Indeterminacy and Imperfect Information[*]

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Abstract

We study equilibrium determination in an environment where two kinds of agents have different information sets: The fully informed agents know the structure of the model and observe histories of all exogenous and endogenous variables. The less informed agents observe only a strict subset of the full information set. All types of agents form expectations rationally, but agents with limited information need to solve a dynamic signal extraction problem to gather information about the variables they do not observe. We show that for parameters values that imply a unique equilibrium under full information, the limited information rational expectations equilibrium can be indeterminate. In a simple application of our framework to a monetary policy problem we show that limited information on part of the central bank implies indeterminate outcomes even when the Taylor Principle holds.

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1 Introduction

We study existence and uniqueness of rational expectations equilibria in environments where agents have nested information sets. Specifically, we consider an environment that contains a fully informed agent and an agent with a limited information set that is nested within the full information set. Throughout, we maintain the assumption that all agents form expectations rationally and focus on linear equilibria in models with Gaussian innovations, which implies that the agent with the limited information set employs the Kalman filter to optimally conduct inference about variables that are outside his information set. In order to provide some structure on our discussion we distinguish between cases where the signal merely reflects a combination of exogenous variables, or where the signal is endogenous in the sense that it also reflects the realization of endogenous variables.

The key finding in our paper is that equilibrium indeterminacy is substantially more prevalent in this environment. Even when the equilibrium is unique in a corresponding model where all agents have full information (or where all agents have partial, but symmetric information, as shown in the appendix), the presence of less informed agents introduces, via the Kalman filter, a feedback mechanism with stable dynamics such that the overall system does not have the saddle-path uniqueness known from the full information case. Conditions such as the Taylor Principle that guarantee equilibrium uniqueness in full information models (and can be controlled by policymakers) have long served as a hallmark of what constitutes 'good' policy. We show that in models with asymmetric information, this connection can not be made and, instead, equilibrium uniqueness crucially depends on the information set of the various agents in the model.

Equilibrium indeterminacy is generic in an environment where the limited information set extends only to exogenous variables, as the use of the Kalman filter by the less informed agents adds indeterminacy-inducing dynamics in this case. In the case where the limited information set contains endogenous variables, the analysis becomes more difficult due to the interplay of endogenous equilibrium dynamics and the projection problems of the less informed, but we continue to find persuasive indeterminacy-inducing dynamics.

In particular, we consider an example where endogenous variables are observed with error, and first construct a specific "benchmark" equilibrium that always exists. In this benchmark equilibrium, agents coordinate on a rational expectations forecast error that rules out sunspot shocks and imposes a unique parametric restriction on the effect of measurement errors and the exogenous shocks of the economic model. Under this parametriza-

tion, information on a subset of unobservable variables is fully revealed. We then show that, depending on the parametrization of the information problem, additional equilibria can exist.

Many models of imperfect information limit attention to cases where lagged state variables are always perfectly known such that informational asymmetries are limited to knowledge about current shocks. In this case, equilibrium uniqueness would be maintained in our introductory example, but this result does not generalize to other, more realistic models. We show that adopting this informational assumption in a typical, small-scale New Keynesian model leads again to indeterminacy.

Our paper touches upon a wide swath of literature that is concerned with the properties of rational expectations equilibria and deviations therefrom. The most immediate precursor to our work is Svensson and Woodford (2004) and a related paper by Aoki (2006). They describe an optimal monetary policy problem in an information environment that is similar to ours: a central bank observes only a subset of all available information, whereas the private sector is fully informed. For a given set of first-order conditions to the optimal policy problem under imperfect information, the approach of Svensson and Woodford falls into the class of expectational linear-difference equations studied here. In contrast to our paper, Svensson and Woodford only consider an equilibrium that corresponds to a minimum-state variable solution, while explicitly assuming away issues of equilibrium indeterminacy. Svensson and Woodford (2004) are not alone in pursing a MSV approach in such models, other examples are given by Aoki (2008), or Nimark (2008). Specifically, they do not address the full set of indeterminate equilibria that can arise in this framework. We show below how our approach compares to Svensson and Woodford (2004) and how the latter maps into our set of solutions.

The paper also draws inspiration from the literature on solving linear rational expectations models, e.g., Sims (2002) whose formalism we employ. In addition, we use the solution technique for indeterminate rational expectations models developed by Lubik and Schorfheide (2003) which focuses on the properties of the rational expectations forecast error. In order to characterize the equilibrium decomposition of this error into fundamental and non-fundamental components we find it convenient to make use of the recent reinterpretation of this earlier paper by Farmer et al. (2015). We show how in an environment with limited information the loadings on the various stochastic components of the endogenous forecast error are related to the endogenous Kalman gain and which further restrictions are therefore imposed on equilibrium selection. In addition, our paper shows that results

obtained in the global games literature (Morris and Shin, 1998), where the introduction of asymmetric information leads to equilibrium uniqueness, do not carry over to all environments that are of interest to macroeconomists.

Our work falls into the general category of recent work that studies the implications of imperfect information on rational expectations equilibria including, for example, Nimark (2011), Mertens (2016), Rondina and Walker (2016), and Lubik and Matthes (2016).

This paper is structured as follows. In the next section, we provide a simple example that details our solution approach and that highlights analytically our findings. We also show how our approach connects with and refines some earlier contributions to the literature. Section 3 discusses the general solution approach, while we apply our framework to a typical model used in the analysis of monetary policy in Section 4. Section 5 concludes with discussing our findings in the light of empirical applicability.

2 A Simple Analytical Example

We begin by developing a simple example for which analytical solutions are available. Our example relies on a bare-bones model of inflation determination and monetary policy, known from Woodford (2003), that is rich enough to develop intuition for the general case. In the first step, we describe equilibrium determination under full information rational expectations which we use as a benchmark to evaluate limited information equilibria against. Within this simple model, we then introduce our framework where agents have different information sets, specifically, where one agent is fully informed and the other agents only observes a subset of the full information set. Finally, we discuss how our framework relates to the set-up and the findings in Svensson and Woodford (2004). In the appendix we show that our results depend on the existence of asymmetric information.

2.1 Determinacy and Indeterminacy under Rational Expectations

We consider a simple model of inflation determination in the vein of the cashless economy without nominal frictions as in Woodford (2003). The model economy consists of a Fisher equation which links the nominal interest rate i_t to the real rate r_t via expected inflation $E_t \pi_{t+1}$ and a monetary policy rule that has the nominal rate respond to current inflation π_t, that is, a Taylor-rule.[1] Furthermore, the real rate of interest is characterized by an

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[1]We also considered the case of a Wicksellian policy rule: $i_t = r_t + \phi \pi_t$, with a time-varying intercept given by the real rate of interest. The steps towards deriving a solution are very much identical to the ones described in the main text. The derivations are available from the authors upon request.

exogenous AR(1) process driven by a Gaussian shock. $\varepsilon_t \sim iid\ \mathcal{N}(0, \sigma_\varepsilon^2)$.

The equation system is thus given by:

$$i_t = r_t + E_t \pi_{t+1}, \tag{1}$$

$$i_t = \phi \pi_t, \tag{2}$$

$$r_t = \rho r_{t-1} + \varepsilon_t. \tag{3}$$

The first equation is the Fisher equation, the second is the policy rule, while the third equation describes the evolution of the real rate as an AR(1) process. All variables can be understood as deviations from their respective steady state values. ϕ is a monetary policy parameter. The autoregressive parameter $|\rho| < 1$ to guarantee stationarity. E_t is the rational expectations operator under full information. It denotes the mathematical expectation conditional on the full information set S^t of all shocks through time t, so that for some variable x_t, $E_t x_{t+h} = E\left(x_{t+h} | S^t\right)$ and $E_t x_t = x_t$.

We can also distinguish two agents in this economy: First, a representative private-sector agent whose behavior is characterized by the Fisher equation (1), and second, a central bank whose behavior is given by the monetary policy rule (2). Under full information rational expectations (FIRE), both agents are assumed to know S^t. This means that they observe all variables in the model without error, that they know the history of all shocks, that they understand the structure of the economy and the solution concepts. We will modify this assumption minimally in the next section.

In order to find a rational expectations equilibrium (REE), we can substitute the policy rule into the Fisher equation to derive a relationship in inflation with driving process r_t:

$$E_t \pi_{t+1} = \phi \pi_t + r_t. \tag{4}$$

The dynamic behavior of inflation depends on the value of the policy coefficient ϕ. Applying covariance stationarity as an equilibrium selection criterion, then it is well known that a unique equilibrium exists if and only $|\phi| > 1$.[2] It is straightforward to establish that the unique REE solution is $\pi_t = \frac{1}{\phi-\rho} r_t$ and $i_t = \frac{\phi}{\phi-\rho} r_t$. This is the case of equilibrium determinacy.

While the remainder of this paper will focus attention on the case where ϕ is inside the unit circle, it is instructive to review the well-studied full-information case of equilibrium

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[2]There has been some recent interest in equilibria that use different selection criteria, accompanied by the development of analytical tools to deal with these cases. A precursor of this literature is the regime-switching model of Davig and Leeper (2007) and the rejoinder by Farmer, Waggoner, and Zha (2009). More recently, Cochrane (2016) looks at equilibria along explosive paths.

indeterminacy that arises when ϕ is inside the unit circle. In this scenario, the equilibrium can be indeterminate in the sense that there are possibly infinitely many solutions that are consistent with equation (4). In order to analyze this case we find it convenient to follow Lubik and Schorfheide (2003) and introduce the rational expectations forecast error $\eta_t = \pi_t - E_{t-1}\pi_t$, whereby $E_{t-1}\eta_t = 0$ by construction. We can use this definition to substitute out the expectations term in (4). This results in the expression:

$$\pi_t = \phi\pi_{t-1} + r_{t-1} + \eta_t. \tag{5}$$

It is easily verifiable that this is a solution to the expectational difference equation (4). In this equilibrium, inflation is a stationary process with autoregressive parameter $|\phi| < 1$ and driving process r_{t-1}.

What makes this equilibrium indeterminate is the fact that the solution imposes no restriction on the evolution of η_t other than it is a martingale difference sequence with $E_{t-1}\eta_t = 0$. Consequently, there can be infinitely many solutions. We can, however, provide some structure, at least in terms of economic intuition, if we are willing to link the process η_t to the shock processes of the economy. To this effect, we find it convenient to follow Lubik and Schorfheide (2003) and rewrite the inflation equation in terms of expectations only. We define $\xi_t = E_t\pi_{t+1}$ and rewrite equation (4) as $\xi_t = \phi\xi_{t-1} + r_t + \phi\eta_t$.

In this specification, the forecast error η_t emerges as an innovation to the conditional expectation ξ_t. Taking an additional step, we can further decompose, without loss of generality, η_t into a fundamental component, namely the policy innovation ε_t and a non-fundamental component, the belief shock b_t; more specifically, $\eta_t = \gamma_\varepsilon\varepsilon_t + \gamma_b b_t$.[3] The coefficients γ_ε and γ_b can be used to index specific equilibria with the set of indeterminate equilibria. In the case of FIRE the choice of these parameters is arbitrary.

Next, we will turn to limited-information settings with rational expectations (LIRE). As in the case of FIRE, we will continue to decompose endogenous forecast errors (η) into fundamental and non-fundamental components (ε_t and b_t, respectively). In the case of LIRE, equilibrium indeterminacy arises even when $|\phi| > 1$, but the set of equilibria will be restricted in the form of additional restrictions on the analogues of γ_ε and γ_b introduced above.

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[3]To be more specific, this is without loss of generality within the set of equilibria that are time-invariant and linear. There are other, non-linear equilibria that can be constructed for this linear model.

2.2 Equilibrium Determinacy under Limited Information

We now introduce the idea that the central bank possesses only limited information. The structure of the model, specifically the form of the equations, remains the same, but we alter the conditioning of the expectations operator with respect to different information sets. We assume that the limited information set of the central bank, denoted Z^t is nested in S^t, the information set of the private sector. That is, the central bank knows less than the private sector, but it still forms expectations rationally.[4] The central bank's projections of any variable x_t are denoted $x_{t|t} = E\left(x_t|Z^t\right)$ and $x_{t+h|t} = E\left(x_{t+h}|Z^t\right)$. Since Z^t is spanned by S^t we can apply the law of iterated expectations to obtain: $E\left(E(x_{t+h}|Z^t)|S^t\right) = x_{t+h|t}$.

We assume that the central bank can set the nominal interest rate only as a function of Z^t. This idea can be captured by imposing the following monetary policy rule:

$$i_t = \phi \pi_{t|t} \tag{6}$$

We can now apply the same conditioning to the other two equations in the system to derive a solution for the central bank's projection for the model's variables. Going through the same steps as above, we can condition down the variables in the system (1) - (3) and derive the projection for inflation. We continue to assume that $|\phi| > 1$, in which case we have $\pi_{t|t} = \frac{1}{\phi - \rho} r_{t|t}$ and $i_t = \frac{\phi}{\phi - \rho} r_{t|t}$. This solution in terms of the central bank's projections for the model variables hold under any definition of the central bank's information set as long as Z^t is spanned by S^t. Under the assumption that policy follows the Taylor principle, $|\phi| > 1$, the central bank projections of the real rate and inflation share the same relationship as the actual variables in the full information model, even though the central bank is aware of the possibility of indeterminacy.

We combine the private sector Fisher equation (1) with the projected policy rule (6):

$$\phi \pi_{t|t} = r_t + E_t \pi_{t+1}. \tag{7}$$

$\pi_{t|t}$ is an endogenous variable, the central bank's projection of inflation, that has to be linked to private sector variables to derive a solution, while $E_t \pi_{t+1}$ is the private sector's expectation. Using the formalism described in the previous section, we can rewrite this expression as:

$$\pi_t = \phi \pi_{t-1|t-1} - r_{t-1} + \eta_t = \frac{\phi}{\phi - \rho} r_{t-1|t-1} - r_{t-1} + \eta_t,$$

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[4]This is a key difference to the framework in Lubik and Matthes (2016) who assume in addition that the central bank engages in least-squares learning to gain information about private-sector outcomes. In our setup, the deviation from the standard rational expectations benchmark is only minor in the sense that the central bank does not observe everything that the private sector does, but is otherwise fully informed.

where $r_{t-1|t-1}$ is an endogenous variable on account of the central bank's projection problem. Finding an REE therefore requires linking projections such as $r_{t-1|t-1}$ to the state variables of the model. Moreover, we need to solve out the endogenous forecast error η_t of the private sector, the solution to which can be unique or indeterminate. The next step therefore requires to specify the information set of the central bank. For purposes of exposition we distinguish between exogenous and endogenous information sets: While exogenous information sets solely reflect the realisation of exogenous variables, endogenous information sets also include (possibly noisy) signals of endogenous variables.[5]

2.2.1 Equilibria with an Exogenous Information Set

Suppose that the central bank observes the real interest rate with measurement error ν_t, where $\nu_t \sim iid \, \mathcal{N}(0, \sigma_\nu^2)$. Therefore, the central bank's information set is such that $Z_t = r_t + \nu_t$. It is exogenous in that the real rate is an exogenous process which does not depend on other endogenous variables. The central bank uses the Kalman filter, the optimal linear filter in this environment, to make projections. Specifically, the projection equation is:

$$r_{t|t} = r_{t|t-1} + \kappa_r \left(r_t - r_{t|t-1} + \nu_t \right). \tag{8}$$

The central bank updates its current projection $r_{t|t}$ relative to its previous one-step ahead projection $r_{t|t-1}$ using the current information $r_t + \nu_t$. The crucial parameter in this updating equation is the (optimal) Kalman gain κ_r which we have to compute separately.

In the specific case of an exogenous information set, the Kalman filtering problem can, however, be solved independently from the rest of model — since both state and measurement equation comprise only exogenous variables — which greatly facilitates the analysis.

We first note that $r_{t|t-1} = \rho r_{t-1|t-1}$ so that we can derive the full equation system which describes the joint evolution of actual inflation π_t, the exogenous real rate r_t, and the central bank projection of the real rate $r_{t|t}$. The system is given by:

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\begin{aligned}
\pi_t &= \frac{\phi}{\phi - \rho} r_{t-1|t-1} - r_{t-1} + \eta_t, \\
r_{t|t} &= \left(1 - \kappa_r \right) \rho r_{t-1|t-1} + \kappa_r \rho r_{t-1} + \kappa_r \varepsilon_t + \kappa_r \nu_t, \\
r_t &= \rho r_{t-1} + \varepsilon_t.
\end{aligned}
\tag{9}
$$

This is a well-specified equation system in three unknowns and can be solved using standard methods for linear rational expectations models that allow for indeterminacy (Lubik and

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[5]In our framework, information sets are defined as linear combinations of variables. For the purpose of our paper, the term "endogenous information" refers to cases where these linear combinations also comprise endogenous variables, which is not to be confused with other settings where the make up of the linear combination itself might be the outcome of an endogenous choice.

Schorfheide, 2003). Specifically, a solution would pin down the endogenous forecast error η_t, which would then determine whether the REE under limited information is determinate or indeterminate. We also note that the equation system is recursive and that the overall dynamic properties depend on the yet unknown value of $(1 - \kappa_r) \rho$; in order to determine the size of this 'root' we need to compute the gain κ_r first.

We find it convenient to define innovations of any variable x_t as its unexpected component relative to the limited information set Z^t: $\widetilde{x}_t = x_t - x_{t|t-1}$. The Kalman gain is given by:

$$\kappa_r = \frac{cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)}{var(\widetilde{Z}_t)}, \tag{10}$$

where the tildes denote the projection innovations, e.g. $\widetilde{r}_t = r_t - r_{t|t-1}$ and $\widetilde{Z}_t = \widetilde{r}_t + \nu_t$. Given the definition of $\Sigma = var\left(\widetilde{r}_t - \widetilde{r}_{t|t}\right) = var\left(\widetilde{r}_t\right) - var\left(\widetilde{r}_{t|t}\right)$ (whereby $cov(\widetilde{r}_t, \widetilde{r}_{t|t}) = var(\widetilde{r}_{t|t})$), we can find that $var\left(\widetilde{r}_t\right) = \rho^2 \Sigma + \sigma_\varepsilon^2$ and that $var(\widetilde{Z}_t) = var\left(\widetilde{r}_t\right) + \sigma_\nu^2$. Similarly, $cov(\widetilde{r}_t, \widetilde{Z}_t) = var\left(\widetilde{r}_t\right)$ and we find the following expression for the Kalman gain:

$$\kappa_r = \frac{\rho^2 \Sigma + \sigma_\varepsilon^2}{\rho^2 \Sigma + \sigma_\varepsilon^2 + \sigma_\nu^2}, \tag{11}$$

which for positive Σ lies within the unit circle.

The variance Σ can be computed from the definition above, whereby $var\left(\widetilde{r}_{t|t}\right) = \kappa_r cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)$ and where we have utilized the expression for the projection equation $\widetilde{r}_{t|t} = \kappa_r \widetilde{Z}_t$. Substituting these expressions into the definition of Σ results in a non-linear Riccati equation:

$$\Sigma = \frac{\rho^2 \Sigma + \sigma_\varepsilon^2}{\rho^2 \Sigma + \sigma_\varepsilon^2 + \sigma_\nu^2} \sigma_\nu^2. \tag{12}$$

The solution to this (quadratic) equation is given by:

$$\Sigma = \frac{1}{2\rho^2} \left[-\left(\sigma_\varepsilon^2 + \left(1 - \rho^2\right) \sigma_\nu^2\right) + \sqrt{\left(\sigma_\varepsilon^2 + \left(1 - \rho^2\right) \sigma_\nu^2\right)^2 + 4\sigma_\varepsilon^2 \sigma_\nu^2 \rho^2} \right]. \tag{13}$$

Since $0 < \kappa_r < 1$, it follows that $0 < (1 - \kappa_r) \rho < 1$ and the law of motion for $r_{t|t}$ in the full equation system is a stable difference equation that delivers a stationary solution. Since the equation for actual inflation does not depend on its own lags, we can conclude that the equilibrium is indeterminate.

As will be discussed next, the equilibrium condition $\pi_{t|t} = \frac{1}{\phi - \rho} r_{t|t}$ imposes restrictions on η_t, without uniquely pinning it down, however. Following Farmer et al. (2015) we can decompose the forecast error $\eta_t = \gamma_\varepsilon \varepsilon_t + \gamma_b b_t + \gamma_\nu \nu_t$, where the weights on the innovations are generally undetermined. Specific values index specific equilibria among the

set of indeterminate equilibria as discussed in Lubik and Schorfheide (2003). The equilibrium condition $\pi_{t|t} = \frac{1}{\phi-\rho} r_{t|t}$ has to hold along every equilibrium path imposes the following restriction on innovations with respect to the central bank's information set: $cov\left(\widetilde{\pi}_t, \widetilde{Z}_t\right) = \frac{1}{\phi-\rho} cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)$. This "subspace-condition" restricts the set of multiple equilibria because it restricts the γ coefficients that determine η_t. This condition is absent from full information models with indeterminacy, and thus differentiates the class of LIRE models from their FIRE counterparts.

We know that $cov\left(\widetilde{r}_t, \widetilde{Z}_t\right) = \rho^2 \Sigma + \sigma_\varepsilon^2$. $cov\left(\widetilde{\pi}_t, \widetilde{Z}_t\right) = cov\left(\widetilde{\pi}_t, \widetilde{r}_t\right) + cov\left(\widetilde{\pi}_t, \nu_t\right)$. Using the innovation representation of the projection equation for π_t we have

$$\widetilde{\pi}_t = -\left(\widetilde{r}_{t-1} - \widetilde{r}_{t-1|t-1}\right) + \eta_t. \tag{14}$$

After some substitution we find that $cov\left(\widetilde{\pi}_t, \widetilde{r}_t\right) = -\rho\Sigma + \gamma_\varepsilon \sigma_\varepsilon^2$. Similarly, we can find $cov\left(\widetilde{\pi}_t, \nu_t\right) = \gamma_\nu \sigma_\nu^2$. Combining all expressions then results in the following linear restriction on the weights in the forecast error: $\eta_t = \gamma_\varepsilon \varepsilon_t + \gamma_b b_t + \gamma_\nu \nu_t$:

$$\gamma_\nu = \frac{\phi}{\phi-\rho}\frac{\Sigma}{\sigma_\nu^2} + \frac{1}{\phi-\rho}\frac{\sigma_\varepsilon^2}{\sigma_\nu^2} - \frac{\sigma_\varepsilon^2}{\sigma_\nu^2}\gamma_\varepsilon. \tag{15}$$

The solution to the simple model under LIRE with an exogenous information set can therefore be written as:

$$\begin{aligned}
\pi_t &= \frac{\phi}{\phi-\rho} r_{t-1|t-1} - r_{t-1} + \gamma_\varepsilon \varepsilon_t + \gamma_b b_t + \gamma_\nu \nu_t, \\
r_{t|t} &= (1-\kappa_r)\rho r_{t-1|t-1} + \kappa_r \rho r_{t-1} + \kappa_r \varepsilon_t + \kappa_r \nu_t, \\
r_t &= \rho r_{t-1} + \varepsilon_t.
\end{aligned} \tag{16}$$

where:

$$\begin{aligned}
\kappa_r &= \frac{\rho^2\Sigma + \sigma_\varepsilon^2}{\rho^2\Sigma + \sigma_\varepsilon^2 + \sigma_\nu^2}, \\
\Sigma &= \frac{1}{2\rho^2}\left[-\left(\sigma_\varepsilon^2 + \left(1-\rho^2\right)\sigma_\nu^2\right) + \sqrt{\left(\sigma_\varepsilon^2 + \left(1-\rho^2\right)\sigma_\nu^2\right)^2 + 4\sigma_\varepsilon^2\sigma_\nu^2\rho^2}\right], \\
-\infty &< \gamma_b < \infty, -\infty < \gamma_\varepsilon < \infty, \gamma_\nu = \frac{\phi}{\phi-\rho}\frac{\Sigma}{\sigma_\nu^2} + \frac{1}{\phi-\rho}\frac{\sigma_\varepsilon^2}{\sigma_\nu^2} - \frac{\sigma_\varepsilon^2}{\sigma_\nu^2}\gamma_\varepsilon
\end{aligned}$$

We can summarize our findings as follows: First, the limited information rational expectations equilibrium is indeterminate even though the full information counterpart has a determinate equilibrium. The optimal filtering procedure employed by the central bank introduces a stable root into the system and thus leaves the endogenous forecast error undetermined. While there is a uniquely determined mapping from the central bank's projections

of state variables to endogenous outcomes, actual equilibrium outcomes - in particular the component that is orthogonal to the central bank's information set - remains indeterminate.

Second, a consistency requirement for Kalman filtering imposes restrictions on the set of multiple equilibria. However, at least in the exogenous-information case, these restrictions do not affect the way belief shocks b_t ("sunspot shocks") may enter the system. In full information solutions under indeterminacy the set of multiple equilibria is typically unrestricted. Optimal filtering in a limited information counterpart restricts how the private agents coordinate on an equilibrium. From an empirical perspective, the FIRE solution results in a reduced-form representation for inflation that is first-order autoregressive. The LIRE solution on the other hand exhibits much richer dynamics. In particular, the resulting inflation process can be quite persistent when the signal-to-noise ratio is small as a large σ_ν^2 translates into a small Kalman gain.

It is also worth pointing out that the equilibria we study in this paper (independent of the exact information structure) can in general *not* be written as a moving average of the forecast errors coming out of the Kalman filter (which is possible in an environment of partial, but symmetric information).

This simple example may be regarded as restrictive as the central bank only observes an exogenous process with error, whereas in practice observed variables are typically endogenous (and measured with error). In the next step, we therefore study the implications of the central bank observing the inflation rate with error. The information set is endogenous in the sense that the variable contained therein is determined in equilibrium. This potentially creates additional feedback within the model.

2.2.2 An Endogenous Information Set

We now assume that the central bank observes the inflation rate with measurement error ν_t such that $Z_t = \pi_t + \nu_t$. The solution in the terms of the central bank projections is identical to the previous case. What changes is however the interdependence between the filtering problem and the equilibrium dynamics. We find it convenient to express the analysis in terms of the projection equation for the real rate to maintain comparability with the previous case. The projection equation is:

$$r_{t|t} = r_{t|t-1} + \kappa_r \left(\pi_t - \pi_{t|t-1} + \nu_t \right). \tag{17}$$

From this we can derive the full equation system as before, namely:

$$\pi_t = \frac{\phi}{\phi - \rho} r_{t-1|t-1} - r_{t-1} + \eta_t \tag{18}$$

$$r_{t|t} = (\rho + \kappa_r) r_{t-1|t-1} + \kappa_r r_{t-1} + \kappa_r \nu_t + \kappa_r \eta_t \tag{19}$$

$$r_t = \rho r_{t-1} + \varepsilon_t \tag{20}$$

This system looks superficially similar to the previous one with exogenous information. What is different is that the coefficient on the lagged real rate projection $(\rho + \kappa_r) r_{t-1|t-1}$ depends now on the endogenous dynamics of π_t, which may or may not lead to an unstable root $|\rho + \kappa_r| > 1$ and, if not, give rise to non-fundamental belief shocks affecting equilibrium dynamics without causing non-stationary variations. The Kalman gain itself may no longer be unique in this setting, a stark contrast from the case of exogenous information.

2.2.3   A Benchmark Solution with Endogenous Information

The observation that the projection equation $r_{t|t} = (\rho + \kappa_r) r_{t-1|t-1} + \kappa_r r_{t-1} + \kappa_r \nu_t + \kappa_r \eta_t$ can be potentially explosive and would therefore pin down the forecast error η_t can be used to find *an* equilibrium. Using the definition $r_t^* = r_t - r_{t|t}$, which is the error from the projection onto the current information set, we can rewrite the projection equation as follows:

$$r_t^* = (\rho + \kappa_r) r_{t-1}^* + \varepsilon_t - \kappa_r \nu_t - \kappa_r \eta_t. \tag{21}$$

This is a first-order difference equation driven by a linear combination of shocks: the exogenous real-rate shock ε_t, the exogenous measurement error ν_t, and the endogenous forecast error η_t. Now suppose that $|\rho + \kappa_r| > 1$. The solution to this explosive equation is $r_t^* \equiv 0$ and $\eta_t = \frac{1}{\kappa_r} \varepsilon_t - \nu_t$. This solution uniquely pins down the forecast error and therefore seems to render the equilibrium determinate. Why is this equilibrium not necessarily unique? The Kalman gain κ_r is endogenous and as such, there can in general be other equilibria with a different Kalman gain. We explore this possibility in the next section The benchmark equilibrium is a full revelation solution in that it implies that $r_{t|t} = r_t$, that is, the real rate projection using current information is exact (the equilibrium inflation rate, however, is not revealed without error to the central bank).

Existence of this solution requires that we can find a κ_r such that $|\rho + \kappa_r| > 1$ and that the subspace condition holds. We can verify that this is in fact the case by substituting the solution into the projection equation which results in $r_t = \rho r_{t-1} + \varepsilon_t$, that is, the exogenous process for the real rate. Substituting the solution in the inflation equation yields $\pi_t = \frac{\rho}{\phi - \rho} r_{t-1} + \frac{1}{\kappa_r} \varepsilon_t - \nu_t$, which depends on the Kalman gain. Since this solution

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achieves full revelation for the real rate, we can hypothesize that it is consistent with the full information solution that we derived above in the following sense: $\pi_t^{FI} = \frac{\rho}{\phi-\rho} r_{t-1} + \frac{1}{\phi-\rho} \varepsilon_t$, which is void of the measurement error. Comparing the LIRE and FIRE solution we find that $\pi_t^{LI} = \pi_t^{FI} - \nu_t$, which implies $\kappa_r = \phi - \rho$. The subspace condition additionally imposes $\kappa_\pi = \kappa_r / (\phi - \rho) = 1$ and always holds.

The nature of this equilibrium is such that in terms of the forecast error decomposition $\eta_t = \gamma_\varepsilon \varepsilon_t + \gamma_b b_t + \gamma_\nu \nu_t$ we have $\gamma_\varepsilon = 1/(\phi - \rho)$, $\gamma_\nu = -1$, and $\gamma_b = 0$. The latter follows since under determinacy sunspot shocks do not affect equilibrium outcomes. As a final step, we need to verify that this Kalman gain is consistent with the Riccati equation. Under this parametrization, the Riccati equation has a positive solution and a solution with $\Sigma = 0$, which implies full revelation of the real rate. This root intersects with the subspace condition at the value $\gamma_\varepsilon = 1/(\phi - \rho)$.[6] Finally, the root of the projection equation $\rho + \kappa_r = \phi > 1$, which validates our original assumption.

2.2.4 Other Equilibria under Endogenous Information

We next turn to finding other equilibria in the model with the endogenous information set described above. As a first step, we again derive the optimal Kalman gain. After some lengthy algebra, we find that:

$$\kappa_r = \frac{cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)}{var(\widetilde{Z}_t)} = \frac{-\rho\Sigma + \gamma_\varepsilon \sigma_\varepsilon^2}{\Sigma + \gamma_\varepsilon^2 \sigma_\varepsilon^2 + \gamma_b^2 \sigma_b^2 + (1 + \gamma_\nu)^2 \sigma_\nu^2}. \tag{22}$$

The projection error variance Σ can be computed from the Riccati equation

$$\Sigma = var\left(\widetilde{r}_t - \widetilde{r}_{t|t}\right) = var\left(\widetilde{r}_t\right) - var\left(\widetilde{r}_{t|t}\right) = \rho^2\Sigma + \sigma_\varepsilon^2 - \frac{-\left(\rho\Sigma + \gamma_\varepsilon \sigma_\varepsilon^2\right)^2}{\Sigma + \gamma_\varepsilon^2 \sigma_\varepsilon^2 + \gamma_b^2 \sigma_b^2 + (1 + \gamma_\nu)^2 \sigma_\nu^2}.$$

Existence of an equilibrium also requires that the subspace condition

$$cov\left(\widetilde{r}_t, \widetilde{Z}_t\right) = (\phi - \rho) cov\left(\widetilde{\pi}_t, \widetilde{Z}_t\right)$$

holds, which imposes the restriction that

$$\phi\Sigma = (1 - (\phi - \rho)\gamma_\varepsilon)\gamma_\varepsilon^2 \sigma_\varepsilon^2 - (\phi - \rho)\gamma_\nu(1 + \gamma_\nu)\sigma_\nu^2 - \gamma_b^2 \sigma_b^2.$$

The solution to the full equation system therefore rests on whether the root $|\rho + \kappa_r| \gtrless 1$. If the root is greater than one, we recover the benchmark equilibrium described above. Using

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[6]In reference to Figure 1 this means that the negative root touches zero where it intersects with the hyperbola of the subspace condition.

numerical simulation we found it generally difficult to find solutions for which the root is less than one if we imposed a small variance of the measurement error. While we can compute solutions to the projection error variance Σ from the Riccati equation in these cases, these generally turn out to be inconsistent with the sub-space condition. Figure 1 highlights the issue. For a typical parametrization and $\gamma_b = 0$ it shows the two roots of the quadratic Riccati equation and the hyperbola of the subspace condition. These values are plotted as functions of the loading on the real rate innovation γ_ε over the range $[-10, 10]$. We also report the Kalman gain for reference purposes.

Existence of a solution requires that the positive root of the Riccati equation and the subspace condition hold at the same time, but they never intersect or even touch. For this given parametrization no equilibrium exists since under these parameters the Kalman filter does not exist in the sense that the projection equations for the variables outside of the central bank's information set are explosive and mutually inconsistent. A wide-ranging exploration of the parameter space does reveal, however, that an equilibrium exists for small ρ, high ϕ, and large σ_ν^2. In this case, the equilibrium would be indeterminate as in the case with an exogenous information set since the Kalman gain is small and negative as already indicated in Figure 1. Moreover, if the equilibrium exists, there is one non-linear restriction on three weights in the forecast error.

If we increase the variance of the measurement error, we can find multiple equilibria. In figure 2, we plot various equilibria (varying the weight on the belief shock b_t and then solving for the other loadings) for the case where the measurement error variance is (unrealistically) large.[7] We show later that with richer models, equilibrium indeterminacy is more prevalent even for substantially smaller values of the measurement error variance.

To summarize, in the LIRE model with an endogenous information set it is possible to find an equilibrium that (almost) replicates the FIRE equilibrium. There is higher inflation volatility on account of the presence of the measurement error. Nevertheless there are potentially other equilibria that lead to indeterminacy but based on numerical solutions we regard them as implausible for this specific model. Other models, because of different equilibrium conditions (and as such subspace conditions) have multiple equilibria with endogenous information sets for reasonable parameters. The scenario described in this section is akin to the outcome described in Lubik and Schorfheide (2003), where an indeterminate equilibrium without sunspots is observationally equivalent to a corresponding determinate equilibrium.

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[7]The dashed black line is the full information response, whereas the red lines are the responses under some of the multiple equilibria.

2.3 MSV Solution: Svensson and Woodford (2004)

In a precursor paper to ours, Svensson and Woodford (2004) study a similar environment, but they solve their model using a minimum-state-variable (MSV) approach. This means that by construction they rule out sunspot shocks. Moreover, the resulting solution resembles a corresponding determinate equilibrium, as discussed in Lubik and Schorfheide (2003), since it abstracts from the additional state introduced by the now-stable root under indeterminacy. We now translate their solution into our framework and show that by imposing MSV-solutions they miss out on the generic nature of indeterminate equilibria under imperfect information.

In terms of the modelling, an important difference between their work and ours is that, in the present model is described by a given rule, whereas Svensson and Woodford endeavour to characterize optimal policy. However, for a given set of first-order conditions to the optimal policy problem under imperfect information, their approach falls into the class of expectational linear-difference equations studied here as well (see Section 3 for a more general discussion). Svensson and Woodford (2004) are not alone in pursing a MSV approach in such models, other examples are given by Aoki (2008), or Nimark (2008); applied to our model, this approach begins with a guess that the equilibrium process for inflation has the following form:

$$\pi_t = g \, r_t^* + \bar{g} \, r_{t|t} \qquad\qquad \bar{g} \equiv \frac{1}{\phi - \rho} \qquad (23)$$
$$= g \, r_t + (\bar{g} - g) \, r_{t|t}$$

For any choice of g, this guess automatically satisfies the sub-space condition $\pi_{t|t} = \bar{g} r_{t|t}$. What remains to be seen is which values for g (if any) would be consistent with the rest of the dynamic system, notably the innovations version of Fisher equation in (14). Notice that the proposed solution excludes belief shocks.

Let us proceed by deriving the dynamics for r_t^* and $r_{t|t}$ implied by (23) for a given value of g. A slight complication for setting up the Kalman filter — encountered also by Svensson and Woodford — is that the guess for inflation in (23) depends on the projected real rate, and thus on the history of measurements (Z^t) which in turn depends on the history of inflation:

$$Z_t = g \, r_t + (\bar{g} - g) \, r_{t|t} + \nu_t \qquad (24)$$

However, notice that the term in $r_{t|t}$ does not add any new information to Z_t. Instead

it provides an implicit definition of an information set spanned by:

$$W_t = g\, r_t + \nu_t \tag{25}$$

in the sense that $E(x_t|Z^t) = E(x_t|W^t)$ for any variable x_t. While projections of variables onto W^t and Z^t are equivalent, the associated Kalman gains will, however, differ by a factor of proportionality.[8]

For starters, consider the Kalman gain involved in projecting the real rate onto W^t, $\tilde{r}_{t|t} = \kappa \tilde{W}_t$:

$$R^2 \equiv g \cdot K \Rightarrow \quad 0 \leq R^2 \leq 1.$$

We can then write

$$
\begin{aligned}
\tilde{\pi}_t &= g \cdot \tilde{r}_t + (\bar{g} - g)\, \kappa\, \tilde{W}_t \\
&= \left(g \cdot (1 - R^2) + \bar{g} \cdot R^2\right) \tilde{r}_t + (\bar{g} - g)\, \kappa\, \nu_t \\
&= \left(g \cdot (1 - R^2) + \bar{g} \cdot R^2\right) \rho\, r^*_{t-1} + \underbrace{\left(g \cdot (1 - R^2) + \bar{g} \cdot R^2\right) \varepsilon_t + (\bar{g} - g)\, \kappa\, \nu_t}_{=\eta_t}
\end{aligned}
\tag{26}
$$

where the last line uses $\tilde{r}_t = \rho\, r^*_{t-1} + \varepsilon_t$.

In order to match (14) we can set η_t equal to the shock components of (26) as indicated above *and* we need to find a value for g that sets the loading on r^*_{t-1} in (26) equal to minus one:

$$\left(g \cdot (1 - R^2) + \bar{g} \cdot R^2\right) \rho = -1 \tag{27}$$

$$\Rightarrow \quad g \leq 0 \tag{28}$$

where the inequality follows from $\bar{g}$, R^2 and ρ being all positive numbers. As a further condition, the solution approach espoused by Svensson and Woodford (2004) would require the roots of the characteristic equation describing the joint dynamics of π_t, $r_{t|t}$ and r_t, see (18)- (20) above, to satisfy the usual counting rule for values inside and outside the unit circle. In the present case, with only one backward-looking variables, r_t, and two forward-looking variables, π_t and $r_{t|t}$, the approach of Svensson and Woodford (2004) would rely on finding one stable and two unstable eigenvalues. However, it can be shown that in the present example, the Kalman filter will always stabilize the dynamics of $r_t - r_{t|t}$ causing the system to have two stable and only one unstable root.

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[8]Let K_r continue to denote the Kalman gain of r_t onto Z_t and we have

$$\tilde{Z}_t = \tilde{W}_t + (\bar{g} - g) \cdot K_r \cdot \tilde{Z}_t = \tilde{W}_t / (1 - (\bar{g} - g) \cdot K_r).$$

Note that the set of MSV candidate solutions — described by (23) for any given value of g — does not span the set of all candidate solutions that we have looked at so far — described by any combination of weights γ for the linear combination of shocks that make up the endogenous forecast error η_t: Furthermore, the set of SW candidates does not even span the restricted set of candidates for η_t where $\gamma_b = 0$. To see this, notice that the MSV candidate is parametrized by a single unknown coefficient, g, which places a restriction on the weights γ_ε and γ_ν implied by the associated specification of η_t as seen in (26).

3 General Framework

Consider the following system of expectational linear-difference equations:

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} E_t \begin{bmatrix} X_{t+1} \\ Y_{t+1} \\ U_{t+1} \end{bmatrix} + J \begin{bmatrix} X_{t+1|t} \\ Y_{t+1|t} \\ U_{t+1|t} \end{bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} & 0 \\ A_{yx} & A_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_t \\ Y_t \\ U_t \end{bmatrix} + \hat{A} \begin{bmatrix} X_{t|t} \\ Y_{t|t} \\ U_{t|t} \end{bmatrix} \tag{29}$$

with

$$U_t \equiv U_{t|t} \tag{30}$$

$$X_t - E_{t-1}X_t = Q\varepsilon_t \qquad \qquad \text{Var}\,(\varepsilon_t) = I \tag{31}$$

where X_t is a vector of backward-looking variables, characterized by exogenous forecast errors $\varepsilon_t \sim N(0, QQ')$, Y_t is a vector of forward-looking variables, U_t is a vector of forward-looking variables that is restricted to lie in a space spanned by a restricted information set that will be discussed shortly.[9] J is a square matrix and may be singular. Expectations $E_t(\cdot)$ are conditional on perfect information about the history of all shocks and variables in the system.[10] Formally, for any variable v_t:

$$E_t\, v_t \equiv E(v_t|S^t) \qquad \quad S^t = \{S_t, S_{t-1}, S_{t-2}, \ldots\} \qquad \quad S_t \equiv \begin{bmatrix} X_t \\ Y_t \end{bmatrix} \tag{32}$$

In contrast, $v_{t|t}$ denotes a conditional expectation derived from a limited information set:

$$v_{t|t} \equiv E(v_t|Z^t) \qquad \quad Z^t = \{Z_t, Z_{t-1}, Z_{t-2}, \ldots\} \qquad \quad Z_t = L\,S_t \tag{33}$$

where L has not full column rank.[11]

Notice that $U_t = U_{t|t}$ is not included in S_t; by definition, U_t merely reflects information contained in S^t and thus does not add any new information.

The linear difference system in (31) does not only arise in models where policy is characterized by Taylor rule under limited information but also extends to the system of first-order conditions for optimal policy under asymmetric information derived by Svensson and Woodford (2004) and Aoki (2006). Notice further that (31) is quite different from the often-cited

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[9]In our monetary-policy applications, U_t will be the vector of policymakers' control variables — typically just the nominal short-term rate. Separating out U_t from X_t and Y_t simply serves to avoid singularities once the system is written in terms of variables that are orthogonal to the policymaker's information set.

[10]Note: Q may have more rows than columns and QQ' may be only positive *semi*-definite (i.e. not necessarily positive definite), as X_t may also track lagged variables.

[11]Notice, this allows also the case of noisy observations $Z_t = \mathcal{C}S_t + \nu_t$ after appropriate redefinition of X_t (and thus S_t) to include also ν_t.

work by Pearlman, Currie and Levine (1986) who study linear, expectational difference systems where *all* expectation operators are conditioned on limited information.[12]

Apart from the term involving $X_{t|t}$ and $Y_{t|t}$, the system in (31) corresponds to the framework(s) known from Blanchard and Kahn (1980), King and Watson (1998), Klein (2000) and Sims (2002). Most of these studies even allow for "singular" systems, where some equations are purely static, without involving any forward-looking terms. For our purpose it will, however, be convenient to assume a non-singular system as in (31).

3.1 Unique Solution for Central-Bank Projections

Throughout, it is assumed that the public information set includes also the central bank's information set — formally: Z^t lies in the span of S^t — which turns out to be highly useful for the next step: Consider conditioning down (31) onto the central bank's information set (Z^t). The resulting system can easily be "solved" in terms of a relationship between $Y_{t|t}$ and $X_{t|t}$ with any of the above-mentioned solution methods for linear rational expectation systems. The resulting system is isomorphic to the full information analogue of (31):

$$\mathcal{J} \begin{bmatrix} X_{t+1|t} \\ Y_{t+1|t} \\ U_{t+1|t} \end{bmatrix} = \mathcal{A} \begin{bmatrix} X_{t|t} \\ Y_{t|t} \\ U_{t|t} \end{bmatrix} \qquad \mathcal{A} = \begin{bmatrix} A_{xx} & A_{xy} & 0 \\ A_{yx} & A_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \hat{A} \qquad \mathcal{J} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} + J \qquad (34)$$

This part of the solution is identical to the analysis of Aoki (2006), Svensson and Woodford (2004).

Notice that the presence of central-bank expectations $Y_{t+1|t}$ and $X_{t+1|t}$ in (31) does not materially alter the problem since (34) is neatly handled by the methods of, for example, Klein (2000) and Sims (2002), independently of whether J (or $\mathcal{J}$) is singular or not.[13]

Provided the conditions for a unique rational expectations under full information are met — which are henceforth assumed to hold — we obtain the following relationship between the central bank's projections of forward- and backward-looking variables:[14]

$$U_{t|t} = F X_{t|t} \qquad Y_{t|t} = G X_{t|t} \qquad X_{t+1|t} = P X_{t|t} \qquad \Rightarrow \qquad S_{t+1|t} = \underbrace{\begin{bmatrix} P & 0 \\ G P & 0 \end{bmatrix}}_{\equiv \mathcal{P}} S_{t|t} \qquad (35)$$

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[12]The framework of Pearlman, Currie and Levine (1986) would thus capture a version of (31) where $X_{t+1|t}$ and $Y_{t+1|t}$ were to replace $E_t X_{t+1}$ and $E_t Y_{t+1}$, respectively.

[13]In anticipation of the Kalman filtering equations introduced further below, we are however assuming that the loadings on $E_t Y_{t+1}$ and $E_t X_{t+1}$ in (31) are non-singular such that the system can be written in the form shown in (31) where these expectations are pre-multiplied by the identity matrix.

[14]A unique RE equilibrium — in terms of mapping projected states $X_{t|t}$ into projected outcomes $Y_{t|t}$ and $U_{t|t}$ — for (34) exists when the number of eigenvalues of $\mathcal{A}$ that lie outside the unit circle is identical to the number of forward-looking variables Y_t and U_t.

where P (and thus also $\mathcal{P}$) is stable. For any sequence of innovations to $X_{t|t}$ *under the central banks' information set Z^t*, denoted

$$\tilde{X}_{t|t} \equiv X_{t|t} - X_{t|t-1},$$

there is is thus a unique, stable solution for the central bank's projections stacked in $S_{t|t}$.[15]

In general, the following notation will be useful: For any variable v_t, define "innovations" $\tilde{v}_t$ and "residuals" v_t^* as

$$\tilde{v}_t \equiv v_t - v_{t|t-1}, \qquad\qquad v_t^* \equiv v_t - v_{t|t} = \tilde{v}_t - \tilde{v}_{t|t}. \qquad (36)$$

and notice that by construction we always have $\tilde{v}_{t|t-1} = 0$.

As all shocks are normally distributed, and assuming a linear, time-invariant equilibrium exists (as will be verified below), we can conjecture that optimal projections will be characterized by a Kalman filter. The Kalman filter (if it exists) then implies

$$S_{t|t} = S_{t|t-1} + K\tilde{Z}_t \qquad (37)$$
$$= \mathcal{P}\, S_{t-1|t-1} + \underbrace{K\,L\,\tilde{S}_t}_{=\tilde{S}_{t|t}} \qquad (38)$$

where K is the Kalman gain, and the second equation follows from (35) and the definition of Z_t. So far, we have found a unique stable solution for the evolution of the central bank's projections, for a given sequence of innovations $\tilde{X}_{t|t} = K\,C\,\tilde{S}_t$. What remains to be shown is how $\tilde{S}_t$ is pinned down in equilibrium and whether the solution for $\tilde{S}_t$ is unique.

3.2 Sub-Space Condition

The endogenous innovations process $\tilde{S}_t$ remains to be derived, and it will have to be seen whether this process is uniquely determined. Importantly, the solution to the system in (34), given in (35), imposes the following restriction on projections of the state innovations onto the information set:

$$\underbrace{\begin{bmatrix} G & -I \end{bmatrix}}_{\equiv \boldsymbol{G}} S_{t|t} = 0 \qquad\Rightarrow\qquad \boldsymbol{G}\tilde{S}_{t|t} = 0 \qquad\Rightarrow\qquad \boldsymbol{G}\,K\,L = 0 \qquad (39)$$

(Notice that (35) implies $\boldsymbol{G}\mathcal{P} = 0$ and thus also $\boldsymbol{G}S_{t|t-1} = 0$.) More succinctly, the solution to (34) implies the following orthogonality condition:

$$E(\tilde{Y}_t - G\tilde{X}_t)\tilde{Z}_t' = 0 \qquad (40)$$

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[15] $\tilde{X}_{t|t}$ is a martingale difference sequence with respect to Z^t but not S^t: By construction we have $\tilde{X}_{t|t-1} = 0$ but typically not $E_{t-1}\tilde{X}_{t|t} = 0$.

3.3 The "Residual System"

From now on, let's work with an expectational difference system expressed in terms of "innovations" $\tilde{S}_t$ and "projection residuals" $S_t^* = \tilde{S}_t - \tilde{S}_{t|t} = S_t - S_{t|t}$. Having solved for the innovations process $\tilde{S}_t$ we can express the solution to the entire system via (35) and (38).

Specifically, we can take expectations of all terms in (34) conditional on Z^t and subtract these on both sides of the equation to obtain[16]

$$E_t(S_{t+1} - S_{t+1|t}) = A\,(S_t - S_{t|t}) \qquad\qquad \Rightarrow \qquad E_t\tilde{S}_{t+1} = A\,S_t^*. \qquad (41)$$

$$\text{where} \quad A \equiv \begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix} \qquad\qquad (42)$$

Clearly, this is not quite a standard difference equation, as the left-hand side involves $\tilde{S}_{t+1}$ and not S_{t+1}^* while the left-hand side involves S_t^* (and not $\tilde{S}_t$).

Define endogenous forecast errors η_t and recall the definition of the exogenous forecast errors ε_t that affect the backward-looking variables:[17]

$$\eta_t \equiv (1 - E_{t-1})Y_t \qquad\qquad\qquad Q\varepsilon_t \equiv (1 - E_{t-1})X_t. \qquad (43)$$

As argued above, the endogenous forecast errors η_t can be written as the sum of a component due to fundamental shocks and a component due to non-fundamental shocks:

$$\eta_t = S\varepsilon_t + Rb_t \qquad\qquad \text{where} \quad E(\varepsilon_t\,b_t') = 0 \qquad\qquad b_t \sim N(0, I) \qquad (44)$$

The residual system (41) can then be written as the state equation of a Kalman filtering system in innovations form:

$$\tilde{S}_{t+1} = A\,(\tilde{S}_t - \tilde{S}_{t|t}) + B\,w_{t+1} \qquad\qquad (45)$$

$$\tilde{Z}_t = L\,\tilde{S}_t. \qquad\qquad (46)$$

$$w_{t+1} \equiv \begin{bmatrix} \varepsilon_{t+1} \\ b_{t+1} \end{bmatrix} \qquad\qquad \text{Var}\,(w_t) = I \qquad\qquad (47)$$

We are interested in restrictions on the forecast errors η_t that ensure a bounded solution to (45) and (46). Strikingly, the Kalman filter, if it exists, ensures bounded innovations $\tilde{S}_{t+1}$ (and thus also bounded residuals $S_t^* = \tilde{S}_t - \tilde{S}_{t|t}$). Furthermore, for the system above, a time-invariant Kalman filter exists under two, fairly mild conditions; see, for example, Anderson and Moore (1979).

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[16]Note that $E_t S_{t+1|t} = S_{t+1|t}$ as Z^t is in the span of S^t.
[17]Note that $(1 - E_{t-1})Y_t = (1 - E_{t-1})\tilde{Y}_t$ and $(1 - E_{t-1})X_t = (1 - E_{t-1})\tilde{X}_t$.

(A, L) **must be "detectable"** which means that every (non-zero) right-eigenvector of A that lies in the nullspace of L must be associated with a stable eigenvalue.[18]

(A, B) **must be "stabilizable"** (A, B) are stabilizable when (A', B') are detectable.

To gain a better intuition for what these conditions entail, recall the notion of "canonical" variables from Blanchard and Kahn (1980). Canonical variables form the basis for solving expectational difference systems. and they are obtained from a rotation $V\tilde{S}_t$ of the original variables such that VAV^{-1} is a (block) diagonal matrix. Intuitively, detectability requires then the measurement equation to load on any unstable "*canonical*" variables of (45).[19] Stabilizability requires that at least every unstable canonical variable is affected by a shock; which is easily satisfied if BB' has full rank, when non-fundamental shocks afflict equilibrium dynamics.

Provided the system given by (45) and (46) is detectable and stabilizable, only the covariance restriction (39) — resulting from $Y_{t|t} = GX_{t|t}$ — can put a lid on the number of equilibria. However, (39) consists of only $N_y \times N_z$ restrictions, which is typically less than the number of $N_y \times (N_\varepsilon + (N_y + 1)/2)$ parameters governing the evolution of endogenous forecast errors.[20]

3.4   A Benchmark Solution

The previous section described a simple model of a cashless economy and characterized a particular benchmark solution, see Section 2.2.3. The generic properties of this solution can also be carried over to the more general framework described in this section, provided we put a little more structure on the measurement vector as well as the number of exogenous and endogenous variables: Specifically, assume that the backward-looking variables are purely exogenous,

$$X_{t+1} = A_{xx}X_t + Q\varepsilon_t$$

and that the policymaker observes all endogenous variables with error:

$$Z_t = Y_t + \nu_t$$

---

[18]Formally: $Av = \lambda v$ and $Lv = 0$ only if $|\lambda| < 1$ or $v = 0$.

[19]Detectability is, for example, violated by an information set that reflects only exogenous variables.

[20]N_y, N_ε, and N_z denote the length of the vectors, Y_t, ε_t and Z_t, respectively. The coefficient matrix S in (44) has $N_y \times N_\varepsilon$ elements and the effects from the $N_y \times N_y$ coefficient matrix R can be identified only up to a rotation.

Furthermore, let there be (at least) as many endogenous variables as exogenous variables, $N_y = N_x$, with an invertible mapping from X_t to Y_t under full information, $|G| \neq 0$, where G is defined in (35).

In this case, the following benchmark solution for Y_t is always consistent with equilibrium:

$$Y_t = GX_t - \nu_t \tag{48}$$

$$\Rightarrow \quad Z_t = GX_t \qquad\qquad \Rightarrow \quad X_{t|t} = X_t \tag{49}$$

$$\Rightarrow \quad K_x = G^{-1} \qquad\qquad\qquad K_y = I \tag{50}$$

which also satisfies the sub-space condition $K_y = G\, K_x$.

3.5 The Case of Lagged Information Revelation

Now consider imperfect information problems of the form where the measurement variable is augmented to remove uncertainty about lagged expectations of current state variables:

$$Z_t = \begin{bmatrix} LS_t \\ E_{t-1}S_t \end{bmatrix} \tag{51}$$

or a different representation with the effect that Z^t spans $E_{t-1}S_t$. Notice that, in this case, Z^t is also spanned by the history of

$$\begin{bmatrix} Le_t \\ E_{t-1}S_t \end{bmatrix} \qquad \text{where} \qquad e_t \equiv S_t - E_{t-1}S_t.$$

Projections onto Z^t can then be written as:

$$S_{t|t} = E_{t-1}S_t + e_{t|t} \qquad S_t^* = e_t - e_{t|t} \tag{52}$$

$$e_{t|t} = Kz_t \qquad z_t = Ce_t \qquad K = \begin{bmatrix} K_\varepsilon \\ K_\eta \end{bmatrix} = \text{Cov}\,(e_t, z_t)\,\text{Var}\,(z_t)^{-1} \tag{53}$$

$$Y_{t|t} = E_{t-1}Y_t + \eta_{t|t} \qquad \eta_{t|t} = K_\eta z_t \tag{54}$$

$$X_{t|t} = E_{t-1}X_t + \varepsilon_{t|t} \qquad \varepsilon_{t|t} = K_\varepsilon z_t \tag{55}$$

In the case of lagged-revelation case filtering $E(S_t|Z^t)$ reduces to a projection on the contemporaneous signal $E(S_t|Z^t) = E(S_t|Z_t)$; due to the presence of $E_{t-1}S_t$ in Z_t, the signal Z_t is a sufficient statistic for $E(S_t|Z^t)$.

In the dynamic Kalman filtering problem we typically employ the decomposition $S_{t|t} = \tilde{S}_{t|t} + S_{t|t-1}$ and — in those settings — we typically have $S_{t|t-1} \neq E_{t-1}S_t$ and thus $\tilde{S}_t \neq e_t$.

However, since we can condition on knowledge of $E_{t-1}S_t$ at time t, we need not bother with $\tilde{S}_{t|t}$ and can focus directly on $e_{t|t}$.[21]

Given the solutions in terms of projections, $Y_{t|t} = \bar{G}\,X_{t|t}$, and $X_{t+1|t} = \bar{P}\,X_{t|t}$, what is left to be determined is thus merely η_t while satisfying the subspace condition. As will be confirmed below, the subspace condition boils down to $\eta_{t|t} = \bar{G}\varepsilon_t$ in this case.

Sub-space condition in the static-information problem: As before, the subspace condition implies $Y_{t|t} = \bar{G}X_{t|t}$ and $X_{t+1} = \bar{P}\,X_{t|t}$. We could still condition this down onto Z^{t-1} and obtain $Y_{t|t-1} = \bar{G}\,X_{t|t-1}$; more relevant, however, are the implications of Z^t spanning $E_{t-1}S_t$ such that the subspace condition now implies

$$Y_{t|t} = \bar{G}X_{t|t} \qquad \Rightarrow \qquad E_{t-1}Y_t = \bar{G}\,E_{t-1}X_t, \qquad \eta_{t|t} = \bar{G}\varepsilon_{t|t} \qquad (56)$$

$$X_{t+1|t} = \bar{P}X_{t|t} \qquad \Rightarrow \qquad E_{t-1}X_{t+1} = \bar{P}\,E_{t-1}X_t \qquad (57)$$

As a result, it is not only the case that the full-information mapping from anticipated states to outcomes applies to forecasts made one step ahead *by the policymaker*, in the static information problem this property also extends to one-step ahead forecasts made by the public. In fact, when the backward-looking variables are purely exogenous, the static-information case can merely contribute to additional volatility of endogenous forecast errors, η_t, but not added persistence. When the backward-looking variables are, however, at least in part endogenous, this additional volatility in η_t may also affect future outcomes.

Implications of sub-space condition for $E_{t-1}Y_t$ From $Y_{t|t} = \bar{G}X_{t|t}$ follows $E_{t-1}Y_t = \bar{G}\,E_{t-1}X_t$ (since E_{t-1} is nested in Z^t). Plugging this back into the original linear-difference system yields the following restriction:

$$A_{yx}X_t + A_{yy}Y_t + \hat{A}_{yx}X_{t|t} + \hat{A}_{yy}Y_{t|t} = \bar{G}\left[A_{xx}X_t + A_{xy}Y_t + \hat{A}_{xx}X_{t|t} + \hat{A}_{xy}Y_{t|t}\right] \quad (58)$$

$$\Rightarrow \quad A_{yx}X_t + A_{yy}Y_t + (\hat{A}_{yx} + \hat{A}_{yy}\bar{G})X_{t|t} = \bar{G}\left[A_{xx}X_t + A_{xy}Y_t + (\hat{A}_{xx} + \hat{A}_{xy}\bar{G})X_{t|t}\right] \quad (59)$$

which further restricts $X_{t|t}$. Using (54) and (55) The last line can be split into:[22]

$$A_{yx}E_{t-1}X_t + A_{yy}E_{t-1}Y_t + (\hat{A}_{yx} + \hat{A}_{yy}\bar{G})E_{t-1}X_t$$
$$= \bar{G}\left[A_{xx}E_{t-1}X_t + A_{xy}E_{t-1}Y_t + (\hat{A}_{xx} + \hat{A}_{xy}\bar{G})E_{t-1}X_t\right] \quad (60)$$

---

[21]Notice that $E_{t-1}S_t$ is spanned by Z^t here but typically not by Z^{t-1}.

[22]Notice that our informational assumptions imply $E_{t-1}X_{t|t} = E_{t-1}X_t$ and $E_{t-1}Y_{t|t} = E_{t-1}Y_t$.

and

$$A_{yx}\varepsilon_{t|t} + A_{yy}\eta_{t|t} + (\hat{A}_{yx} + \hat{A}_{yy}\bar{G})\varepsilon_{t|t} = \bar{G}\left[A_{xx}\varepsilon_{t|t} + A_{xy}\eta_{t|t} + (\hat{A}_{xx} + \hat{A}_{xy}\bar{G})\varepsilon_{t|t}\right] \quad (61)$$

Together with $E_{t-1}Y_t = \bar{G}E_{t-1}X_{t|t}$ (60) requires:

$$A_{yx} + A_{yy}\bar{G} + (\hat{A}_{yx} + \hat{A}_{yy}\bar{G}) = \bar{G}\left[A_{xx} + A_{xy}\bar{G} + (\hat{A}_{xx} + \hat{A}_{xy}\bar{G})\right] \quad (62)$$

which is satisfied by $\bar{G}$ by construction and we are left with the implications of (61).[23]

Considering (61) note that this equation can be rewritten as

$$(A_{yy} - \bar{G}A_{xy})\eta_{t|t} = \left[\bar{G}(A_{xx} + \hat{A}_{xx} + \hat{A}_{xy}\bar{G}) - A_{yx} - (\hat{A}_{yx} + \hat{A}_{yy}\bar{G})\right]\varepsilon_{t|t} \quad (63)$$

$$(A_{yy} - \bar{G}A_{xy})K_\eta z_t = \left[\bar{G}(A_{xx} + \hat{A}_{xx} + \hat{A}_{xy}\bar{G}) - A_{yx} - (\hat{A}_{yx} + \hat{A}_{yy}\bar{G})\right]K_\varepsilon z_t \quad (64)$$

which imposes $N_y \times N_z$ restrictions on the projections $\eta_{t|t}$ and $\varepsilon_{t|t}$, or, respectively, the associated Kalman gains K_x and K_y. Based on (62), it is tempting to conclude that (61) is always satisfied when $\eta_{t|t} = \bar{\varepsilon}_{t|t}$. However, such a conclusion would neglect cases where the matrices premultiplying $\eta_{t|t}$ and $\varepsilon_{t|t}$ in (61) are not of full rank.

In particular, consider the case where $A_{yy} = 0$ and $A_{xy} = 0$ (corresponding to features of the simple Fisher model described above). In this case (61) does not impose a linear restriction between $\eta_{t|t}$ and $\varepsilon_{t|t}$ but rather places a linear restriction on the elements of $\varepsilon_{t|t}$. This reduces the set of possible solutions to the Fisher model to a single case.

In contrast, when $|(A_{yy} - \bar{G}A_{xy})| \neq 0$ we get

$$K_\eta = (A_{yy} - \bar{G}A_{xy})^{-1}\left[\bar{G}(A_{xx} + \hat{A}_{xx} + \hat{A}_{xy}\bar{G}) - A_{yx} - (\hat{A}_{yx} + \hat{A}_{yy}\bar{G})\right]K_\varepsilon \quad (65)$$

$$= \bar{G}K_\varepsilon \quad (66)$$

$$\Rightarrow \eta_{t|t} = \bar{G}\varepsilon_{t|t} \quad (67)$$

where the last step follows from the definition of $\bar{G}$ as solving the full-information problem.[24]

Assuming the above-described rank condition holds: Define $\gamma_t = \eta_t - \bar{G}\varepsilon_t$ such that the subspace condition requires $\gamma_{t|t} = 0$. Without loss of generality let $\gamma_t = G\varepsilon_t + b_t$ where $\text{Cov}(b_t, \varepsilon_t) = 0$, and $b_t \sim N(0, \Omega_{bb})$ for some G and Ω_{bb} which are yet to be determined. In addition, let us specialize the problem as follows:

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[23]To see this argument recall that $\bar{G}$ is defined by the solution to the full-information version of (31), where $X_{t|t} = X_t$ and $Y_{t|t} = Y_t$ which requires (62) to hold.

[24]So see this, set up the full-information model as an undetermined coefficients problem and obtain (62) from which it is straightforward to deduce (66) provided that $|(A_{yy} - \bar{G}A_{xy})| \neq 0$.

- Assume $|\Omega_{\varepsilon\varepsilon}| \neq 0$ such that we can rewrite the problem in a form where $\Omega_{\varepsilon\varepsilon} = I$.[25]

- Let $z_t = C_\varepsilon \varepsilon_t + C_\gamma \gamma_t$ and limit attention to problems where C_γ is square and of full rank; in that case the problem can be rewritten in a form where $C_y = I$.[26]

- With $C_y = I$ and $\Omega_{\varepsilon\varepsilon} = I$ the subspace condition boils down to finding G and Ω_{bb} such that

$$GG' + C_\varepsilon G' + \Omega_{bb} = 0 \tag{68}$$

Notice that (68) has four terms, three of which are clearly symmetric; consequently, $C_\varepsilon G'$ must be symmetric as well and we can also write

$$GG' + \frac{1}{2}C_\varepsilon G' + \frac{1}{2}G C_\varepsilon' + \Omega_{bb} = 0 \tag{69}$$

$$\Rightarrow GG' + \frac{1}{2}C_\varepsilon G' + \frac{1}{2}G C_\varepsilon' + \frac{1}{4}C_\varepsilon C_\varepsilon' = \frac{1}{4}C_\varepsilon C_\varepsilon' - \Omega_{bb} > 0 \tag{70}$$

where the inequality is understood as requiring positive (semi) definiteness. Since $C_\varepsilon C_\varepsilon'$ is known, the inequality thus puts clear bounds on Ω_{bb}.

While continuing to solve (70), we will ignore for a moment the required symmetry of GC_ε' and return to this restriction shortly. Ignoring the symmetry constraint, the remainder of the problem can then be solved as follows:

1. Factorize left- and right-hand side of (70). the left-hand side can be factorized into the "square" of an $N_y \times N_x$ vector:

$$\left[\tfrac{1}{2}C_\varepsilon + G\right]\left[\tfrac{1}{2}C_\varepsilon + G\right]'.$$

the right-hand side can be factorized with the (lower-triangular) Choleski decomposition, which shall be denoted $\Psi^{0.5}$:

$$\Psi = \Psi^{0.5}(\Psi^{0.5})'$$

2. Consider the QR decomposition

$$\left[\tfrac{1}{2}C + G\right] = \begin{bmatrix} R & 0 \end{bmatrix} Q$$

where R is lower triangular and QQ' is orthonormal.[27]

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[25]Alternatively, continue the problem with $\bar{\mathcal{G}} = \bar{G}(\mathrm{chol}\,(\Omega_{\varepsilon\varepsilon}))^{-1}$, $\mathcal{G} = G(\mathrm{chol}\,(\Omega_{\varepsilon\varepsilon}))^{-1}$ and $\mathcal{C} = C\begin{bmatrix} (\mathrm{chol}\,(\Omega_{\varepsilon\varepsilon}))^{-1} & 0 \\ 0 & I \end{bmatrix}$ in lieu of $\bar{G}$, G, and C, respectively.

[26]Alternatively, continue the problem with $\mathcal{Z}_t = (C_\gamma)^{-1}z_t$ as upper portion of the measurement variable.

[27]Notice that the QR is typically computed over the transposes of the above matrices:

$$\left[\tfrac{1}{2}C_\varepsilon + G\right]' = Q'\begin{bmatrix} R' \\ 0 \end{bmatrix}$$

3. Recognize that (70) implies $R = \Psi^{0.5}$. We can thus choose any from any orthonormal matrix Q of dimension $N_x \times N_x$ and compute:

$$G = \begin{bmatrix} \Psi^{0.5} & 0 \end{bmatrix} Q - \frac{1}{2}C_\varepsilon = \Psi^{0.5}Q_1 - \frac{1}{2}C_\varepsilon \tag{71}$$

where

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \qquad Q_1 Q_1' = I, \qquad Q_1 Q_2' = 0, \qquad Q_2 Q_2' = I. \tag{72}$$

The above suggests that for a given Ω_{bb} that is smaller (in the p.d. sense) than $0.25 C_\varepsilon C_\varepsilon'$ we could trace out infinitely many G by picking *any* orthonormal Q. Indeed for solving (70) that is correct. However, (70) does not impose symmetry on GC_ε'; any solution to (69) solves (70) but not vice versa. We need to choose an orthonormal Q that solves (70) *and* ensures symmetry of GC'. As will be shown next, this is easily done by constructing Q from the singular-value decomposition (SVD) of an appropriate rotation of C_ε.

We seek an $N_y \times N_x$ matrix Q_1 (with $Q_1 Q_1' = I$) and such that $\Psi^{0.5}Q_1 C_\varepsilon'$ is symmetric.[28] Equivalently, let's find Q_1 such that $Q_1(\tilde{C}_\varepsilon)'$ is symmetric where $\tilde{C}_\varepsilon = \Psi^{0.5}C_\varepsilon$. The singular-value decomposition of $\tilde{C}_\varepsilon$ yields:

$$\tilde{C}_\varepsilon = USV' = U \begin{bmatrix} S_1 & 0 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}' = US_1 V_1' \tag{73}$$

where U and V are unitary matrices of dimensions $N_y \times N_y$ and $N_x \times N_x$, respectively — i.e. $UU' = I$ and $VV' = I$ — and S_1 a diagonal $N_y \times N_y$ matrix. We thus seek Q_1 such that

$$Q_1 V_1 S_1 U' = US_1 V_1' Q_1' \tag{74}$$

which is can be satisfied in different ways, based on constructing Q_1 from the SVD matrices U, V_1 and V_2 while assuring $Q_1 Q_1' = I$:

1. $Q_1 = UV_1'$

2. $Q_1 = -UV_1'$

3. $Q_1 = BV_2'$ for some B of dimension $N_y \times (N_x - N_y)$, such that $BB' = I$. Such a B can only exist if $N_x \geq 2 \cdot N_y$.[29]

The first two solutions yield $Q_1 \tilde{C}_\varepsilon' = \pm(USU') = \pm(\tilde{C}_\varepsilon \tilde{C}_\varepsilon')$ whereas the third generates $Q_1 \tilde{C}_\varepsilon' = 0$.

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and Q' (R') is then typically denoted "Q" (R); in either case Q is orthonormal and R' is upper triangular.

[28] Note that symmetry of $\Psi^{0.5}Q_1 C_\varepsilon'$ ensures symmetry of $(\Psi^{0.5}Q_1 - 0.5C_\varepsilon)C_\varepsilon'$.

[29] Recall that $V_2'V_2 = I$ and $V_2'V_1 = 0$ follows from $VV' = I$ and that we need $Q_1 Q_1' = I$.

To recap: For a given $|\Omega_{\varepsilon\varepsilon}| \neq 0$, construct $\tilde{C}_{\varepsilon}$ and take the SVD, then choose any of the three ways listed above to construct Q_1 and form G as in (71).

3.6 Set of Indeterminate Solutions in the General Case

<center>[TO BE WRITTEN]</center>

4 Some Numerical Applications

4.1 Lagged information Revelation in a New Keynesian Model

Consider the following New Keynesian model:

$$\pi_t = \frac{\beta}{1+\beta}E_t\pi_{t+1} + \frac{1}{1+\beta}\pi_{t-1} + \frac{\kappa}{1+\beta}y_t \tag{75}$$

$$y_t = E_t y_{t+1} - \sigma(i_t - E_t\pi_{t+1}) + g_t \tag{76}$$

$$i_t = \phi^{\pi}\pi_{t|t} + \phi^y y_{t|t} \tag{77}$$

$$g_t = \rho g_{t-1} + \varepsilon_t \tag{78}$$

The central bank receives noisy signals of inflation and output. With lagged information revelation, all agents know perfectly what shocks occurred in the past. Nonetheless, with lagged inflation appearing in the Phillips curve, even this limited information friction can have persistent effects. Figures 3 and 4 show the impulse reposes of this model to a shock in g_t (which is akin to the real rate shock in our simple example) and belief shocks. We vary the loadings on the belief shocks and then solve for the other coefficients determining endogenous forecast errors. The dashed black lines represent full information responses and the red lines are responses for various equilibria under indeterminacy in this model.[30]

<center>[REMAINDER OF THIS SECTION TO BE WRITTEN]</center>

5 Conclusion

The existence of multiple equilibria in dynamic general equilibrium models has important effects on policy analysis and statistical inference, as well as many other applications. Considering linear rational expectations models the existence and computation of such indeterminate equilibria is well understood when all agents are perfectly informed (Lubik & Schorfheide, 2004). In corresponding models with limited and asymmetric information

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[30]The parameter values are calibrated to standard values, but in this section we just want to show that qualitatively the multiplicity of equilibria that we find can be substantial.

sets, the existence of multiple equilibria has not been much studied. We take a step in this direction and show that in a linear Gaussian setting (i) multiple equilibria in asymmetric information settings can arise even if the corresponding economy under full information would deliver a unique equilibrium, (ii) when multiple equilibria exist, the set of such multiple equilibria is restricted by the structure of the economy, in particular the nature of the less informed agent's filtering problem and (iii) the structure of these multiple equilibria depends crucially on the information set of the less informed agents, in particular whether these agents have access to noisy estimates of endogenous or exogenous variables.
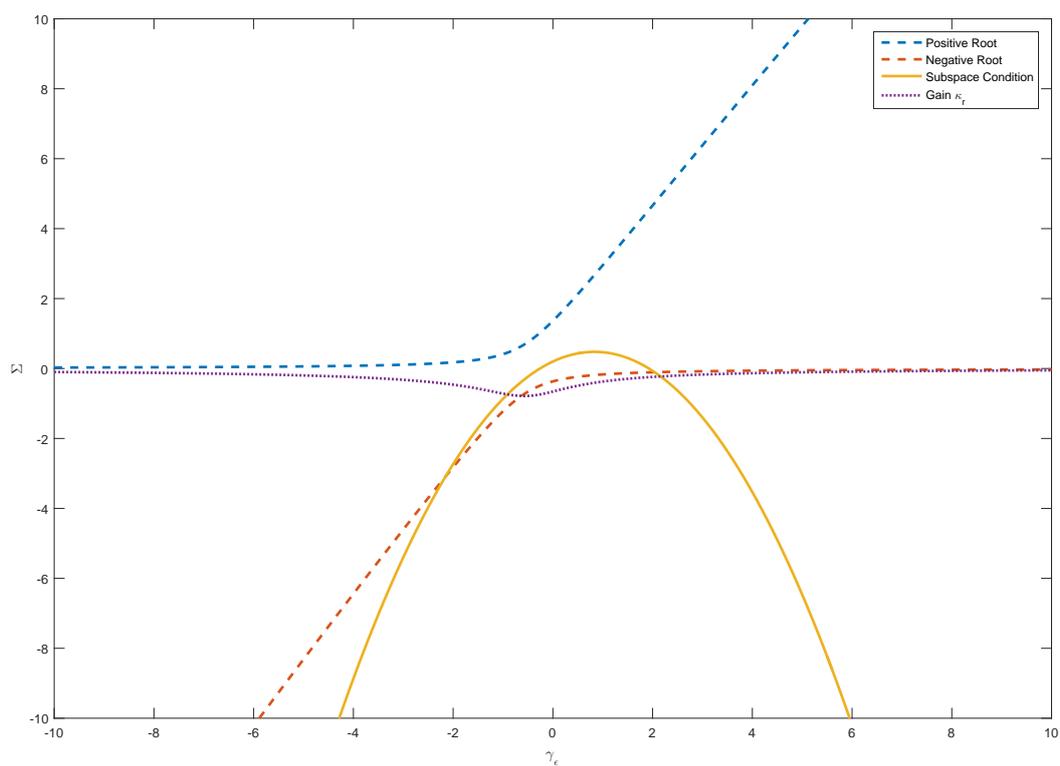
Figure 1: Equilibrium conditions with the endogenous information set
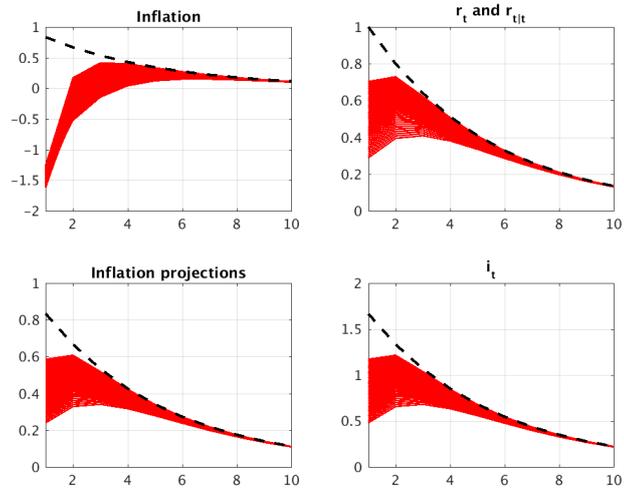
Figure 2: Impulse responses under various equilibria.
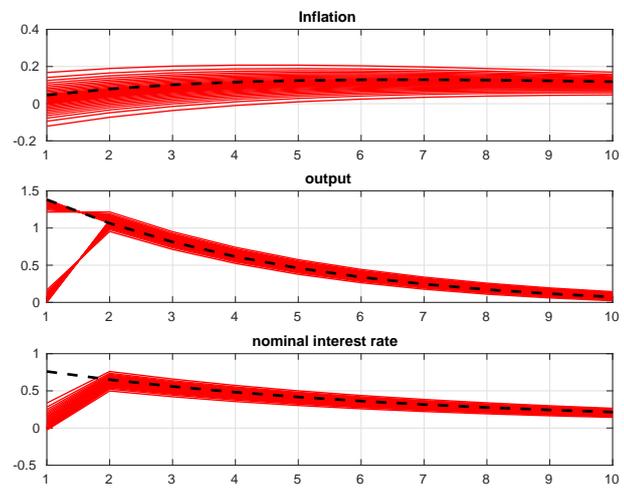
Figure 3: Impulse responses in New Keynesian model to a g_t shock.

30
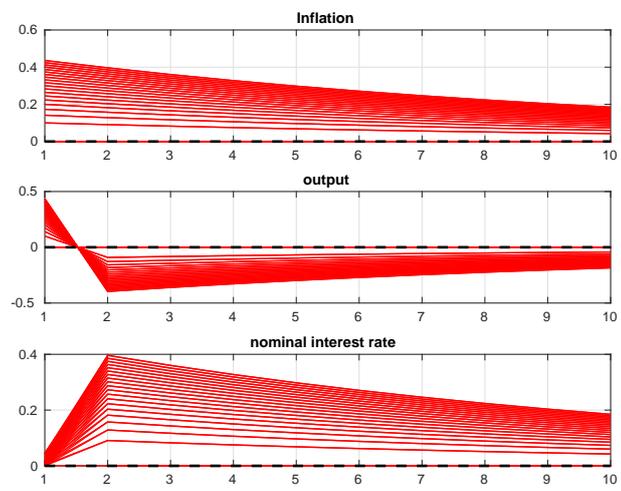
Figure 4: Impulse responses in New Keynesian model to a belief shock.

A   Partial, but Symmetric Information

In this appendix, we show that with partial, but symmetric information, equilibrium unique-ness results of full information settings prevail in our example with exogenous information.

A.1   Equilibrium Conditions

Consider first the following set of equilibrium conditions (which could be considered a natural modification of the full information conditions).

$$i_t = r_t + \pi_{t+1|t} \tag{79}$$

$$i_t = \phi \pi_{t|t} \tag{80}$$

$$r_t = \rho r_{t-1} + \varepsilon_t \tag{81}$$

$$Z_t = r_t + \nu_t \tag{82}$$

$$\tag{83}$$

ν_t and ε_t are iid Gaussian random variables that are independent of each other at all lags and leads.

A.2   The Fisher Equation And The Monetary Policy Rule

What is the Fisher equation? It is an equilibrium condition that tells us how markets price the nominal bond in the economy. Both actors in the market have the same information set (which does *not* include r_t as long as $var(\nu_t) > 0$, which we assume here). Thus, having r_t enter the Fisher equation does not make sense. Mathematically, use the monetary policy rule to substitute out i_t:

$$\phi \pi_{t|t} = r_t + \pi_{t+1|t}$$

Note that both expectations are conditional on the same limited information set which does not include r_t. Yet, the equation above says that $\phi \pi_{t|t} - \pi_{t+1|t} = r_t$, so that r_t should be in the information set spanned by Z_t. This contradicts the definition of Z_t. As such, we don't have a fixed point in information sets.

A.3   The Modified Model

Thinking of the Fisher equation as the equilibrium bond pricing relationship, all variables have to be in the agents' information set. As such, the Fisher equation should read

$$i_t = r_{t|t} + \pi_{t+1|t}$$

With this modified Fisher equation, the model is actually isomorphic to a full information model with the exogenous shock process being given by $r_{t|t}$ instead of r_t. Then standard determinacy conditions apply.

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