Financial Vulnerability and Monetary Policy *

Tobias Adrian Fernando Duarte

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Abstract

We present a parsimonious New Keynesian model that features financial vulnerabilities. The vulnerabilities generate time varying downside risk of GDP growth by driving the dynamics of risk premia. Monetary policy impacts the output gap directly via the IS curve, and indirectly via its impact on financial vulnerabilities. The optimal monetary policy rule always depends on financial vulnerabilities in addition to output, inflation, and the real rate. We show that a classic Taylor rule exacerbates downside risk of GDP growth relative to an optimal Taylor rule, thus generating welfare losses associated with negative skewness of GDP growth.

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1 Introduction

FOMC statements mention financial stability and financial conditions with increasing frequency (see Peek, Rosengren, and Tootell (2015)). Additionally, the notion of downside risks to growth has become more prevalent. In the academic literature, authors increasingly consider the roles of financial conditions and vulnerabilities in monetary policy settings (see Adrian and Shin (2010), Borio and Zhu (2012), Curdia and Woodford (2010), and Gambacorta and Signoretti (2014)).

An important early strand of the literature triggered by Bernanke and Gertler (1989) and Bernanke and Blinder (1992) argues for the credit channel of monetary policy. In the credit channel, financial frictions of borrowers or lenders shift credit demand and supply curves when monetary policy changes, thus giving financial conditions a role in monetary policy via the "external finance premium." However, Bernanke and Gertler (2000) argue that financial vulnerabilities should only impact central banks' monetary policy actions only to the extent that vulnerabilities change the forecasts for inflation and real activity, thus promoting a form of the Taylor (1993) and Taylor (1999) rule within an inflation targeting framework. Financial vulnerabilities refer to downside risks to GDP growth caused by risks to asset valuations, the level of leverage in the financial and nonfinancial sectors, and the degree of maturity transformation (see Adrian, Covitz, and Liang (2015) for a framework on measuring financial vulnerability).

More recent literature has argued for a Taylor rule with financial variables. In the setups of Curdia and Woodford (2010) and Gambacorta and Signoretti (2014), the optimal monetary policy rule is augmented to take financial conditions into account. Limited capital in the financial intermediary sector leads to a distortion in aggregate activity that optimal monetary policy takes into account. However, just as the literature on the credit channel focused on financial conditions, and not financial vulnerabilities, the setups of Curdia and Woodford (2010) and Gambacorta and Signoretti (2014) do not feature financial vulnerabilities either. While financial conditions refer to the notion that the pricing of risk impacts aggregate activity, settings with financial vulnerability give rise to downside risks to growth from financial frictions that impact aggregate economic welfare.

In a series of influential of papers, Svensson (2016b,a) explicitly incorporates financial vulnerabilities in the form of financial crises. Svensson develops a cost benefit framework, argues that financial stability considerations should not enter monetary policy decisions if costs exceed benefits, and finds that, for existing empirical estimates, costs exceed benefits by a substantial margin. In Svensson's setup, the costs consist of distorting the dual mandate objectives, while the benefits are measured in terms of the reduction of the likelihood or severity of financial crises. In contrast, Adrian and Liang (2016) present a comprehensive literature review of the role of financial conditions and vulnerabilities in monetary policy and argue that there is compelling empirical evidence that financial vulnerabilities should affect monetary policy considerations[1]. Adrian and Shin (2010) are early proponents of the risk taking channel of monetary policy.

In this paper, we present a parsimonious macroeconomic framework for incorporating financial vulnerabilities in monetary policy. We extend the standard two equation New Keynesian model (NK model) to incorporate financial vulnerabilities. We model the impact of these underlying vulnerabilities on aggregate macroeconomic activity as a shift in the Euler equation due to changes in the risk premium that households face. Those shifts in the risk premium can be viewed as disturbances to financial intermediation and preference shocks (we present a microfoundation of the reduced form model in a companion paper). As output is a function of the risk premium, and financial vulnerability is defined in terms of risks to output, vulnerability is endogenous to the model.

Our modeling approach is motivated by the empirical evidence that financial conditions forecast tail risks. Estrella and Hardouvelis (1991) and Estrella and Mishkin (1998) show that the term spread, an indicator of the pricing of interest rate risk, forecasts recessions. Gilchrist and Zakrajšek (2012), López-Salido, Stein, and Zakrajšek (2016), and Krishnamurthy and Muir (2016) find that credit spreads forecast downside risks to GDP growth. More generally, Adrian, Boyarchenko, and Giannone (2016) document that financial conditions are strong forecasters of downside risks to GDP growth. Deteriorating financial conditions give rise to an increase in the conditional volatility of GDP and a decline in the conditional mean of GDP in such a way that upper quantiles of GDP growth are more or less constant, while lower quantiles are varying sharply. Hence the unconditional distribution of GDP is highly skewed to the left as a function of financial conditions.

Motivated by these empirical observations, we present a New Keynesian model augmented with financial vulnerabilities that captures these stylized facts. Financial vulnerability is defined as GDP volatility multiplied by an indicator of risk aversion, and adjusted by expected GDP growth. This can be interpreted as the value-at-risk of GDP. Because vulnerability depends on the conditional expectation of GDP growth, it is im-

[1] See Svensson (2017) for a response.

pacted by monetary policy. Furthermore, because the volatility of GDP depends on financial vulnerability, monetary policy also impacts volatility. Importantly, the setup is such that certain parameter restrictions ensure that deteriorating financial conditions are associated with an increase in GDP volatility and a decline in the conditional mean of GDP, thus generating the negative correlations between mean and volatility that Adrian, Boyarchenko, and Giannone (2016) document.

The central bank is assumed to minimize a standard loss function with the squared output gap and the squared inflation rate entering as arguments. We can solve for the optimal policy rules in closed form using dynamic programming. We consider optimal monetary policy rules with flexible prices (no Phillips curve) and sticky prices.

Our optimal monetary policy rule can be cast in the language of a flexible inflation targeting framework, such as the one in Giannoni and Woodford (2012), Svensson (1999), Svensson (2002) and Rudebusch and Svensson (1999). Relative to the standard New Keynesian model, there are two important differences. First, vulnerability becomes a target variable. Second, the coefficients in the linear optimal targeting criterion rule that trade off deviations in output, inflation and financial vulnerability from their desired levels, depend on the parameters that govern GDP vulnerability.

Optimal monetary policy can also be expressed as an augmented Taylor rule. The nominal interest rate not only depends on inflation and output, but also on financial vulnerability. The optimal coefficients on output and inflation are taking the parameters that govern GDP vulnerability into account.

The NKV model (New Keynesian Vulnerability model) is straightforward to calibrate from GDP, inflation, and financial conditions data. When we simulate the model under the optimal monetary policy rule, and the alternative standard New Keynesian Taylor rule, we find that the latter generates higher equilibrium GDP skewness. Therefore, policy makers that take vulnerability into account mitigate downside movements of output.

Our model captures the intuition that in recent years monetary policy has explicitly taken into account and influenced financial conditions.[2] A deterioration of financial conditions corresponds to an increase in tail risk, as conditional GDP volatility rises, while the conditional growth forecast deteriorates. As a result of such an increase in financial vulnerability, i.e. an increase in the downside risk to GDP growth, monetary policy is relatively easier than under the classic Taylor rule. This results in a lowering

[2]Dudley (2015); Yellen (2016).

of vulnerability, and hence in less severe left skewness of GDP.

We also study an extension with a zero lower bound on nominal interest rates. The zero lower bound implies a flexible inflation targeting rule when interest rates are away from the bound, and a forward guidance rule when the zero lower bound is reached. Therefore, the New Keynesian model with financial vulnerability can be extended to settings with a zero lower bound.

In the setting of our paper, monetary policy *always* takes GDP vulnerability into account, even though the policy maker only cares about output and inflation. Incorporating financial vulnerability in the flexible inflation targeting framework strictly dominates the standard Taylor rules that condition only on output and inflation. This result is in stark contrast to Svensson (2016b,a) who argues that financial stability considerations should not be taken into account if costs exceed benefits. Despite the contrasting conclusions, the framework of analysis that we are using here is similar to the framework that Svensson is using. However, while Svensson focuses on tail risks to GDP growth that only occur very rarely, we focus on vulnerabilities that are present most of the time. Importantly, tail risks can naturally occur within our setup, due to a particular form of nonlinearity involving first and second moments. Hence our setting also captures the extreme tail events that Svensson studies.

An important contribution of our setup is that it is closely motivated by the empirical facts of Adrian, Boyarchenko, and Giannone (2016) on the evolution of the conditional GDP distribution. Those authors show that U.S. GDP has strongly time varying downside risk as a function of financial conditions, while upside risks are more or less constant. A parsimonious model that captures these stylized facts is one where financial conditions impact both the conditional mean and the conditional volatility of output.

The remainder of the paper is organized as follows. Section 2 provides the motivation for our model from the existing empirical and theoretical literature on financial stability in a macroeconomic context. Section 3 presents the model together with the optimal policy rules. The model with flexible prices is presented in Section 4. Section 5 presents an extension with a zero lower bound constraint. Section 6 concludes.

2 Financial Vulnerability

Financial vulnerability refers to the presence of amplification mechanisms that are caused by leverage, maturity transformation, or asset valuations. When financial vulnerability is large, small shocks can have severe aggregate macroeconomic consequences. Adrian, Covitz, and Liang (2015) present a framework for the monitoring of financial vulnerability. They measure leverage, maturity transformation, and asset valuations across four sectors: asset markets, the banking system, the market based financial system, and the nonfinanical system. Aikman, Kiley, Lee, Palumbo, and Warusawitharana (2015) propose a quantitative indicator for the vulnerabilities in this framework.

In this paper, following Adrian, Boyarchenko, and Giannone (2016), we construct a measure of financial vulnerability by using the National Financial Conditions Index (NFCI) of the Federal Reserve Bank of Chicago. That index aggregates 105 financial market, money market, credit supply, and shadow bank indicators to compute a single index using the filtering methodology of Stock and Watson (1998). Adrian, Boyarchenko, and Giannone (2016) show that the conditional GDP distribution features strong downside risk as a function of financial conditions. We reproduce the main results of Adrian, Boyarchenko, and Giannone (2016) here using a conditionally heteroskedastic model to estimate the conditional first and second moments of GDP growth

$$y_t = \gamma_0^y + \gamma_1^y y_{t-1} + \gamma_2^y \pi_{t-1} + \gamma_3^y x_{t-1} + \sigma_t^y \varepsilon_t^y \tag{1}$$
$$\ln(\sigma_t^y) = \delta_0^y + \delta_1^y x_{t-1} \tag{2}$$

where $\varepsilon_t^y \sim N(0,1)$, x_t denotes the NFCI financial conditions index, and y_t is the quarterly growth rate of GDP. Mean GDP also depends on the lagged quarterly core PCE inflation rate π and on lagged GDP growth. In addition to estimating the conditional mean and conditional volatility of GDP growth, we also estimate an analogous equation for the inflation rate:

$$\pi_t = \gamma_0^\pi + \gamma_1^\pi y_{t-1} + \gamma_2^\pi \pi_{t-1} + \sigma_t \varepsilon_t^\pi \tag{3}$$
$$\ln(\sigma_t^\pi) = \delta_0^\pi + \delta_1^\pi \pi_{t-1} \tag{4}$$

The model is estimated via maximum likelihood.

The estimation results are in Figure 1 and Table 1. In Panel (a) of Figure 1, we present the conditional mean of GDP growth, actual GDP growth, and the 5th and 95th quantiles. The distribution is left skewed as deteriorating financial conditions are associated with an increase in conditional volatility, and at the same time a decline in the conditional mean of GDP growth (see Table 1). Due the negative correlation of mean and volatility the unconditional distribution is negatively skewed, even though

Figure 1: **Estimated Conditional Distribution of One Quarter Ahead GDP Growth and PCE Inflation**. The figure reports estimates from equations (1), (2), (3), and (4). Panel (a) shows the actual GDP growth, the conditional mean of GDP growth, and the 5th and 95th quantile. Panel (b) shows the actual PCE inflation, the conditional mean of inflation, and the 5th and 95th quantile.

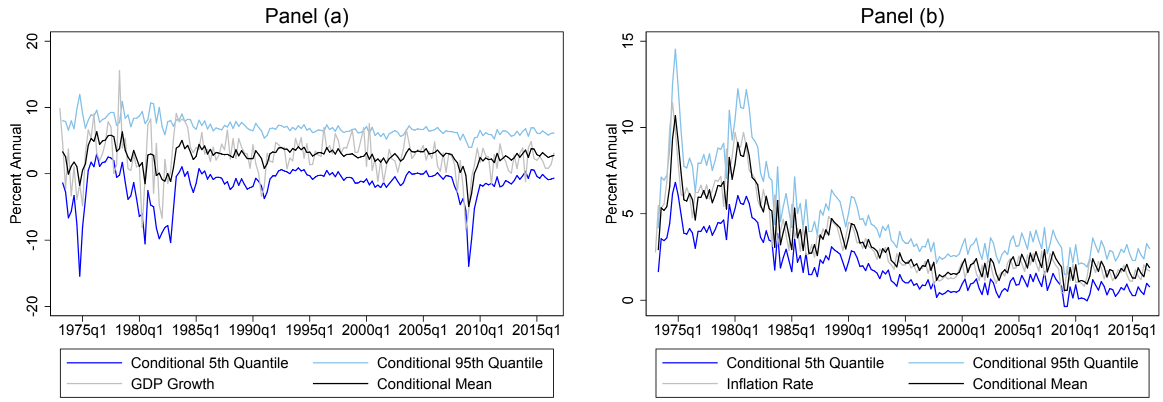


Table 1: **GDP and Inflation Conditional Mean and Volatility Estimates**

VARIABLES	(1) GDP	(2) HET
Financial Conditions (lag)	-1.715***	0.551***
	[-5.096]	[3.765]
GDP Growth (lag)	-0.000356	
	[-1.510]	
Inflation Rate (lag)	0.00277	
	[0.0842]	
Constant	6.213***	1.785***
	[11.02]	[21.69]
Observations	173	173

*** p<0.01, ** p<0.05, * p<0.1

the conditional distribution is conditionally Gaussian. For inflation, financial conditions aren't significant for either the conditional mean equation or the conditional volatility. However, the volatility of inflation scales in the level of inflation. Hence the conditional mean and the conditional volatility are positively correlated. Importantly, financial conditions play no role for inflation dynamics.

The estimates for GDP have the unusual property that the shift in the conditional mean and volatility of GDP offset each other in such a way that the 95th quantile is close to constant. In contrast, the 5th quantile strongly varies as a function of financial conditions. Importantly, this property only arises when the GDP distribution is estimated as a function of financial conditions—real economic indicators do not contain significant information for the tail of the GDP distribution. This is shown more generally by Adrian, Boyarchenko, and Giannone (2016).

These results suggest that GDP vulnerability is related to the time varying left tail of the GDP distribution, as a function of financial conditions. In this paper, we define GDP vulnerability V_t as

$$V_t = \gamma \mathbb{V}\left[dy_t|\mathcal{F}_t\right] - \mathbb{E}\left[dy_t|\mathcal{F}_t\right] \tag{5}$$

where $\mathcal{F}_t$ denotes the filtration generated by the underlying stochastic processes, $\mathbb{V}$ denotes the volatility operator (the square root of the instantaneous variance of dy_t), and $\mathbb{E}$ denotes the expectations operator. In the definition of vulnerability, volatility is multiplied by a coefficient of risk aversion γ. That risk aversion times the volatility is adjusted by the expected value of GDP growth, as a higher expected value can compensate for higher volatility when assessing the left tail risk of GDP growth. This definition of vulnerability is tightly linked to the definition of value-at-risk, as can be seen in the next lemma.

Lemma 1 *For the stochastic process* $dy_t = \mathbb{E}\left[dy_t|\mathcal{F}_t\right]dt + \mathbb{V}\left[dy_t|\mathcal{F}_t\right]dZ_t$*, where* Z_t *denotes a standard Brownian motion, the value-at-risk for confidence level* p *is defined as*

$$\mathbb{P}_t\left[dy_t \leq -VaR_t\right] = pdt \tag{6}$$

Then the value-at-risk VaR_t *is*

$$VaR_t dt = -\tilde{\Phi}^{-1}(p)\,\mathbb{V}\left[dy_t|\mathcal{F}_t\right]dt - \mathbb{E}\left[dy_t|\mathcal{F}_t\right]dt \tag{7}$$

where $\tilde{\Phi}^{-1}(p) = \Phi^{-1}(p) + O\left((p\ dt)^{3/2}\right)$ *and* $\Phi^{-1}(p)$ *denotes the inverse cumulative normal distribution at confidence level* p*. The proof is in the appendix.*

As vulnerability measures the left tail of the GDP distribution, p is small, and therefore $\tilde{\Phi}^{-1}(p)$ is negative. For example, $\tilde{\Phi}^{-1}(5\%) = -1.96$. Therefore, by comparing (7) with (5), we can see that if $\gamma = -\tilde{\Phi}^{-1}(p)$, GDP vulnerability V_t is the value at risk of GDP with confidence level $p = \tilde{\Phi}(-\gamma)$.

We model GDP with the following data generating process

$$dy_t = \sigma^{-1}\left(i_t - r_t - \pi_t\right)dt + d(rp_t) \tag{8}$$

$$d(rp_t) = -\rho V_t dt + \xi\left(V_t - x_t\right)dZ_t \tag{9}$$

$$dx_t = \rho_x\left(\mu_x - x_t\right)dt + \sigma_x dZ_{x,t} \tag{10}$$

Equation (8) is the Euler equation (or IS curve) of a standard NK model augmented with a risk premium. Equation (9) is the key contribution of this setup relative to standard models. It says that the risk premium rp_t evolves as a function of GDP vulnerability V_t. In particular, vulnerability determines the conditional mean $-\rho V_t$ and the conditional volatility $\xi\left(V_t - x_t\right)$ of the risk premium. Importantly, vulnerability is *endogenous* to the stochastic evolution of GDP as a function of shocks to the risk premium. The shocks dZ_t and $dZ_{x,t}$ are uncorrelated standard Brownian motions. We assume that $\rho > 0$ and $\xi > 0$. x_t is a shock to volatility that is modeled as a simple mean reverting process.

Replacing (9) in (8) gives

$$dy_t = \left(\sigma^{-1}\left(i_t - r_t - \pi_t\right) - \rho V_t\right)dt + \xi\left(V_t - x_t\right)dZ_t$$

From this equation, we see that

$$\begin{aligned}
\mathbb{E}\left[dy_t|\mathcal{F}_t\right] &= \sigma^{-1}\left(i_t - r_t - \pi_t\right) - \rho V_t \\
\mathbb{V}\left[dy_t|\mathcal{F}_t\right] &= \xi\left(V_t - x_t\right)
\end{aligned}$$

Using these last two expressions in equation (5) and solving for V_t gives vulnerability as a function of the nominal interest rate

$$V_t = \frac{\gamma\xi x_t + \sigma^{-1}\left(i_t - r_t - \pi_t\right)}{-\left(1 - \gamma\xi - \rho\right)} \tag{11}$$

The standard IS curve of the NK model is thus augmented with an additional term $-\rho V_t$ in the drift, and a shock $\xi\left(V_t - x_t\right)dZ_t$, both of which depend on GDP vulnerability V_t. Vulnerability, in turn, depends on the interest rate $i_t - r_t - \pi_t$ and the process x_t. We can thus interpret x_t as a shock to vulnerability.

The sign of the dependence of V_t on the interest rate i_t depends on the sign of $-\left(1 - \gamma\xi - \rho\right)$. The empirical results can help us pin down the sign of these parameters. The mean-variance tradeoff for y_t follows by writing $\mathbb{E}\left[dy_t|\mathcal{F}_t\right]$ and $\mathbb{V}\left[dy_t|\mathcal{F}_t\right]$ as functions of vulnerability V_t and the shock to vulnerability x_t

$$\mathbb{E}\left[dy_t|\mathcal{F}_t\right] = \left(\gamma\xi - 1\right)V_t - \gamma\xi x_t \tag{12}$$

$$\mathbb{V}\left[dy_t|\mathcal{F}_t\right] = \xi\left(V_t - x_t\right) \tag{13}$$

and then eliminating V_t to get

$$\mathbb{V}_t\left[dy_t\right] = -\left(\frac{\xi}{1-\gamma\xi}\right)\mathbb{E}_t\left[dy_t\right] - \left(\frac{\xi}{1-\gamma\xi}\right)x_t \tag{14}$$

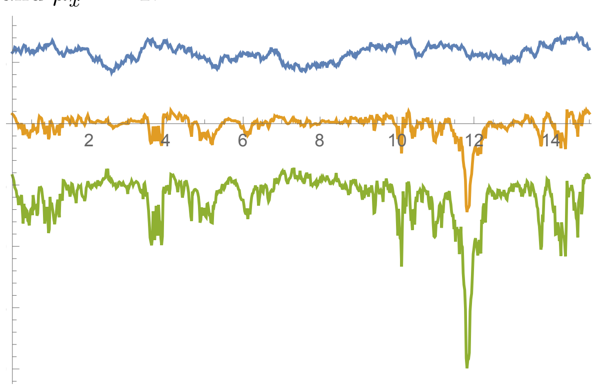
Empirically, to match the negative slope of the mean-variance tradeoff, we thus must have $\frac{1-\gamma\xi}{\xi} > 0$ which implies, since $\gamma > 0$, that $\xi > 0$ and $1 - \gamma\xi > 0$. In addition, to match the positive intercept on average, we must have $\mathbb{E}\left[x_t\right] = \mu_x < 0$.

We can thus draw conclusions about the dependence of vulnerability on the interest rate. As ρ denotes the degree of mean reversion of the risk premium, which is a very persistent variable, it is likely to be very small. Theoretically, we thus assume

$$1 - \gamma\xi - \rho > 0. \tag{15}$$

This assumption implies GDP vulnerability V_t and interest rates i_t are negatively correlated. This correlation is consistent with the empirical observation that when financial conditions deteriorate, GDP vulnerability increases, and short-term interest rates decline. Figure 2 shows a simulated path of (8), (9), (10) setting $i - \pi - r$ to zero, for simplicity. The simulation clearly features the stylized facts of Figure 1.

Figure 2: **Simulated Conditional Distribution of One Quarter Ahead GDP Growth**. The figure shows simulated conditional mean of GDP, and the 5th and 95th quantile of model (8), (9), (10) with $\gamma = 2$, $\sigma = 1$, $\xi = 1$, and $\mu_x = -1$.



The IS curve augmented with the shocks to risk premia that depend on vulnerability lead to an additional channel for monetary policy. The traditional transmission channel is via the drift of the IS curve: higher interest rates are associated with a higher growth rate of output. This is because a higher interest rate shifts consumption from the present to the future, via increased savings. The additional channel that arises in the current setup is the impact of monetary policy on vulnerability, and hence on the volatility of the risk premium. Hence monetary policy impacts total risk in the economy. This channel is sometimes called the "risk taking channel of monetary policy" (see Adrian and Shin (2010) and Borio and Zhu (2012)). When we study optimal monetary policy in the next section, this tradeoff is going to emerge prominently.

The model also features sticky prices, represented by a Phillips curve

$$d\pi_t = (\beta\pi_t - \kappa y_t)\, dt. \tag{16}$$

Vulnerability does not appear in the Phillips curve, as we did not find any statistical evidence of financial conditions influencing the mean or volatility of inflation.

3 Monetary Policy

3.1 Optimal Monetary Policy

The central bank is minimizing a quadratic loss function over the output gap and inflation

$$L(y_t, \pi_t) = \min_{i_t} \mathbb{E}_t \int_t^\infty e^{-t\beta} \left(y_t^2 + \pi_t^2\right) dt. \tag{17}$$

subject to the dynamics of the economy (2), (11), (10), (16). Minimizing the quadratic loss function is a standard approach in the NK literature, as Rotemberg and Woodford (1997), Rotemberg and Woodford (1999) and Woodford (2001) have shown that aggregate welfare can be approximated by such a loss function.

We note that the definition of vulnerability from (5) gives rise to a relationship between the interest rate i and vulnerability V. As a result, one can view the central banks optimization problem as minimizing the welfare loss by choosing a certain level of vulnerability. The central bank's problem can thus be written as

$$L\left(y_t, \pi_t, x_t\right) = \min_{\{V_s\}_{s=t}^{\infty}} \mathbb{E}_t \int_t^{\infty} e^{-s\beta}\left(y_s^2 + \lambda \pi_s^2\right) ds \tag{18}$$

$$s.t.$$

$$dy_t = \left(\left(\gamma\xi - 1\right) V_t - \gamma\xi x_t\right) dt + \xi\left(V_t - x_t\right) dZ_t \tag{19}$$

$$d\pi_t = \left(\beta\pi_t - \kappa y_t\right) dt \tag{20}$$

$$dx_t = -\rho_x\left(x_t - \mu_x\right) dt + \sigma_x dZ_{x,t} \tag{21}$$

We also use the restriction $\beta, \lambda, \alpha, \sigma, r, \kappa, \rho_x, \sigma_x, \rho, \gamma, \xi > 0$, $\mu_x < 0$, and $\gamma\xi - 1 < 0$. The optimal interest rate path i_t can the be recovered using equation (11). The Hamilton-Jacobi-Bellman (HJB) equation for the central banker's optimization is

$$\begin{aligned} 0 \quad = \quad & \min_V \left\{ y^2 + \lambda\pi^2 - \beta L + \frac{\partial L}{\partial y}\left(\left(\gamma\xi - 1\right) V - \gamma\xi x\right) + \frac{1}{2}\frac{\partial^2 L}{\partial y^2}\xi^2\left(V - x\right)^2 + \right. \\ & \left. + \frac{\partial L}{\partial \pi}\left(\beta\pi - \kappa y\right) - \rho_x\left(x - \mu_x\right)\frac{\partial L}{\partial x} + \frac{1}{2}\frac{\partial^2 L}{\partial x^2}\sigma_x^2 \right\} \end{aligned}$$

Intuitively, the HJB takes into account the current value of welfare, as well as the change in welfare associated with changes in the state variables y, π, and x. The first order condition is

$$V = \left(\frac{\partial^2 L}{\partial y^2}\right)^{-1} \frac{\partial L}{\partial y}\frac{1 - \gamma\xi}{\xi^2} + x$$

Hence at the optimum, vulnerability is proportional to x, and depends on the first and second derivative of welfare with respect to output. It is also noteworthy that $\frac{1-\gamma\xi}{\xi^2}$, which defines the slope of output volatility with respect to expected output, appears in the FOC. The second order condition (SOC) is

$$\frac{\partial^2 L}{\partial y^2} > 0$$

Plugging the FOC into the HJB, we get the "concentrated HJB":

$$0 \quad = \quad y^2 + \lambda\pi^2 - \beta L - \frac{\partial L}{\partial y}x - \frac{1}{2}\left(\frac{\partial^2 L}{\partial y^2}\right)^{-1}\left(\frac{\partial L}{\partial y}\right)^2\frac{\left(1 - \gamma\xi\right)^2}{\xi^2} \tag{22}$$

$$+\frac{\partial L}{\partial \pi}\left(\beta\pi - \kappa y\right) - \rho_x\left(x - \mu_x\right)\frac{\partial L}{\partial x} + \frac{1}{2}\frac{\partial^2 L}{\partial x^2}\sigma_x^2 \tag{23}$$

We look for a quadratic solution of the form

$$L(y,\pi,x) = c_0 + c_1 y + c_2 y^2 + c_3 x + c_4 x^2 + c_5 yx + c_6 \pi + c_7 \pi^2 + c_8 y\pi + c_9 \pi x$$

where $c.$ are constants. It is straightforward to solve for the constants using (23), as shown in Appendix A, where we also derive the optimal initial conditions. Optimal level of vulnerability that the central bank picks can be expressed as

$$V_t = \frac{(1-\gamma\xi)}{\xi^2} y_t + \left(\frac{(1-\gamma\xi)}{2\xi^2} \frac{c_8}{c_2} \right) \pi_t + \left(1 - \frac{(\gamma\xi - 1)}{2\xi^2} \frac{c_5}{c_2} \right) x_t + \frac{(1-\gamma\xi)}{2\xi^2} \frac{c_1}{c_2} \tag{24}$$

This can be viewed as a "flexible inflation targeting rule" (see Svensson (1999), Svensson (2002) and Rudebusch and Svensson (1999)) or, more generally, as a linear optimal targeting criterion (Giannoni and Woodford (2012)). Even though vulnerability and its shocks, V_t and x_t, are not target variables, i.e., they do not appear in the loss function equation (18), they still enter the inflation targeting rule, the first-order condition given by equation (24). There are no independent target values for V_t and x_t that the central bank hopes to achieve. The reason V_t and x_t enter the targeting rule is that they forecast the conditional means and variances of y_t and π_t even after controlling for the information already contained in y_t and π_t themselves. This is consistent with the empirical results in Table 1 and with the findings in Adrian, Boyarchenko, and Giannone (2016), who show that financial conditions are excellent predictors of the tail of the GDP distribution in a way that non-financial variables are not. Alternatively, equation (24) can be interpreted as a traditional flexible inflation targeting rule in which the targets for inflation and/or output are time-varying and depend on V_t and x_t. It also important to note that even if a central bank decided not to condition its actions on V_t and x_t, the tradeoff between inflation and output –reflected in the coefficients of the rule in equation (24)– now depends on γ and ξ, the parameters that dictate the strenght of the mean-variance tradeoff of output.

In the appendix, we show that the optimal monetary policy rule (24) satisfies the transversality condition. The optimal rule thus satisfies the first and second order conditions of the HJB, and the transversality condition, and is thus a solution to the dynamic programming problem of the central bank.

We can also express monetary policy as an interest rate rule by writing i_t in terms of V_t and y_t

$$i_t = r + a_0 + a_y y_t + a_\pi \pi_t + a_V V_t \tag{25}$$

where $a_y = \frac{2\sigma\gamma\xi(1-\gamma\xi)}{2\xi^2+(1-\gamma\xi)\frac{c_5}{c_2}}$, $a_\pi = \frac{2\xi^2+\frac{c_5}{c_2}(1-\gamma\xi)+\sigma\gamma\xi(1-\gamma\xi)\frac{c_8}{c_2}}{2\xi^2+\frac{c_5}{c_2}(1-\gamma\xi)}$, $a_V = \frac{\sigma\left(2\xi^2(\rho-1)+(1-\gamma\xi)(\rho+\gamma\xi-1)\frac{c_5}{c_2}\right)}{2\xi^2+\frac{c_5}{c_2}(1-\gamma\xi)}V_t$, and $a_0 = \frac{\sigma\gamma\xi(1-\gamma\xi)\frac{c_1}{c_2}}{2\xi^2+(1-\gamma\xi)\frac{c_5}{c_2}}$. The optimal interest rule can thus be viewed as an augmented Taylor rule. In addition to the output gap y, the level of inflation π, and the equilibrium rate of interest r, the level of vulnerability V enters the optimal rule. As before, the coefficients on y and π depend on the parameters that define vulnerability ξ and γ and thus monetary policy is different from the typical NK model without vulnerabilities not only because vulnerability enters the augmented Taylor rule directly, but also because the presence of vulnerabilities alter the optimal response of interest rates to changes in output and inflation.

3.2 Alternative Monetary Policy Rules

In general, the central bank might follow other monetary policy rules. Consider the following general linear functions

$$\begin{aligned} i_t &= \psi_0 + \psi_y y_t + \psi_\pi \pi_t + \psi_x x_t \\ V_t &= \phi_0 + \phi_y y_t + \phi_\pi \pi_t + \phi_x x_t \end{aligned}$$

which we will use to analyze non-optimal behavior. By equation (11) we obtain the following parameter restrictions:

$$\begin{aligned} \phi_0 &= \frac{\psi_0 - r}{\sigma(\gamma\xi - 1 + \rho)} \qquad & \phi_y &= \frac{\psi_y}{\sigma(\gamma\xi - 1 + \rho)} \\ \phi_\pi &= \frac{\psi_\pi - 1}{\sigma(\gamma\xi - 1 + \rho)} \qquad & \phi_x &= \frac{\psi_x + \sigma\gamma\xi}{\sigma(\gamma\xi - 1 + \rho)} \end{aligned}$$

As a result, we can express the level of vulnerability as

$$V_t = \frac{\psi_0 - r}{\sigma(\gamma\xi - 1 + \rho)} + \frac{\psi_y}{\sigma(\gamma\xi - 1 + \rho)} y_t + \frac{\psi_\pi - 1}{\sigma(\gamma\xi - 1 + \rho)} \pi_t + \frac{\psi_x + \sigma\gamma\xi}{\sigma(\gamma\xi - 1 + \rho)} x_t$$

So we can find the process for output by plugging these expressions into equations (19)-(21) to obtain:

$$dy_t = \left(\frac{(\gamma\xi - 1)(\psi_0 - r)}{\sigma(\gamma\xi - 1 + \rho)} + \frac{(\gamma\xi - 1)\psi_y}{\sigma(\gamma\xi - 1 + \rho)} y_t + \left(\frac{(\gamma\xi - 1)\psi_x - \sigma\gamma\xi\rho}{\sigma(\gamma\xi - 1 + \rho)} \right) x_t \right) dt \tag{26}$$

$$+ \left(\frac{\xi(\psi_0 - r)}{\sigma(\gamma\xi - 1 + \rho)} + \frac{\xi\psi_y}{\sigma(\gamma\xi - 1 + \rho)} y_t + \xi \left(\frac{\psi_x + (1 - \rho)\sigma}{\sigma(\gamma\xi - 1 + \rho)} \right) x_t \right) dZ_t \tag{27}$$

We can then solve for the probability distribution function for y denoted by $p_t(y, \pi, x)$ using the Kolmogorov equation:

$$\frac{\partial}{\partial t} p_t(y,\pi,x) = -\frac{\partial}{\partial y}\left[\left(D_0 + D_y y + D_\pi \pi + D_x x\right) p_t(y,\pi,x)\right] \tag{28}$$

$$+\frac{1}{2}\frac{\partial^2}{\partial y^2}\left[\left(S_0 + S_y y + S_\pi \pi + S_x x\right)^2 p_t(y,\pi,x)\right] \tag{29}$$

$$-\frac{\partial}{\partial \pi}\left[\left(\beta\pi_t - \kappa y_t\right) p_t(y,\pi,x)\right] \tag{30}$$

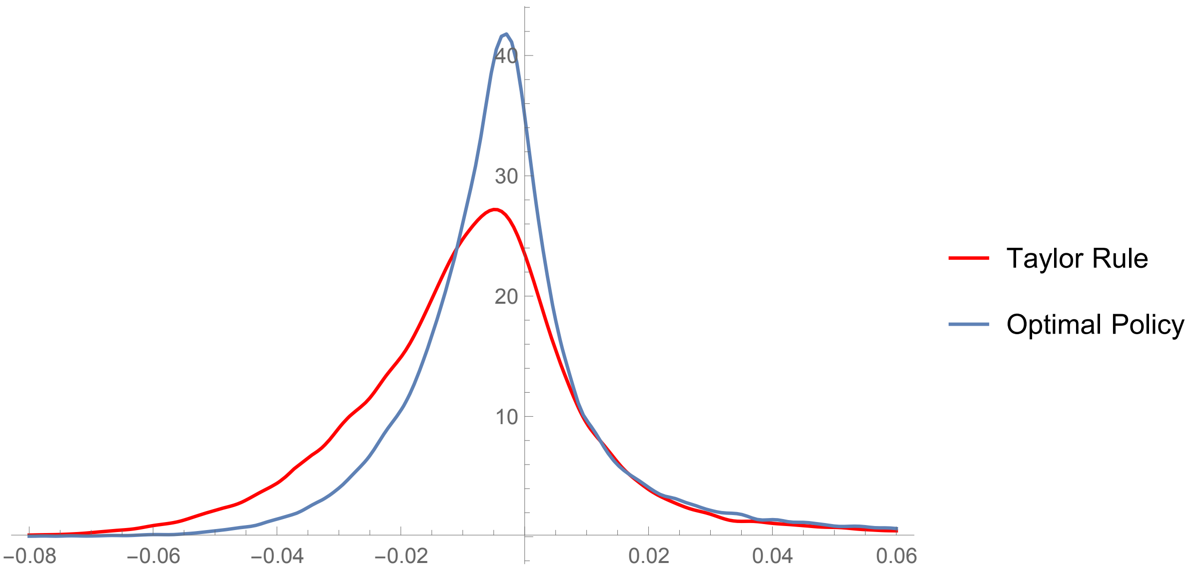
$$-\frac{\partial}{\partial x}\left[\left(-\rho_x\left(x_t - \mu_x\right)\right) p_t(y,\pi,x)\right] \tag{31}$$

$$+\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma_x^2 p_t(y,\pi,x)\right] \tag{32}$$

where $D_0 = \frac{(\gamma\xi-1)(\psi_0-r)}{\sigma(\gamma\xi-1+\rho)}$, $D_y = \frac{(\gamma\xi-1)\psi_y}{\sigma(\gamma\xi-1+\rho)}$, $D_\pi = \frac{(\gamma\xi-1)(\psi_\pi-1)}{\sigma(\gamma\xi-1+\rho)}$, $D_x = \frac{(\gamma\xi-1)\psi_x-\sigma\gamma\xi\rho}{\sigma(\gamma\xi-1+\rho)}$ and $S_0 = \frac{\xi(\psi_0-r)}{\sigma(\gamma\xi-1+\rho)}$, $S_y = \frac{\xi\psi_y}{\sigma(\gamma\xi-1+\rho)}$, $S_\pi = \frac{\xi(\psi_\pi-1)}{\sigma(\gamma\xi-1+\rho)}$, $S_x = \xi\left(\frac{\psi_x+(1-\rho)\sigma}{\sigma(\gamma\xi-1+\rho)}\right)$. The Kolmogorov equation can be solved numerically in a straightforward manner, resulting in the probability distribution function of y conditional on π and x.

Figure 3 shows the PDFs of output y using the optimal policy rule that takes vulnerability into account, and a standard Taylor rule that ignores the presence of vulnerability, for comparison. Clearly, the optimal policy rule generates less skewness of the unconditional output distribution. Intuitively, shocks to vulnerability x contain information about the conditional distribution of the output gap that the policy maker should take into account in setting optimal policy. For a given level of inflation and output, a higher vulnerability –a larger VaR of output– calls for higher interest rates. Higher interest rates induce the private sector to save more and consume less, thus shifting the conditional future distribution of y_t upwards by shifting its conditional mean upwards. Given the link between the expected mean and the expected volatility of output induced by the presence of vulnerability, a higher conditional mean induces a lower volatility of y_t. Together, higher mean and lower volatility mean lower vulnerability – lower VaR for output. For the suboptimal Taylor rule that ignores vulnerability, interest rates remain unchanged when, for a given level of y_t and π_t, V_t changes. Compared to the optimal rule, when V_t increases but i_t remains unchanged, the conditional mean of output is lower and its conditional volatility is higher. Over time, more frequent visit to states of lower mean and higher volatility create an unconditional distribution that is more negatively skewed. When instead V_t decreases, the optimal rule and the suboptimal Taylor rule produce similar right tails for the unconditional distribution of output. The reason is that lower V_t induces both higher mean and lower volatility of output. Therefore, even though the changes in mean and volatility of y_t are different for the two different rules, the actual differences in outcomes for y_t are small because the lower volatility minimizes all fluctuations. Indeed, as V_t approaches x_t, the stochastic part of dy_t vanishes and output becomes (locally) deterministic, diminishing the effects of any changes in interest rates, whether optimal or not.

Figure 3: **Probability Density Functions of the Output under the Optimal Policy Rule and a Standard Taylor Rule**. The figure shows the PDFs using the Kolmogorov equation (32) using the optimal policy rule and the standard Taylor rule. The standard Taylor rule coefficients are calculated for the economy assuming that the policy maker is ignoring the presence of financial vulnerability.



In order to have an equilibrium, $\mathbb{E}\left[e^{-\beta t}y_t^2\right]$ must be bounded. The next lemma characterizes when this is the case.

Lemma 2 *When $i_t = \psi_0 + \psi_y y_t + \psi_x x_t$, the loss function $L(y_0, 0)$ is bounded for all y_0 if and only if*

$$\underline{\psi}_y < \psi_y < \bar{\psi}_y$$

where

$$\begin{aligned}\bar{\psi}_y &= \min\left\{\frac{\beta\sigma(\gamma\xi - 1 + \rho)}{(\gamma\xi - 1)}, \frac{\sigma(\rho + \gamma\xi - 1)}{\xi^2}\left(1 - \gamma\xi - \sqrt{(1-\gamma\xi)^2 + \beta\xi^2}\right)\right\} \\ \underline{\psi}_y &= \frac{\sigma(\rho + \gamma\xi - 1)}{\xi^2}\left(1 - \gamma\xi + \sqrt{(1-\gamma\xi)^2 + \beta\xi^2}\right)\end{aligned}$$

The proof is in the appendix.

4 Optimal Monetary Policy with Flexible Prices

In order to gain further understanding of the role of vulnerabilities for monetary policy, we next turn to a specification with flexible prices. This setup can be alternatively viewed as a flexible price economy, or as the long run solution to an economy with sticky prices. As the algebra is simpler than in the previous section, we present it here in more detail. The central bank's problem is

$$L\left(y_t, \pi_t, x_t\right) = \min_{\{i_s\}_{s=t}^{\infty}} \mathbb{E}_t \int_t^{\infty} e^{-s\beta} y_s^2 ds \tag{33}$$

$$s.t.$$

$$dy_t = \sigma^{-1}\left(i_t - r\right) dt + d(rp_t) \tag{34}$$

$$dx_t = -\rho_x\left(x_t - \mu_x\right) dt + \sigma_x dZ_{x,t} \tag{35}$$

$$d(rp_t) = -\rho V_t dt + \xi\left(V_t - x_t\right) dZ_t \tag{36}$$

$$V_t \equiv \gamma \mathbb{V}\left[dy_t | \mathcal{F}_t\right] - \mathbb{E}\left[dy_t | \mathcal{F}_t\right] \tag{37}$$

$$dZ_{x,t} dZ_t = 0 \tag{38}$$

Parameters have the following sign restrictions $\beta, \lambda, \sigma, r, \kappa, \rho_x, \sigma_x, \rho, \gamma > 0$ and no a priori restrictions on μ_x, ξ. Re-arranging equations (34)-(37) we get

$$dy_t = \left(\left(\gamma\xi - 1\right) V_t - \gamma\xi x_t\right) dt + \xi\left(V_t - x_t\right) dZ_t \tag{39}$$

$$dx_t = -\rho_x\left(x_t - \mu_x\right) dt + \sigma_x dZ_{x,t} \tag{40}$$

$$i_t = \sigma\left(\gamma\xi - 1 + \rho\right) V_t - \sigma\gamma\xi x_t + r \tag{41}$$

It is straightforward to verify that in any equilibrium, irrespective of optimality, there is a long-run level of vulnerability given by

$$E\left[V_t\right] = \frac{\gamma\xi\mu_x}{\gamma\xi - 1} > 0 \text{ as } t \to \infty \text{ a.s}$$

which is obtained by noting that for any non-explosive path of y_t, $\mathbb{E}\left[dy_t\right] = 0$. The long-run mean of vulnerability is positive, a direct implication of having the mean-variance tradeoff for output observed empirically ($\gamma\xi - 1 < 0$ and $\mu_x < 0$). There is nothing the central bank can do to change the existence of this tradeoff or the steady-state level of vulnerability. What the central bank can do is to pick interest rates to achieve its desired level of V_t along the path that approaches the exogenous $E\left[V_t\right]$. The steady-state volatility of the output gap is also positive, for the same exact reasons that $E[V_t] > 0$:

$$\mathbb{V}[dy_t] = \frac{\xi\mu_x}{\gamma\xi - 1} > 0 \text{ as } t \to \infty \text{ a.s}$$

The long-run behavior of V_t and the mean-variance tradeoff imply

$$E[i_t] = \sigma\rho E[V_t] + r > r \text{ as } t \to \infty \text{ a.s} \tag{42}$$

$$E[y_t] > 0 \text{ as } t \to \infty \text{ a.s} \tag{43}$$

Thus, introducing vulnerability makes the economy sustain a positive output gap in the long run, supported by average interest rates higher than the natural rate r. The central bank's problem is

$$L(y_t, \pi_t, x_t) = \min_{\{V_s\}_{s=t}^{\infty}} \mathbb{E}_t \int_t^{\infty} e^{-s\beta} y_s^2 ds \tag{44}$$

$$s.t.$$

$$dy_t = ((\gamma\xi - 1)V_t - \gamma\xi x_t)\,dt + \xi(V_t - x_t)\,dZ_t \tag{45}$$

$$dx_t = -\rho_x(x_t - \mu_x)\,dt + \sigma_x dZ_{x,t} \tag{46}$$

and the optimal i_t can the be recovered using equation (41). The HJB is

$$\begin{aligned} 0 \;=\; & \min_V \left\{ y^2 - \beta L + \frac{\partial L}{\partial y}((\gamma\xi - 1)V - \gamma\xi x) + \frac{1}{2}\frac{\partial^2 L}{\partial y^2}\xi^2 (V - x)^2 \right. \\ & \left. -\rho_x(x - \mu_x)\frac{\partial L}{\partial x} + \frac{1}{2}\frac{\partial^2 L}{\partial x^2}\sigma_x^2 \right\} \end{aligned}$$

The FOC is

$$(1 - \gamma\xi)\frac{\partial L}{\partial y} = \xi^2 (V - x)\frac{\partial^2 L}{\partial y^2} \tag{47}$$

When the central bank increases V_t, because of the output mean-variance tradeoff, it not only lowers the drift of y_t, but it also increases its volatility. The FOC equalizes the marginal losses and gains in the objective function stemming from these two sources. Another way to see it is to note that

$$\mathbb{E}_t\left[dy_t^2\right] = \mathbb{E}_t[dy_t]^2 + \mathbb{V}_t[dy_t]$$

Changing V_t always moves $\mathbb{E}_t\left[dy_t\right]$ and $\mathbb{V}_t\left[dy_t\right]$ in opposite directions, since

$$\begin{aligned}
\frac{\partial}{\partial V_t}\mathbb{E}\left[dy_t|\mathcal{F}_t\right] &= \gamma\xi - 1 < 0 \\
\frac{\partial}{\partial V_t}\mathbb{V}\left[dy_t|\mathcal{F}_t\right] &= \xi > 0
\end{aligned}$$

Thus, minimizing $\mathbb{E}_t\left[dy_t^2\right]$ always involves compromising deviations in $\mathbb{E}_t\left[dy_t\right]$ and $\mathbb{V}_t\left[dy_t\right]$. In fact, the FOC can be written as

$$-\left(\frac{\partial}{\partial V_t}\mathbb{E}_t\left[dy_t\right]\right)\frac{\partial L}{\partial y} = \left(\frac{1}{2}\frac{\partial}{\partial V_t}\mathbb{V}_t\left[dy_t\right]^2\right)\frac{\partial^2 L}{\partial y^2}$$

which shows the precise tradeoff of marginal changes of V_t on the drift and volatility of y_t and, in turn, how these translate to losses in the objective function. The SOC is

$$\frac{\partial^2 L}{\partial y^2} > 0$$

so that increases in y^2 are always detrimental for welfare. Plugging the FOC into the HJB, we get the "concentrated" HJB

$$\begin{aligned}
0 &= y^2 - \beta L - \frac{\partial L}{\partial y}x - \frac{1}{2}\left(\frac{\partial^2 L}{\partial y^2}\right)^{-1}\left(\frac{\partial L}{\partial y}\right)^2\frac{\left(1-\gamma\xi\right)^2}{\xi^2} \\
&\quad -\rho_x\left(x-\mu_x\right)\frac{\partial L}{\partial x} + \frac{1}{2}\frac{\partial^2 L}{\partial x^2}\sigma_x^2
\end{aligned}$$

We look for a quadratic solution

$$L\left(y,x\right) = c_0 + c_1 y + c_2 y^2 + c_3 x + c_4 x^2 + c_5 yx$$

Plugging into the HJB, and using $\frac{\partial L}{\partial y} = c_1 + 2c_2 y + c_5 x$, $\frac{\partial^2 L}{\partial y^2} = 2c_2$, $\frac{\partial L}{\partial x} = c_3 + 2c_4 x + c_5 y$, $\frac{\partial^2 L}{\partial x^2} = 2c_4$, we get the following system of equations on the coefficients $c_0, ..., c_5$

$$
\begin{aligned}
(const) &:\ 0=\sigma_x^2 c_4-\beta c_0+\mu_x\rho_x c_3-\frac{(\gamma\xi-1)^2}{4\xi^2}\frac{c_1^2}{c_2}\\
(y) &:\ 0=\mu_x\rho_x c_5-\left(\beta+\frac{(\gamma\xi-1)^2}{\xi^2}\right)c_1\\
\left(y^2\right) &:\ 0=1-\left(\frac{(\gamma\xi-1)^2}{\xi^2}+\beta\right)c_2\\
(x) &:\ 0=c_1+(\beta+\rho_x)\,c_3-2\rho_x\mu_x c_4+\frac{(\gamma\xi-1)^2}{2\xi^2}\frac{c_5c_1}{c_2}\\
\left(x^2\right) &:\ 0=c_5+(\beta+2\rho_x)\,c_4+\frac{(\gamma\xi-1)^2}{4\xi^2}\frac{c_5^2}{c_2}\\
(yx) &:\ 0=2c_2+\left(\beta+\rho_x+\frac{(\gamma\xi-1)^2}{\xi^2}\right)c_5
\end{aligned}
$$

The solution (omitting c_0) is

$$
\begin{aligned}
c_1 &= -\frac{2\xi^6\mu_x\rho_x}{\left((\gamma\xi-1)^2+\beta\xi^2\right)^2\left((\gamma\xi-1)^2+\xi^2(\rho_x+\beta)\right)}>0\\
c_2 &= \frac{\xi^2}{\left((\gamma\xi-1)^2+\beta\xi^2\right)}>0\\
c_3 &= \frac{2\xi^4\mu_x\rho_x\left((\gamma\xi-1)^4+\xi^2(3\beta+2\rho_x)\left((\gamma\xi-1)^2+\xi^2(\beta+\rho_x)\right)\right)}{(\beta+\rho_x)(\beta+2\rho_x)\left((\gamma\xi-1)^2+\beta\xi^2\right)^2\left((\gamma\xi-1)^2+\xi^2(\beta+\rho_x)\right)^2}<0\\
c_4 &= \frac{\xi^4\left((\gamma\xi-1)^2+2\xi^2(\beta+\rho_x)\right)}{(\beta+2\rho_x)\left((\gamma\xi-1)^2+\beta\xi^2\right)\left((\gamma\xi-1)^2+\xi^2(\rho_x+\beta)\right)^2}>0\\
c_5 &= -\frac{2\xi^4}{\left((\gamma\xi-1)^2+\beta\xi^2\right)\left((\gamma\xi-1)^2+\xi^2(\rho_x+\beta)\right)}<0
\end{aligned}
$$

To pick the optimal initial conditions, we minimize L with respect to y_0 taking x_0 as given

$$
\begin{aligned}
L(y_0,\pi_0,x_0) &= c_0+c_1y_0+c_2y_0^2+c_3x_0+c_4x_0^2+c_5y_0x_0\\
FOC &:\ \frac{\partial L}{\partial y_0}=0\\
SOC &:\ \frac{\partial^2 L}{\partial y_0^2}>0
\end{aligned}
$$

The FOC and SOC can be solved to get

$$\begin{aligned}
y_0^* &= -\left(\frac{c_1}{2c_2}+\frac{c_5}{2c_2}x_0\right)\\
&= \frac{\xi^4\mu_x\rho_x}{\left((\gamma\xi-1)^2+\beta\xi^2\right)\left((\gamma\xi-1)^2+\xi^2\left(\rho_x+\beta\right)\right)}+\frac{\xi^2}{\left((\gamma\xi-1)^2+\xi^2\left(\rho_x+\beta\right)\right)}x_0\\
c_2 &> 0
\end{aligned}$$

The optimal policy in terms of V_t is given by plugging in the optimal solution into the FOC in equation (47):

$$\begin{aligned}
V_t &= \left(\frac{\xi^2\left(\rho_x+\beta\right)-\xi\gamma\left(1-\gamma\xi\right)}{\left((\gamma\xi-1)^2+\xi^2\left(\rho_x+\beta\right)\right)}\right)x_t+\left(\frac{1-\gamma\xi}{\xi^2}\right)y_t \\
&\quad+\frac{\xi^2\mu_x\rho_x\left(\gamma\xi-1\right)}{\left((\gamma\xi-1)^2+\beta\xi^2\right)\left((\gamma\xi-1)^2+\xi^2\left(\rho_x+\beta\right)\right)}
\end{aligned} \tag{48}$$

$$\begin{aligned}
-\left(\frac{\partial}{\partial V_t}\mathbb{E}_t\left[dy_t\right]\right)\frac{\partial L}{\partial y} &= \left(\frac{1}{2}\frac{\partial}{\partial V_t}\mathbb{V}_t\left[dy_t\right]^2\right)\frac{\partial^2 L}{\partial y^2}\\
0 &= \frac{\partial}{\partial V_t}\frac{\partial}{\partial y}\left[\xi\left(V_t-x_t\right)\left(\frac{1}{2}\frac{\partial L}{\partial y}+\gamma L\right)-V_tL\right]\\
dy_t &= \left(\left(\gamma\xi-1\right)V_t-\gamma\xi x_t\right)dt+\xi\left(V_t-x_t\right)dZ_t\\
dy_t &= \left(\gamma\xi\left(V_t-x_t\right)-V_t\right)dt+\xi\left(V_t-x_t\right)dZ_t
\end{aligned}$$

For a given x_t, when the central bank observes an increase in the output gap, it is optimal to *increase* vulnerability. Increasing vulnerability is a tool to cool down the economy, counteracting the increase in y_t. Increasing vulnerability to reduce the drift of y_t must be traded off with the resulting increases in the volatility of y_t. Equation (48) states that, in terms of welfare, the decrese in the drift of y_t always dominates the corresponding increase in volatility. The increase in volatility is not immaterial: the denominator ξ^2 in the coefficient $\left(1-\gamma\xi\right)/\xi^2$ in front of y_t attenuates the response on the central bank because of the volatility effect. Why does the drift component always dominate?

The central bank achieves the increase in vulnerability by adjusting interest rates.

The implied optimal rule for the interest rate is

$$\begin{aligned}
i_t &= \sigma(\gamma\xi - 1 + \rho)V_t - \sigma\gamma\xi x_t + r \\
&= \left(\frac{\sigma(\gamma\xi - 1 + \rho)(1-\gamma\xi)}{\xi^2}\right) y_t \\
&\quad + \left(\frac{\sigma(\gamma\xi - 1 + \rho)\left(\xi^2(\rho_x + \beta) - \xi\gamma(1-\gamma\xi)\right)}{\left((\gamma\xi - 1)^2 + \xi^2(\rho_x + \beta)\right)} - \sigma\gamma\xi\right) x_t \\
&\quad + \frac{\sigma\xi^2\mu_x\rho_x(\gamma\xi - 1 + \rho)(\gamma\xi - 1)}{\left((\gamma\xi - 1)^2 + \beta\xi^2\right)\left((\gamma\xi - 1)^2 + \xi^2(\rho_x + \beta)\right)} + r
\end{aligned}$$

Therefore, whether interest rates increase or decrease with y_t and V_t depends on the sign of $(\gamma\xi - 1 + \rho)$. Note that the signs of $\partial i_t/\partial y_t$ and $\partial V_t/\partial y_t$ are the same.

If $\xi^2(\rho_x + \beta) - \xi\gamma(1-\gamma\xi) = 0$, then V_t does not depend on x_t and

$$\begin{aligned}
V_t &= \left(\frac{(1-\gamma\xi)}{\xi^2}\right) y_t - \frac{\xi^2\mu_x\rho_x}{(\gamma\xi - 1)^2 + \beta\xi^2} \\
i_t &= \left(\frac{\sigma(\gamma\xi - 1 + \rho)(1-\gamma\xi)}{\xi^2}\right) y_t - \sigma\gamma\xi x_t - \frac{\sigma(\gamma\xi - 1 + \rho)\xi^2\mu_x\rho_x}{\left((\gamma\xi - 1)^2 + \beta\xi^2\right)} + r
\end{aligned}$$

If $\xi^2(\rho_x + \beta) - \xi\gamma(1-\gamma\xi) \neq 0$ we can substitute x_t in the Taylor rule by V_t

$$\begin{aligned}
i_t &= \frac{\sigma(\gamma\rho(\gamma\xi - 1) + \xi(\rho - 1)(\beta + \rho_x))}{(\xi\gamma - 1)\gamma + \xi(\beta + \rho_x)} V_t \\
&\quad + \frac{\sigma\gamma}{\xi^2}\frac{(1-\gamma\xi)\left((\gamma\xi - 1)^2 + \xi^2(\beta + \rho_x)\right)}{(\xi\gamma - 1)\gamma + \xi(\beta + \rho_x)} y_t \\
&\quad + \frac{\sigma\gamma\xi^2\mu_x\rho_x(\gamma\xi - 1)}{\left((\xi\gamma - 1)\gamma + \xi(\beta + \rho_x)\right)\left((\gamma\xi - 1)^2 + \beta\xi^2\right)} + r
\end{aligned}$$

We can define some constants to make notation easier

$$\begin{aligned}
i_t &= (\sigma(\rho - 1) - \varphi)V_t + (\varphi\chi)y_t + r - \frac{\varphi\mu_x\rho_x}{\chi} \\
\varphi &= \frac{\sigma\gamma(1-\gamma\xi)}{(\xi\gamma - 1)\gamma + \xi(\beta + \rho_x)} \\
\chi &= \frac{\left((\gamma\xi - 1)^2 + \xi^2(\beta + \rho_x)\right)}{\xi^2} > 0
\end{aligned}$$

Whether interest rates increase or decrease in response to y_t when V_t is held constant depends on the sign of φ. On the other hand, how interest rates respond to V_t keeping

y_t constant depends on the sign of $(\sigma(\rho-1)-\varphi)$. Together, the empirically observed signs are

$$\begin{aligned} \varphi &> 0 \\ \sigma(\rho-1)-\varphi &< 0 \end{aligned}$$

These two inequalities impose the following restrictions on parameters

$$\begin{aligned} \rho &< 1 \\ \xi(\rho_x+\beta) &> -\gamma(\gamma\xi-1) \end{aligned}$$

Now consider the following general linear functions

$$\begin{aligned} i_t &= \psi_0+\psi_y y_t+\psi_x x_t \\ V_t &= \phi_0+\phi_y y_t+\phi_x x_t \end{aligned}$$

which we will use to analyze non-optimal behavior. By equation (41)

$$\begin{aligned} \phi_0 &= \frac{\psi_0-r}{\sigma(\gamma\xi-1+\rho)} \\ \phi_y &= \frac{\psi_y}{\sigma(\gamma\xi-1+\rho)} \\ \phi_x &= \frac{\psi_x+\sigma\gamma\xi}{\sigma(\gamma\xi-1+\rho)} \end{aligned}$$

and

$$V_t=\frac{\psi_0-r}{\sigma(\gamma\xi-1+\rho)}+\frac{\psi_y}{\sigma(\gamma\xi-1+\rho)}y_t+\frac{\psi_x+\sigma\gamma\xi}{\sigma(\gamma\xi-1+\rho)}x_t$$

Plugging into equations (19)-(21) we get

$$\begin{aligned} dy_t &= \left(\frac{(\gamma\xi-1)(\psi_0-r)}{\sigma(\gamma\xi-1+\rho)}+\frac{(\gamma\xi-1)\psi_y}{\sigma(\gamma\xi-1+\rho)}y_t+\left(\frac{(\gamma\xi-1)\psi_x-\sigma\gamma\xi\rho}{\sigma(\gamma\xi-1+\rho)}\right)x_t\right)dt \\ &\quad+\left(\frac{\xi(\psi_0-r)}{\sigma(\gamma\xi-1+\rho)}+\frac{\xi\psi_y}{\sigma(\gamma\xi-1+\rho)}y_t+\xi\left(\frac{\psi_x+(1-\rho)\sigma}{\sigma(\gamma\xi-1+\rho)}\right)x_t\right)dZ_t \qquad (49) \\ dx_t &= -\rho_x(x_t-\mu_x)\,dt+\sigma_x dZ_{x,t} \qquad (50) \end{aligned}$$

The process for x_t can be solved to get

$$\begin{aligned}
x_t &= \mu_x + (x_s - \mu_x)\exp(-\rho_x(t-s)) + \sigma_x \int_s^t \exp(-\theta(t-u))\, dZ_{x,u} \\
E_s[x_t] &= \mu_x + (x_s - \mu_x)\exp(-\rho_x(t-s)) \\
var_s(x_t) &= \sigma_x^2 \int_s^t \exp(-2\theta(t-u))\, du = \frac{\sigma_x^2}{2\theta}\left(1 - \exp(-2\theta(t-s))\right)
\end{aligned}$$

The pdf for x_t is

$$p_{t,x}(x) = \frac{\sqrt{\theta}}{\sqrt{\pi\sigma_x^2\left(1-\exp(-2\theta(t-s))\right)}} \exp\left(-\frac{\theta\left[x - \mu_x + (x_s-\mu_x)\exp(-\rho_x(t-s))\right]^2}{\sigma_x^2\left(1-\exp(-2\theta(t-s))\right)}\right)$$

The Kolmogorov equation for the joint distribution $p_t(y,\pi,x)$ is

$$\begin{aligned}
\frac{\partial}{\partial t} p_t(y,\pi,x) &= -\frac{\partial}{\partial y}\left[(D_0 + D_y y + D_x x)\, p_t(y,\pi,x)\right] \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[(S_0 + S_y y + S_x x)^2 p_t(y,\pi,x)\right] \\
&\quad - \frac{\partial}{\partial x}\left[(-\rho_x(x_t - \mu_x))\, p_t(y,\pi,x)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma_x^2 p_t(y,\pi,x)\right]
\end{aligned}$$

where

$$\begin{aligned}
D_0 &= \frac{(\gamma\xi - 1)(\psi_0 - r)}{\sigma(\gamma\xi - 1 + \rho)}, D_y = \frac{(\gamma\xi - 1)\psi_y}{\sigma(\gamma\xi - 1 + \rho)}, D_x = \frac{(\gamma\xi - 1)\psi_x - \sigma\gamma\xi\rho}{\sigma(\gamma\xi - 1 + \rho)} \\
S_0 &= \frac{\xi(\psi_0 - r)}{\sigma(\gamma\xi - 1 + \rho)}, S_y = \frac{\xi\psi_y}{\sigma(\gamma\xi - 1 + \rho)}, S_x = \xi\left(\frac{\psi_x + (1-\rho)\sigma}{\sigma(\gamma\xi - 1 + \rho)}\right)
\end{aligned}$$

This is therefore fully characterizing the distribution of output under optimal monetary policy. It is noteworthy that optimal monetary policy features a policy rule that conditions on the output gap and on vulnerability. Despite the fact that prices are flexible, monetary policy takes vulnerability and output gap into account. Intuitively, those correspond to the first two conditional moments of the output gap. Lower interest rates boost aggregate demand, but they also increase vulnerability. Hence the central bank faces a tradeoff, even when prices are fully flexible. This tradeoff between growth and vulnerability is at the heart of our New Keynesian Vulnerability model.

5 Optimal Monetary Policy with a Zero Lower Bound

In our analysis so far, we have abstracted from a zero lower bound (ZLB) on nominal interest rates. In this section, we will solve for optimal monetary policy with a zero lower bound. In order to keep the problem analytically tractable, we study a slightly simplified problem. We assume that there is only one shock by setting $dZ^x = 0$ so that $x_t = x$. We also assume that prices are flexible, so that the central bank only considers output deviations in its loss function. The optimal control problem for the central bank is therefore:

$$L(y_t) = \min_{\{V_s\}_{s=t}^{\infty}} \mathbb{E}_t \int_t^{\infty} e^{-s\beta} y_s^2 ds \tag{51}$$

$$s.t.$$

$$dy_t = ((\gamma\xi - 1) V_t - \gamma\xi x) dt + \xi (V_t - x) dZ_t \tag{52}$$

$$i_t \geq 0. \tag{53}$$

We maintain the same parameter sign restrictions as before: $x < 0$, $\gamma\xi - 1 < 0$, and $\gamma\xi - 1 + \rho < 0$. The novelty relative to the earlier analysis is the constraint $i_t \geq 0$.

There is of course a well established literature that studies the existence of the ZLB in settings without uncertainty or with uncertainty but perfect foresight. Eggertsson and Woodford (2003), Eggertsson and Woodford (2004), Werning (2012) show that forward guidance is the optimal policy when the ZLB and study accompanying fiscal policy that can mitigate the liquidity trap. Correia, Farhi, Nicolini, and Teles (2013) show that tax policy can solve the ZLB problems in the standard New Keynesian model. Duarte (2016) analyzes optimal monetary policy rules that implement the optimal forward guidance equilibrium of Eggertsson and Woodford (2003) and Werning (2012) without the need for active fiscal policy. We instead find the optimal (constrained) equilibrium with the added ingredient of vulnerability. In our setup, fiscal policy is passive. We focus on the optimal path and not on implementation. In addition, we solve for the first time in closed form a NK model with a ZLB and uncertainty (and no perfect foresight).

We will start by previewing the solution, and then derive it. Using equation (11) with $\pi_t = 0$, we can express the ZLB in terms of an upper bound on V_t

$$i_t = \sigma (\gamma\xi - 1 + \rho) V_t - \sigma\gamma\xi x + r \geq 0$$

Thus, the bound on V_t is

$$V_t \leq V_b$$

where

$$V_b \equiv \frac{\sigma\gamma\xi x - r}{\sigma\left(\gamma\xi - 1 + \rho\right)} > 0$$

and the subscript b stands for "bound". Therefore, the ZLB implies that vulnerability cannot be "too high" (since interest rates cannot be "too low"). We assume for technical reasons that $V_b \neq \frac{\gamma\xi+1}{\gamma\xi-1}x$, $V_b \neq x$, $V_b \neq \frac{\gamma\xi x}{\gamma\xi-1}$ in order to simplify the analysis (without these conditions, we would have to keep track of many different cases, which would make the computations substantially more complicated, without adding economic instight). The HJB is

$$0 = \min_{V \leq V_b} \left\{ y^2 - \beta L + \frac{\partial L}{\partial y}\left(\left(\gamma\xi - 1\right)V - \gamma\xi x\right) + \frac{1}{2}\frac{\partial^2 L}{\partial y^2}\xi^2\left(V - x\right)^2 \right\}$$

with solution

$$L = \begin{cases} c_0 + c_1 y + c_2 y^2 & , \quad y < y_b \\ d_0 + d_1 y + d_2 y^2 + d_3 \exp\left(\eta_1 y\right) + d_4 \exp\left(\eta_2 y\right) & , \quad y \geq y_b \end{cases}$$

and optimal policy in terms of V

$$V = \begin{cases} \left(\frac{1-\gamma\xi}{2c_2\xi^2}c_1 + x\right) + \left(\frac{1-\gamma\xi}{\xi^2}\right)y & , \quad y < y_b \\ V_b & , \quad y \geq y_b \end{cases}$$

where y_b determines the value of y for which the ZLB (and V_b) is hit

$$y_b = \xi^2 \left(\frac{x\gamma\xi}{\left(\gamma\xi - 1\right)^2 + \beta\xi^2} - \frac{V_b}{\left(\gamma\xi - 1\right)} + \frac{\beta\xi^2 x}{\left(\gamma\xi - 1\right)\left(\left(\gamma\xi - 1\right)^2 + \beta\xi^2\right)} \right)$$

At the ZLB, optimal policy is therefore independent of economic or financial conditions. The level of vulnerability V_b is directly pinned down by the level of the interest rate $i = 0$. Outside of the zero lower bound, when $y < y_b$, optimal monetary policy follows a flexible inflation targeting rule

$$i_t = \sigma\left(\gamma\xi - 1 + \rho\right)\left(\frac{1-\gamma\xi}{2c_2\xi^2}c_1 + x + \frac{1-\gamma\xi}{\xi^2}y_t\right) - \sigma\gamma\xi x + r \tag{54}$$

where the interest rate depends on the level of the output gap y_t. In a more general setup with a time varying x_t, the flexible inflation targeting rule would of course also depend on the level of the shock to vulnerability x_t.

The optimal behavior of the central bank is different from the one studied by Werning (2012) (and, in discrete time, Eggertsson and Woodford (2003)), who considers the same model without vulnerability (dp_t=0 for all t) but with a natural rate that switches from negative to positive, so that the economy starts in a liquidity trap. Without vulnerability, Eggertsson and Woodford (2003)) shows that the optimal policy with commitment is to keep nominal interest rates at zero for longer than the natural rate is negative –in other words, forward guidance. Once the period of zero interest rates is over, optimal policy is characterized by $i_t = (1 - \kappa\sigma\lambda)\pi + r > 0$, so that interest rates jump from zero to a positive value and then react less than one-for-one to inflation, since $(1 - \kappa\sigma\lambda) < 1$.

The constants for the quadratic part in the solution are given by

$$c_0 = \frac{x^2}{\beta}\left(\frac{(\gamma\xi-1)^2}{\xi^2}+2\beta\right)\left(\frac{(\gamma\xi-1)^2}{\xi^2}+\beta\right)^{-3} > 0 \tag{55}$$

$$c_1 = -2x\left(\frac{(\gamma\xi-1)^2}{\xi^2}+\beta\right)^{-2} > 0 \tag{56}$$

$$c_2 = \left(\frac{(\gamma\xi-1)^2}{\xi^2}+\beta\right)^{-1} > 0 \tag{57}$$

$$d_0 = \frac{2}{\beta^2}\left(\frac{\xi^2(V_b-x)^2}{2}+\frac{((\gamma\xi-1)V_b-\gamma\xi x)^2}{\beta}\right) > 0 \tag{58}$$

$$d_1 = \frac{2}{\beta^2}((\gamma\xi-1)V_b-\gamma\xi x) \tag{59}$$

$$d_2 = \frac{1}{\beta} > 0 \tag{60}$$

where the signs for all constants except d_1 are indicated since d_1 can have any sign. The constants d_4 and d_5 are the solution to the following system of two linear equations in two unknowns

$$\begin{bmatrix} \left(\frac{1}{\eta_1}-\frac{\xi^2(V_b-x)}{(1-\gamma\xi)}\right)\eta_1^2\exp(\eta_1 y_b) & \left(\frac{1}{\eta_2}-\frac{\xi^2(V_b-x)}{(1-\gamma\xi)}\right)\eta_2^2\exp(\eta_2 y_b) \\ \exp(\eta_1 y_b) & \exp(\eta_2 y_b) \end{bmatrix}\begin{bmatrix} K_5 \\ K_6 \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

where

$$
\begin{aligned}
M_1 &= \frac{2}{\beta}\left(\frac{\xi^2 (V_b - x)}{(1-\gamma\xi)} - \frac{A_1}{\beta} - y_b\right) \\
M_2 &= \left(\frac{2x^2 (A_3 - \beta)}{(2A_3 - \beta)^3 \beta} - \frac{2A_1^2}{\beta^3} - \frac{2A_2}{\beta^2}\right) - \left(\frac{2x}{(2A_3 - \beta)^2} + \frac{2A_1}{\beta^2}\right) y_b \\
&\quad - \left(\frac{1}{2A_3 - \beta} + \frac{1}{\beta}\right) y_b^2 \\
\eta_1 &= \frac{(1-\gamma\xi) V_b + \gamma\xi x}{\xi^2 (V_b - x)^2} - \frac{\sqrt{((\gamma\xi - 1) V_b - \gamma\xi x)^2 + 2\xi^2 (V_b - x)^2 \beta}}{\xi^2 (V_b - x)^2} \\
\eta_2 &= \frac{(1-\gamma\xi) V_b + \gamma\xi x}{\xi^2 (V_b - x)^2} + \frac{\sqrt{((\gamma\xi - 1) V_b - \gamma\xi x)^2 + 2\xi^2 (V_b - x)^2 \beta}}{\xi^2 (V_b - x)^2}
\end{aligned}
$$

This system has a unique solution if and only if equation (5) holds, which is given by

$$
\begin{aligned}
K_5 &= \frac{\exp(-\eta_1 y_b)}{\Lambda}\left(M_1 - \left(\frac{1}{\eta_2} - \frac{\xi^2 (V_b - x)}{(1-\gamma\xi)}\right)\eta_2^2 M_2\right) \\
K_6 &= \frac{\exp(-\eta_2 y_b)}{\Lambda}\left(\left(\frac{1}{\eta_1} - \frac{\xi^2 (V_b - x)}{(1-\gamma\xi)}\right)\eta_1^2 M_2 - M_1\right)
\end{aligned}
$$

where

$$
\Lambda = \left(\frac{1}{\eta_1} - \frac{\xi^2 (V_b - x)}{(1-\gamma\xi)}\right)\eta_1^2 - \left(\frac{1}{\eta_2} - \frac{\xi^2 (V_b - x)}{(1-\gamma\xi)}\right)\eta_2^2
$$

The solution features a value function L that is continuous. Indeed, L is smooth everywhere except at y_b. At y_b, the first and second derivatives do not exist in the classical sense, so we use the concept of viscosity solution to solve the HJB. Also note that the solution without a ZLB can be recovered by taking $V_b \to \infty$. Equivalently, the solution without the ZLB is given by

$$
\begin{aligned}
L &= c_0 + c_1 y + c_2 y^2 \\
V &= \left(\frac{1-\gamma\xi}{2c_2\xi^2} c_1 + x\right) + \left(\frac{1-\gamma\xi}{\xi^2}\right) y
\end{aligned}
$$

which is exactly the solution in the unconstrained region in the problem with the ZLB, with the difference that it now applies for all y. The constants c_0, c_1 and c_2 in the case without the ZLB are still given by equations (55)-(57).

6 Conclusion

The degree to which financial stability considerations should be incorporated in the conduct of monetary policy has long been debated Adrian and Shin (2010), Adrian and Liang (2016). In this paper, we extend the basic, two equation New Keynesian model to incorporate a notion of financial vulnerability. Shocks to risk premia impact aggregate demand via the Euler equation. The shocks to risk premia are assumed to impact the volatility of output, which is motivated from the empirical observation by Adrian, Boyarchenko, and Giannone (2016) that financial conditions forecast both the mean and the volatility of output. Importantly, our framework reproduces the stylized fact that the conditional mean and the conditional volatility of output are strongly negatively correlated, giving rise to a sharply negatively skewed unconditional output distribution. Vulnerability thus captures movements in the conditional GDP distribution that correspond to the downside risk of growth.

We further assume that the central bank minimizes the expected discounted sum of squared output gaps and squared inflation, which is standard in the literature. This is therefore a central bank that is subject to a dual mandate, without an independent financial stability objective. Despite that narrow objective function, the optimal flexible inflation targeting rule conditions on the level of vulnerability. Intuitively, all variables that provide information about the conditional distribution of GDP should be taken into account in setting optimal monetary policy. This translates into an augmented Taylor rule, where financial vulnerability—as measured by output gap tailrisk as a function of financial variables—is an input into the Taylor rule. Furthermore, the magnitude of the Taylor rule coefficients on output gap and inflation depend on the parameters that determine vulnerability.

The striking result from our setup is that the central bank should **always** condition monetary policy on financial vulnerability. Relative to earlier literature that has made similar arguments (e.g. Curdia and Woodford (2010) and Gambacorta and Signoretti (2014), our modeling approach is deeply rooted in empirical observations which capture macoreconomic shocks of the 2008 crisis very well. Through the negative correlation between conditional mean and conditional variance, our setup captures nonlinearity in macro dynamics in a tractable linear-quadratic setting. The implications of our results for the conduct of monetary policy are in line with the arguments or Adrian and Shin (2010) and Adrian and Liang (2016).

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A Appendix: Solution to the New Keynesian Model with Vulnerabilities

A.1 The coefficients of the Welfare Function

Plugging into the concentrated HJB (23) we find:

$$\begin{aligned}
\frac{\partial L}{\partial y} &= c_1 + 2c_2 y + c_5 x + c_8 \pi \\
\frac{\partial^2 L}{\partial y^2} &= 2c_2 \\
\frac{\partial L}{\partial \pi} &= c_6 + 2c_7 \pi + c_8 y + c_9 x \\
\frac{\partial L}{\partial x} &= c_3 + 2c_4 x + c_5 y + c_9 \pi \\
\frac{\partial^2 L}{\partial x^2} &= 2c_4
\end{aligned}$$

and setting the coefficients in front of combinations of the state variables to zero, we get the following system of equations in $c_0, ..., c_9$:

$$\begin{aligned}
(const) \quad &: \quad 0 = -\beta c_0 + \mu_x \rho_x c_3 + \sigma_x^2 c_4 - \frac{(1-\gamma\xi)^2}{4\xi^2}\frac{c_1^2}{c_2} \\
(y) \quad &: \quad 0 = \left(-\left(\beta + \frac{(1-\gamma\xi)^2}{\xi^2}\right) c_1 + \mu_x \rho_x c_5 - \kappa c_6\right) \\
\left(y^2\right) \quad &: \quad 0 = \left(1 - \left(\beta + \frac{(1-\gamma\xi)^2}{\xi^2}\right) c_2 - \kappa c_8\right) \\
(x) \quad &: \quad 0 = \left(-c_1 - (\beta + \rho_x) c_3 + 2\rho_x \mu_x c_4 - \frac{(1-\gamma\xi)^2}{2\xi^2}\frac{c_1 c_5}{c_2}\right) \\
\left(x^2\right) \quad &: \quad 0 = \left(-(\beta + 2\rho_x) c_4 - c_5 - \frac{1}{4}\frac{(1-\gamma\xi)^2}{\xi^2}\frac{c_5^2}{c_2}\right) \\
(yx) \quad &: \quad 0 = \left(-2c_2 - \left(\beta + \frac{(1-\gamma\xi)^2}{\xi^2} + \rho_x\right) c_5 - \kappa c_9\right) \\
(\pi) \quad &: \quad 0 = \left(\mu_x \rho_x c_9 - \frac{(1-\gamma\xi)^2}{2\xi^2}\frac{c_1 c_8}{c_2}\right) \\
\left(\pi^2\right) \quad &: \quad 0 = \left(\lambda + \beta c_7 - \frac{(1-\gamma\xi)^2}{4\xi^2}\frac{c_8^2}{c_2}\right) \\
(y\pi) \quad &: \quad 0 = \left(-2\kappa c_7 - \frac{(1-\gamma\xi)^2}{\xi^2} c_8\right) \\
(x\pi) \quad &: \quad 0 = \left(-c_8 - \rho_x c_9 - \frac{1}{2}\frac{(1-\gamma\xi)^2}{\xi^2}\frac{c_5 c_8}{c_2}\right)
\end{aligned}$$

If $(1-\gamma\xi)^2 - \beta\xi^2 \neq 0$, there are two solutions for c_2:

$$c_2 = \frac{1}{\left(\frac{(1-\gamma\xi)^2}{\xi^2}\right)^2 - \beta^2}\left(\frac{2\xi^2\kappa^2\lambda}{(1-\gamma\xi)^2} + \frac{(1-\gamma\xi)^2}{\xi^2} \pm \sqrt{4\kappa^2\lambda\left(\frac{\xi^4\kappa^2\lambda}{(\gamma\xi-1)^4} + 1\right) + \beta^2}\right)$$

If $(1-\gamma\xi)^2 - \beta\xi^2 = 0$ then there is a unique solution for c_2:

$$c_2 = \left(\frac{4\lambda\xi^2\kappa^2}{(1-\gamma\xi)^2} + \frac{2(1-\gamma\xi)^2}{\xi^2}\right)^{-1}$$

Given c_2, the other coefficients are determined uniquely:

$$
\begin{aligned}
c_5 &= \left(-\frac{2}{\kappa}c_2 + \frac{1}{\rho_x}\left(\frac{1}{\kappa} - \frac{1}{\kappa}\left(\beta + \frac{(1-\gamma\xi)^2}{\xi^2}\right)c_2\right)\right) \times \\
&\quad \times \left(-\frac{(1-\gamma\xi)^2}{2\rho_x\xi^2}\left(\frac{1}{\kappa c_2} - \frac{1}{\kappa}\left(\beta + \frac{(1-\gamma\xi)^2}{\xi^2}\right)\right) + \frac{1}{\kappa}\left(\beta + \rho_x + \frac{(1-\gamma\xi)^2}{\xi^2}\right)\right)^{-1} \\
c_7 &= -\frac{(1-\gamma\xi)^2}{2\xi^2\kappa}\left(\frac{1}{\kappa} - \frac{1}{\kappa}\left(\beta + \frac{(1-\gamma\xi)^2}{\xi^2}\right)c_2\right) \\
c_8 &= \left(\frac{1}{\kappa} - \frac{1}{\kappa}\left(\beta + \frac{(1-\gamma\xi)^2}{\xi^2}\right)c_2\right) \\
c_9 &= -\frac{2}{\kappa}c_2 - \frac{1}{\kappa}\left(\beta + \frac{(1-\gamma\xi)^2}{\xi^2} + \rho_x\right)c_5 \\
c_4 &= -\frac{1}{(\beta + 2\rho_x)}c_5 - \frac{(1-\gamma\xi)^2}{4(\beta + 2\rho_x)\xi^2}\frac{c_5^2}{c_2} \\
c_1 &= \frac{2\xi^2\mu_x\rho_x}{(1-\gamma\xi)^2}\frac{c_2 c_9}{c_8} \\
c_6 &= -\frac{1}{\kappa}\left(\beta + \frac{(1-\gamma\xi)^2}{\xi^2}\right)c_1 + \frac{\mu_x\rho_x}{\kappa}c_5 \\
c_3 &= -\frac{1}{\beta + \rho_x}c_1 + \frac{2\rho_x\mu_x}{\beta + \rho_x}c_4 - \frac{(1-\gamma\xi)^2}{2(\beta + \rho_x)\xi^2}\frac{c_1 c_5}{c_2} \\
c_0 &= \frac{\mu_x\rho_x}{\beta}c_3 + \frac{\sigma_x^2}{\beta}c_4 - \frac{(1-\gamma\xi)^2}{4\beta\xi^2}\frac{c_1^2}{c_2}
\end{aligned}
$$

where I have arranged the coefficients in "triangular" form so that they can be solved sequentially from c_5 to c_0 in the order in which they are listed.

A.2 The Optimal Initial Conditions

To pick the optimal initial conditions, we minimize L with respect to y_0, π_0 taking x_0 as given

$$\begin{aligned}
L(y_0, \pi_0, x_0) &= c_0 + c_1 y_0 + c_2 y_0^2 + c_3 x_0 + c_4 x_0^2 + c_5 y_0 x_0 + c_6 \pi_0 \\
&\quad + c_7 \pi_0^2 + c_8 y_0 \pi_0 + c_9 \pi_0 x_0 \\
FOC &: \frac{\partial L}{\partial y_0} = 0 \\
FOC &: \frac{\partial L}{\partial \pi_0} = 0 \\
SOC &: \frac{\partial^2 L}{\partial y_0^2} \frac{\partial^2 L}{\partial \pi_0^2} - \frac{\partial^2 L}{\partial y_0 \partial \pi_0} > 0 \\
SOC &: \frac{\partial^2 L}{\partial y_0^2} > 0
\end{aligned}$$

The FOC and SOC are

$$\begin{aligned}
0 &= c_1 + 2c_2 y_0 + c_5 x_0 + c_8 \pi_0 \\
0 &= c_6 + 2c_7 \pi_0 + c_8 y_0 + c_9 x_0 \\
4c_7 c_2 - c_8 &> 0 \\
c_2 &> 0
\end{aligned}$$

The last SOC determines the sign of c_2 and hence which of the two solutions to pick (if $(1 - \gamma\xi)^2 - \beta\xi^2 \neq 0$).

B Appendix: The New Keynesian Model wit a ZLB Constraint

In this Appendix, we derive the solution to the ZLB problem. If $\partial^2 L / \partial y^2 < 0$, the problem is not well defined since the minimum is achieved by $V = -\infty$ and the value function is undefined. Thus, we must have that

$$\frac{\partial^2 L}{\partial y^2} < 0$$

Similarly, if $\partial^2 L / \partial y^2 = 0$ and $\partial L / \partial y > 0$, the minimum is achieved by $V = -\infty$ and the value function is undefined, so we also exclude this case.

The FOC for V in the HJB is

$$V=\begin{cases} [-\infty,V_b] & , \quad \text{if } \frac{\partial^2 L}{\partial y^2}=0 \text{ and } \frac{\partial L}{\partial y}=0 \\ V_b & , \quad \text{if } \frac{\partial^2 L}{\partial y^2}=0 \text{ and } \frac{\partial L}{\partial y}>0 \\ \left(\frac{\partial^2 L}{\partial y^2}\right)^{-1}\frac{\partial L}{\partial y}\frac{(1-\gamma\xi)}{\xi^2}+x & , \quad \text{if } \frac{\partial^2 L}{\partial y^2}>0 \text{ and } \left(\frac{\partial^2 L}{\partial y^2}\right)^{-1}\frac{\partial L}{\partial y}\frac{(1-\gamma\xi)}{\xi^2}+x<V_b \\ V_b & , \quad \text{if } \frac{\partial^2 L}{\partial y^2}>0 \text{ and } \left(\frac{\partial^2 L}{\partial y^2}\right)^{-1}\frac{\partial L}{\partial y}\frac{(1-\gamma\xi)}{\xi^2}+x\geq V_b \end{cases}$$

The concentrated HJB is thus

$$0=\begin{cases} y^2-\beta L & , \quad \text{if } \frac{\partial^2 L}{\partial y^2}=0 \text{ and } \frac{\partial L}{\partial y}=0 \\ y^2-\beta L+((\gamma\xi-1)V_b-\gamma\xi x)\frac{\partial L}{\partial y} & , \quad \text{if } \frac{\partial^2 L}{\partial y^2}=0 \text{ and } \frac{\partial L}{\partial y}>0 \\ y^2-\beta L-x\frac{\partial L}{\partial y}-\frac{(\gamma\xi-1)^2}{2\xi^2}\left(\frac{\partial L}{\partial y}\right)^2\left(\frac{\partial^2 L}{\partial y^2}\right)^{-1} & , \quad \begin{array}{l}\text{if } \frac{\partial^2 L}{\partial y^2}>0 \text{ and} \\ \left(\frac{\partial^2 L}{\partial y^2}\right)^{-1}\frac{\partial L}{\partial y}\frac{(1-\gamma\xi)}{\xi^2}+x<V_b\end{array} \\ \begin{array}{l} y^2-\beta L+((\gamma\xi-1)V_b-\gamma\xi x)\frac{\partial L}{\partial y}+ \\ +\frac{1}{2}\xi^2(V_b-x)^2\frac{\partial^2 L}{\partial y^2}\end{array} & , \quad \begin{array}{l}\text{if } \frac{\partial^2 L}{\partial y^2}>0 \text{ and} \\ \left(\frac{\partial^2 L}{\partial y^2}\right)^{-1}\frac{\partial L}{\partial y}\frac{(1-\gamma\xi)}{\xi^2}+x\geq V_b\end{array} \end{cases}$$

Conjecture that

$$L=\begin{cases} \frac{y^2}{\beta} & , \quad \text{if } \frac{\partial^2 L}{\partial y^2}=0 \text{ and } \frac{\partial L}{\partial y}=0 \\ \frac{y^2}{\beta} & , \quad \begin{array}{l}\text{if } \frac{\partial^2 L}{\partial y^2}=0 \text{ and } \frac{\partial L}{\partial y}>0 \\ \text{and } (\gamma\xi-1)V_b-\gamma\xi x=0\end{array} \\ \frac{2}{\beta^3}A_1^2+\frac{2}{\beta^2}A_1y+\frac{1}{\beta}y^2+K_1\exp\left(\frac{\beta}{A_1}y\right) & , \quad \begin{array}{l}\text{if } \frac{\partial^2 L}{\partial y^2}=0 \text{ and } \frac{\partial L}{\partial y}>0 \\ \text{and } (\gamma\xi-1)V_b-\gamma\xi x\neq 0\end{array} \\ \frac{2x^2(A_3-\beta)}{(2A_3-\beta)^3\beta}-\frac{2x}{(2A_3-\beta)^2}y+\left(-\frac{1}{2A_3-\beta}\right)y^2 & , \quad \begin{array}{l}\text{if } \frac{\partial^2 L}{\partial y^2}>0 \text{ and} \\ \left(\frac{\partial^2 L}{\partial y^2}\right)^{-1}\frac{\partial L}{\partial y}\frac{(1-\gamma\xi)}{\xi^2}+x<V_b\end{array} \\ \begin{array}{l}\frac{2A_2}{\beta^2}+\frac{y^2}{\beta}+K_2\exp\left(\sqrt{\frac{\beta}{A_2}}y\right)+ \\ +K_3\exp\left(-\sqrt{\frac{\beta}{A_2}}y\right)\end{array} & , \quad \begin{array}{l}\text{if } \frac{\partial^2 L}{\partial y^2}>0 \text{ and} \\ \left(\frac{\partial^2 L}{\partial y^2}\right)^{-1}\frac{\partial L}{\partial y}\frac{(1-\gamma\xi)}{\xi^2}+x\geq V_b \\ \text{and } V_b-x\neq 0 \text{ and} \\ \quad (\gamma\xi-1)V_b-\gamma\xi x=0\end{array} \\ \frac{2A_1^2}{\beta^3}+\frac{2A_1}{\beta^2}y+\frac{1}{\beta}y^2+K_4\exp\left(\frac{\beta}{A_1}y\right) & , \quad \begin{array}{l}\text{if } \frac{\partial^2 L}{\partial y^2}>0 \text{ and} \\ \left(\frac{\partial^2 L}{\partial y^2}\right)^{-1}\frac{\partial L}{\partial y}\frac{(1-\gamma\xi)}{\xi^2}+x\geq V_b \\ \text{and } V_b-x=0 \text{ and} \\ (\gamma\xi-1)V_b-\gamma\xi x\neq 0\end{array} \\ \begin{array}{l}\frac{2A_1^2}{\beta^3}+\frac{2A_2}{\beta^2}+\frac{2A_1}{\beta^2}y+\frac{1}{\beta}y^2+ \\ +K_5\exp(\eta_1 y)+K_6\exp(\eta_2 y)\end{array} & , \quad \begin{array}{l}\text{if } \frac{\partial^2 L}{\partial y^2}>0 \text{ and} \\ \left(\frac{\partial^2 L}{\partial y^2}\right)^{-1}\frac{\partial L}{\partial y}\frac{(1-\gamma\xi)}{\xi^2}+x\geq V_b \text{ and} \\ V_b-x\neq 0 \text{ and} \\ (\gamma\xi-1)V_b-\gamma\xi x\neq 0\end{array} \end{cases}$$

where

$$K_1, K_2, K_3, K_4, K_5, K_6$$

are constants of integration and

$$\begin{aligned}
A_1 &= (\gamma\xi - 1) V_b - \gamma\xi x \\
A_2 &= \frac{1}{2}\xi^2 (V_b - x)^2 > 0 \\
A_3 &= -\frac{(\gamma\xi - 1)^2}{2\xi^2} < 0 \\
\eta_1 &= -\frac{1}{2}\left(\frac{A_1}{A_2} + \frac{\sqrt{A_1^2 + 4A_2\beta}}{A_2}\right) \\
&= -\frac{(\gamma\xi - 1) V_b - \gamma\xi x}{\xi^2 (V_b - x)^2} \\
&\quad -\frac{\sqrt{((\gamma\xi - 1) V_b - \gamma\xi x)^2 + 4\frac{1}{2}\xi^2 (V_b - x)^2 \beta}}{\xi^2 (V_b - x)^2} \\
\eta_2 &= -\frac{1}{2}\left(\frac{A_1}{A_2} - \frac{\sqrt{A_1^2 + 4A_2\beta}}{A_2}\right) \\
&= -\frac{(\gamma\xi - 1) V_b - \gamma\xi x}{\xi^2 (V_b - x)^2} \\
&\quad +\frac{\sqrt{\left((\gamma\xi - 1)^2 + 2\beta\xi^2\right) V_b^2 - 2x\xi \left(2\beta\xi + \gamma (\gamma\xi - 1)\right) V_b + x^2\xi^2 (2\beta + \gamma^2)}}{\xi^2 (V_b - x)^2}
\end{aligned}$$

Note that the case

$$\begin{aligned}
V_b - x &= 0 \\
(\gamma\xi - 1) V_b - \gamma\xi x &= 0
\end{aligned}$$

cannot happen since $x \neq 0$. In addition, η_1 and η_2 are always real since

$$A_1^2 + 4A_2\beta = \left((\gamma\xi - 1)^2 + 2\beta\xi^2\right) V_b^2 - 2x\xi \left(2\beta\xi + \gamma (\gamma\xi - 1)\right) V_b + x^2\xi^2 \left(2\beta + \gamma^2\right)$$

and the roots of

$$0 = \left((\gamma\xi - 1)^2 + 2\beta\xi^2\right) V_b^2 - 2x\xi \left(2\beta\xi + \gamma (\gamma\xi - 1)\right) V_b + x^2\xi^2 \left(2\beta + \gamma^2\right)$$

are

$$\left\{ x\xi \frac{2\beta\xi + (\gamma\xi - 1)\gamma + \sqrt{-2\beta}}{(\gamma\xi - 1)^2 + 2\beta\xi^2}, x\xi \frac{2\beta\xi + (\gamma\xi - 1)\gamma - \sqrt{-2\beta}}{(\gamma\xi - 1)^2 + 2\beta\xi^2} \right\}$$

which are always complex, imply $A_1^2 + 4A_2\beta > 0$.

Differentiating, we get

$$L' = \begin{cases} \frac{2}{\beta}y & , \text{ if } \frac{\partial^2 L}{\partial y^2} = 0 \text{ and } \frac{\partial L}{\partial y} = 0 \\ \frac{2}{\beta}y & , \begin{array}{l} \text{if } \frac{\partial^2 L}{\partial y^2} = 0 \text{ and } \frac{\partial L}{\partial y} > 0 \\ \text{and } \left(\gamma\xi - 1\right) V_b - \gamma\xi x = 0 \end{array} \\ \frac{2}{\beta^2}A_1 + \frac{2}{\beta}y + \frac{\beta}{A_1}K_1 \exp\left(\frac{\beta}{A_1}y\right) & , \begin{array}{l} \text{if } \frac{\partial^2 L}{\partial y^2} = 0 \text{ and } \frac{\partial L}{\partial y} > 0 \\ \text{and } \left(\gamma\xi - 1\right) V_b - \gamma\xi x \neq 0 \end{array} \\ -\frac{2x}{(2A_3-\beta)^2} + \left(-\frac{2}{2A_3-\beta}\right) y & , \begin{array}{l} \text{if } \frac{\partial^2 L}{\partial y^2} > 0 \text{ and} \\ \left(\frac{\partial^2 L}{\partial y^2}\right)^{-1} \frac{\partial L}{\partial y} \frac{(1-\gamma\xi)}{\xi^2} + x < V_b \end{array} \\ \begin{array}{l} \frac{2}{\beta}y + \sqrt{\frac{\beta}{A_2}}K_2 \exp\left(\sqrt{\frac{\beta}{A_2}}y\right) - \\ -\sqrt{\frac{\beta}{A_2}}K_3 \exp\left(-\sqrt{\frac{\beta}{A_2}}y\right) \end{array} & , \begin{array}{l} \text{if } \frac{\partial^2 L}{\partial y^2} > 0 \text{ and} \\ \left(\frac{\partial^2 L}{\partial y^2}\right)^{-1} \frac{\partial L}{\partial y} \frac{(1-\gamma\xi)}{\xi^2} + x \geq V_b \text{ and} \\ V_b - x \neq 0 \text{ and} \\ \left(\gamma\xi - 1\right) V_b - \gamma\xi x = 0 \end{array} \\ \frac{2A_1}{\beta^2} + \frac{2}{\beta}y + \frac{\beta}{A_1}K_4 \exp\left(\frac{\beta}{A_1}y\right) & , \begin{array}{l} \text{if } \frac{\partial^2 L}{\partial y^2} > 0 \text{ and} \\ \left(\frac{\partial^2 L}{\partial y^2}\right)^{-1} \frac{\partial L}{\partial y} \frac{(1-\gamma\xi)}{\xi^2} + x \geq V_b \text{ and} \\ V_b - x = 0 \text{ and} \\ \left(\gamma\xi - 1\right) V_b - \gamma\xi x \neq 0 \end{array} \\ \frac{2A_1}{\beta^2} + \frac{2}{\beta}y + \eta_1 K_5 \exp\left(\eta_1 y\right) + \eta_2 K_6 \exp\left(\eta_2 y\right) & , \begin{array}{l} \text{if } \frac{\partial^2 L}{\partial y^2} > 0 \text{ and} \\ \left(\frac{\partial^2 L}{\partial y^2}\right)^{-1} \frac{\partial L}{\partial y} \frac{(1-\gamma\xi)}{\xi^2} + x \geq V_b \text{ and} \\ V_b - x \neq 0 \text{ and } \left(\gamma\xi - 1\right) V_b - \gamma\xi x \neq 0 \end{array} \end{cases}$$

and

$$L'' = \begin{cases} \frac{2}{\beta} & , \text{ if } \frac{\partial^2 L}{\partial y^2} = 0 \text{ and } \frac{\partial L}{\partial y} = 0 \\ \frac{2}{\beta} & , \begin{array}{l} \text{if } \frac{\partial^2 L}{\partial y^2} = 0 \text{ and} \\ \frac{\partial L}{\partial y} > 0 \text{ and } (\gamma\xi - 1) V_b - \gamma\xi x = 0 \end{array} \\ \frac{2}{\beta} + \left(\frac{\beta}{A_1}\right)^2 K_1 \exp\left(\frac{\beta}{A_1} y\right) & , \begin{array}{l} \text{if } \frac{\partial^2 L}{\partial y^2} = 0 \text{ and} \\ \frac{\partial L}{\partial y} > 0 \text{ and } (\gamma\xi - 1) V_b - \gamma\xi x \neq 0 \end{array} \\ \left(-\frac{2}{2A_3 - \beta}\right) & , \begin{array}{l} \text{if } \frac{\partial^2 L}{\partial y^2} > 0 \text{ and} \\ \left(\frac{\partial^2 L}{\partial y^2}\right)^{-1} \frac{\partial L}{\partial y} \frac{(1-\gamma\xi)}{\xi^2} + x < V_b \end{array} \\ \begin{array}{l} \frac{2}{\beta} + \frac{\beta}{A_2} K_2 \exp\left(\sqrt{\frac{\beta}{A_2}} y\right) + \\ + \frac{\beta}{A_2} K_3 \exp\left(-\sqrt{\frac{\beta}{A_2}} y\right) \end{array} & , \begin{array}{l} \text{if } \frac{\partial^2 L}{\partial y^2} > 0 \text{ and} \\ \left(\frac{\partial^2 L}{\partial y^2}\right)^{-1} \frac{\partial L}{\partial y} \frac{(1-\gamma\xi)}{\xi^2} + x \geq V_b \\ \text{and } V_b - x \neq 0 \text{ and} \\ (\gamma\xi - 1) V_b - \gamma\xi x = 0 \end{array} \\ \frac{2}{\beta} + \left(\frac{\beta}{A_1}\right)^2 K_4 \exp\left(\frac{\beta}{A_1} y\right) & , \begin{array}{l} \text{if } \frac{\partial^2 L}{\partial y^2} > 0 \text{ and} \\ \left(\frac{\partial^2 L}{\partial y^2}\right)^{-1} \frac{\partial L}{\partial y} \frac{(1-\gamma\xi)}{\xi^2} + x \geq V_b \\ \text{and } V_b - x = 0 \text{ and} \\ (\gamma\xi - 1) V_b - \gamma\xi x \neq 0 \end{array} \\ \begin{array}{l} \frac{2}{\beta} + \eta_1^2 K_5 \exp(\eta_1 y) + \\ + \eta_2^2 K_6 \exp(\eta_2 y) \end{array} & , \begin{array}{l} \text{if } \frac{\partial^2 L}{\partial y^2} > 0 \text{ and} \\ \left(\frac{\partial^2 L}{\partial y^2}\right)^{-1} \frac{\partial L}{\partial y} \frac{(1-\gamma\xi)}{\xi^2} + x \geq V_b \\ \text{and } V_b - x \neq 0 \text{ and} \\ (\gamma\xi - 1) V_b - \gamma\xi x \neq 0 \end{array} \end{cases}$$

The first two cases, given by

$$\begin{aligned} \frac{\partial^2 L}{\partial y^2} &= 0 \text{ and } \frac{\partial L}{\partial y} = 0 \\ \frac{\partial^2 L}{\partial y^2} &= 0 \text{ and } \frac{\partial L}{\partial y} > 0 \text{ and } (\gamma\xi - 1) V_b - \gamma\xi x = 0 \end{aligned}$$

can never occur, since $L'' = 2/\beta > 0$. Using the computations above, we can solve for the conditions that define the rest of the cases. For the case listed third

$$\begin{aligned}
\frac{\partial^2 L}{\partial y^2} &= 0 \text{ and } \frac{\partial L}{\partial y} > 0 \text{ and } \left(\gamma\xi - 1\right) V_b - \gamma\xi x \neq 0 \\
&\iff \\
K_1 &< 0 \\
y &= \frac{A_1}{\beta} \log\left(-\frac{2}{K_1}\frac{A_1^2}{\beta^3}\right) \\
0 &< \frac{A_1}{\beta} \log\left(-\frac{2}{K_1}\frac{A_1^2}{\beta^3}\right) \\
0 &\neq \left(\gamma\xi - 1\right) V_b - \gamma\xi x
\end{aligned} \tag{61}$$

For the case listed fourth,

$$\begin{aligned}
\frac{\partial^2 L}{\partial y^2} &> 0 \text{ and } \left(\frac{\partial^2 L}{\partial y^2}\right)^{-1} \frac{\partial L}{\partial y}\frac{(1-\gamma\xi)}{\xi^2} + x < V_b \\
&\iff \\
y &< \xi^2 \left(\frac{x\gamma\xi}{\left(\gamma\xi - 1\right)^2 + \beta\xi^2} - \frac{V_b}{\left(\gamma\xi - 1\right)} + \frac{\beta\xi^2 x}{\left(\gamma\xi - 1\right)\left(\left(\gamma\xi - 1\right)^2 + \beta\xi^2\right)}\right)
\end{aligned} \tag{62}$$

For the case listed fifth

$$\begin{aligned}
\frac{\partial^2 L}{\partial y^2} &> 0 \text{ and } \left(\frac{\partial^2 L}{\partial y^2}\right)^{-1} \frac{\partial L}{\partial y}\frac{(1-\gamma\xi)}{\xi^2} + x \geq V_b \\
\text{and } V_b - x &\neq 0 \text{ and } \left(\gamma\xi - 1\right) V_b - \gamma\xi x = 0 \\
&\iff \\
-\left(\frac{\xi x}{\left(\gamma\xi - 1\right)\beta}\right)^2 &< K_2 \exp\left(\frac{\sqrt{2\beta}\left(\gamma\xi - 1\right)}{\xi x} y\right) \\
&\quad + K_3 \exp\left(-\frac{\sqrt{2\beta}\left(\gamma\xi - 1\right)}{\xi x} y\right) \qquad (63) \\
-\frac{2x}{\beta} y - 2\beta\left(\frac{\xi x}{\left(\gamma\xi - 1\right)\beta}\right)^2 &\geq \left(x + 2\beta\right) K_2 \exp\left(\frac{\sqrt{2\beta}\left(\gamma\xi - 1\right)}{\xi x} y\right) + \\
&\quad \left(\left(\frac{1-\gamma\xi}{\xi}\right)\sqrt{2\beta} + 2\beta\right) K_3 \exp\left(-\frac{\sqrt{2\beta}\left(\gamma\xi - 1\right)}{\xi x} y\right) \\
0 &\neq V_b - x \\
V_b &= \frac{\gamma\xi x}{\gamma\xi - 1} \qquad (64)
\end{aligned}$$

For the case listed sixth

$$
\begin{aligned}
\frac{\partial^2 L}{\partial y^2} &> 0 \text{ and } \left(\frac{\partial^2 L}{\partial y^2}\right)^{-1} \frac{\partial L}{\partial y} \frac{(1-\gamma\xi)}{\xi^2} + x \geq V_b \\
\text{and } V_b - x &= 0 \text{ and } (\gamma\xi - 1) V_b - \gamma\xi x \neq 0 \\
&\Longleftrightarrow \\
0 &< \frac{2}{\beta} + \left(\frac{\beta}{A_1}\right)^2 K_4 \exp\left(\frac{\beta}{A_1} y\right) \\
0 &\leq \frac{2A_1}{\beta^2} + \frac{2}{\beta} y + \frac{\beta K_4}{A_1} \exp\left(\frac{\beta}{A_1} y\right) \\
0 &= V_b - x \\
0 &\neq (\gamma\xi - 1) V_b - \gamma\xi x
\end{aligned} \tag{65}
$$

For the case listed seventh

$$
\begin{aligned}
\frac{\partial^2 L}{\partial y^2} &> 0 \text{ and } \left(\frac{\partial^2 L}{\partial y^2}\right)^{-1} \frac{\partial L}{\partial y} \frac{(1-\gamma\xi)}{\xi^2} + x \geq V_b \\
\text{and } V_b - x &\neq 0 \text{ and } (\gamma\xi - 1) V_b - \gamma\xi x \neq 0 \\
&\Longleftrightarrow \\
0 &< \frac{2}{\beta} + \eta_1^2 K_5 \exp(\eta_1 y) + \eta_2^2 K_6 \exp(\eta_2 y) \\
0 &\leq \frac{2}{\beta}\left(\frac{A_1}{\beta} - \frac{\xi^2 (V_b - x)}{(1-\gamma\xi)}\right) + \frac{2}{\beta} y \\
&\quad + \left(1 - \frac{\xi^2 (V_b - x)}{(1-\gamma\xi)} \eta_1\right) \eta_1 K_5 \exp(\eta_1 y) \\
&\quad + \left(1 - \frac{\xi^2 (V_b - x)}{(1-\gamma\xi)} \eta_2\right) \eta_2 K_6 \exp(\eta_2 y) \\
0 &\neq V_b - x \\
0 &\neq (\gamma\xi - 1) V_b - \gamma\xi x
\end{aligned} \tag{66}
$$

We invoke equation (5) to discard the cases in equations (64) and (65). We look for a solution using the cases in equations (62) and (66), and show at the end that the case in equation (61) does not apply in this case.

Using the cases in equations (62) and (66), we get

$$
L = \begin{cases}
\frac{2x^2(A_3-\beta)}{(2A_3-\beta)^3\beta} - \frac{2x}{(2A_3-\beta)^2} y + \left(-\frac{1}{2A_3-\beta}\right) y^2 & , \quad y < y_b \\
\begin{array}{l} \frac{2A_1^2}{\beta^3} + \frac{2A_2}{\beta^2} + \frac{2A_1}{\beta^2} y + \frac{1}{\beta} y^2 \\ + K_5 \exp(\eta_1 y) + K_6 \exp(\eta_2 y) \end{array} & , \quad \begin{array}{l} \frac{2}{\beta}\left(\frac{A_1}{\beta} - \frac{\xi^2 (V_b - x)}{(1-\gamma\xi)}\right) + \frac{2}{\beta} y \\ + \left(\frac{1}{\eta_1} - \frac{\xi^2 (V_b - x)}{(1-\gamma\xi)}\right) \eta_1^2 K_5 \exp(\eta_1 y) \\ + \left(\frac{1}{\eta_2} - \frac{\xi^2 (V_b - x)}{(1-\gamma\xi)}\right) \eta_2^2 K_6 \exp(\eta_2 y) \geq 0 \end{array}
\end{cases}
$$

where

$$y_b = \xi^2 \left(\frac{x\gamma\xi}{(\gamma\xi-1)^2+\beta\xi^2} - \frac{V_b}{(\gamma\xi-1)} + \frac{\beta\xi^2 x}{(\gamma\xi-1)\left((\gamma\xi-1)^2+\beta\xi^2\right)} \right)$$

We now pick K_5 and K_6 to make the boundaries of the two regions coincide and to make L continuous there. The boundaries coincide iff

$$\begin{aligned} 0 &= \frac{2}{\beta}\left(\frac{A_1}{\beta} - \frac{\xi^2(V_b-x)}{(1-\gamma\xi)}\right) + \frac{2}{\beta}y_b + \left(\frac{1}{\eta_1} - \frac{\xi^2(V_b-x)}{(1-\gamma\xi)}\right)\eta_1^2 K_5 \exp(\eta_1 y_b) \\ &\quad + \left(\frac{1}{\eta_2} - \frac{\xi^2(V_b-x)}{(1-\gamma\xi)}\right)\eta_2^2 K_6 \exp(\eta_2 y_b) \end{aligned}$$

Continuity at the boundary requires

$$\begin{aligned} \frac{2x^2(A_3-\beta)}{(2A_3-\beta)^3\beta} - \frac{2x}{(2A_3-\beta)^2}y_b + \left(-\frac{1}{2A_3-\beta}\right)y_b^2 &= \frac{2A_1^2}{\beta^3} + \frac{2A_2}{\beta^2} + \frac{2A_1}{\beta^2}y_b \\ &\quad + \frac{1}{\beta}y_b^2 + K_5\exp(\eta_1 y_b) + K_6\exp(\eta_2 y_b) \end{aligned}$$

We can write these two equations in matrix form

$$\begin{bmatrix} \left(\frac{1}{\eta_1} - \frac{\xi^2(V_b-x)}{(1-\gamma\xi)}\right)\eta_1^2\exp(\eta_1 y_b) & \left(\frac{1}{\eta_2} - \frac{\xi^2(V_b-x)}{(1-\gamma\xi)}\right)\eta_2^2\exp(\eta_2 y_b) \\ \exp(\eta_1 y_b) & \exp(\eta_2 y_b) \end{bmatrix} \begin{bmatrix} K_5 \\ K_6 \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

where

$$\begin{aligned} M_1 &= \frac{2}{\beta}\left(\frac{\xi^2(V_b-x)}{(1-\gamma\xi)} - \frac{A_1}{\beta} - y_b\right) \\ M_2 &= \left(\frac{2x^2(A_3-\beta)}{(2A_3-\beta)^3\beta} - \frac{2A_1^2}{\beta^3} - \frac{2A_2}{\beta^2}\right) - \left(\frac{2x}{(2A_3-\beta)^2} + \frac{2A_1}{\beta^2}\right)y_b \\ &\quad - \left(\frac{1}{2A_3-\beta} + \frac{1}{\beta}\right)y_b^2 \end{aligned}$$

The first line in equation (5) guarantees a unique solution to this sytem, given by

$$\begin{aligned} \begin{matrix} K_5 \\ K_6 \end{matrix} &= \begin{bmatrix} \left(\frac{1}{\eta_1} - \frac{\xi^2(V_b-x)}{(1-\gamma\xi)}\right)\eta_1^2\exp(\eta_1 y_b) & \left(\frac{1}{\eta_2} - \frac{\xi^2(V_b-x)}{(1-\gamma\xi)}\right)\eta_2^2\exp(\eta_2 y_b) \\ \exp(\eta_1 y_b) & \exp(\eta_2 y_b) \end{bmatrix}^{-1} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \\ &= \frac{1}{\Lambda}\begin{bmatrix} \exp(-\eta_1 y_b) & -\left(\frac{1}{\eta_2} - \frac{\xi^2(V_b-x)}{(1-\gamma\xi)}\right)\eta_2^2\exp(-\eta_1 y_b) \\ -\exp(-\eta_2 y_b) & \left(\frac{1}{\eta_1} - \frac{\xi^2(V_b-x)}{(1-\gamma\xi)}\right)\eta_1^2\exp(-\eta_2 y_b) \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \end{aligned}$$

where

$$\Lambda = \left(\frac{1}{\eta_1} - \frac{\xi^2 (V_b - x)}{(1-\gamma\xi)}\right)\eta_1^2 - \left(\frac{1}{\eta_2} - \frac{\xi^2 (V_b - x)}{(1-\gamma\xi)}\right)\eta_2^2$$

It can be checked that the first and second derivatives in the viscosity sense statisfy the viscosity solution conditions at y_b.

C Appendix: Proofs

Lemma 3 *When $i_t = \psi_0 + \psi_y y_t + \psi_x x_t$, the loss function $L(y_0, 0)$ is bounded for all y_0 if and only if*

$$\underline{\psi}_y < \psi_y < \bar{\psi}_y$$

where

$$\begin{aligned} \bar{\psi}_y &= \min\left\{\frac{\beta\sigma(\gamma\xi - 1 + \rho)}{(\gamma\xi - 1)}, \frac{\sigma(\rho + \gamma\xi - 1)}{\xi^2}\left(1 - \gamma\xi - \sqrt{(1-\gamma\xi)^2 + \beta\xi^2}\right)\right\} \\ \underline{\psi}_y &= \frac{\sigma(\rho + \gamma\xi - 1)}{\xi^2}\left(1 - \gamma\xi + \sqrt{(1-\gamma\xi)^2 + \beta\xi^2}\right) \end{aligned}$$

Proof. Consider the following system of SDE for y_t and x_t

$$\begin{aligned} dy_t &= (A_0 + A_y y_t + A_x x_t)\,dt + (B_0 + B_y y_t + B_x x_t)\,dZ_{y,t} \\ dx_t &= -\kappa_x (x_t - \mu_x)\,dt + \sigma_x dZ_{x,t} \\ dZ_{y,t} dZ_{x,t} &= 0 \\ x_0 &= 0 \\ y_0 &\in \mathbb{R} \end{aligned}$$

where $A_0, A_y, A_x, B_0, B_y, B_x, \kappa_x, \mu_x > 0, \sigma_x > 0$ are constants and $dZ_{y,t}$, $dZ_{x,t}$ are standard brownian motions. The process for x_t is an Ornstein-Uhlenbeck process with standard solution

$$\begin{aligned} x_t &= x_0 e^{-\kappa_x t} + \mu_x(1 - e^{-\kappa_x t}) + \sigma_x \int_0^t e^{-\kappa_x (t-s)} dZ_{x,s} \\ &= \mu_x(1 - e^{-\kappa_x t}) + \sigma_x \int_0^t e^{-\kappa_x (t-s)} dZ_{x,s} \end{aligned}$$

Let

$$\phi_t = \exp\left(\left(A_y - \frac{B_y^2}{2}\right)t + B_y Z_{y,t}\right)$$

be the solution to the homogenous SDE

$$\begin{aligned}\frac{d\phi_t}{\phi_t} &= A_y dt + B_y dZ_{y,t} \\ \phi_0 &= 1\end{aligned}$$

Then

$$\begin{aligned} y_t &= \phi_t y_0 + \phi_t \int_0^t \phi_s^{-1} \left[(A_0 + A_x x_s) - B_y (B_0 + B_x x_s) \right] ds \qquad (67) \\ &\quad + \phi_t \int_0^t \phi_s^{-1} (B_0 + B_x x_s) \, dZ_{y,s} \end{aligned}$$

Using that

$$\begin{aligned}
\mathbb{E}[\phi_t] &= \exp(A_y t) && (68) \\
\mathbb{E}\left[\phi_t^2\right] &= \exp\left(\left(B_y^2 + 2A_y\right) t\right) && (69) \\
\mathbb{E}\left[\phi_t \phi_s^{-1}\right] &= \exp(A_y (t - s)) && (70) \\
\mathbb{E}\left[\phi_t^2 \phi_s^{-1}\right] &= \exp\left(\left(B_y^2 + 2A_y\right) t\right) \exp\left(\left(-B_y^2 - A_y\right) s\right) && (71) \\
\mathbb{E}\left[\phi_t^2 \phi_s^{-2}\right] &= \exp\left(\left(B_y^2 + 2A_y\right) t\right) \exp\left(-\left(B_y^2 + 2A_y\right) s\right) && (72) \\
\mathbb{E}\left[(B_0 + B_x x_s)^2\right] &= \frac{B_x^2 \sigma_x^2}{2\rho_x} + B_x^2 \mu_x^2 + 2B_0 B_x \mu_x + B_0^2 - \frac{B_x^2 \sigma_x^2}{2\rho_x} \exp(-2\rho_x s) && (73) \\
\mathbb{E}[x_s x_u] &= \frac{\sigma_x^2}{2\rho_x} \exp(-\rho_x (s + u)) (\exp(2\rho_x \min(s, u)) - 1) + \mu_x^2 && (74)
\end{aligned}$$

we can compute

$$\mathbb{E}[y_t] = \begin{cases} \left(y_0 + \frac{(A_x - B_x B_y)\mu_x + (A_0 - B_0 B_y)}{A_y}\right) \exp(A_y t) \\ \quad - \frac{(A_x - B_x B_y)\mu_x + (A_0 - B_0 B_y)}{A_y} & , \quad \text{if } A_y \neq 0 \\ E[y_t] = y_0 \\ + ((A_x - B_x B_y) \mu_x + (A_0 - B_0 B_y)) t & , \quad \text{if } A_y = 0 \end{cases} \qquad (75)$$

In addition, using equations (68)-(74),

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t \phi_t\phi_s^{-1}\left(\left(A_x - B_xB_y\right)x_s + \left(A_0 - B_0B_y\right)\right)ds\right)^2\right] \\
= \; &\mathbb{E}\left[\begin{array}{c} \left(\int_0^t \phi_t\phi_s^{-1}\left(\left(A_x - B_xB_y\right)x_s + \left(A_0 - B_0B_y\right)\right)ds\right)\times \\ \times\left(\int_0^t \phi_t\phi_u^{-1}\left(\left(A_x - B_xB_y\right)x_u + \left(A_0 - B_0B_y\right)\right)du\right) \end{array}\right] \\
= \; &\int_0^t\int_0^t \mathbb{E}\left[\begin{array}{c} \phi_t\phi_s^{-1}\left(\left(A_x - B_xB_y\right)x_s + \left(A_0 - B_0B_y\right)\right)\phi_t\times \\ \times\phi_u^{-1}\left(\left(A_x - B_xB_y\right)x_u + \left(A_0 - B_0B_y\right)\right) \end{array}\right]duds
\end{aligned}$$

and

$$\begin{aligned}
y_t^2 &= \left(\phi_t y_0 + \int_0^t \phi_t\phi_s^{-1}\left(\left(A_x - B_xB_y\right)x_s + \left(A_0 - B_0B_y\right)\right)ds + \int_0^t \phi_t\phi_s^{-1}\left(B_0 + B_xx_s\right)dZ_{y,s}\right)^2 \\
&= \left(\phi_t y_0\right)^2 + \left(\int_0^t \phi_t\phi_s^{-1}\left(\left(A_x - B_xB_y\right)x_s + \left(A_0 - B_0B_y\right)\right)ds\right)^2 \\
&\quad +2\left(\phi_t y_0\right)\left(\int_0^t \phi_t\phi_s^{-1}\left(\left(A_x - B_xB_y\right)x_s + \left(A_0 - B_0B_y\right)\right)ds\right) \\
&\quad +2\left(\phi_t y_0\right)\left(\int_0^t \phi_t\phi_s^{-1}\left(B_0 + B_xx_s\right)dZ_{y,s}\right) \\
&\quad +2\left(\int_0^t \phi_t\phi_s^{-1}\left(\left(A_x - B_xB_y\right)x_s + \left(A_0 - B_0B_y\right)\right)ds\right)\left(\int_0^t \phi_t\phi_s^{-1}\left(B_0 + B_xx_s\right)dZ_{y,s}\right) \\
&\quad +\left(\int_0^t \phi_t\phi_s^{-1}\left(B_0 + B_xx_s\right)dZ_{y,s}\right)^2
\end{aligned}$$

we get

$$
\begin{aligned}
\mathbb{E}\left[y_t^2\right] \;=\;& -\frac{\left(A_x-B_xB_y\right)^2\sigma_x^2}{2A_y\rho_x\left(\rho_x-A_y\right)}-\frac{1}{\left(B_y^2+2A_y\right)}\left(\frac{B_x^2\sigma_x^2}{2\rho_x}+B_x^2\mu_x^2+2B_0B_x\mu_x+B_0^2\right)\\
&+\frac{\left(A_x-B_xB_y\right)^2\mu_x^2+\left(A_0-B_0B_y\right)\left(A_x-B_xB_y\right)\mu_x}{A_y^2}\\
&+\frac{\left(A_0-B_0B_y\right)\left(A_x-B_xB_y\right)\mu_x+\left(A_0-B_0B_y\right)^2}{A_y^2}\\
&+\frac{2\sigma_x^2\left(A_x-B_xB_y\right)^2}{\left(\rho_x+A_y\right)^2\left(\rho_x-A_y\right)}\exp\left(-\left(\rho_x-A_y\right)t\right) &\quad(76)\\
&+\left(\begin{array}{c}\frac{1}{2}\frac{\sigma_x^2}{A_y}\frac{\left(A_x-B_xB_y\right)^2}{\left(\rho_x+A_y\right)^2}\\ +\frac{\left(A_x-B_xB_y\right)^2\mu_x^2+\left(A_0-B_0B_y\right)\left(A_x-B_xB_y\right)\mu_x}{A_y^2}\\ +\frac{\left(A_0-B_0B_y\right)\left(A_x-B_xB_y\right)\mu_x+\left(A_0-B_0B_y\right)^2}{A_y^2}\end{array}\right)\exp\left(2A_yt\right) &\quad(77)\\
&+\left(-\frac{\left(A_x-B_xB_y\right)^2\sigma_x^2}{2\rho_x\left(\rho_x+A_y\right)^2}+\frac{1}{\left(2\rho_x+2A_y+B_y^2\right)}\frac{B_x^2\sigma_x^2}{2\rho_x}\right)\exp\left(-2\rho_xt\right)\\
&+\left(\begin{array}{c}\left(-\frac{2}{B_y^2+A_y}\left(A_0-B_0B_y+\mu_x\left(A_x-B_xB_y\right)\right)\right)y_0\\ -2\left(A_0+\mu_xA_x-B_0B_y-\mu_xB_xB_y\right)\times\\ \times\frac{\left(A_0B_y^2-B_0B_y^3+A_0A_y+\mu_xA_xB_y^2-\mu_xB_xB_y^3+\mu_xA_xA_y-B_0A_yB_y-\mu_xA_yB_xB_y\right)}{A_y^2\left(B_y^2+A_y\right)}\end{array}\right)\exp\left(A_yt\right)\\
&+\left(\begin{array}{c}y_0^2+\frac{2}{B_y^2+A_y}\left(A_0-B_0B_y+\mu_x\left(A_x-B_xB_y\right)\right)y_0\\ +\frac{1}{B_y^2+2A_y}\left(B_0^2+\mu_x^2B_x^2+\frac{1}{2}\frac{\sigma_x^2}{\rho_x}B_x^2+2\mu_xB_0B_x\right)\\ -\frac{1}{2}\frac{\sigma_x^2}{\rho_x}\frac{B_x^2}{B_y^2+2\rho_x+2A_y}\end{array}\right)\exp\left(\left(B_y^2+2A_y\right)t\right) &\quad(78)
\end{aligned}
$$

It follows that

$$
\lim_{t\to\infty}\mathbb{E}\left[e^{-t\beta}y_t^2\right]=0
$$

for all y_0 if and only if

$$
\begin{aligned}
-\left(\rho_x-A_y\right)-\beta &< 0 &\quad(79)\\
A_y-\beta &< 0 &\quad(80)\\
-2\rho_x-\beta &< 0 &\quad(81)\\
\left(B_y^2+2A_y\right)-\beta &< 0 &\quad(82)
\end{aligned}
$$

Using equation (26), we identify

$$\begin{aligned}
A_0 &= \frac{(\gamma\xi - 1)(\psi_0 - r)}{\sigma(\gamma\xi - 1 + \rho)} && (83)\\
A_y &= \frac{(\gamma\xi - 1)\psi_y}{\sigma(\gamma\xi - 1 + \rho)} && (84)\\
A_x &= \left(\frac{(\gamma\xi - 1)\psi_x - \sigma\gamma\xi\rho}{\sigma(\gamma\xi - 1 + \rho)}\right) && (85)\\
B_0 &= \frac{\xi(\psi_0 - r)}{\sigma(\gamma\xi - 1 + \rho)} && (86)\\
B_y &= \frac{\xi\psi_y}{\sigma(\gamma\xi - 1 + \rho)} && (87)\\
B_x &= \xi\left(\frac{\psi_x + (1-\rho)\sigma}{\sigma(\gamma\xi - 1 + \rho)}\right) && (88)
\end{aligned}$$

Using that $(\rho + \gamma\xi - 1) < 0$, inequalities (79)-(82) can then be solved for ψ_y to get

$$\begin{aligned}
\psi_y &> \frac{\sigma(\rho + \gamma\xi - 1)}{\xi^2}\left(1 - \gamma\xi + \sqrt{(1-\gamma\xi)^2 + \beta\xi^2}\right) && (89)\\
\psi_y &< \min\left\{\frac{\beta\sigma(\gamma\xi - 1 + \rho)}{(\gamma\xi - 1)}, \frac{\sigma(\rho + \gamma\xi - 1)}{\xi^2}\left(1 - \gamma\xi - \sqrt{(1-\gamma\xi)^2 + \beta\xi^2}\right)\right\} && (90)
\end{aligned}$$