2$ and $\sigma_{x,i} = \sigma_{X,i}$ are the drift and diffusion coefficient of $x = \log X$ under the risk-adjusted measure.

In order to solve this system of ODEs, define

$$z_{ij} = q_{D,ij}, i,j \in \{1,\ldots,N\} \tag{A.29}$$

$$z_{N+i,j} = \frac{dq_{D,ij}}{dx}, i,j \in \{1,\ldots,N\}. \tag{A.30}$$

Then, we obtain the following first order linear system

$$\frac{dz_{ij}}{dx} - z_{N+i,j} = 0,\ i,j \in \{1,\ldots,N\}, \tag{A.31}$$

$$\frac{dz_{N+i,j}}{dx} + \frac{2\widehat{\mu}_{x,i}}{\sigma_{x,i}^2}z_{N+i,j} + \sum_{k\neq i}\frac{2\widehat{\lambda}_{ik}}{\sigma_{x,i}^2}\left(z_{kj} - z_{ij}\right) - \frac{2r_i}{\sigma_{x,i}^2}z_{ij} = 0,\ i,j \in \{1,\ldots,N\}.$$

[17] Note that since the puts are perpetual, $\frac{\partial q}{\partial t} = 0$. Hence, q is solely a function of the stochastic process $x = \log X$.

Expressing the above equation system in matrix form gives

$$Z' + AZ = 0_{2N\times N}, \tag{A.32}$$

where the ij'th element of the $2N$ by N matrix, Z, is

$$[Z]_{ij} = z_{ij}, i \in \{1, \ldots, 2N\}, j \in \{1, \ldots, N\}, \tag{A.33}$$

and $Z' = \frac{dZ}{dx}$.

To solve eq. (A.32), one first finds the eigenvectors and eigenvalues of A_0. Their defining equation is

$$A_0\underline{e}_i = \omega_i \underline{e}_i,\ i \in \{1, \ldots, 2N\}, \tag{A.34}$$

where ω_i is the i'th eigenvalue and $\underline{e}_i$ is the corresponding eigenvector. Note that A_0 has N positive and N negative eigenvalues (Jobert and Williams, 2006).

It follows from (A.34) that the eigenvalues of A_0 are the roots of its characteristic polynomial; that is, any eigenvalue ω is a solution to the following $2N$'th-order polynomial:

$$\det(A_0 - \omega I) = 0,$$

To simplify the above expression for the characteristic polynomial, we then use the following identity from Silvester: If $F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$, where $F_{ij}, i, j \in \{1, 2\}$ are N by N matrices, any two of which commute with each other, then

$$\det F = \det(F_{11}F_{22} - F_{12}F_{21}). \tag{A.35}$$

Since

$$A_0 - \omega I = \begin{pmatrix} -\omega I & -I \\ 2S_x^{-1}(\widehat{\Lambda} - R) & 2S_x^{-1}M_x - \omega I \end{pmatrix}$$

and diagonal matrices commute with all other matrices of the same size, any pair of the 4 submatrices in A_0 commute. Therefore, one can apply (A.35) and

$$0 = \det(A_0 - \omega I) = \det(\omega(\omega I - 2S_x^{-1}M_x) - 2S_x^{-1}(\widehat{\Lambda} - R)). \tag{A.36}$$

When $N \leq 2$ the above polynomial is of order 4 or less and can be solved exactly in closed-form. When $N \geq 3$, it must be solved numerically. Once the eigenvalues have been obtained, the eigenvectors are obtained by solving (A.34). We then define the $2N$ by $2N$ matrix of eigenvectors, E, by stacking the eigenvectors as follows

$$E = (\underline{e}_1, \ldots, \underline{e}_{2N}).$$

Hence, the ij'th component of E is the i'th element of the j'th eigenvector, i.e.

$$E_{ij} = (\underline{e}_j)_i.$$

Given the E matrix, we can define the $2N$ by N matrix W via

$$EW = Z. \tag{A.37}$$

We can then rewrite eq. (A.32) as

$$EW' + A_0EW = 0_{2N\times N}, \tag{A.38}$$

$$\Leftrightarrow E^{-1}(EW' + A_0EW) = W' + E^{-1}A_0EW = 0_{2N\times N}, \tag{A.39}$$

$$\Leftrightarrow W' + DW = 0_{2N\times N}, \tag{A.40}$$

where

$$D = E^{-1}A_0E = \mathrm{diag}(\omega_1, \dots, \omega_{2N}). \tag{A.41}$$

The first order differential equation system of eq. (A.40) is similar to that of eq. (A.32), with the notable difference that D is a diagonal matrix while A_0 isn't. Making use of this, we premultiply both sides of eq. (A.40) by the integrating factor e^{Dx}, which yields

$$e^{Dx}W' + e^{Dx}DW = (e^{Dx}W)' = 0_{2N\times N}.$$

Integrating the above equation gives

$$e^{Dx}W = K,$$

where K is a $2N$ by N matrix of constants of integration. Therefore, the general solution of eq. (A.40) is

$$W = e^{-Dx}K,$$

which, given eq. (A.37), implies

$$Z = Ee^{-Dx}K = e^{-Dx}EK, \tag{A.42}$$

given that D is $2N \times 2N$ and diagonal, and that E is $2N \times 2N$.

Thus,

$$q_{D,ij}(x) = \sum_{l=1}^{2N} h_{ij}(\omega_l)e^{-\omega_l x}, \tag{A.43}$$

where the $h_{ij}(\omega_l)$ are constants of integration that depend on the eigenvalues.

Note that, for any eigenvalue ω of A_0, the particular solution $q_{D,ij} = h_{ij}(\omega)e^{-\omega x}$, with $h_{ij}(\omega_l) = 0, \forall\omega_l \neq \omega$, solves (A.43). Indeed, we then have

$$Z = \begin{bmatrix} H(\omega) \\ -\omega H(\omega) \end{bmatrix} e^{-\omega x}, \quad \text{where} \quad H(\omega) = \begin{pmatrix} h_{1,\bullet}(\omega) \\ \overline{H}_0(\omega) \end{pmatrix},$$

and

$$Z' = -\omega Z.$$

Hence, (A.32) implies that

$$(-\omega I_{2N\times 2N} + A_0)Z = 0_{2N\times N},$$

or, equivalently,

$$\begin{pmatrix} -\omega I_{N\times N} & -I_{N\times N} \\ 2S_x^{-1}(\widehat{\Lambda} - R) & 2S_x^{-1}M_x - \omega I_{N\times N} \end{pmatrix} \begin{bmatrix} H(\omega) \\ -\omega H(\omega) \end{bmatrix} = 0_{2N\times N}. \tag{A.44}$$

This particular solution is important since it allows us to express $N(N-1)$ of the N^2 integation constants in terms of the first N ones. Indeed, simplifying (A.44) gives

$$\begin{aligned} -\omega I_{N\times N}H(\omega) + I_{N\times N}\omega H(\omega) &= 0_{N\times N}, \\ (2S_x^{-1}(\widehat{\Lambda} - R) - \omega(2S_x^{-1}M_x - \omega I_{N\times N}))H(\omega) &= 0_{N\times N}, \end{aligned}$$

where the first equation is trivial. To solve the second equation, we first consider

$$G(\omega)\,(h_{1j}(\omega), \ldots, h_{Nj}(\omega))^T = 0_{N\times 1}, \tag{A.45}$$

where

$$G(\omega) = 2S_x^{-1}(\widehat{\Lambda} - R) - \omega(2S_x^{-1}M_x - \omega I_{N\times N}).$$

We denote the ij'th element of $G(\omega)$ by $g_{ij}(\omega)$. We know from (A.36) that $\det G(\omega) = 0$. Thus, the equations

$$\sum_{k=1}^{N} g_{ik}(\omega)h_{kj}(\omega),\ i \in \{1, \ldots, N\} \tag{A.46}$$

are linearly dependent. However, the system

$$\sum_{k=1}^{N} g_{ik}(\omega)h_{kj}(\omega),\ i \in \{2, \ldots, N\} \tag{A.47}$$

is linearly independent, allowing us to solve for $h_{kj}(\omega)$, $k \in \{2, \ldots, N\}$ in terms of $h_{1j}(\omega)$, for $j \in \{1, \ldots, N\}$, that is

$$\overline{H}_k(\omega) = \overline{G}_k^{-1}(\omega)\ g_{k+\bullet,1}(\omega)\ h_{k+1,\bullet}(\omega)\,.$$

Solving the above linear equation system gives us $h_{ij}(\omega)$, $i \in \{2, \ldots, N\}$, $j \in \{1, \ldots, N\}$ in terms of $h_{1j}(\omega)$, $j \in \{1, \ldots, N\}$. Hence, for each eigenvalue ω of A_0, this leaves us with N free integration constants. Then, to ensure the finiteness of the Arrow-Debreu prices as $x \to \infty$, focus on the N positive eigenvalues of A_0. That is, we obtain $q_{D,0,ij}$, the Arrow-Debreu prices in region 0, where

$X > X_{D,1}$:

$$q_{D,0,ij}(x) = \sum_{l=1}^{N} h_{ij}(\omega_{0,l}) e^{-\omega_{0,l}x}, \tag{A.48}$$

where, without loss of generality, $\omega_{0,1} > \ldots > \omega_{0,N} > 0$ are the N positive eigenvalues of A_0. Note that (A.48) contains N^2 free integrations constants $h_{1j}(\omega_{0,l})$, which will be identified by the value and smooth pasting conditions (Section A.4.3).

A.4.2 Region k: $X_{D,k+1} \leq X < X_{D,k}$

We now turn to the analysis of Arrow-Debreu securities in region k, i.e. when current earnings are above default boundary $k+1$, but below default boundary k. Note that the above analysis in region $k = 0$ can be seen as a special case of the analysis below, with $X_{D,0} \to \infty$.

First, note that $(q_{D,k})_{ij} = \delta_{ij}, \forall i \leq k$. Indeed, if earnings are currently lower than $X_{D,k} < X_{D,k-1} < \ldots$, and if the current state is $i \leq k$, then the firm is in default and the present value of a dollar when the firm defaults in state j is 1 if $i = j$, and 0 otherwise. In particular, this means that, in region N, where $X \leq X_{D,N}$, we have

$$q_{D,N,ij} = \delta_{ij}.$$

Applying (A.27) to the unknown $q_{D,k,ij}, i > k$, yields the following system of ODEs

$$\begin{aligned}
\frac{dz_{k+1,j}}{dx} - z_{N+k+1,j} &= 0, \\
\frac{dz_{k+2,j}}{dx} - z_{N+k+2,j} &= 0, \\
&\vdots \\
\frac{dz_{N,j}}{dx} - z_{2N,j} &= 0, \\
\frac{dz_{N+k+1,j}}{dx} + \frac{2\widehat{\mu}_{y,k+1}}{\sigma^2_{y,k+1}} z_{N+k+1,j} + \sum_{l=1}^{k} \frac{2\widehat{\lambda}_{k+1,l}}{\sigma^2_{y,k+1}} (\delta_{lj} - z_{k+1,j}) & \\
+ \sum_{l=k+2}^{N} \frac{2\widehat{\lambda}_{k+1,l}}{\sigma^2_{y,k+1}} (z_{lj} - z_{k+1,j}) - \frac{2r_{k+1}}{\sigma^2_{y,k+1}} z_{k+1,j} &= 0, \\
\frac{dz_{N+k+2,j}}{dx} + \frac{2\widehat{\mu}_{y,k+2}}{\sigma^2_{y,k+2}} z_{N+k+2,j} + \sum_{l=1}^{k} \frac{2\widehat{\lambda}_{k+2,l}}{\sigma^2_{y,k+2}} (\delta_{l,j} - z_{k+2,j}) & \\
+ \sum_{l=k+1, l\neq k+2}^{N} \frac{2\widehat{\lambda}_{k+2,l}}{\sigma^2_{y,k+2}} (z_{l,j} - z_{k+2,j}) - \frac{2r_{k+2}}{\sigma^2_{y,k+2}} z_{k+2,j} &= 0 \\
&\vdots
\end{aligned}$$

$$\frac{dz_{2N,j}}{dx} + \frac{2\widehat{\mu}_{y,N}}{\sigma_{y,N}^2} z_{2N,j} + \sum_{l=1}^{k} \frac{2\widehat{\lambda}_{N,l}}{\sigma_{y,N}^2} (\delta_{l,j} - z_{N,j}) + \sum_{l=k+1}^{N-1} \frac{2\widehat{\lambda}_{N,l}}{\sigma_{y,N}^2} (z_{l,j} - z_{N,j}) - \frac{2r_N}{\sigma_{y,N}^2} z_{N,j} \quad = \quad 0,$$

for $j = \{1, \ldots, N\}$. Rewriting the above equation system in matrix form, we obtain

$$Z'_k + A_k Z_k + B_k \; = \; Z'_k + A_k(Z_k + A_k^{-1} B_k) \; = \; \widetilde{Z}'_k + A_k \widetilde{Z}_k = 0, \tag{A.49}$$

where $\widetilde{Z}_k = (Z_k + A_k^{-1} B_k)$, Z_k is the following $2(N-k)$ by N matrix

$$Z_k = \begin{pmatrix} z_{k+1,1} & z_{k+1,2} & \cdots & z_{k+1,N} \\ z_{k+2,1} & z_{k+2,2} & \cdots & z_{k+2,N} \\ \vdots & \vdots & \cdots & \vdots \\ z_{N,1} & z_{N,2} & \cdots & z_{N,N} \\ z_{N+k+1,1} & z_{N+k+1,2} & \cdots & z_{N+k+1,N} \\ z_{N+k+2,1} & z_{N+k+2,2} & \cdots & z_{N+k+2,N} \\ \vdots & \vdots & \cdots & \vdots \\ z_{2N,1} & z_{2N,2} & \cdots & z_{2N,N} \end{pmatrix}.$$

Thereafter, the development made, in region 0, between equations (A.34) and (A.42) can be applied to $\widetilde{Z}_k$ in (A.49) to yield

$$\widetilde{Z}_k = e^{-D_k x} E_k K_k, \tag{A.50}$$

or, equivalently,

$$Z_k = e^{-D_k x} E_k K_k - A_k^{-1} B_k. \tag{A.51}$$

Therefore,

$$\begin{aligned} q_{D,k,ij}(x) &= \delta_{ij}, i \in \{1, \ldots, k\}, j \in \{1, \ldots, N\} \\ q_{D,k,i,j}(x) &= \sum_{l=1}^{2(N-k)} h_{ij}(\omega_l) e^{-\omega_l x} - [A_k^{-1} B_k]_{i-k,j}, i \in \{k+1, \ldots, N\}, j \in \{1, \ldots, N\}. \end{aligned}$$

Once more, for each eigenvalue ω of A_k, the particular solution

$$Z_k = \begin{pmatrix} H_k(\omega) \\ -\omega H_k(\omega) \end{pmatrix} e^{-\omega y} - A_k^{-1} B_k, \quad \text{where} \quad H_k(\omega) = \begin{pmatrix} h_{k+1,\bullet}(\omega) \\ \overline{H}_k(\omega) \end{pmatrix},$$

can be used to express the constants in all lines of H_k but the first, as functions of the $h_{k+1,1}, \ldots, h_{k+1,N}$ constants. This leaves us with N free constants for each of the $2(N-k)$ eigenvalues.

A.4.3 Boundary conditions

Given the above development, we are left with N^2 free integration constants in region 0, and $2(N-k)N$ free constants in region $k \in \{1, \ldots, N-1\}$. Hence, we still have to solve for the N^3 constants,

$$N^2 + \sum_{k=1}^{N-1} 2(N-k)N = N^2 + 2N \sum_{k=1}^{N-1} (N-k) = N^3, \tag{A.52}$$

that satisfy the N^3 boundary conditions (value matching & smooth pasting) of the problem at hand. For each default boundary $X_{D,k}, k \in \{1, \ldots, N\}$, we have that:

(VM) The value of the $N + (N-k)N$ Arrow-Debreu securities with $i \geq k$, must be the same on both sides of the default boundary, i.e.

$$q_{D,k-1,ij}(x_{D,k}) = q_{D,k,ij}(x_{D,k}), \text{ where } , i \in \{k, \ldots, N\}, j \in \{1, \ldots, N\}; \tag{A.53}$$

(SP) The dynamics of the $(N-k)N$ Arrow-Debreu securities with $i > k$, must be the same on both side of the default boundary, i.e.

$$\left.\frac{dq_{D,k-1,ij}}{dx}\right|_{x_{D,k}} = \left.\frac{dq_{D,k,ij}}{dx}\right|_{x_{D,k}}, \text{ where } , i \in \{k+1, \ldots, N\}, j \in \{1, \ldots, N\}. \tag{A.54}$$

A.5 Modified Arrow-Debreu securities

Arrow-Debreu securities provide the expected value of a 1\$ cash flow conditional on the state of the world in which they occur. In particular, in region k at time t, Arrow-Debreu prices

$$q_{D,k,ij,t}(x) = E_t \left[\frac{\pi_{\tau_D}}{\pi_t} 1_{\nu_{\tau_D}=j} | \nu_t = i \right] \tag{A.55}$$

$$= E_t^{\mathbb{Q}} \left[1_{\nu_{\tau_D}=j} | \nu_t = i \right] \tag{A.56}$$

$$= \mathbb{Q} \left[\nu_{\tau_D} = j | \nu_t = i \right], \tag{A.57}$$

can be interpreted risk-adjusted probability that default, occurring at unknown time τ_D, will occur in state j conditional on the current state of the economy, ν_t, being i. For cash flows that do not depend on the level of earnings when the firm defaults, $X_{\tau_D} = e^{x_{\tau_D}}$, these Arrow-Debreu securities yield a straightforward approach to derive the cash flows' expected values. If a cash flow does depend on X_{τ_D}, the Arrow-Debreu securities may not be as useful.

In a continuous model, the earnings always approach the default boundary from above and default occur when $X_{\tau_D} = X_D$; that is, there is no uncertain with respect to the level of earnings upon default and Arrow-Debreu securities can readily be used to compute expected cash flows. In our economy, however, "deep defaults" can occur when the state of the economy jumps from its current state to a worse state.

Recall that we ordered the default boundaries such that $X_{D,1} > \ldots > X_{D,N}$; hence, state N is the best state of the economy, state 1 is the worst. When the state of the economy jumps toward a better state, the default boundary decreases as growth opportunities improve; hence, if the firm was not in default, it is even further away from default after the jump. However, if the level of earnings is $X_{D,j+1} < X_{\tau_D^-} \leq X_{D,j}$ prior to a jump to state j at time τ_D, the firm automatically defaults. The level of earnings X_{τ_D} is then only a fraction of the default boundary $X_{D,j}$, and the firm will thus be able to honor its obligations to the debtholders, for instance, only partially.

We thus introduce "modified" Arrow-Debreu securities

$$\tilde{q}_{D,k,ij,t}(x) = E_t\left[\frac{\pi_{\tau_D}}{\pi_t}\frac{X_{\tau_D}}{X_{D,j}}1_{\nu_{\tau_D}=j}\middle|\nu_t = i\right] \tag{A.58}$$

$$= E_t\left[\frac{\pi_{\tau_D}}{\pi_t}e^{x_{\tau_D}-x_{D,j}}1_{\nu_{\tau_D}=j}\middle|\nu_t = i\right] \tag{A.59}$$

$$= E_t^{\mathbb{Q}}\left[e^{x_{\tau_D}-x_{D,j}}1_{\nu_{\tau_D}=j}\middle|\nu_t = i\right], \tag{A.60}$$

to account for the uncertainty surrounding the recovery rate. Note that, as long as shareholders have no bargaining power, they shouldn't care about the depth of deep defaults.

Technically, the (standard) Arrow-Debreu securities are special cases of their modified counterparts, with $\frac{x_{\tau_D}}{x_{D,j}} = 1$. Moreover, when in region 0, deep defaults are not a direct concern as the firm would survive even to a jump to the worse state, state 1. Hence, the general solution in (A.48) holds. However, in region $k > 0$, applying (A.27) to the unknown $\tilde{q}_{D,k,ij}$, $i > k$, accounting for deep defaults, yields the following system of ODEs

$$
\begin{aligned}
\frac{dz_{k+1,j}}{dx} - z_{N+k+1,j} &= 0, \\
\frac{dz_{k+2,j}}{dx} - z_{N+k+2,j} &= 0, \\
&\vdots \\
\frac{dz_{N,j}}{dx} - z_{2N,j} &= 0, \\
\frac{dz_{N+k+1,j}}{dx} + \frac{2\widehat{\mu}_{y,k+1}}{\sigma^2_{y,k+1}} z_{N+k+1,j} + \sum_{l=1}^{k} \frac{2\widehat{\lambda}_{k+1,l}}{\sigma^2_{y,k+1}} \left(e^{x-x_{D,j}}\delta_{lj} - z_{k+1,j}\right) & \\
+ \sum_{l=k+2}^{N} \frac{2\widehat{\lambda}_{k+1,l}}{\sigma^2_{y,k+1}} \left(z_{lj} - z_{k+1,j}\right) - \frac{2r_{k+1}}{\sigma^2_{y,k+1}} z_{k+1,j} &= 0, \\
\frac{dz_{N+k+2,j}}{dx} + \frac{2\widehat{\mu}_{y,k+2}}{\sigma^2_{y,k+2}} z_{N+k+2,j} + \sum_{l=1}^{k} \frac{2\widehat{\lambda}_{k+2,l}}{\sigma^2_{y,k+2}} \left(e^{x-x_{D,j}}\delta_{lj} - z_{k+2,j}\right) & \\
+ \sum_{l=k+1, l\neq k+2}^{N} \frac{2\widehat{\lambda}_{k+2,l}}{\sigma^2_{y,k+2}} \left(z_{l,j} - z_{k+2,j}\right) - \frac{2r_{k+2}}{\sigma^2_{y,k+2}} z_{k+2,j} &= 0 \\
&\vdots \\
\frac{dz_{2N,j}}{dx} + \frac{2\widehat{\mu}_{y,N}}{\sigma^2_{y,N}} z_{2N,j} + \sum_{l=1}^{k} \frac{2\widehat{\lambda}_{N,l}}{\sigma^2_{y,N}} \left(e^{x-x_{D,j}}\delta_{lj} - z_{N,j}\right) + \sum_{l=k+1}^{N-1} \frac{2\widehat{\lambda}_{N,l}}{\sigma^2_{y,N}} \left(z_{l,j} - z_{N,j}\right) - \frac{2r_N}{\sigma^2_{y,N}} z_{N,j} &= 0,
\end{aligned}
$$

or, equivalently,

$$
Z_k' + A_k Z_k + \widetilde{B}_k = 0, \tag{A.61}
$$

where

$$
\widetilde{B}_k = \begin{pmatrix} 0_{N-k\times k} & 0_{N-k\times N-k} \\ \widetilde{B}_k^N & 0_{N-k\times N-k} \end{pmatrix},
$$

and

$$
\widetilde{B}_k^N = \begin{pmatrix}
2\frac{\widehat{\lambda}_{k+1,1}}{\sigma^2_{k+1}} e^{x-x_{D,1}} & 2\frac{\widehat{\lambda}_{k+1,2}}{\sigma^2_{k+1}} e^{x-x_{D,2}} & \cdots & 2\frac{\widehat{\lambda}_{k+1,k}}{\sigma^2_{k+1}} e^{x-x_{D,k}} \\
2\frac{\widehat{\lambda}_{k+2,1}}{\sigma^2_{k+2}} e^{x-x_{D,1}} & 2\frac{\widehat{\lambda}_{k+2,2}}{\sigma^2_{k+2}} e^{x-x_{D,2}} & \cdots & 2\frac{\widehat{\lambda}_{k+2,k}}{\sigma^2_{k+2}} e^{x-x_{D,k}} \\
\vdots & \vdots & \cdots & \vdots \\
2\frac{\widehat{\lambda}_{N,1}}{\sigma^2_{N}} e^{x-x_{D,1}} & 2\frac{\widehat{\lambda}_{N,2}}{\sigma^2_{N}} e^{x-x_{D,2}} & \cdots & 2\frac{\widehat{\lambda}_{N,k}}{\sigma^2_{N}} e^{x-x_{D,k}}
\end{pmatrix}.
$$

Now, $\widetilde{B}_k$ is not constant with respect to x anymore, but $\widetilde{B}_k' = \widetilde{B}_k$. Hence, letting $\widetilde{Z}_k = Z_k + (A_k + I)^{-1}\widetilde{B}_k$, we have $\widetilde{Z}_k' = Z_k' + (A_k + I)^{-1}\widetilde{B}_k$

$$\widetilde{Z}_k' + A_k\widetilde{Z}_k = Z_k' + (A_k + I)^{-1}\widetilde{B}_k + A_k Z_k + A_k(A_k + I)^{-1}\widetilde{B}_k \tag{A.62}$$

$$= Z_k' + A_k Z_k + \widetilde{B}_k = 0. \tag{A.63}$$

Once more, the development made between equations (A.34) and (A.42) can be applied to $\widetilde{Z}_k$ in (A.63) to yield

$$\widetilde{Z}_k = e^{-D_k x} E_k K_k, \tag{A.64}$$

or, equivalently,

$$Z_k = e^{-D_k x} E_k K_k - (A_k + I)^{-1}\widetilde{B}_k. \tag{A.65}$$

Therefore,

$$\tilde{q}_{D,k,ij}(x) = \delta_{ij} e^{x - x_{Dj}} = \delta_{ij}\frac{X}{X_{Di}}, \quad i \in \{1, \ldots, k\}, j \in \{1, \ldots, N\},$$

$$\tilde{q}_{D,k,i,j}(x) = \sum_{l=1}^{2(N-k)} h_{ij}(\omega_l) e^{-\omega_l x} - [(A_k + I)^{-1}\widetilde{B}_k]_{i-k,j}, \quad i \in \{k+1, \ldots, N\}, j \in \{1, \ldots, N\}.$$

For $1 \leq i \leq k$, if the earnings at current time t are $X_t < X_{Di}$ while the current state is i, it must be that the state just jumped to state i at time t^- and the firm is now in (deep) default, hence the first equation in the above system.

A.6 Bond Prices

We need to assume some ordering of default boundaries. If we assume that the earnings default boundary is always higher in bad real states (as opposed to nominal states), then we have the following possibilities:

$$X_{D,1} > X_{D,2} > X_{D,3} > X_{D,4} \tag{A.66}$$

$$X_{D,2} > X_{D,1} > X_{D,3} > X_{D,4} \tag{A.67}$$

$$X_{D,1} > X_{D,2} > X_{D,4} > X_{D,3} \tag{A.68}$$

$$X_{D,2} > X_{D,1} > X_{D,4} > X_{D,3}. \tag{A.69}$$

If deflation is worse than inflation, the overall ordering of states will be $1, 2, 3, 4$, going from worst to best. It therefore makes sense, at least at the outset to work with the following ordering of default boundaries:

$$X_{D,1} > X_{D,2} > X_{D,3} > X_{D,4}. \tag{A.70}$$

In this proof it is not necessary to distinguish between the state of the economy at dates $t-$ and t. The central part of our proof consists of proving that

$$E_t\left[\int_t^{\tau_D}\frac{\pi_s^\$}{\pi_t^\$}ds\right| s_t=i\left.\right]=\frac{1}{r_{P,i}^\$}-\sum_{j=1}^{4}\frac{q_{D,ij}^\$}{r_{P,j}^\$}, \tag{A.71}$$

where $r_{P,i}^\$$, the discount rate for a fixed nominal perpetuity, when the economy is in state i, is given by

$$r_{P,i}^\$=\left(E_t\left[\int_t^{\infty}\frac{\pi_s^\$}{\pi_t^\$}ds\right| s_t=i\left.\right]\right)^{-1}, \tag{A.72}$$

and

$$E_t\left[\frac{\pi_{\tau_D}^\$}{\pi_t^\$}\alpha_{\tau_D}A_{\tau_D}^\$\left(X_{\tau_D}\right)\right| s_t=i\left.\right] \quad = \quad \sum_{j=1}^{4}\alpha_j A_j^\$\left(X_{D,j}\right)\tilde{q}_{D,ij}^\$. \tag{A.73}$$

To prove (A.71), we note that

$$E_t\left[\int_t^{\tau_D}\frac{\pi_s^\$}{\pi_t^\$}ds\right| s_t=i\left.\right]=E_t\left[\int_t^{\infty}\frac{\pi_s^\$}{\pi_t^\$}ds\right| \nu_t=i\left.\right]-E_t\left[\frac{\pi_{\tau_D}^\$}{\pi_t^\$}\int_{\tau_D}^{\infty}\frac{\pi_s^\$}{\pi_{\tau_D}^\$}ds\right| s_t=i\left.\right],$$

and conditioning on the event $\{\nu_{\tau_D}=j\}$, we obtain

$$E_t\left[\frac{\pi_{\tau_D}^\$}{\pi_t^\$}\int_{\tau_D}^{\infty}\frac{\pi_s^\$}{\pi_{\tau_D}^\$}ds\right| s_t=i\left.\right]=\sum_{j=1}^{4}E_t\left[\Pr\left(\nu_{\tau_D}=j\right| s_t=i\left.\right)\frac{\pi_{\tau_D}^\$}{\pi_t^\$}\int_{\tau_D}^{\infty}\frac{\pi_s^\$}{\pi_{\tau_D}^\$}ds\right| s_t=i\left.\right].$$

Since consumption is Markovian, so is the state-price density, which implies that

$$\begin{aligned}
& E_t\left[\Pr\left(s_{\tau_D}=j\right| s_t=i\left.\right)\frac{\pi_{\tau_D}^\$}{\pi_t^\$}\int_{\tau_D}^{\infty}\frac{\pi_s^\$}{\pi_{\tau_D}^\$}ds\right| s_t=i\left.\right] \\
= \quad & E_t\left[\Pr\left(s_{\tau_D}=j\right| s_t=i\left.\right)\frac{\pi_{\tau_D}^\$}{\pi_t^\$}\right| s_t=i\left.\right]E_t\left[\int_{\tau_D}^{\infty}\frac{\pi_s^\$}{\pi_{\tau_D}^\$}ds\right| s_{\tau_D}=j\left.\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
E_t\left[\int_t^{\tau_D}\frac{\pi_s^\$}{\pi_t^\$}ds\right| s_t=i\left.\right] \quad = \quad & E_t\left[\int_t^{\infty}\frac{\pi_s^\$}{\pi_t^\$}ds\right| s_t=i\left.\right] \\
& -\sum_{j=1}^{4}E_t\left[\Pr\left(\nu_{\tau_D}=j\right| \nu_t=i\left.\right)\frac{\pi_{\tau_D}^\$}{\pi_t^\$}\right| \nu_t=i\left.\right]E_t\left[\int_{\tau_D}^{\infty}\frac{\pi_s^\$}{\pi_{\tau_D}^\$}ds\right| s_{\tau_D}=j\left.\right].
\end{aligned} \tag{A.74}$$

Conditional on being in state i, the value of a claim which pays one risk-free unit of consumption in perpetuity is $E_t\left[\int_t^\infty \frac{\pi_s^\$}{\pi_t^\$} ds \middle| s_t = i\right]$, so the discount rate for this perpetuity, $r_{P,i}^\$$, is given by (A.72). Consequently, (A.74) implies

$$E_t\left[\int_t^{\tau_D} \frac{\pi_s^\$}{\pi_t^\$} ds \middle| s_t = i\right] = \frac{1}{r_{P,i}^\$} - \sum_{j=1}^4 \frac{E_t\left[\Pr\left(s_t = i | s_{\tau_D} = j\right) \frac{\pi_{\tau_D}^\$}{\pi_t^\$} \middle| s_t = i\right]}{r_{P,j}^\$}. \tag{A.75}$$

To obtain (A.71) from the above expression, we note that

$$q_{D,ij,t}^\$ = E_t\left[\Pr\left(s_{\tau_D} = j | s_{\tau_t} = i\right) \frac{\pi_{\tau_D}^\$}{\pi_t^\$} \middle| s_t = i,\right]. \tag{A.76}$$

To prove (A.73), we condition on the event $\{s_{\tau_D} = j\}$ to obtain

$$E_t\left[\frac{\pi_{\tau_D}^\$}{\pi_t^\$} \alpha_{\tau_D} A_{\tau_D}^\$ \left(X_{\tau_D}\right) \middle| s_t = i\right] = \sum_{j=1}^4 \alpha_j A_j^\$ \left(X_{D,j}\right) E_t\left[\frac{X_{\tau_D}}{X_{D,j}} \Pr\left(s_{\tau_D} = s_j | s_t = i\right) \frac{\pi_{\tau_D}^\$}{\pi_t^\$} \middle| s_t = i\right].$$

Using (A.76) to simplify the above expression we obtain (24).

A.7 Equity risk premium and equity volatility

Applying Ito's Lemma to $S_{i,t}^\$$ gives

$$\frac{dS_{i,t}^\$ + (X_t - c)dt}{S_{i,t}^\$} = \frac{S_{i,t}^\$}{X_t} \frac{\partial S_{i,t}^\$}{\partial X_t} \frac{dX_t}{X_t} + \frac{1}{2} \frac{X_t^2}{S_{i,t}} \frac{\partial^2 S_{i,t}^\$}{\partial X_t^2} \left(\frac{dX_t}{X_t}\right)^2 + \sum_{j \neq i} \frac{S_{j,t}^\$ - S_{i,t}^\$}{S_{i,t}^\$} dN_{ij,t} + \frac{(X_t - c)dt}{S_{i,t}^\$} \tag{A.77}$$

Observe that

$$\frac{\partial S_{i,t}}{\partial X_t} = (1-\eta)\frac{1}{r_{A,i}^\$} - \sum_{j=1}^4 \left(A_j^\$(X_{D,j}) \frac{\partial \tilde{q}_{D,ij,t}^\$}{\partial X_t} - (1-\eta)\frac{\partial q_{D,ij,t}}{\partial X_t} \frac{c}{r_{P,j}^\$}\right) \tag{A.78}$$

$$\frac{\partial^2 S_{i,t}}{\partial X_t^2} = -\sum_{j=1}^4 \left(A_j^\$(X_{D,j}) \frac{\partial^2 \tilde{q}_{D,ij,t}^\$}{\partial X_t^2} - (1-\eta)\frac{\partial^2 q_{D,ij,t}}{\partial X_t^2} \frac{c}{r_{P,j}^\$}\right) \tag{A.79}$$

Define the date-t conditional nominal expected return

$$\mu_{R,i,t}^\$ = E_t\left[\frac{dS_{s_{t-},t}^\$ + (X_t - c)dt}{S_{s_{t-},t}^\$} \middle| s_{t-} = i\right]. \tag{A.80}$$

and the date-t conditional real expected return

$$\mu_{R,i,t} = E_t \left[\left. \frac{dS_{s_{t-},t} + (Y_t - c/P_t)dt}{S_{s_{t-},t}} \right| s_{t-} = i \right]. \tag{A.81}$$

The basic asset pricing equation is

$$\mu^{\$}_{R,i,t} - r^{\$}_{i,t} = -E_t \left[\left. \frac{d\pi^{\$}_t}{\pi^{\$}_t} \frac{dS^{\$}_{s_{t-},t}}{S^{\$}_{s_{t-},t}} \right| s_{t-} = i \right] \tag{A.82}$$

The real SDF is given by

$$\frac{d\pi_t}{\pi_{t-}}|_{s_{t-}=i} = -r_i dt - \gamma \sigma_{C,i} dZ_t - \sum_{j \neq i} (\omega_{ij} - 1) dN^P_{ij,t}. \tag{A.83}$$

and the nominal SDF by

$$\frac{d\pi^{\$}_t}{\pi^{\$}_{t-}}|_{s_{t-}=i} = \frac{d\pi_t}{\pi_{t-}}|_{s_{t-}=i} + \mu_{P,i} dt \tag{A.84}$$

$$= -r^{\$}_i dt - \gamma \sigma_{C,i} dZ_t - \sum_{j \neq i} (\omega_{ij} - 1) dN^P_{ij,t}. \tag{A.85}$$

Hence

$$\mu^{\$}_{R,i,t} - r^{\$}_{i,t} = \sum_{j \neq i} (1 - \omega_{ij}) \frac{S^{\$}_j - S^{\$}_i}{S^{\$}_i} \lambda_{ij} \tag{A.86}$$

Observe that because

$$\mu^{\$}_{R,i,t} = \mu_{R,i,t} + \mu_{P,i} \tag{A.87}$$

and

$$r^{\$}_{i,t} = r_{i,t} + \mu_{P,i}, \tag{A.88}$$

we have

$$\mu_{R,i,t} - r_{i,t} = \mu^{\$}_{R,i,t} - r^{\$}_{i,t} \tag{A.89}$$

The unexpected stock return in state i is given by

$$\sum_{j \neq i} \sigma^P_{R,ij} dN^P_{ij,t}, \tag{A.90}$$

where

$$\sigma^P_{R,ij} = \frac{S_j}{S_i} - 1 \tag{A.91}$$

The real SDF is given by

$$\frac{d\pi_t}{\pi_{t-}}|_{\nu_{t-}=i} = -r_i dt - \gamma\sigma_{C,i} dZ_t - \sum_{j\neq i}(\omega_{ij}-1)dN^P_{ij,t}. \tag{A.92}$$

Now the risk premium in state i is

$$\mu_{R,i} - r_i = \sum_{j\neq i}(\omega_{ij}-1)\sigma^P_{R,ij}\lambda_{ij}. \tag{A.93}$$

First, note that if $s \equiv \log S = \log f(X)$ and $x \equiv \log X$, then

$$\frac{\partial \log S}{\partial \log X} = \frac{\partial f(e^x)}{\partial x} = \frac{1}{f(e^x)} f'(e^x)e^x = \frac{X}{S} f'(X). \tag{A.94}$$

Second, recall that

$$S^{\$}_{i,t} = A^{\$}_i(X_t) - (1-\eta)v_{B,i}c - \sum_{j=1}^{4}\left(A^{\$}_j(X_{D,j})\tilde{q}^{\$}_{D,ij,t}(X_t) - (1-\eta)q^{\$}_{D,ij,t}(X_t)v_{B,i}c\right) \tag{A.95}$$

$$= (1-\eta)X_t v_{A,i} - (1-\eta)v_{B,i}c - \sum_{j=1}^{4}\left(A^{\$}_j(X_{D,j})\tilde{q}^{\$}_{D,ij,t}(X_t) - (1-\eta)q^{\$}_{D,ij,t}(X_t)v_{B,i}c\right), \tag{A.96}$$

where we made explicit that the Arrow-Debreu prices, $q^{\$}_{D,\cdot,ij,t}$, depend on the current value of X_t. Hence,

$$\frac{\partial \log S_{i,t}}{\partial \log X_t} = \frac{X_t}{S_{i,t}}\left[\frac{A^{\$}_i(X_t)}{X_t} - \sum_{j=1}^{4}\left(A^{\$}_j(X_{D,j})\frac{\partial \tilde{q}^{\$}_{D,ij,t}}{\partial X_t} - (1-\eta)\frac{\partial q^{\$}_{D,ij,t}}{\partial X_t}v_{B,i}c\right)\right], \tag{A.97}$$

Table 1: Model Calibration

This table presents the parameter values of the model. Real consumption data (non-durable goods plus service consumption expenditures) and the Producer Price Index (PPI) are used to calibrate the model to the aggregate U.S. economy, while aggregate earning data from S&P are considered to calibrate the model to a benchmark U.S. firm. We retrieved quarterly consumption data from the Bureau of Economic Analysis, quarterly earning series from Robert J. Shiller's website, and the monthly PPI from the US. Bureau of Labor Statistics. The personal consumption expenditure chain-type price index is used to deflate nominal earnings. The real and nominal conditions are determined by the estimation of two separate Markov-regime switching models over the period 1952Q1-2015Q4. All estimates are annualized when applicable. The calibration is detailed in Section 3.

	Unconditional	Conditional			
		State 1 Recession & Deflation	**State 2 Recession & Inflation**	**State 3 Boom & Deflation**	**State 4 Boom & Inflation**
Economic environment					
Stationary probability		6.63%	16.70%	21.78%	54.89%
Consumption growth rate	1.96%	0.17%	0.17%	2.50%	2.50%
Consumption growth volatility	0.78%	0.92%	0.92%	0.74%	0.74%
Inflation rate	2.50%	-6.12%	5.92%	-6.12%	5.92%
Real interest rate	3.88%	2.92%	2.92%	4.17%	4.17%
Nominal interest rate	6.38%	-3.19%	8.85%	-1.95%	10.09%
Riskless perpet. discount rate	6.13%	5.82%	6.17%	5.89%	6.25%
Risky flow discount rate	2.80%	3.13%	3.13%	2.71%	2.71%
Firm characteristics					
Growth rate of real earnings	1.94%	-9.76%	-9.76%	5.50%	5.50%
Growth rate of nominal earnings	4.45%	-15.87%	-3.84%	-0.61%	11.43%
Volatility of earnings	25.16%	29.70%	29.70%	23.78%	23.78%
Liquidation value	50.00%	50.00%	50.00%	50.00%	50.00%
Tax rate	15.00%	15.00%	15.00%	15.00%	15.00%

Table 2: Conditional Firm Characteristics

This table presents the state-contingent firm characteristics in a model with real and nominal risk. The different columns report the values of the variables for various current states of the economy, whereas the different rows report the values of the variables depending on the state at which the firm has selected its financing policy. The nominal earnings level is standardized at 1 in each scenario and equity volatility is annualized. Leverage is the ratio of the market value of debt to the sum of the market values of debt and equity. The parameter values of the model are reported in Table 1 and discussed in Section 3.

		State 1	**State 2**	**State 3**	**State 4**
		Recession & Deflation	**Recession & Inflation**	**Boom & Deflation**	**Boom & Inflation**
Decision Variables	Coupon	Default boundary			
State 1 financing	0.6735	0.1833	0.1752	0.1711	0.1635
State 2 financing	0.7172	0.1952	0.1866	0.1823	0.1741
State 3 financing	0.9193	0.2502	0.2391	0.2336	0.2231
State 4 financing	0.9790	0.2664	0.2547	0.2488	0.2377
As % of coupon		*0.2721*	*0.2601*	*0.2541*	*0.2427*
Optimal Leverage	Unconditional	Conditional			
State 1 financing	0.3141	0.3580	0.3412	0.3153	0.3000
State 2 financing	0.3314	0.3771	0.3597	0.3327	0.3168
State 3 financing	0.4074	0.4599	0.4398	0.4092	0.3904
State 4 financing	0.4286	0.4828	0.4620	0.4306	0.4111
Unconditional	*0.4002*	*0.4519*	*0.4321*	*0.4020*	*0.3835*
Equity	Unconditional	Conditional			
State 1 financing	21.6612	18.1047	18.5686	22.3012	22.7779
State 2 financing	21.1365	17.5739	18.0610	21.7550	22.2572
State 3 financing	18.7897	15.2226	15.8048	19.3113	19.9220
State 4 financing	18.1192	14.5581	15.1648	18.6128	19.2526
Unconditional	*19.0041*	*15.4417*	*16.0136*	*19.5343*	*20.1340*
Debt	Unconditional	Conditional			
State 1 financing	9.8709	10.0964	9.6188	10.2695	9.7622
State 2 financing	10.4268	10.6403	10.1451	10.8490	10.3192
State 3 financing	12.8485	12.9624	12.4081	13.3754	12.7597
State 4 financing	13.5193	13.5896	13.0250	14.0759	13.4403
Unconditional	*12.6148*	*12.7288*	*12.1838*	*13.1320*	*12.5269*
Equity Volatility	Unconditional	Conditional			
State 1 financing	0.3678	0.4787	0.4679	0.3413	0.3345
State 2 financing	0.3760	0.4912	0.4793	0.3489	0.3414
State 3 financing	0.4169	0.5538	0.5362	0.3870	0.3760
State 4 financing	0.4301	0.5741	0.5546	0.3993	0.3870
Unconditional	*0.4141*	*0.5495*	*0.5323*	*0.3843*	*0.3735*

Table 3: Nominal Risk Implications for Asset Prices

This table presents the impact of nominal risk on corporate asset prices. Panel A shows how the risk of changes in the inflation rate affects a firm's equity value, its volatility and risk premium, as well as its debt value. Panels B and C disentangle nominal risk into two separate effects: deflation risk and the risk of facing high inflation. In presence of nominal risk, the inflation rate is $(\mu_D, \mu_I) = (-6.12\%, 5.92\%)$, while it is $(\mu_D, \overline{\mu}) = (-6.12\%, 2.50\%)$ in the case of deflation risk only, and $(\overline{\mu}, \mu_I) = (2.50\%, 5.92\%)$ in the case of high inflation risk only. Reported results are in percentage point difference compared to the asset pricing predictions obtained without uncertainty about the nominal state (i.e., the inflation rate is constant at $\overline{\mu} = p_D\mu_D + p_I\mu_I = 2.50\%$). For comparison purposes, predictions in the case without nominal risk are obtained with the same debt and default policies as those of the baseline firm described in Table 2. Reported differences in debt and equity values are in relative terms, whereas those for equity volatility and risk premium are in absolute terms. Each column reports model predictions for a different current state of the economy. The parameter values of the model are reported in Table 1 and discussed in Section 3.

	Unconditional	State 1 Recession & Deflation	State 2 Recession & Inflation	State 3 Boom & Deflation	State 4 Boom & Inflation
Panel A: Nominal risk					
Equity value	-0.75	-3.67	0.35	-3.12	0.21
Equity volatility	0.27	1.70	-0.26	1.14	-0.09
Equity risk premium	0.01	0.12	-0.03	0.03	-0.01
Debt value	0.60	3.51	-0.80	4.01	-0.69
Panel B: Deflation risk					
Equity value	-23.2	-27.9	-24.7	-24.4	-21.7
Equity volatility	8.2	13.1	10.9	8.2	6.8
Equity risk premium	0.4	1.0	0.9	0.2	0.2
Debt value	29.4	29.2	26.0	32.8	29.1
Panel C: Inflation risk					
Equity value	15.6	17.1	18.1	14.3	15.1
Equity volatility	-4.6	-6.1	-6.5	-3.8	-4.1
Equity risk premium	-0.2	-0.5	-0.5	-0.1	-0.1
Debt value	-21.3	-19.4	-20.5	-20.8	-21.9

Table 4: Cross-Sectional Implications of Nominal Risk

This table reports cross-sectional predictions of the impact of nominal risk on asset prices. The different panels show how the risk of changes in the inflation rate affects firm asset prices for different levels of leverage. Panel A presents the results for the firm value, Panel B for equity value, Panel C for debt value, Panel D for the equity return volatility, and Panel E for the equity risk premium (in basis points). Reported results are in percentage point difference compared to the asset pricing predictions obtained without uncertainty about the nominal state (i.e., the inflation rate is constant at $\overline{\mu} = p_D\mu_D + p_I\mu_I = 2.50\%$). For comparison purposes, predictions in the case without nominal risk are obtained with the same debt and default policies as those of the baseline firm described in Table 2. Reported differences in debt and equity values are in relative terms, whereas those for equity volatility and risk premium are in absolute terms. The parameter values of the model are reported in Table 1 and discussed in Section 3.

	Low Leverage (Liquid. Value = 10%)		**Baseline** (Liquid. Value = 50%)		**High Leverage** (Liquid. Value = 90%)		**High-minus-low Leverage effect**	
Current state	Rec.	Exp.	Rec.	Exp.	Rec.	Exp.	Rec.	Exp.
Panel A: Relative Effect on Firm Value (%)								
Financing state								
Recession	-0.12	-0.08	-0.13	-0.08	-0.12	-0.08	0.01	0.00
Expansion	-0.23	-0.16	-0.23	-0.17	-0.18	-0.14	0.05	0.02
Panel B: Relative Effect on Equity Value (%)								
Financing state								
Recession	-0.47	-0.42	-0.63	-0.57	-0.92	-0.90	-0.45	-0.48
Expansion	-0.61	-0.55	-0.78	-0.73	-0.95	-1.02	-0.35	-0.47
Panel C: Relative Effect on Debt Value (%)								
Financing state								
Recession	0.92	1.12	0.79	0.99	0.44	0.64	-0.48	-0.48
Expansion	0.55	0.78	0.43	0.65	0.13	0.31	-0.42	-0.47
Panel D: Effect on Equity Volatility (%)								
Financing state								
Recession	0.16	0.12	0.23	0.18	0.42	0.40	0.26	0.28
Expansion	0.22	0.17	0.31	0.26	0.59	0.61	0.38	0.44
Panel E: Effect on Equity Risk Premium (bps)								
Financing state								
Recession	1.06	0.28	1.27	0.32	1.15	0.22	0.10	-0.05
Expansion	1.25	0.32	1.36	0.32	0.39	0.00	-0.86	-0.32

Table 5: Nominal Risk, Optimal Capital Structure, and Financial Leverage

This table reports the impact of nominal risk on a firm's optimal capital structure and financial leverage. The results compare the predictions of the full model, which incorporates the risk of being in high inflation or in deflation, with those obtained without uncertainty about the nominal state (i.e., the inflation rate is constant at $\overline{\mu} = p_D \mu_D + p_I \mu_I = 2.50\%$). The results are presented for either an optimal or an exogenous capital structure. In the latter case, the debt coupon is set equal to the one obtained in the full model. Panels A, B, and C provide the results for various liquidation values to reflect different optimal capital structure choices. The parameter values of the model are reported in Table 1 and discussed in Section 3.

	Full Model			**No Nominal Risk** Endogenous Coupon			**No Nominal Risk** Exogenous Coupon		
	Coupon	Leverage		Coupon	Leverage		Coupon	Leverage	
Current state		Rec.	Exp.		Rec.	Exp.		Rec.	Exp.
Panel A: Low Leverage (Liquidation Value = 10%)									
Financing state									
Recession	0.49	25.83	22.61	0.52	27.01	23.64	0.49	25.56	22.34
Expansion	0.67	33.80	29.85	0.71	35.23	31.11	0.67	33.54	29.57
Panel B: Baseline (Liquidation Value = 50%)									
Financing state									
Recession	0.70	35.93	31.65	0.74	36.99	32.57	0.70	35.61	31.31
Expansion	0.96	46.16	41.07	1.00	47.39	42.17	0.96	45.86	40.74
Panel C: High Leverage (Liquidation Value = 90%)									
Financing state									
Recession	1.32	60.09	53.96	1.34	60.66	54.43	1.32	59.77	53.57
Expansion	1.74	72.62	66.24	1.92	76.97	70.57	1.74	72.41	65.94

Table 6: Fama & French Factor Alphas of 10 Portfolios sorted on Duration

This table reports equally-weighted monthly excess returns for ten portfolios sorted inflation and deflation betas with OLS standard errors in parentheses. We sort all common stocks listed on NYSE, AMEX, and NASDAQ at the end of each month from January 1931 to December 2014 into deciles based on their inflation beta in Panel A and their deflation beta in Panel B using weighted least squares regressions with expanding windows.

	Panel A. Inflation Betas									
	Low β_inf (1)	I2 (2)	I3 (3)	I4 (4)	I5 (5)	I6 (6)	I7 (7)	I8 (8)	I9 (9)	High β_inf (10)
Mean return	13.84	15.56	15.88	15.51	15.85	15.38	15.82	15.38	17.14	21.38
S.E.	(3.06)	(2.43)	(2.25)	(2.19)	(2.10)	(2.17)	(2.30)	(2.47)	(2.86)	(4.02)
	Panel B. Deflation Betas									
	Low β_def (1)	D2 (2)	D3 (3)	D4 (4)	D5 (5)	D6 (6)	D7 (7)	D8 (8)	D9 (9)	High β_def (10)
Mean return	20.56	17.14	16.33	15.78	15.86	15.57	15.76	15.60	15.70	15.74
S.E.	(3.14)	(2.37)	(2.20)	(2.15)	(2.16)	(2.18)	(2.30)	(2.44)	(2.74)	(3.42)

Figure 1: Nominal price variations in the U.S., 1913-2015.

This figure illustrates the fluctuations in nominal prices in the U.S. We consider monthly series of the producer price index (PPI) and compute the year-on-year percent (YoY) changes. Periods of deflation are identified in red, whereas the grey areas denote the NBER recessions. The data are retrieved from Robert Shiller's website.

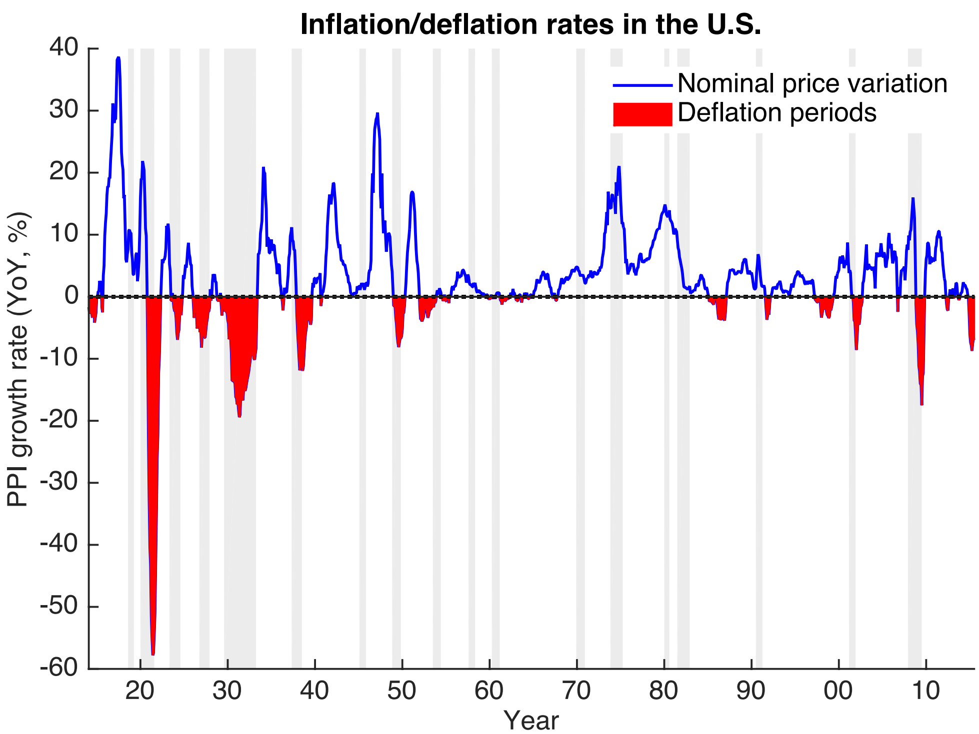
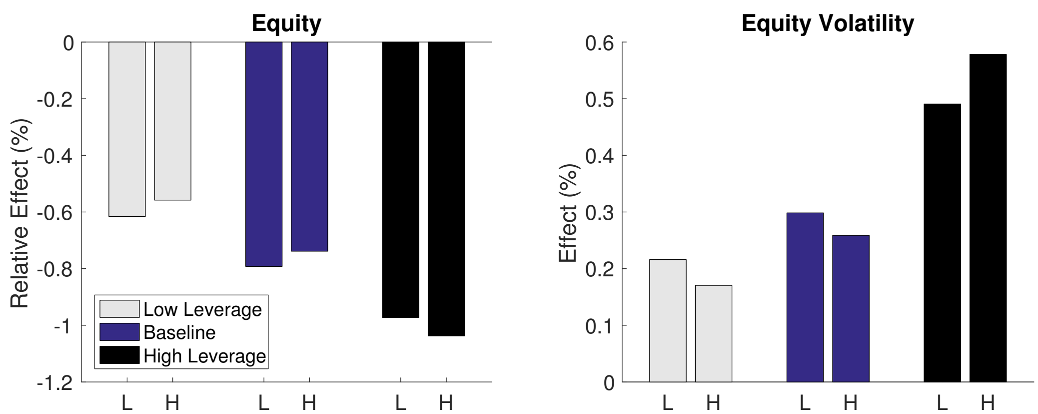


Figure 2: Conditional Impact of Nominal Risk on Equity

This figure illustrates how nominal risk affects a firm's equity value and its volatility across different states of the economy. The top two panels shows the predictions by real economic conditions, as characterized by periods of recession (L) or expansion (H). The bottom two panels display the predictions across the different nominal states of the economy, as characterized by periods of deflation (D) and high inflation (I). Reported results are in percentage point difference compared to the predictions obtained without uncertainty about the nominal state (i.e., the inflation rate is constant at $\overline{\mu} = p_D \mu_D + p_I \mu_I = 2.50\%$). For comparison purposes, predictions in the case without nominal risk are obtained with the same debt and default policies as those of the baseline firm described in Table 2. Reported differences are in relative (absolute) terms for equity value (volatility). Low and high leverage predictions are obtained with liquidation values set at 10% and 90%, respectively. The parameter values of the model are reported in Table 1 and discussed in Section 3.

((a)) Conditional on the real state



((b)) Conditional on the nominal state

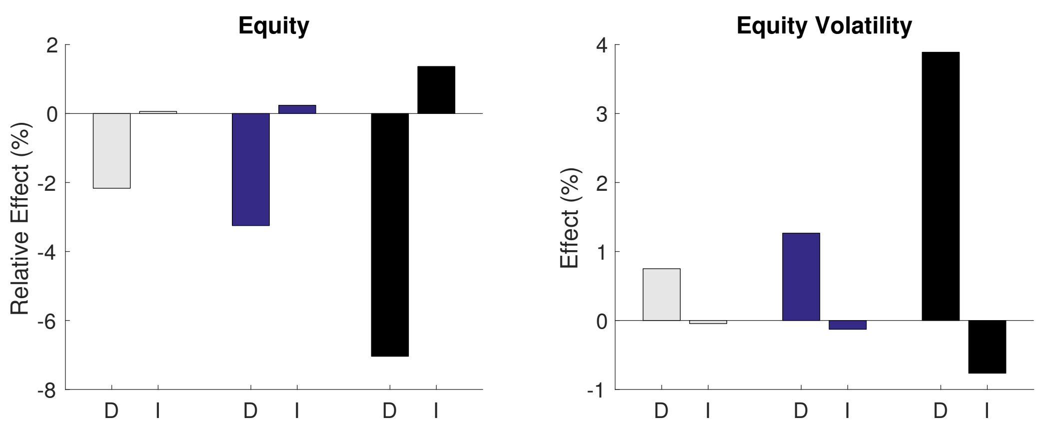


Figure 3: Nominal Risk Impact on Equity by Financial Leverage

This figure illustrates the equity sensitivity to nominal risk for different levels of financial leverage. The top two panels show how the risk of changes in the inflation rate affects a firm's equity value and its volatility. The bottom two panels disentangle nominal risk into two separate effects: deflation risk (*Defl*) and the risk of facing high inflation (*Infl*). In presence of nominal risk, the inflation rate is $(\mu_D, \mu_I) = (-6.12\%, 5.92\%)$, while it is $(\mu_D, \overline{\mu}) = (-6.12\%, 2.50\%)$ in the case of deflation risk only, and $(\overline{\mu}, \mu_I) = (2.50\%, 5.92\%)$ in the case of high inflation risk only. Reported results are in percentage point difference compared to the predictions obtained without uncertainty about the nominal state (i.e., the inflation rate is constant at $\overline{\mu} = p_D\mu_D + p_I\mu_I = 2.50\%$). For comparison purposes, predictions in the case without nominal risk are obtained with the same debt and default policies as those of the baseline firm described in Table 2. Reported differences are in relative (absolute) terms for equity value (volatility). Low and high leverage predictions are obtained with liquidation values set at 10% and 90%, respectively. The parameter values of the model are reported in Table 1 and discussed in Section 3.