A Theory of Multi-Period Debt Structure

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Abstract

We develop a model of multi-period debt structure. A simple trade-off between the termination threat required to make repayments incentive compatible and the desire to avoid early liquidation determines the number of repayments, their timing, and repayment amounts. For mature firms with risky cash flows, frequent repayments maximize pledgeable income—for example, by rolling over short-term debt. In contrast, for firms with cash-flow growth or significant risk-free cash flows, adding risky repayments can decrease pledgeable income. In some cases, a single risky bullet repayment maximizes pledgeable income, effectively a long-term debt contract.

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How do firms choose the term structure of their debt? While a large literature has investigated why firms use debt to raise financing for investments,[1] we know much less about the determinants of the number of repayment dates, their timing, and the respective repayment amounts. To shed light on these issues, this paper develops a model of multi-period debt financing in a setting with unverifiable cash flow. In our model, a rich term structure of debt emerges from a simple trade-off between providing the firm with incentives to repay and preventing costly early liquidation.

The main friction in our model is classic incomplete contracting: Cash flow is non-verifiable, such that the entrepreneur can abscond with the cash flow instead of repaying debt. As in Bolton and Scharfstein (1990, 1996) and Hart and Moore (1998), debt induces a termination threat that makes repayment incentive compatible. However, in contrast to the two-period nature of these papers, in our model the firm produces a sequence of non-contractible cash flows over many periods. This multi-period setup allows us to study the optimal debt structure: How many repayment dates should there be? What should be the timing and size of those repayments?

The key trade-off that determines the optimal debt structure balances default risk with the incentives necessary to ensure repayment. Repayment incentives derive from the threat of early termination. Specifically, the firm's creditors commit to liquidate if the firm defaults on any of its contractual repayments. Early liquidation is costly because it leads to the loss of a longer stream of future cash flows. The entrepreneur therefore would like to schedule debt payments as late as possible. However, there is a limit to how late repayments can credibly be made to investors: Towards the end of the project, the entrepreneur's continuation value is lower, leading to larger incentives to divert the cash flow and default.

We first develop a baseline model, in which the firm generates a risky cash flow every period, drawn independently from the same binary distribution (zero or positive). In this setting, we show that a repayment profile with constantly spaced debt payments towards the end of the project is

[1]Classic contributions to this literature include models of costly state verifiation (Townsend (1979); Gale and Hellwig (1985)), termination threat models of debt (Bolton and Scharfstein (1990, 1996); Hart and Moore (1994, 1998)), incentive-based theories of debt (Innes 1990), and theories based on information sensitivity (Gorton and Pennacchi (1990); Dang et al. (2012)).

optimal, where the spacing of repayment dates is determined by the riskiness of the firm's cash flows. On each repayment date, the firm pays back the entire realized period cash flow, in order to minimize the number of risky payments. The cash flows between payment dates accrue to the entrepreneur, thereby providing incentives to honor each of the contractual payments. All else equal, the larger the amount of outside financing that the firm needs to raise, the larger the number of repayment dates and the more front-loaded the repayment schedule.

This baseline model generates a number of key comparative statics on how cash-flow risk, profitability, and leverage affect the optimal debt structure. For example, when period cash flows are riskier, the average repayment time decreases, consistent with classic empirical evidence on debt maturity in Barclay and Smith (1995) and Stohs and Mauer (1996) and the more recent empirical work on corporate debt maturity profiles by Choi et al. (2016). Higher profitability, on the other hand, is associated with more backloaded repayments, consistent with the evidence in Guedes and Opler (1996) and Qian and Strahan (2007). Higher leverage is associated with earlier repayment, consistent with evidence on the debt structure of leveraged buyout deals in Axelson et al. (2013).

A key feature of our baseline model is that pledgeable income is maximized by scheduling as many repayments as possible, subject to spacing these repayments such that they satisfy incentive compatibility. As a consequence, mature firms (i.e., risky period cash flows but no cash-flow growth) with large outside financing needs opt for earlier and more frequent debt payments. One natural implementation is to roll over a sequence of short-term debt contracts, where the rollover frequency is determined by the riskiness of the firm's cash flows. This result therefore echoes and extends the classic insight (Bolton and Scharfstein (1990, 1996) and Hart and Moore (1998)) that short-term debt alleviates financing constraints arising from non-verifiable cash flows.

Interestingly, the result that a sequence of short-term debt contracts maximizes pledgeability no longer holds under more general cash flow distributions. Our model therefore provides a unified framework that can capture both incentives to finance with short-term debt (leading to maturity mismatch) and incentives to finance with longer term debt (approximate matching of the maturities of assets and liabilities). Specifically, we point out two situations, in which limiting the number of risky repayments maximizes pledgeable income.

First, when there is growth in the firm's expected cash flows, pledgeable income is generally maximized by a debt contract with relatively few risky repayments towards the end of the project's life (even though the firm could in principle add more repayment dates). In fact, in some cases a single (bullet) repayment maximizes pledgeability. For growth firms, the optimal debt structure therefore resembles long-term debt with debt maturity closer to the maturity of the firm's assets.

Second, when the firm generates a positive minimum cash flow in each period, the debt contract that maximizes pledgeable income depends on the riskiness of the firm. When the safe cash flow component is large relative to total cash flow, pledgeability is maximized by offering a safe repayment in every period—essentially safe short-term debt. If, on the other hand, the risky part of the cash flow makes up a significant fraction of the firm's overall cash flow, pledgeability is maximized by alternating between safe and risky repayments. While safe repayments occur throughout the lifetime of the firm's assets, risky repayments are scheduled towards the end of the project and need to be appropriately spaced to preserve incentive compatibility. Moreover, increasing the number of risky payments only raises pledgeability up to a point: Pledgeability is generally maximized with a fixed number of risky repayments that is independent of the total number of periods. In this case, the optimal debt structure resembles a combination of safe short-term debt and a number of risky long-term bonds or loans. One natural implementation of this debt structure is a sequence of coupon bonds with safe coupons and risky principal repayments.

Our paper contributes to the literature on optimal debt contracts. We build on the literature on debt as a termination threat (in particular, Bolton and Scharfstein (1990, 1996); Hart and Moore (1995, 1998); Berglöf and von Thadden (1994)). While these papers highlight the importance of short-term debt (relative to asset maturity), the two-period nature of these models does not lend itself to study the optimal repayment structure when multiple repayment dates are possible. The papers that have extended termination-threat models to more periods, generally do not focus on the optimal term structure of debt repayments. For example, Hart and Moore (1994) characterize the fastest and slowest way to repay in a deterministic multi-period setting, but because of the absence of default risk, their model does not pin down the number and timing of repayments, which is the focus of our paper.

Our approach also differs from the literature on optimal financial contracting in dynamic settings (e.g., DeMarzo and Fishman (2007); DeMarzo and Sannikov (2006)). These papers derive the optimal financing contract in dynamic settings. In contrast, we restrict the contracting space to debt contracts, which allows us to derive a rich set of novel predictions on optimal multi-period debt structure.

More broadly, our paper is also related to the literature on debt maturity, albeit with a different focus. We study how a firm's debt stucture (including maturity) emerges from the inability to verify cash flows. In contrast, classic theories of debt maturity have focused on private information (Flannery (1986); Diamond (1991, 1993)), whereas the more recent literature has highlighted strategic interaction among creditors (Cheng and Milbradt (2012)), the inability to commit to financing policies (Brunnermeier and Oehmke (2013), He and Milbradt (2016)), and debt overhang (Diamond and He (2014)).

Finally, our paper is related to a series of papers by Rampini and Viswanathan (2010, 2013), who develop a multi-period model of financing subject to enforcement constraints. A key difference to our paper is the assumption regarding exclusion. In Rampini and Viswanathan (2010, 2013), no exclusion is possible, such that the optimal contract can be implemented by one-period state-contingent debt contracts. In our paper, liquidation by creditors effectively excludes the entrepreneur from future investment, creating a non-trivial role for debt contracts of different maturities. Moreover, in these models, as in the models of dynamic financing by Albuquerque and Hopenhayn (2004) and Clementi and Hopenhayn (2006), there is no default on the equilibrium path. The possibility of equilibrium default is a key feature of our model.

1 Model Setup

We consider an entrepreneur who can set up a firm to undertake an investment project. The investment requires an initial outlay of I at date $t = 0$. If funded, the project lasts for T discrete periods. At each $t \in \mathcal{T}$ the project generates a cash flow X_t, where $\mathcal{T} = \{1, 2, \ldots, T\}$ denotes the set of potential cash flow dates. The cash flow distribution at each date t is binary. With probability $\frac{1}{K}$, the project generates positive cash flow of $K\Delta$, where $\Delta > 0$ and $K \in \mathbb{Z}_+$. With probability $1 - \frac{1}{K}$, the cash flow at date t is zero. Therefore, at each date t, the project yields an expected cash flow of Δ, while the parameter K captures the riskiness of the project's cash flows.[2] We assume that the cash flow realizations X_t are serially uncorrelated. The assumption of binary cash flows makes the analysis tractable and highlights the key tradeoffs. We extend our analysis to more general cash flow distributions in Section 5.2. The entrepreneur has cash at hand c and must therefore finance the remainder $D \equiv I - c$ by raising outside financing.

We make two key assumptions about the contracting environment. First, we restrict our attention to debt contracts with commitment to liquidate as the only means of outside financing available to the firm. This distinguishes our analysis apart from the literature on optimal contracting in dynamic settings (e.g., DeMarzo and Sannikov (2006); DeMarzo and Fishman (2007)). The commitment to liquidate implies that the issues of renegotiation analyzed in Gromb (1994) do not arise in our setting.[3] Second, we assume that the entrepreneur cannot save. Rather, at each date t, the entrepreneur consumes the cash flow net of any debt payments made at that date. This no-savings assumption greatly simplifies our analysis, but it is not crucial for our main results. We relax this assumption in Section 5.1, where we show that adding savings to the model does not affect the main economic insights from our model.

[2] As will become clear below, the assumption that K is an integer is for mathematical convenience only. Our results are virtually unchanged without that assumption, except for the added notational complexity of having to deal with integer constraints.

[3] Gromb (1994) shows that, in a multi-period setting, the ability to repeatedly renegotiate the debt contract can severely constrain pledgeability. Due to the creditors' ability to commit to liquidate, this issue does not arise in our framework.

A debt contract is then characterized by a sequence of promised repayments $\mathcal{R} = \{R_t\}$, $t \in \mathcal{T}$, where, if at any date t the entrepreneur has promised a positive repayment $R_t > 0$ but does not pay, the project is liquidated by the firm's creditors. When the project is liquidated, neither the investor nor the entrepreneur receive any future cash flows. In other words, the project's liquidation value is normalized zero and the entrepreneur cannot undertake another investment after liquidation, either because he is excluded from credit markets or because, without the original lender, he has lost access to his investment project. As long as liquidation has not occurred, at each date t the entrepreneur consumes the cash flow net of any debt payments made at that date, $X_t - R_t$. Because the entrepreneur cannot save, the firm can only use contemporaneous cash flow to make the debt payment, so that feasibility of any debt repayment requires that $R_t \leq X_t$.[4]

The main contracting friction in our model is standard non-verifiability of cash flows. At any date t the entrepreneur can abscond with the cash flow that realized in that period, allowing the entrepreneur to default even when the realized cash flow is sufficient to make the promised repayment R_t. Any credible sequence of debt repayments $\mathcal{R}$ must be therefore be incentive compatible. Formally, denote by V_t the entrepreneur's value function (which is also the firm's equity value) at the beginning of period t. Then, a debt contract $\mathcal{R}$ is incentive compatible if and only if $R_t \leq V_{t+1}$ for every t.

The debt contract $\mathcal{R}$ can be interpreted in a number of ways. In the most narrow interpretation, $\mathcal{R}$ is the payment schedule of a single debt contract that specifies multiple repayments over time. Interpreted more broadly, $\mathcal{R}$ captures the firms aggregate debt structure, where individual repayments R_t are potentially separate contracts (i.e., a portfolio of loans or bonds). While from a theoretical perspective these two interpretations are equivalent, the broader interpretation will be useful in linking our model to empirical evidence. Finally, as we will show below, the contract $\mathcal{R}$ can also be implemented by a sequence of one-repayment-date (bullet) contracts that are rolled

[4]Note that our setup rules out making early repayments (for example, through callable bonds or loans). When debt is callable, the borrower may have an incentive to call the debt and make early repayments whenever a cash flow realizes on a non-repayment date. Callable instruments therefore effectively allow the borrower to repay debt using prior cash flow and are economically similar to the savings extension in Section 5.1.

over at each repayment date.

We assume that both the entrepreneur and the investors are risk neutral. In addition, investors are perfectly competitive, such that any incentive compatible contract that provides an total expected repayment of D is acceptable to investors. Given this assumption, the entrepreneur then chooses a repayment schedule $\mathcal{R}$ to maximize her expected payoff at day 0. Formally, the entrepreneur's maximization problem is:

$$\begin{aligned} & \max\nolimits_{\mathcal{R}} V_0 \\ \text{s.t.} \quad & R_t \leq V_{t+1} \quad \text{(IC)} \\ & \mathcal{D}(\mathcal{R}) = D \quad \text{(IR)}. \end{aligned} \tag{1}$$

The entrepreneur's payoff V_t satisfies the following recursive and explicit formulations:

$$V_t = \Delta + \Pr(X_t \geq R_t)\left(-R_t + V_{t+1}\right) \tag{2}$$

$$= \sum_{i=t}^{T} \prod_{s=t}^{i-1} \Pr\left(X_s \geq R_s\right) \Delta - \sum_{i=t}^{T} \prod_{s=t}^{i} \Pr\left(X_s \geq R_s\right) R_i, \tag{3}$$

and $\mathcal{D}(\mathcal{R})$ denotes the value to investors of a debt contract $\mathcal{R}$, which is given by

$$\mathcal{D}(\mathcal{R}) = \sum_{t=0}^{T} \prod_{s \leq t} \Pr(X_s \geq R_s) R_t. \tag{4}$$

Using investors' IR constraint (1), as well as the expressions for the entrepreneur's payoff V_t and the value of the debt contract to investors $\mathcal{D}(\mathcal{R})$ given in (2) and (4), we can then simplify the entrepreneur's value function V_0 to

$$V_0 = \sum_{i=0}^{T} \prod_{s=0}^{i-1} \Pr\left(X_s \geq R_s\right) \Delta - D. \tag{5}$$

Clearly, it is not in the interest of the firm to offer a debt contract that defaults with probability one in any period. Therefore, in equilibrium any promised payment R_t is weakly smaller than the cash flow in that period, which we will refer to as the feasibility condition:[5]

$$R_t \leq K\Delta. \tag{6}$$

The main choice variable for the entrepreneur is whether he should promise a positive repayment at any particular date t. To see this, denote the set of dates with positive repayments by $\mathcal{Q} \equiv \{t \in \mathcal{T} | R_t > 0\}$. Then, as the following lemma shows, it is the timing of repayments that matters for the firm, while the exact size of each repayment can usually not be uniquely determined.

Lemma 1 *For any two incentive compatible repayment schedules, $\mathcal{R}$ and $\mathcal{R}'$, if $\mathcal{Q}(\mathcal{R}) = \mathcal{Q}(\mathcal{R}')$ and $\mathcal{D}(\mathcal{R}) = \mathcal{D}(\mathcal{R}')$, then the entrepreneur is indifferent between $\mathcal{R}$ and $\mathcal{R}'$.*

Intuitively, Lemma 1 states that if two debt contracts $\mathcal{R}$ and $\mathcal{R}'$ have identical repayment dates and same expected values, then they yield the same expected payoffs to the entrepreneur. This result follows directly from the binary cash flow assumption: The probability of making a repayment $R_t \in (0, K\Delta]$ is $\Pr(X = K\Delta) = \frac{1}{K}$, the probability of positive cash flow realization, regardless of the size of the repayment. This is reflected in equation (5) in that any positive repayment R_t enters the entrepreneur's payoff only through the probability of default at date t. Therefore, only the number and timing of payments $\mathcal{Q}$ is important, but not the size of each individual repayment.

[5]Suppose, in contrast, that the optimal contract contains a promised repayment $R_t > K\Delta$ for some t. Then the entrepreneur will default with certainty at date t, even if the positive cash flow $K\Delta$ realizes. Then, the investor's IR constraint (1) holds only if the expected total repayments before period t are equal to D. However, if this is the case, then the entrepreneur would adjust R_t to 0. This adjustment would not change the investors IR constraint (1), but would give the entrepreneur a weakly larger payoff (strictly larger if $t \leq T - 1$).

2 Optimal Debt Structure

2.1 Optimal Debt Structure: An Intuitive Derivation

The main trade-off that determines optimal debt structure is between early liquidation and sufficient pledgeability. On the one hand, the entrepreneur likes to make debt payments as late as possible. By doing so, the project is less likely to be terminated early on, providing the entrepreneur higher expected cash flows. On the other hand, the entrepreneur faces limits to how late he can credibly promise to make repayments to investors. Towards the end of the project, the entrepreneur's continuation value is lower, such that he has larger incentives to divert the cash flow and default.

To see how this trade-off shapes the firm's optimal debt structure, it is instructive to start by considering a firm with low outside financing needs D. The nature of the optimal debt structure then emerges as we gradually increase the amount of required outside financing D. We build up these results using several cases before moving to a general proposition that fully characterizes the optimal debt repayment structure. The numbering of the cases will become clear as we move from case to case.

Case 1-1: $D \in \left(0, \frac{\Delta}{K}\right]$. We start by assuming that the amount of required outside financing D is weakly less than $\frac{\Delta}{K}$. Clearly, because $V_{T+1} = 0$, the entrepreneur cannot credibly promise to make a payment to investors at date T. Therefore, incentive compatibility requires that $R_T = 0$. However, when $D \leq \frac{\Delta}{K}$, the entrepreneur can raise D by offering a single repayment of $R_{T-1} = KD$ to raise D. This payment is incentive compatible because the entrepreneur's continuation value at $T-1$ is given by Δ (the expected cash flow at date T), which, given $D \in \left(0, \frac{\Delta}{K}\right]$, exceeds the date $T-1$ payment of KD:

$$R_{T-1} = KD \leq V_T = \Delta. \tag{7}$$

A single repayment of KD at date $T-1$ also satisfies the investor's IR constraint (1), as $\mathcal{D}(\mathcal{R}) = \frac{1}{K}KD = D$.

From equation (2), we then see that the entrepreneur's continuation value at the beginning of date $T-1$ is given by

$$V_{T-1} = \Delta + \frac{1}{K}(\Delta - KD),$$

The overall payoff to the entrepreneur by

$$V_0 = \Delta + V_1 = ... = (T-1)\Delta + V_{T-1} = T\Delta + \frac{1}{K}(\Delta - KD). \tag{8}$$

Note that even though the entrepreneur can potentially choose to make multiple repayments or offer a single repayment before period $T-1$, both of these options are not optimal. Intuitively, promising multiple risky repayments inefficiently increases default risk. Promising repayment earlier than date $T-1$ risks that the project is terminated unnecessarily prematurely. Therefore, any alternative schedule with a single repayment of KD at $t' < T-1$, which yields a payoff to the entrepreneur of $(t'+1)\Delta + \frac{1}{K}[(T-t')\Delta - KD]$, is dominated by (8).

Case 1-2: $D \in \left(\frac{\Delta}{K}, \frac{2\Delta}{K}\right]$. When the required amount of outside financing D exceeds $\frac{\Delta}{K}$, a single repayment of KD at date $T-1$ is no longer incentive compatible: When $KD > \Delta$, this payment would violate the IC constraint (7). To support a higher repayment, the entrepreneur then optimally moves the single repayment date forward to $T-2$. Because now the final two periods' cash flows are left to the entrepreneur, the entrepreneur's payoff from continuing past date $T-2$ is given by $V_{T-1} = 2\Delta$, which provides the upper bound for the incentive compatible repayment at $T-2$:

$$R_{T-2} = KD \leq V_{T-1} = 2\Delta.$$

Therefore, when the required amount of financing D lies in the interval $\left(\frac{\Delta}{K}, \frac{2\Delta}{K}\right]$, the entrepreneur can raise the required financing with a single repayment at date $T-2$. Because any additional repayment date would create unnecessary default risk, a single payment at date $T-2$ is the optimal way to finance the project.

Case 1-K: $D \in \left(\frac{(K-1)\Delta}{K}, \Delta\right]$. It is easy to see that as the amount of required outside financing D continues to increase, the entrepreneur optimally keeps moving the single repayment forward

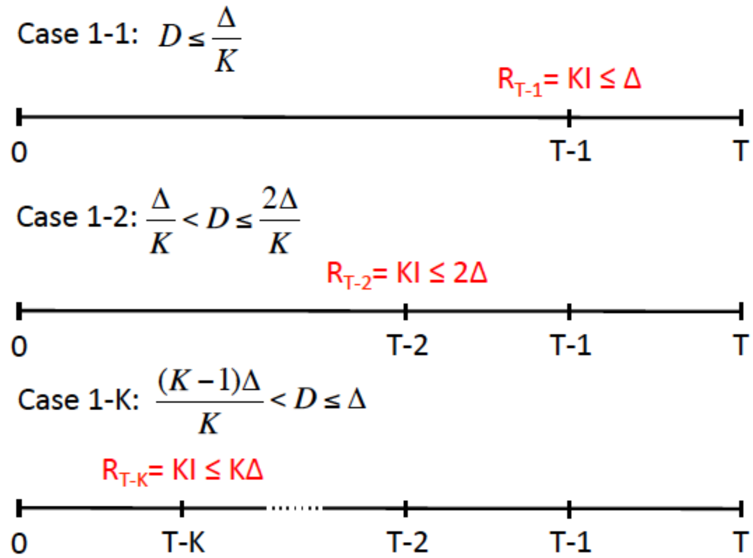

Figure 1: This figure illustrates the range of outside financing needs for which financing is possible with one repayment date, cases 1-1 to 1-K.

to maintain incentive compatibility. This is possible as long as the required single repayment satisfies the feasibility constraint (6). This leads us to the last case in which financing with one repayment date is possible, case 1-K, with a repayment of KD at date $T-K$. Analogous to before, the entrepreneur's continuation value $V_{T-K+1} = K\Delta$ allows for a maximum incentive-compatible repayment $R_{T-K} = K\Delta$. However, note that at this point also the feasibility constraint (6) binds. Therefore, $D = \Delta$ is the maximum amount of outside financing that can be raised with a single repayment.

Figure 1 summarizes the cases in which financing with only one repayment date is possible (cases 1-1 to 1-K).

Case 2-1: $D \in \left(\Delta, \Delta + \frac{\Delta}{K^2}\right]$. When the required amount of outside financing D exceeds Δ, a single repayment of KD (at any date) is no longer a feasible way to finance the project; the required repayment would violate the feasibility condition (6). Therefore, the entrepreneur must now promise repayments at two dates. The optimal way to do this is to move forward the existing repayment date from date $T-K$ to $T-K-1$ and to add a second repayment at date $T-1$, resulting in optimal repayment dates $\mathcal{Q} = \{T-K-1, T-1\}$. Note that once there are two repayment dates, the size of each repayment is no longer uniquely determined, except when D is at the upper boundary of the interval (i.e., $D = \Delta + \frac{\Delta}{K^2}$). One possible contract (the fastest way to repay) is to set the earlier repayment such that it just satisfies the feasibility constraint, $R_{T-K-1} = K\Delta$, and set the second repayment to raise the remainder, $R_{T-1} = K^2(D-\Delta)$.[6] One can easily verify that this contract satisfies investor's IR condition (1),

$$\mathcal{D}(\mathcal{R}) = \frac{1}{K}R_{T-K-1} + \frac{1}{K^2}R_{T-1} = D.$$

Moreover, the contract is incentive compatible: For any $D \in \left(\Delta, \Delta + \frac{\Delta}{K^2}\right]$ the IC constraint at date $T-1$ is clearly satisfied,

$$R_{T-1} = K^2(D-\Delta) \leq \Delta = V_T.$$

To check the IC constraint at date $T-K-1$, note that, using (2), we can write the continuation value after date $T-K-1$ as

$$V_{T-K} = \Delta + V_{T-K+1} = ... = (K-1)\Delta + V_{T-1} = K\Delta + \frac{1}{K}(-R_{T-1} + \Delta).$$

Because R_{T-1} is incentive compatible, we have $V_{T-K} \geq K\Delta = R_{T-K-1}$, such that R_{T-K-1} is incentive compatible. Intuitively, leaving K periods of cash flow between the two repayment dates to the entrepreneur makes sure that the repayment of $K\Delta$ at date $T-K-1$ is incentive compatible. The second repayment at date $T-1$ is bounded by Δ, by exactly the same intuition in case 1-1. Finally, it is also easy to verify that the schedule $\mathcal{R}$ with $R_{T-K-1} = K\Delta$ and $R_{T-1} = \Delta$ attains the upper bound of $D = \Delta + \frac{\Delta}{K^2}$ in this case.

Case 2-2: $D \in \left(\Delta + \frac{\Delta}{K^2}, \Delta + \frac{2\Delta}{K^2}\right]$. As we increase D further, the optimal repayment dates

[6] Alternatively, the slowest way to repay is to set the final repayment to the maximum incentive compatible amount, $R_{T-1} = \Delta$, and to then set $R_{T-K-1} = KD - \frac{\Delta}{K}$. As a result, any equilibrium contract satisfies $R_{T-K-1} \in \left(K\Delta - \frac{\Delta}{K}, K\Delta\right]$, imposing relatively tight bounds on the size of repayments.

shift forward to $\mathcal{Q} = \{T-K-2, T-2\}$. Specifically, compared with case 2-1, both repayment dates are moved forward by one period. This increases pledgeability at the second (and last) repayment date, while maintaining incentives to repay at the first repayment date. Similar to case 1-2, the maximum incentive compatible repayment at $T-2$ is $V_{T-1} = 2\Delta$. By keeping K periods between repayments maintains the incentive compatibility of the first repayment, which is at most $K\Delta$. The debt contract with $R_{T-K-2} = K\Delta$ and $R_{T-2} = 2\Delta$ then attains the upper bound of this case, $D = \Delta + \frac{2\Delta}{K^2}$.

Case 2-K: $D \in \left(\Delta + \frac{(K-1)\Delta}{K^2}, \Delta + \frac{\Delta}{K}\right]$. As we continue to increase D, at some point we arrive at Case 2-K, which is the last case in which the required amount of financing can be raised with two repayment dates, which occur at dates $\mathcal{Q} = \{T-2K, T-K\}$. To raise the maximum amount of outside financing with two repayment dates, the entrepreneur offers repayments of $R_{T-2K} = R_{T-K} = K\Delta$, which attains the maximum debt value of $\Delta + \frac{\Delta}{K}$, the upper boundary of case 2-K. At this point, the feasibility condition for both repayments binds. To borrow more, the entrepreneur has to again increase the number of repayment dates. Cases 2-1 to 2-K are illustrated in Figure 2.

Based on the pattern that emerges above, we are now in a position to characterize the **general Case N-j**, which has N repayments with the final repayment occuring at date $T-j$. Assume that the amount of required outside financing falls into the interval

$$D \in \left(\sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{(j-1)\Delta}{K^N}, \sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{j\Delta}{K^N}\right].$$

Then the optimal repayment dates are given by $\mathcal{Q} = \{T-(N-1)K-j, T-(N-2)K-j, ..., T-j\}$ and the maximum feasible and incentive compatible repayments are $R_{T-nK-j} = K\Delta$ for all $n = 1, 2, ..., N-1$, and $R_{T-j} = j\Delta$ for the final repayment at date $T-j$. The expected value of this debt contract is

$$\mathcal{D} = \frac{1}{K}K\Delta + \frac{1}{K^2}K\Delta + ... + \frac{1}{K^{N-1}}K\Delta + \frac{1}{K^N}j\Delta,$$

which is equal to the maximum amount of financing that can be raised in case N-j. The general

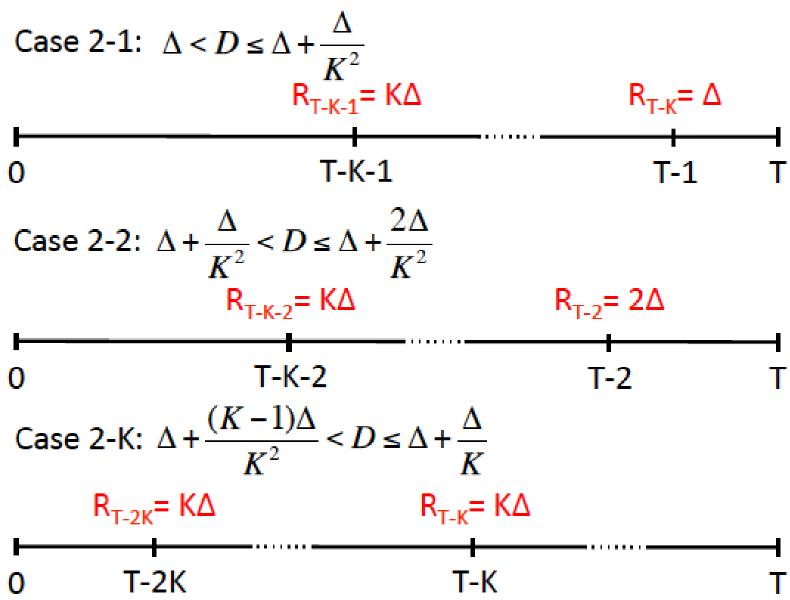

Figure 2: This figure illustrates the range of outside financing needs for which financing is possible with two repayment dates, Case 2-1 to Case 2-K.

case N-j is illustrated in Figure 3.

2.2 Optimal Debt Structure: General Characterization

Based on case N-j, we can now give a full characterization of the optimal repayment schedule $\mathcal{Q}$.

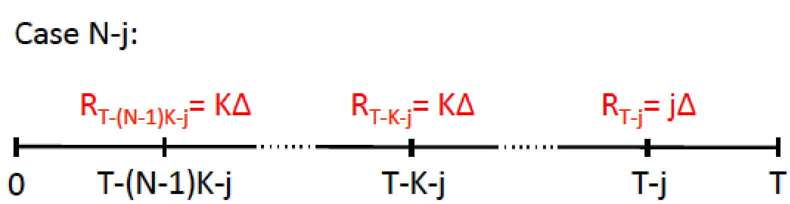

Figure 3: This figure illustrates the general case N-j.

Proposition 1 *In an optimal debt contract, the set of repayment dates is*

$$\mathcal{Q}_{N,j} \equiv \{T-j, T-K-j, T-2K-j, ..., T-(N-1)K-j\}$$

if and only if the required investment

$$D \in \left(\sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{(j-1)\Delta}{K^N}, \sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{j\Delta}{K^N} \right], \tag{9}$$

which is a partition of all feasible investment amounts when (N, j) is any pair of positive integers such that

$$T-(N-1)K-j \geq 0.$$

In addition, when D equals any one of the cutoff values $\sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{j\Delta}{K^N}$, the unique optimal debt repayment schedule is given by

$$R_t = \begin{cases} K\Delta, & \text{if } t \in \mathcal{Q} \backslash \{T-j\} \\ j\Delta, & \text{if } t = T-j \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 1 pins down the firm's optimal debt structure. For a given required amount of required outside financing D, the proposition uniquely characterizes the optimal repayment dates, as well as the optimal payment amounts at each repayment date at the boundary of each of the intervals in (9). At the boundaries, the incentive compatibility constraints bind at each of the repayment dates, such that it is impossible to shift repayments between repayment dates in $\mathcal{Q}$. In between the boundaries of the intervals in (9), the repayment dates are still uniquely determined, but the repayment amounts are not uniquely determined. As shown in Lemma 1, the entrepreneur is then indifferent between all incentive compatible (and feasible) repayment patterns based on the repayment dates $\mathcal{Q}$.

One key feature of the optimal repayment schedule is that, once the firm starts making repayments, these repayments are constantly spaced, separated by K periods to ensure incentive compatibility. Intuitively, because each additional repayment date adds a discrete amount of additional default risk, firms do not smooth their repayments across all periods. Instead, it is optimal to minimize the number of repayments, subject to incentive compatibility and feasibility constraints.[7] When a repayment is missed, the firm is liquidated.[8]

Denoting by $PI(N)$ the maximum pledgeable income of a debt contract with $N \leq \frac{T}{K}$ risky repayment dates, pledgeable income takes the form of a simple geometric sum,

$$PI(N) = \Delta \sum_{n=0}^{N-1} \frac{1}{K^n}. \tag{10}$$

Intuitively, the maximum the firm can pledge with N repayment dates is N repayments of $R_{T-nK} = K\Delta$, each weighted by the probability of making the n^{th} repayment $\frac{1}{K^n}$, where $n = 1, 2, ..., N$. Pledgeable income is therefore increasing in the number of repayments offered by the firm. However, because each additional repayment induces default risk, the firm chooses a repayment schedule that minimizes the number of risky repayments subject to raising the required amount D. Therefore, $PI(N)$ allows us to pin down the number of payment dates of the optimal debt contract:

Corollary 1 *The optimal debt contract has exactly N repayments, if*

$$D \in \left(\Delta \sum_{i=0}^{N-2} \frac{1}{K^i}, \Delta \sum_{i=0}^{N-1} \frac{1}{K^i} \right]. \tag{11}$$

[7] As we show in Section 4.2, firms partially smooth repayments when there is a safe cash flow component, by making a safe repayment every period and risky repayments periodically.

[8] Note that if the cash flow on a repayment date is not sufficient to make the contractual repayment, the firm is generally not able to raise additional financing from another lender, unless is can directly dilute its existing debt. The reason is that once repayments start, the firm has already promised almost all pledgeable future cash flows to the original lender. In particular, when D is at any of the boundaries of (9), all pledgeable future cash flows are exhausted at each repayment.

2.3 Optimal Debt Structure: Implementation

While the optimal debt contract characterized above can be interpreted narrowly as one grand contract that specifies all repayments, a second natural interpretation is that it reflects a firm's aggregate debt structure. For example, individual repayments could correspond to separate loans or bonds. The repayment schedule $\mathcal{R}$ then captures the firm's aggregate maturity profile (Choi et al. (2016)). This second interpretation will be particularly useful when linking the predictions of our model to the empirical literature (see Section 3).

Another interesting observation is that the optimal debt contract $\mathcal{R}$ can always be implemented via a sequence of zero coupon bonds. Under this implementation, the firm borrows D at date zero via a single zero coupon bond, with maturity equal to the first repayment date of the optimal debt contract $\mathcal{R}$. At the first repayment date, the firm makes a net repayment of R_t and rolls over the remaining amount owed to lenders by issuing a new zero coupon bond that matures on the second repayment date.[9]

Corollary 2 *For any debt contract $\mathcal{R}$ with repayment dates $\mathcal{Q} = \{t_1, t_2, ..., t_N\}$, there is a rollover implementation by a sequence of zero coupon bonds. The first bond has a face value of $F_{t_1} = KD$ maturing at date t_1. The ith ($i \geq 2$) bond has a face value of $F_{t_i} = K(F_{t_{i-1}} - R_{t_{i-1}})$ maturing at date t_i. This implementation is dynamically consistent in the sense that at each rollover date, the firm has no strict incentive to issue a different debt contract.*

One implication of Corollary 2 is that in our baseline model, debt capacity is maximized by rolling over short-term debt. This result therefore echoes and extends the classic insight (Bolton and Scharfstein (1990, 1996) and Hart and Moore (1998)) that short-term debt alleviates financing constraints arising from non-verifiable cash flows. Empirically, this result implies that mature firms (i.e., risky period cash flows but no cash-flow growth) maximize debt capacity by issuing short-term

[9] As stated in Corollary 2, this rollover implementation is dynamically consistent. For models in which there are incentives to shorten maturity ex post to dilute existing creditors, see Brunnermeier and Oehmke (2013) and He and Milbradt (2016).

debt. Interestingly, as we show in Section 4, this result no longer holds under more general cash flow distributions, where limiting short-term debt may be necessary to maximize debt capacity.

3 Empirical Implications

In this section, we discuss the key empirical implications of our baseline model. Specifically, we investigate how the cash flow risk, profitability, and leverage affect debt structure: the number of repayments, the spacing between repayments, and the duration of the optimal repayment schedule.

To analyze debt duration, we calculate the expected average repayment time of the optimal debt contract (i.e., similar to Macaulay duration), which is given by

$$ART \equiv \frac{\sum_{i=1}^{N} t_i \frac{R_{t_i}}{K^i}}{D}. \tag{12}$$

One slight complication when analyzing debt duration is that, as shown in Proposition 1, the exact repayment amounts at each repayment date, and therefore the average repayment time ART, are generally not pinned down uniquely.[10] We therefore calculate both the longest and shortest ART as follows.

Lemma 2 *Given any sustainable amount D of outside financing, the longest average repayment time*

$$\sup ART = [T - j - (N-1)K] + \frac{1}{D}\left[\sum_{i=1}^{N-2} \frac{\Delta}{K^{i-1}} i + \frac{j\Delta}{K^{N-1}}(N-1)\right] \tag{13}$$

is attained by back loading repayments: $R_{t_i} = K\Delta$ for all $2 \leq i \leq N-1$, $R_{t_N} = j\Delta$, and $R_{t_1} = K(D - \sum_{i=1}^{N-2} \frac{\Delta}{K^i} - \frac{j\Delta}{K^N})$. The shortest average repayment time

$$\inf ART = (T - j) - \frac{1}{D}\sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}}(N-1-i) \tag{14}$$

[10] Repayment dates are uniquely determined, but except at the boundaries between cases there are a number of repayment amount schedules that raise the required amount of outside financing.

is attained by front loading repayments: $R_{t_i} = K\Delta$ *for all* $i \leq N-1$ *and* $R_{t_N} = K^N \left(D - \sum_{i=0}^{N-2} \frac{\Delta}{K^i} \right)$.

3.1 Cash Flow Risk

Our first set of empirical predictions relates to the riskiness of cash flows, which in our model is captured by the parameter K. The effect of cash flow risk on the optimal debt contract is an immediate corollary of Proposition 1: When cash flow becomes riskier, the repayment profile becomes lumpier, with higher individual promised payments ($K\Delta$) and longer time intervals (K) between two repayment dates. In addition, maximum pledgeable income with N repayments ($PI(N)$) is decreasing in cash-flow risk. As a result, to raise the same amount of outside financing, a firm with riskier cash flows needs to spread its debt repayments across more repayment dates. Combined with the longer intervals between repayments, this implies that the entire repayment profile of the optimal debt contract extends forward. Consequently, debt duration decreases when risk increases.

Proposition 2 *As cash-flow risk* K *increases, holding all other parameters including* D *constant, the number of repayments* $\#Q$ *weakly increases; the time between two repayments increases; and debt duration (in terms of both* $\inf ART$ *and* $\sup ART$*) decreases.*

The prediction that cash flow risk is associated with earlier debt repayment has broad support in the empirical literature on debt maturity. Stohs and Mauer (1996) find that riskier firms (as measured by lower EBITDA volatility) have shorter maturity debt. Barclay and Smith (1995) document that higher volatility of asset returns (implied from equity returns) correlates negatively with the fraction of debt that matures in more than three years. Guedes and Opler (1996) document that higher industry volatility of ROA growth is negatively correlated with maturity.

Our model also provides a potential framework to assess the dispersion in corporate debt maturity profiles documented in Choi et al. (2016), who show that higher rollover risk after the Ford/GM downgrade in 2005 led to more dispersed new debt issuance. Consistent with this finding, in our model higher K, which increases the likelihood for the firm of not being able to roll over its debt, leads to a debt structure with more repayment dates. However, note that the firm does not, in fact, reduce rollover risk by offering more repayment dates. Rather, to raise the same amount of outside financing in the presence of higher rollover risk, firms have to offer more repayments because each individual repayment is less likely to be made.

3.2 Profitability

Second, we examine the effect of the profitability of the investment project on debt structure. The expected period cash flow Δ is a natural measure for profitability. However, given the binary cash flow structure, a change in Δ also affects the variance of the period cash flow, $\Delta^2(K-1)$. To analyze the marginal effect of higher profitability, we therefore increase Δ while holding cash flow variance constant by reducing K. This reduction in K means that in this comparative static the results in Proposition 2 are flipped. In particular, the debt profile of a more profitable company features fewer repayments and shorter intervals between repayment dates. Consequently, profitable firms structure their debt into fewer repayments that are concentrated towards the end of the project's life. Debt duration of profitable firms is therefore longer.

Proposition 3 *As the expected period cash flow Δ increases, holding cash flow variance $\Delta^2(K-1)$ and all other parameters constant, the number of repayments weakly decreases, the time between risky repayments decreases, and* $\sup ART$ *increases. Finally, when $K \geq 4$, also* $\inf ART$ *increases.*

The main empirical prediction of Proposition 3 is that higher profitability is associated with more backloaded repayments. This prediction is consistent with the evidence that profitability is generally associated with longer debt maturity. For example, Qian and Strahan (2007) find that more profitable firms (as measured by net income divided by assets) borrow longer-term. Similarly, Guedes and Opler (1996) show that less profitable firms (as measured by larger operating loss carryforwards) tend to have debt of shorter maturity.

3.3 Leverage

Finally, we analyze the effect of leverage. The easiest way to analyze higher leverage in our model is through a reduction of the firm's cash resources c. Less cash at hand directly translates into a higher require amount of outside financing $D = I - c$, while leaving the NPV of the firm's project unchanged. From Proposition 1, we know that the entrepreneur can increase the amount raised by issuing debt in three ways: (i) by increasing promised repayments within a given case N-j; (ii) by moving existing repayment dates forward (an increase in j); and (iii) by adding more repayment dates (an increase in N). In the latter two scenarios, the optimal debt contract becomes more short-term, in the sense that the average repayment time of the optimal debt contract decreases. Debt duration therefore unambiguously decreases whenever an increase in leverage requires adding a new repayment date or moving forward an existing repayment date . When the entrepreneur simply raises face values for a given set of repayment dates (i.e., within a given case N-j), things are slightly more complicated. In this case, the longest average repayment time $\sup ART$ is decreasing in D, while the shortest average repayment time $\inf ART$ is increasing in D. This asymmetry happens because to attain $\sup ART$, repayments are back loaded and a higher D increases the first repayment, which shortens ART. On the other hand, repayments are front loaded under $\inf ART$, and a higher D therefore increases the last repayment, which lengthens ART. Taken together, we arrive at the following proposition.

Proposition 4 *Holding all other parameters constant, an increase in leverage (higher D) weakly increases the number of repayments N. Moreover, higher leverage leads to a decrease in the average repayment time of the optimal debt contract across cases N-j, i.e., whenever the increase in D changes the payment dates Q. Within a given case N-j, the longest average repayment time* $\sup ART$ *is decreasing in D while the shortest average repayment* $\inf ART$ *is increasing in D.*

Empirically, our model therefore predicts that the debt structures of highly levered firms are more front-loaded, with more repayment dates. As a result, high leverage is predicted to be associated with shorter debt duration. This finding is consistent with the evidence on the debt structure of leveraged buyout deals in Axelson et al. (2013), who document that repayment profiles of buyout deals are more frontloaded during times when deals are highly levered. In contrast, Barclay and Smith (1995) and Stohs and Mauer (1996) document a negative correlation between leverage and debt maturity. One reason for this difference may be that buyouts, where the entire repayment structure is optimized at the time of the deal, more closely correspond to the setting in our paper than looking at snapshots of the average maturity of existing debt or new debt issuances.

4 Cash Flow Growth and Positive Low Cash Flow

In this section, we add two features to our baseline model. First, in Section 4.1 we allow for growth in the project's cash flow. Second, in Section 4.2 we allow for positive cash flow in the low cash-flow state. We will show that, in contrast to the results in Section 2.3, in both of these cases, pledgeable income is generally no longer maximized by offering as many risky repayments as possible. Rather, pledgeable income is generally largest under a contract that limits the number of risky repayment dates to strictly less than the maximum feasible number. Depending on the situation, the resulting contract then resembles risky long-term debt or a combination of safe short-term and risky long-term debt.

4.1 Cash Flow Growth

Suppose that the positive cash flow realizations grow at the rate $\mu > 1$. Specifically, at any date $t \in \mathcal{T}$, the cash flow is given by $X_t \in \{K\mu^t\Delta, 0\}$. As in the baseline model, the probability of receiving a positive cash flow $K\mu^t\Delta$ at date t is $\frac{1}{K}$, such that the expected cash flow at date t is $\mu^t\Delta$.

This cash flow distribution differs from the baseline model mainly in that the maximum feasible repayment now depends on the time when the particular repayment is made. In contrast, in the baseline model, the maximum feasible repayment is $K\Delta$, which is time-invariant. The specification with growth in cash flow may be particularly relevant for young firms, growth firms, and other situations in which the firm's capacity to produce cash flow increases over time.

Similar to the baseline model, where we assumed that K is an integer, we now make an analogous assumption on the pair of (K, μ).

Assumption 1 *There exists $m \in \mathbb{Z}_+$ such that*

$$K = \sum_{s=1}^{m} \mu^s. \tag{15}$$

Assumption 1 ensures that it is incentive compatible for the firm to repay the maximum feasible amount $K\mu^t\Delta$ at t, if the next m periods' cash flows are left to the entrepreneur,

$$K\mu^t\Delta = \mu^{t+1}\Delta + \mu^{t+2}\Delta + ... + \mu^{t+m}\Delta,$$

where m is an integer. As a result, it is incentive compatible for the entrepreneur to repay $K\mu^t\Delta$ every m periods.

Some of the main insights from the baseline model remain valid with growth in cash flow. As before, the entrepreneur would like minimize the number of risky repayments and schedule them as late as possible, subject to maintaining incentive compatibility. Moreover, once the firm starts making repayments, these are constantly spaced. The slight difference is that cash-flow growth allows risky repayments to be scheduled closer to each other, at intervals of $m < K$.

Despite these similarities, one key implication of the baseline model changes when we allow for growth in cash flow. Whereas in the baseline model pledgeable income is maximized by scheduling as many repayments as possible (recall equation (10)), in the presence of cash flow growth, increasing the number of repayments no longer necessarily increases pledgeable income. To see this, note that the key to increasing pledgeable income by introducing an additional repayment is that the value of existing repayments, which are shifted forward to accommodate the additional repayment, remains unchanged. As illustrated in Figure 4, this is no longer the case when there is cash flow

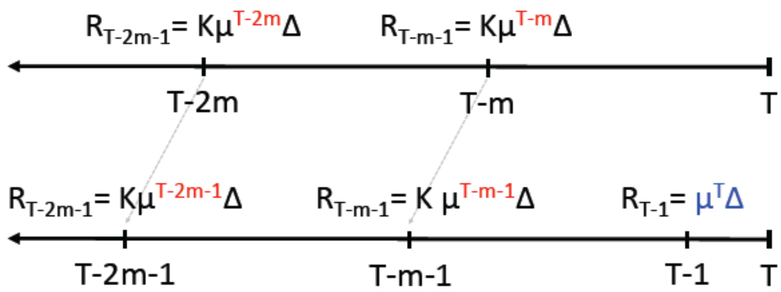

Figure 4: In the presence of cash-flow growth, shifting forward existing repayments and adding an additional repayment can reduce pledgeable income. Existing repayments that are shifted forward need to be reduced by the growth factor μ to preserve feasibility. The resultant reduction in the value of all existing repayments outweighs the extra pledgeable income generated by the additional repayment (equal to $\frac{1}{K^N}\mu^T\Delta$) when the number of existing repayments N is sufficiently large.

growth. In particular, when cash flows grow over time, positive cash-flow realizations are smaller in earlier periods, such that existing repayments have to be scaled down when they are shifted forward. Therefore, whether adding an additional repayment increases pledgeable income depends on which effect dominates, the decrease in the value of existing repayments that are being shifted forward or the value of the additional repayment that is added at the end.

When the number of existing repayments N is large, the reduction in expected repayments from the first N repayment dates dominates: Shifting forward the existing N repayments by one period reduces their values by the growth factor μ. On the other hand, the value of the additional repayment (e.g. $\mu^T\Delta$ at date T in Figure 4) is weighted by the probability that the firm survives past the first N repayments, $\frac{1}{K^N}$, and therefore becomes arbitrarily small when N is large. As a result, for large N a further increase in the number of repayments decreases pledgeable income. Pledgeability is then maximized with N^* repayments, where $N^*>0$ is given by the smallest integer such that

$$\left[(\mu^{-1}-1)K\sum_{j=1}^{N^*+1}\left(K\mu^{-m}\right)^j\right]+1<0. \tag{16}$$

Because $\mu^{-1}-1<0$ and $K\mu^{-m}>1$, N^* is well defined and unique. Importantly, N^* is independent

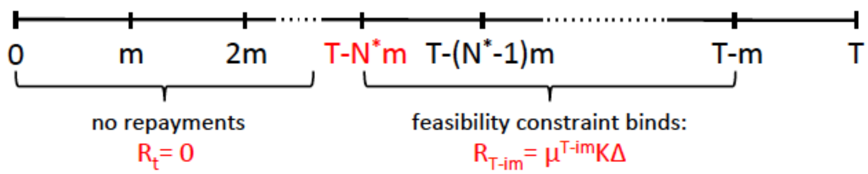

Figure 5: When there is growth in cash flow, pledgeable income is generally maximized with a fixed number of N^* repayments towards the end of the project.

of T. Therefore, even when the number of possible repayment dates T grows large, pledgeability continues to be maximized with a fixed number of N^* repayments, as illustrated in Figure 5.

Proposition 5 *In the model with growing cash flows ($\mu > 1$),*

1. *the maximum pledgeable income $PI(N)$ is maximized at N^* for any T sufficiently large;*

2. *for any $N \leq N^*$, the maximum pledgeable income with N repayments $PI(N)$ is*

$$PI(N) = \sum_{i=0}^{N-1} \frac{\mu^{T-(N-i)m}}{K^i} \Delta;$$

3. *for any $N \leq N^*$, if $D \in (PI(N-1), PI(N)]$, the optimal debt contract has N repayment dates and has the first repayment date $t_1 \geq T - Nm$;*

4. *for any $N \leq N^*$, if $D = PI(N)$, there is a unique optimal debt contract characterized by*

$$R_t = \begin{cases} \mu^t K \Delta, & \text{if } t \in \{T-m, T-2m, ..., T-Nm\} \\ 0, & \text{otherwise.} \end{cases}$$

One implication of Proposition 5 is that for some firms it may never be optimal to schedule more than one repayment, regardless of the financing need D:

Corollary 3 *When $1 > \frac{1}{K} + \mu^{-m}$, the maximum number of repayment dates N^* is 1.*

From Proposition 5 and Corollary (3), we see that in the presence of cash-flow growth, the optimal debt contract resembles long-term debt. Independent of the project's horizon T, all repayments occur in the final N^*m periods. In particular, as T becomes large, the earliest possible repayment date under the optimal debt contract, $T - N^*m$, approaches T, in the sense that $\lim_{T\to\infty} \frac{T-N^*m}{T} = 1$.

The finding that long-term debt can maximize pledgeability puts an interesting twist on our understanding of the role of short-term debt in increaseing pledgeability via a termination threat. In two period models, for the threat of termination to be credible, debt essentially has to be short-term, one-period debt. When many repayment dates are possible, on the other hand, it is possible that the debt maturity that maximizes pledgeability roughly matches the project's horizon, especially when T is large. Depending on parameters, our model can therefore capture both, incentives to finance with short-term debt (leading to maturity mismatch) and incentives to finance with longer term debt (approximate matching of the maturities of assets and liabilities).

4.2 Positive Low Cash Flow

In this section, we extend the model to allow for a risk-free cash flow component L. Specifically, assume that the cash flow distribution X_t is binary with a high cash flow of $L+K\Delta$ with probability $\frac{1}{K}$ and a low (but positive) cash flow of L with complementary probability. The average per-period cash flow is $\Delta + L$. As before, we assume that $\Delta > 0$, and that $K > 1$ is an integer.

Obviously, if $D \leq (T-1)L$, the optimal debt structure is to repay by risk-free cash flows up to L at every $t \in \{1, 2, ..., T-1\}$. One can also easily verify that any risk-free repayment profile is indeed incentive compatible. More generally, Proposition 6 shows that when the risk-free cash-flow component is sufficiently large, it is never optimal to use risky debt, and pledgeable income is maximized by making a risk-free repayments at every date, as illustrated in Figure 6.

Proposition 6 *If*

$$L \geq \frac{\Delta}{K-1}, \tag{17}$$

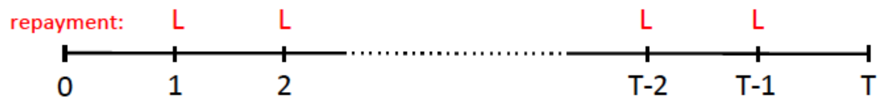

Figure 6: When $L \geq \frac{\Delta}{K-1}$, pledgeable income is maximized by making a risk-free repayment of L every period.

then the risk-free schedule $R_t = L$ for all $t \in \{1, 2, ..., T-1\}$ maximizes pledgeable income.

The intuition behind Proposition 6 is as follows. The benefit of increasing the repayment beyond the risk-free level is that the entrepreneur pays back more when the high cash flow realizes, which improves pledgeable income. However, this risky repayment also generates default risk, which hurts the expected value of the current as well as all subsequently scheduled repayments. Therefore, risky repayments are never optimal when the risk-free cash flow L is large compared to the expected benefit of adding a risky repayment component of Δ, as implied by condition (17). In addition, condition (17) is more likely to hold when cash flow risk is large (high K). In this case, default risk is higher, so that risk-free debt is more likely to be optimal.

For the remainder of this section, we assume (17) does not hold, in order to focus on the case in which introducing risky repayments can increase pledgeability. As in the case with cash-flow growth analyzed in Section 4.1, some of the baseline results continue to hold. Specifically, all risky repayments continue to be scheduled towards the end of the project. In addition, in order to minimize the number of risky repayments, every risky repayment is set to the entire high cash flow realization of $K\Delta + L$, and risky repayments are spaced K periods apart. However, similar to the case with cash-flow growth, we again find that, when there is a risk-free cash-flow component, pledgable income is usually maximized by limiting the number of risky repayments, in this case to $N^{**} > 0$, where N^{**} is the smallest integer such that

$$\frac{\Delta + L}{K^{N^{**}+1}} < L.$$

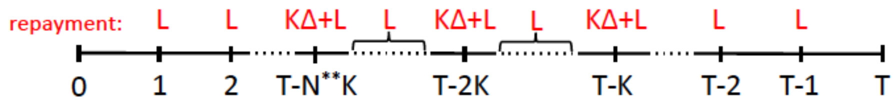

Figure 7: When $L < \frac{\Delta}{K-1}$, pledgeability is maximized by limiting the number of risky repayment dates to N^{**}. Risky repayments are scheduled towards the end of the project and spaced K periods apart.

We summarize these finding in the following proposition.

Proposition 7 *In the presence of a risk-free cash flow component L,*

1. *the maximum pledgeable income $PI(N)$ is maximized with N^{**} repayment dates for any T sufficiently large;*

2. *for any $N \le N^{**}$, the maximum pledgeable income is*

$$PI(N) = (T - NK)L + \sum_{j=1}^{N} \frac{\Delta + L}{K^{j-1}};$$

3. *for any $N \le N^{**}$, if $D \in (PI(N-1), PI(N)]$, the optimal debt contract has N repayment dates, and the first risky repayment will be made at $t_1 \ge T - NK$;*

4. *for any $N \le N^{**}$, if $D = PI(N)$, there exists a unique optimal contract that is characterzied by*

$$R_t = \begin{cases} K\Delta + L, & \text{if } t \in \{T-K, T-2K, ..., T-NK\} \\ L & \text{otherwise.} \end{cases}$$

Part 3 of Proposition 7 shows that in the presence of a risk-free cash flow component, scheduling as many risky repayments as possible does not generally maximize pledgeability. While this result is similar to the case with cash-flow growth, the intuition for limiting the number of risky repayments is different. As the entrepreneur moves the repayment schedule forward by one period to increase risky repayments, she sacrifices one period with a risk-free repayment of L. The contribution to the firm's pledgeable income from the final risky repayment is weighted by the probability of making this repayment $\frac{1}{K^N}$ (if there are N risky repayments). Therefore as N becomes very large, the benefit from the last risky repayment diminishes exponentially, and the cost of sacrificing a risk-free repayment of L dominates.[11]

Part 4 of Proposition 7 shows that a risk-free cash-flow component leads to incentives to partially smooth repayments over time. While the firm continues to only offer periodic risky repayments (if any), the firm pays out the risk-free cash flow component L every period.

As in the baseline model, the optimal debt structure can be implemented in a number of ways. In particular, analogous to the rollover implementation discussed in Corollary 2, in the presence of a (relatively small) safe cash-flow component, the optimal debt structure can be implemented by rolling over a sequence of coupon bonds with a fixed coupon of L and declining face values F_t, where t denotes the maturity date of the bond (at the maturity date, both a coupon and the face value are paid).

Corollary 4 *For any debt contract $\mathcal{R}$ characterized by part 4 of Proposition 7, there exists a rollover implementation by a sequence of coupon bonds with a fixed coupon L. The first coupon bond has a face value of $F_{T-NK} = K\left[D - (T-NK)L\right] - L$ maturing at date $T - NK$. The ith ($i \geq 2$) bond has a face value of $F_{T-(N-i+1)K} = K\left[F_{T-(N-i+2)K} - K\left(\Delta + L\right)\right] - L$ maturing at date t_i. This implementation is dynamically consistent in the sense that at each rollover date, the firm has no strict incentive to issue a different debt contract.*

[11] This result is quite robust. For example, we show in Section 5.2 that, for large class of general cash flow distributions, pledgeability is maximized by limiting the number of risky repayments. In particular, the entrepreneur generally does not "smooth out" risky repayments over the entire lifespan of the project.

5 Extensions

5.1 Allowing for Savings

Up to now, we have assumed that the entrepreneur can only use contemporaneous cash flow to make repayments. In this section, we relax this assumption and sketch how our analysis generalizes to the case in which the entrepreneur can save.

The main difference relative to the the baseline model in Section 1 is that the entrepreneur can now use cash flows realized in previous periods to make repayments at a later date.[12] This has two key implications. First, savings make it feasible for the entrepreneur to make payments that exceed the period cash flow; R_t could be strictly greater than $K\Delta$. Second, savings introduce a non-trivial trade-off between repayment amounts and default risk. Specifically, when the entrepreneur can save, it can be optimal to offer smaller repayments at a later date in order to allow the firm to accumulate cash, making these repayments less risky. Consequently, the firm may offer repayments smaller than $K\Delta$, spaced strictly less than K periods apart.

In the remainder of this section, we discuss these two effects of savings in turn and show that, under certain conditions, the two key features of the model without savings, repayments of $K\Delta$ spaced K periods apart, remain optimal even when the entrepreneur can save.

First, we show that it is never optimal to offer repayments strictly greater than $K\Delta$, even when the ability to save makes this possible. To see this, consider a single, larger repayment of $R_t = 2K\Delta$. For simplicity, suppose this is the last repayment. For this repayment to be incentive compatible, it must occur on or before date $T-2K$, otherwise the surplus left to the entrepreneur after making this repayment would be less than $2K\Delta$, violating incentive compatibility. Now consider splitting up this single repayment into two repayments of $K\Delta$ at dates $T-2K$ and $T-K$. It is easy to see this new schedule is incentive compatible. More importantly, though, splitting the repayment strictly improves the payoff to the entrepreneur: For any realization of cash flows, if the entrepreneur can

[12]In that respect, introducing savings is similar to allowing for callable debt, something we also ruled out in our baseline model.

repay $2K\Delta$ at date $T-2K$, she can also make the two repayments of $K\Delta$ under the new schedule. But the new schedule allows the entrepreneur strictly more time (and thereby a higher chance) to repay the second $K\Delta$. Therefore, there are cash flow realizations under which the entrepreneur is liquidated under the original schedule, but not under the adjusted schedule. By this logic, one can show that any debt contract with any individual repayment greater than $K\Delta$ is strictly dominated, as stated by the following proposition.

Proposition 8 *In the optimal debt contract with savings, any individual repayment R_t is weakly smaller than $K\Delta$.*

Second, we consider the firm's incentive to offer payments of less than $K\Delta$ spaced less than K periods apart. To do so, we start with the contract that maximizes pledgeable income in the baseline model, namely repayments of $K\Delta$ every K periods, starting from date 1. Note that the expected value of the first repayment, which is made with probability $\frac{1}{K}$, is Δ. Now consider moving this first repayment to date 2, leaving the remaining repayment schedule unchanged and reducing the first repayment to $(K-1)\Delta$ to preserve incentive compatibility. Even though the first repayment is now smaller, the probability of making this smaller repayment at date 2 is strictly higher: $1-\left(1-\frac{1}{K}\right)^2 > \frac{1}{K}$. As a result, the present value of the first repayment $\left[1-\left(1-\frac{1}{K}\right)^2\right](K-1)\Delta$ is greater than that under the original schedule Δ when $K>2$. In addition, under the adjusted repayment schedule, the continuation probability at the first repayment date is higher, making all future repayments more valuable.

The previous example shows that, under some conditions, savings can alter the structure of repayments of size $K\Delta$ every K periods. In particular, when leverage is high, such that the first repayment is relatively early, an additional period to accumulate savings has a large effect on the probability of making the first repayment. In this case, the optimal repayment schedule with savings will differ from the optimal repayment schedule in the baseline model. For lower leverage, on the other hand, the first repayment is not that early and the basic repayment structure of the baseline model (repayments of $K\Delta$ every K periods once repayments start) is still optimal, even when the entrepreneur can save. Of course, the ability to save leads to less default along the equilibrium path relative to the baseline model.

5.2 General Cash Flow Distribution

In this section, we briefly discuss the role of the binary cash flow distribution. In particular, we show relatively weak conditions under which the key result in section 4, that the entrepreneur wants to limit the number of repayment dates, carries over to more general cash flow distributions.

Proposition 9 *Suppose the cash flow distribution $F(X)$ has the following properties:*

There exists some positive cash flow level $L > 0$ such that

1. *$X \geq L$ holds with probability 1;*
2. *$F(L) = \epsilon > 0$;*
3. *X has finite expectation.*

Then for any T sufficiently large, any repayment profile that maximizes pledgeable income has strictly less than $2N^*$ risky repayments, where $N^* > 0$ is the smallest integer that satisfies

$$\frac{L}{E(X)} > \max\{(1-\epsilon)^{N^*-2}, 2(1-\epsilon)^{2N^*-1}\}. \tag{18}$$

According to Proposition 9, the cash flow distribution needs to satisfy two main properties such that limiting the number of repayment dates is optimal. First, the period cash flow distribution has a positive lower bound L and, second, there is a mass point at L. (The third conditions simply makes sure that the expected period cash flow is well defined.) Note that these conditions are relatively weak. For example, any discrete cash flow distribution with strictly positive support satisfies these requirements.

The intuition for Proposition 9 is essentially the same as Part 3 of Proposition 7: The benefit of an additional risky repayment is weighted by the survival probability, which decreases exponentially with the number of risky repayments. The cost of an additional risky repayment, on the other hand, is constant; the entrepreneur sacrifices one a risk-free repayment of L. Therefore, at some point adding another risky repayment reduces pledgeable income.

6 Conclusion

This paper provides a model of optimal debt structure. Building on the insights of the literature on debt as a termination threat, which as mostly worked in two-date settings, our multi-period model generates rich implications on the optimal number, timing, and size of payments to creditors. The optimal debt structure is determined by a simple trade-off between providing the firm with incentives to repay and preventing costly early liquidation.

The model generates a rich set of empirical predictions: Depending on the required amount of outside financing and the cash-flow characteristics of the firm, the resulting debt structures can resemble a sequence of risky short-term debt contracts (firms with stable expected cash flow and large outside financing needs), long-term debt (growth firms) and safe short-term debt (firms with a significant safe cash-flow component), or a combination of safe short-term debt and risky bonds or loans (firms with a moderate safe cash-flow component).

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A Omitted Proofs

Proof of Lemma 1: Because $\mathcal{Q}(\mathcal{R}) = \mathcal{Q}(\mathcal{R}')$, and X_t is either $K\Delta$ or 0, equation (6) implies that $\Pr(X_t \geq R_t) = \Pr(X_t \geq R'_t)$ for any t. Therefore,

$$\sum_{t=0}^{T}\prod_{s=0}^{t-1}\Pr\left(X_s \geq R_s\right)\Delta = \sum_{t=0}^{T}\prod_{s=0}^{t-1}\Pr\left(X_s \geq R'_s\right)\Delta.$$

In addition, since $\mathcal{D}(\mathcal{R}) = \mathcal{D}(\mathcal{R}')$, it follows from equation (5) that $\mathcal{R}$ and $\mathcal{R}'$ will lead to the same V_0. Therefore, the entrepreneur is indifferent between $\mathcal{R}$ and $\mathcal{R}'$.

Proof of Proposition 1: We prove this proposition by a series of claims.

Claim 1: For any two incentive compatible debt contracts, $\mathcal{R}$ and $\mathcal{R}'$, if $\mathcal{D}(\mathcal{R}) = \mathcal{D}(\mathcal{R}')$ and $\mathcal{Q} \subset \mathcal{Q}'$, then the entrepreneur strictly prefers $\mathcal{R}$. Put differently, other things equal, the entrepreneur wants to reduce the number of repayments.

To see this, first note that since $\mathcal{D}(\mathcal{R}) = \mathcal{D}(\mathcal{R}')$, it follows from Lemma 1 that $V_0(\mathcal{R}) > V_0(\mathcal{R}')$ if and only if

$$\sum_{t=0}^{T}\prod_{s=0}^{t-1}\Pr\left(X_s \geq R_s\right)\Delta - \sum_{t=0}^{T}\prod_{s=0}^{t-1}\Pr\left(X_s \geq R'_s\right)\Delta > 0.$$

Because $\mathcal{Q} \subset \mathcal{Q}'$, $\Pr(X_s \geq R_s) \geq \Pr(X_s \geq R_s')$ for all $s \in \mathcal{T}$. However, since at least one element in $\mathcal{Q}'$ does not belong to $\mathcal{Q}$, there is at least one $s' \in \mathcal{T}$ such that $R_{s'} = 0$ and $R_{s'}' \in (0, K\Delta]$. Hence, at s', $\Pr(X_{s'} \geq R_{s'}) = 1 > 1/K = \Pr(X_{s'} \geq R_{s'}')$. Therefore, the entrepreneur strictly prefers $\mathcal{R}$.

Claim 2: Denote by $\#\mathcal{Q}(\mathcal{R})$ the number of repayments of the debt contract $\mathcal{R}$ and by ϱ the vector of the repayment dates. If $\mathcal{D}(\mathcal{R}) = \mathcal{D}(\mathcal{R}')$, $\#\mathcal{Q}(\mathcal{R}) = \#\mathcal{Q}(\mathcal{R}')$, and $\varrho > \varrho'$ (that is, any element of ϱ is greater than or equal to ϱ', and at least one element of ϱ is strictly greater than the corresponding element of ϱ'), then the entrepreneur strictly prefers $\mathcal{R}$. Put differently, if two incentive compatible debt contracts have the same value and the same number of repayments, the entrepreneur prefers the one with late repayments.

Because $\varrho > \varrho'$, for any t, $\prod_{s=0}^{t-1} \Pr(X_s \geq R_s) \geq \prod_{s=0}^{t-1} \Pr(X_s \geq R_s')$, and there exists a repayment date $t_j \in \mathcal{Q}(\mathcal{R})$ that comes strictly later than the corresponding repayment date $t_j' \in \mathcal{Q}(\mathcal{R}')$. Then, at t_j, $\prod_{s=0}^{t_j-1} \Pr(X_s \geq R_s) > \prod_{s=0}^{t_j-1} \Pr(X_s \geq R_s')$. Therefore, the entrepreneur strictly prefers $\mathcal{R}$.

We next prove Corollary 1 as a lemma for Proposition 1, even though for exhibition purposes, the result is stated in the paper as a corollary.

Before proving the corollary, we state a repeatedly used adjustment procedure to the debt contract as a lemma.

Lemma 3 *Suppose $t_i, t_j \in \mathcal{Q}(\mathcal{R})$ are two repayment dates, with $R_{t_j} < K\Delta$. Define "(t_i, t_j, ϵ) adjustment" to be the following procedure to construct a new contract $\mathcal{R}'$: $R_{t_i}' = R_{t_i} - \epsilon$ and $R_{t_j}' = R_{t_j} + \frac{\epsilon}{K^{i-j}}$, leaving all other repayments unchanged. Then, the value of debt is unchanged, $\mathcal{D}(\mathcal{R}') = \mathcal{D}(\mathcal{R})$. In addition, if $t_i > t_j$, then $\mathcal{R}'$ is also incentive compatible.*

Proof of Lemma 3: First, it is straight forward from (4) that $\mathcal{D}(\mathcal{R}') = \mathcal{D}(\mathcal{R})$. Next, if $t_i > t_j$, let V' be the entrepreneur's continuation value under contract $\mathcal{R}'$. It is clear from (3) that $V_t' \geq V_t$ holds for all t and strictly for $t_j < t \leq t_i$. In particular, $V_{t_j+1}' = V_{t_j+1} + \frac{\epsilon}{K^{i-j}}$, so condition $R_{t_j}' \leq V_{t_j+1}'$ still holds. IC conditions for other repayments are trivially satisfied.

Proof of Corollary 1: Consider any debt contract $\mathcal{R}$ with $\#\mathcal{Q}(\mathcal{R}) \leq n-1$ first. Let the i^{th} repayment date be $t_i \in \mathcal{Q}$. Note that the maximum amount of any single repayment is $K\Delta$; and R_{t_i} is actually paid if and only if $X_{t_\tau} = K\Delta$ for all $\tau \leq i$, which happens with probability $1/K^i$. Therefore, the maximum total expected repayments of $\mathcal{R}$ with at most $n-1$ repayments is

$$\sum_{i=1}^{n-1} \left[\frac{1}{K^i}(K\Delta)\right] = \Delta \sum_{j=0}^{n-2} \frac{1}{K^i} < D$$

by equation (11). Hence, investor's IR constraint (1) implies that there must be at least n repayments: $\#\mathcal{Q}(\mathcal{R}) \geq n$.

Next, we show that any contract $\mathcal{R}$ with $\#\mathcal{Q}(\mathcal{R}) = n+k$, where $k \geq 1$, can be strictly improved. Suppose there is a positive integer $j < n+k$ such that $R_{t_j} < K\Delta$. Then, we can apply $(n+k, j, \epsilon)$ adjustment until either all initial $n+k-1$ repayments equal $K\Delta$ or the last one repayment $R_{t_{n+k}}' = 0$. In the first case,

$$\sum_{j=1}^{n+k} \left[\frac{1}{K^j} R_{t_j}\right] > \sum_{j=1}^{n+k-1} \left[\frac{1}{K^j}(K\Delta)\right] \geq D,$$

so the total expected value of repayments exceeds D, contradicting the IR constraint (1). In the second case, the entrepreneur can eliminate the last repayment without affecting the value of debt. By Claim 1, the adjusted repayment schedule is strictly preferred. Hence, we establish Corollary 1.

Now, we prove Proposition 1. If D satisfies (9), then it automatically satisfies (11). Corollary 1 implies that the optimal debt contract will include exactly N repayments. Denote the optimal debt contract by $\mathcal{R}^*$.

Next, we inductively prove that the ith repayment occurs at $t_i^* = T - j - (N-i)K$. We first establish the statement for $i = N$, namely $t_N^* = T - j$. Consider any debt contract $\mathcal{R}$ whose last repayment date is later than $T - j$. The incentive compatibility constraint implies that $R_{t_N} \leq (j-1)\Delta$. However, since the expected value of the first $N-1$ repayments is at most $\sum_{i=0}^{N-2} \frac{\Delta}{K^i}$ (attained when every repayment is $K\Delta$), $D \leq \sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{(j-1)\Delta}{K^N}$, violating equation (9).

Consider any debt contract $\mathcal{R}$ with the last repayment date $t_N < T - j$. We can apply (t_N, t_i, ϵ) adjustment until $R_{t_i} = K\Delta$ for all $i < N$. After such an adjustment, condition (9) implies $R_{t_N} \leq j\Delta$. So, the entrepreneur can delay R_{t_N} to $T - j$ without affecting incentive compatibility and the value of debt, which by Claim 2, makes entrepreneur strictly better off. Hence, in $\mathcal{R}^*$, $t_N^* = T - j$.

Suppose $t_s^* = T - j - (N-s)K$ for all $s \geq i+1$. We now prove the statement for t_i^*. First, starting from $\mathcal{R}^*$, we can apply (t_i^*, t_l^*, ϵ) adjustment for all $l < i$, until $R_{t_l} = K\Delta$. Next, apply (t_i^*, t_l^*, ϵ) adjustment for all $l > i$, until $R_{t_l^*} = K\Delta$ (if $l < N$) or $j\Delta$ (if $l = N$). By Lemma (3), both adjustments do not affect the value of debt, and the first one is incentive compatible. It is easy to see that the second adjustment is also incentive compatible because the induction assumption implies that $V_{t_l^*} = K\Delta$ (or $j\Delta$) for all $i < l < N$ (or $l = N$).

After the adjustment,

$$R_{t_i^*} = \left(D - \sum_{l=0, l \neq i-1}^{N-2} \frac{\Delta}{K^l} - \frac{j\Delta}{K^N} \right) K^i.$$

From (9), $R_{t_i^*} \in \left(K\Delta - \frac{\Delta}{K^{N-i}}, K\Delta\right] \subset ((K-1)\Delta, K\Delta]$. Therefore, IC condition for $R_{t_i^*}$ implies that $t_i^* \leq t_{i+1}^* - K$. If $t_i^* < t_{i+1}^* - K$, we can simply move $R_{t_i^*}$ to a later date: $t_{i+1}^* - K$. It is easy to see that such an adjustment does not affect the value of debt and is still incentive compatible. By Claim 2, the new contract dominates $\mathcal{R}^*$, contracting with the optimality of $\mathcal{R}^*$. Therefore, the induction conclusion holds for i and $t_n^* = T - j - (N-n)K$ for any $n \leq N$ in $\mathcal{R}^*$.

Finally, when $D = \sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{j\Delta}{K^N}$, we know from the previous proof that $\mathcal{Q}(\mathcal{R}^*) = \{T-j, T-j-K, \ldots, T-j-(N-1)K\}$. IC conditions imply that $R_{t_i}^* \leq K\Delta$ for $i < N$ and $R_{t_N}^* \leq j\Delta$. As a result, $\mathcal{D}(\mathcal{R}^*) \leq D$, with equality holding if and only if $R_{t_i}^* = K\Delta$ for $i < N$ and $R_{t_N}^* = j\Delta$. This establishes the uniqueness and completes the proof.

Proof of Lemma 2: Suppose D satisfies (9). Proposition (1) uniquely determines the set of repayment dates. Suppose $\sup ART$ is attained by some schedule $\mathcal{R}$ other than the one specified in the lemma. Then, there must exist an $i \in [2, N-1]$ such that $R_{t_i} < K\Delta$, or $i = N$ and $R_{t_i} < j\Delta$. Given such an i, consider an alternative schedule $\mathcal{R}'$ given by a (t_1, t_i, ϵ) adjustment. It is clear that when $\epsilon < \frac{K\Delta - R_{t_i}}{K^N}$ (or $\frac{j\Delta - R_{t_N}}{K^N}$ if $i = N$), schedule $\mathcal{R}'$ is still incentive compatible. By Lemma 3, $\mathcal{D}(\mathcal{R}') = \mathcal{D}(\mathcal{R})$. The adjusted schedule $\mathcal{R}'$ increases ART:

$$ART(\mathcal{R}') - ART(\mathcal{R}) = \frac{\epsilon(t_i - t_1)}{KD} > 0.$$

Contradiction! So $\sup ART$ is uniquely attained by $R_{t_N} = j\Delta$, $R_{t_i} = K\Delta$ for all $i \in [2, N-1]$, and

$R_{t_1} = K(D - \sum_{i=1}^{N-2} \frac{\Delta}{K^i} - \frac{j\Delta}{K^N})$. Substituting these into equation (12) implies

$$\begin{aligned}
&\sup ART \\
=&\frac{1}{D}\left\{(D - \sum_{i=1}^{N-2} \frac{\Delta}{K^i} - \frac{j\Delta}{K^N})(T - j - (N-1)K)\right. \\
&\left. + \sum_{i=1}^{N-2} \frac{\Delta}{K^i}\left(T - j - (N-1-i)K\right) + \frac{j\Delta}{K^N}(T - j)\right\} \\
=&(T - j) - \frac{1}{D}\left[\sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}}\left(N - i - 1\right) - \left(N - 1\right)\left(K\Delta - R_{t_1}\right)\right] \\
=&(T - j) - \frac{1}{D}\left[\sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}}\left(N - i - 1\right) + K\left(N - 1\right)\left(D - \sum_{i=0}^{N-2} \frac{\Delta}{K^i} - \frac{j\Delta}{K^N}\right)\right],
\end{aligned}$$

which is equivalent to equation (13).

Similarly, in order to attain $\inf ART$, the entrepreneur wants to front-load repayments, so that the weights on earlier repayment dates are as large as possible. This is done by setting $R_{t_i} = K\Delta$ for all $i \leq N-1$ and $R_{t_N} = K^N\left(D - \sum_{i=0}^{N-2} \frac{\Delta}{K^i}\right)$. Substituting these into equation (12) implies

$$\begin{aligned}
&\inf ART(D) \\
=&\frac{\sum_{i=0}^{N-2} \frac{\Delta}{K^i}\left(T - j - (N-1-i)K\right) + (D - \sum_{i=0}^{N-2} \frac{\Delta}{K^i})(T - j)}{D} \\
=&(T - j) - \frac{1}{D}\sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}}(N - 1 - i),
\end{aligned}$$

which is exactly equation (14).

Proof of Proposition 2: Corollary 3 together with the fact that $\Delta \sum_{i=0}^{N-1} \frac{1}{K^i}$ is decreasing in K directly imply that $\#\mathcal{Q}$ is weakly increasing in K. It follows from Proposition 1 that if $t_i, t_{i+1} \in \mathcal{Q}$, then $t_{i+1} - t_i = K$. So, it is obvious that the time interval between two consecutive repayments is strictly increasing in K.

We now study $\inf ART$ and $\sup ART$ as K increases to $K+1$. Let's first consider $\inf ART$. Similarly to Proposition 4, there are two cases.

Case 1: Suppose $\mathcal{Q}$ does not change as K increases to $K+1$. When $N = 1$, $\inf ART$ does not change, because j does not change. For $N \geq 2$, we have (noting that $K \geq 2$ by assumption)

$$
\begin{aligned}
& D\left[\inf ART(K) - \inf ART(K+1)\right] \\
=& \sum_{i=0}^{N-2} \frac{\Delta}{(K+1)^{i-1}}(N-1-i) - \sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}}(N-1-i) \qquad (19)\\
=&(N-1)\Delta + \sum_{i=2}^{N-2}\left[\frac{\Delta}{(K+1)^{i-1}} - \frac{\Delta}{K^{i-1}}\right](N-1-i) \\
\geq& \Delta(N-1)\left[1 + \sum_{i=2}^{N-2}\left[\frac{1}{(K+1)^{i-1}} - \frac{1}{K^{i-1}}\right]\right] \\
>& \Delta(N-1)\left[1 - \sum_{i=1}^{\infty}\frac{1}{K^i}\right] \geq 0. \qquad (20)
\end{aligned}
$$

Therefore, when K increases to $K+1$ and $\mathcal{Q}$ does not change, $\inf ART$ decreases, strictly so if $N \geq 2$.

Case 2: Suppose $\mathcal{Q}$ changes as K increases to $K+1$. It directly follows equation (14) that $\inf ART$ is a decreasing with respect to N and j respectively. Denote by N_K and j_K the equilibrium outcome in Proposition 1 given K. If $N_{K+1} \geq N_K$ and $j_{K+1} \geq j_K$, then $\inf ART(K+1) < \inf ART(K)$. Because we have established $N_{K+1} \geq N_K$, so we only need to show that if $N_{K+1} > N_K$ and $j_{K+1} < j_K$, $\inf ART(K+1) < \inf ART(K)$. This is proved below. From equation (14), we have

$$
\begin{aligned}
& \inf ART(K+1) - \inf ART(K) \\
=&(j_K - j_{K+1}) - \frac{1}{D}\left[\sum_{i=0}^{N_{K+1}-2}\frac{\Delta}{(K+1)^{i-1}}(N_{K+1}-1-i) - \sum_{i=0}^{N_K-2}\frac{\Delta}{K^{i-1}}(N_K-1-i)\right] \\
=&(j_K - j_{K+1}) - \frac{1}{D}\left[\sum_{i=N_K-1}^{N_{K+1}-2}\frac{\Delta}{(K+1)^{i-1}}(N_{K+1}-1-i) + \sum_{i=0}^{N_K-2}\frac{\Delta}{(K+1)^{i-1}}(N_{K+1}-N_K)\right. \\
& \left.\sum_{i=0}^{N_K-2}\frac{\Delta}{(K+1)^{i-1}}(N_K-1-i) - \sum_{i=0}^{N_K-2}\frac{\Delta}{K^{i-1}}(N_K-1-i)\right] \\
<& K - \frac{1}{D}\left[\sum_{i=N_K-1}^{N_{K+1}-2}\frac{\Delta}{(K+1)^{i-1}}(N_{K+1}-1-i) + \sum_{i=0}^{N_K-2}\frac{\Delta}{(K+1)^{i-1}}(N_{K+1}-N_K)\right] \\
\leq& K - \frac{K}{D}\left[\sum_{i=0}^{N_{K+1}-2}\frac{\Delta}{(K+1)^{i}} + \frac{1}{K}\sum_{i=0}^{N_{K+1}-2}\frac{\Delta}{(K+1)^{i}}\right] \\
\leq& K - \frac{K}{D}\left[\sum_{i=0}^{N_{K+1}-1}\frac{\Delta}{(K+1)^{i}}\right] \leq 0
\end{aligned}
$$

Here, the first inequality is true because of equation (20), and the last inequality is due to the fact that $D \leq \sum_{i=0}^{N_{K+1}-1}\frac{\Delta}{(K+1)^i}$. Therefore, $\inf ART$ decreases in K. This concludes the proof of Case 2 and the analysis of $\inf ART$.

We now turn to $\sup ART$. It follows from equation (13) that $\sup ART$ can be rewritten as

$$\sup ART = [T - j - (N-1)K] + \frac{1}{D}\left[\sum_{i=1}^{N-2} \frac{\Delta}{K^{i-1}} i + \frac{j\Delta}{K^{N-1}}(N-1)\right].$$

Obviously, if the increase in K does not change $\mathcal{Q}$, $\sup ART$ will decrease. Otherwise, suppose the increase in K leads to a change of $\mathcal{Q}$. Similar to the proof for $\inf ART$, one can verify $\sup ART$ is decreasing in N and j respectively. As a result, if $N_{K+1} \geq N_K$ and $j_{K+1} \geq j_K$, then $\sup ART(K+1) < \sup ART(K)$. Finally, we prove the result for $N_{K+1} > N_K$ and $1 \leq j_K - j_{K+1} \leq K-1$:

$$\begin{aligned}
&\sup ART(K+1) - \sup ART(K)\\
\leq & K(N_K+1) - N_{K+1}(K+1)\\
&+ \frac{1}{D}\left[\sum_{i=1}^{N_{K+1}-2} \frac{\Delta}{(K+1)^{i-1}} i + \frac{j_{K+1}\Delta}{(K+1)^{N_{K+1}-1}}(N_{K+1}-1) - \sum_{i=1}^{N_K-2} \frac{\Delta}{K^{i-1}} i - \frac{j_K\Delta}{K^{N_K-1}}(N_K-1)\right].
\end{aligned}$$

Let's consider the terms in the bracket. First,

$$\begin{aligned}
&\frac{1}{D}\left[\sum_{i=1}^{N_{K+1}-2} \frac{\Delta}{(K+1)^{i-1}} i + \frac{j_{K+1}\Delta}{(K+1)^{N_{K+1}-1}}(N_{K+1}-1)\right]\\
=&(K+1)N_{K+1}\frac{1}{D}\left(\sum_{i=0}^{N_{K+1}-2} \frac{\Delta}{(K+1)^i} + \frac{(j_{K+1}-1)\Delta}{(K+1)^{N_{K+1}}}\right)\\
&\qquad - \frac{1}{D}(K+1)\left(\sum_{i=0}^{N_{K+1}-2} \frac{\Delta}{(K+1)^i}(N_{K+1}-i) + \frac{(N_{K+1}-j_{K+1})\Delta}{(K+1)^{N_{K+1}}}\right)\\
<&(K+1)N_{K+1} - \frac{1}{D}(K+1)\left(\sum_{i=0}^{N_{K+1}-2} \frac{\Delta}{(K+1)^i}(N_{K+1}-i) + \frac{(N_{K+1}-j_{K+1})\Delta}{(K+1)^{N_{K+1}}}\right).
\end{aligned}$$

Second,

$$\begin{aligned}
&-\frac{1}{D}\left[\sum_{i=1}^{N_K-2} \frac{\Delta}{K^{i-1}} i + \frac{j_K\Delta}{K^{N_K-1}}(N_K-1)\right]\\
=&-K(N_K+1)\frac{1}{D}\left(\sum_{i=0}^{N_K-2} \frac{\Delta}{K^i} + \frac{j_K\Delta}{K^{N_K}}\right)\\
&\qquad + \frac{1}{D}K\left(\sum_{i=0}^{N_K-2} \frac{\Delta}{K^i}(N_K+1-i) + \frac{2j_K\Delta}{K^{N_K}}\right)\\
\leq&-K(N_K+1) + \frac{1}{D}K\left(\sum_{i=0}^{N_K-2} \frac{\Delta}{K^i}(N_K+1-i) + \frac{2j_K\Delta}{K^{N_K}}\right).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\sup ART(K+1) - \sup ART(K)\\
<&\frac{1}{D}\left[K\left(\sum_{i=0}^{N_K-2}\frac{\Delta}{K^i}(N_K+1-i)+\frac{2j_K\Delta}{K^{N_K}}\right)-(K+1)\left(\sum_{i=0}^{N_{K+1}-2}\frac{\Delta}{(K+1)^i}(N_{K+1}-i)+\frac{(N_{K+1}-j_{K+1})\Delta}{(K+1)^{N_{K+1}}}\right)\right]\\
\leq&\frac{1}{D}\left[K\left(\sum_{i=0}^{N_K-2}\frac{\Delta}{K^i}(N_{K+1}-i)+\frac{2j_K\Delta}{K^{N_K}}\right)-(K+1)\left(\sum_{i=0}^{N_{K+1}-2}\frac{\Delta}{(K+1)^i}(N_{K+1}-i)+\frac{(N_{K+1}-j_{K+1})\Delta}{(K+1)^{N_{K+1}}}\right)\right]\\
\leq&\frac{1}{D}\left[\sum_{i=0}^{N_K-2}\left(\frac{K\Delta}{K^i}-\frac{(K+1)\Delta}{(K+1)^i}\right)(N_{K+1}-i)-\sum_{i=N_K-1}^{N_{K+1}-2}\frac{\Delta}{(K+1)^{i-1}}(N_{K+1}-i)+\frac{2Kj_K\Delta}{K^{N_K}}-\frac{(N_{K+1}-j_{K+1})\Delta}{(K+1)^{N_{K+1}-1}}\right]\\
=&\frac{1}{D}\left[-N_{K+1}\Delta+\sum_{i=1}^{N_K-3}\left(\frac{\Delta}{K^i}-\frac{\Delta}{(K+1)^i}\right)(N_{K+1}-1-i)\right.\\
&\qquad\left.-\sum_{i=N_K-1}^{N_{K+1}-2}\frac{\Delta}{(K+1)^{i-1}}(N_{K+1}-i)+\frac{2Kj_K\Delta}{K^{N_K}}-\frac{(N_{K+1}-j_{K+1})\Delta}{(K+1)^{N_{K+1}-1}}\right].
\end{aligned}
$$

Note, the second term in the bracket is 0 if $N_K \leq 3$; in such a case, simple mathematical induction can show that $\sup ART(K+1) < \sup ART(K)$. When $N_K \geq 4$, we first have

$$
\begin{aligned}
&-N_{K+1}\Delta+\sum_{i=1}^{N_K-3}\left(\frac{\Delta}{K^i}-\frac{\Delta}{(K+1)^i}\right)(N_{K+1}-1-i)+\frac{2Kj_K\Delta}{K^{N_K}}\\
\leq&-N_{K+1}\Delta+(N_{K+1}-2)\sum_{i=1}^{\infty}\left(\frac{\Delta}{K^i}-\frac{\Delta}{(K+1)^i}\right)+\frac{2\Delta}{K^{N_K-2}}\\
=&\frac{N_{K+1}(1-(K-1)K)\Delta}{(K-1)K}-2\Delta\left[\frac{1}{(K-1)K}-\frac{1}{K^{N_K-2}}\right]<0.
\end{aligned}
$$

This inequality, combined with the fact that

$$
-\frac{2\Delta}{(K+1)^{N_{K+1}-3}}+\frac{j_{K+1}\Delta}{(K+1)^{N_{K+1}-1}}<-\frac{2\Delta}{(K+1)^{N_{K+1}-3}}+\frac{(K+1)\Delta}{(K+1)^{N_{K+1}-1}}<0,
$$

implies that $\sup ART(K+1) < \sup ART(K)$. Hence, the conclusion holds for any $N_{K+1} > N_K$ and $j_{K+1} < j_K$. In all, $\sup ART$ decreases in K.

Proof of Proposition 3: Assume the variance of per-period cash flow is a constant: $\Delta^2(K-1) = \alpha^2$ for some constant $\alpha > 0$. Denote the solution by $\Delta_K = \frac{\alpha}{\sqrt{K-1}}$, which is decreasing in K. Therefore, and $t_{i+1} - t_i = K$ is decreasing in Δ. In addition, $\Delta \sum_{i=0}^{N-1} \frac{1}{K^i} = \Delta \sum_{i=0}^{N-1} \frac{1}{\left(\frac{\alpha^2}{\Delta^2}+1\right)^i}$ is increasing in Δ, so Corollary 3 implies that $\#\mathcal{Q}$ is decreasing in Δ.

In the remainder of the proof, we show both $\inf ART$ and $\sup ART$ increase as Δ increases. We only consider the increase in Δ that decreases K to a smaller integer. Without loss of generality, we focus on the comparison between $ART(\Delta_{K+1})$ and $ART(\Delta_K)$.

We first show that if $K \geq 4$, and an increase in Δ from Δ_{K+1} to Δ_K does not affect $\mathcal{Q}$, then $\inf ART$ increases. From (14),

$$
\begin{aligned}
&\inf ART(\Delta_{K+1}) - \inf ART(\Delta_K)\\
=&-\frac{1}{D}\left[\sum_{i=0}^{N-2}\frac{\Delta_{K+1}}{(K+1)^{i-1}}(N-1-i) - \sum_{i=0}^{N-2}\frac{\Delta_K}{K^{i-1}}(N-1-i)\right]\\
=&-\frac{\alpha}{D}\left[(N-1)\left(\frac{K+1}{\sqrt{K}} - \frac{K}{\sqrt{K-1}}\right) + \sum_{i=1}^{N-2}\left(\frac{1}{\sqrt{K}(K+1)^{i-1}} - \frac{1}{\sqrt{K-1}K^{i-1}}\right)(N-1-i)\right]\\
<&-\frac{\alpha}{D}(N-1)\left[\left(\frac{K+1}{\sqrt{K}} - \frac{K}{\sqrt{K-1}}\right) + \sum_{i=1}^{\infty}\left(\frac{1}{\sqrt{K}(K+1)^{i-1}} - \frac{1}{\sqrt{K-1}K^{i-1}}\right)\right]\\
=&-\frac{\alpha}{D}(N-1)\left[\frac{1}{\sqrt{K}}\frac{K+1}{1-\frac{1}{K+1}} - \frac{1}{\sqrt{K-1}}\frac{K}{1-\frac{1}{K}}\right]\\
=&-\frac{\alpha}{D}(N-1)\left[\frac{1}{\sqrt{K}}\frac{(K+1)^2}{K} - \frac{1}{\sqrt{K-1}}\frac{K^2}{K-1}\right]
\end{aligned}
$$

One can verify that the last expression as a function of K is negative when $K \geq 4$. Therefore, for fixed N and j, when Δ increases from Δ_{K+1} to Δ_K, $\inf ART$ increases.

Next, we consider the case when an increasing in Δ changes $\mathcal{Q}$. By the same arguments in Proposition 2, we only need to show that when $j_K - j_{K+1} \leq K-1$ and $N_{K+1} \geq N_K + 1$, $\inf ART(\Delta_{K+1}) < \inf ART(\Delta_K)$. This is established below:

$$
\begin{aligned}
&\inf ART(\Delta_{K+1}) - \inf ART(\Delta_K)\\
=&(j_K - j_{K+1}) - \frac{1}{D}\left[\sum_{i=0}^{N_{K+1}-2}\frac{\Delta_{K+1}}{(K+1)^{i-1}}(N_{K+1}-1-i) - \sum_{i=0}^{N_K-2}\frac{\Delta_K}{K^{i-1}}(N_K-1-i)\right]\\
\leq&(K-1) - \frac{1}{D}\left[\sum_{i=0}^{N_K-2}\frac{\Delta_{K+1}}{(K+1)^{i-1}}(N_K-1-i) + \sum_{i=0}^{N_{K+1}-2}\frac{\Delta_{K+1}}{(K+1)^{i-1}}(N_{K+1}-N_K)\right.\\
&\qquad\left. + \sum_{i=N_K-1}^{N_{K+1}-2}\frac{\Delta_{K+1}}{(K+1)^{i-1}}(N_K-1-i) - \sum_{i=0}^{N_K-2}\frac{\Delta_K}{K^{i-1}}(N_K-1-i)\right]\\
<&(K-1) - \frac{1}{D}\left[(N_{K+1}-N_K)\sum_{i=0}^{N_K-2}\frac{\Delta_{K+1}}{(K+1)^{i-1}} + \sum_{i=N_K-1}^{N_{K+1}-2}\frac{\Delta_{K+1}}{(K+1)^{i-1}}\right]\\
<&(K-1) - \frac{K}{D}\left[\sum_{i=0}^{N_{K+1}-2}\frac{\Delta_{K+1}}{(K+1)^i} + \frac{1}{K}\sum_{i=0}^{N_{K+1}-2}\frac{\Delta_{K+1}}{(K+1)^i}\right]\\
<&(K-1) - \frac{K}{D}\left[\sum_{i=0}^{N_{K+1}-2}\frac{\Delta_{K+1}}{(K+1)^i} + \frac{\Delta_{K+1}}{(K+1)^{N_{K+1}-1}}\right] < 0.
\end{aligned}
$$

Therefore, when Δ increases from Δ_{K+1} to Δ_K, $\inf ART$ increases.

We now show that the same property holds for $\sup ART$. Let's first consider the case that $\mathcal{Q}$ does not change. From (13),

$$\sup ART = [T - j - (N-1)K] + \frac{\alpha}{D}\left[\sum_{i=1}^{N-2} \frac{1}{K^{i-1}\sqrt{K-1}} i + \frac{j}{K^{N-1}\sqrt{K-1}}(N-1)\right].$$

When Δ increases from Δ_{K+1} to Δ_K without changing $\mathcal{Q}$, $\sup ART$ increases, because $\sup ART$ is strictly decreasing in K. Similarly, if $N_{K+1} \geq N_K$ and $j_{K+1} \geq j_K$, $\sup ART$ also increases in Δ. Hence, we only need to show that the same property holds in the case that $j_K - j_{K+1} \leq K-1$ and $N_{K+1} \geq N_K + 1$. Similarly to the proof of Proposition 2, we have

$$\begin{aligned}
&\sup ART(K+1) - \sup ART(K)\\
\leq &\frac{1}{D}\left[K\left(\sum_{i=0}^{N_K-2} \frac{\Delta_K}{K^i}(N_{K+1}-i) + \frac{2j_K\Delta_K}{K^{N_K}}\right) - (K+1)\left(\sum_{i=0}^{N_{K+1}-2} \frac{\Delta_{K+1}}{(K+1)^i}(N_{K+1}-i) + \frac{(N_{K+1}-j_{K+1})\Delta_{K+1}}{(K+1)^{N_{K+1}}}\right)\right]\\
\leq &\frac{1}{D}\left[\sum_{i=0}^{N_K-2}\left(\frac{K\Delta}{K^i} - \frac{(K+1)\Delta}{(K+1)^i}\right)(N_{K+1}-i) - \sum_{i=N_K-1}^{N_{K+1}-2} \frac{\Delta}{(K+1)^{i-1}}(N_{K+1}-i) + \frac{2Kj_K\Delta}{K^{N_K}} - \frac{(N_{K+1}-j_{K+1})\Delta}{(K+1)^{N_{K+1}-1}}\right]\\
< &\frac{\alpha}{D}\left[N_{K+1}(\sqrt{K} - \sqrt{K+1}) + \sum_{i=0}^{\infty}\left(\frac{1}{K^i\sqrt{K}} - \frac{1}{(K+1)^i\sqrt{K+1}}\right)(N_{K+1}-1-i)\right.\\
&\qquad\left. - \sum_{i=N_K-1}^{N_{K+1}-2} \frac{1}{(K+1)^{i-1}\sqrt{K+1}}(N_{K+1}-i) + \frac{2Kj_K}{K^{N_K}\sqrt{K}} - \frac{(N_{K+1}-j_{K+1})}{(K+1)^{N_{K+1}-1}\sqrt{K+1}}\right]\\
\leq &\frac{\alpha}{D}\left[N_{K+1}\left(\frac{K\sqrt{K}}{K-1} - \frac{(K+1)\sqrt{K+1}}{K}\right) - \left(\frac{\sqrt{K}}{K-1} - \frac{\sqrt{K+1}}{K}\right) + \frac{2\sqrt{K}}{K^{N_K-1}}\right.\\
&\qquad\left. - \sum_{i=N_K-1}^{N_{K+1}-2} \frac{1}{(K+1)^{i-1}\sqrt{K+1}}(N_{K+1}-i) - \frac{(N_{K+1}-j_{K+1})}{(K+1)^{N_{K+1}-1}\sqrt{K+1}}\right].
\end{aligned}$$

In the last inequality, the sum of the first three terms is negative when $K \geq 4$, and the sum of the last term is less than 0 as in the proof of Proposition 2. Therefore, under the condition that $K \geq 4$, $\sup ART(\Delta_{K+1}) > \sup ART(\Delta_K)$. This completes the proof.

Proof of Proposition 4: Corollary 3 directly implies that $\#\mathcal{Q}$ is weakly increasing in D. In the rest of the proof, we study $\inf ART(D)$ and $\sup ART(D)$ as D increases.

Let's begin with $\inf ART$. There are two cases depending on whether D is at the boundary of (9).

Case 1: Both D and $D+\epsilon$ satisfy (9) for some common N and j; that is, the increase in D does not change $\mathcal{Q}$. From equation (14), we have

$$\inf ART(D+\epsilon) = (T-j) - \frac{1}{D+\epsilon}\sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}}(N-1-i) > \inf ART(D).$$

Hence, when D increases and the set of repayment dates does not change, $\inf ART(D)$ is strictly increasing in D.

Case 2: Suppose $D = \sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{j\Delta}{K^N}$ for some N and $j = 0, 1, ..., K-1$. In this case, when D increases to $D+\epsilon$, the last repayment moves one period forward from $T-j$ to $T-(j+1)$. Note that here we slightly abuse notation by equivalencing Case (N-1)-K in (9) and “Case N-0”. We then have

$$\lim_{\epsilon\to 0}\inf ART(D+\epsilon) = (T-(j+1)) - \frac{1}{D}\sum_{i=0}^{N-2}\frac{\Delta}{K^{i-1}}(N-1-i) = \inf ART(D) - 1.$$

Therefore, when the increase in D leads to an earlier last repayment date (but keeps the number of repayments), $\inf ART$ discretely drops by 1.

In sum, $\inf ART$ is not continuous in D. In particular, when D is in the interior of a case, $\inf ART(D)$ is continuously increasing in D. At any boundary of (9), $\inf ART(D)$ is left-continuous, but drops discretely by one when D increases marginally.

Let's now turn to $\sup ART$. Similarly, there are again two cases.

Case 1: Both D and $D+\epsilon$ satisfy (9) for some common N and j. It then follows from equation (13) that

$$\sup ART(D+\epsilon) - \sup ART(D) < 0,$$

because

$$\sum_{i=0}^{N-2}\frac{\Delta}{K^{i-1}}(N-i-1) + K(N-1)\left(-\sum_{i=0}^{N-2}\frac{\Delta}{K^{i}} - \frac{j\Delta}{K^N}\right) < 0.$$

Therefore, if the increase in D does not change $\mathcal{Q}$, $\sup ART(D)$ is strictly decreasing in D.

Case 2: Suppose $D = \sum_{i=0}^{N-2}\frac{\Delta}{K^i} + \frac{j\Delta}{K^N}$ for some N and $j = 0, 1, ..., K-1$. When D marginally increases to $D+\epsilon$, the equilibrium debt contract then features $\#\mathcal{Q} = N$ and $t_N = T-(j+1)$. Therefore,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\sup ART(D+\epsilon)\\
=&\lim_{\epsilon\to 0}(T-(j+1)) - \frac{1}{D+\epsilon}\left[\sum_{i=0}^{N-2}\frac{\Delta}{K^{i-1}}(N-i-1) + K(N-1)\left((D+\epsilon) - \sum_{i=0}^{N-2}\frac{\Delta}{K^i} - \frac{(j+1)\Delta}{K^N}\right)\right]\\
=&\sup ART(D) - 1 + \frac{\frac{\Delta}{K^N}(N-1)K}{D}.
\end{aligned}$$

Therefore,

$$\lim_{\epsilon\to 0}\sup ART(D+\epsilon) \in (\inf ART(D)-1, \inf ART(D))$$

because $\sup ART(D) = \inf ART(D)$ and $\frac{\frac{\Delta}{K^N}(N-1)K}{D} < 1$.

Therefore, we conclude that $\sup ART(D)$ is strictly decreasing in D if the marginal change of D does not change $\mathcal{Q}$; however, when the marginal increase in D leads to a different set of repayment dates, $\sup ART$ has a discrete drop. The magnitude of the drop, however, is smaller than that of $\inf ART$.

Finally, we prove the claim that a higher leverage leads to a decrease in the average repayment time of the optimal debt contract across cases $N-j$. Formally, denote by $D_{N,j}$ the financing needs that lead to a contract with N repayments and the last repayment at $T-j$; then, $\sup ART(D_{N,j+1}) < \inf ART(D_{N,j})$.

Consider two boundary financing needs $\bar{D}_{N,j-1} = \sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{(j-1)\Delta}{K^N}$ and $\bar{D}_{N,j} = \sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{j\Delta}{K^N}$. We have

$$\begin{aligned}
&\inf ART(D_{N,j}) - \sup ART(D_{N,j+1}) \\
=&\lim_{\epsilon\to 0} \inf ART(\bar{D}_{N,j-1}+\epsilon) - \lim_{\epsilon\to 0} \sup ART(\bar{D}_{N,j}+\epsilon) \\
=&ART(\bar{D}_{N,j-1}) - ART(\bar{D}_{N,j}) - \frac{1}{\bar{D}_{N,j}} \frac{\Delta}{K^N}(N-1)K \\
=&1 + \frac{1}{\bar{D}_{N,j}} \sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}}(N-1-i) - \frac{1}{\bar{D}_{N,j-1}} \sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}}(N-1-i) - \frac{1}{\bar{D}_{N,j}} \frac{\Delta}{K^N}(N-1)K \\
=&1 - \frac{\frac{\Delta}{K^N}}{\bar{D}_{N,j}\bar{D}_{N,j-1}} \left[\sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}}(N-1-i)\right] - \frac{1}{\bar{D}_{N,j}} \frac{\Delta}{K^N}(N-1)K \\
=&1 + \frac{\frac{\Delta}{K^N}}{\bar{D}_{N,j}\bar{D}_{N,j-1}} \left[\sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}} i\right] + \frac{\frac{K\Delta}{K^N}(N-1)(j-1)\Delta}{\bar{D}_{N,j}\bar{D}_{N,j-1}} - \frac{2\frac{K\Delta}{K^N}(N-1)}{\bar{D}_{N,j}} \\
>&0.
\end{aligned}$$

Here, the last inequality is due to the fact that

$$\begin{aligned}
&1 - \frac{2\frac{K\Delta}{K^N}(N-1)}{\bar{D}_{N,j}} \\
>&\frac{1}{\bar{D}_{N,j}} \left[\sum_{i=0}^{N-2} \frac{\Delta}{K^i} - \frac{2(N-1)\Delta}{K^{N-1}}\right] \\
=&\frac{1}{\bar{D}_{N,j}} \left[\frac{K(N-1)\Delta}{K^{N-1}} - \frac{2(N-1)\Delta}{K^{N-1}}\right] \\
\geq&0,
\end{aligned}$$

because $K \geq 2$. This completes the proof.

Proof of Proposition 5:

Part 1: Consider a debt contract $\mathcal{R}$ with $\#\mathcal{Q}(\mathcal{R}) = 1$ first. We claim that the repayment date $t = T-m$ maximizes $PI(1)$. To see this, we note that R_t has two upper bounds. First, $R_t \leq K\mu^t\Delta$, due to the feasibility constraint; and second, $R_t \leq V_{t+1}$ for the contract to be incentive compatible. Note that

$$V_{t+1} = \sum_{s=t+1}^{T} \mu^s \Delta = \mu^t \sum_{s=1}^{T-t} \mu^s \Delta.$$

Hence, $\forall t \in [T-m, T)$,

$$V_{t+1} \leq \left(\sum_{s=1}^{m} \mu^s\right) \mu^t \Delta = K\mu^t\Delta,$$

and so the incentive compatibility constraint must be binding to attain the maximum pledgeable income; that is, $R_t = V_{t+1}$. Since V_{t+1} is strictly decreasing in t, when $t \in [T-m, T)$, in order to achieve the maximum pledgeability, the repayment date must be $T-m$, and the maximum pledgeable income is $K\mu^{T-m}\Delta$.

Now, consider $t \le T-m$. Then,

$$V_{t+1} \ge \left(\sum_{s=1}^{m} \mu^s\right) \mu^t \Delta \ge K\mu^t \Delta.$$

So the feasibility constraint must be binding to maximize $PI(1)$. Because $K\mu^t\Delta$ is strictly increasing in t, in order to achieve the maximum pledgeability, the repayment date must be $T-m$, and the maximum pledgeable income is $K\mu^{T-m}\Delta$. Combining both cases, if $\#\mathcal{Q}(\mathcal{R}) = 1$, $PI(1) = K\mu^{T-m}\Delta$, which is attained by making the only repayment at date $T-m$. By the same arguments, we show that for a debt contract $\mathcal{R}$, if $t_j \in \mathcal{Q}$ and $R_{t_j} = V_{t_j+1}$ (i.e., $V_{t_j} = \mu^{t_j}\Delta$), then the maximum repayment at the $(j-1)^{\text{th}}$ repayment date occurs at $t_j - m$.

Now, let's consider $PI(2)$. For any debt contract $\mathcal{R}$ with $\#\mathcal{Q}(\mathcal{R}) = 2$, suppose $t_2 = T-q$. It follows from the proof of the one repayment contract that $q \le m$. Then, in order to attain the maximum pledgeable income, the entrepreneur can first set

$$R_{t_2} = V_{t_2+1} = \sum_{s=T-q+1}^{T} \mu^s \Delta.$$

As a consequence, $t_1 = t_2 - m$. Denote by $PI^q(N)$ the maximum pledgeable income of a contract with N repayments and the last repayment occurs at date $T-q$. Then,

$$PI^q(2) = \frac{K\mu^{T-q-m}\Delta}{K} + \frac{1}{K^2} \sum_{s=T-q+1}^{T} \mu^s \Delta.$$

So,

$$PI^{q+1}(2) - PI^q(2) > 0 \Leftrightarrow \mu^{-1} + \frac{\mu^m}{K^2} > 1. \tag{21}$$

Suppose equation (21) holds, then $PI^q(2)$ is strictly increasing in q. Hence,

$$PI(2) = PI^m(2) = \mu^{T-2m}\Delta + \frac{\mu^{T-m}\Delta}{K}.$$

Let's now compare $PI(2)$ and $PI(1)$ under equation (21).

$$I(2) > I(1) \Leftrightarrow \mu^{-m} + \frac{1}{K} > 1. \tag{22}$$

Note that, by equation (15), we have

$$K - \frac{K}{\mu} = \sum_{s=1}^{m} \mu^s - \sum_{s=0}^{m-1} \mu^s = \mu^m - 1. \tag{23}$$

Then, equation (22) is equivalent to $1 - \mu^m + \frac{\mu^m}{K} > 0$, which holds if and only if $\frac{K}{\mu} - K + \frac{\mu^m}{K} > 0$. The last inequality is equivalent to equation (21). So, when $n = 2$, $PI(2) = PI^m(2)$ if and only if $PI(2) > PI(1)$.

We now use induction. Assume that $PI(n) = PI^m(n)$ if and only if $PI(n) > PI(n-1)$, where $n \ge 2$. Let's consider $n+1$. Fix any $t_{n+1} = T-q$. When the contract can attain the largest pledgeable income,

$R_{t_{n+1}} = V_{T-q+1}$. Then, by the assumption that $PI(n) = PI^m(n)$, we have

$$PI^q(n+1) = \left[\sum_{j=1}^{n} \frac{\mu^{T-q-(n-j+1)m}\Delta}{K^{j-1}} + \frac{1}{K^{n+1}} \sum_{j=T-q+1}^{T} \mu^j \Delta\right].$$

Then,

$$\begin{aligned}
& PI^{q+1}(n+1) - PI^q(n+1) \\
= & \left[\sum_{j=1}^{n} \frac{\mu^{T-(q+1)-(n-j+1)m}\Delta}{K^{j-1}} + \frac{1}{K^{n+1}} \sum_{j=T-(q+1)+1}^{T} \mu^j \Delta\right] \\
& \qquad - \left[\sum_{j=1}^{n} \frac{\mu^{T-q-(n-j+1)m}\Delta}{K^{j-1}} + \frac{1}{K^{n+1}} \sum_{j=T-q+1}^{T} \mu^j \Delta\right] \\
= & \mu^{T-q}\left[(\mu^{-1}-1)\sum_{j=1}^{n} \frac{\mu^{-(n-j+1)m}\Delta}{K^{j-1}}\right] + \frac{\mu^{T-q}\Delta}{K^{n+1}}.
\end{aligned}$$

So, $PI^{q+1}(n+1) > PI^q(n+1)$ if and only if

$$\left[(\mu^{-1}-1)\sum_{j=1}^{n} \frac{\mu^{-(n-j+1)m}}{K^{j-1}}\right] + \frac{1}{K^{n+1}} > 0. \tag{24}$$

Now, suppose equation (24) holds, then $PI(n+1) = PI^m(n+1)$. We then have

$$PI(n+1) - PI(n) = \sum_{j=1}^{n+1} \frac{\mu^{T-(n-j+2)m}}{K^{j-1}} - \sum_{j=1}^{n} \frac{\mu^{T-(n-j+1)m}}{K^{j-1}} > 0$$

if and only if

$$(1-\mu^m)\sum_{j=1}^{m} \frac{\mu^{-(n-j+1)m}}{K^{j-1}} + \frac{1}{K^n} > 0. \tag{25}$$

It then follows from equation (23) that

$$1 - \mu^m = \frac{K}{\mu} - K.$$

So, equation (25) and equation (24) are equivalent. Therefore, if $PI(n+1) = PI^m(n+1)$, $PI(n+1) > PI(n)$.

Note that equation (24) is equivalent to

$$\left[(\mu^{-1}-1)K\sum_{j=1}^{n} \left(K\mu^{-m}\right)^{n-j+1}\right] + 1 > 0.$$

So, it follows from equation (16) that if and only if $N \leq N^*$, $PI(N) \geq PI(N-1)$. Therefore, $PI(N)$ is maximized at $N = N^*$.

Part 2: Because for any $N \leq N^*$, $PI(N) = PI^m(N)$. Therefore,

$$PI(N) = \sum_{i=0}^{N-1} \frac{\mu^{T-(N-i)m}}{K^i} \Delta.$$

Part 3: Now, suppose $D \in (PI(N-1), PI(N)]$. By the definition of $PI(N)$, since $D > PI(N-1)$, it is impossible to design a contract with at most $N-1$ repayment dates such that the investor's IR constraint holds. But $D \leq PI(N)$, so there exists a contract with N repayment dates such that the investor's participation constraint holds.

Consider any contract $\mathcal{R}$ with $\#\mathcal{Q} = N + p$ ($p \in \mathbb{Z}_+$). Without loss of generality, we only consider contracts with $R_t = K\mu^t\Delta$, $\forall t \in \mathcal{Q} \setminus \{t_{N+p}\}$. Otherwise, if $R_{t_j} < K\mu^{t_j}\Delta$, the entrepreneur can make the (t_{N+p}, t_j, ϵ) adjustment, until either $R_{t_j} = K\mu^{t_j}\Delta$ or $R_{t_{N+p}} = 0$. The former case is under consideration, while in the latter case, the entrepreneur is strictly better off. Note, in this process, the contract's incentive compatibility and the investor's participation constraint are preserved.

We now first claim that for any $t_j, t_{j+1} \in \mathcal{Q} \setminus \{t_1\}$, $t_{j+1} - t_j \geq m$. Let $t_j \in \mathcal{Q}$ be the last repayment date at which $t_{j+1} - t_j < m$. Because the original debt contract $\mathcal{R}$ is incentive compatible, it follows from the definition of m in equation (15) and $R_s = K\mu^s\Delta$ for all $s \in \mathcal{Q} \setminus \{t_{N+p}\}$ that

$$\begin{aligned}
& R_{t_j} \leq V_{t_j+1}, \\
\Leftrightarrow & \sum_{s=t_j+1}^{t_j+m} \mu^s\Delta \leq \sum_{s=t_j+1}^{t_{j+1}} \mu^s\Delta + \frac{1}{K^{N+p-j}} \left(\sum_{s=t_{N+p}+1}^{T} \mu^s\Delta - R_{t_{N+p}} \right) \\
\Leftrightarrow & R_{t_{N+p}} \leq \sum_{s=t_{N+p}+1}^{T} \mu^s\Delta - K^{N+p-j} \sum_{s=t_{j+1}+1}^{t_{j+1}+\ell} \mu^s\Delta,
\end{aligned}$$

where $\ell = t_j + m - t_{j+1} \geq 1$. If $t_{N+p} + 1 = T$,

$$\begin{aligned}
& R_{t_{N+p}} \\
\leq & \sum_{s=t_{N+p}+1}^{T} \mu^s\Delta - K^{N+p-j} \sum_{s=t_{j+1}+1}^{t_{j+1}+\ell} \mu^s\Delta \\
= & \mu^{t_{j+1}} \left[\mu^{T-t_{j+1}}\Delta - K^{N+p-j} \sum_{s=1}^{\ell} \mu^s\Delta \right] \\
< & \Delta\mu^{t_{j+1}} \left[\mu^{(N+p-(j+1))m+1} - \mu^{(N+p-j)m+1} \right] < 0,
\end{aligned}$$

where the first strict inequality is again due to the definition of m in equation (15). Such an inequality contradicts to the assumption that $R_{t_{N+p}} > 0$ in the original contract, and so $t_{N+p} + 1 < T$.

Then, the entrepreneur can construct a new contract $\mathcal{R}'$ such that $R'_{t_{N+p}} = 0$, $R'_{t_{N+p}+1} = R_{t_{N+p}}$, and $R'_s = R_s$ for all other s. That is, the last repayment is delayed to date $t_{N+p} + 1$. Let's check the incentive compatibility of the new contract $\mathcal{R}'$. Under $\mathcal{R}'$, at date $t_{N+p}+1$, we have

$$\begin{aligned}
& V'_{t_{N+p}+2} - R'_{t_{N+p}+1} \\
=& V'_{t_{N+p}+2} - R_{t_{N+p}} \\
\geq & \sum_{s=t_{N+p}+2}^{T} \mu^s \Delta - \left(\sum_{s=t_{N+p}+1}^{T} \mu^s \Delta - K^{N+p-j} \sum_{s=t_{j+1}+1}^{t_{j+1}+\ell} \mu^s \Delta \right) \\
> & \left(K^{N+p-j} - \mu^{(N+p-j-1)m} \right) \mu^{t_{j+1}+1} \Delta > 0.
\end{aligned}$$

Hence, the incentive compatibility constraint holds at date $t_{N+p}+1$. Because $t_{s+1}-t_s = m$ for all $s \geq j+1$, the incentive compatibility constraint holds at all these dates. At t_j, because $V'_{t_j+1} > V_{t_j+1}$ (since all subsequent repayment amounts are unchanged, and the last repayment is delayed), $V'_{t_j} > V_{t_j}$; then, the incentive compatibility of $\mathcal{R}$ implies the incentive compatibility of $\mathcal{R}'$. Importantly, the investor's participation constraint does not change. Therefore, $\mathcal{R}'$ satisfies both incentive compatibility constraint and investor participation constraint. Finally, because $V'_{t_j} > V_{t_j}$, and all repayments before t_j are the same in $\mathcal{R}$ and $\mathcal{R}'$, $\mathcal{R}$' is strictly better than $\mathcal{R}$. This implies that our original assumption $t_{j+1} - t_j < m$ is invalid.

We then only need to consider a contract with $t_{j+1} - t_j \geq m$ for any $t_j, t_{j+1} \in \mathcal{Q} \setminus \{t_1\}$. Since $t_{N+p} < T$ and $p \geq 1$, $t_{N+p} - (N+p-j)m < T - (N-j+1)m$. Hence, if the entrepreneur offers a contract $\mathcal{R}'$ with $\#\mathcal{Q}' = N$, the j^{th} repayment day could be $T-(N-j+1)m$, which is late than the j^{th} repayment date in the original contract $\mathcal{R}$. Therefore, there exists $\mathcal{R}'$ with $\#\mathcal{Q}' = N$ that is strictly better than $\mathcal{R}$. Therefore, when $D \in (PI(N-1), PI(N)]$, the optimal contract has exactly N repayment dates. In addition, since $t_{j+1} - t_j = m$ and $T - t_N \leq m$, the first repayment date $t_1 \geq T - Nm$.

Part 4: When $D = PI(N)$, the optimal debt contract must have $t_N = T-m$, because $PI(N) = PI^m(N)$. Then, $\#\mathcal{Q} = \{T-m, T-2m, ..., T-Nm\}$. In addition, to attain the maximum pledgeable income, the entrepreneur must make the largest repayment at each repayment date, so $R_t = K\mu^t\Delta$, implying that the schedule proposed is the unique one to attain $D = PI(N)$, which can be attained by the repayment schedule

$$R_t = \begin{cases} \mu^t K \Delta, & \text{if } t \in \{T-m, T-2m, ..., T-Nm\} \\ 0, & \text{otherwise.} \end{cases}$$

Proof of Corollary 3: Consider the left-hand side of equation (16) when $N = 1$, we have

$$\left[(\mu^{-1}-1)K \sum_{j=1}^{N} \left(K\mu^{-m} \right)^{N-j+1} \right] + 1 = (\mu^{-1}-1)K(K\mu^{-m}). \tag{26}$$

Note that $(\mu^{-1}-1)K = 1 - \mu^m$, equation (26) becomes $K\mu^{-m} - K + 1$, which is negative because $1 > \frac{1}{K} + \mu^{-m}$. So when $1 > \frac{1}{K} + \mu^{-m}$, $N^* = 1$.

Proof of Proposition 6: Suppose on the contrary that a risky contract $\mathcal{R}$ maximizes the value of debt. Denote by $t \leq T-1$ the last risky repayment date. Construct a new contract $\mathcal{R}'$ such that $R'_s = R_s$ for all $s < t$ or $s > t+1$, $R'_t = L$, and $R'_{t+1} = R_{t+1} + \max(0, R_t - \Delta - L) < R_t$.

The new contract $\mathcal{R}'$ is incentive compatible because for any s,

$$V'_s \geq (\Delta + L) + \frac{1}{K}(V'_{s+1} - R_s) \geq \Delta + L,$$

so all risk-free repayments $R'_s \leq L$ when $s > t+1$ are automatically incentive compatible. In addition, R'_{t+1} is incentive compatible because either $R'_{t+1} = R_{t+1}$, in which case IC trivially holds, or $R'_{t+1} = R_{t+1} + R_t - \Delta - L$ and

$$V'_{t+2} = V_{t+2} = V_{t+1} + R_{t+1} - (\Delta + L) \geq R_t + R_{t+1} - (\Delta + L) = R'_{t+1}.$$

Finally, note that

$$\begin{aligned}
&V'_t - V_t \\
\geq& \left[2(\Delta + L) - L + \frac{V'_{t+2} - R'_{t+1}}{K}\right] - \left[(\Delta + L) + \frac{V_{t+2} + (\Delta + L) - R_{t+1} - R_t}{K}\right] \\
=&\Delta + \frac{R_{t+1} + R_t - (\Delta + L) - R'_{t+1}}{K} > 0,
\end{aligned}$$

Therefore, we can recursively show that $V'_s > V_s$ for all $s \leq t$ and $R'_s = R_s$ is thereby incentive compatible. Next we show $\mathcal{D}(\mathcal{R}') > \mathcal{D}(\mathcal{R})$. This is because when (17) holds,

$$\mathcal{D}(\mathcal{R}') - \mathcal{D}(\mathcal{R}) \geq L + \frac{R_{t+2} + \max(0, R_t - \Delta - L)}{K} - \frac{R_{t+1} + R_{t+2}}{K} \geq L - \frac{\Delta + L}{K} > 0.$$

Contradiction with the maximality of $\mathcal{R}$. Therefore, risk-free schedule $R_s = L$ maximizes pledgeability.

Proof of Proposition 7: We prove this proposition in four parts.

Part 1: Let N be the number of risky repayments. We show that the value of any repayment profile with $N \neq N^{**}$ can be strictly improved. Let's first consider a contract $\mathcal{R}$ with $N > N^{**}$. Suppose $t_N < T$ is the last risky repayment date. If there is a $t < t_1 \in \mathcal{Q}$, such that $R_t < L$, the entrepreneur can simply set $R'_t = L$ to increase the value of the contract. If $t \in (t_j, t_{j+1})$, the entrepreneur can apply the (t_j, t, ϵ) adjustment, until either $R'_t = L$ or $R'_{t_j} = 0$. In the former case, the value of the contract does not change; in the latter case, the value of the contract will increase, because all repayment after t_j become less risky. Note that in the adjustment process, the incentive compatibility is preserved. Hence, without loss of generality, we can consider the debt contract $\mathcal{R}$, in which at any $t \notin \mathcal{Q}$, $R_t = L$. The entrepreneur can then apply the (t_j, t_{j+1}, ϵ) adjustment for $t_j, t_{j+1} \in \mathcal{Q}$, such that the incentive compatibility constraint is binding at each risky repayment date. (Here, we take $t_{N+1} = T$. It is possible that $t_{j+1} - t_j > K$, and so the incentive compatibility at t_{j+1} cannot be binding. However, in this case, the entrepreneur can just set $R'_{t_{j+1}} = L$ and $R'_{t_{j+1}+1} = K\Delta + L$ to increase the value of the contract.) This is guaranteed one-by-one from the last risky repayment date. In addition, the value of the contract does not change in the adjustment process. Therefore, below, we only need to show that a contract $\mathcal{R}$ with $R_s = L$ for all $s \notin \mathcal{Q}$ and $R_t = V_{t+1}$ for all $t \in \mathcal{Q}$ can be strictly improved.

We can construct a new contract $\mathcal{R}'$: $R'_s = R_s$ for all $s < t_1$ and $s > t_N + 1$, $R'_{t_1} = L$, $R'_{s+1} = R_s$ for all $s \in [t_1, t_N - 1]$, and $R'_{t_N+1} = (T - (t_N + 1))\,\Delta + L$. (If $t_N + 1 = T$, $R'_{t_N+1} = 0$. This case will be nested in the following proof.) Since $R_s = L$ for all $s \in [t_N + 2, T - 1]$ and $R_T = 0$, $V'_{t_N+2} = (T - (t_N + 1))\,\Delta + L$. Hence, the incentive compatibility constraint is binding at $t_N + 1$. What's more, $V'_{t_N+1} = V_{t_N} = \Delta + L$. Therefore, at each previous date, the incentive compatibility constraint is binding.

Hence, we just need to show that $\mathcal{D}(\mathcal{R}') > \mathcal{D}(\mathcal{R})$. Note that

$$\begin{aligned}
&\mathcal{D}(\mathcal{R}') - \mathcal{D}(\mathcal{R}) \\
\geq &R'_{t_1} + \frac{R'_{t_N+1}}{K^N} - \frac{R_{t_N} + R_{t_N+1}}{K^N} \\
= &L + \frac{(T-(t_N+1))\,\Delta - ((T-t_N)\,\Delta + L)}{K^N} \\
= &L - \frac{\Delta + L}{K^N} > 0
\end{aligned}$$

Here, the first inequality is strict when $t_N + 1 = T$, and the last inequality is due to the definition of N^{**}.

Next, consider the case with $N < N^{**}$. Without loss of generality, we assume that $R_s = L$ for all $s < t_1$. The entrepreneur can construct the following new contract $\mathcal{R}'$: $R'_s = R_s$ for all $s < t_1 - K$, $R'_s = R_{s+K}$ for all $s \in [t_1 - K, T-1-K]$, $R'_{T-K} = K\Delta + L$, and $R'_s = L$ for all $s > T-K$. Since there are K periods after date $T-K$, R'_{T-K} is incentive compatible. In addition, because $R'_{T-K} = K\Delta + L$ and $R'_s = L$ for all $s > T-K$, $V'_{T-K} = \Delta + L$; hence, at any repayment date $s < T-K$, $\mathcal{R}'$ is incentive compatible because $\mathcal{R}$ is incentive compatible.

Note that $N < N^{**}$, and so $L < (\Delta + L)/K^{N+1}$. Therefore, we have

$$\mathcal{D}(\mathcal{R}') - \mathcal{D}(\mathcal{R}) = \frac{K(\Delta + L)}{K^{N+1}} - KL > 0,$$

Hence, the pledgeable income after adding the extra risky repayment increases. Therefore, $PI(N)$ is maximized at N^{**}.

Part 2: In the remainder of the proof, we assume $N \leq N^{**}$. When $N = 1$, to maximize the value of the debt, obviously IC condition must be binding at the risky repayment date t_1, and that at any date $s < t_1$, $R_s = L$. First, it is easy to see that $t_1 \geq T-K$; otherwise, because $R_s \leq L$ for all $s > t_1$, so

$$V_{t_1+1} \geq (T-t_1)\Delta + L > K\Delta + L \geq R_{t_1}$$

and IC condition cannot possibly bind. For any $t_1 > T-K$, we can construct a new contract $\mathcal{R}'$: $R_s = L$ for all $s \neq T-K$, and $R'_{T-K} = K\Delta + L$. This is obviously incentive compatible. Now,

$$\begin{aligned}
&\mathcal{D}(\mathcal{R}') - \mathcal{D}(\mathcal{R}) \\
= &\frac{K(\Delta + L)}{K} - \left[(t_1 - (T-K))L + \frac{(T-t_1)(\Delta + L)}{K}\right] \\
= &(K - (T-t_1))\left(\frac{\Delta + L}{K} - L\right).
\end{aligned}$$

Hence, $\mathcal{D}(\mathcal{R}') - \mathcal{D}(\mathcal{R}) \geq 0$, because $t_1 \geq T-K$ and $N \leq N^{**}$. Note that the inequality is strict if $t_1 > T-K$; hence, $PI(1)$ is uniquely attained by $R_{t_1} = K\Delta + L$ at date $t_1 = T-K$, and $R_s = L$ at date $s \neq t_1$. Then, $PI(1) = (T-K)L + (\Delta + L)$.

Now, let's consider any $N \leq N^{**}$ and consider a contract $\mathcal{R}$ with $\#\mathcal{Q} = N$. Fixing all repayments at dates $s \geq t_2$, we have $V_{t_2+1} \geq R_{t_2}$ by the incentive compatibility of $\mathcal{R}$. Then, as the same argument of the case $N = 1$, the value of the contract is maximized when $t_1 = t_2 - K$, $R_{t_1} = K\Delta + L$, $R_s = L$ for all $s < t_2$ but $s \neq t_1$. We can now increase the value of the contract by increasing R_{t_2} to V_{t_2+1}. Similarly, fixing all repayments at dates $s \geq t_n$ for all $n \leq N$, the value of the contract is maximized by setting $t_j = t_j - K$ for all $j < n$, $R_{t_j} = K\Delta + L$ at date t_j for all $j < n$, and $R_s = L$ at dates $s < t_n$ but $s \neq t_j$ for any $t_j < t_n$. Then, finally, because $R_T = V_{T+1} = 0$, the contract's value is maximized by setting $t_N = T-K$.

Therefore, in order to maximize the pledgeable income, the entrepreneur needs to make risky repayment at dates $t = T - iK$, where $i = 1, 2, \ldots, N$. In addition, at each risky repayment date, the risky repayment should be $K\Delta + L$, such that both the feasibility constraint and the incentive compatibility constraint binding; at other dates, the entrepreneur needs to repay L. Therefore, the maximum pledgeable income of a debt contract with N repayment dates is

$$PI(N) = (T - NK)L + \sum_{j=1}^{N} \frac{\Delta + L}{K^{j-1}}.$$

Part 3: For any $N \leq N^{**}$, if $D \in (PI(N-1), PI(N)]$, by the definition of $PI(N)$, the entrepreneur cannot use a contract with at most $N-1$ repayment dates to attain D, but she can use a contract with N repayment dates to attain D.

Now, we show that a contract with $N + p$ repayments, where $p \geq 1$, can be strictly improved. The first step is to show that at all dates when the entrepreneur does not make risky repayment, the entrepreneur repays L. Suppose there is one date t such that $R_t < L$. If t is earlier than the first risky repayment t_1, the entrepreneur apply the $(t, t_{1,}\epsilon)$ adjustment by setting $R'_t = R_t + \epsilon$ and $R'_{t_1} = R_{t_1} - K\epsilon$. Then, the investor's participation constraint does not change, and the entrepreneur is at least as good as before (strictly better if $R'_{t_1} \leq L$ after the adjustment). If $t \in (t_j, t_{j+1})$, then the entrepreneur can make the (t_j, t, ϵ) adjustment by setting $R'_{t_j} = R_{t_j} - \epsilon$ and $R'_t = R_t + \epsilon$. Such an adjustment will not change the incentive compatibility and the investor's participation constraint either; again, the entrepreneur is at least not worse off. Hence, when consider the optimal debt contract, the entrepreneur will repay L when she does not make risky repayments. Then, the rest of the proof is the same as that of Corollary 1 by iterated application of the (t_{N+p}, t_i, ϵ) adjustments.

As shown above, in the optimal debt contract, $t_{j+1} - t_j \leq K$ and $t_N \geq T - K$; otherwise, if some $t_j < t_{j+1} - K$ (we can denote by $t_{N+1} = T$), it is strictly better for the entrepreneur to set $R'_{t_j} = L$ and $R'_{t_j+1} = R_{t_j}$. Such a new contract is still incentive compatible, because the value between $t_j + 1$ and t_j will be greater than or equal to $K(\Delta + L)$. Therefore, $t_1 \geq T - NK$.

Part 4: For any $N \leq N^{**}$, if $D = PI(N)$, the optimal debt contract will have exactly N risky repayment dates. Then, by Part 2, to attain D by a contract with N risky repayment dates, the entrepreneur has to repay $K\Delta + L$ at dates $t_i = T - iK$ (for $i = 1, 2, \ldots, N$) and repay L at all other dates. Therefore, there is a unique optimal contract that attains D, which is

$$R_t = \begin{cases} K\Delta + L, & \text{if } t \in \{T-K, T-2K, ..., T-NK\} \\ L & \text{otherwise.} \end{cases}$$

Proof of Proposition 8: Consider a repayment schedule with $R_t > K\Delta$ for some $t \in \mathcal{Q}$. Let R_{t_j} be the last repayment with $R_{t_j} > K\Delta$. There are two cases. First, the repayment schedule has exactly j repayments, and so R_{t_j} is also the last repayment. Then,

$$T - t_j = \left\lceil \frac{R_{t_j}}{\Delta} \right\rceil > K;$$

otherwise, $R_{t_j} > V_{t_j+1}$, violating the incentive compatibility constraint. Then, fix all previous $j-1$ repayments (the time and the amount), the entrepreneur may consider the following adjustment: $R'_{t_j} = K\Delta$ and $R'_{t_j+K} = R_{t_j} - K\Delta$. Such a new repayment schedule is still incentive compatible; otherwise, if $R'_{t_j+K} > V_{t_j+K+1} = (T - t_j - K)\Delta$, $R_{t_j} > (T - t_j)\Delta = V_{t_j+1}$, violating the assumption that the original repayment schedule is incentive compatible.

Because the first $j-1$ repayments do not change, if the entrepreneur can repay R_{t_j} at date t_j, he is able to make the repayments R'_{t_j} and R'_{t_j+K} (because of saving). Indeed, there is a positive probability that the entrepreneur cannot repay R_{t_j} but can make the repayments R'_{t_j} and R'_{t_j+K}, because the project may generate positive cash flows between t_j+1 and t_j+K. In addition, because of saving, the investor's participation constraint is also satisfied. Therefore, the entrepreneur can even reduce R'_{t_j+K} to a certain R''_{t_j+K} and still keep the investor's participation constraint satisfied.

Now, let's compare V_{t_j} and V'_{t_j}. Denote by S_t the total funds the entrepreneur can use to make repayment at date t. We can calculate

$$V_{t_j} = \Delta + \Pr\left(S_{t_j} \geq R_{t_j}\right)\left(-R_{t_j} + (T-t_j)\Delta\right)$$

and

$$\begin{aligned}
&V'_{t_j} \\
=&\Delta + \Pr\left(S'_{t_j} \geq R'_{t_j}\right)\Big\{-R'_{t_j} + K\Delta \\
&\qquad + \Pr\left(S'_{t_j+K} \geq R''_{t_j+K} | S'_{t_j} \geq R'_{t_j}\right)\left(-R''_{t_j+K} + (T-t_j-K)\Delta\right)\Big\} \\
=&\Delta + \Pr\left(S'_{t_j} \geq R'_{t_j}\right)\left(-R'_{t_j} + K\Delta\right) \\
&\quad + \Pr\left(S'_{t_j} \geq R'_{t_j}\right)\Pr\left(S'_{t_j+K} \geq R''_{t_j+K} | S'_{t_j} \geq R'_{t_j}\right)\left(-R''_{t_j+K} + (T-t_j-K)\Delta\right)
\end{aligned}$$

As we argued above, the same first $j-1$ repayments imply that $S_{t_j} = S'_{t_j}$, and then because of $R_{t_j} = R'_{t_j} + R'_{t_j+K} > R'_{t_j}$, if $\Pr\left(S_{t_j} \geq R'_{t_j}\right) > 0$,

$$\Pr\left(S_{t_j} \geq R'_{t_j}\right) \geq \Pr\left(S_{t_j} \geq R_{t_j}\right).$$

In addition, if $S_{t_j} \geq R_{t_j}$, then $S_{t_j} - R'_{t_j} = R'_{t_j+K} > R''_{t_j+K}$, and so

$$\Pr\left(S_{t_j} \geq R'_{t_j}\right)\Pr\left(S'_{t_j+K} \geq R''_{t_j+K} | S_{t_j} \geq R'_{t_j}\right) > \Pr\left(S_{t_j} \geq R_{t_j}\right).$$

These imply that

$$\begin{aligned}
&V'_{t_j} \\
>&\Delta + \Pr\left(S_{t_j} \geq R_{t_j}\right)\left(-R'_{t_j} + K\Delta\right) + \Pr\left(S_{t_j} \geq R_{t_j}\right)\left(-R''_{t_j+K} + (T-t_j-K)\Delta\right) \\
>&\Delta + \Pr\left(S_{t_j} \geq R_{t_j}\right)\left(-R'_{t_j} + K\Delta - R'_{t_j+K} + (T-t_j-K)\Delta\right) \\
=&V_{t_j}
\end{aligned}$$

Hence, the adjustment makes the entrepreneur strictly better off.

In the second case, the repayment schedule has more than j repayments. Then, by assumption, all repayments after date t_j are at most $K\Delta$. The entrepreneur can then make all the repayments after date t_j as late as possible, until that delaying one of these repayments one date will violate the incentive compatibility. Then, the entrepreneur can iteratedly apply the (t_{i-1}, t_i, ϵ) adjustment for all $i \geq j$ beginning from $i = N$ (the last repayment), such that the incentive compatibility constraint binding at each repayment date after date t_j. The entrepreneur will not be worse off by this adjustment, for the same reason as in the first case. Suppose now that R_{t_j} is still strictly greater than $K\Delta$. Then, $V_{t_{j+1}} = (t_{j+1} - t_j)\,\Delta$, and $t_{j+1} - t_j = \left\lceil \frac{R_{t_j}}{\Delta} \right\rceil > K$. Hence, the same argument in the first case will prove that such a contract is not optimal for the entrepreneur.

Proof of Proposition 9: First, N^* is well defined because the LHS of (18) is a constant and the RHS decreases to 0, as $N^* \to \infty$.

The proof is by contradiction. Let $\{R_i | i = 0, 1, 2, ..., T\}$ be any repayment profile that attains the maximum pledgeable income, with risky payments (strictly greater than L) at dates $\{t_1, t_2, ..., t_N\}$. Suppose on the contrary that $N \geq 2N^*$. Before proving the proposition, we first introduce a key procedure that we use repeatedly.

For any fixed K, define risk-free modification with respect to day K to be the following procedure that constructs a new repayment profile $\tilde{\mathcal{R}}_i(K)$: For all $t > K$ or $t < t_1$, $\tilde{R}_t \equiv R_t$; $\tilde{R}_{t_1} \equiv L$; and for all $t_1 < t \leq K$, $\tilde{R}_t \equiv R_{t-1}$. Essentially, this modification removes repayment R_K; shifts all repayments between t_1 and K one period backward; and inserts a risk-free repayment of L at date t_1. The repayment profile changes from

$$\mathcal{R} = \{R_0, R_1, ..., R_{t_1-1}, R_{t_1}, R_{t_1+1}, ..., R_{t_2}, ..., R_{K-1}, R_K, R_{K+1}, ..., R_T\}$$

to

$$\tilde{\mathcal{R}}(K) = \{R_0, R_1, ..., R_{t_1-1}, L, R_{t_1}, ..., R_{t_2}, ..., R_{K-2}, R_{K-1}, R_{K+1}, ..., R_T\}.$$

The expected payoff to the borrower at the beginning of date t is

$$V_t = \sum_{i=t}^{T} \left[E(X_i) \prod_{s=t}^{i-1} Prob(X_s \geq R_s) - R_i \prod_{s=t}^{i} Prob(X_s \geq R_s) \right],$$

and the expected payoff to the lender is still defined by equation (4).

Let $\tilde{R}(t_n)$ be the risk-free modification w.r.t. t_n with $n > N^*$, i.e. there are at least N^* prior risky payments. By the definition of such a modification and equation (4), the expected value of the modified repayment profile is

$$\begin{aligned}
\mathcal{D}(\tilde{R}(t_n)) \quad = \quad & \textstyle\sum_{t=0}^{t_1-1} R_t + L \\
& + \textstyle\sum_{t=t_1}^{t_n-1} R_t \prod_{i=0}^{t} Prob(X_i \geq R_i) \\
& + \textstyle\sum_{t=t_n+1}^{T} R_t \prod_{i=0, i\neq t_n}^{t} Prob(X_i \geq R_i).
\end{aligned}$$

Compare it with $\mathcal{D}(\mathcal{R})$ in (4), the difference is

$$\begin{aligned}
\mathcal{D}(\tilde{\mathcal{R}}(t_n)) - \mathcal{D}(\mathcal{R}) \quad = \quad & L - R_{t_n} \textstyle\prod_{i=0}^{t_n} Prob(X_i \geq R_i) \\
& + \textstyle\sum_{t=t_n+1}^{T} R_t \left[\prod_{i=0, i\neq t_n}^{t} Prob(X_i \geq R_i) - \prod_{i=0}^{t} Prob(X_i \geq R_i) \right] \\
\geq \quad & L - R_{t_n} \textstyle\prod_{i=0}^{t_n} Prob(X_i \geq R_i).
\end{aligned}$$

First note that $R_{t_n} Prob(X_{t_n} \geq R_{t_n})$ is weakly dominated by $E(X_{t_n})$. In addition, there are at least N^* risky repayment before t_n and the survival probability with each risky payment $Prob(X_t > R_t) \leq Prob(X_t > L) = 1 - \epsilon$. As a result, the difference $\mathcal{D}(\tilde{\mathcal{R}}(t_n)) - \mathcal{D}(\mathcal{R})$ is at least

$$\mathcal{D}(\tilde{\mathcal{R}}(t_n)) - \mathcal{D}(\mathcal{R}) \geq L - (1-\epsilon)^{N^*} E(X_{t_n}) > 0 \tag{27}$$

by the definition of N^*. Therefore, $\tilde{\mathcal{R}}(t_n)$ is a repayment profile that is strictly more valuable to the investor. However, it may not be incentive compatible.

Now we prove the original proposition. There are two cases depending whether or not there exists a risky payment $R_t \geq E(X_t)$ for some $t \in \{t_{N^*+1}, t_{N^*+2}, ..., t_N\}$.

Case 1: Suppose there exists an $t_n \in \{t_{N^*+1}, t_{N^*+2}, ..., t_N\}$ such that $R_{t_n} \geq E(X_{t_n})$.

In this case, we show that $\tilde{\mathcal{R}}(t_n)$ is actually incentive compatible, namely $\tilde{R}_t \leq \tilde{V}_{t+1}$ for all t, where $\tilde{V}$ is the payoff to the borrower with the modified schedule. Combined with condition (27), the modified schedule $\tilde{\mathcal{R}}(t_n)$ gives the desired contradiction to the optimality of $\mathcal{R}$.

First, the IC conditions $\tilde{R}_t \leq \tilde{V}_{t+1}$ are not affected when $t > t_n$ because the payments $\tilde{R}_t = R_t$ and consequently $\tilde{V}_t = V_t$. Next, we show $\tilde{V}_{t+1} \geq V_t \geq E(X_t)$ for all $t_1 < t \leq t_n$ by induction method. Note that for any t, $V_t \geq E(X_t)$, which is a direct result from IC ($R_t \leq V_{t+1}$) and the recursive formulation of V_t:

$$V_t = E(X_t) + Prob(X_t \geq R_t)(-R_t + V_{t+1}) \geq E(X_t).$$

Combined with the presumption that we are in case 1, we have:

$$\begin{aligned} V_{t_n} &= E(X_{t_n}) + Prob(X_{t_n} \geq R_{t_n})(-R_{t_n} + V_{t_n+1}) \\ &\leq E(X_{t_n})(1 - Prob(X_{t_n} \geq R_{t_n})) + Prob(X_{t_n} \geq R_{t_n})V_{t_n+1} \\ &\leq V_{t_n+1} = \tilde{V}_{t_n+1}. \end{aligned}$$

This establishes the initial step of the induction. Now suppose $V_s \leq \tilde{V}_{s+1}$ holds for some $t_1 + 1 < s \leq t_n$, and we want to show $V_{s-1} \leq \tilde{V}_s$. From the induction assumption and the fact that $\tilde{R}_s = R_{s-1}$ in this region, we have

$$\begin{aligned} V_{s-1} &= E(X_{s-1}) + Prob(X_{s-1} \geq R_{s-1})(-R_{s-1} + V_s) \\ &\leq E(X_s) + Prob(X_s \geq \tilde{R}_s)(-\tilde{R}_s + \hat{V}_{s+1}) \\ &= \tilde{V}_s. \end{aligned}$$

This completes the induction proof. IC for all $t_1 < t \leq t_n$ follows immediately:

$$\tilde{R}_t = R_{t-1} \leq V_t \leq \tilde{V}_{t+1}.$$

Finally, by definition of t_1, all repayments before t_1 are risk free. Mathematically, for all $t \leq t_1$, $\tilde{R}_t \leq L \leq E(X_t)$. It is easy to show by induction that $\tilde{V}_t \geq E(X_t)$ also holds for all $t \leq t_1$. Therefore, IC also holds when $t \leq t_1$. We have verified the IC condition for the modified schedule $\tilde{\mathcal{R}}(t_n)$ for all $t = 1, 2, ..., T$, and thereby completing the proof of case 1.

Case 2: Suppose all repayments at t_{N^*+1} are strictly smaller than $E(X_t)$, i.e. $R_t < E(X_t)$ for all $t \geq t_{N^*+1}$.
Let $\hat{\mathcal{R}}$ be the risk-free modification to R_{T-1}, except for $\hat{R}_{t_{N^*+1}+1}$, which is alternatively defined as

$$\hat{R}_{t_{N^*+1}+1} = R_{t_{N^*+1}} - (1-\epsilon)^{N^*-2} E(X_{T-1}). \tag{28}$$

This is well defined because $R_{t_{N^*+1}} \geq L$ is a risky payment, and condition (18) guarantees that $\hat{R}_{t_{N^*+1}+1} > 0$. Denote by $\hat{V}_t$ the corresponding expected payoff to the borrower.

First, we show that the modified schedule $\hat{\mathcal{R}}$ is incentive compatible in three exhaustive cases, $t > t_{N^*+1}$, $t_{N^*+1} \geq t > t_1$, and $t \leq t_1$. Similar to Case 1, it is easy to show by induction that $\hat{V}_t \geq E(X_t)$ for all $t > t_{N^*+1}$. Because we are in Case 2, this result also means that IC holds strictly for all $t > t_{N^*+1}$.

Next we establish IC for $t_1 < t \leq t_{N^*+1}$. First, we show $V_{t_{N^*+1}} \leq \hat{V}_{t_{N^*+1}+1}$. When $t_1 \leq t \leq T-2$ and $t \neq t_{N^*+1}$, recall $\hat{R}_{t+1} = R_t$, so

$$\begin{aligned} \hat{V}_{t+1} - V_t &= E(X_{t+1}) + Prob(X_{t+1} \geq \hat{R}_{t+1})(-\hat{R}_{t+1} + \hat{V}_{t+2}) \\ &\quad - [E(X_t) + Prob(X_t \geq R_t)(-R_t + V_{t+1})] \\ &= Prob(X_t \geq R_t)(\hat{V}_{t+2} - V_{t+1}). \end{aligned} \tag{29}$$

By iteration, we have

$$\hat{V}_{t_{N^*+1}+2} - V_{t_{N^*+1}+1} = \prod_{s=t_{N^*+1}+1}^{T-2} Prob(X_s \geq R_s)(\hat{V}_T - V_{T-1}).$$

Because $\hat{V}_T = E(X_T)$ and $V_{T-1} \leq E(X_{T-1}) + E(X_T)$, together with the fact that there are at least $N - N^* - 2 \geq N^* - 2$ risky repayments, so the above difference is bounded below by

$$\hat{V}_{t_{N^*+1}+2} - V_{t_{N^*+1}+1} \geq -(1-\epsilon)^{N^*-2} E(X_{T-1}).$$

Now consider $\hat{V}_{t_{N^*+1}+1} - V_{t_{N^*+1}}$. From the above lower bound and definition (28), we have

$$\begin{aligned}
\hat{V}_{t_{N^*+1}+1} - V_{t_{N^*+1}} &= E(X_{t_{N^*+1}+1}) + Prob(X_{t_{N^*+1}+1} \geq \hat{R}_{t_{N^*+1}+1})(-\hat{R}_{t_{N^*+1}+1} + \hat{V}_{t_{N^*+1}+2}) \\
&\quad - \left[E(X_{t_{N^*+1}}) + Prob(X_{t_{N^*+1}} \geq R_{t_{N^*+1}})(-R_{t_{N^*+1}} + V_{t_{N^*+1}+1})\right] \\
&\geq Prob(X_{t_{N^*+1}} \geq R_{t_{N^*+1}})(-\hat{R}_{t_{N^*+1}+1} + R_{t_{N^*+1}} + \hat{V}_{t+2} - V_{t+1}) \\
&\geq Prob(X_{t_{N^*+1}} \geq R_{t_{N^*+1}})[(1-\epsilon)^{N^*-2} E(X_{T-1}) - (1-\epsilon)^{N^*-2} E(X_{T-1})] \\
&= 0
\end{aligned}$$

Therefore, we have shown $\hat{V}_{t_{N^*+1}+1} \geq V_{t_{N^*+1}}$. Combined with the iterative formula (29), we have $\hat{V}_{t+1} \geq V_t$ hold for all $t \geq t_1 + 1$, which in turn implies the desired IC condition:

$$\hat{V}_{t+1} \geq V_t \geq R_{t-1} \geq \hat{R}_t.$$

Finally, similar to case 1, when $t \leq t_1$, all repayments $\hat{R}_t$ are risk free which are in turn dominated by $E(X_t)$. It is also easy to inductively prove that $\hat{V}_t \geq E(X_t)$. This completes the verification of incentive compatibility of $\hat{\mathcal{R}}$.

Next, we prove that the modified schedule has an expected value $\mathcal{D}(\hat{\mathcal{R}})$ that strictly dominates $\mathcal{D}(\mathcal{R})$, thereby contradicting with the optimality of $\mathcal{R}$ and completing the proof. Write out $\mathcal{D}(\hat{\mathcal{R}})$ explicitly:

$$\begin{aligned}
\mathcal{D}(\hat{\mathcal{R}}) &= \textstyle\sum_{t=0}^{t_1-1} R_t + L + \sum_{t=t_1}^{t_{N^*+1}-1} R_t \prod_{i=1}^{t} Prob(X_i \geq R_i) \\
&\quad + \hat{R}_{t_{N^*+1}+1} Prob(X_{t_{N^*+1}+1} \geq \hat{R}_{t_{N^*+1}+1}) \textstyle\prod_{i=1}^{t_{N^*+1}-1} Prob(X_i \geq R_i) \\
&\quad + \textstyle\sum_{t=t_{N^*+1}+1}^{T-2} R_t Prob(X_{t_{N^*+1}+1} \geq \hat{R}_{t_{N^*+1}+1}) \prod_{i=1, i\neq t_{N^*+1}}^{t} Prob(X_i \geq R_i),
\end{aligned}$$

where $\hat{R}_{t_{N^*+1}+1}$ is related to $R_{t_{N^*+1}}$ by (28). Because $\hat{R}_{t_{N^*+1}+1} < R_{t_{N^*+1}}$, so

$$Prob(X_{t_{N^*+1}+1} \geq \hat{R}_{t_{N^*+1}+1}) \geq Prob(X_{t_{N^*+1}} \geq R_{t_{N^*+1}}).$$

Thus $\mathcal{D}(\hat{\mathcal{R}})$ is bounded below by

$$\begin{aligned}
\mathcal{D}(\hat{\mathcal{R}}) &\geq \textstyle\sum_{t=1}^{t_1-1} R_t + L + \sum_{t=t_1}^{t_{N^*+1}-1} R_t \prod_{i=0}^{t} Prob(X_i \geq R_i) \\
&\quad + \hat{R}_{t_{N^*+1}+1} \textstyle\prod_{i=0}^{t_{N^*+1}} Prob(X_i \geq R_i) \\
&\quad + \textstyle\sum_{t=t_{N^*+1}+1}^{T-2} R_t \prod_{i=0}^{t} Prob(X_i \geq R_i),
\end{aligned} \tag{30}$$

From (4), (30), and (28), we have:

$$\begin{aligned}\mathcal{D}(\hat{\mathcal{R}})-\mathcal{D}(\mathcal{R}) &\geq L-R_{T-1}\prod_{i=0}^{T-1}Prob(X_i\geq R_i)\\ &\quad +(\hat{R}_{t_{N^*+1}+1}-R_{t_{N^*+1}})\prod_{i=0}^{t_{N^*+1}}Prob(X_i\geq R_i)\\ &= L-R_{T-1}\prod_{i=0}^{T-1}Prob(X_i\geq R_i)\\ &\quad -(1-\epsilon)^{N^*-2}E(X_{T-1})\prod_{i=0}^{t_{N^*+1}}Prob(X_i\geq R_i).\end{aligned}$$

Because $R_{T-1} < V_T = E(X_T)$, and there are at least $N \geq 2N^*$ and N^*+1 risky repayment in $[1, T-1]$ and $[1, t_{N^*+1}]$ respectively, the above lower bounded is in turn greater than

$$\mathcal{D}(\hat{\mathcal{R}})-\mathcal{D}(\mathcal{R}) \geq L - E(X_T)(1-\epsilon)^{2N^*} - E(X_{T-1})(1-\epsilon)^{2N^*-1} > 0,$$

where the last inequality is from the definition of N^* in (18). This completes the proof of case 2 and the proposition.