Tuning in RBC Growth Spectra

Szilard Benk* Tamas Csabafi† Jing Dang‡ Max Gillman§
Michal Kejak¶

February 16, 2017

Abstract

For US postwar data, the paper explains an array of RBC puzzles by adding to the standard RBC model external margins for both physical capital and human capital, and examining model fit with data across business cycle (BC) and low frequency (LF) as well as Medium Cycle (MC) windows. The model results in a goods sector productivity shock with a 7500 times smaller variance than the standard RBC model, implying greatly improved amplification of the shock. In addition, output growth persistence autocorrelation profiles are modeled as in data, thus improving upon the propagation puzzle. The model produces a consumption-output ratio as in the business cycle data, a labor share of output that is countercyclic as in data, and human capital investment time that is countercyclic as in data. Also the capacity utilization rate is procyclic within BC, LF and MC windows as in data; including labor moments, a wide array of moments are explained for correlations, volatilities and growth persistence across these business cycle and lower frequency windows. Using a metric of fit, along with a uniform grid search, measures of fit are presented by window and category. In the BC window, key correlations have only an average 15% deviation from the data moments; the LF growth persistence has only an average 8% deviation from the data moments.

Keywords: Real Business Cycle, amplification, growth persistence, data moments, human capital.

JEL Code: E13, E32, F10, O41.

*International Monetary Fund (IMF).
†University of Missouri at St. Louis.
‡SGCC, China.
§University of Missouri at St. Louis.
¶CERGE-EI, Czech Republic.

1 Introduction

Real business cycle (RBC) models face many puzzles, such as matching the fundamental consumption-output ratio data, improving shock amplification, matching output growth propagation, and matching labor data. The challenge now also includes explaining the unusual US post-2008 data of a deep recession and prolonged, low frequency, below-trend, recovery. One approach to this is to step back to the overview of RBC problems as given by King & Rebelo (2000). They peel back the RBC edges by revealing in particular how adding external margins for labor and capital (variable utilization rates) allows a reduction in the variance of the total factor productivity (TFP) shock. They argue that adding external margins provides better amplification with "much smaller shocks", that these shocks can be measured in a way similar to the backing out of shocks in a model with home production by Ingram et al. (1997), and that explaining low frequency fluctuations are important.

This paper contributes a cohesive improvement on an array of puzzles across frequencies by adding the two external margins: through a variable physical capital utilization rate and a human capital investment sector (a "home production" sector that enables endogenous growth). While including a productivity shock to the human capital sector as well as to the goods sector, the paper exploits the method of backing out the shocks from data series by additionally demanding that the model calibration has the same variance-covariance matrix as the backed-out productivity shocks (Benk et al., 2005). From the three extensions of having both capital and labor external margins, forcing the model to be consistent with the backed-out shock properties, and including a human capital sectoral shock, the first startling result is a huge "amplification" of the goods sector TFP productivity shock. Rather than the dynamics of the model being driven by a large variation in the TFP shock, with a coincident weak shock amplification, instead the goods sector productivity shock has a variance 7500 times smaller than the 0.007 King & Rebelo (2000) standard, implying strong shock amplification.[1]

Second, while dramatically lowering the average amplitude of the goods sector TFP shock, the model captures key moments across both low and business cycle frequencies. The results follow from a human capital investment sector productivity shock that has a mean some 25 times less than the goods sector productivity shock, and a three times smaller variance, giving it a 22000 times smaller variance than the standard RBC TFP shock variance of 0.007. This small but potent shock to the growth rate of human capital creates a permanent income effect that does not overwhelm the goods sector temporary income TFP shock, but instead supplements it so as to allow the latter to be greatly reduced. In effect, rather than founding the model dynamics upon only a temporary TFP effect, the adding of the human capital sector shock, with a near-one correlation with the goods sector

[1] In comparison, King & Rebelo (2000) report a reduction from 0.007 to 0.001 from adding external margins.

TFP shock, boosts the temporary income shock with just enough of a permanent income effect as needed to explain the basic moments at the heart of the RBC challenges, including includes the consumption-output puzzle. Consumption varies by more because it follows the permanent income hypothesis of consumption built within Ramsey-based models by which consumption follows permanent income which in turn now also rises by more. To demonstrate this, the paper backs out of the model's historical consumption-output ratio and shows its surprisingly good reproduction of US data,

Extending the well-known but minimalist two-sector approach, several data puzzles simultaneously are "resolved" in as succinct a way as can be found here. Besides strong shock amplification and matching well the many correlation and volatility moments, the paper contributes as well a propagation of the growth rate of output, consumption and investment as found in the autocorrelation profiles of the data (Benhabib et al. 2006). In addition, with a variable physical capital utilization rate, similar to Hansen & Prescott (2005), the model's "labor's share of income is counter-cyclical, as it is in US data" (Hansen and Prescott, p. 850). We also find an asymmetric business cycle in the TFP backed-out shock from a Harding & Pagan (2002) and Hansen and Prescott perspective. Other notable puzzle features are that: an initial slightly negative labor impulse feature results, as has been reported in data and in theory (Benhabib et al. 2006, Gali 1999); human capital investment time is a countercyclical as suggested by data (Dellas & Sakellaris 2003); a Chari et al. (2007)-related labor wedge is explained using human capital time similar to McGrattan's (2016) result using intangible capital investment time; and the model captures procyclic physical capital utilization rates, as demonstrated here to be consistent with data.

The best-case "nesting" model for accomplishing the simultaneous puzzle-tasks is the general model with the two external margins, and so named Model 2. Results are compared to a special case called Model 1 that fixes the physical capital utilization rate equal at one. Model 1 does reasonably well except for matching the growth rate autocorrelation profiles, known as the propagation puzzle. Model 1 also has less shock amplification with only a 46 times smaller goods sector TFP shock variance, as compared to 0.007, rather than the 7500 times decrease of Model 2. Results are presented for a standard RBC model without either external margin (which can be specified as a special case of Model 2 in which the human capital growth rate exogenous). This model is seen to fall well short on both volatility and propagation moments.

Model 2 results are presented for an extended set of the standard cyclic correlations, volatilities and the output growth autocorrelation with high model fit not only within the RBC window but also at the low frequency and the Comin & Gertler (2006) "Medium Term Cycle" frequency that combines high, business cycle and low frequency windows.[2] Model fit is measured by using an extension of the Jermann (1998) metric, which provides

[2] The windows are similar to those in Hansen & Ohanian (2016) except that they combine the high frequency and business cycle windows.

an aggregate numeric value of the average fractional deviation of the model's simulated moments from the data's moments for any target set. This quantifies the degree of success in the moment comparison, which has been used since Kydland & Prescott (1982) and Long & Plosser (1983), so as to facilitate choice of the best calibration. The new metric design can be applied using any model.

The aggregate average metric is also broken down for results within each of the four frequencies (high, business cycle, low and Medium), as well as separately for each correlations, volatilities and growth persistence. As a preview, consider that in the business cycle window, the average aggregate metric for Model 2 is a small 15 percent deviation from the data moments. In the low frequency window, the Model 2 growth persistence shows an even smaller 8 percent deviation from data. For the broadest set of 59 targets, which include the high frequency window, Model 2 results in an aggregate average 46 percent deviation of model moments from data moments.

The paper also contributes how the backed-out economy wide shock compares to the standard RBC TFP shock, with historical data as in Nolan & Thoenissen (2009), and with a focus given to post-2008 data. The backed-out goods sector TFP shock rather closely tracks the standard Solow residual except after the Great Recession. Instead of falling continuously after 2010 as does the Solow residual, the model's shock begins rising as does actual US GDP growth, while in addition both the model's backed out shock and the US economy's GDP growth remain below trend. This may reflect the King & Rebelo (2000) and McGrattan (2015) criticism that the Solow residual is not an exact measure of the economy's TFP; the smaller shocks of the two sector economy in this paper may offer some improvement on a TFP measure.[3]

1.1 Related Literature

Cogley & Nason (1995), Rotemberg & Woodford (1996), Perli & Sakellaris (1998) and Benhabib et al. (2006) highlight the weak internal propagation of the standard RBC model in terms of matching the data profile of a falling output growth persistence. Benhabib et al. (2006) match this profile by adding additional physical capital investment sectors with independent shocks; at the same time they find an initially negative labor impulse response in support of Gali (1999), who in turn argues that data is consistent with a negative labor TFP response rather than the standard RBC positive labor TFP response.[4] Our paper achieves the propagation explanation, along with the initially negative (slightly) labor impulse, in a way similar to Perli & Sakellaris (1998) who add a human capital investment sector

[3] King & Rebelo (2000) call the smaller resulting residuals when also using a second, home, production sector the "Crucini residuals" (in their footnote 60); McGrattan (2015) finds adding a second investment sector "quantitatively important for analyzing U.S. aggregate fluctuations".

[4] See Benhabib et al. (2006) Figures 1 and 5 for the match of output growth's autocorrelation profile and see Figure 7 for their generation of a Gali (1999) type labor impulse response.

for improved propagation. And like Maffezzoli (2000), who also adds a human capital investment sector and a second independent shock for matching international RBC data moments, we match moments better by using the human capital sectoral shock, but one that is correlated (near one) with the goods sector, and without externalities. Like DeJong & Ingram (2001) who use a human capital investment sector along to model a countercyclical human capital investment time, which they support with evidence that is consistent with both Perli & Sakellaris (1998) and Dellas & Sakellaris (2003), we also use this sector in that way but without including as well additional investment sectors as does DeJong & Ingram (2001).

McGrattan (2015) explains the RBC "labor wedge" with labor spent in an intangible capital investment sector, with multiple output sectors and shocks, and an economy-wide TFP shock; Hansen & Ohanian (2016) use correlated sectoral shocks but not a separate investment sector; and Benhabib & Wen (2004) uses demand shocks. Like McGrattan (2015), the paper here analogously fills this wedge with the extra investment sector time except that it is human capital rather than intangible capital investment. And abstracting from the numerous sector specific shocks, this paper is closer to Maffezzoli (2000) in using only two, productivity-only, sectoral shocks, here being the two highly correlated goods and human capital investment shocks with a covariance-variance matrix that breaks the mold of identical shocks that Maffezzoli (2000) uses to good effect.

Relative to the seminal literature on external margins, the paper's external labor margin builds upon Hansen (1985), Rogerson (1988), Benhabib et al. (1991), Greenwood & Hercowitz (1991), Lucas (1988) and Perli (1998). The paper's external physical capital margin extends Greenwood et al. (1988) by adding a variable physical capital utilization rate across both sectors. This in turn builds upon the one sector RBC economies of Cooley et al. (1995) and Hansen & Prescott (2005) with a variable capacity utilization rate and a focus on business cycle asymmetries, as is the focus of Harding & Pagan (2002).

King et al. (1988) extend cyclic analysis to growth spectra at the low frequency, as does Comin & Gertler (2006); this paper is closer to the latter approach but an appendix shows its consistency with the former's. Kehoe and Prescott (2007) help explain depressions with RBC productivity shocks; and Hansen & Ohanian (2016) extend the RBC model to multiple sectors for explaining low frequency data; our paper differs from the latter two by using two sectors, as opposed to one-sector our numerous sector models respectively. Buera & Moll (2016) use financial frictions and heterogeneous agents to explain RBC "wedges"; our paper extends a representative agent approach while including the financial crisis period of the Great Recession.[5]

The paper differs these latter two with an extended array of moment targets, the use of general equilibrium trade theory for intuition, the iterative convergence of the calibration to

[5] And as also found in our previous work: Benk et al. (2008, 2010), and Basu et al. (2012).

the backed-out data-based shocks, and use of human capital and variable capacity utilization. Grossman et al. (2016) focus on long term model properties with human capital, but without application to data. Christiano et al. (2001) solves basic RBC puzzles, including the equity risk premium that this paper does not address, but this paper instead uses homothetic utility and production without an adjustment cost of the physical capital stock. McGrattan & Prescott (2014) provide alternative RBC approaches to using human capital.

Section 2 describes the full model and balanced growth path behavior and shocks to it. Section 3 describes the calibration, the backed-out shocks and impulse responses. Section 4 presents moment results, Section 6 discusses the dynamics of the two sector model as related to a dual Stolper & Samuelson (1941)-Rybczynski (1955) theorem as proved in the Appendix C, and Section 7 concludes.

2 The Model

The general model with the two external margins added is called Model 2. Model 1 is the special case that sets the physical capital utilitzation rate to 1. Also included in several of the model comparisons below is the case with neither external margin, which is the standard one-sector RBC model as simply referred to as the RBC case (using the King and Plosser, 2000, calibration).

For the general model, the representative agent maximizes its expected sum of discounted utility U. The agent derives utility from consumption, c_t, leisure, x_t, and a function $s(u_t) \equiv (1-u_t)^B$, where $u_t \in [0,1)$ is the physical capital capacity utilization rate at each time period t. With $A \in R_+$, $B \in R$, and $\sigma \in R_+$, the time t period utility is given by

$$U(c_t, x_t, u_t) = \frac{\left[c_t x_t^A (1-u_t)^B\right]^{1-\sigma} - 1}{1-\sigma}, \tag{1}$$

which satisfies the necessary conditions for the existence of a balanced growth path (BGP) equilibrium.[6] The unused physical capital capacity utilization, in terms of $1-u_t$, is included in utility to be symmetric with including leisure x_t, which is the unused human capital utilization rate. The symmetry follows in the sense of DeJong et al. (1996) such that the human capital capacity utilization rate is $1-x_t$, while u_t is the physical capital utilization rate, although they exclude u_t from the utility function.

The case of $B = 0$, so that u_t drops out of the utility function, is fulled allowed as a possible calibration choice; B is allowed in general to be positive or negative. In the calibration below, it ends up robustly negative, but only slightly so, as is consistent with the negative analogue found in Otani (1996). This implies a utility gain from more fully

[6] For more, see King et al. (1988).

utilizing physical capital. In Model 1, $(1-u_t)^B = 1$, so that the a variable utilization rate is eliminated everywhere from the model, is implicitly set to one, and no longer affects utility. This is analogous to a goods-only model without leisure or time allocation decisions, in which leisure time does not enter utility and labor time is implicitly equal to one.

The representative agent time endowment for each period t, is allocated to l_{gt}, the fraction of time spent in goods production, to l_{ht}, the fraction of time spent in human capital investment production, and to x_t, leisure:

$$1 = x_t + l_{gt} + l_{ht}. \tag{2}$$

Physical capital investment, i_{kt}, determines the capital stock k_t accumulation as in DeJong et al. (1996):

$$k_{t+1} = k_t - \delta(u_t) k_t + i_{kt}, \tag{3}$$

where $\delta(u_t)$ is a function. It is the endogenous depreciation rate of physical capital that depends on u_t. The functional form for $\delta(u_t)$ is assumed to be[7]

$$\delta(u_t) = \frac{\delta_k}{\psi} u_t^{\psi}, \tag{4}$$

with $\psi > 1$ and $\delta_k > 0$; a faster rate of utilization results in a higher rate of depreciation. It follows that $\delta'(u) > 0$ and $\delta''(u) > 0$ so that the marginal cost of utilization of the physical capital stock is increasing in the utilization rate.

Denote by y_t the real goods output that corresponds to the data notion of GDP. For the goods production function A_g is a positive factor productivity parameter, z_t^g the total factor productivity shock, v_{gt} the share of the physical capital stock being allocated to the goods sector and $v_{gt}u_t k_t$ the amount of physical capital in the goods sector that is utilized for production purposes. Let h_t denote the stock of human capital at the beginning of time period t; then $l_{gt}h_t$ represents the effective labor input, or the share of human capital used in goods production. With $\phi_1 \in [0,1]$ denoting the share of physical capital used in goods production, the output function is

$$y_t = A_g e^{z_t^g} (v_{gt} u_t k_t)^{\phi_1} (l_{gt} h_t)^{1-\phi_1}. \tag{5}$$

[7] Others with similar endogenous depreciation rate as a function of the utilization rate include Greenwood et al. (1988), DeJong et al. (1996), and Benhabib & Wen (2004).

The human capital stock is then accumulated over time according to the following standard law of motion,

$$h_{t+1} = (1-\delta_h)h_t + i_{ht}, \tag{6}$$

where δ_h is the assumed constant depreciation rate of human capital and i_{ht} is the per period investment in human capital. For the human capital investment technology, A_h denotes a positive factor productivity parameter in the human sector; z_t^h represents the sectorial productivity shock in natural logarithms; $v_{ht} = 1 - v_{gt}$ is the remaining fraction of physical capital allocated to the human sector; and $v_{ht}u_t k_t$ is the amount of physical capital in the human sector that is utilized for human investment production. With $\phi_2 \in [0,1]$ being the share of physical capital in the human capital investment, the production function is

$$i_{ht} = A_h e^{z_t^h}(v_{ht}u_t k_t)^{\phi_2}(l_{ht}h_t)^{1-\phi_2}. \tag{7}$$

2.1 Shock Structure

In the economy are two random shocks following first-order autoregressive processes:

the goods productivity shock z_t^g, where

$$z_t^g = \rho_g z_{t-1}^g + \varepsilon_t^g, \quad 0 < \rho_g < 1, \tag{8}$$

and the human capital investment sector productivity shock z_t^h, where

$$z_t^h = \rho_h z_{t-1}^h + \varepsilon_t^h, \quad 0 < \rho_h < 1 \tag{9}$$

and the innovations are normally distributed according to

$$\begin{pmatrix} \varepsilon_t^g \\ \varepsilon_t^h \end{pmatrix} \sim N(\mathbf{0}, \mathbf{\Sigma}), \tag{10}$$

where the general structure of the second-order moments is the variance-covariance matrix $\mathbf{\Sigma}$, with individual variances denoted by σ_g^2 and σ_h^2. This allows for any degree of covariance between the shocks.

Given no externalities, the competitive equilibrium of the economy coincides with the result of the social planner's problem, which can be stated as[8]

$$\max_{\{c_t,l_{gt},l_{ht},x_t,v_{gt},v_{ht},u_t,k_{t+1},h_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \frac{[c_t x_t^A (1-u_t)^B]^{1-\sigma} - 1}{1-\sigma}, \tag{11}$$

subject to (2)-(10).

2.2 Social Planner Equilibrium

Definition 1 *A general equilibrium of this model is a set of contingent plans* $\{c_t,\ k_{t+1},\ h_{t+1},\ v_{gt},\ v_{ht},\ u_t,\ x_t,\ l_{gt},\ l_{ht}\}$ *that solve the social planner's maximization problem in (11) for the initial endowment* $\{k_0,\ h_0\}$ *and exogenous stochastic technology processes* $\{z_t^g,\ z_t^h\}$*, with initial conditions* $\{z_0^g,\ z_0^h\}$ *and variance-covariance matrix* $\mathbf{\Sigma}$*.*

Definition 2 *A deterministic balanced growth path (BGP) equilibrium of this model is a set of paths* $\{c_t,\ k_{t+1},\ h_{t+1},\ v_{gt},\ v_{ht},\ u_t,\ x_t,\ l_{gt},\ l_{ht}\}$ *that solve the central planner's maximization problem in (11) for the initial endowment* $\{k_0,\ h_0\}$ *and exogenous technology parameters* $\{z_t^g = 0, z_t^h = 0\}$*, such that* $\{c_t, k_t, h_t\}$ *grow at a common trend, and* $\{v_g, v_h, u, x, l_g, l_h\}$ *are constant.*

Appendix A.1 presents the dynamic equilibrium conditions and Appendix A.2 the BGP equilibrium conditions.

Proposition 3 *The social planner equilibrium is the same as the decentralized representative agent's equilibrium.*

Proof. This follows immediately as there are no (Lucas, 1988; Maffozolli, 2000) externalities, as in Gomme (1993); Appendix A.3 shows that the same equilibrium conditions result. ■

2.3 Balanced Growth Path Behavior

There are two main sets of margins: the intratemporal and intertemporal. For the former, the marginal rate of substitution between goods and leisure as usual equal to the marginal product of labor, denoted by w_t, or $\frac{A}{x_t}\frac{c_t}{h_t} = w_t$. For the latter margin there are two capital stocks, k_t and h_t, and so two intertemporal "Euler" relations.

The Euler conditions for k_t and h_t in terms of the BGP growth rate g, time preference $\beta \equiv \frac{1}{1+\rho}$, and the marginal products and the utilization rates of each respective capital, . With the goods sector marginal product of physical capital given as $MP_G^k \equiv$

[8] The first order conditions of the representative agent can be found in Appendix A.1.

$\phi_1 A_g \left(\frac{l_g h_t}{v_g u k_t}\right)^{1-\phi_1}$, and the human capital investment sector marginal product of human capital given as $MP_H^h \equiv (1-\phi_2)\, A_h \left(\frac{v_h u k_t}{l_h h_t}\right)^{\phi_2}$, the BGP intertemporal conditions are

$$1+g = \left[\frac{1+(u)\, MP_G^k - \frac{\delta_k}{\psi} u^{\psi}}{1+\rho}\right]^{1/\sigma}; \tag{12}$$

$$1+g = \left[\frac{1+(1-x)\, MP_H^h - \delta_h}{1+\rho}\right]^{1/\sigma}. \tag{13}$$

The consumer equalizes the return on capitals across sectors along the BGP so that $(u)\, MP_G^k - \frac{\delta_k}{\psi} u^{\psi} = (1-x) MP_H^h - \delta_h$.

On the firm side, the relative price of human capital investment to goods output is denoted by p_{ht}, and defined in terms of the ratio of the shadow value of human to physical capital. Therefore $p_{ht} \equiv \frac{\chi_t}{\lambda_t}$, with the respective shadow values given in Appendix equations (5) and (6) of the decentralized problem, respectively. Using the equilibrium conditions, the relative output price can be expressed as a constant multiplied by the stationary BGP factor input ratio in either the goods or human capital investment sector respectively, so that the time index on p_{ht} may be dropped without loss of generality:

$$p_h = \left[\frac{A_g}{A_h}\right] \left[\frac{1-\phi_1}{1-\phi_2}\right]^{1-\phi_2} \left[\frac{\phi_1}{\phi_2}\right]^{\phi_2} \left[\frac{v_g u k_t}{l_g h_t}\right]^{\phi_1-\phi_2}, \tag{14}$$

$$= \left[\frac{A_g}{A_h}\right] \left[\frac{1-\phi_1}{1-\phi_2}\right]^{1-\phi_1} \left[\frac{\phi_1}{\phi_2}\right]^{\phi_1} \left[\frac{v_h u k_t}{l_h h_t}\right]^{\phi_1-\phi_2}. \tag{15}$$

Now using factor rental prices, whereby along the BGP, $w \equiv (1-\phi_1) A_g (\frac{v_{gt} u_t k_t}{l_{gt} h_t})^{\phi_1}$ and $r \equiv \phi_1 A_g (\frac{v_{gt} u_t k_t}{l_{gt} h_t})^{\phi_1 - 1}$, it is also true that the stationary factor rental price ratio is proportional to the stationary sectoral input ratios:

$$\frac{w}{r} = \frac{1-\phi_1}{\phi_1} \frac{v_{gt} u k_t}{l_{gt} h_t} = \frac{1-\phi_2}{\phi_2} \frac{v_h u k_t}{l_h h_t}. \tag{16}$$

This implies that the relative price of output is a simple function of the input rental price ratio. And since the goods sector is more physical capital intensive than the human capital investment sector, it is true that $\phi_1 > \phi_2$, and that p_h rises when the wage to interest rate ratio w/r rises:

$$p_h = \left[\frac{A_g}{A_h}\right] \left[\frac{1-\phi_1}{1-\phi_2}\right]^{1-\phi_2} \left[\frac{\phi_1}{\phi_2}\right]^{\phi_2} \left[\frac{\phi_1}{(1-\phi_1)} \frac{w}{r}\right]^{\phi_1-\phi_2}. \tag{17}$$

2.4 Dynamics

The actual Model 2 simulated shock involves a smaller magnitude of the human capital shock relative to the goods sector shock, rather than an equi-proportional shock, even though their near-one correlation (0.995) makes the two shocks act as, what will be termed here, the "economy-wide" shock. This results because the ratio $\frac{A_g}{A_h}$ is equal to 25 and because the variance in the human capital shock is lower. The consequent economy-wide positive shock acts as a "tempored" goods sector TFP shock, as tempored by the way in which an increased human capital productivity acts to send resources in the opposite direction, towards the human capital investment sector, relative to the goods sector TFP productivity shock, which directs resources towards the goods sector.

Given the temporing by the human capital sectoral shock, which adds a permanent income effect to the economy-wide shock, the goods sector shock is of much smaller variance than found typically. And the result of the economy-wide shock is to cause the input factor price ratio, $\frac{w}{r}$, to rise, to cause the relative price of human capital investment, p_h, to rise, and to induce resources towards the goods sector. Because leisure time falls significantly, the "economy-wide" shock also allows more labor time in the human capital investment sector. Given the correlated nature of the shocks, this impulse response behavior can be thought of in terms of beginning the movement from a trough in the business cycle back towards a peak, as output growth begins rising again, using the Pagan and Harding (2002) focus on output growth.

The magnitude of the change in $\frac{w}{r}$ is much smaller than that within the standard RBC model as the greater amplification of the shock requires less relative input price change to result. This coincides with a smaller impulse up in labor time in the goods sector compared to the standard RBC, and a bigger decrease in leisure time as compared to the standard RBC. In this way, the greater leisure decrease allows for increased labor time allocation in both the human capital investment sector and the goods sector.

Meanwhile economy-wide positive shock causes the goods sector share of physical capital to actually pulse downwards, as the human capital share of physical capital pulses up. This is because the shock causes the physical capital capacity utilization rate to rise significantly and so result in an overall increase in the employed physical capital within the goods sector. The upshot is that the physical to human capital rises across both sectors, in proportions dictated by equation (16), and the actual change in the factor input ratio is much smaller than in the standard RBC model. This means that the greater shock amplitude demands in effect a much smaller reallocation of resources in order to fits the business cycle facts, thus thereby greatly decreasing the stress normally put upon resources to reallocate much more dramatically as in the standard RBC model.

There results a much smaller demand on resources to move towards the now cheaper physical capital in production of sectoral outputs, as a result of both human capital and physical capital being increased by the economy-wide shock. Since the physical capital stock, in effect by the nature of the relative productivity factor magnitudes between sectors, or $\frac{A_g}{A_h}$, is increased by more than is human capital stock, and because of the increased utilization rate of physical capital, the relative price of human capital p_h falls. Over the business cycle and low frequencies, in fact as can be consistent with the equation (17) linking of p_h and w/r, the input price ratio w/r is countercyclic at the business cycle, low frequency and at the Medium Cycle, as is supported by data presented below.

A related, seemingly counter-intuitive, result is that the relative magnitude of the output increase of each sector, as a result of the economy-wide shock, is that the goods sector output (physical capital, or goods) rises by less than does the human capital sectors output (human capital), so that y_t/i_{ht} falls. This results because of the increased physical capital utilization rate that acts by itself to cause a greater amount of physical capital to surge in the goods sector relative to the human capital sector since the goods sector is more intensive in human capital. As a result, the economy-wide shock generates a procyclic expansion of the goods sector and a counter-cyclic movement of labor in the human capital sector results (as in empirical evidence).

Cumulating from these sectoral input and output changes, the economy-wide shock now pushes up consumption and output permanently because of the permanent effect of positively shocking the human capital growth rate. This compares to the temporary rise in both c and y of the standard RBC model that is now no longer a feature of the Model 2. Instead, the economy-wide shock causes consumption to move much more closely with output, thereby greatly minimizing the fundamental consumption-output puzzle of the standard RBC model. These results are seen below in the impulse responses.

3 Calibration, Impulse Responses, and Backed-out Shocks

By normalizing the variables that grow along the balanced growth path (BGP), and then log-linearizing all the equilibrium conditions of the two models around their normalized growth paths, two stochastic systems of linear equations result. Using the human capital stock , h_t, for normalization as in Benk et al. (2008, 2010), for both models the respective systems are solved for in terms of the state variable k_t/h_t and the two shock processes, z_t^g and z_t^h. The calibrated models are solved by the method of undetermined coefficients Uhlig (1998).[9]

3.1 Calibration

Table 1 presents the calibrated structural and exogenous shock parameters for Model 1 and 2. Table 2 presents the calibration grid ranges, in which 5,000 steps within the ranges were

[9]Please see Appendix A for solution methodology.

employed. For Model 1, there are 53 targets including two BGP equilibrium values (balanced growth rate and leisure) as BGP targets. For Model 2, there are 59 targets, of which three are BGP equilibrium values (adding the BGP physical capital utilization rate). The high target number resulted from experiments which found a better fit with more targets, but with a diminishing return to adding targets.

The calibration methodology of Jermann (1998) is modified and combined with the shock identification scheme of Benk et al. (2005) building on Ingram et al. (1997). For the shock identification, the data period is 1959Q1 to 12015Q4 for Model 1, and 1972Q1 to 2015Q4 for Model 2, with a shorter period for Model 2 due to the lack of physical capital utilization data before 1972Q1. Please see Appendix B for a data description.

To calibrate the models, first long-run BGP targets are restricted based on US data. Targets for Model 1 are the balanced growth rate of the economy, g, and leisure time, x, set at 0.0035 and 0.5 respectively, following Gomme & Rupert (2007) These imply the weight of leisure in the agent's utility function, A, and the scale parameter of the human capital investment sector, A_h through the intratemporal and the second intertemporal margins.

For Model 2 the BGP targets are leisure time, again set at 0.5, the growth rate g being 0.0035 for the shorter data period, and the physical capital utilization rate, u, which is 0.785 as calculated from US data. Following Gomme & Rupert (2007), the long-run value of the endogenous physical capital depreciation rate is set to 0.025. These targets imply the leisure preference, A, through the first intertemporal margin in equation (37), the utilization rate's weight, B, in the utility function through the second intratemporal margin in equation (33), the human sector scale parameter, A_h, via the second intertemporal condition in (38), and the depreciation parameter of the endogenous depreciation rate, δ_k, through the physical capital law of motion in (3).

To calibrate the remaining seven structural parameters for both models, a grid in a bounded parameter space is established with lower and upper bounds for parameters as set out in Table 2.[10] For each possible combination of the grid coordinates the models are solved with iterative convergence of the backed-out shock's properties to the model's assumed shock properties, as in Benk et al. (2005). This extends the method of Jermann (1998) by iterative convergence of the shocks and a mean normalization of the distance metric to transform each individual distance measure into percentage deviations of the simulated moments from the US data targets.[11]

The resulting metric is used by limiting the examination of the results to the top 200

[10] This uses similar features to *Bayesian* estimation by setting bounds with prior information; instead of the Bayesian estimation of a parameter within each set of bounds, the grid search here computes distance metrics uniformily across each bounded parameter space.

[11] The approach is alternative to use of a simulated annealing algorithm, which was also explored, but which gives a different calibration with each run. Simulated annealing is also embedded in Bayesian estimation of the calibration parameters. It gives a different calibration with each run because of its "temperature-gauge" property. Complete Matlab codes of the grid search approach as well as simulated annealing, both with iterative convergence of the model shocks to data, are available with detailed descriptions upon request.

best (lowest measures) metric, out of $9x10^{36}$ successfully convergent runs. There are a total of 59 US data based targets for Model 2 and 53 for Model 1. The lowest obtained metric for Model 2 was 0.41, while the one presented in the Tables has a value of 0.46; this can be interpreted as on average a 46% deviation of the 59 targets from their model-achieved values.[12]

For the grid ranges of Table 2, for each calibrated parameter of Models 1 and 2, the lower bound of the discount factor β is set to 0.95 and the upper bound to 0.99. The parameter for of the constant elasticity of substitution (CES) in utility is bounded between 0.40 and 2.00 as found for example in the quarterly estimates of Hall (1988) and Mehra & Prescott (1985) respectively. The share of physical capital in the goods producing sector, ϕ_1, has a lower bound of 0.30 and upper bound of 0.40, with the range for the scale parameter of the goods sector, A_g, residually set between 0.50 and 2.00.

In Model 1 the bounds for the constant depreciation rate of physical capital are set to 0.015 and 0.03, which coincide with annual rates of 6 percent at the lower bound and a 12 percent depreciation rate at the upper bound. In Model 2 the depreciation parameter δ_k is determined through the physical capital law of motion when given the long-run quarterly target depreciation rate of 0.025. Then the convexity parameter ψ is bounded between 2.00 and 4.00. The share of physical capital in human investment production, ϕ_2, is given a range of 0.08 and 0.29 in line with Jorgenson & Fraumeni (1991) and Jones et al. (2005). The bounds for the constant depreciation rate of human capital, δ_h, are set to 0.001 and 0.015, where estimates that serve as a basis are those of DeJong & Ingram (2001), Jorgenson & Fraumeni (1991), and Jones et al. (2005).

The range for the persistence parameters, ρ_g and ρ_h, of the goods and human capital sectorial shocks is identical with the lower bound set to 0.01 and the upper bound set to 0.99. In order to reduce computational intensity, the initial guess for each of the shock variances is set to an initial value of 0.007 as found in King and Rebelo (2000). The cross-correlation between these two sectorial shocks is given a range between -0.99 and 0.99, avoiding -1 and 1 because of the positive semi-definite requirement for solving the underlying models.

Within the defined grid, each point represents a possible combination of parameters, with all permutations investigated. Iterative convergence is imposed such that the shock parameter variance-covariance matrix that is assumed in the calibration is the same as that variance-covariance matrix of the backed-out shocks that result from use of US data. This identity between assumed shock parameters and backed-out shock parameters is implemented for each grid point following the Benk et al. (2005) methodology of using seemingly unrelated regression estimation to find the variance-covariance matrix $\mathbf{\Sigma}$ of the backed-out shocks.

Then a normalized vector distance metric is constructed for each grid point and used to

[12] We thank Viktor Huszar, DWO LLC., for the use of a massive parallel processing system.

select the best calibration from the entire set of grid points. The metric is constructed by using the simulation-based moment vector, denoted by Θ, along with the corresponding US data-based target moment vector, $\hat{\Theta}$. Denoting the distance metric by D, it is defined so as to give the average fractional deviation of the model moment from the data moment across all targeted moments. This is found by summing up each of the fractional deviations of model moment from data moment, and dividing by the total number of targeted moments; call the latter T. For each Model 1 and 2, with $z = 1, 2$, then the definition of D_z is $D_z \equiv \left(\sum_i \sum_j \sum_k \left|\hat{\Theta}_{ijk} - \Theta_{ijk}\right| / \left|\hat{\Theta}_{ijk}\right|\right) \Big/ T_z$, with $i = 1, 2, 3$ for the targeted moment categories of each 1) correlations, 2) volatilities, and 3) autocorrelation lags; $j = 1, ..., 5$ represents the four band-pass filtered frequencies (HF, BC, LF, MC) plus the unfiltered data used only for the autocorrelation lags (as in the literature); and $k(i, j, z)$ varies as it represents the number of targets used within each moment category and data frequency, for each of Models 1 and 2.[13]

The calibration and shock construction procedure yield a 7500 times smaller shock variance for the goods sector productivity shock and 22000 times smaller for the human capital investment sector shock , as reported in Table 1, and as compared to the standard RBC 0.007 (King & Rebelo 2000). This indicates improved amplification compared to standard one-sector RBC models.

Parameter	Description	Model 1	Model 2
β	Discount Factor	0.972	0.986
σ	CES Parameter	0.850	0.412
A	Weight of Leisure	1.11	1.10
B	Weight of Capacity Util.	−	−0.159
A_g	Scale Parameter of Goods Sector	1.65.	0.80
A_h	Scale Parameter of Human Sector	0.065	0.032
ϕ_1	Physical Capital Share in Goods Production	0.319	0.36
ϕ_2	Physical Capital Share in Human Investment	0.162	0.20
δ_k	Depreciation Parameter (Physical Capital)	0.018	0.19
ψ	Convexity of Endog. Depr. Rate	−	3.34
δ_h	Depreciation Rate of Human Capital	0.010	0.001
ρ_g	Auto-correlation of TFP	0.98	0.98
ρ_h	Auto-correlation of Human Shock	0.99	0.98
σ_g^2	Variance of TFP	$1.52x10^{-4}$	$9.4x10^{-7}$
σ_h^2	Variance of Human Productivity Shock	$1.47x10^{-4}$	$3.2x10^{-7}$
$\sigma_{g,h}$	Correlation of Shock Innovations	0.994	0.995

Table 1: Model 1 and 2 calibration parameter values.

3.2 Impulse Responses

Simulation is done using Uhlig's (1998) method, where the extra state variable of h_t is included along with k_t. So it is a standard extension as used in Benk et al. (2010); see also Mulligan and Sali-i-Martin (1993). The Online Appendix B provides all of the Matlab

[13] Alternatively, a 0.99 correlated metric is $D_{alt} = [(\hat{\Theta} - \Theta)/\hat{\Theta}]'\Omega[(\hat{\Theta} - \Theta)/\hat{\Theta}]$, where Ω is an identity matrix of the size of the number of targets k, and D_{alt} is a squared Euclidean distance; D_z in contrast is an average fractional deviation of model from data moments. D_{alt} is of interest as it is a special case of the Mahalanobis (1936) distance.

Parameter	Description	Grid Range	
		Model 1	Model 2
β	Discount Factor	$0.95 - 0.99$	$0.95 - 0.99$
σ	CES Parameter	$0.40 - 2.00$	$0.40 - 2.00$
A	Weight of Leisure	BGP^*	BGP^*
B	Weight of Capacity Util.	$-$	BGP^*
A_g	Scale Parameter of Goods Sector	$0.50 - 2.00$	$0.50 - 2.00$
A_h	Scale Parameter of Human Sector	BGP^*	BGP^*
ϕ_1	Physical Capital Share in Goods Production	$0.30 - 0.40$	$0.30 - 0.40$
ϕ_2	Physical Capital Share in Human Investment	$0.08 - 0.29$	$0.08 - 0.29$
δ_k	Depreciation Parameter (Physical Capital)	$0.015 - 0.030$	BGP^*
ψ	Convexity of Endog. Depr. Rate	$-$	$2.00 - 4.00$
δ_h	Depreciation Rate of Human Capital	$0.001 - 0.015$	$0.001 - 0.015$
ρ_g	Auto-correlation of TFP	$0.01 - 0.99$	$0.01 - 0.99$
ρ_h	Auto-correlation of Human Shock	$0.01 - 0.99$	$0.01 - 0.99$
σ_g^2	Variance of TFP	0.007(initial)	0.007(initial)
σ_h^2	Variance of Human Productivity Shock	0.007(initial)	0.007(initial)
$\sigma_{g,h}$	Correlation of Shock Innovations	$(-0.99) - 0.99$	$(-0.99) - 0.99$

Table 2: Model 1 and 2 grid search ranges. (* BGP refers to calibrated values for parameters obtained through use of BGP conditions).

codes. To contrast the results with the standard RBC model, Figure 3.2's impulse responses to an economy-side shock include both the full model (Model 2) in blue and a King and Rebelo (2000) calibration of the RBC shock in red.

The economy-wide shock, defined as a simultaneous increase in the goods sector productivity (the TFP shock) and the human capital sector productivity with a goods sector TFP shock of one percent and a human capital productivity shock of 1/25th percent, the same ratio as the means of A_g and A_h. The simultaneity is comparable to the model calibration of a correlation equal to more than 0.99.

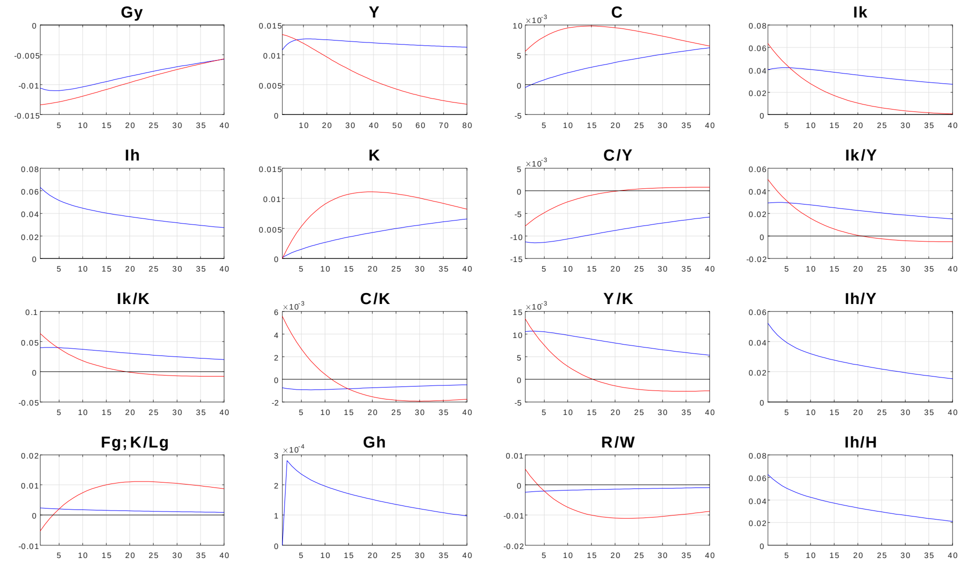
Figure 3.2 shows the impulses to the model's variables for 200 periods, longer than the standard 40 used in the RBC literature, in order to see results covering the full Medium cycle, as defined from 2 to 200 periods (Comin and Gertler, 2006). This enables clarity on how the results show a permanent increase in consumption, output, the physical capital stock, the human capital stock, and investment in both the physical and human capital stocks. These latter investment rates rise because the capital stocks rise and cause a greater BGP depreciation of capital that needs to be replaced. These permanent effects result from the increase in the growth rate of human capital, Gh in the figure, that drives the permanent income up as this growth rate is impulsed up and then returns to zero.

All other stationary variables such as time allocations and the physical capital capacity utilization rate, as well as input prices w and r and relative output prices, p_h, going back to zero. In comparison, the RBC model has all variables going back to zero as there is no permanent income effect in that model.

Observe first that the propagation of the model is apparent in Figure 3.2, by the much more prolonged increase in the growth rates of physical and human capital, as given by Ik/K compared to the RBC response in Ik, and by the similarly long profile of Ih/H, in that both Ik/K and Ih/H do not fall to zero until about the 150th period. In contrast the RBC

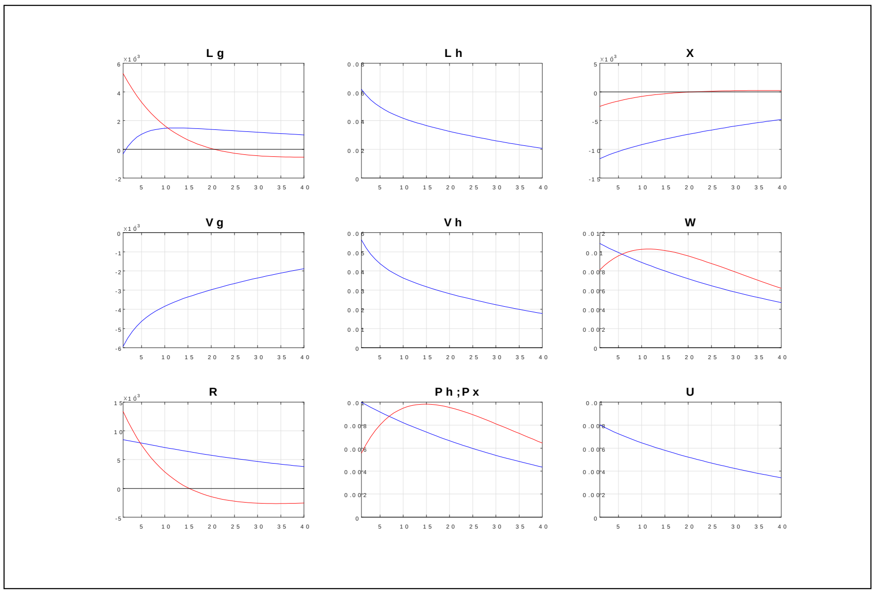
Ik/K increase falls slightly below zero in about the 20th period. The same relative profile results for the output to capital ratio, Y/K. Second, note that the consumption to output ratio, C/Y, stays below zero all the way until near the 200th period, while in contrast the RBC ratio of C/Y (red) only stays below zero until about the 20th period. This shows the more prolonged countercyclical nature of this key c/y ratio in Model 2.

Finally note the much smaller changes "required" of the model in relative prices and factor ratios as compared to the standard RBC model. The interest to rate rate ratio, R/W in Figure 3.2, falls slightly as compared to the RBC, and similarly the physical capital to human capital ratio in the goods sector, Fg in the figure, rises only slightly as compared to the 10 fold higher rise in the capital to labor ratio, K/Lg in the figure, for the RBC model.



Effect of a Temporary Economy-Wide Shock on Output, Consumption, Ratios, and Growth Rates

Figure 3.2 has two marked contrasting results of the Model 2 relative to the RBC model. The interest rate, as given by R in the figure, jumps up and falls gradually towards a zero change somewhat beyond the 200 periods for Model 2, while R starts falling before the 20th period in the RBC model. The contrast results because the return on human capital in Model 2 (the marginal product of human capital in the human capital investment sector; not shown) rises in profile just as does R (but about one-tenth of the magnitude) and then stays positive as it falls very gradually to zero at about 150 periods. This also explains in part why leisure falls so much more in Model 2 as compared to the RBC model, why time in the human capital sector, Lh in the figure, rises and stays positive until near to the 200th period. In addition, the result for labor time in the goods sector, Lg in the figure, is a nice looking (compared to data-based expectations) hump-shaped labor impulse (with a very small initial drop as in the Gali, 1999, effect). Finally the physical capital shifts to the human capital sector from the physical capital sector because both sectors are expanding, the capacity utilization rate of physical capital, U in the figure, jumps up and slowly falls, and so there is extra physical capital in the physical capital intensive sector, the goods sector, that can be reallocated to the human capital investment sector so as to enable both sectors to expand. This relative scarcity of human capital that induces reallocation of physical capital to that sector is indicated by the higher relative price of human to physical capital investment, p_h which is given by Ph in the figure, and which jumps up and gradually falls down to zero at around 200 periods.



Effect of a Temporary Economy-Wide Shock on Sectoral Resources and Relative Prices

3.3 Estimated, Backed-out, Shocks

Figures zzzzz and zzzzz2 graph the unfiltered goods sector shock ("TFP") series obtained from Models 2 and 1, separately, along with the respective, traditional, Solow-residual, goods sector TFP in each graph. The model-based goods TFP shock series is constructed as in Benk et al. (2005) by matching the implicit equilibrium solution for a set of the model's decision variables to the data for each variable in that set.

Note that for Model 2, seven data series are used, the consumption-output ratio, the investment-output ratio, labor hours, ratios of output, consumption, investment relative to human capital plus the utilization rate series. The Model 1 TFP is constructed the same set except for the utilization rate. Every quarterly data series used is for the 1972:1 to 2015:4; the starting date of 1972 is a result of using the utilization rate data which only starts then.

The method for backing-out the shocks is to match each of the chosen data series to each of the model's respective variable solutions, in terms of the known parameters, the state variable (k_t/h_t), and the shocks $\{z_t^g, z_t^h\}$. This gives a set of equations, one for each of the data-matched variables, in terms of the state variable and the two shocks. Using US data for the state variable (k_t/h_t), this leaves a set of seven equations in the two unknown shocks $\{z_t^g, z_t^h\}$, an overidentification of the shocks. Overidentification allows for a relatively data-invariant backed-out shock, using different combinations of variables, as opposed to an exactly identified shock set from using any two of the data series alone. Then the two backed-out shock series are constructed from the set of variable soultions by following the Benk et al. (2005) method of ordinary least squares. This estimates each shock at each time period t using the seven data points for each time period t, for each shock. More data series is "better" than two because this gives a larger "data sample" from which to estimate each shock at each point in time.

Figure 1's model based goods TFP shock has a 0.78 correlation with the Solow residual. The high correlation compares to a similar magnitude found in Nolan & Thoenissen (2009), who also back out and compare their TFP model shock to the Solow residual, although using instead a different model that has a financial shock, a money supply growth shock, and a goods sector TFP shock. One very noticeable departure of the two series is that the model TFP sharply falls in 2008 and then gradually begins rising, while the Solow residual continues to fall.

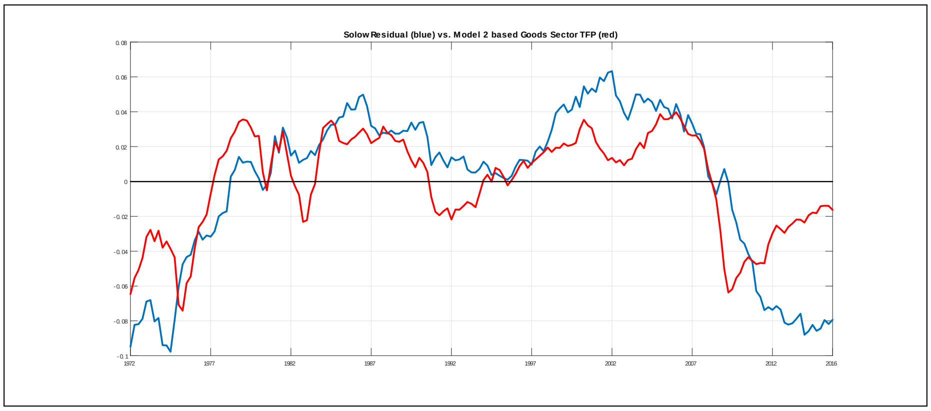


Figure 1: Total Factor Productivity: Model 2 (red) vs Solow residual (blue).

By examining the differences across filtered frequencies, further detail emerges as to where the two depart. Figure 2 shows that in the Business cycle window the Solow residual rises in 2008-9, while the model goods TFP shock moves sharply down, as consistent with the Great Recession. In the low frequency, the model TFP begins rising after 2010 and gradually approaching the baseline (zero) in 2016, while the low frequency Solow residual only rises slightly and remains well below (zero); the same holds in the Medium cycle. This is circumstantial evidence that the model shock is doing a better job of explaining how the US economy cycled sharply down in 2008 and then began a slow gradual recovery, with almost a whole "lost decade" before getting back (to zero here); the model seems to be closer to experience than the Solow residual.

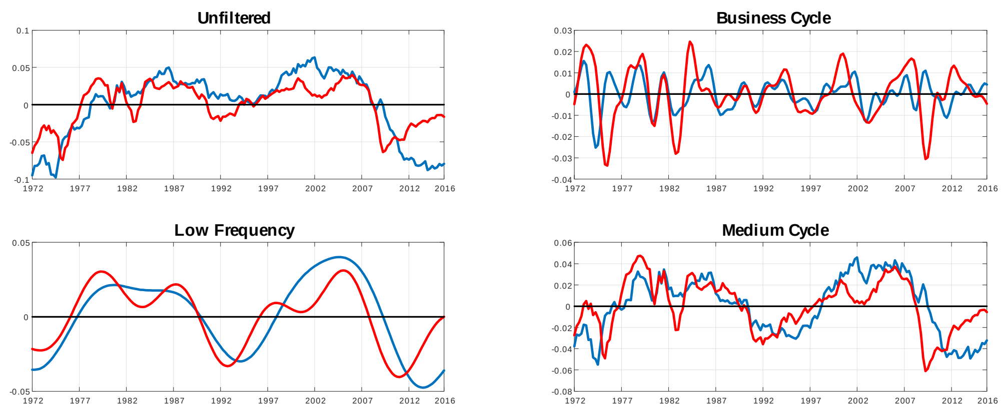


Figure 2: Total Factor Productivity - Model 2 derived (red) versus Solow residual (blue) at different frequencies.

Figure 3 shows the Model 1 backed out, unfiltered, shock compared to the Solow residual in the upper lefthand tile and the correlation is only 0.27. Tracking the Solow residual better does not prove that Model 2 is better than Model 1, but it provides some circumstantial evidence as to how the models are performing relative to more standard measures of TFP. Model 1 better captures the post 2010 upturn than the Solow residual, it appears, but it does not return to zero in 2016.

And Figure 4 shows that Model 1, for example, seems to miss a lot of low frequency growth in the 1970's to 1980's, as well as not providing as sharp an upturn post 2010 compared to Model 2.

Model 2 appears as the preferred model. For example Feenstra et al. (2015) show a positive rate of increase in TFP post 2010 but one that is well-below its historical trend, as is consistent with Model 2. [14] Also consider how the Wage Growth Tracker data graphed in Figure 5 shows that the median US wage growth fell sharply during the Great Recession

[14] "Total Factor Productivity at Constant National Prices for United States"; sourced from Feenstra et al. (2015); retrieved from FRED, https://fred.stlouisfed.org/series/RTFPNAUSA632NRUG, August 25, 2016.

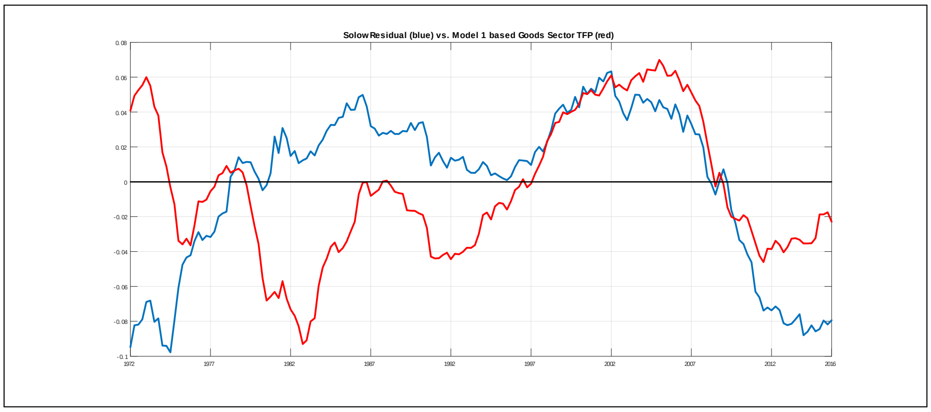


Figure 3: Total Factor Productivty: Model 1 (red) vs Solow residual (blue).

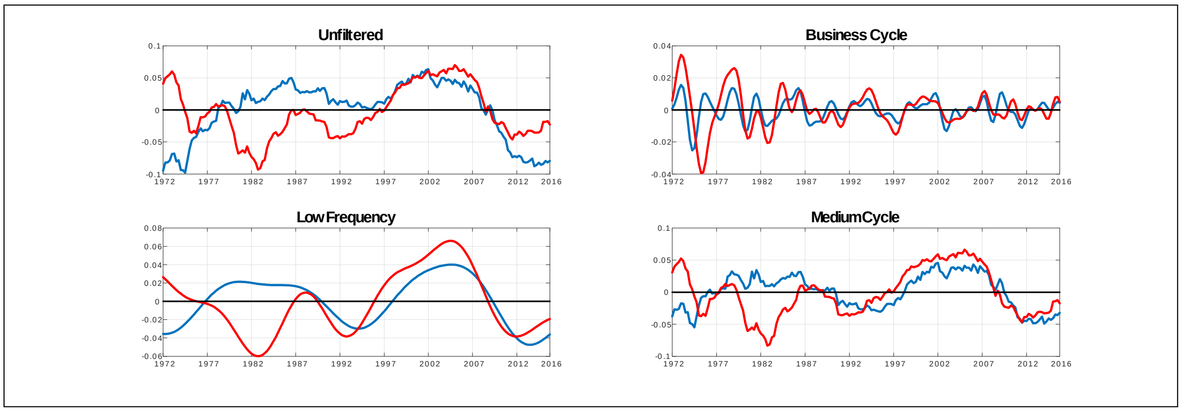


Figure 4: Total Factor Productivity - Model 1 derived (red) versus Solow residual (blue) at different frequencies.

and began steadily rising after 2010, very similar to the Figure zzzzz post-2010 period for Model 2's goods TFP shock.[15]

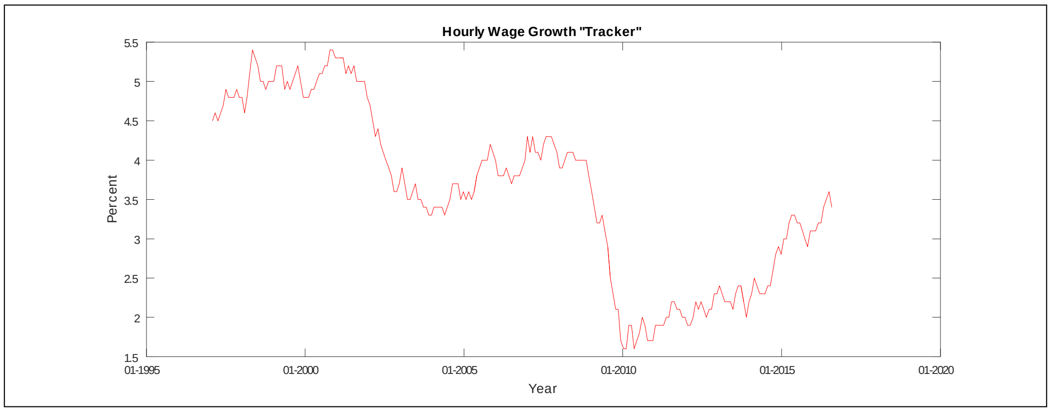


Figure 5: Wage Growth "Tracker": Three-month moving average of median hourly wage growth, 1997:3-2016:7 (Source: Federal Reserve Bank of Atlanta).

Finally consider that in Appendix D the negative backed-out shock components for the goods sector shock and graphed. In addition, the positive backed-out shock components for the goods sector shock are graphed. In comparison it is clear that the magnitude of the backed out shocks are much smaller for the negative shocks than for the positive shocks. This implies the type of assymmetry that Hansen and Prescott (2005) identify for the US and that Harding and Pagan (2002) identify for both the US and in a broader sample set.

4 Results

Moment results based on model simulation are presented at different frequencies for key correlations with output, own volatilities, and persistence of growth rates. The simulation methodology follows Restrepo-Ochoa & Vazquez (2004).[16] The frequencies are found using the Christiano & Fitzgerald (2003) band-pass filter. The windows are high frequency (HF: 2 - 6 quarters), business cycle frequency (BC: 6 - 32 quarters), low frequency (LF: 32 - 200 quarters), and the Comin & Gertler (2006) 'Medium Cycle' that combines these frequencies (MC: 2 - 200 quarters).

These windows are also used to exhibit the economy's ability to explain the consumption to output ratio that stands at the heart of the many puzzles facing real business cycle theory. Starting with the c/y ratio of the US data, the first results reported are a comparison of

[15] The Atlanta Fed "Wage Tracker" uses BLS CPS household data, and can act as a proxy of productivity.

[16] Online Appendix D.4 sets how this Restrepo-Ochoa & Vazquez (2004) simulation methodology is equivalent to King et al. (1988).

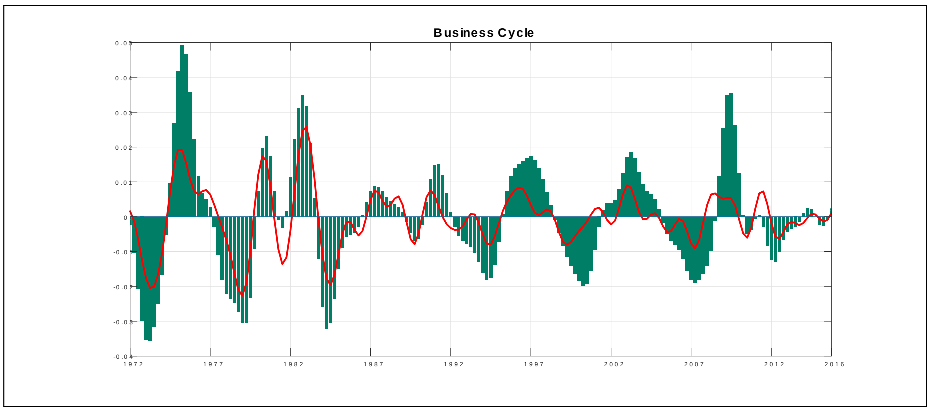


Figure 6: Consumption-Output Ratio (C/Y) at the Business Cycle Frequency: Model 2 total shock contribution (green area) versus US Data (red line).

the economy's explanation of c/y relative to the data. Added for additional comparison is a construction of c/y using the standard RBC.

4.1 Consumption-Output Ratio

For the quarterly historical US data from 1971:4 to 2015:4, Figure 6 shows the Model 2 constructed c/y for the business cycle (BC) window in green compared to the data for c/y given by the red line.[17] This construction is made from the backed out shocks and the models solution for c/y in terms of k/h and the shocks. This requires using the US historical data on k/h as well as the backed out shocks, which in turn used data as described in the last section. The figure shows that the data is tracked rather closely by Model 2, including during the Great Recession and its aftermath.

In contrast, Figure 7 shows in blue the standard RBC model fit of c/y, in the business cycle window, using the backed out TFP shock for the RBC model and the RBC solution of c/y as a function of the shocks and the state variable k. The same actual c/y data is included in red. It is clear that the standard RBC model fits the data less well than does Model 2 above, in that it have a much less exacting fit of the data, with too much volatility compared to the data and to Model 2.

To see a sampling of the low frequency results, Figure 8 shows the Model 2 generated c/y in the Medium Cycle frequency, as compared to the c/y data filtered to the same frequency. Model 2 appears to capture fairly well the actual Medium Term Cycle c/y.

Again to see the comparison to the RBC at the lower frequencies, consider Figure 9. This shows across frequencies the RBC model value for c/y, as given by the black line, compared

[17]The data description is given in Appendix B.

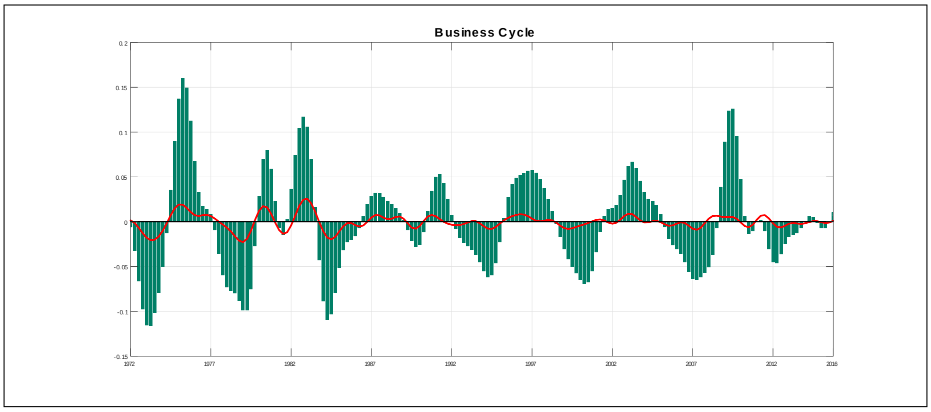


Figure 7: Consumption-Output Ratio (C/Y) at the Business Cycle Frequency: RBC model total shock contribution (green area) versus US Data (red line).

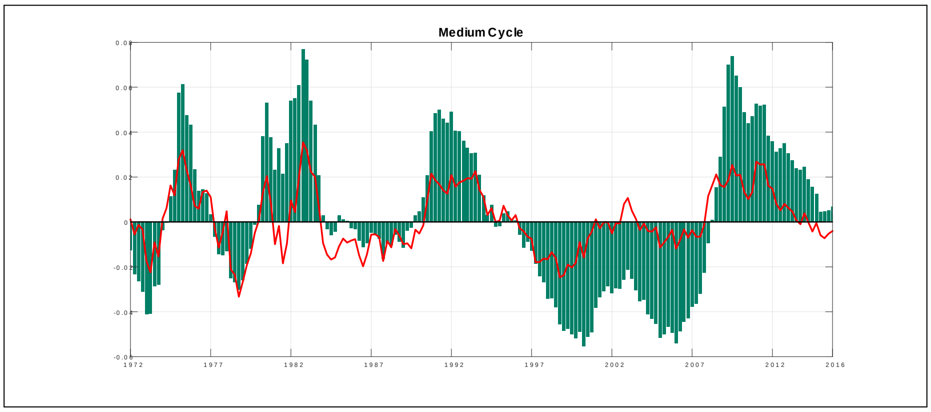


Figure 8: Consumption-Output Ratio (C/Y) at the Medium Cycle Frequency: Model 2 total shock contribution (green area) versus US Data (red line).

to the red line for the actual c/y data and to the green shading for the Model 2 c/y. For the unfiltered data, the business cycle, the low frequency and the Medium Cycle, it is clear that the RBC model (black line) is much more volatile compared to the data (red line) and Model 2 (green).

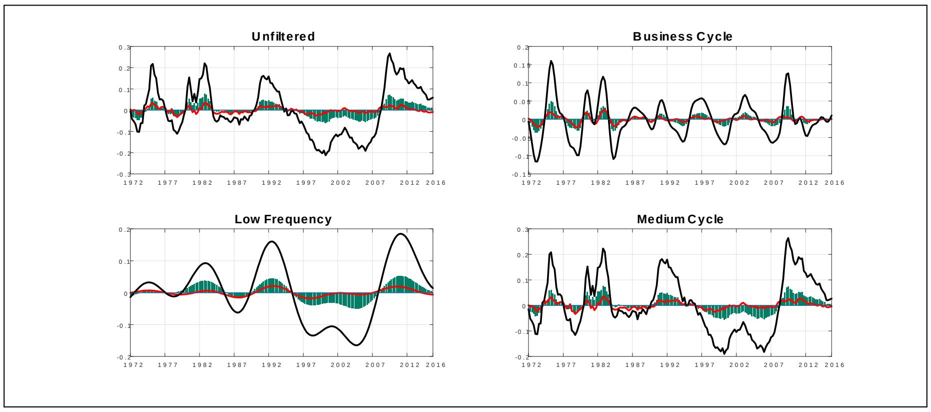


Figure 9: Consumption-Output Ratio (C/Y): Model 2 total shock contribution (green area), RBC model TFP contribution (black line) versus US Data (red line).

4.2 Key Cyclic Correlations

Tables 3 - 6 report moments for US data, Model 1 and Model 2. Table 3 shows that the comovement of consumption and investment with output is closely matched by Models 1 and 2 at the business cycle frequency. Both models are able to capture a positive correlation between labor hours and output as suggested by US data at the business cycle, low frequency, and the Comin & Gertler (2006) Medium Cycle, with Model 2 closer to the data. Both models capture the positive business cycle correlation between labor hours and consumption, unlike the standard RBC model. Both models generate a strong negative theoretical correlation between human capital investment time hours and output as suggested in DeJong et al. (1996), and as consistent with certain limited evidence. Model 2 is also able to capture the positive correlation of physical capital utilization rate and output at the business cycle frequency and the lower frequencies, although doing best at the business cycle.[18]

Figure 10 shows using FRED how the real interest rate, defined as the 3 month Treasury bill minus the CPI inflation rate, divided by the real wage rate, defined as Compensation of Employees as Wage and Salary Disbursements divided by the CPI index, is very procyclic since 1957. In turn this supports finding a countercyclic w/r in the model, at business cycle

[18] Please see the Online Appendix A.4 for a description of the simulation methodology.

Variable		High freq. 2 -6 qrs.	Bus. cyc. 6 - 32 qrs.	Low freq. 32 - 200 qrs.	Med. term 2 - 200 qrs.
$corr(c_t, y_t)$	Data	0.475	0.891	0.980	0.963
	Model 1	0.893	0.776	0.931	0.927
	Model 2	0.989	0.928	0.856	0.837
$corr(i_{kt}, y_t)$	Data	0.809	0.939	0.834	0.833
	Model 1	0.784	0.841	0.691	0.696
	Model 2	0.997	0.991	0.936	0.939
$corr(l_{gt}, y_t)$	Data	0.394	0.732	0.589	0.595
	Model 1	-0.196	0.200	0.027	0.036
	Model 2	-0.141	0.874	0.823	0.819
$corr(l_{ht}, y_t)$	Data	-	-	-	-
	Model 1	0.214	-0.016	0.131	0.111
	Model 2	0.119	-0.891	-0.833	-0.827
$corr(u_t, y_t)$	Data	0.432	0.797	0.447	0.483
	Model 1	-	-	-	-
	Model 2	0.001	0.926	0.871	0.819
$corr(c_t, l_{gt})$	Data	0.206	0.766	0.592	0.596
	Model 1	-0.077	0.672	0.362	0.319
	Model 2	-0.229	0.651	0.378	0.383

Table 3: Matching Correlations (US Data 1959Q1-2015Q4, Model 1 & 2).

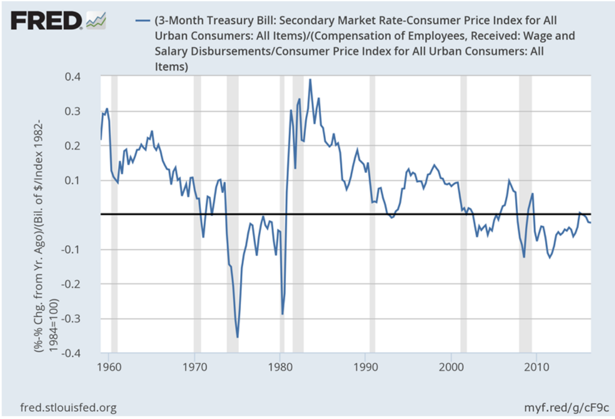


Figure 10: A procycle r/w for the US, 1957-2017.

and lower frequencies, as reported in the next Table. Also reported is the Table is how the labor share of the goods sector is countercyclic in the model, in the BC window, consistent with data according to Hansen and Prescott (2005), as well in the LF and MC windows.[19]

Variable Model 2 Moments	High freq. 2 -6 qrs.	Bus. cyc. 6 - 32 qrs.	Low freq. 32 - 200 qrs.	Med. term 2 - 200 qrs.
$corr(\frac{w_t l_{gt} h_t}{y_t}, y_t)$	0.48	-0.72	-0.74	-0.73
$corr(\frac{w_t}{r_t}, y_t)$	0.53	-0.47	-0.67	-0.65

Table 4: Matching Correlations (US Data 1959Q1-2015Q4, Model 1 & 2).

4.3 Volatilities

Tables 5 show that the volatility moments of the data are captured relatively well with Model 1 being better in some cases and Model 2 in others. For example, both models are very close to the data for output growth volatility. The volatility physical capacity utilization rate is matched only in Model 2, albeit improvement here is possible given too little volatility in Model 2 compared to the data.

[19] Hansen and Prescott (2005) report a -0.33 business cycle correlation between the labor share and output for the US 1954:1- 1999:3.

Variable		High freq. 2 -6 qrs.	Bus. cyc. 6 - 32 qrs.	Low freq. 32 - 200 qrs.	Med. term 2 - 200 qrs.
$vol(g_{y,t})$	Data	0.0068	0.0064	0.0038	0.0100
	Model 1	0.0047	0.0037	0.0034	0.0068
	Model 2	0.0050	0.0043	0.0034	0.0074
$vol(g_{c,t})$	Data	0.0038	0.0036	0.0029	0.0059
	Model 1	0.0036	0.0031	0.0069	0.0079
	Model 2	0.0022	0.0017	0.0015	0.0031
$vol(g_{i_k,t})$	Data	0.0200	0.0207	0.0105	0.0302
	Model 1	0.0160	0.0150	0.0260	0.0330
	Model 2	0.0120	0.0110	0.0091	0.0190
$vol(y_t)$	Data	0.0044	0.0166	0.0469	0.0500
	Model 1	0.0034	0.0100	0.0590	0.0600
	Model 2	0.0033	0.0100	0.0360	0.0380
$vol(c_t)$	Data	0.0024	0.0097	0.0382	0.0396
	Model 1	0.0028	0.0068	0.0550	0.0550
	Model 2	0.0015	0.0038	0.0200	0.0200
$vol(i_{kt})$	Data	0.0129	0.0540	0.0912	0.1076
	Model 1	0.0110	0.0420	0.1500	0.1500
	Model 2	0.0081	0.0290	0.0910	0.0960
$vol(l_{gt})$	Data	0.0017	0.0049	0.0221	0.0227
	Model 1	0.0070	0.0112	0.0090	0.0158
	Model 2	0.0037	0.0120	0.0350	0.0370
$vol(u_t)$	Data	0.0055	0.0254	0.0318	0.0420
	Model 1	-	-	-	-
	Model 2	0.0011	0.0023	0.0039	0.0047

Table 5: Countercyclic Data Moments for Labor Share of Output and Input Price Ratio

4.4 Persistence

Table 6 shows model persistence in two ways. First, to follow the literature of Benhabib et al. (2006) and Cogley & Nason (1995), the unfiltered simulated model data is used to generate teh autocorrelation profile $\rho(\cdot)$, and compared to the unfiltered actual data. This is done for output growth, consumption growth, physical capital investment growth, and the levels of goods sector labor and the physical capital capacity utilization rate. The main failing of Model 1 is that it only gets the initial level of growth persistence, but not the falling autocorrelation profile as seen in the data. Model 2 better captures both the level and the autocorrelation profile across the four data growth autocorrelations with three lags reported.

With an extension to 16 lags, Figure 11 graphs the three growth autocorrelation profiles, plus the profile for labor, for the data (in blue), for Model 1 (in red) and for Model 2 (in yellow). The four tiles are A: output growth; B: consumption growth; C: physical capital investment growth; and D: goods sector labor. In contrast, as reported by Benhabib et al. (2006), traditional RBC models fail to reproduce the output growth persistence beyond the first lag.

Variable		Lag 1	Lag 2	Lag 3
$\rho(g_{y,t})$	Data	0.270	0.216	0.160
	Model 1	0.636	0.605	0.596
	Model 2	0.271	0.220	0.188
$\rho(g_{c,t})$	Data	0.369	0.284	0.305
	Model 1	0.631	0.610	0.608
	Model 2	0.380	0.361	0.347
$\rho(g_{ik,t})$	Data	0.264	0.177	0.082
	Model 1	0.329	0.265	0.225
	Model 2	0.282	0.213	0.170
$\rho(l_{gt})$	Data	0.987	0.975	0.962
	Model 1	0.956	0.917	0.883
	Model 2	0.993	0.983	0.971
$\rho(u_t)$	Data	0.956	0.863	0.751
	Model 1	-	-	-
	Model 2	0.956	0.919	0.887

Table 6: Simulated Autocorrelation Functions vs. Data (US Data 1959Q1 - 2015Q4).

A second measure of persistence was computed across frequency using filtered data, again with three lags. The results for this are reported in the following section using the metric

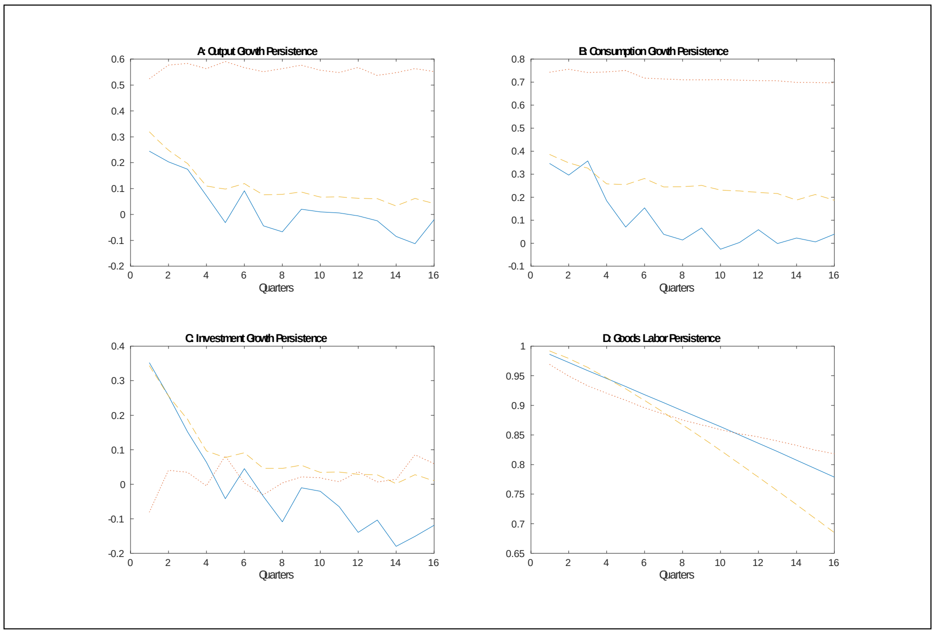


Figure 11: Autocorrelation profiles of variables for 15 quarters: US data-based, 1972Q1-2015Q4, solid blue line; Model 1 simulated data, dotted red line; Model 2 simulated data, dashed yellow line.

measure of the average distance of the model moments from the data moments. This second filtered set of persistence results are denoted by Persistence *, while unfiltered persistence moments are denoted by Persistence **.

5 Metric for Model Comparison

Besides its use in the calibration choice, the other advantage of the distance metric is that it represents the average percentage point deviation of the simulated moments from the US data-based moments. Therefore, it allows further information of model performance for the choice of the calibration, as well as the ability to compare the performances of different DSGE models relative to the data across all moments and/or across subsets of moments. Table 7 presents both a set of moment results for the largest, "Overall", set of moment comparisons (which includes the high frequency moments) and a set of subsets of results across frequencies and within categories of moment types. Adding up the metric for each target and dividing by the number of targets gives the corresponding average metric, for each set of moments reported.

Table 7A shows the "Overall" average metric across all moments that are reported in Tables 3, 5 and 6 for which there was data for comparison (human capital investment time was excluded). For Model 2, there are 20 Correlation moments, 32 Volatility moments, and 15 unfiltered Persistence ** moments for a total of 67 "Overall"; all of these 67 model moments are targeted in the calibration grid search. The corresponding average metrics respectively are 0.50 for correlations, 0.51 for volatilities, 0.15 for unfiltered persistence, and 0.46 for the overall average.

For Model 1, there are 16 Correlation moments, 28 Volatility moments, and 12 (unfiltered) Persistence ** moments for a total of 56 targeted moments (fewer in number than Model 2 because there is not a varable physical capital capacity utilization rate); again all of these reported model moments are targeted in the Model 1 grid search for the calibration. The corresponding average metrics are 0.50 for correlations, 0.53 for volatilities, 0.95 for unfiltered persistence, and 0.59 "Overall". The filtered persistence (Persistence *) constitutes the remaining row of Tables 7A.

This is included in the Table (but not used to calculate the Overall metric) as an alternative measure of persistence to that of using the unfiltered data as is the focus of the literature. For Model 2, also included is the capacity utilization growth with 3 lags, so as to give 5x3=15 moments that are averaged within each of the four frequencies for Model 2; this results in a total of 60 targets with an average metric of 0.38 for Model 2.

Table 7B breaks the results down by frequency, with four windows of HF, BC, LF, and MC (detailed results here not reported). It shows the average metric for the categories of Correlation, Volatilities, and Persistence*, by frequency. There are 12 moment metrics averaged within each frequency for Model 1 and 15 moment metrics averaged within each frequency for Model 2.[20]

Results in Table 7A show that Model 2 has a lower average distance metric for the "Overall" calculation, as well as for the correlations and both unfiltered and filtered persistence. Results in Table 7B show that Model 2 correlations, within the BC, LF and MC windows, have an average 15%, 39% and 33% average deviation, respectively. Model 2 volatilities in the LF window have a 36% average deviation. These Model 2 results are better than Model 1 comparable results. Model 2 also has lower average deviations than Model 1 for the filtered Persistence* in the BC, LF and MC windows, including a very low 8% deviation for Model 2 in the LF window. Model 1 does beat Model 2 for filtered persistence in the LF window, with a 2% average deviation.[21]

In addition, for one robustness check, similar metrics were computed for the Model 2

[20] For the alternative, filtered, Persistence*, there are 3 autocorrelation lags for each of the 4 growth series of Model 1 (output, consumption, investment, and labor); when all 48 are added together and divided by 48, the average metric of 0.73 results for Model 1's filtered Persistence* metric. For the comparable measures in Model 2, the capacity utilitzation rate is added in the metric averages.

[21] Model 1 and 2 were extended with a government sector and corresponding shock, in the fashion of Chari et al. (2007), but did not improve overall on the performance in terms of the distance metric; for example the government model was marginally better in BC volatilities but worse in capturing BC and LF correlations and the autocorrelation profiles of growth rates in Figure 11.

TABLE 6A	Average Metric Across All Moments	
	Model 1	Model 2
Overall	0.59	0.46
Correlations	0.50	0.50
Volatilities	0.53	0.51
Persistence*	0.73	0.38
Persistence**	0.95	0.15

TABLE 6B	Average Metric Across Four Frequencies							
	Model 1 HF	Model 2 HF	Model 1 BC	Model 2 BC	Model 1 LF	Model 2 LF	Model 1 MC	Model 2 MC
Correlations	0.95	1.15	0.27	0.15	0.39	0.39	0.40	0.33
Volatilities	0.60	0.51	0.43	0.64	0.70	0.36	0.38	0.51
Persistence*	0.41	0.75	1.91	0.42	0.02	0.08	0.59	0.27

Table 7: Model 1 and 2 percentage deviation based metric for moments.

with the assumption of a 1.0 correlation between shocks. The metric tables change little with this exercise. And although a 1.0 correlation did not emerge as the preferred calibration, these results indicate that such a simplification of the shock structure would be plausible.

6 Discussion

Human capital plays a stealth role in a standard accounting sense in the Section 3 model, similar to that played by McGrattan's (2015) intangible capital Both of these capitals in their respective settings are not accounted for using Solow growth accounting in the sense that the so-called "labor wedge" of Chari et al. (2007) is exactly equal to the time spent in human capital investment here or in intangible capital investment in McGrattan (2016). To see this, consider that for a prototype exogenous growth economy as in Chari et al. (2007) a wedge τ_{lt} exists between the firm's marginal product of labor and the consumer's marginal rate of substitution between leisure and goods ($MRS_{c,x}$) is described by :

$$(1-\tau_{lt})\tilde{w}_t = \frac{A\tilde{c}_t}{(1-l_{gt})} = MRS_{c,x}, \tag{18}$$

where $\tilde{w}_t = (1-\phi_1)A_g e^{z_t^g}\left[\frac{\tilde{k}_t}{z_t l_{gt}}\right]^{\phi_1} z_t$ and variables with a tilde represent normalized variables by the exogenous growth trend. The wedge is the share of productive time that is not used towards goods production.

The Model 2 marginal rate of substitution between goods and leisure can be written as [Online Appendix equation (89)] :

$$\begin{aligned} w_t x_t &= A\tilde{c}_t; & (19)\\ w_t\left(1-l_{gt}-l_{ht}\right) &= A\tilde{c}_t. & (20)\end{aligned}$$

where variables with tildes for Model 2 represent h_t- normalized variables. Dividing both sides of the above equation by the sum of non-market time, $(1-l_{gt})$ yields an equivalent equation to (18) as

$$(1-\tau_{ht}) = \frac{A\tilde{c}_t}{(1-l_{gt})}; \tag{21}$$

where $\tau_{ht} \equiv \frac{l_{ht}}{1-l_{gt}}$. McGrattan (2016) similarly shows that intangible capital time comprises the labor wedge in her economy.

Yet the human capital sector and its near simultaneous productivity shock with the goods sector shock is what drives the shock amplification and what delivers the business cycle and well as low frequency fact-fitting. In essence, with the second sector besides goods, there are now two sectors each producing the input in which the respective sectors are relatively intensive in. The physical capital intensive goods sector produces output which is instaneously convertible into physical capital investment; the human capital intensive human capital investment sector produces human capital investment.

The production of inputs from the outputs of each respective sector sets up a dynamic that reverberates through the Rybczynski (1955) theorem and dually with the Stolper & Samuelson (1941) theorem. The change in sectoral output from a change in an input is equal to the change in the sector's relative price relative to a change in its input price; this duality is proven in the Appendix C.[22]. What this means is that an economy-wide productivity shock causes a series of simultaneous reallocations as expected in such a two sector general equilibrium.

As more of both factor inputs are created by the productivity shocks in Model 2, but the magnitude of the shock is greater in the goods sector, there ends up with more physical capital relative to human capital. This occurs also as both physical and human capital utilization rates rise (u rises, and leisure falls). The resulting change in relative factor input prices is that the wage to interest rate ratio rises on impulse and so in turn the physical to human capital ratio rises across sectors in a proportional fashion (see Fg in Figure 3.2 and the related BGP equation (16)).

The rise in w/r makes the human capital sector more expensive since it is intensive in human capital and so p_h, the relative price of human to physical capital, rises (Ph rises in Figure 3.2). The goods sector ends up expanding in a procyclic fashion and for a negative productivity shock the goods sector contracts. During these dynamics the human capital investment time on average shows a negative correlation with output over both the business cycle and at lower frequencies (Table 3, fourth row down). In contrast, note that the countercyclical human capital investment time that seems to exist in the data does not result in Model 1. This gives an enhanced role to the second external margin in Model 2, of a variable physical capital utilization rate, in that this appears necessary for several

[22] Mulligan & Sala-i Martin (1993), with a linear production function for human capital, and Bond et al. (1996), with a continuous time version of this paper's Model 1, prove a related Stolper & Samuelson (1941) theorem but not its duality to Rybczynski (1955) or its RBC application as shown here.

related results: the downward autocorrelation growth profile in Figure 10, reproducing a countercyclical human capital investment time, and a goods sector TFP shock amplified so substantially as to have a 7500 smaller variance than a standard RBC model. The Model 2 capacity utilization rate also results in a countercyclic movement in the labor share as in Hansen and Prescott (2005).

The role of implicit relative input and output prices, such as w/r and p_h, and even including u, in explaining business cycles comovement might be made more clear by considering the perspective of Harding and Pagan (2002, p.380). They conclude that "it follows that information upon the evolutionary process for the growth rate in activity needs to be gathered in order to describe the cycle. In particular, the output from theoretical models that is needed relates to the growth rate in output..." Given the role of human capital in making the growth rate of output endogenous„ and that the model appears to do well in matching autocorrelation profiles of the growth in variables (Figure 11 and Table 6), it stands to reason that a focus of stylized facts might include the growth rate of output, as well perhaps as the growth rate for say consumption and human capital.

Tables 8, 9, and 10 show the Model 2 correlation moments of output growth, consumption growth and human capital growth rates with respect to w/r, w, r, and p_h. The set of w/r, w and p_h are all procyclic with the three aforementioned growth rates at the BC, LF, and MC windows. The rental price r is procycle in the LF and MC windows, but countercyclic in the BC window for output and consumption growth, but again procyclic in the BC window for human capital growth. In addition the rental rate r and the marginal product of human capital (MP_H^h, in equation 13) are strongly positively correlated with each other in all windows (0.97:HF; 0.88:BC; 0.81:LF; 0.81:MC), as the model is driven by the return on capital staying the same across human and physical capital. Taken together the growth moment results here support the role of positive comovement of w/r and p_h with the growth rate of output, as the return to human capital and physical capital rise along the BC, LF and MC windows (see equations 12 to 17).

Variable Model 2 Moments	High freq. 2 -6 qrs.	Bus. cyc. 6 - 32 qrs.	Low freq. 32 - 200 qrs.	Med. term 2 - 200 qrs.
$corr(\frac{w_t}{r_t}, g_{yt})$	-0.0006	0.66	0.52	0.32
$corr(w_t, g_{yt})$	0.67	0.10	0.34	0.21
$corr(r_t, g_{yt})$	0.64	-0.27	0.06	0.06
$corr(p_{ht}, g_{yt})$	0.52	0.09	0.33	0.20
$corr(u_t, g_{yt})$	-0.70	0.19	0.11	-0.04

Table 8: Model 2 Simulated Correlation Moments of Output Growth with Relative Prices and Capacity Utilization.

So the typically not well-accounted for human capital plays a large role in this model, especially when the second external margin is added. Human capital investment adds the

Variable Model 2 Moments	High freq. 2 -6 qrs.	Bus. cyc. 6 - 32 qrs.	Low freq. 32 - 200 qrs.	Med. term 2 - 200 qrs.
$corr(\frac{w_t}{r_t}, g_{ct})$	-0.05	0.78	0.47	0.32
$corr(w_t, g_{ct})$	0.91	0.69	0.47	0.34
$corr(r_t, g_{ct})$	0.72	-0.01	0.10	0.19
$corr(p_{ht}, g_{ct})$	0.60	0.33	0.23	0.23
$corr(u_t, g_{ct})$	-0.74	-0.08	-0.33	-0.29

Table 9: Model 2 Simulated Correlation Moments of Consumption Growth with Relative Prices and Capacity Utilization.

Variable Model 2 Moments	High freq. 2 -6 qrs.	Bus. cyc. 6 - 32 qrs.	Low freq. 32 - 200 qrs.	Med. term 2 - 200 qrs.
$corr(\frac{w_t}{r_t}, g_{ht})$	0.77	0.89	0.61	0.39
$corr(w_t, g_{ht})$	-0.25	0.82	0.93	0.90
$corr(r_t, g_{ht})$	-0.36	0.52	0.72	0.62
$corr(p_{ht}, g_{ht})$	-0.19	0.73	0.81	0.78
$corr(u_t, g_{ht})$	0.34	-0.59	-0.89	-0.77

Table 10: Model 2 Simulated Correlation Moments of Human Capital Growth with Relative Prices and Capacity Utilization.

second sector and brings into play the two-sector dynamics. It allows for an external labor margin by being able to reallocate to the human capital sector and by also being able to increase the utilization rate of productive time by decreasing leisure time along what might be considered the internal margin. The human capital sector shock adds the permanent income effect that drives a hugely amplified shock mechanism, which becomes even stronger when combined with the variable physical capital utilization rate. And the human capital sector allows for a level of output growth persistence as in the data, and when adding a variable physical capital utilization rate, a persistence that falls in autocorrelation profile as in the data.

Even though we do not find human capital explicitly in our national income and product accounts, or in our capital wealth accounts such as by the OECD, human capital has been helping to explain a set of parallel processes. It can explain why capital does not flow freely into less developed economies (Lucas, 1990) that lack sufficient quantities of it; it can explain isomorphically the goods sector TFP from a Lucas (1988) BGP accounting point of view; and it can begin to untangle the structural transformation towards human capital intensive sectors. So it may be consistent with these budding literatures that it can also explain low frequency as well as business cycle dynamics once the growth rate is a function of the resources devoted to human capital accumulation.

7 Conclusion

The paper's dramatic shock amplification results because the economy-wide temporary shock creates a permanent income effect that raises consumption, output and the capital stocks permanently. This is a result of shocking the investment rate of human capital as well the goods sector TFP with an above 99% correlation that defines what is called the economy-wide shock. The magnitude of the variance of the Model 2 economy-wide shock is some 7500 times smaller than the traditional RBC TFP shock variance, and the human capital shock variance 22000 times smaller. The paper shows that the model can match well traditional RBC data moments for correlations, volatilities and output growth persistence, even when including the data of the Great Recession and the post-2010 below-trend recovery. In addition, physical capital utilization rate procyclic moments and human capital time's countercyclical moments are captured as is the level and autocorrelation shape of the growth persistence of output, consumption and investment, along with that profile for labor. The Model 2 provides a simultaneous tuning into both growth, or low frequency, spectra and the business cycle, while reproducing well the historical data on the consumption to output ratio, across frequencies, a major challenge at the heart of the many RBC puzzles.

Key to producing the model results is a calibration that employs a deep grid search while demanding both state variable convergence (Blanchard & Quah 1989) and iterative convergence of the model's shock properties to those properties of the backed-out shocks (Benk et al. 2005). It is an interesting result, from historically backing out the model's goods sector productivity shock from US data, that this backed out productivity shock rises sharply post 2010 albeit at a below trend rate , similar to Feenstra et al. (2015), but unlike the traditionally constructed TFP shock which rises much less post 2010. This may be indicating that the model is better capturing the post 2010 recovery through its human capital channel which is hard to account for in the standard RBC accounting that lies behind the Solow residual.

The metric used in the grid seach is the sum of all of the fractional moment deviations of model from data, as normalized by dividing by the total number of moment targets. This makes the aggregate metric the average fractional deviation of the model's moments from the data moments across four spectra of a large moment set. The use of the metric resulted in the leap in the amplification of the shock and the other salient results. The grid search enables a comprehensive calibration space search with the aggregate metric providing a measure of the model's performance as well as a tool to limit the focus of the parameter space to that in which the lowest fractional deviations of the metric were found.

Future research includes estimating confidence intervals for the calibration methodology building on the Simulated Method of Moments (SMM) literature, and a full comparison to Bayesian methods. Better explaining the volatility of the capacity utilization rate of physical capital may be viable and difficult but possible explanation of the equity premium with the human capital model (Li 2000) are left for future research. Better accounting may be found following McGrattan & Prescott (2014). Further grid search work could include eliminating the high frequency targets in order to sharpen the focus on business cycle and low frequencies, or alternatively, selectively weighting certain desired targets, which we did not do here.

References

Baier, S., Mulholland, S., Turner, C. & Tamura, R. (2004), Income and education of the states of the United States: 1840–2000, Working Paper 2004-31, Federal Reserve Bank of Atlanta.

Barro, R. J. & Lee, J. W. (2013), 'A new data set of educational attainment in the world, 1950–2010', *Journal of Development Economics* **104**(C), 184–198.

Basu, P., Gillman, M. & Pearlman, J. (2012), 'Inflation, human capital and Tobin's q', *Journal of Economic Dynamics and Control* **36**(7), 1057–1074.

Benhabib, J., Perli, R. & Sakellaris, P. (2006), 'Persistence of business cycles in multisector real business cycle models', *International Journal of Economic Theory* **2**(3-4), 181–197.

Benhabib, J., Rogerson, R. & Wright, R. (1991), 'Homework in Macroeconomics: Household Production and Aggregate Fluctuations', *Journal of Political Economy* **99**(6), 1166–87.

Benhabib, J. & Wen, Y. (2004), 'Indeterminacy, aggregate demand, and the real business cycle', *Journal of Monetary Economics* **51**(3), 503–530.

Benk, S., Gillman, M. & Kejak, M. (2005), 'Credit Shocks in the Financial Deregulatory Era: Not the Usual Suspects', *Review of Economic Dynamics* **8**(3), 668–687.

Benk, S., Gillman, M. & Kejak, M. (2008), 'Money Velocity in an Endogenous Growth Business Cycle with Credit Shocks', *Journal of Money, Credit and Banking* **40**(6), 1281–1293.

Benk, S., Gillman, M. & Kejak, M. (2010), 'A banking explanation of the US velocity of money: 1919-2004', *Journal of Economic Dynamics and Control* **34**(4), 765–779.

Blanchard, O. J. & Quah, D. (1989), 'The Dynamic Effects of Aggregate Demand and Supply Disturbances', *American Economic Review* **79**(4), 655–73.

Bond, E. W., Wang, P. & Yip, C. K. (1996), 'A General Two-Sector Model of Endogenous Growth with Human and Physical Capital: Balanced Growth and Transitional Dynamics', *Journal of Economic Theory* **68**(1), 149–173.

Buera, F. J. & Moll, B. (2016), 'Aggregate Implications of a Credit Crunch: The Importance of Heterogeneity', *American Economic Journal: Macroeconomics* **7**(3), 1–42.

Chari, V. V., Kehoe, P. J. & McGrattan, E. R. (2007), 'Business Cycle Accounting', *Econometrica* **75**(3), 781–836.

Christiano, L. J., Boldrin, M. & Fisher, J. D. M. (2001), 'Habit Persistence, Asset Returns, and the Business Cycle', *American Economic Review* **91**(1), 149–166.

Christiano, L. J. & Fitzgerald, T. J. (2003), 'The Band Pass Filter', *International Economic Review* **44**(2), 435–465.

Cogley, T. & Nason, J. M. (1995), 'Output Dynamics in Real-Business-Cycle Models', *American Economic Review* **85**(3), 492–511.

Comin, D. & Gertler, M. (2006), 'Medium-Term Business Cycles', *American Economic Review* **96**(3), 523–551.

Cooley, T. F., Hansen, G. D. & Prescott, E. C. (1995), 'Equilibrium business cycles with idle resources and variable capacity utilization', *Economic Theory* **6**(1), 35–49.

DeJong, D. N. & Ingram, B. F. (2001), 'The Cyclical Behavior of Skill Acquisition', *Review of Economic Dynamics* **4**(3), 536–561.

DeJong, D. N., Ingram, B. F., Wen, Y. & Whiteman, C. H. (1996), Cyclical Implications of the Variable Utilization of Physical and Human Capital, Macroeconomics 9609004, EconWPA.

Dellas, H. & Sakellaris, P. (2003), 'On the cyclicality of schooling: theory and evidence', *Oxford Economic Papers* **55**(1), 148–172.

Feenstra, R. C., Inklaar, R. & Timmer, M. (2013), The Next Generation of the Penn World Table, NBER Working Papers 19255, National Bureau of Economic Research, Inc.

Feenstra, R. C., Inklaar, R. & Timmer, M. (2015), 'The Next Generation of the Penn World Table', *American Economic Review* **105**(10), 3150–3182.

Gali, J. (1999), 'Technology, Employment, and the Business Cycle: Do Technology Shocks Explain Aggregate Fluctuations?', *American Economic Review* **89**(1), 249–271.

Gomme, P. & Rupert, P. (2007), 'Theory, measurement and calibration of macroeconomic models', *Journal of Monetary Economics* **54**(2), 460–497.

Greene, W. H. (2003), *Econometric Analysis*, 5 edn, Prentice Hall, Upper Saddle River, NJ.

Greenwood, J. & Hercowitz, Z. (1991), 'The Allocation of Capital and Time over the Business Cycle', *Journal of Political Economy* **99**(6), 1188–214.

Greenwood, J., Hercowitz, Z. & Huffman, G. W. (1988), 'Investment, Capacity Utilization, and the Real Business Cycle', *American Economic Review* **78**(3), 402–17.

Grossman, G. M., Helpman, E., Oberfield, E. & Sampson, T. (2016), Balanced Growth Despite Uzawa, NBER Working Papers 21861, National Bureau of Economic Research, Inc.

Hall, R. E. (1988), 'Intertemporal Substitution in Consumption', *Journal of Political Economy* **96**(2), 339–357.

Hamilton, J. (1994), *Time series analysis*, Princeton Univ. Press, Princeton, NJ.

Hansen, G. D. (1985), 'Indivisible Labor and the Business Cycle', *Journal of Monetary Economics* **16**, 309–327.

Hansen, G. D. & Ohanian, L. E. (2016), Neoclassical Models in Macroeconomics, NBER Working Papers 22122, National Bureau of Economic Research, Inc.

Hansen, G. D. & Prescott, E. C. (2005), 'Capacity constraints, asymmetries, and the business cycle', *Review of Economic Dynamics* **8**(4), 850–865.

Harding, D. & Pagan, A. (2002), 'Dissecting the cycle: a methodological investigation', *Journal of Monetary Economics* **49**(2), 365–381.

Ingram, B. F., Kocherlakota, N. R. & Savin, N. E. (1997), 'Using theory for measurement: An analysis of the cyclical behavior of home production', *Journal of Monetary Economics* **40**(3), 435–456.

Jermann, U. J. (1998), 'Asset pricing in production economies', *Journal of Monetary Economics* **41**(2), 257–275.

Jones, L. E., Manuelli, R. E. & Siu, H. E. (2005), 'Fluctuations in Convex Models of Endogenous Growth II: Business Cycle Properties', *Review of Economic Dynamics* **8**(4), 805–828.

Jorgenson, D. & Fraumeni, B. (1991), The Output Of The Education Sector, Harvard Institute of Economic Research Working Papers 1543, Harvard - Institute of Economic Research.

King, R. G., Plosser, C. I. & Rebelo, S. T. (1988), 'Production, growth and business cycles : I. The basic neoclassical model', *Journal of Monetary Economics* **21**(2-3), 195–232.

King, R. G. & Rebelo, S. T. (2000), Resuscitating Real Business Cycles, NBER Working Papers 7534, National Bureau of Economic Research, Inc.

Kydland, F. E. & Prescott, E. C. (1982), 'Time to Build and Aggregate Fluctuations', *Econometrica* **50**(6), 1345–70.

Li, V. E. (2000), 'Household Credit and the Monetary Transmission Mechanism', *Journal of Money, Credit and Banking* **32**(3), 335–356.

Long, John B, J. & Plosser, C. I. (1983), 'Real Business Cycles', *Journal of Political Economy* **91**(1), 39–69.

Maffezzoli, M. (2000), 'Human Capital and International Real Business Cycles', *Review of Economic Dynamics* **3**(1), 137–165.

Mahalanobis, P. C. (1936), 'On the generalised distance in statistics', *Proceedings of the National Institute of Sciences of India* **2**(1), 49–55.

McGrattan, E. R. (2015), Intangible Capital and Measured Productivity, manuscript, University of Minnesota, June.

McGrattan, E. R. & Prescott, E. C. (2014), 'A Reassessment of Real Business Cycle Theory', *American Economic Review* **104**(5), 177–82.

Mehra, R. & Prescott, E. C. (1985), 'The equity premium: A puzzle', *Journal of Monetary Economics* **15**(2), 145–161.

Mulligan, C. B. & Sala-i Martin, X. (1993), 'Transitional Dynamics in Two-Sector Models of Endogenous Growth', *The Quarterly Journal of Economics* **108**(3), 739–73.

Nelson, C. R. & Plosser, C. I. (1982), 'Trends and random walks in macroeconmic time series : Some evidence and implications', *Journal of Monetary Economics* **10**(2), 139–162.

Nolan, C. & Thoenissen, C. (2009), 'Financial shocks and the US business cycle', *Journal of Monetary Economics* **56**(4), 596–604.

Otani, K. (1996), 'A Human Capital Approach to Entrepreneurial Capacity', *Economica* **63**(250), 273–289.

Perli, R. (1998), 'Increasing returns, home production and persistence of business cycles', *Journal of Economic Dynamics and Control* **22**(4), 519–543.

Perli, R. & Sakellaris, P. (1998), 'Human capital formation and business cycle persistence', *Journal of Monetary Economics* **42**(1), 67–92.

Psacharopoulos, G. (1994), 'Returns to investment in education: A global update', *World Development* **22**(9), 1325–1343.

Restrepo-Ochoa, S. I. & Vazquez, J. (2004), 'Cyclical features of the Uzawa-Lucas endogenous growth model', *Economic Modelling* **21**(2), 285–322.

Rogerson, R. (1988), 'Indivisible Labor, Lotteries and Equilibrium', *Journal of Monetary Economics* **21**, 3–16.

Rotemberg, J. J. & Woodford, M. (1996), 'Real-Business-Cycle Models and the Forecastable Movements in Output, Hours, and Consumption', *American Economic Review* **86**(1), 71–89.

Rybczynski, T. (1955), 'Factor Endowment and Relative Commodity Prices', *Economica* **22**(88), 336–341.

Stolper, W. & Samuelson, P. (1941), 'Protection and real wages', *Review of Economic Studies* **9**(1), 58–73.

Uhlig, H. (1998), A toolkit for analyzing nonlinear dynamic stochastic models easily, Discussion paper / institute for empirical macroeconomics, QM&RBC Codes, Quantitative Macroeconomics & Real Business Cycles.

Van Long, N. (1992), 'The Dynamic Rybczynski Theorem and its Dual', *Journal of Economic Integration* **7**(2), 137–150.

Appendices

A Equilibrium Condtions and the Steady State

A.1 Equilibrium Conditions

Define the Lagrange multiplier of the representative agent's budget constraint as λ_t, and that of the human capital accumulation's as χ_t. Then agent's first order conditions are the following,

$$c_t: \quad c_t^{-\sigma} x_t^{A(1-\sigma)} (1-u_t)^{B(1-\sigma)} = \lambda_t; \tag{22}$$

$$l_{gt}: \quad A c_t^{1-\sigma} x_t^{A(1-\sigma)-1} (1-u_t)^{B(1-\sigma)} = \lambda_t w_t h_t; \tag{23}$$

$$l_{ht}: \quad A c_t^{1-\sigma} x_t^{A(1-\sigma)-1} (1-u_t)^{B(1-\sigma)} = \chi_t (1-\phi_2) A_h e^{z_t^h} \left[\frac{v_{ht} u_t k_t}{l_{ht} h_t}\right]^{\phi_2} h_t; \tag{24}$$

$$\begin{aligned} u_t: \quad B c_t^{1-\sigma} x_t^{A(1-\sigma)} (1-u_t)^{B(1-\sigma)} &= \lambda_t \phi_1 A_g e^{z_t^g} \left[\frac{v_{gt} u_t k_t}{l_{gt} h_t}\right]^{\phi_1 - 1} (v_{gt} k_t) + \\ + \chi_t \phi_2 A_h e^{z_t^h} \left[\frac{v_{ht} u_t k_t}{l_{ht} h_t}\right]^{\phi_2 - 1} (v_{ht} k_t) &- \lambda_t \delta_k u_t^{\psi - 1} k_t; \end{aligned} \tag{25}$$

$$v_{gt}: \quad \lambda_t \phi_1 A_g e^{z_t^g} \left[\frac{v_{gt} u_t k_t}{l_{gt} h_t}\right]^{\phi_1 - 1} (u_t k_t) = \chi_t \phi_2 A_h e^{z_t^h} \left[\frac{(1-v_{gt}) u_t k_t}{l_{ht} h_t}\right]^{\phi_2 - 1} (u_t k_t); \tag{26}$$

$$\begin{aligned} k_{t+1}: \quad \lambda_t = \beta E_t \lambda_{t+1} &\left[1 + r_{t+1} u_{t+1} v_{gt+1} - \frac{\delta_k}{\psi} u_{t+1}^{\psi}\right] + \\ &+ \beta E_t \chi_{t+1} \phi_2 A_h e^{z_{t+1}^h} \left[\frac{v_{ht+1} u_{t+1} k_{t+1}}{l_{ht+1} h_{t+1}}\right]^{\phi_2 - 1} (u_{t+1} v_{ht+1}); \end{aligned} \tag{27}$$

$$h_{t+1}: \quad \chi_t = \beta E_t \chi_{t+1} \left[1 + (1-\phi_2) A_h e^{z_{t+1}^h} \left[\frac{v_{ht+1} u_{t+1} k_{t+1}}{l_{ht+1} h_{t+1}}\right]^{\phi_2} l_{ht+1} - \delta_h\right] + \beta E_t \lambda_{t+1} w_{t+1} l_{gt+1};$$

$$(28)$$

where r_t and w_t denote the own marginal productivity conditions of physical and human capital such that $r_t \equiv \phi_1 A_g e^{z_t^g}(v_{gt}u_t k_t)^{\phi_1 - 1}(l_{gt}h_t)^{1-\phi_1}$ and $w_t \equiv (1-\phi_1) A_g e^{z_t^h}(v_{gt}u_t k_t)^{\phi_1}(l_{gt}h_t)^{-\phi_1}$. Also, $p_{ht} \equiv \frac{\chi_t}{\lambda_t}$ denotes the relative price of human capital in terms of consumption goods. Then the representative agent's equilibrium conditions can be stated as,

$$A_g e^{z_t^g}(v_{gt}u_t k_t)^{\phi_1}(l_{gt}h_t)^{1-\phi_1} = c_t + i_{kt}; \tag{29}$$

$$A_h e^{z_t^h}((1-v_{gt})u_t k_t)^{\phi_2}(l_{ht}h_t)^{1-\phi_2} = h_{t+1} - (1-\delta_h)h_t; \tag{30}$$

$$p_{ht} = \left[\frac{A_g e^{z_t^g}}{A_h e^{z_t^h}}\right]\left[\frac{1-\phi_1}{1-\phi_2}\right]^{1-\phi_2}\left[\frac{\phi_1}{\phi_2}\right]^{\phi_2}\left[\frac{v_{gt}u_t k_t}{l_{gt}h_t}\right]^{\phi_1-\phi_2}; \tag{31}$$

$$\frac{A}{x_t}\frac{c_t}{h_t} = w_t; \tag{32}$$

$$\frac{B}{(1-u_t)}\frac{c_t}{k_t} = r_t - \delta_k u_t^{\psi-1}; \tag{33}$$

$$x_t = 1 - l_{gt} - l_{ht}; \tag{34}$$

$$\frac{1-\phi_1}{\phi_1}\frac{v_{gt}u_t k_t}{l_{gt}h_t} = \frac{1-\phi_2}{\phi_2}\frac{(1-v_{gt})u_t k_t}{l_{ht}h_t}; \tag{35}$$

$$i_{kt} = k_{t+1} - k_t + \frac{\delta_k}{\psi}u_t^{\psi}k_t; \tag{36}$$

$$1 = \beta E_t \left[\left(\frac{c_t}{c_{t+1}} \right)^{\sigma} \left(\frac{x_{t+1}}{x_t} \right)^{A(1-\sigma)} \left(\frac{1-u_{t+1}}{1-u_t} \right)^{B(1-\sigma)} \left[1 + r_{t+1} u_{t+1} - \frac{\delta_k}{\psi} u_{t+1}^{\psi} \right] \right]; \quad (37)$$

$$\begin{aligned} 1 = \beta E_t \left(\frac{c_t}{c_{t+1}} \right) \left(\frac{x_{t+1}}{x_t} \right)^{A(1-\sigma)} \left(\frac{1-u_{t+1}}{1-u_t} \right)^{B(1-\sigma)} \frac{p_{ht+1}}{p_{ht}} \\ \left[1 + (l_{gt+1} + l_{ht+1})(1-\phi_2) A_h e^{z_t^h} \left[\frac{(1-v_{gt+1}) u_{t+1} k_{t+1}}{l_{ht+1} h_{t+1}} \right]^{\phi_2} - \delta_h \right]. \end{aligned} \quad (38)$$

Equation (29) is the goods market clearing condition; equation (30) is the human capital law of motion; equation (31) defines the relative price of human capital in units of consumption goods; equation (32) is the intratemporal condition that governs the substitution between leisure and consumption; meanwhile, equation (33) is the second intratemporal condition governing the substitution between managerial capacity and consumption. Equation (34) is the time constraint; equation (35) equates weighted factor intensities across sectors; and (36) is the physical capital law of motion. Equations (37) and (38) are the inter-temporal capital efficiency conditions with respect to physical and human capital, where the capacity utilization of physical capital is the equivalent of used entrepreneurial capacity, u_t, and the capacity utilization of human capital is equivalent to total working time, $(1 - x_t)$.

The set of 10 equations in (29) - (38) and the marginal efficiency conditions fully describe Model 2. Altogether, there are 12 equations in 12 unknowns $\{k_{t+1}, h_{t+1}, i_{kt}, c_t, u_t, l_{gt}, l_{ht}, x_t, v_{gt}, p_{ht}, r_t^k, w_t\}$. Furthermore, the exogenous variables $\{z_t^g, z_t^h\}$ are governed by the AR(1) processes defined in equations (8), and (9).[23]

A.2 Solution for BGP Economy

First, express the first order conditions for Model 2 in Appendix A.1 in terms of the variables' long-run values. Then the first order conditions become:

$$A_g (v_g u k)^{\phi_1} (l_g h)^{1-\phi_1} = c + i_k; \quad (39)$$

$$A_h ((1-v_g) u k)^{\phi_2} (l_h h)^{1-\phi_2} = h(1+g) - (1-\delta_h) h; \quad (40)$$

[23] Model 1 equilibrium conditions are identical except for being stripped off of the utilization rate and there is no second intratemporal condition as in (33).

$$p_h = \left[\frac{A_g}{A_h}\right]\left[\frac{1-\phi_1}{1-\phi_2}\right]^{1-\phi_2}\left[\frac{\phi_1}{\phi_2}\right]^{\phi_2}\left[\frac{v_g uk}{l_g h}\right]^{\phi_1-\phi_2}; \tag{41}$$

$$\frac{A}{x}\frac{c}{h} = w; \tag{42}$$

$$\frac{B}{(1-u)}\frac{c}{k} = r - \delta_k u^{\psi-1}; \tag{43}$$

$$x = 1 - l_g - l_h; \tag{44}$$

$$\frac{1-\phi_1}{\phi_1}\frac{v_g uk_t}{l_g h_t} = \frac{1-\phi_2}{\phi_2}\frac{v_h uk_t}{l_h h_t}; \tag{45}$$

$$i_k = k(1+g) - k + \frac{\delta_k}{\psi}u^{\psi}k; \tag{46}$$

$$(1+g)^{\sigma} = \beta\left[1 + ru - \frac{\delta_k}{\psi}u^{\psi}\right]; \tag{47}$$

$$(1+g)^{\sigma} = \beta\left[1 + (l_g+l_h)(1-\phi_2)A_h\left[\frac{(1-v_g)uk}{l_h h}\right]^{\phi_2} - \delta_h\right]; \tag{48}$$

where $1+g$ is the gross balanced growth rate of the economy. Define $f_g \equiv \frac{v_g uk}{l_g h}$ and $f_h \equiv \frac{(1-v_g)uk}{l_h h}$. Then the above system can be narrowed down to 8 equations in 8 unknowns, $\{f_g,\, f_h,\, g,\, u,\, \frac{c}{k},\, l_g,\, l_h,\, p_h\}$, by also using that $r_t = \phi_1 A_g f_g^{\phi_1-1}$ and $w_t = (1-\phi_1)A_g f_g^{\phi_1}$, as

$$A_g f_g^{\phi_1-1}u\left[\frac{l_g f_g}{l_g f_g + l_h f_h}\right] = \frac{c}{k} + g + \frac{\delta_k}{\psi}u^{\psi}, \tag{49}$$

$$A_h f_h^{\phi_2} l_h = g + \delta_h; \tag{50}$$

$$p_h = \left[\frac{A_g}{A_h}\right]\left[\frac{1-\phi_1}{1-\phi_2}\right]^{1-\phi_2}\left[\frac{\phi_1}{\phi_2}\right]^{\phi_2} f_g^{\phi_1-\phi_2}; \tag{51}$$

$$\frac{A}{x}\frac{c}{k}\left[\frac{l_g f_g + l_h f_h}{u}\right] = (1-\phi_1)A_g f_g^{\phi_1}; \tag{52}$$

$$\frac{B}{(1-u)}\frac{c}{k} = \phi_1 A_g f_g^{\phi_1-1} - \delta_k u^{\psi-1}; \tag{53}$$

$$\frac{1-\phi_1}{\phi_1} f_g = \frac{1-\phi_2}{\phi_2} f_h; \tag{54}$$

$$(1+g)^\sigma = \beta\left[1 + \phi_1 A_g f_g^{\phi_1-1} u - \frac{\delta_k}{\psi} u^\psi\right]; \tag{55}$$

$$(1+g)^\sigma = \beta\left[1 + (l_g + l_h)(1-\phi_2)A_h f_h^{\phi_2} - \delta_h\right]. \tag{56}$$

Given the exogenous information set of parameters (ϕ_1, ϕ_2, A_g, A_h, δ_k, ψ, δ_h, β, σ, A, B), the uniqueness of the solution to the system in (49) - (56) can be narrowed down to the uniqueness of the variables g and u. In order to show this, one can solve for f_g, f_h, $\frac{c}{k}$, l_g, l_h, p_h in terms of g and u, which leaves a system of two equations, (52) and (53), in two unknowns g and u. First, one may solve for f_g using (55) as,

$$f_g = \left[\frac{\frac{(1+g)^\sigma}{\beta} - 1 + \frac{\delta_k}{\psi} u^\psi}{\phi_1 A_g}\right]^{\frac{1}{\phi_1-1}u}. \tag{57}$$

Then f_h directly follows from (54),

$$f_h = \frac{1-\phi_1}{1-\phi_2}\frac{\phi_2}{\phi_1} f_g. \tag{58}$$

Then one can express total labor time $(l_g + l_h) \equiv D$ from equation (56):

$$D = (l_g + l_h) = \left[\frac{\frac{(1+g)^\sigma}{\beta} - 1 + \delta_h}{(1-\phi_2) A_h f_h^{\phi_2}} \right]. \tag{59}$$

To express the time shares one can express l_h in terms of g and u from equation (50) and then use the solution for total labor time in equation (59),

$$l_h = \left[\frac{g + \delta_h}{A_h} \right] f_h^{-\phi_2}, \tag{60}$$

$$l_g = D - l_h. \tag{61}$$

Next by using equation (51) and the obtained expression for f_g it follows that the relative price of human capital in terms of g and u is,

$$p_h = \left[\frac{A_g}{A_h} \right] \left[\frac{1-\phi_1}{1-\phi_2} \right]^{1-\phi_2} \left[\frac{\phi_1}{\phi_2} \right]^{\phi_2} f_g^{\phi_1 - \phi_2}. \tag{62}$$

Now one can obtain an expression in g and u for c/k from equation (49),

$$\frac{c}{k} = A_g f_g^{\phi_1 - 1} \left[\frac{l_g f_g}{l_g f_g + l_h f_h} \right] - g - \frac{\delta_k}{\psi} u^\psi. \tag{63}$$

Then after substituting (57) - (63) into equation (52) and (53) one obtains a system of two highly nonlinear equations in g and u: $\Omega(g, u) = 0$. This system of two equations then can be solved numerically for the baseline calibration of parameters defined in Table 1.[24]

[24] For Model 1 the steady state solution can be obtained in a similar fashion where the solution can be narrowed down to one equation (intratemporal margin) in one unknown (the BGP growth rate).

A.3 The Decentralized Economy

The representative consumer maximizes its discounted lifetime utility time t period utility is given by

$$\max E_t \sum_{t=0}^{\infty} \beta^t \frac{\left[c_t x_t^A (1-u_t)^B\right]^{1-\sigma} - 1}{1-\sigma}, \tag{64}$$

subject to the time constraint in equation (2), the physical accumulation constraint in (3), the human capital law of motion and human capital investment technology in equations (6) and (7), given the exogenous processes in (8)-(10). Setting up to constrained optimization problem of the consumer yields the following equilibrium conditions that are identical to the social planner's equilibrium conditions in Appendix A.1:

$$\frac{A}{x_t}\frac{c_t}{h_t} = w_t; \tag{65}$$

$$\frac{B}{1-u_t}\frac{c_t}{h_t} = \left[r_t - \delta_k u_t^{\psi-1}\right]\frac{k_t}{h_t}; \tag{66}$$

$$p_{ht} = \frac{w_t}{(1-\phi_2)A_h e^{z_t^h}\left[\frac{v_{ht}u_t k_t}{l_{ht}h_t}\right]^{\phi_2}} = \frac{r_t}{\phi_2 A_h e^{z_t^h}\left[\frac{v_{ht}u_t k_t}{l_{ht}h_t}\right]^{\phi_2-1}}; \tag{67}$$

$$1 = \beta E_t \left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\left(\frac{x_{t+1}}{x_t}\right)^{A(1-\sigma)}\left(\frac{1-u_{t+1}}{1-u_t}\right)^{B(1-\sigma)}\left[1 + r_{t+1} - \frac{\delta_k}{\psi}u_{t+1}^{\psi}\right]; \tag{68}$$

$$\begin{aligned} 1 \;\; = \;\; & \beta E_t \left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\left(\frac{x_{t+1}}{x_t}\right)^{A(1-\sigma)}\left(\frac{1-u_{t+1}}{1-u_t}\right)^{B(1-\sigma)} \\ & \left(\frac{p_{ht+1}}{p_{ht}}\right)\left[1 + (1-x_{t+1})(1-\phi_2)A_h e^{z_{t+1}^h}\left[\frac{v_{ht+1}u_{t+1}k_{t+1}}{l_{ht+1}h_{t+1}}\right]^{\phi_2} - \delta_h\right]. \end{aligned} \tag{69}$$

The firm maximizes profit Π_{gt}, given by $\Pi_{gt} = y_t - w_t l_{gt} h_t - r_t v_{gt} u_t k_t$, subject to the Cobb-Douglas production technology in equation (5), which yields the equilibrium conditions that equate marginal products with real factor input prices as

$$r_t = \phi_1 A_g e^{z_t^g} \left[\frac{v_{gt} u_t k_t}{l_{gt} h_t}\right]^{\phi_1 - 1}; \tag{70}$$

$$w_t = (1-\phi_1) A_g e^{z_t^g} \left[\frac{v_{gt} u_t k_t}{l_{gt} h_t}\right]^{\phi_1}. \tag{71}$$

B Data Description

The US data used in this paper is from 1959:Q1 until 2015:Q4 except for that of the physical capital utilization rate, which is only available from 1971:Q4, and human and physical capital data, which is available only until the end of 2012. In constructing real data series for US macroeconomic variables Gomme & Rupert (2007) have been followed. Analogously to their methodology the aggregate series are constructed as:[25]

1. *Nominal Market Investment* = Non-residential Fixed Investment + Change in Private Inventories
2. *Nominal Home Investment* = Residential Fixed Investment + PCE on Durables
3. *Nominal Investment* = Nominal Home Investment + Nominal Market Investment
4. *Real Investment*= Nominal Investment / (Average Price Deflator / 100)
5. *Nominal Market Output* = Gross Domestic Product - PCE: Housing Services
6. *Nominal Private Market Output* = Nominal Market Output - Employee Compensation: Government
7. *Real Market Output* = Nominal Market Output / (Average Price Deflator / 100)
8. *Real Private Market Output* = Nominal Private Market Output / (Average Price Deflator / 100)
9. *Physical Capital Utilization Rate* = Total Capacity Utilization: Manufacturing
10. *Labor Hours* = Non-farm Business Sector: Average Weekly Hours
11. *Nominal Market Consumption* = PCE on Non-durable Goods + PCE on Services - PCE on Housing Services

[25] The raw series and the construction of the underlying data series can be found in *data.xls* included with the Matlab files upon request.

12. *Real Market Consumption*= Nominal Market Consumption / (Average Price Deflator/100)

13. *Average Price Deflator* = (Implicit Price Deflator:Non-durables + Implicit Price Deflator: Services)/2

According to Gomme & Rupert (2007), output (y) is measured by real per capita GDP less real per capita Gross Housing Product as defined above. It is due to the argument that home sector production should be removed when calculating market output using the *National Income and Product Accounts* (NIPA). The price deflator is constructed by taking the average of the implicit price deflators on non-durables and services. Population is measured by the number of non-institutionalized persons aged over 16 years. Consumption (c) is measured by real personal expenditures on non-durables and services less Gross Housing Services. Investment is measured by the sum of real Non-residential Fixed Investment, the Change in Private Inventories, Residential Fixed Investment, and Personal Consumption Expenditures on durables. Lastly, working hours are measured by the average weekly labor hours.

The annual index of human capital per person data series is based on years of schooling [Barro & Lee (2013)], and returns to education [Psacharopoulos (1994)]. The series have been constructed by Feenstra et al. (2013) using the perpetual inventory method. Quarterly human capital data has been interpolated using the annual data of Feenstra et al. (2013) by following Baier et al. (2004) where they define the depreciation rate to human capital as the average of death rates in different age groups for which the data has been obtained from the Center for Disease Control (CDC) database. Also, for the period after 2012 the human capital data has been forecasted by fitting it to an AR1 process. The quarterly physical capital data is constructed from Bureau of Economic Analysis (BEA) annual US capital stock estimates and quarterly data on investment expenditures.

Description	Units	Seasonally Adjusted	BEA / BLS Code / Source	Frequency	Time Range
PCE on Nondurable Goods	Billions of Dollars	SAAR	DNDGRC1	Quarterly	1959Q1 - 2015Q4
PCE on Services	Billions of Dollars	SAAR	DSERRC1	Quarterly	1959Q1 - 2015Q4
PCE on Housing Services	Billions of Dollars	SAAR	DHUTRC1	Quarterly	1959Q1 - 2015Q4
Implicit Price Deflator (Nondurables)	Index 2009=100	SA	DNDGRD3	Quarterly	1959Q1 - 2015Q4
Implicit Price Deflator (Services)	Index 2009=100	SA	DSERRD3	Quarterly	1959Q1 - 2015Q4
Gross Domestic Product	Billions of Dollars	SAAR	A191RC1	Quarterly	1959Q1 - 2015Q4
Employee Compensation: Government	Billions of Dollars	SAAR	B202RC1	Quarterly	1959Q1 - 2015Q4
Civ. Noninst. Pop. 16 and over	1000s of Persons	NSA	CNP160V	Quarterly	1959Q1 - 2015Q4
Capacity Utilization: Manufacturing	Percentage	SA	TCU:MAN.	Quarterly	1971Q4 - 2015Q4
Nonfarm Bus. Sector: Avg. Weekly Hours	Index 2009=100	SA	PRS585006023	Quarterly	1959Q1 - 2015Q4
Nonres. Fixed Investment	Billions of Dollars	SAAR	A008RC1	Quarterly	1959Q1 - 2015Q4
Residential Fixed Investment	Billions of Dollars	SAAR	A011RC1	Quarterly	1959Q1 - 2015Q4
Change in Private Inventories	Billions of Dollars	SAAR	A014RC1	Quarterly	1959Q1 - 2015Q4
PCE on Durables	Billions of Dollars	SAAR	DDURRC1	Quarterly	1959Q1 - 2015Q4
Index of Human Capital per Person	Index	NSA	Penn World Tables 8.0	Annual	1950 - 2012

Table 11: Raw Data Sources.

C Duality Theorems

Stolper & Samuelson (1941) and Rybczynski (1955) duality theorems underlie the movement of resources between sectors of the general Model 2. This section provides the proof of the dual theorems.

Duality results because as the Ryb input factors increase, they cause a change in the relative price of sectoral outputs, at which point, immediately, the stol-sam theorem kicks in and determines in addition how input prices should change. The next proposition establishes first that a change in the relative price of outputs, denoted her by $p_{ht,}$ causes a change in rental prices w_t and r_t in accordance to the relative factor intensities, as in the Stolper-Samuelson theorem. The Corollary then applies this to the notion that in general equilibrium an exogenous parameter must change in order for the relative price to change. In this economy, only the productivity parameters change, so the Corollary's application is to an identical productivity shock to both sectors.

Let p_{ht} denote the relative price of human capital investment to physical capital investment, as defined by $p_{ht} \equiv \frac{\chi_t}{\lambda_t}$, where χ_t is the shadow price of human capital investment and λ_t is the shadow price of physical capital investment, as given in equations (27) and (28) of Appendix A.1.

Proposition 4 *In the two-sector economy of Section 2.2, when the price of human capital investment, relative to the price of physical capital investment, rises then the real wage w_t rises and the real interest r_t^k falls because the human capital investment sector is relatively intensive in human capital ($\phi_1 > \phi_2$), and the real wage is the factor rental price of human capital. This means that the change in r_t and w_t with respect to p_{ht} depends only on the factor intensity ranking.*

Proof. From equations (130) and (131) of Appendix A.1 and denoting the implicit factor rental prices by r_t^k and w_t, as also given in Appendix A.1 for physical capital and human capital respectively, the following relations hold between the rental prices and the relative price of capital p_{ht} :

$$r_t \equiv \phi_1 (z_t^h)^{\frac{\phi_1-1}{\phi_1-\phi_2}} (z_t^g)^{\frac{1-\phi_2}{\phi_1-\phi_2}} A_h^{\frac{\phi_1-1}{\phi_1-\phi_2}} A_g^{\frac{1-\phi_2}{\phi_1-\phi_2}} \left[\frac{\phi_2}{\phi_1}\right]^{\frac{\phi_2(\phi_1-1)}{\phi_1-\phi_2}} \left[\frac{1-\phi_2}{1-\phi_1}\right]^{\frac{(1-\phi_2)(\phi_1-1)}{\phi_1-\phi_2}} p_{ht}^{\frac{\phi_1-1}{\phi_1-\phi_2}}; \quad (72)$$

$$w_t \equiv (1-\phi_1)(z_t^h)^{\frac{\phi_1}{\phi_1-\phi_2}} (z_t^g)^{\frac{1-\phi_2}{\phi_1-\phi_2}} A_h^{\frac{\phi_1}{\phi_1-\phi_2}} A_g^{\frac{-\phi_2}{\phi_1-\phi_2}} \left[\frac{\phi_2}{\phi_1}\right]^{\frac{\phi_2\phi_1}{\phi_1-\phi_2}} \left[\frac{1-\phi_2}{1-\phi_1}\right]^{\frac{(1-\phi_2)\phi_1}{\phi_1-\phi_2}} p_{ht}^{\frac{\phi_1}{\phi_1-\phi_2}}. \quad (73)$$

Given that $\phi_1 > \phi_2$ by assumption, making the human capital investment relatively more intensive in human capital relative to the goods sector, then equations (72) and (73) imply $\frac{\partial r_t}{\partial p_{ht}} < 0$ and $\frac{\partial w_t}{\partial p_{ht}} > 0$. ■

Corollary 5 *In the two-sector economy of Section 2.2, the Stolper & Samuelson (1941) effect in the two-shock economy: as a special case of equations (8)-(10), assume identical shocks such that in log-linear (deviations from the steady state) form* $\hat{z}_t^g = \hat{z}_t^h$*; after such an economy-wide shock, an increase in the relative price of a sector's output will cause a relatively bigger increase in the implicit competitive price of the unit of the input that is used relatively intensively in that sector.*

Proof. Combine equations (72) and (73) and log-linearize along the models' respective steady states, with log-linearized variables denoted by a hat, to find that

$$\hat{w}_t - \hat{r}_t = \frac{\hat{p}_{ht} - \hat{z}_t^g + \hat{z}_t^h}{\phi_1 - \phi_2}. \tag{74}$$

Given $\phi_1 > \phi_2$ and identical shocks such that $\hat{z}_t^g = \hat{z}_t^h$, an increase in p_{ht} causes $[\hat{w}_t - \hat{r}_t]$ to increase, so that $\hat{w}_t$ rises more than $\hat{r}_t$, the human capital investment sector is relatively more factor intensive in human capital than in physical capital, and so the relative price of the more intensively used factor within the human capital sector rises when the relative price of human capital investment rises; conversely for the goods sector. ■

In a dual way and following Van Long (1992), consider the next proposition.

Proposition 6 *Rybczynski (1955) effect: In the two-sector economy of Section 2.2, an increase in the allocation of a factor input in a sector will expand the output of that sector if it is more intensive in the increased input; the output of the other sector more intensive in the other factor input will decrease or increase by a relatively lower quantity.*

Proof. One sector produces goods y_t (or alternatively physical capital investment) with a price normalized to unity; the second sector produces human capital investment i_{ht} at a relative price p_{ht}, in terms of the goods output. Appendix A.1 in equation 31 derives the relative price as the ratio of the marginal products with respect to human capital of each the goods and human capital investment investment sectors:

$$p_{ht} = \frac{(1-\phi_1)A_g z_t^g \left[\frac{v_{gt}u_t k_t}{l_{gt}h_t}\right]^{\phi_1}}{(1-\phi_2)A_h z_t^h \left[\frac{v_{ht}u_t k_t}{l_{ht}h_t}\right]^{\phi_2}}. \tag{75}$$

Consider the change in human capital investment with respect to each factor of physical capital given by $v_{ht}u_t k_t$ and of human capital given by $l_{ht}h_t$, where these are notated by R_1^h and R_2^h respectively, so that $R_1^h \equiv \frac{\partial i_{ht}}{\partial v_{ht}u_t k_t}$, and $R_2^h \equiv \frac{\partial i_{ht}}{\partial l_{ht}h_t}$, and these are standard marginal products. It follows the relative price p_{ht} can be expressed in terms of only the factor prices r_t and w_t, and the marginal products R_1^h and R_2^h:

$$R_1^h = \phi_2 A_h z_t^h \left[\frac{v_{ht} u_t k_t}{l_{ht} h_t}\right]^{\phi_2 - 1} = \frac{r_t}{p_{ht}}, \tag{76}$$

$$R_2^h = (1-\phi_2) A_h z_t^h \left[\frac{v_{ht} u_t k_t}{l_{ht} h_t}\right]^{\phi_2} = \frac{w_t}{p_{ht}}. \tag{77}$$

Then given the assumption that $(1-\phi_2) > \phi_2$ it follows that $R_1^h < R_2^h$. This implies that in the human capital investment sector, increasing human capital by a unit will increase output of human capital investment by more than would increasing physical capital by a unit; conversely for the goods sector. ■

Proposition 7 *Define* S_1^h *and* S_2^h *as the change in the real interest rate and the wage rate with respect to a change in the relative price of human capital, so that* $S_1^h \equiv \frac{\partial r_t}{\partial p_{ht}}$ *and* $S_2^h \equiv \frac{\partial w_t}{\partial p_{ht}}$. *It follows that* $S_1^h = \frac{r_t}{p_{ht}}$ *and* $S_2^h = \frac{w_t}{p_{ht}}$.

Proof. For S_1^h consider equation (26) in Appendix A.1:

$$\lambda_t \phi_1 A_g e^{z_t^g} \left[\frac{v_{gt} u_t k_t}{l_{gt} h_t}\right]^{\phi_1 - 1} (u_t k_t) = \chi_t \phi_2 A_h e^{z_t^h} \left[\frac{(1-v_{gt}) u_t k_t}{l_{ht} h_t}\right]^{\phi_2 - 1} (u_t k_t). \tag{78}$$

Dividing both sides by λ_t and by $(u_t k_t)$ one obtains

$$\phi_1 A_g e^{z_t^g} \left[\frac{v_{gt} u_t k_t}{l_{gt} h_t}\right]^{\phi_1 - 1} = \frac{\chi_t}{\lambda_t} \phi_2 A_h e^{z_t^h} \left[\frac{(1-v_{gt}) u_t k_t}{l_{ht} h_t}\right]^{\phi_2 - 1}. \tag{79}$$

Then using the definitions for p_{ht} and r_t^k in Appendix A one can write equation (79) as

$$r_t^k = p_{ht} \phi_2 A_h e^{z_t^h} \left[\frac{(1-v_{gt}) u_t k_t}{l_{ht} h_t}\right]^{\phi_2 - 1}. \tag{80}$$

By taking the partial derivative of the right hand side of equation (80) with respect to $p_{ht} \equiv \chi_t / \lambda_t$ it directly follows that

$$S_1^h = \frac{\partial r_t^k}{\partial p_{ht}} = \phi_2 A_h z_t^h \left[\frac{v_{ht} u_t k_t}{l_{ht} h}\right]^{\phi_2 - 1} = \frac{r_t^k}{p_{ht}}. \tag{81}$$

For S_2^h equation the right hand side of equations (130) and (131) in Appendix A.1:

$$\lambda_t w_t h_t = \chi_t (1-\phi_2) A_h e^{z_t^h} \left[\frac{v_{ht} u_t k_t}{l_{ht} h_t}\right]^{\phi_2} h_t. \tag{82}$$

After simplifying with h_t and dividing both sides of (82) with λ_t one gets

$$w_t = \frac{\chi_t}{\lambda_t}(1-\phi_2)A_h e^{z_t^h}\left[\frac{v_{ht}u_t k_t}{l_{ht}h_t}\right]^{\phi_2}; \tag{83}$$

Then using the definition of $p_{ht} \equiv \chi_t/\lambda_t$ yields

$$w_t = p_{ht}(1-\phi_2)A_h e^{z_t^h}\left[\frac{v_{ht}u_t k_t}{l_{ht}h_t}\right]^{\phi_2}; \tag{84}$$

By taking the partial derivative of the right hand side of equation (84) with respect to p_{ht} it directly follows that

$$S_2^h = \frac{\partial w_t}{\partial p_{ht}} = (1-\phi_2)A_h z_t^h\left[\frac{v_{ht}u_t k_t}{l_{ht}h_t}\right]^{\phi_2} = \frac{r_t^k}{p_{ht}}. \tag{85}$$

■

Corollary 8 *Duality between Stolper-Sameulson and Rybczynski effects: The change in the output of the human capital investment sector with respect to a change in an input is equal respectively to the change in that input's implicit competitive price with respect to a change in the implicit relative price of human capital investment to goods output.*

Proof. Propositions 6 and 7 imply directly that $R_1^h = S_1^h$ and $R_2^h = S_2^h$. ■

D Backed-Out Shock Positive and Negative Properties

Figure 12 show the positive range of the backed out goods sector TFP shock, as compared to the negative range in Figure 13. Figure 12 has an average median probability density around 3000 while Figure 13 has one around 1000. Positive values cause increases upwards from the Trough to the Peak, using Harding and Pagan (2002) terminology. These are substantially higher than the negative values that cause "Peak to Trough" movement. This indicates higher, or stronger, positive upswings in the economy and less strong, negative, downturns in the economy, an asymmetry found in US data by Harding and Pagan (2002) and Hansen and Prescott (2005).

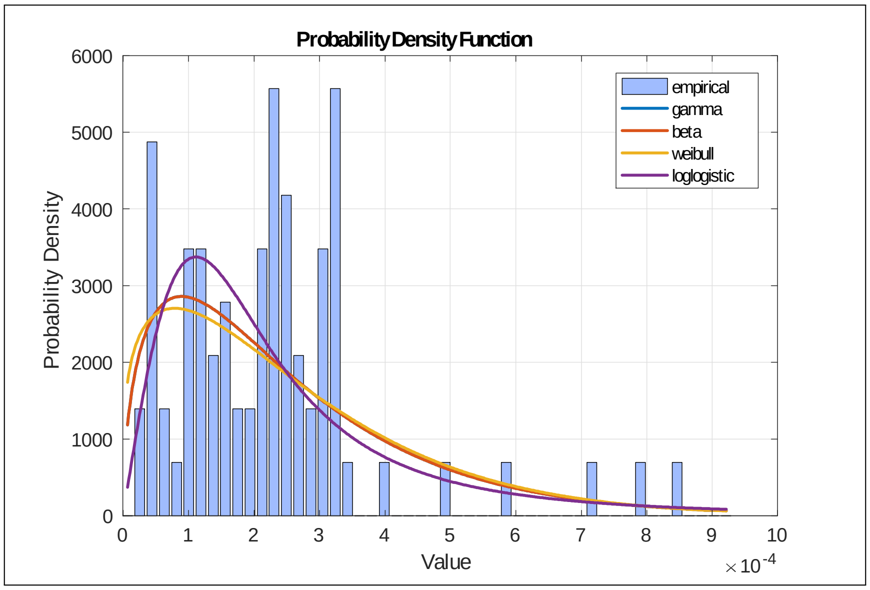


Figure 12: Positive Range: Magnitude of Probability Density and Estimated Distribution of Positive "Backed-out" Goods Sector TFP Shock.

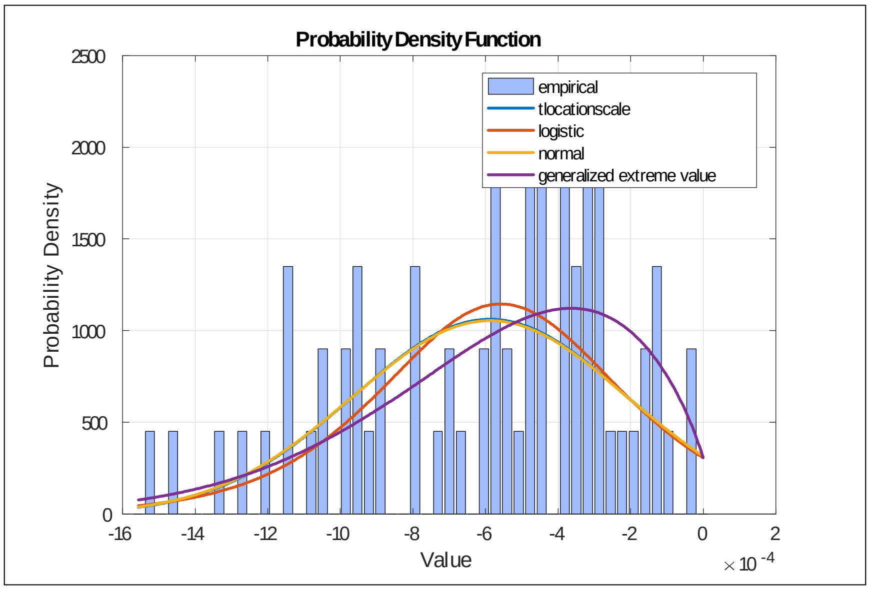


Figure 13: Negative TFP Range: Magnitude of Probability Density and Estimated Distribution of Negative "Backed-out" Goods Sector TFP Shock.

Online Appendices

A Stochastic Discounting and Log-linear Solution Methodology

Such endogenous growth models exhibit non-stationary features. Endogenous variables $\{k_{t+1}, h_{t+1}, c_t, i_{kt}, i_{ht}\}$ grow with a common rate along the BGP. In order to be able to solve the model in (29) - (38), respectively, one has to rewrite the systems of equations by using the following newly defined stationary variables $g_{ht+1} \equiv \frac{h_{t+1}}{h_t}$; $\tilde{k}_t \equiv \frac{k_t}{h_t}$; $\tilde{\imath}_{kt} \equiv \frac{i_{kt}}{h_t}$;$\tilde{\imath}_{kt} \equiv \frac{i_{kt}}{h_t}$; and $\tilde{c}_t \equiv \frac{c_t}{h_t}$.

Then by using the factor reward definitions in Appendix A.1, the stationary model is log-linearized along its steady state, after which the method of undetermined coefficients is applied as in Uhlig (1998) to solve for the recursive policy functions of the models. Any variables with a hat represent the variable in log-deviation from its steady state and variables without time subscripts represent steady state values.

A.1 Model 2 - Stochastic Discounting

After normalizing the growing endogenous variables with the human capital stock, h_t, the equilibrium conditions in (29) - (38) become:

$$A_g e^{z_t^g} (v_{gt} u_t \tilde{k}_t)^{\phi_1} l_{gt}^{1-\phi_1} = \tilde{c}_t + \tilde{k}_{t+1}(1 + g_{ht+1}) - \tilde{k}_t + \frac{\delta_k}{\psi} u_t^{\psi} \tilde{k}_t; \tag{86}$$

$$A_h e^{z_t^h} ((1 - v_{gt}) u_t \tilde{k}_t)^{\phi_2} l_{ht}^{1-\phi_2} = g_{ht+1} + \delta_h; \tag{87}$$

$$p_{ht} = \left[\frac{A_g}{A_h}\right] \left[\frac{e^{z_t^g}}{e^{z_t^h}}\right] \left[\frac{1-\phi_1}{1-\phi_2}\right]^{1-\phi_2} \left[\frac{\phi_1}{\phi_2}\right]^{\phi_2} \left[\frac{v_{gt} u_t \tilde{k}_t}{l_{gt}}\right]^{\phi_1 - \phi_2}; \tag{88}$$

$$\frac{A}{x_t} \tilde{c}_t = (1 - \phi_1) A_g e^{z_t^g} \left[\frac{v_{gt} u_t \tilde{k}_t}{l_{gt}}\right]^{\phi_1}; \tag{89}$$

$$\frac{B}{(1-u_t)}\tilde{c}_t = \left[\phi_1 A_g e^{z_t^g}\left[\frac{v_{gt}u_t\tilde{k}_t}{l_{gt}}\right]^{\phi_1-1} - \delta_k u_t^{\psi-1}\right]\tilde{k}_t; \tag{90}$$

$$x_t = 1 - l_{gt} - l_{ht}; \tag{91}$$

$$\frac{1-\phi_1}{\phi_1}\frac{v_{gt}u_t\tilde{k}_t}{l_{gt}} = \frac{1-\phi_2}{\phi_2}\frac{(1-v_{gt})u_t\tilde{k}_t}{l_{ht}}; \tag{92}$$

$$\begin{aligned}1 = \beta E_t\left(\frac{\tilde{c}_t}{\tilde{c}_{t+1}}\right)^{\sigma}\left(\frac{1}{1+g_{ht+1}}\right)^{\sigma}\left(\frac{x_{t+1}}{x_t}\right)^{A(1-\sigma)}\left(\frac{1-u_{t+1}}{1-u_t}\right)^{B(1-\sigma)}\\ \left[1+\phi_1 A_g e^{z_{t+1}^g}\left[\frac{v_{gt+1}u_{t+1}\tilde{k}_{t+1}}{l_{gt+1}}\right]^{\phi_1-1} u_{t+1} - \frac{\delta_k}{\psi}u_{t+1}^{\psi}\right];\end{aligned} \tag{93}$$

$$\begin{aligned}1 = \beta E_t\left(\frac{\tilde{c}_t}{\tilde{c}_{t+1}}\right)\left(\frac{1}{1+g_{ht+1}}\right)^{\sigma}\left(\frac{x_{t+1}}{x_t}\right)^{A(1-\sigma)}\left(\frac{1-u_{t+1}}{1-u_t}\right)^{B(1-\sigma)}\frac{p_{ht+1}}{p_{ht}}\\ \left[1+(l_{gt+1}+l_{ht+1})(1-\phi_2)A_h e^{z_{t+1}^g}\left[\frac{(1-v_{gt+1})u_{t+1}\tilde{k}_{t+1}}{l_{ht+1}}\right]^{\phi_2} - \delta_h\right].\end{aligned} \tag{94}$$

A.2 Log-Linearized System of Model 2

Here the log-linearized system of equations is presented as implemented in Matlab. Variables with a hat represent variables transformed to log-linear deviations from their steady states. The variable $\hat{G}_{ht}$ denotes the log-linearized version of the gross growth rate of human capital.

$$0 \approx -\hat{\tilde{y}}_t + \left[\frac{\tilde{c}}{\tilde{y}}\right]\hat{\tilde{c}}_t + \left[\frac{\tilde{i}_k}{y}\right]\hat{\tilde{i}}_{kt}; \tag{95}$$

$$0 = -\hat{\tilde{y}}_t + \hat{z}_t^g + \phi_1\hat{v}_{gt} + \phi_1\hat{u}_t + \phi_1\hat{\tilde{k}}_{t-1} + (1-\phi_1)\hat{l}_{gt}; \tag{96}$$

$$0 \approx -\hat{\tilde{i}}_{kt} + \left[\frac{(1+g)k}{\tilde{i}_k}\right]\hat{\tilde{k}}_t + \left[\frac{(1+g)k}{\tilde{i}_k}\right]\hat{G}_{ht} + \left[\frac{\frac{\delta_k}{\psi}\tilde{k}u^{\psi} - \tilde{k}}{\tilde{i}_k}\right]\hat{\tilde{k}}_{t-1} + \left[\frac{\delta_k \tilde{k} u^{\psi}}{\tilde{i}_k}\right]\hat{u}_t \quad (97)$$

$$0 \approx -\hat{\tilde{i}}_{ht} + \left[\frac{(1+g)}{\tilde{i}_h}\right]\hat{G}_{ht} \quad (98)$$

$$0 = -\hat{\tilde{i}}_{ht} + \hat{z}_t^h + \phi_2\hat{v}_{ht} + \phi_2\hat{u}_t + \phi_2\hat{\tilde{k}}_{t-1} + (1-\phi_2)\hat{l}_{ht}; \quad (99)$$

$$0 \approx \hat{x}_t + \left[\frac{l_g}{x}\right]\hat{l}_{gt} + \left[\frac{l_h}{x}\right]\hat{l}_{ht}; \quad (100)$$

$$0 \approx \hat{v}_{gt} + \left[\frac{v_h}{v_g}\right]\hat{v}_{ht}; \quad (101)$$

$$0 = -\hat{\tilde{c}}_t + \hat{w}_t + \hat{x}_t; \quad (102)$$

$$0 = -\hat{\tilde{c}}_t + \hat{\tilde{k}}_{t-1} + \hat{s}_t + \left[\frac{r^k}{r^k - \delta_k u^{\psi-1}}\right]\hat{r}_t^k - \left[\frac{\delta_k(\psi-1)u^{\psi-1}}{r^k - \delta_k u^{\psi-1}}\right]\hat{u}_t \quad (103)$$

$$0 \approx \hat{s}_t + \left[\frac{u}{s}\right]\hat{u}_t; \quad (104)$$

$$0 = \hat{v}_{gt} - \hat{v}_{ht} + \hat{l}_{ht} - \hat{l}_{gt}; \quad (105)$$

$$0 = -\hat{p}_{ht} + \hat{z}_t^g - \hat{z}_t^h + [\phi_1 - \phi_2]\hat{v}_{gt} + [\phi_1 - \phi_2]\hat{u}_t + [\phi_1 - \phi_2]\hat{\tilde{k}}_{t-1} + [\phi_2 - \phi_1]\hat{l}_{gt}; \quad (106)$$

$$0 = -\hat{r}^k_t + \hat{z}^g_t + [\phi_1 - 1]\hat{v}_{gt} + [\phi_1 - 1]\hat{u}_t + [\phi_1 - 1]\hat{\hat{k}}_{t-1} + [1 - \phi_1]\hat{l}_{gt}; \tag{107}$$

$$0 = -\hat{w}_t + \hat{z}^g_t + \phi_1\hat{v}_{gt} + \phi_1\hat{u}_t + \phi_1\hat{\hat{k}}_{t-1} - \phi_1\hat{l}_{gt}; \tag{108}$$

$$\begin{aligned} 0 = E_t\{\sigma\hat{\hat{c}}_t - \sigma\hat{\hat{c}}_{t+1} - \sigma\hat{G}_{ht+1} + A(1-\sigma)\hat{x}_{t+1} - A(1-\sigma)\hat{x}_t + \frac{uB(1-\sigma)}{1-u}\hat{u}_t \\ + \left[\frac{r^k u}{1 + r^k u - \frac{\delta_k}{\psi}u^\psi}\right]\hat{v}_{gt+1} \\ + \left[\frac{r^k u - \delta_k u^\psi}{1 + r^k u - \frac{\delta_k}{\psi}u^\psi} - \frac{uB(1-\sigma)}{1-u}\right]\hat{u}_{t+1}\}; \end{aligned} \tag{109}$$

$$\begin{aligned} 0 = E_t\{\sigma\hat{\hat{c}}_t - \sigma\hat{\hat{c}}_{t+1} - \sigma\hat{G}_{ht+1} + A(1-\sigma)\hat{x}_{t+1} - A(1-\sigma)\hat{x}_t - \frac{uB(1-\sigma)}{1-u}\hat{u}_{t+1} \\ + \frac{uB(1-\sigma)}{1-u}\hat{u}_t - \hat{p}_{ht} \\ + \left[\frac{(1-\delta_h)}{1 + \frac{w}{p_h}(l_g + l_h) - \delta_h}\right]\hat{p}_{ht+1} \\ + \left[\frac{\frac{w}{p_h}(l_g + l_h)}{1 + \frac{w}{p_h}(l_g + l_h) - \delta_h}\right]\hat{w}_{t+1} \\ + \left[\frac{\frac{w}{p_h}l_g}{1 + \frac{w}{p_h}(l_g + l_h) - \delta_h}\right]\hat{l}_{gt+1} \\ + \left[\frac{\frac{w}{p_h}l_h}{1 + \frac{w}{p_h}(l_g + l_h) - \delta_h}\right]\hat{l}_{ht+1}\} \end{aligned} \tag{110}$$

$$\hat{z}^g_{t+1} = \rho_g\hat{z}^g_t + \epsilon^g_{t+1}; \tag{111}$$

$$\hat{z}^h_{t+1} = \rho_h\hat{z}^h_t + \epsilon^h_{t+1}; \tag{112}$$

A.3 Solution Methodology

After obtaining the log-linear systems in equations (95) to (112) the method proposed by Uhlig (1998) is applied to solve the model and obtain the recursive policy functions. The Uhlig (1998) method based on the method of undetermined coefficients following King et al. (1988) is chosen, as it is relatively simple to implement. In order to be able to apply this solution method one has to rewrite the above log-linear first order conditions in the following matrix form:

$$Ax_t + Bx_{t-1} + Cy_t + Dz_t = 0, \tag{113}$$

$$E_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t Lz_{t+1} + Mz_t] = 0, \tag{114}$$

$$z_{t+1} = Nz_t + \epsilon_{t+1}, \tag{115}$$

where $E_t(\epsilon_{t+1}) = 0$; the vector x_t (size $mx1$) contains the endogenous state variables; y_t (size $nx1$) is the vector of all other endogenous variables; meanwhile, z_t (size $kx1$) is the vector of exogenous stochastic variables. It is assumed that the coefficient matrix C is of size lxn, where $l \geq n$ and of rank n. l is the number of deterministic equations, F is a coefficient matrix of size $(m+n-l)xm$, and N has only stable eigenvalues. In the underlying baseline model x_t contains the log-linear versions of $\hat{k}_t$ and $\hat{g}_{ht}$. There are two exogenous variables in z_t, namely, $\hat{z}_t^g$ and $\hat{z}_t^h$. All other six and seven endogenous variables in y_t, are $\hat{c}_t$, $\hat{u}_t$, $\hat{l}_{gt}$, $\hat{l}_{ht}$, $\hat{v}_{gt}$, $\hat{p}_{ht}$.[26]

The log-linear solution method by Uhlig (1998) is seeking to find a recursive equilibrium law of motion of the following form:

$$x_t = Px_{t-1} + Qz_t; \tag{116}$$

$$y_t = Rx_{t-1} + Sz_t. \tag{117}$$

[26] The growth rate of human capital $g_{ht} \equiv h_{t+1}/h_t$ is defined as a state variable in order to satisfy the requirement that the $l \geq n$ condition is imposed by the log-linear approximation. Since it is not a state variable in the proper sense it will vanish from the recursive policy functions.

Therefore, the underlying solution method is looking for P, Q, R, and S so that the equilibrium described by these rules is stable in nature. In the case of Model 1 and 2, $l = n$, then, [27]

(i) P must satisfy the following quadratic matrix equation:

$$(F - JC^{-1}A)P^2 - (JC^{-1}B - G + KC^{-1}A)P - KC^{-1}B + H = 0. \tag{118}$$

(ii) R is given by

$$R = -C^{-1}(AP + B). \tag{119}$$

(iii) Q satisfies

$$vec(Q) = (N'^{-1}A) + I_k \otimes (JR + FP + G - KC^{-1}A))^{-1} vec((JC^{-1}D - L)N + KC^{-1}D - M), \tag{120}$$

where $vec(.)$ denotes column wise vectorization.

(iv) Lastly, S is given by

$$S = -C^{-1}(AQ + D). \tag{121}$$

In order to have a stationary recursive solution, the key is to pick up the solution for P, whose eigenvalues are both smaller than unity. Given P the solution to R, Q, and S directly follows.

A.4 Simulation Methodology

In order to characterize the cyclical and long-run components of the simulated data an asymmetric Christiano & Fitzgerald (2003) type band-pass filter is applied at different frequencies defined in Section 4. Since Model 1 and 2 are solved for human capital normalized variables we use the method described by Restrepo-Ochoa & Vazquez (2004) to construct log-level simulated series by using the model solution given by equations (116) and (117).Using the method of Restrepo-Ochoa & Vazquez (2004) non-stationary log-level series are constructed for output, consumption, and physical investment, while. Then these series are used to

[27] For more details see Corollary 1 in Uhlig (1998).

calculate simulated RBC correlations; volatilities; and growth persistence as it can be seen in Section 4.

Consider that n_t denotes any non-stationary variable of the model and $\tilde{n}_t \equiv n_t/h_t$. Then the logarithm of n_t can be written as

$$\log n_t = \log \tilde{n}_t + \log h_t. \tag{122}$$

Then one may observe about the growth rate of human capital, $g_{h,t} = (1 + g_t)$, that

$$\frac{h_{t+1}}{h_t} = g_h e^{g_{ht}} \approx g_h(1 + g_t), \tag{123}$$

where $g_h = 1 + g$ is the steady state value of h_{t+1}/h_t. From the log-linear solution of the model in (116) the law of motion of the log-deviations from the steady state values of $\tilde{k}_{t+1}$ and g_{ht+1} are

$$\hat{g}_{ht} = P_{21}\hat{k}_t + Q_{21}z_t^g + Q_{22}z_t^g, \tag{124}$$

$$\hat{k}_{t+1} = P_{11}\hat{k}_t + Q_{11}z_t^g + Q_{12}z_t^g, \tag{125}$$

where P_{ij} and Q_{ij} denote the generic elements of matrices P and Q respectively and variables with hats represent variables in terms of log-deviations from their respective steady states. Next take natural logarithm of equation (123) to obtain,

$$\log h_{t+1} = \log h_t + \log g_h + \hat{g}_{ht}. \tag{126}$$

Then combining (126), (125), and (124) one can obtain an expression for the solution of the model implied synthetic human capital stock in log levels as,

$$\log h_{t+1} = \log h_t + \log g_h + P_{21}P_{11}\hat{k}_{t-1} + [P_{21}Q_{11} + Q_{21}]z_t^g + [P_{21}Q_{12} + Q_{22}]z_t^h. \tag{127}$$

From equation (127) one can observe that the synthetic time series of the human capital stock in log-levels has a unit root with a drift. From this it follows that by using equation (122) and (127) to obtain time series for consumption, investment, and output, will generate series with a unit root with a drift. This as pointed out by Restrepo-Ochoa & Vazquez (2004) is in line with evidence in the literature over the presence of a unit root in aggregate time series also highlighted by Nelson & Plosser (1982). Given the non-stationary time series obtained from the model for the log-levels of output, consumption, and investment, now the band pass filter can be applied at different frequencies as defined earlier following Basu et al. (2012), and Comin & Gertler (2006).

Then to construct the simulated growth rates for output, consumption, and physical investment the obtained non-stationary logarithmic series for output, consumption, and investment are first differenced. Construction of the growth rates this way is identical to constructing growth rates by using directly the model decision rules as suggested by King et al. (1988). This direct method is shown for consumption growth,

$$\begin{aligned} g_{c,t+1} &= \log c_{t+1} - \log c_t \\ &= \log \tilde{c}_{t+1} - \log \tilde{c}_t + \log h_{t+1} - \log h_t \\ &= (\log \tilde{c}_{t+1} - \log \tilde{c}) - (\log \tilde{c}_t - \log \tilde{c}) + \log \frac{h_{t+1}}{h_t} \\ &= \hat{c}_{t+1} - \hat{c}_t + (\log g_{h,t+1} - \log g_{ht}) + \log g_h \\ &= \hat{c}_{t+1} - \hat{c}_t + \hat{g}_{ht+1} + \log g_h, \end{aligned} \tag{128}$$

where the key is to express any growth rate in terms of variables of the normalized stationary model, after which one can substitute out $\hat{c}_t$ and $\hat{g}_{ht}$ using the solutions for them in equations (116) and (117)

The equivalence between the two methods of obtaining the growth rates of non-stationary variables can be easily seen if one moves equation (122) one period forward to $t+1$ and then subtracts the time period t version of the same equation for consumption growth in this instance:

$$g_{c,t} = \log c_{t+1} - \log c_t = \log \tilde{c}_{t+1} - \log \tilde{c}_t + \log h_{t+1} - \log h_t = \hat{c}_{t+1} - \hat{c}_t + \hat{g}_{ht+1} + \log g_h; \tag{129}$$

Equation (128) is identical to the final expression in (129) for the growth rate of consumption.

B Matlab Code Description

This section gives a detailed description of the Matlab codes used to obtain the results in the main body and appendices of this paper. The collection of the Matlab codes that produce the figures and results for this paper are available in the '*Tune in RBC Growth Spectra Matlab*' zip file .

B.1 Calibration Codes with Grid Search

The folders *Model 1 Calibration with Grid System* and *Model 2 Calibration with Grid System* contain the set of codes that perform the grid point based calibration procedure.

1. *main.m*: In this Matlab script one sets the lower and upper bounds and the number of steps between those to set up the grid point system. By running this script the calibration procedure is initiated and results are sorted in the matrix file called *Parameter_ Combinations2.m*.

2. *distance.m*: This Matlab function file defines the calibration metric, which is a weighted vector distance between a US data and a simulated data based moment vector.

3. *SUR.m*: This function file implements the extraction of shock process series using the method of Ingram et al. (1997), Benk et al. (2005, 2008, 2010), and Nolan & Thoenissen (2009); also implements the estimation of shock parameters by using the Seemingly Unrelated Regression (SUR) estimator as described in Greene (2003). For the extraction and estimation it calls the function called *solution.m* and the data file *data.xls*. The set of data used is set in this function.

4. *solution.m*: The Matlab function solves for the recursive solution of the underlying model by using the method of undetermined coefficients as in Uhlig (1998) for each iteration of the convergence and shock estimation.

5. *data.xls*: This MS Excel file contains the raw log-level US data from 1959Q1 to 2015Q4 as described in Appendix B.

6. *bpass.m*: This is a band-pass filter function file as in Christiano & Fitzgerald (2003).[28]

B.2 Calibration Codes with Simulated Annealing

The folders *Model 1 Calibration with SA* and *Model 2 Calibration with SA* contain the set of codes that perform the Simulated Annealing based calibration procedure.

1. *main.m*: In this Matlab script one sets the lower and upper bounds and initial values for each parameter. By running this script the calibration procedure is initiated.

2. *search.m*: This Matlab function file defines the calibration metric, which is a weighted vector distance between a US data and a simulated data based moment vector.

   (a) This function file initiates the shock extraction and convergence procedure using US data by calling *iteration.m*.

[28] This band pass filter Matlad function has been created by Eduard Pelz of the Federal Reserve Bank of Cleveland also available at: http://www.clev.frb.org/research/workpaper/1999/bpass.txt.

(b) If shock parameter convergence occurs the *search.m* uses the estimated and convergent shock parameters to obtain simulated data and its moments.

(c) The simulated data based moments are then used to calculate the distance between the data based moments and the simulated data based one.

3. *iteration.m*: This Matlab function file performs an iterative process as described in Benk et al. (2005, 2008, 2010). The function uses to estimate shock processes by calling the function *SUR.m*. Then the new estimates are fed back into the iterative loop until the estimated parameters converge or it stops when the shock persistence parameters reach unity, e.g. explode.

4. *SUR.m*: This function file implements the extraction of shock process series using the method of Ingram et al. (1997), Benk et al. (2005, 2008, 2010), and Nolan & Thoenissen (2009); also implements the estimation of shock parameters by using the Seemingly Unrelated Regression (SUR) estimator as described in Greene (2003). For the extraction and estimation it calls the function called *solution.m* and the data file *data.mat*. The set of data used is set in this function.

5. *solution.m*: The Matlab function solves for the recursive solution of the underlying model by using the method of undetermined coefficients as in Uhlig (1998) for each iteration of the convergence and shock estimation.

6. *data.mat*: This matrix contains the raw log-level US data from 1959Q1 to 2015Q4 as described in Appendix B.

7. *bpass.m*: This is a band-pass filter function file as in Christiano & Fitzgerald (2003).[29]

B.3 Simulation Codes

The folder *Simulation Codes* contains the set of codes that carry out the simulation of the baseline Models 1 and 2.

1. *main.m*: This Matlab script produces the figures and the tables in the command window of Matlab that are equivalent to the ones in the paper body and appendices. This file also calls all simulation files to obtain the results from the baseline models of the paper and their variants.

2. *sim M1.m*: This Matlab function performs a number of tasks all related to results obtained from Model 1. These are in order:

   (a) It calculates the steady state of Model 1 and stores it in the *StSt M1* matrix.

[29] This band pass filter Matlad function has been created by Eduard Pelz of the Federal Reserve Bank of Cleveland also available at: http://www.clev.frb.org/research/workpaper/1999/bpass.txt.

(b) Solves for the recursive policy functions of Model 1 using the method of undetermined coefficients as in Uhlig (1998).

(c) Generates simulated time series for the model variables using the recursive solution of Model 1 then constructs the synthetic non-stationary log-level series of key variables and growth rates as in Restrepo-Ochoa & Vazquez (2004) and King et al. (1988).

(d) Calls the band-pass filter function '*bpass.m*' as described in Christiano & Fitzgerald (2003), and filters the simulated time series for key variables at four different frequencies. The frequency bands are the high frequency [2-6 quarters]; the medium or business cycle frequency [6-32 quarters]; the low frequency [32-200 quarters]; and the Comin & Gertler (2006) type medium cycle [2-200 quarters].

(e) Using the Statistical toolbox of Matlab, more specifically its *crosscorr*, *autocorr*, and *std* functions, it calculates key cross-correlations between variables, variables' auto-correlation functions, and variables' standard deviations [i.e. volatility]. The results are then stored in the *CORR M1*, *ACORR M1*, and *VOL M1* matrices.

(f) Lastly, it calculates the impulse response functions to all variables to a goods sector TFP shock stored in the *IRFa M1* matrix; to a human sector productivity shock stored in the *IRFb M1* matrix; and to an economy wide shock stored in matrix *IRFc m.*

3. *sim M2.m*: This Matlab function performs a number of tasks all related to results obtained from Model 2. These are in order:

   (a) It calculates the steady state of Model 2 and stores it in the *StSt M2* matrix.

   (b) Solves for the recursive policy functions of Model 1 using the method of undetermined coefficients as in Uhlig (1998).

   (c) Generates simulated time series for the model variables using the recursive solution of Model 2 then constructs the synthetic non-stationary log-level series of key variables and growth rates.

   (d) Calls the band-pass filter function '*bpass.m*'.

   (e) It calculates key cross-correlations between variables, variables' auto-correlation functions, and variables' standard deviations [i.e. volatility]. The result are then stored in the *CORR M2*, *ACORR M2*, and *VOL M2* matrices.

   (f) Lastly, it calculates the impulse response functions to all variables to a goods sector TFP shock stored in the *IRFa M2* matrix; to a human sector productivity shock stored in the *IRFb M2* matrix; and to an economy wide shock stored in matrix *IRFc M2*.

4. *bpass.m*: This function contains the default asymmetric band-pass filter.

5. *data moments.m*: This function imports the raw data file *data.xls* and calculates all US business cycle moments at the earlier defined four frequencies. Then it stores the correlations, auto-correlations and volatilities *CORR Data*, *ACORR Data*, and *VOL Data* respectively.

6. *parameters.m*: This files stores the baseline calibrations of Model 1 and 2 in the vectors named *Parameters M1* and *Parameters M2* respectively.

7. *solow.m*: This file extracts the Solow residual, which is then compared to an alternative measure of the goods sector technology shock.

8. *TFP M1.m*: This file extracts the goods sector TFP shock series from using the recursive solution of Model 1. It stores the TFP series in *ZG M1*, meanwhile, it calculates the auto-correlation functions of output, investment, and consumption growth after feeding the extracted TFP series back to the recursive solution of Model 1.

9. *TFP M2.m*: This file extracts the goods sector TFP shock series from using the recursive solution of Model 2. It stores the TFP series in *ZG M2*, meanwhile, it calculates the auto-correlation functions of output, investment, and consumption growth after feeding the extracted TFP series back to the recursive solution of Model 2

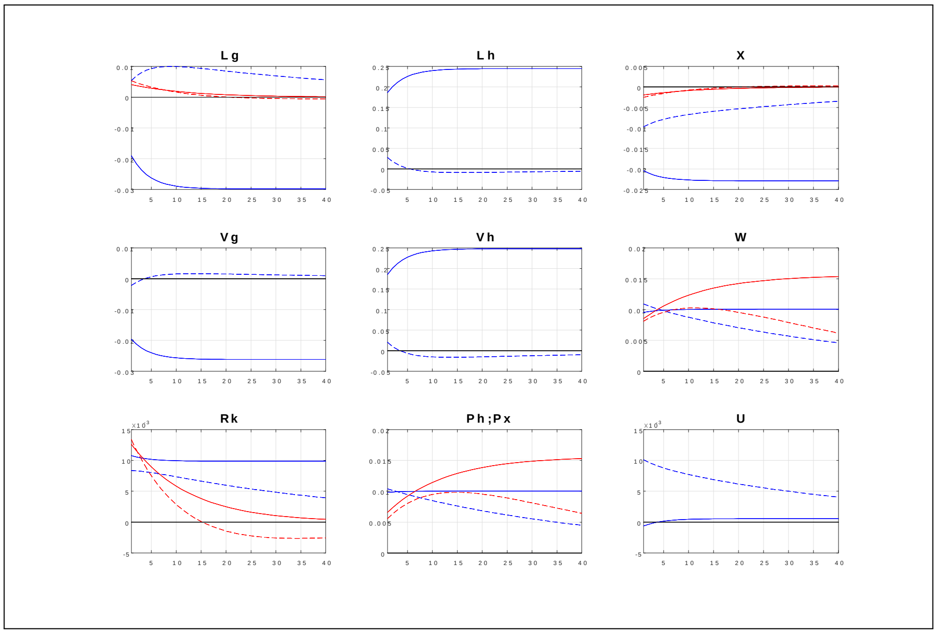
10. *data.xls*: This file contains raw log-level US data series from 1959Q1 to 2015Q4 as described in Appendix B. It is used to calculate data moments.

C Transitional Dynamics and Impulse Responses

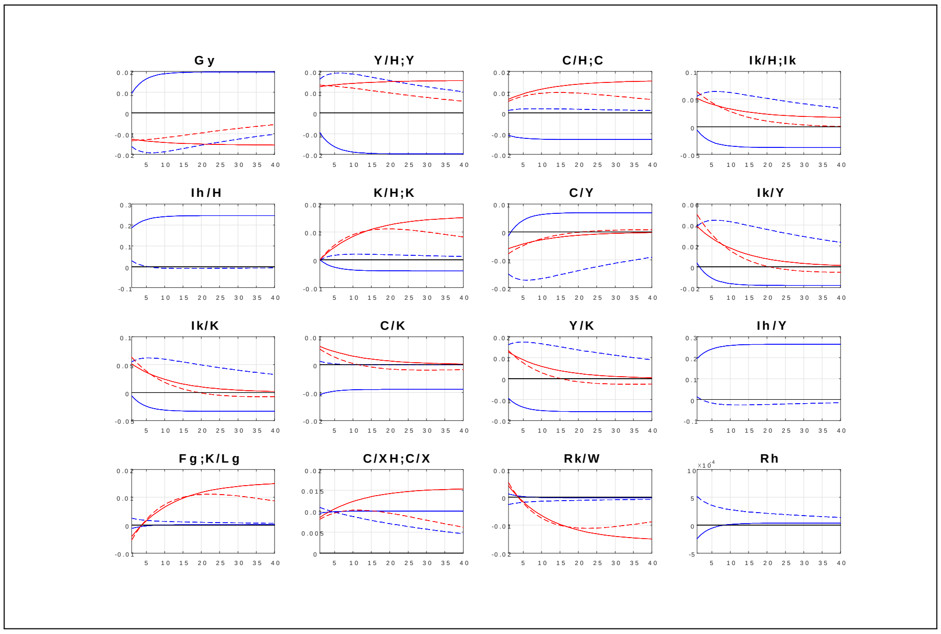
Footnote about Orthogonal Impulse Responses: To obtain the impulse responses of correlated shocks, when a single shock hits the econmy, one has to obtain the orthogonalized impulse responses as suggested by Hamilton (1994) in Chapter 8. This method does not apply here as the impulse responses presented here are to an 'economy-wide' shock, in which case both the goods and human sector productivity shocks hit at the same time.

Footnote about r_t^h: Define the return on using one extra unit of human capital in the human capital investment sector as r_t^h, which is the marginal product of human investment production with respect to human capital, i.e. $r_t^h \equiv \frac{\partial i_{ht}}{\partial l_{ht} h_t} = (1-\phi_2) A_h e^{z_t^h} \left[\frac{v_{ht} u_t k_t}{l_{ht} h_t}\right]^{\phi_2}$.

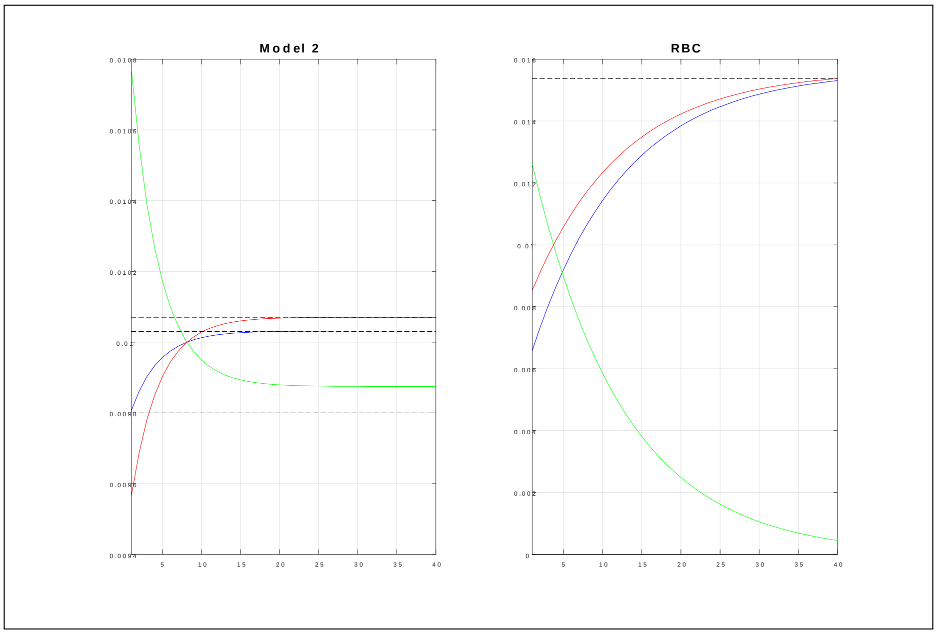
C.1 Goods Sector TFP Shock



Transitional dynamics and impulse responses of labor and physical capital sectoral shares, the utilization rate, and factor prices to a permanent and a temporary'goods sector TFP shock. (solid blue - Model 2 perm. shock, dashed blue - Model 2 temp. shock; solid red - Simple RBC perm. shock; dashed red - Simple RBC temp. shock)

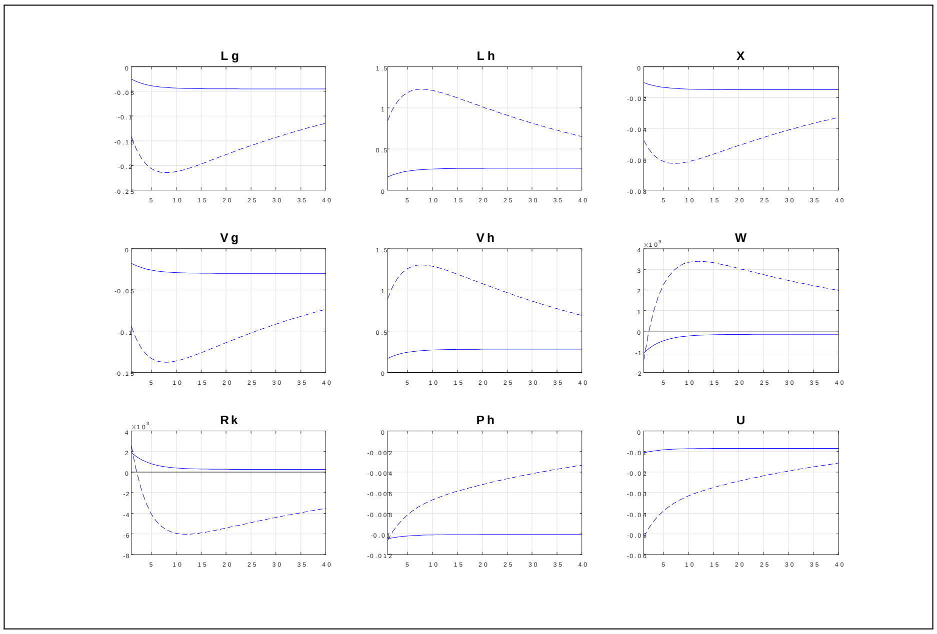


Transitional dynamics and impulse responses of selected variables to a permanent and a temporary goods sector TFP shock, where $F_g = (v_g uk)/(l_g h)$. (solid blue - Model 2 perm. shock, dashed blue - Model 2 temp. shock; solid red - Simple RBC perm. shock; dashed red - Simple RBC temp. shock)

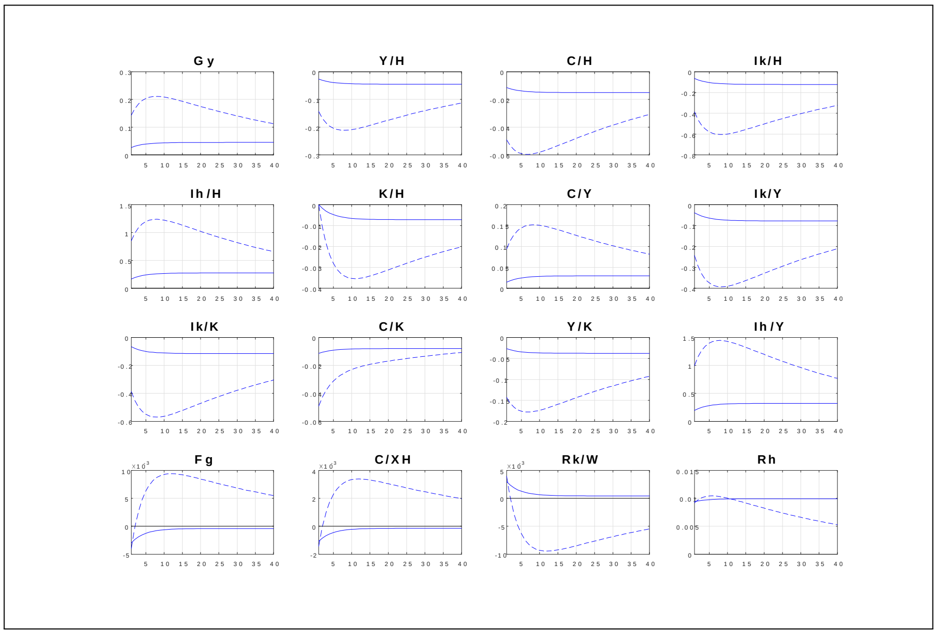


Transitional dynamics of factor prices and the relative price of the human investment good to a permanennt goods sector TFP shock.(dashed black lines are respective new steady states).

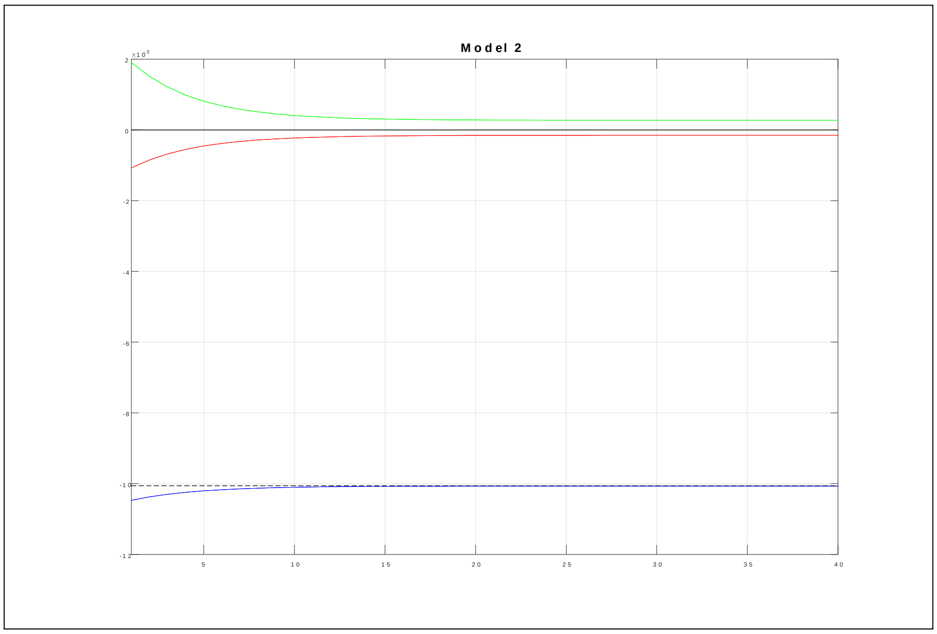
C.2 Human Sector Productivty Shock



Transitional dynamics and impulse responses of labor and physical capital sectoral shares, the utilization rate, and factor prices to a permanent and a temporary'human sector productivity shock. (solid blue - Model 2 perm. shock, dashed blue - Model 2 temp. shock; solid red - Simple RBC perm. shock; dashed red - Simple RBC temp. shock)



Transitional dynamics and impulse responses of selected variables to a permanent and a temporary human sector productivity shock, where $F_g = (v_g uk)/(l_g h)$. (solid blue - Model 2 perm. shock, dashed blue - Model 2 temp. shock; solid red - Simple RBC perm. shock; dashed red - Simple RBC temp. shock)



Transitional dynamics of factor prices and the relative price of the human investment good to a permanennt human sector productivty shock.(dashed black lines are respective new steady states).

C.3 Relative Price Analogue Between Model 2 and RBC Model

In Model 2 the first order conditions with respect to goods sector labor, l_{gt}, and human investment time, l_{ht}, are

$$l_{gt}: \quad Ac_t^{1-\sigma}x_t^{A(1-\sigma)-1}(1-u_t)^{B(1-\sigma)} = \lambda_t w_t h_t; \tag{130}$$

$$l_{ht}: \quad Ac_t^{1-\sigma}x_t^{A(1-\sigma)-1}(1-u_t)^{B(1-\sigma)} = \chi_t(1-\phi_2)A_h e^{z_t^h}\left[\frac{v_{ht}u_t k_t}{l_{ht}h_t}\right]^{\phi_2} h_t, \tag{131}$$

where $w_t = MP^y_{l_{gt}h_t} = (1-\phi_1)A_g e^{z_t^g}\left[\frac{v_{gt}u_t k_t}{l_{gt}h_t}\right]^{\phi_1}$. Equating the right hand side (RHS) of (130) and (131) yields

$$\lambda_t w_t h_t = \chi_t (1-\phi_2) A_h e^{z_t^h} \left[\frac{v_{ht} u_t k_t}{l_{ht} h_t}\right]^{\phi_2} h_t. \tag{132}$$

Dividing both sides with the human capital stock, h_t, yields an expression that equates the marginal product of goods production adjusted by the shadow price of physical capital with the marginal product of human investment production, $MP^{i_h}_{l_{ht}h_t} = (1-\phi_2) A_h e^{z_t^h} \left[\frac{v_{ht} u_t k_t}{l_{ht} h_t}\right]^{\phi_2}$, adjusted by the shadow price of the human investment good:

$$\lambda_t w_t = \chi_t (1-\phi_2) A_h e^{z_t^h} \left[\frac{v_{ht} u_t k_t}{l_{ht} h_t}\right]^{\phi_2}. \tag{133}$$

Then defining the relative price of the human investment good relative to physical capital as the ratio of the shadow prices, $p_{ht} = \frac{\chi_t}{\lambda_t}$, and rearranging (133) to express p_{ht} one gets the following:

$$p_{ht} = \frac{\chi_t}{\lambda_t} = \frac{MP^{y}_{l_{gt}h_t}}{MP^{i_h}_{l_{ht}h_t}} = \frac{w_t}{(1-\phi_2) A_h e^{z_t^h} \left[\frac{v_{ht} u_t k_t}{l_{ht} h_t}\right]^{\phi_2}}. \tag{134}$$

One can show in a simple RBC model that there is an analogous relative price of leisure relative to consumption (or physical capital). For this one may seek an expression that is the ratio of the marginal product of labor in the goods sector and a marginal benefit of leisure. Now consider the first order condition of a simple RBC model with respect to leisure:

$$x_t: \quad A c_t^{(1-\sigma)} x_t^{A(1-\sigma)-1} = \lambda_t w_t. \tag{135}$$

In the simple RBC model the wage rate as in Model 2 is defined as the marginal product of goods production with respect to labor, i.e. $w_t = MP^{y}_{l_{gt}} = (1-\phi_1) A_g e^{z_t^g} \left[\frac{k_t}{l_{gt}}\right]^{\phi_1}$. The RHS of (135) is the marginal utility gain of the agent by increasing leisure by a unit. Therefore, in the absence of another home sector leisure is analogous to human investment production and the marginal utility with respect to leisure is analogous to the marginal product with respect to human capital in Model 2. Then rearranging (135) one may obtain an analogous expression for the relative price of leisure, which is denoted by p_{xt}:

$$p_{xt} = \frac{1}{\lambda_t} = \frac{MP^y_{l_{gt}}}{MU_x} = \frac{w_t}{Ac_t^{(1-\sigma)} x_t^{A(1-\sigma)-1}}.$$