

Sustainable Intergenerational Insurance[‡]

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Abstract

This paper studies the dynamic and steady state properties of optimal intergenerational insurance when enforcement is limited. It considers a pure exchange and stochastic overlapping generations economy. The optimal allocation is chosen by a benevolent government whose welfare function values the initial old and places a positive, but vanishing weight on the welfare of future generations. The optimal allocation is constrained to be self-enforceable. That is, generations must have no incentive to default on the consumption allocation at any history of states. We show that the optimal intergenerational insurance when enforcement is limited takes the form of a history-dependent pension plan payable by the young to the old generation. In a simple two-state example we show how the degree of insurance depends on the history of states, in particular, insurance falls with more consecutive good states for the young but reverts whenever the bad state occurs. Finally, we solve for the optimal time-dependent and stationary contracts and numerically compare the welfare loss of these schemes relative to the fully optimal history-dependent scheme.

Keywords: Intergenerational transfers, risk sharing, stochastic overlapping generations, limited commitment

JEL: D64, E21, H55

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1. Introduction

In modern economies, one of the functions of government is to provide security and insurance for the elderly. Most provisions to the elderly are based on a pay-as-you-go scheme that requires contributions from younger generations. Political economy requires the younger generations to tacitly agree to contributions, with the expectation that they will in turn receive support in their old age. The following questions then naturally arise: How should social security policies be designed when there is earning or endowment risk? How can such schemes be implemented or approximated?

The aim of this paper is to characterize a sustainable intergenerational insurance that has two objectives. First, intergenerational insurance must be politically determined by a policymaker (social planner), that values the ex-ante welfare of all current and future generations according to a utilitarian criterion. Second, an intergenerational insurance must be sustainable in that in each state and time all generations must agree to support the transfer scheme, in anticipation that future generations will do the same. Their support is determined by a comparison between the gains expected from the social security system and the intertemporal utility they would attain in the absence of intergenerational insurance (autarky). Since the autarky allocation is inefficient, the politically determined and sustainable insurance promotes efficiency through intergenerational risk-sharing.

To this aim, we study intergenerational risk sharing in a dynamic economy à la Samuelson (1958), but with uncertainty and limited commitment. There are overlapping generations of individuals that live for two periods: young and old. In each period, an agent receives an endowment of a non-storable consumption good. There is uncertainty about the distribution of the aggregate endowment between young and old. In the absence of a risk-sharing arrangement, agents consume their individual endowment. A competitive market with contingent claim would allow agents to transfer resources and increase the welfare of every generation. However, since there are only two periods of life, agents belonging to different cohorts meet only once, so that intergenerational exchange through contingent claims is impossible because the young are born after the realization of shock, which is too late to be able to share with their elders. To focus on the intergenerational aspects of insurance, we suppose that shocks affect agents belonging to the same generation equally, therefore there can no be intra-generational exchange. In this case, a competitive insurance market is necessarily incomplete and cannot allocate consumption efficiently.

We characterize the optimal consumption allocation chosen by a policymaker who values the welfare of the initial old and places a positive but vanishing weight on the welfare of future generations. The constraints on this optimization are due to the limited ability of the planner to enforce the intergenerational insurance. Transfers can only be sustained by means of indirect penalties, which can include the threat of future exclusion

from insurance possibilities. This means that generations called upon to make a transfer to the elderly, will only do so in the anticipation that future generations will reciprocate.

We find that the optimal sustainable intergenerational insurance is non-stationary: it depends on the history of past shocks. This intergenerational transmission of welfare improves individual incentives to share the risk, but it exposes future generations to the risk of their dynasty's history. Specifically, the insurance scheme requires the young to pay transfers to the old, which are larger the longer is the history of his dynasty that is influenced by positive shocks. However, the insurance scheme is characterized by amnesia if a negative shock hits, which means that the past income realizations of the constrained generations will not affect the consumption of currently living generations. Furthermore, stronger aggregate shock on income improves welfare by reducing the consumption inequality among generations that share risks under limited enforcement.

The policymaker who sets intergenerational transfers as functions of the history of past shocks cannot do better than with these consumption allocations, if the agents are free to opt out and cannot be forced to participate. However, the social planner can certainly perform worse if there are “realistic” constraints to the allocations that are implemented. We therefore also consider two relevant suboptimal cases and propose institutions that might be able to implement such schemes. In the first scenario, the policymaker can set intergenerational transfers only contingent on time and the current realization of shock. This will be defined as time-dependent allocations. In the second scenario, intergenerational transfers can be made contingent only on the current realization of shocks. This is the case for stationary allocation. In both cases, we evaluate the welfare losses in terms of equivalent consumptions compared to the optimal history-dependent scheme. The consumption equivalent welfare change measures by how much a generation's consumption (in both periods and in all states of the world) would need to change to obtain the same level of welfare as in the optimal history-dependent case. This welfare assessment will depend on the weight attached to the initial old generation.

This paper is related to several strands of literature. First, it is connected with the literature on Pareto-improving transfers in economies with intergenerational conflict and reputational concerns. In this literature, intergenerational transfers can be sustained in every period as an equilibrium by means of a chain of rewards and penalties spanning over generations (see, e.g., Hammond, 1975; Cremer, 1986; and Kandori, 1992). To study the political economy of intergenerational transfers, several papers extend this idea to a setting in which decisions are made by majority rule (see, e.g., Cooley and Soares, 1999; Azariadis and Galasso, 2002; and Rangel, 2003). In this positive approach, the link between current and future benefits arises because successive median-voters coordinate on grim trigger strategies. We depart from those models by embedding uncertainty within a pension game where transfers are centrally determined by policymakers and political supported by each voter in each time and state.

This links the model to the literature on intergenerational risk sharing in endowment economies with overlapping generations. This literature has stressed the ability of governments with full commitment to facilitate the distribution of aggregate risk across generations through time-invariant fiscal policies (see, e.g., Enders and Lapan, 1982; Shiller, 1999; Chattopadhyay and Gottardi, 1999; and Rangel and Zeckhauser, 2001).¹ These papers, however, assume that participation is mandatory and policies are fully enforceable, while we focus on voluntary participation and self-enforcing intergenerational insurance.² This leads to a completely new result—namely that the optimal allocation necessary exhibit history dependence if it is required to be sustainable, i.e., no government would ever decide to default on its obligations. This result rationalizes Gordon and Varian (1988)’s claim that to be time-consistent an optimal intergenerational risk-sharing agreement must be non-stationary.³

The paper closest to us is Ball and Mankiw (2007) who study the insurance role of social security system in a two-period partial equilibrium model in which households only consume in the last period of life. Different from us, however, the optimal allocation derived for state contingent transfers makes consumption a random walk. The reason is that full risk sharing causes each shock to be spread equally over all current and future generations. It implies that economy’s total asset will eventually be negative. Thus a government that would actually implement these policies would default in finite time with certainty. We depart from this paper by characterizing optimal intergenerational insurances that are politically implementable through voluntarily inter-vivos transfers.⁴

By analyzing an economy with limited enforceability, this paper connects methodologically to literature on participation that characterizes the optimal insurance of a dynamic game in terms of promised utilities for future consumptions. While Thomas and Worrall (1988) and Kocherlakota (1996) focus on risk sharing among a finite number of infinitely living identical agents, Thomas and Worrall (2007) and Krueger and Perri (2011) allow for a continuum of infinitely-lived agents. However, our setup differs

¹On optimal social security scheme in a stochastic economy see the review by Weil (2008).

²The introduction of political institutions to promote the emergence of an unfunded social security system in a stochastic economy are investigated by Demange (2009) and Galasso and D’Amato (2010). While these papers remove the assumption of full commitment in the government decision-making when participation is mandatory, we allow voters to voluntarily participate to the social security compact.

³It is worth mentioning the literature on intergenerational risk sharing in economies with overlapping generation and capital accumulation. This literature considers stationary environment and focusses on time invariant allocations. If agents’ welfare is evaluated at an ex-ante stage, the presence of a durable good like land provides a normative justification of a pay-as-you-go scheme (see, e.g., Bohn, 2009; Gottardi and Kubler, 2011; Olovsson, 2010). If agents’ utility is evaluated at an interim stage, however, a Pareto improvement can only be obtained if some financial markets are missing or the economy is dynamically inefficient (see, e.g., Demange and Laroque, 1999; Imrohoroglu, Imrohoroglu and Joines, 1999; Krueger and Kubler, 2006). See Demange (2002) for a formal discussion of interim and ex ante optimality.

⁴Our approach is also related to the literature the studies the effects of other forms of fiscal policy interventions like, for example, debt (see, e.g., Fisher, 1983; Gale, 1994), and capital taxation (see Smetters, 2004), on intergenerational risk sharing according to an ex ante welfare criterion.

from these contributions in that there are an infinite number of agents who live for two periods and transfers go from young to old.⁵

The paper is organized as follows. Section 2 sets up the model and defines the concept of sustainable intergenerational insurance, and the first-best benchmark case. Section 3 studies the existence and properties of the optimal intergenerational insurance and shows how the problem can be recursively formulated. It discusses the case with two states. In Section 4 we consider two restricted cases of contracts that are not fully optimal: one where intergenerational insurance can depend on the time as well as the state, but not on the previous history and another where intergenerational insurance depends only on the state, but not on time or previous history. Section 5 shows our results in a simple parametric case. Section 6 concludes. The proofs that are not in the text are reported in the Appendix.

2. The Model

We consider an infinitely-repeated, stochastic, overlapping generations model in which agents live for two periods: young and old. In each period, $t = 0, 1, 2, \dots, \infty$, agents receive a stochastic endowment of a perishable consumption good. There is no technology for savings. Endowments of the young and old agents in state $s \in \mathcal{S} := \{1, 2, \dots, S\}$ are $e^y(s)$ and $e^o(s)$. Endowments are finite and strictly positive. We assume $S \geq 2$ and denote the aggregate endowment by $e(s) := e^y(s) + e^o(s)$.⁶ We will assume that states are identically and independently distributed and denote the probability of state s by $\pi(s)$. The preferences of an agent are time separable with a common generational discount factor of $\beta \in (0, 1]$. We denote the lifetime endowment utility of an agent born in state q , who consumes $e^y(q)$ when young, as

$$\hat{v}(q) := u(e^y(q)) + \beta \hat{\omega}$$

where $\hat{\omega} := \sum_{s=1}^S \pi(s)u(e^o(s))$ is the expected endowment utility of the old and u is the common utility function that is strictly increasing, strictly concave and satisfies the Inada condition $\lim_{c \rightarrow 0} u_c(c) = \infty$.⁷

⁵In the absence of intergenerational conflicts, Alvarez and Jermann (2000), Kehoe and Levine (2001), and Krueger and Perri (2006) study how an increase in income volatility affects consumption inequality within groups that share income risks. Like us, they conclude that higher income volatility reduces the incentives to default and improves risk sharing.

⁶We are focussing on intergenerational transfers and therefore assume that there is a representative agent in each generation. We ignore questions about inequality within generations and how this changes over time.

⁷We use subscripts on functions to indicate derivatives.

All information about endowments is public: there is complete information. We let

$$\lambda(s) := \frac{u_c(e^y(s))}{u_c(e^o(s))}$$

denote the ratio of marginal utilities of the young and old at the autarky values where agents consume just their endowments. If $\lambda(s) > 1$, then it follows, from that the assumption that the utility of consumption is independent of age, that $e^y(s) < e^o(s)$ and the old are wealthier in that state. We order states such that $\lambda(s) \geq \lambda(s-1) \geq \dots \geq \lambda(1)$. Thus, a higher state indicates that the old are relatively wealthy compared to young compared to a lower state.⁸

Since the $\lambda(s)$ varies across states, it is desirable to share risk across generations. We will also assume that $\lambda(1) < \beta/\delta$, which will ensure that there is at least one state in which it is desirable to transfer consumption from the young to the old. Since there is no storage or savings the only possibility for risk-sharing is via intergenerational transfers between young and old agents. However, we assume that all transfers must be voluntary. That is, agents will only make a transfer if making a transfer is in their interest.⁹ It follows that the old will never make a transfer to the young. Indeed, the old have no future and, if they are required to pay for transfers to younger generations, they would default on their obligations. However, the young may make a transfer to the old in state s if the loss in utility when young is sufficiently compensated by the transfers that are expected to be received when old. We can interpret such transfers as a public pay-as-you-go pension plan.¹⁰

The economy starts at date $t = 0$ with an initial old and initial young generation. At every subsequent date, $t = 1, 2, \dots, \infty$ a new generation of young are born.¹¹ We consider the problem of a fictitious and benevolent social planner who chooses intergenerational insurance, that is, consumption of the young and old, to maximize expected discounted welfare. We assume that the planner discounts the utility of future generations at a geometric discount factor of $\delta \in (0, 1)$.¹² The consumptions chosen may depend on the current and past history of states. We denote $s_t \in \mathcal{S}$ as the state at date $t = 0, 1, 2 \dots \infty$ and $s^t := (s_0, s_1, \dots, s_t) \in \mathcal{S}^t$ as the history of states up to and including date t . The probability of reaching history s^t is denoted $\pi(s^t)$ where $\pi(s^{t-1})\pi(s_t | s^{t-1}) = \pi(s^{t-1}, s_t)$. Since states are assumed to be i.i.d., $\pi(s^t) = \pi(s_0) \dots \pi(s_t)$ where $\pi(s_t)$ is the probability of state s_t and $\pi(s_{t+1} | s^t) = \pi(s_{t+1})$. We let $c^y(s^t)$ denote the consumption of the young

⁸Where two states have the same value of λ , we will use the convention that the states are ordered by the aggregate endowment (higher states associated with higher aggregate endowment).

⁹We ignore any altruistic motives for making transfers.

¹⁰The assumption that the transfer is voluntary can be interpreted as a plan that requires support of both generations if put to a vote.

¹¹Since time is infinite, the initial old are quite special in being the only generation that lives for just one period.

¹²Such an objective is common in the literature, see, e.g., Farhi and Werning, 2007.

after history s^t and $c^o(s^t)$ denote the consumption of the old after history s^t .

The constraints on the planner's choice are threefold. First, there is the aggregate constraint that aggregate consumption cannot exceed the total endowment:

$$(2.1) \quad c^y(s^{t-1}, s_t) + c^o(s^{t-1}, s_t) \leq e(s_t) \quad \forall (s^{t-1}, s_t).$$

Next is the constraint that the utility of the old exceeds the utility they would get by consuming their endowment:

$$(2.2) \quad u(c^o(s^{t-1}, s_t)) \geq u(e(s_t)) \quad \forall (s^{t-1}, s_t).$$

As mentioned, this constraint means that any transfer must be from the young to the old and not vice-versa. To capture this constraint we will, from now on, simply assume that the consumption of the young cannot exceed the endowment of the young: $c^y(s^{t-1}, s_t) \in [0, e^y(s_t)]$ and write $C^y(s) := [0, e^y(s)]$, or correspondingly, $c^o(s^{t-1}, s_t) \in C^o(s_t) := [e^o(s), e(s)]$. Third, there is the analogous constraint for the young generation:

$$(2.3) \quad u(c^y(s^{t-1}, s_t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1} | s^{t-1}, s_t) u(c^o(s^{t-1}, s_t, s_{t+1})) \geq \hat{v}(s_t) \quad \forall (s^{t-1}, s_t)$$

where $\hat{v}(s_t)$ denotes the endowment utility of a young agent born in state s_t .

It is clear that even with the constraint (2.3) there is the possibility of a transfer from the young to the old in some state provided the choice of consumption allocation includes some future transfers from the future young in the next period.

We first define sustainability and optimality of intergenerational insurance.

Definition 1. *An Intergenerational Insurance is a history-dependent sequence $\{c^y(s^t), c^o(s^t)\}$. It is sustainable if and only if it belongs to the set Λ where*

$$\Lambda := \{ \{c^y(s^t), c^o(s^t)\} \mid (2.1) \text{ and } (2.3), c^y(s^{t-1}, s_t) \in C^y(s_t), c^o(s^{t-1}, s_t) \in C^o(s_t) \forall (s^{t-1}, s_t) \}.$$

Definition 2. *A Sustainable Intergenerational Insurance $\{c^y(s^t), c^o(s^t)\} \in \Lambda$ is optimal if it maximizes*

$$[P1] \quad \frac{\beta}{\delta} \sum_{s_0} \pi(s_0) u(c^o(s_0)) + \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \delta^t \left(u(c^y(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1} | s^t) u(c^o(s^t, s_{t+1})) \right) \right],$$

where $\mathbb{E}_0$ is the expectation over all future histories, subject to the constraint

$$(2.4) \quad \sum_{s_0} \pi(s_0) u(c^o(s_0)) \geq \omega_0.$$

We denote the maximum value function corresponding to a solution to [P1] by $V(\omega_0)$. The constraint (2.4) requires that the planner, in making the choice at date $t = 0$ before the state s_0 is known, to offer the initial old generation an expected utility of at least ω_0 . Here ω_0 is treated as a parameter of the problem. From the constraints (2.2) it follows that $\sum_{s_0} \pi(s_0) u(c^o(s_0)) \geq \hat{\omega}$, the expected endowment utility of the old, and therefore we will consider only $\omega_0 \geq \hat{\omega}$. The weight of the planner on the initial old generation is β/δ . This is consistent with the weight placed by the planner on subsequent old generations and allows us to consider the initial old symmetrically with future generations.¹³

It is easily checked that the constraint set Λ is compact and convex. It is possible however, that the constraint set is empty for $\omega_0 > \hat{\omega}$. In this case, we set $V(\omega_0) = -\infty$. If $\omega_0 = \hat{\omega}$, then it is clear that the autarkic allocation $c^y(s^{t-1}, s_t) = e^y(s_t)$ and $c^o(s^{t-1}, s_t) = e^o(s_t)$ for all (s^{t-1}, s_t) is sustainable because it belongs to the set Λ . However, we will be interested mainly in non-autarkic allocations that are sustainable and sustainable allocations where $\omega_0 > \hat{\omega}$. We will give a simple and well-known condition for the existence of non-autarkic sustainable allocations when $\omega_0 = \hat{\omega}$ in the next paragraph. When this condition is satisfied, we can show (see Appendix) that there will exist a solution to the above maximization problem, so that $V(\omega_0)$ is finite and that the range of ω_0 where $V(\omega_0)$ is finite is a compact interval $\Omega := [\hat{\omega}, \bar{\omega}]$. Moreover, that $V(\omega_0)$ is non-increasing and concave on Ω .¹⁴

The existence of a non-autarkic sustainable allocation for $\omega_0 = \hat{\omega}$ can be addressed by considering small stationary (depending only on the state) transfers from the young starting from the endowment allocation. If there is a vector of transfers such that the marginal utility loss in making a transfer when young is compensated by the expected marginal utility gain when old, then a non-trivial intergenerational insurance is sustainable. Let $\hat{m}_{qs} := \beta \pi(s) u_c(e^o(s)) / u_c(e^y(q))$ denote the marginal rate of substitution of consumption when young in state q for consumption when old in state s , evaluated at the autarky allocation. Let $\hat{M} = (\hat{m}_{qs})$ denote the matrix of these marginal rates of substitution. It is well known (see, e.g., Aiyagari and Peled, 1991 and Chattopadhyay and Gottardi, 1999) that a non-trivial sustainable allocation satisfying the constraints (2.1), (2.2) and (2.3) exists whenever the Perron root (largest eigenvalue) of the matrix $\hat{M}$ is greater than one (see the Appendix for a proof). In this case there exists a non-negative

¹³This weight will not affect the solution when the constraint (2.4) binds.

¹⁴In particular, we will show that there is an $\bar{\omega} \in \Omega$ such that $V(\omega_0)$ is constant for $\omega_0 \leq \bar{\omega}$ and strictly decreasing and strictly concave for $\omega_0 > \bar{\omega}$. We will discuss the determination of $\bar{\omega}$ in the subsequent section.

set of transfers that improve lifetime utility of the young in each state. A simple sufficient condition for this to be true is that the Frobenius lower bound, given by the minimum row sum of $\hat{M}$, is greater than one. We shall assume this from now on.

Assumption 1. *For each state $q = 1, 2, \dots, S$*

$$\sum_{s=1}^S \hat{m}_{qs} = \beta \sum_{s=1}^S \pi(s) \frac{u_c(e^o(s))}{u_c(e^y(q))} > 1.$$

This assumption says that at the autarky allocation the young would, if they could, prefer to save for their old age even at a zero interest rate in every state.¹⁵ This is a natural assumption given that our focus is on pension provision. However, it is stronger than needed to ensure the existence of a sustainable allocation and we will be able to relax it in the two-state example we consider below. We will also give examples of when Assumption 1 is satisfied. It should be noted that Assumption 1 does not depend on δ . It will be easier to satisfy the larger is the discount factor β . If Assumption 1 is satisfied and $\omega_0 = \hat{\omega}$, then a non-trivial sustainable allocation exists.¹⁶

The next section will reformulate the programming problem in [P1] recursively and characterize optimal intergenerational insurance. We will also consider the first-best benchmark without the incentive constraints (2.3).

Definition 3. *An Intergenerational Insurance is first-best if it maximizes [P1] subject to (2.4) and $\{c^y(s^t), c^o(s^t)\} \in \Lambda^*$ where*

$$\Lambda^* := \{ \{c^y(s^t), c^o(s^t)\} \mid (2.1), c^y(s^{t-1}, s_t) \in C^y(s_t), c^o(s^{t-1}, s_t) \in C^o(s_t) \forall (s^{t-1}, s_t) \}.$$

That is the first-best is an allocation that ignores the incentive constraint of the young.¹⁷

3. Optimal Intergenerational Insurance

To understand how the optimal intergenerational insurance evolves, we reformulate [P1] recursively. This can be done because states are i.i.d. and all constraints are

¹⁵Recall, that there are no savings in this model and the good is non-storable. The only way for agents to consume more in old age is to share risk across generations.

¹⁶By continuity, if Assumption 1 is satisfied, then it will be also be satisfied for some $\omega_0 > \hat{\omega}$.

¹⁷It would be possible to define the first-best as the allocation that also ignores the incentive constraint of the old, (2.2). Although this would be natural, as a point of comparison, Definition 3 will allow us to highlight the role played by the incentive constraints of the young in generating a non-stationary allocation.

forward looking. Hereafter, unless otherwise specified, we omit t indices and use recursive notation, with primes denoting next-period variables. The expected utility promised ω is the state variable in the recursive formulation and we let $\omega'(s)$ denote the expected utility promised to the old next period when the current state is s . The constraint for the young (2.3) may be rewritten as:

$$(3.1) \quad u(c^y(s)) + \beta\omega'(s) \geq \hat{v}(s), \quad \forall s \in \mathcal{S}.$$

In each period, the expected utility given to the current old generation must be at least that previously promised:

$$(3.2) \quad \sum_s \pi(s)u(e(s) - c^y(s)) \geq \omega.$$

This constraint is analogous to constraint (2.4) in the previous section, but it is now required to hold at every date. The constraint for the old is unchanged and subsumed in the requirement that $c^y(s) \in C^y(s)$. That is, the old make no transfers. The expected utility promised to the old next period, $\omega'(s)$, is now a choice variable that must be chosen from the set Ω such that a sustainable allocation exists. It is standard that Ω is an interval.¹⁸ The lower endpoint is given by $\hat{\omega}$ where the old consume their endowment and we shall denote the upper endpoint by $\bar{\omega}$. This upper endpoint is endogenous and determined such that there is no feasible sustainable allocation for $\omega > \bar{\omega}$. In each period, we have a constraint set Γ_S that is the set of state contingent consumptions $c^y(s)$ for the young and the set of state contingent promises, $\omega'(s) \in \Omega$ such that the constraints (3.1) and (3.2) are satisfied. That is

$$\Gamma_S := \{ \{c^y(s) \in C^y(s), \omega'(s) \in \Omega\}_{s \in \mathcal{S}} \mid (3.1) \text{ and } (3.2) \}.$$

It is easily seen that the constraint set Γ_S is convex because the utility function is concave and Ω is an interval. It follows from Assumption 1 that a sustainable allocation exists and therefore that Ω and Γ_S are non-empty for $\omega \in \Omega$. The functional equation is therefore defined by:

$$[P2] \quad V(\omega) = \max_{\{c^y(s), \omega'(s)\} \in \Gamma_S} \left[\sum_{s=1}^S \pi(s) \left(\frac{\beta}{\delta} u(e(s) - c^y(s)) + u(c^y(s)) \right) + \delta V(\omega'(s)) \right].$$

The optimal solution can be solved by beginning with $\omega_0 \in \Omega$, solving for the young consumption in the initial period together with the future promised utilities. The future promised utilities are then used to resolve the problem for the next period and so on.

¹⁸It is clear that Ω is an interval because if there is a sustainable allocation for ω , then there is a sustainable allocation for ω_1 where $\hat{\omega} \leq \omega_1 < \omega$ because the constraint set is larger. It is also standard to establish that the interval Ω is closed by considering a sequence ω_n converging to ω and the corresponding sequence of allocations and showing that the limiting allocation is sustainable and delivers ω to the current old.

The optimal solution exists from the compactness of Ω and Γ_S and the consumption allocation is unique from the strict concavity of the utility function.

The function $V(\omega)$ is non-increasing in ω because a decrease in ω relaxes the constraint set.¹⁹ It can also be shown that $V(\omega)$ is concave (see Appendix). Although $V(\omega)$ cannot be found by a standard contraction mapping argument starting from an arbitrary value function (because the value function associated with the autarkic allocation also satisfies the functional equation as well as the optimal value function), a similar approach can be used to iterate on the value function starting from a function $V^*(\omega) \geq V(\omega)$. Following the arguments of Thomas and Worrall (1994), it can be shown that the limit of this iteration is the optimal value function $V(\omega)$ (see Appendix).²⁰

Let $\mu(s)$ be the multiplier associated with the constraints in (3.1), ν be the multiplier associated with the constraint (3.2) and $\eta(s)$ be the multipliers associated with the constraints $c^y(s) \leq e^y(s)$. It can be shown that the value function is differentiable (see Appendix) and hence, the first-order conditions are:

$$(3.3) \quad \chi(s) := \frac{u_c(c^y(s))}{u_c(e(s) - c^y(s))} = \frac{\frac{\beta}{\delta}(1 + \nu) + \eta(s)}{1 + \mu(s)},$$

$$(3.4) \quad V_\omega(\omega'(s)) = -\frac{\beta}{\delta}\mu(s).$$

The envelope condition is

$$(3.5) \quad V_\omega(\omega) = -\frac{\beta}{\delta}\nu.$$

The recursive formulation shows that the expected utility promised to the old is sufficient to describe the evolution of the consumption allocation and encompasses information about the past history. As already stated $V(\omega)$ is non-increasing and concave. Furthermore, it can also be shown that there is an $\tilde{\omega} \in \Omega$ such that $V_\omega(\omega) = 0$ for $\omega < \tilde{\omega}$ and $V(\omega)$ is strictly decreasing and strictly concave for $\omega > \tilde{\omega}$ and that $\tilde{\omega} > \hat{\omega}$. The evolution of the allocation starts from ω_0 . This determines the initial value of ν from equation (3.5). For each date t and each value of ω , the solutions to (3.1), (3.3) and (3.4) together with the constraint $c^y(s) \leq e^y(s)$ determine $c^y(s)$ and $\omega'(s)$ for each state s as functions of ω . If state $\hat{s}$ occurs, then the ω for the next period is equal to $\omega'(\hat{s})$.

Before discussing the solution to problem [P2], we consider the first-best benchmark

¹⁹The function $V(\omega)$ is not a Pareto-frontier. However, the function $\tilde{V}(\omega) := V(\omega) - \omega$ can be considered as a Pareto-frontier that trades off the expected utility of the current old against the planner's valuation of the expected discounted utility of all other generations.

²⁰Indeed, it is precisely this iteration starting from the first-best value function $V^*(\omega)$, defined below, that is used to compute the optimal value function and solution when we solve examples numerically in Section 5.

described in Definition 3 that ignores the young's incentive constraint. Let Γ_S^* denote the superset of Γ_S that excludes the incentive constraint (3.1). There is a corresponding first-best frontier $V^*(\omega)$. The solution to this first-best problem satisfies (3.3) with $\mu(s) = 0$ and satisfies $V_\omega^*(\omega'(s)) = 0$. It follows from this that the first-best allocation is stationary, $\omega'(s)$ independent of the state, and independent of time after the first period. In particular after the initial period, $\nu = 0$ in equation (3.3). Thus, we can define the first-best consumption for the young for periods $t \geq 1$ as $c^{y,*}(s)$ where

$$(3.6) \quad \frac{u_c(c^{y,*}(s))}{u_c(e(s) - c^{y,*}(s))} = \max \left\{ \frac{\beta}{\delta}, \lambda(s) \right\}.$$

Since $\lambda(1) < \beta/\delta$, there is a state $s^* \in \mathcal{S}$ such that for states $q > s^*$, $u_c(c^{y,*}(q))/u_c(e(q) - c^{y,*}(q)) = \lambda(q)$ and no transfer is made to the old in that state.²¹ Let $\omega^* := \sum_s \pi(s)u(e(s) - c^{y,*}(s))$ and let $v^* := \sum_s \pi(s)u(c^{y,*}(s)) + (\beta/\delta)\omega^*$. Again, since $\lambda(1) < \beta/\delta$, $\omega^* > \hat{\omega}$. With consumption determined by equation (3.6), we set $\omega'(s) = \omega^*$ for each state. Hence, the first-best frontier can be found by solving

$$V^*(\omega) = \max_{\{c^y(s)\}} \left[\sum_{s=1}^S \pi(s) \left(u(c^y(s)) + \frac{\beta}{\delta} u(e(s) - c^y(s)) \right) + \frac{\delta}{1 - \delta} v^* \right]$$

subject to (3.2). The solution to this problem determines the consumption in the initial period, $c_0^{y,*}$. For $\omega \leq \omega^*$, $c_0^{y,*}(s) = c^{y,*}(s)$. For $\omega > \omega^*$, in the first period we have

$$(3.7) \quad \frac{u_c(c^{y,*}(s))}{u_c(e(s) - c^{y,*}(s))} = \max \left\{ \frac{\beta}{\delta} (1 + \nu), \lambda(s) \right\},$$

and $\sum_s \pi(s)u(e(s) - c_0^{y,*}(s)) = \omega_0$. In this case, the young make a larger transfer to the old in the initial period compared to that made in subsequent periods and the ratio of marginal utilities varies with the state only if the constraint $c^y(s) \leq e^y(s)$ is binding (no transfer is made). It is easily checked that $V^*(\omega)$ is strictly decreasing and strictly concave for $\omega > \omega^*$ and the derivative at ω^* is zero. We summarize the solution to the first-best problem in the following proposition.

Proposition 1. *The first-best Intergenerational Insurance has, for all histories and dates $t \geq 1$, that the consumption of the young is $c^{y,*}(s)$ given by equation (3.6). Transfers to the old are made only in states $s \leq s^*$. At date $t = 0$, $c_0^{y,*}(s) = c^{y,*}(s)$ for $\omega_0 \leq \omega^*$. Otherwise, $c_0^{y,*}(s) \leq c^{y,*}(s)$ and $c_0^{y,*}(s)$ satisfies equation (3.7) and $\sum_s \pi(s)u(e(s) - c_0^{y,*}(s)) = \omega_0$.*

We will now assume that this first best allocation is not sustainable. That is, for at least one state, this first-best allocation violates an incentive constraint (3.1). In

²¹Formally, s^* is the largest state such that $u_c(c^{y,*}(s))/u_c(e(s) - c^{y,*}(s)) < \lambda(s)$. This exists because $\lambda(1) < \beta/\delta$. It is possible that $s^* = S$ and that transfers to the old are made in each state.

particular, we assume that for some state s

$$u(c^y(s)) - u(c^{y,*}(s)) > \beta(\omega^* - \hat{\omega}).$$

This condition will always hold for β small enough. Let $S^b(\omega)$ denote the set of states where the incentive constraint (3.1) binds for a given ω and let $\bar{S}^b(\omega)$ denote its complement. We are interested in the case where $S^b(\omega)$ for $\omega \geq \tilde{\omega}$ is non-empty.²² The complement set $\bar{S}^b(\omega)$ may be empty for some $\omega \geq \tilde{\omega}$. We first show that a similar property on transfers holds in the constrained case as well as in the first best. In particular, if no transfer is made in a given state, then no transfer will be made in a higher state. This is because it is apparent that the old and the young can not be simultaneously constrained (strictly positive multipliers) in the same state for a given ω if the allocation next period is non-trivial. When the allocation next period is non-trivial, the young can only be constrained if the transfer today for that state is positive. Thus, the multipliers $\mu(s)$ and $\eta(s)$ hold with complementary slackness. If the old are constrained, then both young and old consume their autarky allocation in that state and the ratio of marginal utilities in (3.3) is equal to the autarky ratio $\lambda(s)$. Hence, $\eta(s) = \lambda(s) - (\beta/\delta)(1 + \mu(s))$. It therefore follows that if there is no transfer in some state s at date t , there is no transfer in any state $q > s$, because by definition λ is non-decreasing in the state. We state this result as a separate lemma.

Lemma 1. *If at any history s^t , the optimal consumption is at the autarky allocation in some state s at date t , then the allocation is also at the autarky allocation for any state $q > s$ at date t .*

The difference to the first-best case is that the set of states in which no transfer will be made to the old will vary with ω . We can now state the main result on the constrained solution where the incentive constraint binds.

Proposition 2. (i) *For states s, q in $S^b(\omega)$ such that $\hat{v}(s) > \hat{v}(q)$, then $c^y(s) > c^y(q)$ and $\omega'(s) > \omega'(q)$. (ii) *The set $S^b(\omega)$ is monotone increasing in ω and $\omega'(s)$ is increasing in ω for any state $s \in S^b(\omega)$. (iii) *For states $s \in \bar{S}^b(\omega)$, $\omega'(s) = \tilde{\omega}$ and $\chi(s) \geq \beta/\delta$.***

These properties are quite intuitive.²³ The larger is larger value $\hat{v}$, the outside option of the young, the larger is both the consumption as well as the expected future utility offered to the young (Part (i) of Proposition 2). An increase in ω means that more has to be offered to the current old to maintain previous promises and therefore meeting the incentive constraints of the young becomes more difficult. The set of states where the

²²We can show that $\tilde{\omega} < \omega^*$.

²³The proof is given in the Appendix.

incentive constraint binds cannot contract. An increase in ω means that there must be an increase in the promised expected utility to the current young for a given state, $\omega'(s)$. The net effect of an increase in ω on $c^y(s)$ is in principle ambiguous but if the incentive constraint is binding before the change in ω , the increase in ω will cause a fall in $c^y(s)$ (Part (ii) of Proposition 2). For lower values of $\hat{v}$, the young will not be constrained and in this case the promised utility is *reset* to $\tilde{\omega}$, independently of the current state. This is an important property of optimal sustainable intergenerational insurance. In any situation in which the incentive constraint on the young does not bind, the contract is independent of past history and restarts from a promised expected utility to the old of $\tilde{\omega}$. If the incentive constraint does bind, then more must be promised to current young, when they become old, in order to induce them to make current contributions (Part (iii) of Proposition 2).

We can now describe how the insurance evolves over time. The chosen value of $\omega'(s)$ and the state determine the promised expected utility of the old next period. Thus, the value of $\mu(s)$ and the state s determine the subsequent value of ν , that is $\nu' = \mu(s)$ if state s occurs. Hence, ν and therefore ω increases if $\mu(s) > \nu$ and state s and ω decreases if state s occurs and $\mu(s) < \nu$. This allows us to conclude the following. Suppose that the same state recurs for two periods in succession. If in the first period $\mu(s) > \nu$, and the same state reoccurs, then in the following period $\mu(s)$ (at the second date) will be higher. That is $c^y(s)$ and ω' will increase. Likewise, if in the first period $\mu(s) < \nu$, then $c^y(s)$ and $\omega'(s)$ decline in the second period. This will be useful when we come to consider the two state case, where the repetition of an individual state will be important in characterizing the solution.

Two State Case

One of the complications of understanding the optimal allocation is that the set of states for which the incentive constraint binds changes with ω and therefore changes over time (as shown in Proposition 2 the set of states for which the incentive constraint binds will increase with ω). Even where this does not happen, the dynamics of the optimal allocation are still non-trivial. To examine this case, we will now consider a simple two state case and make the following simplifying assumption.

Assumption 2. (i) *In state 1 a movement toward the first-best involves a transfer from the young and in state 2 a movement toward the first-best involves a transfer from the old, that is, $\lambda(1) < \beta/\delta < \lambda(2)$; Moreover: (ii) the young generation are never constrained in state 2; (iii) the old generation are never constrained in state 1.*

Assumption 2(i) means that in autarky the old are relatively wealthy compared to young in state 2: $\lambda(1) < \lambda(2)$. Assumptions 2(ii) and (iii) are assumptions on endogenous properties of the optimum allocation. In terms of the notation used above,

Assumptions 2(ii) requires that $S^b(\omega) = \{1\}$ and $\bar{S}^b(\omega) = \{2\}$ for all ω . Despite these assumptions on endogenous properties, we will show in the numerical solutions that there are parameter configurations where these assumptions hold.

Given Assumptions 2(ii) and (iii), the multiplier on the constraint of the young in state 2 is zero and the multiplier on the constraint of the old in state 1 is zero. Therefore, we can simplify notation by letting μ refer to the multiplier on the young's constraint in state 1 and η refer to the multiplier on the constraint of the old in state 2. For further notational convenience, let π denote the probability of state 1 (consequently the probability of state 2 is $1 - \pi$). With Γ_2 as the constraint set in this two-state case, the functional equation can then be written as

$$V(\omega) = \max_{\{(c^y(s), \omega'(s))_{s=1,2}\} \in \Gamma_2} \pi \left(\frac{\beta}{\delta} u(e(1) - c^y(1)) + u(c^y(1)) + \delta V(\omega'(1)) \right) \\ + (1 - \pi) \left(\frac{\beta}{\delta} u(e(2) - c^y(2)) + u(c^y(2)) + \delta V(\omega'(2)) \right).$$

The first-order conditions can be also written as

$$(3.8) \quad x := \frac{u_c(c^y(1))}{u_c(e(1) - c^y(1))} = \frac{\beta}{\delta} \frac{1 + \nu}{1 + \mu},$$

$$(3.9) \quad y := \frac{u_c(c^y(2))}{u_c(e(2) - c^y(2))} = \frac{\beta}{\delta} (1 + \nu) + \eta = \max\left\{\frac{\beta}{\delta} (1 + \nu), \lambda(2)\right\},$$

$$(3.10) \quad V_\omega(\omega'(1)) = -\frac{\beta}{\delta} \mu,$$

$$(3.11) \quad V_\omega(\omega'(2)) = 0.$$

Together with the two constraints

$$\pi u(e(1) - c^y(1)) + (1 - \pi) u(e(2) - c^y(2)) \geq \omega, \\ u(c^y(1)) + \beta \omega'(1) \geq \hat{v}(1),$$

there are six equations in six unknowns $\{c^y(1), c^y(2), \omega'(1), \omega'(2), \mu, \nu\}$. There is also an envelope condition (3.5). It is immediate that $y \geq \beta/\delta$ and $x \leq y$.²⁴ However, $x \geq \beta/\delta$ as $\nu \geq \mu$ (we will argue in the next paragraph that $x \leq \beta/\delta$). It is clear from (3.11) that the promised utility for the old reverts to a constant level $\tilde{\omega}$, the largest value of ω such that $V_\omega(\omega) = 0$, whenever state 2 occurs. This resetting condition shows that the evolution of the optimal allocation will depend solely on the number of consecutive state 1s in the recent history. The evolution of x (in state 1) and y (in state 2) and ω is illustrated in Figure 1. The diagram illustrates how the ratio of marginal utilities, x and y , and the expected utility promised to the old ω' depend on the history of the states. For example, $x(1, 1)$ indicates the value of the ratio of marginal utilities in state 1 in period 3 after the history of state 1 at both dates 1 and 2, and $\omega'(1, 1, 1)$ is the

²⁴We will use x^* and y^* to designate the first-best values.

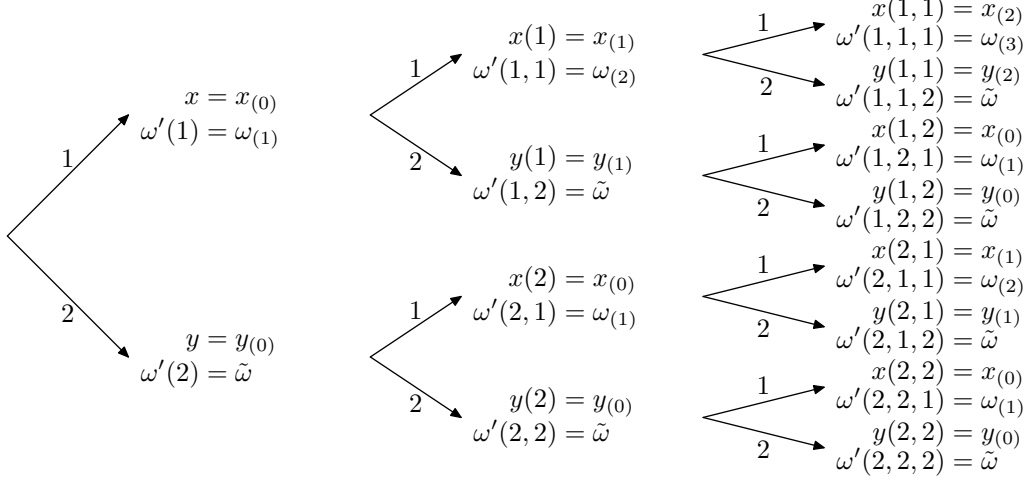


Figure 1: Illustration of Two-State Case

promised expected utility of the old after the same history. We know from the recursive structure that the allocation depends only on the state and the promised utility in the previous period. Also from Assumption 2 we know that the promised utility is reset to $\tilde{\omega}$ after every state 2 occurrence. We assume that the initial promised utility is $\tilde{\omega}$, which implies $\nu_{(0)} = 0$ and therefore that $y = \max\{\beta/\delta, \lambda(2)\}$ (for simplicity Figure 1 illustrates the case where $\lambda(2) \leq \beta/\delta$). Every time that state 2 occurs, the evolution of the intergenerational insurance is the same as starting from the initial date. Therefore, the allocation depends only on the number of consecutive state 1's that have occurred in the most recent history. This reduces the complexity of the allocation completely and it is only necessary to keep track of the number of consecutive 1's in the most recent history. For example, $x_{(2)}$ is the ratio of marginal utilities in state 1 after two previous state 1s and $\omega_{(3)}$ is the value of the promised expected utility of the old in the same history. The allocation is therefore determined by the sequence $\omega_{(n)}$ for $n > 0$ from $\omega_{(0)} = \tilde{\omega}$.²⁵

Consider then starting with an initial promised expected utility of $\tilde{\omega}$. Let $\mu_{(n)}$ denote the multipliers associated with n consecutive state 1s (corresponding to the sequence $\omega_{(n)}$). By assumption $\nu_{(0)} = 0$ and $\mu_{(0)} > 0$. Thus $\mu_{(0)} > \nu_{(0)}$, $x_{(0)} < \beta/\delta$ and $y_{(0)} \geq \beta/\delta$. As discussed above, if state 1 is repeated $\mu_{(1)} > \mu_{(0)}$ meaning a greater expected utility must be offered to the old if state 1 is repeated. By induction, the sequence $\mu_{(n)}$ is strictly increasing in n and hence $y_{(n)}$ is increasing in n (is constant when the non-negativity constraint binds and strictly increasing when the non-negativity constraint does not bind). It also follows that the ratio $x_{(n)}/y_{(n)} = 1/(1 + \mu_{(n)}) < 1$ is decreasing in

²⁵We use the notation $\omega_{(n)}$ to emphasize that n is not time, but it does indicate a length of period of state 1s in the recent history. The intergenerational insurance going forward for, e.g., $n = 5$ whether $t = 5$ or $t = 5000$, say.

n whenever the young make a positive transfer.²⁶ It also follows that $x_{(n)} < \beta/\delta \leq y_{(n)}$. Since $x_{(n)} < y_{(n)}$, it follows that if utility exhibits constant relative risk aversion, then the share of young consumption in state 2 is always less than the share of young consumption in state 1, that is $c_{(n)}^y(2)/e(2) < c_{(n)}^y(1)/e(1)$ for any n . We can measure the degree of risk-sharing across states by $\vartheta_{(n)} := (x_{(n)}/y_{(n)} - \lambda(1)/\lambda(2))/(x^*/y^* - \lambda(1)/\lambda(2))$. When risk-sharing is at the first-best level, $x_{(n)} = x^*$ and $y_{(n)} = y^*$, then $\vartheta_{(n)} = 1$. When no risk is shared across states, $x_{(n)} = \lambda(1)$, $y_{(n)} = \lambda(2)$ and $\vartheta_{(n)} = 0$. It is possible to show that $\vartheta_{(n)}$ is increasing in n when the young make no transfer in state 2 and strictly decreasing whenever the young make a strictly positive transfer in state 2. That is risk-sharing declines with a longer sequence of good shocks for the young. This can be summarized by

Proposition 3. *Intergenerational Insurance is determined by the number of consecutive state 1s in the most recent history. Starting from $\omega_{(0)} = \bar{\omega}$, the sequence $\omega_{(n)}$ is increasing in n . The corresponding ratio of marginal utilities satisfies $x_{(n)} < \beta/\delta \leq y_{(n)}$ with $\vartheta_{(n)}$ increasing in n when the young make no transfer in state 2 and strictly decreasing in n whenever the young make a strictly positive transfer in state 2.*

Long-run Implications of Optimal Allocations

Now we examine the convergence properties in the two-state case. The sequence $\omega_{(n)}$ is increasing and bounded above so will converge to a limit $\bar{\omega}$. In the long-run $x_{(n)}$ converges to β/δ such that $c^y(1)$ converges to $c^{y,*}(1)$, the first-best allocation for state 1, and hence, $\omega_{(n)}$ converges to

$$\bar{\omega} = \frac{1}{\beta} (\hat{v}(1) - u(c^{y,*}(1)))$$

As just described the optimal allocation depends on the number of consecutive state 1's in the most recent history. We can now identify a state with the length of the most recent string of state 1s. The evolution of these strings is given by the following transition matrix:

$$P_{nn'} = \begin{cases} 1 - \pi & \text{if } n' = 0 \text{ for } n = 0, 1, 2, \dots, \\ \pi & \text{if } n' = n + 1 \text{ for } n = 0, 1, 2, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

That is $P_{nn'}$ gives the probability of moving from a string of length n to a string of length n' . There are two possibilities: either state 2 occurs (with probability $1 - \pi$) and the new string length of 1s becomes $n' = 0$, or state 1 occurs (with probability π) and the string length increase by one, $n' = n + 1$.

²⁶In the example we compute below $x_{(n)}$ is also increasing in n .

At any date $T \geq 1$, the probability of having a string of length n is given by the probabilities

$$\rho_n^T = (1 - \pi)\pi^n, \quad \text{for } n = 0, 1, 2, 3, \dots, T - 1, \quad \text{and} \quad \rho_T^T = \pi^T.$$

Since the probability of state 2 is $1 - \pi$, the probability that the expected promised utility is to be reset to $\tilde{\omega}$ is $1 - \pi$ irrespective of the date or the history. The probability therefore that in state 2, consumption is set to the first-best (or that no transfer is made) $y = y_{(0)} = \max\{\beta/\delta, \lambda(2)\}$ is $(1 - \pi)$ irrespective of the date or history. Likewise, at date T the probability of the outcome $x_{(n)}$ conditional on state 1 is $(1 - \pi)\pi^n$ for $n = 0, 1, \dots, T - 2$ and is π^n for $n = T - 1$, and the probability of the outcome $y_{(n)}$ conditional on state 2 is $(1 - \pi)\pi^n$ for $n = 0, 1, \dots, T - 2$ and is π^n for $n = T - 1$.

Proposition 4. *The sequence $\omega_{(n)}$ converges to $\bar{\omega}$ as $n \rightarrow \infty$. The long-run distribution of $\omega_{(n)}$ is given by the probability distribution*

$$\rho_n = (1 - \pi)\pi^n, \quad \text{for } n = 0, 1, 2, 3, \dots$$

That this is the long-run distribution is easily checked because in this stationary state, the probability of having $n = 0$ (where the last state is state 2) is $(1 - \pi) \sum_{n=0}^{\infty} \rho_n = (1 - \pi) = \rho_0$ and the probability of being in state $n \geq 1$ is $\pi\rho_{n-1} = \pi(1 - \pi)\pi^{n-1} = \rho_n$.²⁷

4. Suboptimal Intergenerational Insurance

Above, we allowed the social planner to set intergenerational transfers as arbitrary functions of the history of past shocks. Thus, the social planner cannot do better than with these allocations, if the agents are free to opt out and cannot be forced to participate. However, the social planner can certainly perform worse if there are constraints to the allocations that are implemented. In this section we consider two relevant cases. The first in which the social planner can set intergenerational transfers only contingent on time and the current realization of shock. This will be defined time-dependent allocation. The second in which intergenerational transfers can be made contingent only on the current realization of shocks. In this case intergenerational insurance is restricted to be stationary. In the numerical section, we will evaluate the welfare losses in terms of certainty-equivalent consumption levels generated by the implementation of such constrained allocations as compared with the long-run optimal allocation characterized in the previous section. For simplicity, we will continue to consider the two state case.

²⁷In our numerical computations, we will approximate the stationary distribution by taking the finite case but with T large and simulating a large number of different paths.

Time-Dependent Solution

Consider first a time-dependent solution where the consumption values depend only on time and the current realization of state, (t, s) , but are not otherwise allowed to depend on past history. The objective function is then equivalent to [P1] under the restriction that $c^y(s^{t-1}, s_t)$ is independent of s^{t-1} . The consumption of the young in state s at date t is denoted $c_t^y(s)$. As above, the planner promises to the initial old generation an expected utility at least equal to ω_0 :

$$(4.1) \quad \pi u(e(1) - c_0^y(1)) + (1 - \pi) u(e(2) - c_0^y(2)) \geq \omega_0.$$

Furthermore, both young and old must participate voluntarily to the risk sharing agreement. Under Assumption 2(ii) and (iii), only the following two constraints must be considered:

$$(4.2) \quad u(c_t^y(1)) + \beta (\pi u(e(1) - c_{t+1}^y(1)) + (1 - \pi) u(e(2) - c_{t+1}^y(2))) \geq \hat{v}(1) \quad \forall t,$$

and

$$(4.3) \quad u(e(2) - c_t^y(2)) \geq u(e(2) - e^y(2)) \quad \forall t.$$

The scaled multiplier associated with these constraints are $(\beta/\delta) \nu_0$, $\pi \delta^t \mu_t$, and $(1 - \pi) \delta^t \eta_t$, respectively. Therefore, the solution of the optimization program involves the following set of first-order conditions:

$$(4.4) \quad x_0 := \frac{u_c(c_0^y(1))}{u_c(e(1) - c_0^y(1))} = \frac{\frac{\beta}{\delta} (1 + \nu_0)}{1 + \mu_0},$$

$$(4.5) \quad y_0 := \frac{u_c(c_0^y(2))}{u_c(e(2) - c_0^y(2))} = \max \left\{ \frac{\beta}{\delta} (1 + \nu_0), \lambda(2) \right\},$$

and

$$(4.6) \quad x_t := \frac{u_c(c_t^y(1))}{u_c(e(1) - c_t^y(1))} = \frac{\beta}{\delta} \frac{1 + \pi \mu_{t-1}}{1 + \mu_t},$$

$$(4.7) \quad y_t := \frac{u_c(c_t^y(2))}{u_c(e(2) - c_t^y(2))} = \max \left\{ \frac{\beta}{\delta} (1 + \pi \mu_{t-1}), \lambda(2) \right\}.$$

These first-order conditions are similar to (3.8) and (3.9), where μ_{t-1} replaces the multiplier associated with the promise-keeping constraint in the history-dependent case. Here the probability π stands from the fact that at time t the case where in the previous history s^{t-1} the last shock was $s = 1$, so that the previous generation of young was constrained in its participation, is occurring with probability π .

To determine the solution, we can proceed by iteration provided that the equilibrium level of the first-period multiplier μ_0 is determined. Indeed, given μ_0 , for any level of ω_0 ,

and therefore ν_0 , we can determine the initial allocation of consumption $c_0^y(s)$. Then, we simply use the first-order conditions (4.6) and (4.7) together with constraints (4.2) and (4.3) to obtain the time-dependent consumption allocation in each state $c_t^y(s)$ and the equilibrium multiplier μ_t as a function of the previous period multiplier μ_{t-1} . In the long run, the solution is converging to a stationary allocation characterized by $\bar{\mu} := \lim_{t \rightarrow \infty} \mu_t$ and constant marginal rates of substitution. Along the transition path, the sequence of multipliers is increasing over time if the expected utility promised to the initial old is sufficiently low. Such a sequence is instead decreasing when ω_0 is high enough. A shooting method can then be developed to determine the initial equilibrium level μ_0 . It can be shown that for any initial level ω_0 , the sequence $\{x_t, y_t\}_{t=0}^\infty$ is converging to the long-run allocation along a saddle path. All this is proved in the following proposition.

Proposition 5. *For any ω_0 , the system of equations (4.4), (4.5), (4.6), and (4.7) generates a sequence $\{x_t, y_t\}_{t=0}^\infty$, which converges along a saddle path to the following values*

$$(4.8) \quad \bar{x} := \frac{\beta}{\delta} \left(\frac{1 + \pi \bar{\mu}}{1 + \bar{\mu}} \right); \quad \bar{y} := \max \left\{ \frac{\beta}{\delta} (1 + \pi \bar{\mu}), \lambda(2) \right\}.$$

Stationary Solution

In this subsection we restrict attention to stationary allocations. This means that whenever the same state occurs, the same consumption allocation is chosen and the past history is irrelevant. Hence, for simplicity, it will be now denoted $c^y(s)$. The stationary allocation is clearly suboptimal, since it does not allow to spread current shocks over future generations. However, the existing literature on optimal intergenerational risk sharing has mainly focused on stationary allocations for reasons of tractability. The aim of this section is therefore to characterize the stationary sustainable intergenerational insurance to evaluate the welfare loss associated with such an agreement when compared to the optimal insurance scheme.

As mentioned above a non-trivial stationary contract exists if the matrix of the agent's marginal rate of substitutions evaluated at the autarky allocation has a maximal eigenvalue greater than 1. With two states and shocks that are i.i.d. the largest eigenvalue is given by the sum of the diagonal elements. It then follows that a stationary equilibrium with intergenerational risk-sharing exists if and only if:

$$(4.9) \quad \frac{\beta\pi}{\lambda(1)} + \frac{\beta(1-\pi)}{\lambda(2)} > 1.$$

If condition (4.9) is satisfied, the optimum stationary solution can be found again by maximizing the value [P1] subject to the constraints (4.1), (4.2), and (4.3), and the additional restriction that the consumption allocation $c^y(s^{t-1}, s_t)$ is independent of s^{t-1}

as well as t . The solution then involves the following system of first-order conditions:

$$(4.10) \quad x := \frac{u_c(c^y(1))}{u_c(e(1) - c^y(1))} = \frac{\beta}{\delta} \frac{1 + \nu_0 - \delta(\nu_0 - \pi\mu)}{1 + \mu},$$

$$(4.11) \quad y := \frac{u_c(c^y(2))}{u_c(e(2) - c^y(2))} = \max \left\{ \frac{\beta}{\delta} (1 + \nu_0 - \delta(\nu_0 - \pi\mu)), \lambda(2) \right\},$$

together with the complementary slackness conditions on the possibly binding incentive constraints

$$u(c^y(1)) + \beta(\pi u(e(1) - c^y(1)) + (1 - \pi)u(e(2) - c^y(2))) \geq \hat{v}(1)$$

and

$$u(e(2) - c_t^y(2)) \geq u(e(2) - e^y(2)),$$

and on the promise-keeping constraint to the generation of initial old:

$$\pi u(e(1) - c^y(1)) + (1 - \pi) u(e(2) - c^y(2)) \geq \omega_0$$

with $\pi\mu/(1 - \delta)$, $\pi\eta/(1 - \delta)$, and $(\beta/\delta)\nu_0$ equal to the scaled multipliers associated with such constraints, respectively.

Proposition 6. *There always exists a level of ω_0 , which implies $\nu_0 = \pi\mu$, that yields a stationary solution $\{x, y\}$ equal to the saddle point (4.4) of the time-dependent solution.*

$$x = \frac{\beta}{\delta} \left(\frac{1 + \pi\bar{\mu}}{1 + \bar{\mu}} \right); \quad y = \max \left\{ \frac{\beta}{\delta} (1 + \pi\bar{\mu}), \lambda(2) \right\}.$$

The proposition above comes directly by inspecting the first-order conditions (4.10) and (4.11). For each level of ω_0 , we determine a different stationary consumption allocation. Such an allocation $c^y(s)$ is larger than the long-run allocation of the time-dependent solution whenever $\nu_0 < \pi\mu$ and smaller otherwise. Quantitatively, a stationary allocation and a time-dependent one differ mainly along the transition path. However, such a difference is small if the convergence of the time-dependent solution toward the stationary allocation occurs in a few periods, which is what we typically find in the examples we have computed.

5. Computed Example

In this section we solve numerically for the optimal and suboptimal intergenerational insurance allocations. We provide a parametric example for which Assumption 2 is

satisfied and illustrate how policy rules, dynamic properties and welfare are affected by the implementability of each of the allocations considered.

We assume that the common utility function is of the CRRA form with coefficient of relative risk aversion equal to 1, i.e., $u(c) = \ln(c)$, and set the common discount factor $\beta = 0.97$ and planner's discount factor $\delta = 0.95$. The economy has two states $\mathcal{S} := \{1, 2\}$ with the probability of state 1 equal to $\pi = 3/5$. Total endowments in the two states are equal and normalized to one. In $s = 1$, the young has a share α of total endowment, while in $s = 2$, the young's share of the endowment is $\alpha - \sigma$. We consider the case with $\alpha = 0.7$ and $\sigma = 0.2$, so that in state 1 the young are relatively wealthier than the old compared to state 2, namely $\lambda(1) = 3/7$ is smaller than $\lambda(2) = 1$. Hence, Part (i) of Assumption 2 is satisfied. Furthermore, the autarky level of intertemporal utility is larger in state 1 compared to the level in state 2, i.e., $\hat{v}(1) > \hat{v}(2)$. In what follows, we will therefore refer to rich young as the young born in state 1 and poor young as the young born in state 2. Table 1 summarizes the parameters.

Table 1: Parameters Values

π	3/5	Probability of Realization of State 1
α	0.7	Total Endowment Share to Young in State 1
$\alpha - \sigma$	0.5	Total Endowment Share to Young in State 2
β	0.97	Common Discount Factor
δ	0.95	Planner's Discount Factor

Note that Assumption 1 is satisfied: $\beta \sum_{s=1}^2 \pi(s) \frac{u_c(e^o(s))}{u_c(e^y(1))} \approx 1.9012 > 1$ and $\beta \sum_{s=1}^2 \pi(s) \frac{u_c(e^o(s))}{u_c(e^y(2))} \approx 1.3580 > 1$, so that in both states the young would like to save if they could. The equilibrium allocation also satisfies Parts (ii) and (iii) of Assumption 2, a result that we obtain in our numerical solutions in wide ranges of parameters.

Optimal Allocation

We now present the numerical solution for the optimal intergenerational insurance. The algorithm uses dynamic programming techniques to evaluate the set of state-dependent policies $\omega'(s)$ and $c^y(s)$ in each of the two states of the world.²⁸

Figure 2 plots the optimal policy rules $\omega'(s)$ for each s as well as the dynamics of $\omega_{(n)}$. Consider first the policy rules reported in Panel (a) where the dotted line is the 45-degree line. The solid blue line represents $\omega'(1)$ and the dashed red line depicts $\omega'(2)$ as functions of the state variable ω . The future promised utility $\omega'(2)$ is constant and equal to $\tilde{\omega} = -0.7046$, where $\tilde{\omega}$ maximizes the value $V(\omega)$ and corresponds to the largest

²⁸See the Appendix for a detailed description of the implemented algorithm.

level of ω at which the promise-keeping constraint does not bind. The policy $\omega'(1)$ is instead an increasing function of ω and converges monotonically to the steady-state level $\bar{\omega} = -0.642$, occurring after an infinite sequence of state 1s. The level $\bar{\omega}$ is the largest attainable ω for which multiplier ν equals multiplier μ . The set $\Omega := [\hat{\omega}, \bar{\omega}]$ is then a compact interval with lower endpoint equal to the autarky level $\hat{\omega} = -0.9996$. The equilibrium consumption generated by these policies exhibits positive correlation with the individual endowment.

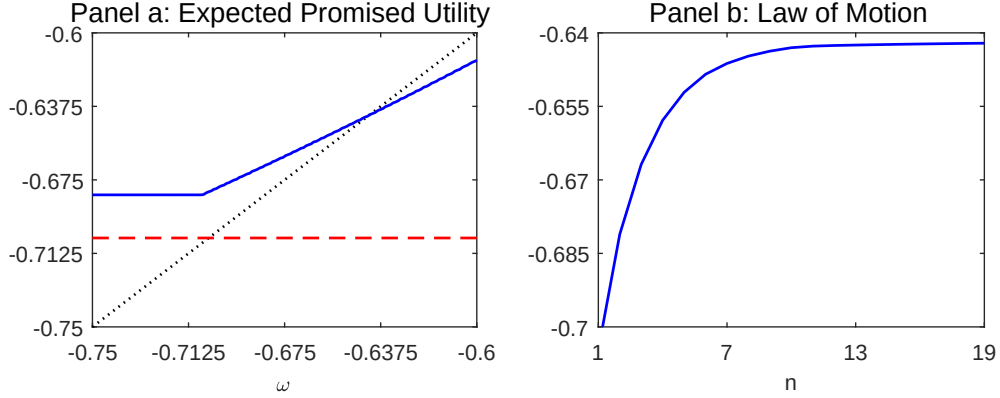


Figure 2: Policy Rules and Dynamics of Future Promised Utilities

In our example, the participation constraint of rich young binds for any level of ω , while the one of poor young turns out to be slack in the set Ω . This implies that the planner must promise more to the young born in state 1 compared to the young born in state 2, i.e., $\omega'(1) > \omega'(2)$, in order to have them participating to the risk-sharing agreement. To understand better the distortions arising under limited commitment, suppose for a moment that the young's participation constraints were slack in either state for ω low enough so that also the promise keeping constraint does not bind. Then, the first-best allocation would be implemented by simply promising $\omega'(s) = \omega^*$ in both states, where $\omega^* = -0.6828$ is the level of ω maximizing the first-best Pareto frontier $V^*(\omega)$. When instead young participation binds in state 1, the planner must discriminate between poor and rich young. This can be optimally implemented by extracting the largest rent from unconstrained agents, namely the old—since the promise keeping constraint does not bind—and the poor young. As a result, for any $\omega \leq \tilde{\omega}$ the optimal policies are such that $c^y(1) > c^{y,*}(1)$ and $\omega'(1) > \omega'(2)$ with $\omega'(2) = \tilde{\omega} < \omega^*$.

How $\omega'(1)$ changes with ω for $\omega > \tilde{\omega}$ then depends on the bindingness of the promise keeping and the old participation constraints. When $\omega > \tilde{\omega}$, the promise-keeping constraint binds. An increase in ω leads to an increase in ν and to more transfers to be paid to current old in order to keep past promises. This in turn makes participation of constrained young more difficult unless the planner also promises them a larger future

utility $\omega'(1)$. The magnitude of such an increase in $\omega'(1)$ depends on η . It can be that for small level of ω , the old should be required to transfer part of their endowment to the young if state 2 occurs, i.e., $\eta > 0$. In this case, the promise-keeping constraint is satisfied only if $c^y(1)$ decreases when ω rises. When instead the level of the past promised utility is so large that the old can consume more than the individual endowment in both states, $c^y(2)$ can be reduced in tandem with $c^y(1)$ to keep past promises. This in turn implies that the future promised level $\omega'(1)$ must increase by a smaller magnitude when old receive transfers in both states compared to the case with $\eta > 0$. Our parametric case is characterized by having $\eta = 0$ for each ω .

In Panel (b) we depict the law of motion of $\omega'(1)$, which is a monotonic increasing function of the number of consecutive state 1s, n . The function is denoted by $\omega_{(n)}$ with initial condition $\omega_{(0)} = \tilde{\omega}$ so that $\nu_{(0)} = 0$ and $\mu_{(0)} > \nu_{(0)}$. The panel also illustrates the dynamics of the contract since any time state 2 occurs, the allocation resets to the initial value $\omega_{(0)} = \tilde{\omega}$.

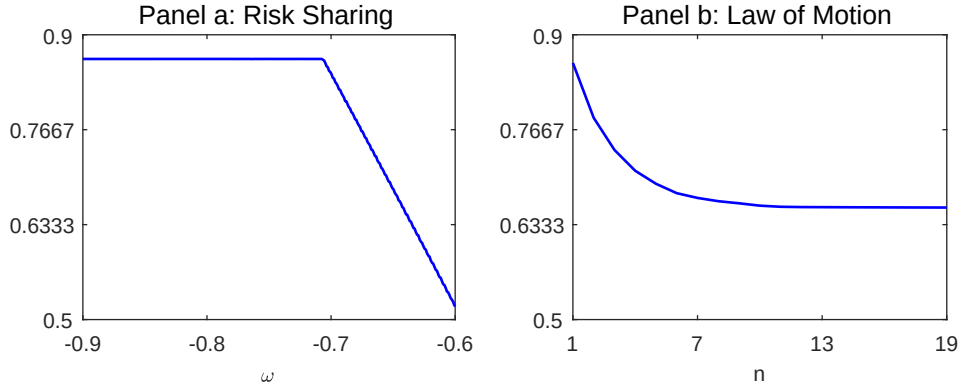


Figure 3: Policy Rules and Dynamics of Risk Sharing

Panel (a) of Figure 3 plots the intergenerational marginal rate of substitution in state 1 over the one in state 2 relative to the first-best ratio, i.e., $\vartheta = [x/y - \lambda(1)/\lambda(2)] / [x^*/y^* - \lambda(1)/\lambda(2)]$. This function provides a measure of intergenerational insurance. Full intergenerational insurance would require the function to be equal one, while no risk sharing implies that the function is equal zero. The optimal allocation under limited commitment is however characterized by partial intergenerational insurance, whose extent varies with ω . Risk sharing is at a constant level when the promise-keeping constraint is slack and decreases in ω under a binding promise-keeping constraint if young make a positive transfers in both states. The fact that risk sharing decreases with ω comes from the observation that a higher ω is associated with a larger number of consecutive states 1s, i.e., a larger n . Hence, a higher ω implies that the current young has inherited a longer sequence of positive shocks from his dynasty and must now transfer a larger share of endowment to

current old, independently of states, to keep past promises. As a consequence, while the intergenerational marginal rates of substitution increase in both states, y increases more than x . In parametric cases where $\eta > 0$ for small enough values of the state variable, risk sharing would be increasing for these values of ω due to the artifact that transfers would be set to zero when state 2 would occur.

Panel (b) illustrates the corresponding dynamics of risk sharing as a function n . The function $x_{(n)}$ increases in n and eventually converges to the first-best level $\beta/\delta = 1.0211$. The function $y_{(n)}$ also increases in n but at a rate higher than that of $x_{(n)}$ and ultimately converges to the level 1.27, which is larger than β/δ . The shape of the dynamics of risk sharing simply reflects the dynamics of these functions.

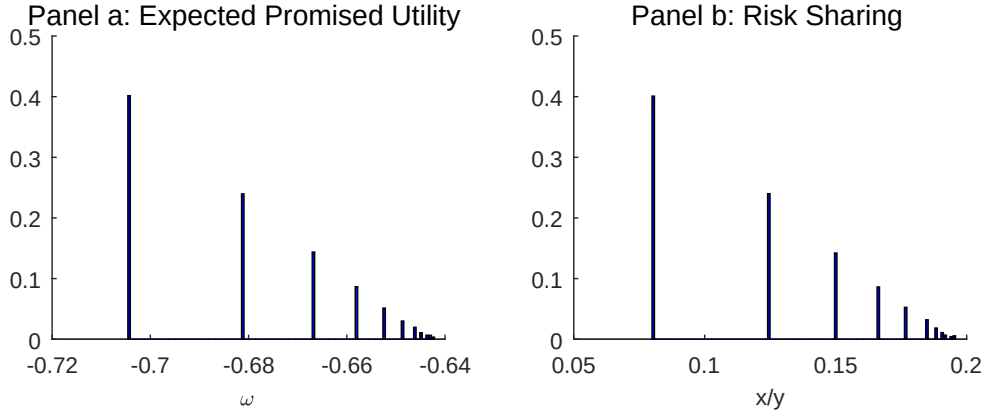


Figure 4: Long Run Distribution

We have shown that in the two state case, there is a limiting stationary distribution for the promised utility ω . In Figure 4, we draw the distribution of $\omega_{(n)}$ and the corresponding risk-sharing measure obtained by performing 20,000 simulated paths of an economy lasting 1000 periods starting from initial condition $\omega_{(0)} = \tilde{\omega}$. This confirms the results on the long-run properties of the optimal allocation reported in Proposition 4, namely the fact that the probability of realization of $\omega_{(n)}$ is $(1 - \pi) \pi^n$ for any number of consecutive state 1s $n = 0, 1, 2, \dots$

Suboptimal Allocations and Welfare

We now present simulation results related to the suboptimal intergenerational insurance schemes. For the time-dependent allocation, the relevant state variables are the time t and the current realized state s . The time-dependent allocation converges along a saddle path to a stationary allocation. The phase diagram is depicted in Figure 5.

Starting from an initial promised utility such that $\nu_0 = 0$, the sequence of marginal rates of substitution $\{x_t, y_t\}_{t=0}^{\infty}$ converges to the pair $(\bar{x}, \bar{y}) = (0.9449, 1.1615)$. We check that Parts (ii) and (iii) of Assumption 2 are satisfied along the path of convergence toward the stationary allocation. Convergence to the stationary allocation occurs after a few periods, which implies that both the consumption pattern and the welfare generated by a stationary allocation and a time-dependent one differ only in a small scale. According to the theory, there exists an initial promised utility ω_0 for which the long-run allocation of a time-dependent solution coincides with the stationary allocation.

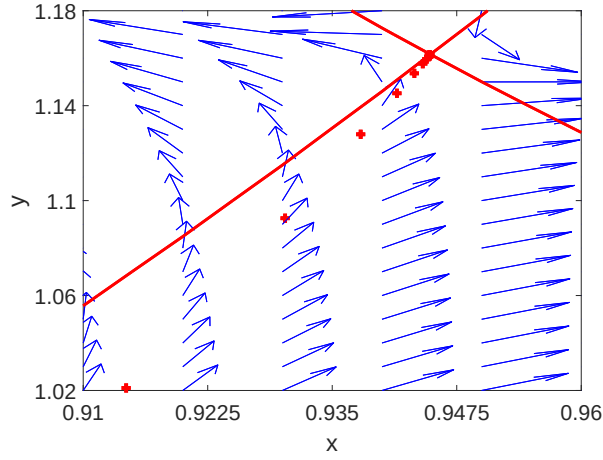


Figure 5: Saddle Path for the Time-Dependent Allocation

To facilitate the quantitative evaluation of welfare, Table 2 reports the percentage variation of individual consumption that is required in a economy implementing a suboptimal risk-sharing agreement to achieve the same level of welfare generated by the history-dependent allocation. We consider as suboptimal allocation the one generated by a stationary allocation with initial promised utility such that $\nu_0 = 0$. We then compare the welfare generated by such an allocation with the long-run welfare generated by the optimal history-dependent allocation. The equivalent variation is given by the value ξ solving

$$\mathbb{E} \left[\frac{\partial}{\partial} u(e(s) - c^{y,OPT}(s)) + u(c^{y,OPT}(s)) \right] = \mathbb{E} \left[\frac{\partial}{\partial} u((e(s) - c^{y,SUB}(s))(1 + \xi)) + u(c^{y,SUB}(s)(1 + \xi)) \right]$$

where “OPT” stands for the optimal long-run allocation in the benchmark reform and the superscript “SUB” stands for the suboptimal long-run allocation in the alternative reform. The planner experiences a welfare gain (loss) from the optimal allocation whenever $\xi > 0$ ($\xi < 0$). We shall consider the following two different values for the endowments’ share $\sigma = \{0.2, 0.3\}$. For $\sigma = 0.3$, the ratio of marginal utilities of the young and old in state 2 is larger than in the benchmark case, i.e., $\lambda(2) = 1.5$, while the one in state 1 is as before, i.e., $\lambda(1) = 3/7$, which implies that intergenerational risk sharing is more

desirable with higher σ .

Table 2: Welfare

	Stationary-Alternative Reform	$\sigma = 0.2 \rightarrow \sigma = 0.3$
Certainty Equivalent ξ	0.05%	0.18%

The certainty equivalent increase of consumption is thereby improved compared to the benchmark case: the more desirable the risk sharing is, the larger is the welfare loss from implementing alternative suboptimal allocations.

6. Conclusion

We have considered the optimal sustainable intergenerational insurance in a simple overlapping generations pure endowment economy. Although a simple environment, the optimal insurance can involve some complex dynamics. For example, in the simple two-state economy, we have shown that with successive good states for the young, the degree of insurance declines though some insurance is offered even in the limit. Equally, when the bad state occurs, the situation reverts and more insurance is provided.

We have considered the welfare cost of implementing simpler insurance arrangements than the optimal sustainable intergenerational insurance. These included a time varying and stationary insurance arrangement.

Appendix

Proof that Assumption 1 implies the existence of a sustainable allocation.

The autarky utility of all agents in state q is given by:

$$\hat{v}(q) := u(e^y(q)) + \beta \sum_{s=1}^S \pi(s) u(e^o(s)).$$

Consider a small transfer of $d\tau(s)$ in state s from the young to the old. The problem of existence of a sustainable allocation can be answered by finding a vector of positive transfer $d\tau$ such that the autarky utility of agents is not decreased in any state and increased in at least one state. Differentiating, the change in the autarky utility is non-negative if

$$(A.1) \quad -u_c(e^y(q))d\tau(q) + \beta \sum_{s=1}^S \pi(s) u_c(e^o(s))d\tau(s) \geq 0.$$

Rearranging (A.1) in terms of the marginal rates of substitution we have

$$-d\tau(q) + \sum_{s=1}^S m_{qs}d\tau(s) \geq 0.$$

Writing in matrix notation, the problem of existence can then be addressed by finding a vector $d\tau > 0$ that solves

$$(A.2) \quad (\hat{M} - I) d\tau \geq 0$$

where I is the identity matrix and $\hat{M}$ is the matrix of marginal rates of substitution evaluated at the endowment allocation. Equation (A.2) has a well-known solution. Using the Perron-Frobenius theorem, that there exists a strictly positive solution for $d\tau$, provided the Perron root, that is the largest eigenvalue, of $\hat{M}$ is greater than one. A lower bound for the Perron root is given by the minimum row sum of $\hat{M}$. Thus, Assumption 1 is sufficient for the Perron root to be greater than one and hence the existence of positive transfers from the young to the old that improve the utility of each generation. Hence, there exists a non-trivial sustainable allocation under Assumption 1.

Proof that $V(\omega)$ is concave.

To show that $V(\omega)$ is concave, consider the mapping T defined by

$$(TP)(\omega) = \max_{\{c^y(s), \omega'(s)\} \in \Gamma_S} \left[\sum_{s=1}^S \pi(s) \left(\frac{\beta}{\delta} u(e - c^y(s)) + u(c^y(s)) \right) + \delta P(\omega'(s)) \right].$$

Let $\mu(s)$, $(\beta/\delta)\nu$, and $\eta(s)$ be the multipliers on the constraints (3.1), (3.2) and the constraint that the consumption of the young cannot exceed the endowment of the young. The solution for the consumption of the young generation in the absence of constraint (3.1) will satisfy:

$$(A.3) \quad \frac{u_c(c^y(s))}{u_c(e - c^y(s))} = \max\left\{\frac{\beta}{\delta}(1 + \nu), \lambda(s)\right\}.$$

We denote the consumption that solves this equation when $\nu = 0$ as $c^{y,*}(s)$. Let

$$\omega^* = \sum_{s=1}^S \pi(s) (u(e - c^{y,*}(s)))$$

and

$$v^* = \sum_{s=1}^S \pi(s) \left(u(c^{y,*}(s)) + \frac{\beta}{\delta} u(e - c^{y,*}(s)) \right)$$

be the expected utility of the old and expected welfare at these consumption levels. The solution to the problem without the constraint (3.1) is stationary, so that $\omega'(s) = \omega^*$. Therefore, we can write

$$V^*(\omega) = \max_{(c^y(s) \in C^y(s))_{s \in S}} \sum_{s=1}^S \pi(s) \left(u(c^y(s)) + \frac{\beta}{\delta} u(e - c^y(s)) \right) + \frac{\delta}{(1-\delta)} v^*$$

subject to (3.2). The objective is concave in $c^y(s)$ and the constraint set convex. Thus, $V^*(\omega)$ is concave. If $\omega \leq \omega^*$, then clearly $c^y(s) = c^{y,*}(s)$ and $V^*(\omega) = v^*/(1 - \delta)$ and $V^{*,'}(\omega) = 0$. For $\omega > \omega^*$, we have that $c^y(s)$ satisfies (A.3). It can be checked that $V^{*,'}(\omega) < 0$ for $\omega > \omega^*$ and $V^{*,'}(\omega^*) = 0$.

Consider again the mapping T and start with $P = V^*$. It follows from the definitions that $TV^*(\omega) \leq V^*(\omega)$ because $V^*(\omega) \leq v^*/(1 - \delta)$ and because the maximization defined above adds the incentive constraint (3.1). Make the induction hypothesis that $T^n V^*(\omega) \leq T^{n-1} V^*(\omega)$ for $n \geq 2$ (we have just established that the inequality holds for $n = 1$). Applying the mapping T to both $T^n V^*(\omega)$ and $T^{n-1} V^*(\omega)$ shows that $T^{n+1} V^*(\omega) \leq T^n V^*(\omega)$ because the constraint set is the same in both cases but the objective is no greater in the former case, by the induction hypothesis. Hence, the sequence $T^n V^*(\omega)$ is non-increasing and converges.

Proof of Proposition 2.

It can be seen that the first-order and envelope conditions (3.4) and (3.5) do not depend on the state directly. State dependence of the solution enters through the incentive constraint (3.1), the aggregate endowment in (3.3) and the non-negativity constraint $c^y(s) \leq e^y(s)$. As we have argued above (Lemma 1), the incentive constraint and the non-negativity constraint cannot bind simultaneously. Therefore, we can partition the state space into three regions depending on whether the incentive and non-negativity constraints bind or not.

First consider the case for the states where the incentive constraint (3.1) binds. Consider a given ω and hence from (3.5) a given ν . It follows from risk aversion and condition (3.3) that c^y and μ are positively related for any given state. Also from risk aversion, c^y is negatively related to the aggregate endowment for a fixed ν and μ . Equally, ω' and μ are also positively related from (3.4) and the concavity of V . Hence, $u(c^y) + \beta\omega'$ is increasing in μ for any given state and decreasing in the aggregate endowment.

It therefore follows that if the incentive constraint (3.1) binds, and the autarky utility $\hat{v}$ is negatively related to the aggregate endowment e , then $\mu(s) > \mu(q)$ whenever $\hat{v}(s) > \hat{v}(q)$.²⁹ That is, for states s and q , such that the incentive constraint is binding, $c^y(s) > c^y(q)$ and $\omega'(s) > \omega'(q)$.

Equally, ω and ν are positively related from equation (3.5). Therefore, an increase in ω lowers c^y for a fixed μ whereas it has no direct effect on ω' . For a given state and value of $\hat{v}(s)$ and $e(s)$, an increase in ω will therefore raise the value of $\mu(s)$ at the solution (if the incentive constraint was binding for the lower value of ω , it will continue to bind for the higher value). The net effect is to increase $\omega'(s)$ through the indirect change in $\mu(s)$, whereas the effect on $c^y(s)$ is ambiguous because an increase in ω leads to an increase in ν and $\mu(s)$.

Now consider the case for states where the incentive constraint does not bind and $\mu(s) = 0$. In this case, it follows from (3.4) that $V_\omega(\omega'(s)) = 0$ and hence, $\omega'(s) = \tilde{\omega}$. (There is no advantage to setting $\omega'(s) < \tilde{\omega}$ and it represents a Pareto improvement to increase the promised expected utility of the old up to $\tilde{\omega}$.) Thus, in such states the promised utility of the old generation is *reset* to the same promised utility of $\tilde{\omega}$ independently of the state. For states where $\mu(s) = 0$, it follows from (3.3) that the ratio of marginal utilities is larger than β/δ and strictly larger if $\nu > 0$ or $\lambda(s) > \beta/\delta$. That is, the consumption of the young is low compared to the first-best allocation and consumption of the old is high compared to the first best allocation. We also know from Lemma 1 that whenever the old are constrained, the consumption is equal to the endowment in that state and the young are not constrained, $\eta(s) > 0$ implies $\mu(s) = 0$, and that if the old are constrained in state s , they are constrained in all states $q > s$. Hence, in these states the promised expected utility to the old is reset to $\tilde{\omega}$.

²⁹If $e(S) \geq e(S-1) \geq \dots \geq e(1)$, then $\hat{v}(s) > \hat{v}(q)$ whenever $s > q$.

Description of algorithm used for the computed example

We describe here the algorithm implemented to solve for the Pareto-frontier and the optimal allocation. The algorithm is based upon dynamic programming technique. While these methods are generally not applicable when solving incentive constrained problems since the constraint set changes after every iteration of the value function when solving for the functional equation [P2], here we start the value function iteration with the first-best Pareto-frontier, V^* . This initial guess can be calculated analytically in a straightforward way. We then use it to define a new maximization problem with value equal to the Pareto frontier and solve the new problem by iterating over the value function. Convergence to the true solution of the functional equation [P2] is monotonic from above. To implement the algorithm, we discretize the state space for ω and, hence, solve the functional equation for a finite number of values for ω in each iteration. After having determined the solution of the iteration program we can then evaluate the set of state-dependent policies $\omega'(s)$ and $c^y(s)$ in each of the two states of the world. The details of the algorithm are as follows:

- *Step 1:* Calculate the initial guess J_0 for the value V , that is the Pareto-frontier describing the first-best solution.
- *Step 2:* Adjust the domain Ω_n of the state variable ω given the guess J_n for the value function V .
- *Step 3:* Solve the static maximization problem for each realization of the state variable ω given J_n . Use this result to update the guess to J_{n+1} .
- *Step 4:* If $\sup_{\omega \in \Omega_n} (J_n(\omega) - J_{n+1}(\omega)) > \epsilon > 0$, go back to Step 2.
- *Step 5:* Use J_{n+1} to calculate policy functions and find the law of motion on Ω_n .

References

- [1] Alvarez, F., and U. Jermann, 2000. Efficiency, Equilibrium, and Asset Pricing with Risk of Default, *Econometrica*, 68, 775-798.
- [2] Azariadis, C., and V. Galasso, 2002. Fiscal Constitutions, *Journal of Economic Theory*, 103(2), 255-281.
- [3] Ball, L., and G. Mankiw, 2007. Intergenerational Risk Sharing in the Spirit of Arrow, Debreu, and Rawls, with Applications to Social Security Design, *Journal of Political Economy*, 115(4), 523-547.
- [4] Bohn, H., 2009. Intergenerational Risk Sharing and Fiscal Policy, *Journal of Monetary Economics*, 56(6), 805-816.
- [5] Chattopadhyay, S., and P. Gottardi, 1999. Stochastic OLG Models, Market Structure, and Optimality, *Journal of Economic Theory*, 89(1), 21-67.
- [6] Cooley, T. F., and J. Soares, 1999. A Positive Theory of Social Security Based on Reputation, *Journal of Political Economy*, 107(1), 135-160.
- [7] Cremer, J., 1986. Cooperation in Ongoing Organizations, *Quarterly Journal of Economics*, 101(1), 33-49.
- [8] Demange, G., 2002. On Optimality in Intergenerational Risk Sharing, *Economic Theory*, 20(1), 1-27.
- [9] Demange, G., 2009. On Sustainable Pay-as-You-Go Contribution Rules, *Journal of Public Economic Theory*, 11(4), 493-527.
- [10] Demange, G., and G. Laroque, 1999. Social Security and Demographic Shocks, *Econometrica*, 67(3), 527-542.
- [11] D'Amato, M., and V. Galasso, 2010. Political Intergenerational Risk Sharing, *Journal of Public Economics*, 94(9-10), 628-637.
- [12] Enders, W., and H. E. Lapan, 1982. Social Security Taxation and Intergenerational Risk Sharing, *International Economic Review*, 23(3), 647-658.
- [13] Farhi, E., and I. Werning, 2007. Inequality and Social Discounting, *Journal of Political Economy*, 115(3), 365 – 402.
- [14] Fisher, S., 1983. Welfare Aspects of Government Issue of Indexed Bonds, in *Inflation, Debt, and Indexation*, ed. R. Dornbusch and M. H. Simonsen, MIT Press.
- [15] Gale, D., 1994. The Efficient Design of Public Debt, in *Financial Innovation and Risk Sharing*, ed. F. Allen and D. Gale, MIT Press.
- [16] Gordon, M., and H. Varian, 1988. Intergenerational Risk Sharing, *Journal of Public Economics*, 37, 185-202.
- [17] Gottardi, P., and F. Kubler, 2010. Social Security and Risk Sharing, *Journal of Economic Theory*, 146(3), 1078-1106.
- [18] Hammond, P., 1975. Charity: Altruism or Cooperative Egoism, in *Altruism, Morality, and Economic Theory*, edited by E. Phelps, Russell Sage Foundation.
- [19] Imrohoroglu, A., S. Imrohoroglu and D. Joines, 1999. Social Security in an Overlapping Generations Economy with Land, *Review of Economic Dynamics*, 2(3), 638-665.
- [20] Kandori, M., 1992. Repeated Games Played by Overlapping Generations of Players, *Review of Economic Studies*, 59(1), 81-92.
- [21] Kehoe, T. and D. Levine, 2001. Liquidity Constrained Markets versus Debt Constrained Markets, *Econometrica*, 69, 575-598.
- [22] Kocherlakota, N., 1996. Implications of Efficient Risk Sharing without Commitment, *Review of Economic Studies*, 63(4), 595-609.
- [23] Krueger, D., and F. Kubler, 2006. Pareto-Improving Social Security Reform when Financial Markets are Incomplete!?, *American Economic Review*, 96(3), 737-755.
- [24] Krueger, D., and F. Perri, 2006. Does Income Inequality Lead to Consumption Inequality? Evidence and Theory, *Review of Economic Studies*, 63(4), 163-193.
- [25] Krueger, D., and F. Perri, 2011. Public versus Private Risk Sharing, *Journal of Economic Theory*, 146(3), 920-956.
- [26] Olovsson, C., 2010. Quantifying the Risk-Sharing Welfare Gains of Social Security, *Journal of*

- Monetary Economics*, 57(3), 364-375.
- [27] Rangel, A., 2003. Forward and Backward Intergenerational Goods: Why is Social Security Good for Environment?, *American Economic Review*, 93(3), 813-834.
 - [28] Rangel, A., and R. Zeckhauser, 2001. Can Market and Voting Institutions Generate Optimal Intergenerational Risk Sharing?, NBER Chapters, in: *Risk Aspects of Investment-Based Social Security Reform*, 113-152.
 - [29] Shiller, R., 1999. Social Security and Institutions for Intergenerational, Intragenerational and International Risk Sharing, *Carnegie-Rochester Conference on Public Policy*, 50, 165-204.
 - [30] Smetters, K. A., 2004. Trading with the Unborn: A New Perspective on Capital Income Taxation, *Michigan Retirement Research Center* working paper 066.
 - [31] Thomas, J., and T. Worrall, 1988. Self-Enforcing Wage Contracts, *Review of Economic Studies*, 55(4), 541-554.
 - [32] Thomas, J., and T. Worrall, 2007. Unemployment Insurance under Moral Hazard and Limited Commitment: Public versus Private Provision, *Journal of Public Economic Theory*, 9(1), 151-181.
 - [33] Weil, P., 2008. Overlapping Generation: First Jubilee, *Journal of Economic Perspectives*, 22(4), 115-134.