

Credit markets, Limited commitment and Optimal monetary policy *

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Abstract

In a dynamic model with credit under limited commitment money can be essential when limited memory weakens the effects of punishment for default. There exist equilibria where both money and credit are used as media of exchange, and default occurs. In this equilibria the Friedman rule is not optimal. Inflation acts to discourage default by raising the cost of holding money, which is primarily held by defaulters. This results in relaxing the limited commitment constraint and raising welfare for all agents, including defaulting ones. The equilibrium is unique if and only if monetary policy and agents' money holdings are chosen sequentially.

1 Introduction

The important characteristics of the model are as in Carapella and Williamson [1], which builds on Lagos and Wright [7], Rocheteau and Wright [8] and Gu et al. [3]. In the model, exchange can be carried out using money and credit, with the latter subject to limited commitment. To support credit in equilibrium, borrowers must face the threat of punishment if they default. But punishment is potentially limited due to the inability of some lenders in decentralized exchange to observe past defaults by would-be borrowers. These limitations on punishment will play a critical role in credit market dysfunction in the model, and provide the a role for money, whose interaction with credit we are interested in. We focus on equilibria with individual-specific punishments – confined to the borrower who defaults.

Symmetric equilibria in this model feature similar characteristics to those of equilibria in Gu et al. [3]: monetary equilibria are such that credit is redundant, in that changes in credit conditions are neutral. When this is the case, the optimal monetary policy is the Friedman rule, if feasible.

More realistically, we analyze asymmetric equilibria in which some borrowers default in equilibrium while others do not. Borrowers in the model are intrinsically

*These are my own views, and not necessarily those of the Board of Governors of the Federal Reserve System or its staff.

identical, but those borrowers who default have no reputation to lose from doing so, while those who repay their debts do it because there is sufficient loss from defaulting. Typically, in models with limited commitment, for example Kehoe and Levine (1993), Kocherlakota (1996), or Sanches and Williamson (2010), there is only potential default, with credit supported by the threat of off-equilibrium punishments. This is problematic if we want to explain regularities in real-world default behavior. In our model, as in Carapella and Williamson [1], asymmetric equilibria display credit market dysfunction that corresponds to features of financial turmoil. In particular, in such equilibria there is an endogenous breakdown in credit relationships, with self-fulfilling default behavior. We are interested in understanding the role of monetary policy in these economies, specifically we analyze the effect of inflation on the terms of trade, incentives to default and welfare.

In asymmetric equilibria, there are instances in which a lender faces an adverse selection problem – he or she cannot tell the difference between a would-be borrower who has defaulted in the past, and one who has not. Such lenders will then charge borrowers a default premium, unless they can separate borrowers on the basis of the terms of trade they are offered. In this respect, there exists asymmetric equilibria that are pooling and separating. In a pooling equilibrium non defaulting borrowers pay the cost of other agents' defaults by being charged a default premium. In a separating equilibrium non defaulting borrowers are charged the intertemporal interest rate, thus enjoying higher consumption due to the absence of a default premium. Even in separating equilibria, however, non defaulting borrowers pay the cost of other agents' defaults by having to keep defaulting borrowers from pooling. This is achieved by making the pooling allocation relatively unappealing to a defaulter, that is by giving up some consumption. Whether a pooling or a separating equilibrium arises depends, among other things, on the growth rate of money, which, in a stationary allocation, translates one to one to inflation.

Inflation has two effects on equilibrium outcomes: it affects directly the cost of holding money, reducing agents' money holdings. Inflation also acts indirectly on the limited commitment constraint, affecting the borrowing ability of every agent subject to this constraint, including non money holders. This indirect mechanism works as follows: inflation taxes defaulters more than non defaulters because in any equilibrium the former hold more money than the latter. Higher inflation is then associated with lower value of defaulting, which relaxes the limited commitment constraint and can induce higher consumption and welfare for non defaulting borrowers at the expenses of defaulting ones. There is a limit, however, to the effectiveness of this mechanism. Consider a separating equilibrium: as inflation increases non defaulting borrowers are able to consume more, which in turn makes pooling attractive to defaulting borrowers. This sets an upper bound on the level of inflation that is consistent with a separating equilibrium. In other words, monetary policy can affect the extent of adverse selection in credit markets by inducing agents to switch from a separating to a pooling equilibrium.

It is not obvious, however, that an optimal monetary policy will try to select a separating equilibrium. In a separating equilibrium inflation is a redistribution from defaulting to non defaulting borrowers, while in a pooling equilibrium even defaulting borrowers benefit from a relaxed borrowing constraint, despite having

to pay higher cost for holding money.

Finally, there is a unique equilibrium in the game between private agents and the central bank if and only if they choose sequentially, regardless of who goes first. If the central bank chooses first the rate of growth of money and runs the Friedman rule, then agents respond by holding only money, and thus achieve the first best allocation. The same argument goes through if the central bank chooses instead a rate of growth of money that exceeds the intertemporal interest rate. This induces agents to economize on money holdings and use credit up to their borrowing constraint. Viceversa, if agents choose first to only hold money, then the central bank runs the Friedman rule. If defaulting and non defaulting agents choose to hold different amounts of money then the Friedman rule is not the optimal monetary policy, but it is still the unique equilibrium.

If instead the central bank and private agents choose simultaneously then multiple equilibria arise. One equilibrium features the central bank running the Friedman rule and agents holding only money, thus achieving the first best. Another monetary equilibrium features the central bank running monetary policy away from the Friedman rule and agents using both money and credit as media of exchange.

2 The Baseline Model

The baseline model we build on is a version of Carapella and Williamson [1], and is related to models of monetary exchange such as Lagos and Wright [7], Rocheteau and Wright [8] and to models that study credit economies with limited commitment, for example Sanches and Williamson [9] or Gu and Wright [2]. Our ideas combine key results from the literature in monetary economics with key results from the literature on endogenous borrowing constraints. We focus on the interaction between money and credit as media of exchange, and on the role of monetary policy in affecting equilibrium outcomes.

Time is indexed by $t = 1, 2, 3, \dots$, and each period consists of two subperiods, in which trade occurs, respectively, in a centralized market (CM) and a decentralized market (DM). There is a continuum of agents with mass 2, half of whom are buyers, with the other half being sellers. Each buyer has preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t [-H_t + u(x_t)] \quad (1)$$

where H_t is labor supply minus consumption during the CM , x_t is consumption in the DM , and $0 < \beta < 1$. Assume that $u(\cdot)$ is strictly concave, strictly increasing, and twice continuously differentiable with $u(0) = 0$, $u'(0) = \infty$, $u'(\infty) = 0$, and $-\frac{x u''(x)}{u'(x)} < 1$. A seller has preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t (X_t - h_t), \quad (2)$$

where X_t is consumption in the CM , and h_t is labor supply in the DM . Buyers can produce only in the CM , and sellers produce only in the DM . When productive, an agent has access to a technology which permits the production of one unit of the perishable consumption good for each unit of labor input.

As in Carapella and Williamson [1], during the DM , each buyer is randomly matched with a seller. A fraction ρ of DM meetings are *limited-information meetings*, in which the seller does not have access to the buyer's history. Even though there is limited information in this sense, the interaction between the buyer and seller in the meeting will be publicly recorded. The remaining fraction $1 - \rho$ of DM meetings are *full-information meetings*, in which the seller has access to the public record and the interaction between buyer and seller is recorded. Thus, the key assumption about "memory" (see Kocherlakota [6]) in the model is that agents engaged in exchange may sometimes not have access to the public record, but all information that could possibly be useful for any agent living in this world always resides in the public record. Note that the public record includes information on whether meetings in the DM were limited information or full information meetings. Credit histories are perfect, but a would-be lender may not have access to credit histories.

Another key credit friction, in addition to imperfect recordkeeping, is limited commitment (Kehoe-Levine [4], Kocherlakota [5], Sanches-Williamson [9]), in that economic agents in the model cannot be forced to work. Thus, a private debt will be repaid only if it is in the debtor's interest to do so.

In a DM meeting between a buyer and a seller, the buyer makes a take-it-or-leave-it offer to the seller. In this baseline credit model, a take-it-or-leave-it offer will be a credit contract, involving goods produced by the seller and given to the buyer in the current DM , in exchange for a promise by the buyer to supply goods to the seller in the next CM . The nature of that contract will depend on the information available to the seller – whether the meeting is limited-information or full-information – and what the buyer stands to lose if he or she should default on the credit contract.

For readers who are unfamiliar with the Lagos-Wright [7] structure, in this credit market context, the nature of heterogeneity among agents provides us with a simple motive for intertemporal exchange and credit contracts. Random matching in the DM is helpful, as this permits the coexistence of credit arrangements with poor information and with good information about credit histories, respectively. This will play an important role in the analysis. Finally, quasilinear preferences for buyers eliminates wealth effects, and makes the CM a period when debts are settled and the problem restarts. This gives some elements of decision making a two-period structure, while maintaining an infinite horizon – the latter being critical for supporting the credit arrangements. Linear preferences for sellers, combined with take-it-or-leave-it offers by buyers in the DM imply that behavior by sellers is trivial, simplifying our analysis by allowing us to focus on the behavior of buyers.

A key element in our model is money, which is model as an intrinsically useless object, perfectly divisible and storable. The total stock of money in period zero is denoted M_0 , and a central bank can produce or destroy money at no cost. Thus, for $t > 0$ the stock of money evolves according to $M_{t+1} = \gamma M_t$, with $\gamma \in \mathbb{R}^{++}$. Money

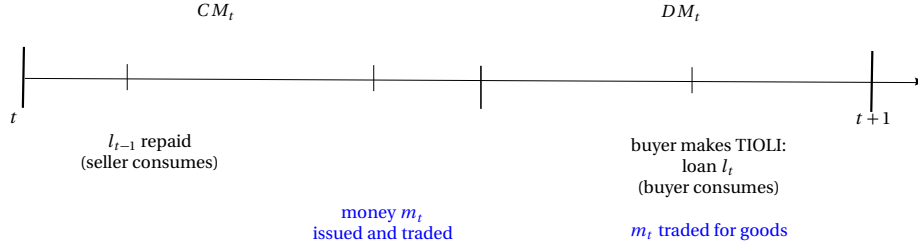


Figure 1: Timing

is issued and sold on a Walrasian market at the end of the CM, where ϕ_t denotes the price of money in terms of CM goods at time t . In the CM, agents first meet in a centralized location, where debts from the previous DM are settled. Then, in the latter part of the CM, anonymity holds. Money is issued and traded on a Walrasian market in exchange for the CM good. Thus, at the end of the CM neither the central bank nor the economic agents know the identities of the buyers and sellers of money. This implies that only lump sum taxes are feasible, although possibly not used in equilibrium. In the DM buyers can use both credit, up to the limit which they are willing to repay, and money to purchase the DM good.

As a visual aid, Figure 1 shows the sequence of activities during a period in the model. In black we represent activities related to credit only, while in blue we represent activities related to money only.

We focus on equilibria that are stationary, in that the real value of money is constant over time and each buyer and each seller receive the same allocation, and consume the same amount in each period. Therefore, in any equilibrium $\phi_t M_t = \phi_{t+1} M_{t+1}$. This implies that when $\frac{M_{t+1}}{M_t} = \gamma$ then $\frac{\phi_t}{\phi_{t+1}} = \gamma$. Typically, in models with limited commitment, credit is supported by the threat of punishment for default. This punishment never occurs in equilibrium, but agents have equilibrium beliefs about how that punishment occurs – off equilibrium. We analyze economies with individual punishments, where a default triggers retribution directed only against the individual defaulter. Note that there are many equilibria, and we confine attention to ones that are particularly interesting for the issues we want to address.

In order to analyze the interaction between money and credit market conditions (incentives to default and equilibrium defaults), we consider equilibria where agents behave asymmetrically, with some buyers defaulting in equilibrium, as in Carapella and Williamson [1]. These are equilibria where a fraction α of buyers (the *good buyers*) never defaults, but a fraction $1 - \alpha$ (*bad buyers*) will default on their debts if anyone chooses to lend to them. This is interesting as buyers are fundamentally identical, but in an asymmetric equilibrium identical economic agents are treated differently. It will in general be possible to support equilibria with differing values for α . Indeed, a special case is equilibria with $\alpha = 1$, which we have

already considered, and such equilibria can coexist with other equilibria with $\alpha < 1$. Thus, note that α is endogenous but, as we hope to make clear, it is indeterminate. In an asymmetric equilibrium, good buyers never default because they would be punished for default by being treated in the same way as bad buyers, thus losing access to exchange in the DM under full information. Bad buyers always default as they have nothing to lose – sellers who know their type will not lend to them.

We analyze equilibria according to the strategy that buyers play in limited information meetings, as in full information meetings access to the public record and individual punishments always guarantee that bad buyers do not obtain credit, while good buyers do. As a consequence, bad buyers consume in full information meetings if and only if they use money.

In limited information meetings, however, good buyers can choose to pool with bad buyers or separate from them. If they pool with bad buyers, they will have to repay a higher loan rate due to the credit risk that the seller face. With probability $1 - \alpha$ the buyer is indeed bad and will not repay the loan. If good buyers play separating strategies, they must make sure that bad buyers are unable or unwilling to imitate them, which would result in a pooling strategy. The allocation that solves a good buyer's problem in separating strategies must satisfy two incentive constraints, one in the CM and one in the DM. In the CM a bad buyer need not prefer to purchase a different amount of money from the one he would purchase in a separating equilibrium, and then, in the DM, imitate the good buyer in the event that he is in a limited information meeting. In the DM a bad buyer need not prefer to pool with the good buyer by possibly transferring a smaller amount of money to the seller, and selling any left over money in the next CM. Clearly, his money holdings must be sufficient to make this deviation feasible. Because pooling and separating strategies involve different trade offs, it is not obvious which strategy a good buyer would prefer to play. We first characterize solutions to the buyers' problem that involve pooling and separating strategies, then we solve for the equilibrium strategy and the optimal monetary policy.

Notice that there always exists an equilibrium where all agents use only money to pay for DM goods and the central bank runs the Friedman rule to achieve the first best consumption allocation. This is in the same spirit of the symmetric equilibria in Carapella and Williamson [1], the only difference being that we now introduce fiat money.

Lemma 1 *There always exists a symmetric equilibrium where every agent accumulates m^* , where m^* solves*

$$\max_m -\phi m + u(x) \quad (3)$$

$$s.t. \quad x \leq \beta \phi' m \quad (4)$$

The consumption allocation in DM meetings is first best if and only if $\gamma = \beta$.

Proof. Consider first the solution to the problem of a social planner's maximizing utility of all agents subject to resource constraints:

$$\max_m u(x) - h \quad (5)$$

$$s.t. \quad x \leq h \quad (6)$$

where x and h denote respectively buyer's consumption and seller's production in the DM. The objective function (5) denotes the net gain in utility from trading in the DM, and (6) the feasibility constraint on the allocation in the DM. Notice the the CM allocation is indeterminate due to the assumption of transferable utility. The solution to the social planner's problem is $x^* = \{x > 0 : u'(x) = 1\}$. The decision problem of a private agent in symmetric strategies, with beliefs that all buyers will default on loans obtained in the DM, is then (3), where agents maximize per period utility from purchasing money in the CM (which costs ϕm in terms of CM good) and consuming in the DM subject to the seller's participation constraint (4). The choice of m^* in (3) satisfies $u'(\beta \phi' m) \beta \phi' = \phi$. In a stationary equilibrium $\frac{\phi}{\phi'} = \gamma$, yielding $u'(\beta \phi' m) = \frac{\gamma}{\beta}$. The consumption allocation is first best if and only if $\beta \phi' m = x^*$, that is equivalent to $\gamma = \beta$. ■

In the remainder of the paper we focus on economies where monetary policy is away from the Friedman rule, i.e. $\gamma > \beta$. This will allow us to study equilibria where money and credit coexist, and to characterize optimal monetary policy in those equilibria.

Let v_t and $\hat{v}_t$ denote the value to good and bad buyers respectively at the end of CM_t when money is issued and traded on a Walrasian market. Then, given a belief system, the seller's set of acceptable offers and feasibility constraint on consumption and production of goods, the good buyer's problem is:

$$v_t = \max_{\{H_{t+1}^i, x_{t,i}^i, l_{t,i}^i, \tau_{t+1,i}^i, m_t\}} \{-\phi_t m_t + (1-\rho)[u(x_{t,F}) + \beta(m_t - d_{t,F}) - \beta H_{t+1,F}] + \rho[u(x_{t,L}) + \beta(m_t - d_{t,L}) - \beta H_{t+1,L}] + \beta v_{t+1}\} \quad (7)$$

$s.t.$

$$H_{t+1,i} \geq \tau_{t+1,i} \quad (8)$$

$$l_{t,i} \leq \beta \tau_{t+1,i} \quad (9)$$

$$x_{t,i} \leq l_{t,i} + \beta \phi' d_{t,i} \quad (10)$$

$$\beta H_{t+1,i} \leq \beta (v_{t+1} - \hat{v}_{t+1}) \quad (11)$$

where:

- $\hat{v}_t$ denotes the punishment continuation value for a buyer who defaults in CM_t
- m_t denotes the amount of money that the buyer purchases in CM_t
- H_{t+1}^i denotes the total amount of labor that the buyer has to exert in CM_{t+1} in order to pay his obligations
- $\tau_{t+1,i}$ denotes the amount of repayment (consumption goods produced by the buyer) in CM_{t+1} for the loan l_t^i obtained in the previous period in DM_t , where $i \in \{F, L\}$ according to whether the offer is in a full or limited information meeting.

(8) is a feasibility constraint on the labor effort of a buyer in CM_{t+1} to pay for his obligations: the repayment on the loan obtained from the seller in the DM_t ($\tau_{t+1,i}$).

(9) is the participation constraint for the seller: the seller in DM_t will extend a loan, l_t , only up to the present expected value of the repayment he will receive in CM_{t+1} , given his beliefs.

(10) is the buyer's feasibility constraint on consumption in DM_t : he can consume as many goods, x_t , as the seller is willing to produce, which equal the unsecured loan, $l_{t,i}$, plus the present value the transfer of money, $\beta\phi'd_{t,i}$.

(11) is the incentive constraint for the buyer in each meeting: the discounted payoff from non defaulting, $\beta v_{t+1} - \beta H_{t+1}^i$, must exceed the payoff from defaulting on the repayment of the loan and taxes requires, $\beta \hat{v}_{t+1}$. If (11) is violated then the buyer's strategy in the next CM at $t+1$ is to default.

A solution to problem (7) is always such that the buyer does not work more than he has to in order to pay his obligations, so (8) binds and such that (9) binds because the buyer makes a take-it-or-leave-it offer to the seller. Because $u(x)$ is strictly increasing then (10) binds. Therefore (7) can be simplified to:

$$\begin{aligned} v_t = & \max_{\{x_{t,i}, \tau_{t+1,i}, m_t\}} \{-\phi_t m_t + (1-\rho)[u(x_{t,F}) + \beta(m_t - d_{t,F}) - \beta\tau_{t+1,F}] \\ & + \rho[u(x_{t,L}) + \beta(m_t - d_{t,L}) - \beta\tau_{t+1,L}] + \beta v_{t+1}\} \\ \text{s.t.} & \\ & x_{t,i} \leq \beta\tau_{t+1,i} \\ & \tau_{t+1,i} \leq v_{t+1} - \hat{v}_{t+1} \end{aligned}$$

The $\hat{v}_t$ punishment continuation value for a defaulting buyer at the end of CM_t :

$$\hat{v}_t = \max_{\{\hat{x}_t, \hat{l}_t, \hat{m}_t, \hat{d}_{t,i}\}} -\phi_t \hat{m}_t + (1-\rho)[u(\hat{x}_{t,F}) + \beta(\hat{m}_t - \hat{d}_{t,F})] \quad (12)$$

$$\begin{aligned} & + \rho[u(\hat{x}_{t,L}) + \beta(\hat{m}_t - \hat{d}_{t,L})] + \beta \hat{v}_{t+1} \\ \text{s.t.} & \\ & \hat{x}_{t,i} \leq \hat{l}_t + \beta\phi'\hat{d}_{t,i} \leq \beta\hat{\tau}_{t+1,i} + \beta\phi'\hat{d}_{t,i} \end{aligned} \quad (13)$$

where:

- ρ is the probability of a limited information meeting
- $\hat{x}_{t,i}$ denotes the consumption of a defaulting buyer in DM_t
- $\hat{l}_{t,i}$ is the loan that the seller extends to a defaulting buyer in DM_t .
- $\hat{m}_t, \hat{d}_{t,i}$ denote, respectively, the units of money that a defaulting buyer purchases in CM_t and the units of money that he transfers to the seller in a type i meeting

(13) is the buyer's feasibility constraint on consumption in DM_t , which is the same constraint as for a non defaulting buyer.

The following sections characterize the different types of equilibria.

3 Pooling equilibria

Suppose first that in LI meetings bad agents imitate good ones and carry over to the next CM any unused units of money. The decision problem of good agents is then

$$\begin{aligned}
 v = \max \quad & \{-\phi m + (1-\rho)u(x_F) + \rho u(x_L) + \beta[(1-\rho)(\phi'(m-d_F) - \tau_F) + \rho(\phi'(m-d_L) - \tau_L)] + \beta v'\} \\
 \text{s.t.} \quad & x_F \leq \beta \phi' d_F + l_F \\
 & x_L \leq \beta \phi' d_L + l_L \\
 & l_F \leq \beta(v - \hat{v}) \\
 & l_L \leq \alpha \beta(v - \hat{v}) \\
 & \tau_F = \frac{l_F}{\beta}, \tau_L = \frac{l_L}{\alpha \beta}
 \end{aligned} \tag{14}$$

Where $l_L \leq l_F$.

Lemma 2 *If the LC constraint is slack in LI meetings then it is slack in FI meetings.*

Thus the possible solutions to (14) are that i) both LC are slack, ii) LC_F is slack and LC_L binds, iii) both LC bind.

Lemma 3 *If $\gamma > \beta$ then there is no solution with both LC slack.*

Idea of proof: because can always use less money, given that both LC is slack, and save on CM labor.

Because we are now focusing on economies where $\gamma > \beta$ then we disregard the case where both constraints are slack.

3.1 Case with LC_F is slack and LC_L binds and $m > 0$.

If this is an equilibrium then it must be that $x_F = x^* > x_L$ and $d_F \leq d_L = m$.

Lemma 4 *If LC_i is slack then $d_{-i} = m$.*

Idea of proof: suppose not, then either a) $d_{-i} < m$ and $d_i = m$, or b) $d_{-i} < m$ and $d_i < m$. In case a) we can construct an allocation such that $m' < m$, $d'_i \leq m$ and $d'_{-i} \leq m'$ so that $x'_i = x_{-i}$. Because LC_i is slack then we can always finance x'_i with more credit replacing the money. Buying less money to start with, in the CM, implies higher utility to the buyer since $\gamma > \beta$. In case b) we can reach a similar contradiction using $m' < m$ and financing the same consumption allocation.

Thus, with $x_F = x^* > x_L$ and $d_F \leq d_L = m$ the good buyer's problem becomes

$$\begin{aligned}
 v = \max \quad & \left\{ -\phi m + (1-\rho)[u(x^*) + \beta \phi'(m-d_F) - l_F] + \rho \left[u(x_L) + \beta \phi'(m-d_L) - \frac{l_L}{\alpha} \right] + \beta v' \right\} \\
 \text{s.t.} \quad & x_L = \beta \phi' d_L + l_L \\
 & l_F \leq \beta(v - \hat{v}) \\
 & l_L = \alpha \beta(v - \hat{v})
 \end{aligned} \tag{15}$$

Lemma 5 *If LC_F is slack then $d_F = 0$.*

Idea of proof: suppose not, then $d_F > 0$ and we can construct an alternative allocation with $d'_F < d_F$ so that LC_F binds for the same $x'_F = x_F$ by using more credit than money. Then the buyer can carry money $d_F - d'_F > 0$ in the next CM, which yields an increase in utility of $\beta\phi' > 0$.

Then the good buyer's choice of m solves

$$v = \max_m \{-\phi m + (1-\rho)[u(x^*) + \beta\phi' m - x^*] + \rho[u(x_L) - \beta(v - \hat{v})] + \beta v'\}$$

with $x_L = \beta\phi' d_L + \alpha\beta(v - \hat{v})$. Thus m satisfies $-\phi + \beta\phi'[(1-\rho) + \rho u'(x_L)] = 0$, implying:

$$m = \frac{u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right) - \alpha\beta(v - \hat{v})}{\beta\phi'} \quad (16)$$

Now, in order to solve for $\hat{v}$, we solve the problem of the bad buyer

$$\hat{v} = \max_{\hat{m}} \{-\phi \hat{m} + (1-\rho)u(\beta\phi' \hat{m}) + \rho[u(x_L) + \beta\phi'(\hat{m} - d_L)] + \beta \hat{v}'\} \quad (17)$$

with $d_L = m \leq \hat{m}$. Their choice of $\hat{m}$ satisfies $-\phi + \beta\phi'[(1-\rho)u'(\beta\phi' \hat{m}) + \rho] = 0$, implying

$$\hat{m} = \frac{u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right)}{\beta\phi'}. \quad (18)$$

Because in this equilibrium $\hat{m} \geq m$ then one necessary and sufficient for this equilibrium is

$$u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right) \geq u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right) - \alpha\beta(v - \hat{v}). \quad (19)$$

Furthermore, for this to be an equilibrium, two incentive constraints need to be satisfied: i) an interim, or DM, incentive constraint for the bad buyer to play pooling in LI meetings, conditional on entering the meeting with money holdings $\hat{m}$, is $u(x_L) + \beta\phi'(\hat{m} - d_L) \geq u(\beta\phi' \hat{m})$, and ii) a CM incentive constraint for the bad buyer to choose money holdings $\hat{m}$ in CM and play pooling in DM.

3.1.1 DM incentive constraint

If the DM constraint was violated, the bad buyer would strictly prefer to separate, conditional on being in a LI meeting with money holdings $\hat{m}$, by spending all his money instead of imitating the good buyer. The DM incentive constraint can also be rewritten as

$$u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right)\right) + \beta\phi'(\hat{m} - m) \geq u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right)\right) \quad (20)$$

where $d_L = m$ has been substituted out. We can further rewrite the DM incentive constraint as

$$\begin{aligned} \alpha\beta(v - \hat{v}) \geq & \frac{\left[u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right)\right) - u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right)\right) \right]}{\beta\phi'} \\ & - u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right) + u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right) \end{aligned}$$

Lemma 6 (19) always implies (20).

Proof. Combining (19) with (20) yields

$$\begin{aligned} \min \left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right), u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right) - \frac{\left[u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right)\right) - u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right)\right) \right]}{\beta\phi'} \right) \geq \\ u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right) - \alpha\beta(v - \hat{v}) \end{aligned}$$

But because the lowest possible value of the right hand side of the LC constraint, $(v - \hat{v})$, is zero, then (19) implies that $u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right) \geq u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right)$. Thus the relevant necessary and sufficient condition is simply (19). ■

The intuition behind this result is quite simple: conditional on entering a LI meeting with given money holdings $\hat{m} \geq m$ the best strategy for a bad buyer is to pool with a good buyer and exploit the larger consumption guaranteed by his reputation. In fact, the DM incentive constraint implies that $x_L \geq \hat{x}_F = \beta\phi'\hat{m}$. Because the money holdings the bad buyer enters the meeting with would allow him to achieve only DM consumption $\hat{x}_F$, then he prefers to pool.

The CM incentive constraint, instead, takes care of whether the bad buyer has incentive to choose $\hat{m}$ in the first place.

3.1.2 CM incentive constraint

The CM incentive constraint for the bad buyer to choose money holdings $\hat{m}$ in CM and play pooling in DM. Differently from the DM incentive constraint, the bad buyer could choose money holdings $\hat{m}$ if the consumption stemming from transferring $\hat{m}$ to the seller, namely $\beta\phi'\hat{m}$ by the seller's participation constraint, is sufficiently larger than x_L .

Thus the CM incentive constraint entails ruling out a double deviation. The choice of $\hat{m}$, if the bad buyer planned on deviating to a separating strategy in a LI meeting, satisfies $-\phi + \beta\phi'u'(\beta\phi'\hat{m}) = 0$, yielding

$$\hat{m} = \frac{u'^{-1}\left(\frac{\phi}{\beta\phi'}\right)}{\beta\phi'} \quad (21)$$

In the following lemma we show that if a bad buyer was to deviated from the pooling strategy and separate by choosing a different money holding, it must be that he would choose to hold more money than a bad buyer who pools. Intuitively, if $\hat{m} > \hat{m}$ then the deviating bad buyer can consume more than $\hat{x}_F = \beta \phi' \hat{m}$ also in LI meetings. It wouldn't be optimal for a bad buyer to deviate to a separating strategy by holding less than $\hat{m}$ because this would imply lower consumption in both FI and LI meetings than by playing the equilibrium pooling strategy, and it would imply simply saving on purchase of money in the CM.

Lemma 7 *If $\gamma > \beta$ then $\hat{m} > \hat{m}$.*

Proof. Combining first order condition (21) with (18) implies that $\hat{m} > \hat{m}$ if and only if $u'^{-1}\left(\frac{\phi}{\beta \phi'}\right) > u'^{-1}\left(\frac{\frac{\phi}{\beta \phi'} - \rho}{1 - \rho}\right)$. This is equivalent to $\frac{\phi}{\beta \phi'}(1 - \rho) < \frac{\phi}{\beta \phi'} - \rho$ due to strict concavity of u , which is always satisfied since $\frac{\phi}{\beta \phi'} = \frac{\gamma}{\beta} > 1$. ■

The CM incentive constraint that rules out the deviation by an individual bad buyer (thus not affecting prices in the CM) is

$$-\phi \hat{m} + (1 - \rho) u(\beta \phi' \hat{m}) + \rho [u(x_L) + \beta \phi'(\hat{m} - m)] \geq -\phi \hat{m} + u(\beta \phi' \hat{m}) \quad (22)$$

where $x_L = u'^{-1}\left(\frac{\frac{\phi}{\beta \phi'} - (1 - \rho)}{\rho}\right)$. In order to fully characterize this incentive constraint we need to solve for $(\hat{m} - m)$.

3.1.3 Equilibrium characterization

In order to characterize the pooling equilibrium allocation, (14) and (17) imply:

$$(1 - \beta)(v - \hat{v}) = \phi(\hat{m} - m) + (1 - \rho)[u(x^*) - u(\beta \phi' \hat{m}) + \beta \phi' m - x^*] + \rho[u(x_L) - u(x_L) - \beta(v - \hat{v}) - \beta \phi'(\hat{m} - d_L)]$$

which, using $d_L = m$, we can rewrite as

$$(1 - \beta + \beta \rho)(v - \hat{v}) = (\phi - \rho \beta \phi')(\hat{m} - m) + (1 - \rho)[u(x^*) - u(\beta \phi' \hat{m}) + \beta \phi' m - x^*]$$

and further as

$$\begin{aligned} & \left[1 - \beta + \beta \rho + \alpha \beta \left(1 - \frac{\phi}{\beta \phi'}\right)\right](v - \hat{v}) = \\ & \left(\frac{\phi}{\beta \phi'} - \rho\right) \left(u'^{-1}\left(\frac{\frac{\phi}{\beta \phi'} - \rho}{1 - \rho}\right) - u'^{-1}\left(\frac{\frac{\phi}{\beta \phi'} - (1 - \rho)}{\rho}\right)\right) + \\ & + (1 - \rho) \left[u(x^*) - x^* - u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta \phi'} - \rho}{1 - \rho}\right)\right) + u'^{-1}\left(\frac{\frac{\phi}{\beta \phi'} - (1 - \rho)}{\rho}\right)\right] \end{aligned}$$

Using $\frac{\phi}{\beta\phi'} = \frac{\gamma}{\beta} > 1$, we then have

$$\begin{aligned} & \left[1 - \beta + \beta\rho + \alpha\beta\left(1 - \frac{\gamma}{\beta}\right)\right](v - \hat{v}) = \\ & \left(\frac{\gamma}{\beta} - \rho\right)\left(u'^{-1}\left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}\right) - u'^{-1}\left(\frac{\frac{\gamma}{\beta} - (1 - \rho)}{\rho}\right)\right) + \\ & + (1 - \rho)\left[u(x^*) - x^* - u\left(u'^{-1}\left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}\right)\right) + u'^{-1}\left(\frac{\frac{\gamma}{\beta} - (1 - \rho)}{\rho}\right)\right] \end{aligned} \quad (23)$$

Thus

$$\begin{aligned} m = & \frac{u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1 - \rho)}{\rho}\right)}{\beta\phi'} - \frac{\alpha\left(\frac{\gamma}{\beta} - \rho\right)\left(u'^{-1}\left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}\right) - u'^{-1}\left(\frac{\frac{\gamma}{\beta} - (1 - \rho)}{\rho}\right)\right)}{\phi'\left[1 - \beta + \beta\rho + \alpha\beta\left(1 - \frac{\gamma}{\beta}\right)\right]} + \\ & - \frac{\alpha(1 - \rho)\left[u(x^*) - x^* - u\left(u'^{-1}\left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}\right)\right) + u'^{-1}\left(\frac{\frac{\gamma}{\beta} - (1 - \rho)}{\rho}\right)\right]}{\phi'\left[1 - \beta + \beta\rho + \alpha\beta\left(1 - \frac{\gamma}{\beta}\right)\right]} \end{aligned}$$

that can be simplified to

$$\begin{aligned} m = & u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1 - \rho)}{\rho}\right)\left\{\frac{1}{\beta\phi'} + \frac{\alpha\left(\frac{\gamma}{\beta} - 1\right)}{\phi'\left[1 - \beta + \beta\rho + \alpha\beta\left(1 - \frac{\gamma}{\beta}\right)\right]}\right\} - \frac{\alpha\left(\frac{\gamma}{\beta} - \rho\right)u'^{-1}\left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}\right)}{\phi'\left[1 - \beta + \beta\rho + \alpha\beta\left(1 - \frac{\gamma}{\beta}\right)\right]} + \\ & - \frac{\alpha(1 - \rho)\left[u(x^*) - x^* - u\left(u'^{-1}\left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}\right)\right)\right]}{\phi'\left[1 - \beta + \beta\rho + \alpha\beta\left(1 - \frac{\gamma}{\beta}\right)\right]} \end{aligned}$$

and further to

$$\begin{aligned} m = & u'^{-1}\left(\frac{\frac{\gamma}{\beta} - (1 - \rho)}{\rho}\right)\left\{\frac{1 - \beta + \beta\rho}{\beta\phi'\left[1 - \beta + \beta\rho + \alpha\beta\left(1 - \frac{\gamma}{\beta}\right)\right]}\right\} \\ & - \frac{\alpha\left\{(1 - \rho)\left[u(x^*) - x^* - u\left(u'^{-1}\left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}\right)\right)\right] + \left(\frac{\gamma}{\beta} - \rho\right)u'^{-1}\left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}\right)\right\}}{\phi'\left[1 - \beta + \beta\rho + \alpha\beta\left(1 - \frac{\gamma}{\beta}\right)\right]} \end{aligned}$$

similarly,

$$\begin{aligned} \hat{m} - m = & \frac{(1 - \beta + \beta\rho)\left[u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1 - \rho}\right) - u'^{-1}\left(\frac{\frac{\gamma}{\beta} - (1 - \rho)}{\rho}\right)\right]}{\beta\phi'\left[1 - \beta + \beta\rho + \alpha\beta\left(1 - \frac{\gamma}{\beta}\right)\right]} \\ & + \frac{\alpha(1 - \rho)\left[u(x^*) - x^* - u\left(u'^{-1}\left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}\right)\right)\right]}{\phi'\left[1 - \beta + \beta\rho + \alpha\beta\left(1 - \frac{\gamma}{\beta}\right)\right]} \end{aligned}$$

Returning to the CM incentive constraint we can now rewrite (22) as

$$\phi(\hat{m} - \hat{m}) + (1 - \rho)u(\beta\phi'\hat{m}) + \rho[u(x_L) + \beta\phi'(\hat{m} - m)] - u(\beta\phi'\hat{m}) \geq 0$$

which, substituting out equilibrium allocations, becomes

$$\begin{aligned} & \frac{\gamma}{\beta} \left(u'^{-1} \left(\frac{\gamma}{\beta} \right) - u'^{-1} \left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho} \right) \right) + (1 - \rho) u \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho} \right) \right) + \rho u \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta} - (1 - \rho)}{\rho} \right) \right) + \\ & \frac{\rho(1 - \beta + \beta\rho) \left[u'^{-1} \left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho} \right) - u'^{-1} \left(\frac{\frac{\gamma}{\beta} - (1 - \rho)}{\rho} \right) \right]}{[1 - \beta + \beta\rho + \alpha\beta(1 - \frac{\gamma}{\beta})]} + \frac{\alpha\beta\rho(1 - \rho) [u(x^*) - x^* - u(u'^{-1}(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}))]}{[1 - \beta + \beta\rho + \alpha\beta(1 - \frac{\gamma}{\beta})]} \geq \\ & u \left(u'^{-1} \left(\frac{\gamma}{\beta} \right) \right) \end{aligned}$$

We can further rewrite then (22) as

$$\begin{aligned} & \frac{\gamma}{\beta} u'^{-1} \left(\frac{\gamma}{\beta} \right) + \left(\frac{\rho(1 - \beta + \beta\rho)}{[1 - \beta + \beta\rho + \alpha\beta(1 - \frac{\gamma}{\beta})]} - \frac{\gamma}{\beta} \right) u'^{-1} \left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho} \right) + \\ & (1 - \rho) \left(\frac{[1 - \beta + \beta\rho + \alpha\beta(1 - \frac{\gamma}{\beta})] - \alpha\beta\rho}{[1 - \beta + \beta\rho + \alpha\beta(1 - \frac{\gamma}{\beta})]} \right) u \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho} \right) \right) + \rho u \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta} - (1 - \rho)}{\rho} \right) \right) + \\ & - \frac{\rho(1 - \beta + \beta\rho) u'^{-1} \left(\frac{\frac{\gamma}{\beta} - (1 - \rho)}{\rho} \right)}{[1 - \beta + \beta\rho + \alpha\beta(1 - \frac{\gamma}{\beta})]} + \frac{\alpha\beta\rho(1 - \rho) [u(x^*) - x^*]}{[1 - \beta + \beta\rho + \alpha\beta(1 - \frac{\gamma}{\beta})]} \geq \\ & u \left(u'^{-1} \left(\frac{\gamma}{\beta} \right) \right) \end{aligned}$$

which is

$$\begin{aligned} & \frac{\gamma}{\beta} \hat{x}_F + \left(\frac{\rho(1 - \beta + \beta\rho) - \frac{\gamma}{\beta} [1 - \beta + \beta\rho + \alpha\beta - \alpha\gamma]}{[1 - \beta + \beta\rho + \alpha\beta - \alpha\gamma]} \right) \hat{x}_F + \\ & (1 - \rho) \left(\frac{1 - \beta + \beta\rho + \alpha\beta(1 - \rho) - \alpha\gamma}{[1 - \beta + \beta\rho + \alpha\beta - \alpha\gamma]} \right) u(\hat{x}_F) + \rho u(x_L) + \\ & + \frac{\alpha\beta\rho(1 - \rho) [u(x^*) - x^*] - \rho(1 - \beta + \beta\rho)x_L}{[1 - \beta + \beta\rho + \alpha\beta - \alpha\gamma]} \geq u(\hat{x}_F) \\ & \frac{\gamma}{\beta} (\hat{x}_F - \hat{x}_F) + (1 - \rho) \left(\frac{1 - \beta + \beta\rho + \alpha\beta(1 - \rho) - \alpha\gamma}{[1 - \beta + \beta\rho + \alpha\beta - \alpha\gamma]} \right) u(\hat{x}_F) + \rho u(x_L) + \\ & + \frac{\alpha\beta\rho(1 - \rho) [u(x^*) - x^*] + \rho(1 - \beta + \beta\rho)(\hat{x}_F - x_L)}{[1 - \beta + \beta\rho + \alpha\beta - \alpha\gamma]} \geq u(\hat{x}_F) \end{aligned}$$

where it must be that $\hat{x}_F \geq x_L > \hat{x}_F$, otherwise $\hat{m} \geq \hat{m}$ would be violated, which we know must be always true because $\frac{\gamma}{\beta} > 1$.¹

¹In fact, $\hat{m} \geq \hat{m}$ if and only if $\frac{\gamma}{\beta}(1 - \rho) < \frac{\gamma}{\beta} - \rho$.

We can rewrite (22) further as

$$\begin{aligned}
& (1-\rho)(1-\beta+\beta\rho+\alpha[\beta(1-\rho)-\gamma])u(\hat{x}_F)+ \\
& +\alpha\beta\rho(1-\rho)[u(x^*)-x^*]+\rho(1-\beta+\beta\rho)(\hat{x}_F-x_L) \geq \\
& \left(u(\hat{x}_F)-\rho u(x_L)-\frac{\gamma}{\beta}(\hat{x}_F-\hat{x}_F)\right)(1-\beta+\beta\rho+\alpha[\beta-\gamma])
\end{aligned} \tag{24}$$

Because $\hat{x}_F$ does not depend on α or ρ then

$$\begin{aligned}
& \alpha(1-\rho)[\beta(1-\rho)-\gamma]u(\hat{x}_F)+ \\
& +\alpha\beta\rho(1-\rho)[u(x^*)-x^*] \\
& +\alpha(\gamma-\beta)\left(u(\hat{x}_F)-\rho u(x_L)-\frac{\gamma}{\beta}(\hat{x}_F-\hat{x}_F)\right) \geq \\
& \left(u(\hat{x}_F)-\rho u(x_L)-\frac{\gamma}{\beta}(\hat{x}_F-\hat{x}_F)\right)(1-\beta+\beta\rho) \\
& -(1-\rho)(1-\beta+\beta\rho)u(\hat{x}_F)-\rho(1-\beta+\beta\rho)(\hat{x}_F-x_L)+
\end{aligned}$$

that can be rearranged as

$$\begin{aligned}
& \alpha\{(1-\rho)[(\beta(1-\rho)-\gamma)u(\hat{x}_F)+\beta\rho(u(x^*)-x^*)]\} \\
& +\alpha(\gamma-\beta)\left(u(\hat{x}_F)-\rho u(x_L)-\frac{\gamma}{\beta}(\hat{x}_F-\hat{x}_F)\right) \geq \\
& (1-\beta+\beta\rho)\left[u(\hat{x}_F)-\rho u(x_L)-\frac{\gamma}{\beta}(\hat{x}_F-\hat{x}_F)-(1-\rho)u(\hat{x}_F)-\rho(\hat{x}_F-x_L)\right]
\end{aligned}$$

Thus, if $(1-\rho)[(\beta(1-\rho)-\gamma)u(\hat{x}_F)+\beta\rho(u(x^*)-x^*)]+(\gamma-\beta)\left(u(\hat{x}_F)-\rho u(x_L)-\frac{\gamma}{\beta}(\hat{x}_F-\hat{x}_F)\right) > 0$ then (24) is a lower bound on α , otherwise it is an upper bound on α .

This inequality is a necessary and sufficient condition for a pooling equilibrium to exits.

Also, assuming $1-\beta+\beta\rho+\alpha\beta\left(1-\frac{\gamma}{\beta}\right) > 0$, $v > \hat{v}$ if and only if

$$\begin{aligned}
& \left(\frac{\gamma}{\beta}-\rho\right)u'^{-1}\left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho}\right)+\left(1-\frac{\gamma}{\beta}\right)u'^{-1}\left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho}\right)+ \\
& +(1-\rho)\left[u(x^*)-x^*-u\left(u'^{-1}\left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho}\right)\right)\right] > 0
\end{aligned} \tag{25}$$

Using (23), if $1-\beta+\beta\rho+\alpha\beta\left(1-\frac{\gamma}{\beta}\right) > 0$, $\hat{m} \geq m$ if and only if:

$$\begin{aligned}
& \alpha\beta\left(\frac{\gamma}{\beta}-\rho\right)\left(u'^{-1}\left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho}\right)-u'^{-1}\left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho}\right)\right)+ \\
& +\alpha\beta(1-\rho)\left[u(x^*)-x^*-u\left(u'^{-1}\left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho}\right)\right)+u'^{-1}\left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho}\right)\right] \geq
\end{aligned}$$

$$\left[1 - \beta + \beta\rho + \alpha\beta\left(1 - \frac{\gamma}{\beta}\right)\right] \left[u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right) - u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right) \right]$$

which is equivalent to (19) being satisfied, and can be rearranged as

$$\begin{aligned} [1 - \beta + \beta\rho + \alpha\beta(1 - \rho)] u'^{-1}\left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}\right) - (1 - \beta + \beta\rho) u'^{-1}\left(\frac{\frac{\gamma}{\beta} - (1 - \rho)}{\rho}\right) + \\ + \alpha\beta(1 - \rho) \left[u(x^*) - x^* - u\left(u'^{-1}\left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}\right)\right) \right] \geq 0 \end{aligned}$$

Proposition 8 *If $1 - \beta + \beta\rho + \alpha\beta\left(1 - \frac{\gamma}{\beta}\right) > 0$ and $\frac{\gamma}{\beta} > 1$, then a pooling equilibrium with good agents carrying into the DM $m > 0$ exists if and only if*

$$\begin{aligned} \left[\frac{1 - \beta + \beta\rho + \alpha\beta(1 - \rho)}{\alpha\beta} \right] \beta\phi'\hat{m} - \frac{1 - \beta + \beta\rho}{\alpha\beta} x_L + (1 - \rho)[u(x^*) - x^*] \geq \\ (1 - \rho)u(\beta\phi'\hat{m}) \geq \\ \frac{(1 - \beta + \beta\rho - \alpha\gamma + \alpha\beta)[- \phi\hat{m} + u(\beta\phi'\hat{m}) - \rho u(x_L)] + \rho(1 - \beta + \beta\rho)x_L}{1 - \beta + \beta\rho - \alpha\gamma + \alpha\beta(1 - \rho)} \\ - \frac{\rho\alpha\beta(1 - \rho)[u(x^*) - x^*]}{1 - \beta + \beta\rho - \alpha\gamma + \alpha\beta(1 - \rho)} + \beta\phi'\hat{m}\left(\frac{\gamma}{\beta} - \rho\right) \end{aligned} \quad (26)$$

where $x_L = u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1 - \rho)}{\rho}\right)$, $\beta\phi'\hat{m} = u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1 - \rho}\right)$, $\beta\phi'\hat{m} = u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'}}{1 - \rho}\right)$, with $\hat{m} > \hat{m}$ and $x_L \geq \beta\phi'\hat{m}$ (from the DM IC).

Proof. If $\frac{\gamma}{\beta} > 1$, the two necessary and sufficient conditions are $v > \hat{v}$:

$$\begin{aligned} \left(\frac{\gamma}{\beta} - \rho\right) u'^{-1}\left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}\right) + \left(1 - \frac{\gamma}{\beta}\right) u'^{-1}\left(\frac{\frac{\gamma}{\beta} - (1 - \rho)}{\rho}\right) + \\ + (1 - \rho) \left[u(x^*) - x^* - u\left(u'^{-1}\left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}\right)\right) \right] > 0 \end{aligned} \quad (27)$$

and $\hat{m} \geq m$:

$$\begin{aligned} [1 - \beta + \beta\rho + \alpha\beta(1 - \rho)] u'^{-1}\left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}\right) - (1 - \beta + \beta\rho) u'^{-1}\left(\frac{\frac{\gamma}{\beta} - (1 - \rho)}{\rho}\right) + \\ + \alpha\beta(1 - \rho) \left[u(x^*) - x^* - u\left(u'^{-1}\left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}\right)\right) \right] \geq 0 \end{aligned}$$

The latter inequality can be rewritten as:

$$\begin{aligned} & \frac{[1-\beta+\beta\rho]}{\alpha\beta} \left[u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) - u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right] + \\ & + (1-\rho) \left[u(x^*) - x^* - u \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) \right) + u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) \right] \geq 0 \end{aligned} \quad (28)$$

Moreover, if $1-\beta+\beta\rho+\alpha\beta\left(1-\frac{\gamma}{\beta}\right) > 0$ then (28) always implies (27).

Notice that (22) can be rewritten as

$$\begin{aligned} & (1-\rho)u(\beta\phi'\hat{m}) \left[\frac{1-\beta+\beta\rho-\alpha\gamma+\alpha\beta(1-\rho)}{(1-\beta+\beta\rho-\alpha\gamma+\alpha\beta)} \right] \geq \\ & -\phi\hat{m} + u(\beta\phi'\hat{m}) - \rho u(x_L) + \rho \left[\frac{1-\beta+\beta\rho}{(1-\beta+\beta\rho-\alpha\gamma+\alpha\beta)} \right] x_L \\ & - \frac{\rho\alpha\beta(1-\rho)[u(x^*)-x^*]}{(1-\beta+\beta\rho-\alpha\gamma+\alpha\beta)} + \left[\left(\frac{\gamma}{\beta} - \rho \right) \beta\phi' - \rho\alpha\beta \frac{(\phi-\rho\beta\phi')}{(1-\beta+\beta\rho-\alpha\gamma+\alpha\beta)} \right] \hat{m} \end{aligned}$$

and further rearranged as

$$\begin{aligned} & (1-\rho)u(\beta\phi'\hat{m}) \geq \\ & \frac{(1-\beta+\beta\rho-\alpha\gamma+\alpha\beta)}{1-\beta+\beta\rho-\alpha\gamma+\alpha\beta(1-\rho)} [-\phi\hat{m} + u(\beta\phi'\hat{m}) - \rho u(x_L)] + \frac{\rho(1-\beta+\beta\rho)}{1-\beta+\beta\rho-\alpha\gamma+\alpha\beta(1-\rho)} x_L \\ & - \frac{\rho\alpha\beta(1-\rho)[u(x^*)-x^*]}{1-\beta+\beta\rho-\alpha\gamma+\alpha\beta(1-\rho)} + \frac{[(1-\beta+\beta\rho-\alpha\gamma+\alpha\beta)\left(\frac{\gamma}{\beta}-\rho\right)\beta\phi' - \rho\alpha\beta(\phi-\rho\beta\phi')]\hat{m}}{(1-\beta+\beta\rho-\alpha\gamma+\alpha\beta(1-\rho))} \end{aligned}$$

where we have substituted $m = \frac{u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'}-(1-\rho)}{\rho}\right) - \alpha\beta(v-\hat{v})}{\beta\phi'}$ from (16), and $(v-\hat{v})$ as defined in (23) and rewritten as:

$$\left(1-\beta+\beta\rho-\left(\frac{\gamma}{\beta}-1\right)\alpha\beta\right)(v-\hat{v}) = (\phi-\rho\beta\phi')\hat{m} - \left(\frac{\gamma}{\beta}-1\right)x_L + (1-\rho)[u(x^*)-x^*-u(\beta\phi'\hat{m})]$$

Likewise, (28) can be rewritten as

$$\begin{aligned} & -\frac{1-\beta+\beta\rho}{\alpha\beta} x_L + (1-\rho)[u(x^*)-x^*] \\ & + \left[1-\rho + \frac{1-\beta+\beta\rho}{\alpha\beta}\right] \beta\phi'\hat{m} \geq (1-\rho)u(\beta\phi'\hat{m}) \end{aligned}$$

Combining (22) with (28) yields (26). ² ■

²Where we have rewritten the last term in the second inequality as

$$\frac{[(1-\beta+\beta\rho-\alpha\gamma+\alpha\beta)\left(\frac{\gamma}{\beta}-\rho\right)\beta\phi' - \rho\alpha\beta(\phi-\rho\beta\phi')]\hat{m}}{(1-\beta+\beta\rho-\alpha\gamma+\alpha\beta(1-\rho))} = \beta\phi'\hat{m}\left(\frac{\gamma}{\beta}-\rho\right)$$

Notice that $1 - \beta + \beta\rho + \alpha\beta\left(1 - \frac{\gamma}{\beta}\right) > 0$ can be interpreted as an upper bound on α . Moreover, under additional assumption, we can characterize whether (28) or (22) is the relevant necessary and sufficient condition for the existence of a pooling equilibrium.

Corollary 9 *Maintain the assumptions $1 - \beta + \beta\rho + \alpha\beta\left(1 - \frac{\gamma}{\beta}\right) > 0$ and $\frac{\gamma}{\beta} > 1$. If*

$$u(\hat{x}_F) - \rho u(x_L) - (1 - \rho)u(\hat{x}_F) - \frac{\gamma}{\beta}\hat{x}_F > -\left[\frac{\alpha\beta\rho(1 - \rho)}{[1 - \beta + \beta\rho + \alpha(\beta - \gamma)]} + \frac{\gamma}{\beta}\right]\hat{x}_F$$

then the pooling equilibrium $m, x^, x_L, \hat{m}, \hat{x}_F, x_L$ exists if and only if (22) is satisfied.*

Proof. Substituting the notation for the equilibrium allocation in (28) yields

$$\begin{aligned} [1 - \beta + \beta\rho](\hat{x}_F - x_L) \\ + \alpha\beta(1 - \rho)[u(x^*) - x^* - u(\hat{x}_F)] \geq -\alpha\beta(1 - \rho)\hat{x}_F \end{aligned}$$

where the right hand side is always negative. Likewise, (22) can be rearranged as

$$\begin{aligned} (1 - \beta + \beta\rho)(\hat{x}_F - x_L) + \alpha\beta(1 - \rho)[u(x^*) - x^* - u(\hat{x}_F)] \geq \\ \left(u(\hat{x}_F) - \rho u(x_L) - \frac{\gamma}{\beta}(\hat{x}_F - \hat{x}_F) - (1 - \rho)u(\hat{x}_F)\right) \frac{[1 - \beta + \beta\rho + \alpha(\beta - \gamma)]}{\rho} \end{aligned}$$

where $\frac{[1 - \beta + \beta\rho + \alpha(\beta - \gamma)]}{\rho} > 0$. Thus, if

$$\left(u(\hat{x}_F) - \rho u(x_L) - \frac{\gamma}{\beta}(\hat{x}_F - \hat{x}_F) - (1 - \rho)u(\hat{x}_F)\right) \frac{[1 - \beta + \beta\rho + \alpha(\beta - \gamma)]}{\rho} > -\alpha\beta(1 - \rho)\hat{x}_F$$

then (22) implies (28), and it is thus the only necessary and sufficient condition for the existence of the pooling equilibrium $m, x^*, x_L, \hat{m}, \hat{x}_F, x_L$. This is true when

$$u(\hat{x}_F) - \rho u(x_L) - (1 - \rho)u(\hat{x}_F) - \frac{\gamma}{\beta}\hat{x}_F > -\left[\frac{\alpha\beta\rho(1 - \rho)}{[1 - \beta + \beta\rho + \alpha(\beta - \gamma)]} + \frac{\gamma}{\beta}\right]\hat{x}_F$$

Otherwise the opposite holds: (28) implies (22), and it is thus the only necessary and sufficient condition for the existence of the pooling equilibrium $m, x^*, x_L, \hat{m}, \hat{x}_F, x_L$.

■

Corollary 10 *If $[1 - \beta + \beta\rho - \alpha\gamma + \alpha\beta(1 - \rho)] > 0$ and*

$$\begin{aligned} [1 - \beta + \beta\rho - \alpha\gamma + \alpha\beta(1 - \rho)]u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1 - \rho}\right) + (1 - \rho)[u(x^*) - x^*] \leq \\ u\left(u'^{-1}\left(\frac{\phi}{\beta\phi'}\right)\right) - \frac{\gamma}{\beta}u'^{-1}\left(\frac{\phi}{\beta\phi'}\right) - \rho u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1 - \rho)}{\rho}\right)\right) + \frac{(1 - \beta + \beta\rho)u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1 - \rho)}{\rho}\right)}{\alpha\beta} \end{aligned}$$

then there is no pooling solution to the buyers' problem with LC_F slack, LC_L binding and $m > 0$.

Proof. It follows from (26) that a necessary condition for a pooling solution to the buyers' problem with LC_F slack, LC_L binding and $m > 0$ is

$$\begin{aligned} & \left[1 - \rho + \frac{1 - \beta + \beta \rho}{\alpha \beta} \right] \beta \phi' \hat{m} - \frac{1 - \beta + \beta \rho}{\alpha \beta} x_L + (1 - \rho)[u(x^*) - x^*] > \\ & \frac{(1 - \beta + \beta \rho - \alpha \gamma + \alpha \beta)[- \phi \hat{m} + u(\beta \phi' \hat{m}) - \rho u(x_L)] + \rho(1 - \beta + \beta \rho)x_L}{1 - \beta + \beta \rho - \alpha \gamma + \alpha \beta(1 - \rho)} \\ & - \frac{\rho \alpha \beta(1 - \rho)[u(x^*) - x^*]}{1 - \beta + \beta \rho - \alpha \gamma + \alpha \beta(1 - \rho)} + \beta \phi' \hat{m} \left(\frac{\gamma}{\beta} - \rho \right) \end{aligned}$$

Which, rearranged, is

$$\begin{aligned} & \left[\frac{1 - \beta + \beta \rho - \alpha \gamma + \alpha \beta}{\alpha \beta} \right] \beta \phi' \hat{m} \\ & + (1 - \rho)[u(x^*) - x^*] \left[\frac{1 - \beta + \beta \rho - \alpha \gamma + \alpha \beta}{1 - \beta + \beta \rho - \alpha \gamma + \alpha \beta(1 - \rho)} \right] > \\ & \frac{(1 - \beta + \beta \rho - \alpha \gamma + \alpha \beta)[- \phi \hat{m} + u(\beta \phi' \hat{m}) - \rho u(x_L)]}{1 - \beta + \beta \rho - \alpha \gamma + \alpha \beta(1 - \rho)} \\ & + \frac{1 - \beta + \beta \rho - \alpha \gamma + \alpha \beta}{\alpha \beta[1 - \beta + \beta \rho - \alpha \gamma + \alpha \beta(1 - \rho)]} (1 - \beta + \beta \rho)x_L \end{aligned}$$

Recall that it is assumed that $1 - \beta + \beta \rho - \alpha \gamma + \alpha \beta > 0$. If also $[1 - \beta + \beta \rho - \alpha \gamma + \alpha \beta(1 - \rho)] > 0$ then dividing both sides by $1 - \beta + \beta \rho - \alpha \gamma + \alpha \beta$ and multiplying both sides by $[1 - \beta + \beta \rho - \alpha \gamma + \alpha \beta(1 - \rho)]$ implies

$$\begin{aligned} & [1 - \beta + \beta \rho - \alpha \gamma + \alpha \beta(1 - \rho)][\beta \phi' \hat{m}] + (1 - \rho)[u(x^*) - x^*] > \\ & u(\beta \phi' \hat{m}) - \phi \hat{m} - \rho u(x_L) + \frac{(1 - \beta + \beta \rho)x_L}{\alpha \beta} \end{aligned}$$

With $x_L = u'^{-1}\left(\frac{\frac{\phi}{\beta \phi'} - (1 - \rho)}{\rho}\right)$, $\beta \phi' \hat{m} = u'^{-1}\left(\frac{\frac{\phi}{\beta \phi'} - \rho}{1 - \rho}\right)$, $\beta \phi' \hat{m} = u'^{-1}\left(\frac{\phi}{\beta \phi'}\right)$, where $\hat{m} > \hat{m}$ and $x_L \geq \beta \phi' \hat{m}$ (from the DM IC), this becomes

$$\begin{aligned} & [1 - \beta + \beta \rho - \alpha \gamma + \alpha \beta(1 - \rho)] u'^{-1}\left(\frac{\frac{\phi}{\beta \phi'} - \rho}{1 - \rho}\right) + (1 - \rho)[u(x^*) - x^*] > \\ & u\left(u'^{-1}\left(\frac{\phi}{\beta \phi'}\right)\right) - \frac{\gamma}{\beta} u'^{-1}\left(\frac{\phi}{\beta \phi'}\right) - \rho u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta \phi'} - (1 - \rho)}{\rho}\right)\right) + \frac{(1 - \beta + \beta \rho) u'^{-1}\left(\frac{\frac{\phi}{\beta \phi'} - (1 - \rho)}{\rho}\right)}{\alpha \beta} \end{aligned}$$

■

3.1.4 Necessary and sufficient conditions in the space ρ, β

Additional assumptions:

- a) $\gamma\alpha \leq 1$ as specified in 1- below.
b) $u'''(x) \leq 0$ for all $x \leq x^*$ (using it for $\frac{\partial^2 \hat{x}_F}{\partial \rho \partial \beta}$ so double check I need it)
c) either i) $1 - \alpha \leq \rho$, or ii) $1 - \alpha > \rho$ and $\beta \leq \bar{\beta} = \frac{1-\alpha\gamma}{(1-\alpha-\rho)}$ (for (30) is decreasing in ρ)
d) $\frac{1-\beta+\beta\rho}{\alpha} + \beta < \gamma$ (for (30) is decreasing in β)
Useful calculations:

1-For $\hat{x}_F = \beta \phi' \hat{m} = u'^{-1}\left(\frac{\gamma}{\beta} - \rho\right)$, $\frac{\partial \hat{x}_F}{\partial \beta} = \frac{\left(-\frac{\gamma}{\beta^2(1-\rho)}\right)}{u''\left(u'^{-1}\left(\frac{\gamma}{\beta} - \rho\right)\right)} > 0$ and $\frac{\partial \hat{x}_F}{\partial \rho} = \frac{\frac{1}{(1-\rho)^2}\left(\frac{\gamma}{\beta} - 1\right)}{u''\left(u'^{-1}\left(\frac{\gamma}{\beta} - \rho\right)\right)} < 0$
and $\frac{\partial^2 \hat{x}_F}{\partial \rho \partial \beta} = \frac{1}{(1-\rho)^2} \left[\frac{-\frac{\gamma}{\beta^2} u''(\hat{x}_F) - \left(\frac{\gamma}{\beta} - 1\right) u'''(\hat{x}_F) \frac{\partial \hat{x}_F}{\partial \beta}}{[u''(\hat{x}_F)]^2} \right] > 0$ if $u'''(x) \leq 0$
2-For $x_L = u'^{-1}\left(\frac{\gamma}{\beta} - (1-\rho)\right)$, we have $\frac{\partial x_L}{\partial \beta} = \frac{-\frac{\gamma}{\beta^2\rho}}{u''\left(u'^{-1}\left(\frac{\gamma}{\beta} - (1-\rho)\right)\right)} > 0$ and $\frac{\partial x_L}{\partial \rho} = -\frac{\frac{1}{\rho^2}\left(\frac{\gamma}{\beta} - 1\right)}{u''\left(u'^{-1}\left(\frac{\gamma}{\beta} - (1-\rho)\right)\right)} > 0$
because $u''(x) < 0$ for all x , and $\frac{\partial^2 x_L}{\partial \rho \partial \beta} = \frac{\frac{1}{\rho^2} \frac{\gamma}{\beta^2} u''(x_L) + \frac{1}{\rho^2} \left(\frac{\gamma}{\beta} - 1\right) u'''(x_L) \frac{\partial x_L}{\partial \beta}}{[u''(x_L)]^2} < 0$ if $u'''(x) \leq 0$
. Notice that

$$\frac{\partial \hat{x}_F}{\partial \beta} - \frac{\partial x_L}{\partial \beta} = -\frac{\gamma}{\beta^2\rho} \left[\frac{\rho u''\left(u'^{-1}\left(\frac{\gamma}{\beta} - (1-\rho)\right)\right) - (1-\rho) u''\left(u'^{-1}\left(\frac{\gamma}{\beta} - \rho\right)\right)}{(1-\rho) u''\left(u'^{-1}\left(\frac{\gamma}{\beta} - (1-\rho)\right)\right) \rho u''\left(u'^{-1}\left(\frac{\gamma}{\beta} - \rho\right)\right)} \right]$$

3-For $\frac{1-\beta+\beta\rho}{\alpha\beta(1-\rho)}$ we have that $\frac{\partial\left(\frac{1-\beta+\beta\rho}{\alpha\beta(1-\rho)}\right)}{\partial \rho} = \frac{1}{\alpha\beta(1-\rho)^2} > 0$ and

$$\frac{\partial\left(\frac{1-\beta+\beta\rho}{\alpha\beta(1-\rho)}\right)}{\partial \beta} = \frac{(1-\beta+\beta\rho - (1-\rho)^2)}{\alpha\beta(1-\rho)^2} = \frac{1-(1-\rho)(1-\rho+\beta)}{\alpha\beta(1-\rho)^2}$$

where $1-(1-\rho)(1-\rho+\beta) > 0$ if and only if $\rho(2-\rho) - \beta(1-\rho) > 0$ that is $\beta < \frac{\rho(2-\rho)}{(1-\rho)}$. Thus $\frac{\partial\left(\frac{1-\beta+\beta\rho}{\alpha\beta(1-\rho)}\right)}{\partial \beta} > 0$ if and only if $\beta < \frac{\rho(2-\rho)}{(1-\rho)}$.

Lemma 11 $\beta < \frac{\rho(2-\rho)}{(1-\rho)}$ is always satisfied for $\rho \geq \frac{1}{2}$.

Proof.

For $\rho = \frac{1}{2}$ inequality $\beta < \frac{\rho(2-\rho)}{(1-\rho)}$ becomes simply $\beta < \frac{3}{2}$ which is always satisfied because $\beta \in (0, 1)$. Moreover, $\frac{\rho(2-\rho)}{(1-\rho)}$ is increasing in ρ (since $\frac{\partial}{\partial \rho} \left(\frac{\rho(2-\rho)}{(1-\rho)} \right) = \frac{1+(1-\rho)^2}{(1-\rho)^2} > 0$) implying the result. $\blacksquare$

Consider the necessary and sufficient conditions in proposition (8) and rearrange each of them in a space ρ, β for comparison with possible coexistence with other solutions to the buyers' problems.

1- Consider $1 - \beta + \beta\rho + \alpha\beta\left(1 - \frac{\gamma}{\beta}\right) > 0$. If $\gamma\alpha > 1$ then this condition is never satisfied. Thus, assuming $\gamma\alpha \leq 1$ we can rewrite it as $\beta \leq f_\beta(\rho) = \frac{1-\gamma\alpha}{1-\alpha(1-\rho)}$. Notice that $f_\beta(0) = \frac{1-\gamma\alpha}{1-\alpha}$ and $f_\beta(1) = 1 - \gamma\alpha$, and that $f'_\beta(\rho) = \frac{-(1-\gamma\alpha)\alpha}{[1-\alpha(1-\rho)]^2} < 0$, and that $f''_\beta(\rho) = \frac{2(1-\gamma\alpha)\alpha^2}{[1-\alpha(1-\rho)]^3} > 0$. Thus it is a strictly decreasing and convex function of ρ , and only values of β, ρ below this curve satisfy the inequality.

2- Consider inequality (28) after substituting out for the endogenous variables $x_L = u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right)$, $\beta\phi'\hat{m} = u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right)$, $\beta\phi'\hat{\hat{m}} = u'^{-1}\left(\frac{\phi}{\beta\phi'}\right)$, where $\hat{m} > \hat{\hat{m}}$ and $x_L \geq \beta\phi'\hat{m}$ (from the DM IC), we have

$$\left[\frac{1 - \beta + \beta\rho + \alpha\beta(1-\rho)}{\alpha\beta} \right] u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right) - \frac{1 - \beta + \beta\rho}{\alpha\beta} u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right) + (1-\rho)[u(x^*) - x^*] \geq (1-\rho)u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right)\right)$$

Notice that this is never satisfied at $\rho = 1$ regardless of the value of β : evaluated at $\rho = 1$ we have that $\lim_{\rho \rightarrow 1} u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right) = 0$ and $\lim_{\rho \rightarrow 1} u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right) = u'^{-1}\left(\frac{\gamma}{\beta}\right) = \beta\phi'\hat{\hat{m}}$. Then (28) becomes $0 \geq u'^{-1}\left(\frac{\gamma}{\beta}\right)$, which is never satisfied because $\beta\phi'\hat{\hat{m}} > 0$.

Analogously, (28) is always satisfied at $\rho = 0$ regardless of the value of β : $\lim_{\rho \rightarrow 0} u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right) = u'^{-1}\left(\frac{\gamma}{\beta}\right) = \beta\phi'\hat{\hat{m}}$ and $\lim_{\rho \rightarrow 0} u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right) = 0$, so that

$$\left[\frac{1-\beta}{\alpha\beta} \right] u'^{-1}\left(\frac{\gamma}{\beta}\right) + [u(x^*) - x^*] \geq u\left(u'^{-1}\left(\frac{\gamma}{\beta}\right)\right) - u'^{-1}\left(\frac{\gamma}{\beta}\right)$$

which is always satisfied because $[u(x^*) - x^*] > u\left(u'^{-1}\left(\frac{\gamma}{\beta}\right)\right) - u'^{-1}\left(\frac{\gamma}{\beta}\right)$ by definition of x^* and $\left[\frac{1-\beta}{\alpha\beta} \right] u'^{-1}\left(\frac{\gamma}{\beta}\right) > 0$.

Similarly, (28) is always satisfied at $\rho = \frac{1}{2}$ regardless of the value of β : $u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right) = u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right)$, so that

$$(1-\rho)[u(x^*) - x^*] \geq (1-\rho) \left[u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right)\right) - u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right) \right]$$

which, again, is always satisfied by definition of x^* .

Now consider $\rho \in \left(\frac{1}{2}, 1\right)$, and rewrite (28) as

$$\left[\frac{1 - \beta + \beta\rho}{\alpha\beta(1-\rho)} \right] u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right) - \frac{1 - \beta + \beta\rho}{\alpha\beta(1-\rho)} u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right)$$

$$+(u(x^*)-x^*)-\left[u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'}-\rho}{1-\rho}\right)\right)-u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'}-\rho}{1-\rho}\right)\right] \geq 0 \quad (29)$$

and differentiating with respect to ρ yields:

$$\begin{aligned} \frac{\partial \{\}}{\partial \rho} &= \frac{(\hat{x}_F - x_L)}{\alpha\beta(1-\rho)^2} + \left[\frac{1-\beta+\beta\rho}{\alpha\beta(1-\rho)} \right] \left[\frac{\partial \hat{x}_F}{\partial \rho} - \frac{\partial x_L}{\partial \rho} \right] - \frac{\partial \hat{x}_F}{\partial \rho} [u'(\hat{x}_F) - 1] \\ &= \frac{(\hat{x}_F - x_L)}{\alpha\beta(1-\rho)^2} - \left[\frac{1-\beta+\beta\rho}{\alpha\beta(1-\rho)} \right] \frac{\partial x_L}{\partial \rho} - \frac{\partial \hat{x}_F}{\partial \rho} \left[u'(\hat{x}_F) - \frac{\alpha\beta(1-\rho)+1-\beta+\beta\rho}{\alpha\beta(1-\rho)} \right] \end{aligned} \quad (30)$$

where, the DM IC implies that $\hat{x}_F - x_L < 0$, previous calculations imply that $\frac{\partial x_L}{\partial \rho} > 0$ and $\frac{\partial \hat{x}_F}{\partial \rho} < 0$.

Lemma 12 *If $\gamma\alpha \leq 1$ and $\gamma > \beta$ then (30) is decreasing in ρ either if i) $1-\alpha \leq \rho$, or if ii) $1-\alpha > \rho$ and $\beta \leq \bar{\beta} = \frac{1-\alpha\gamma}{(1-\alpha-\rho)}$. Alternatively, (30) is decreasing in ρ if and only if*

$$\frac{(\hat{x}_F - x_L)}{\alpha\beta(1-\rho)^2} - \left[\frac{1-\beta+\beta\rho}{\alpha\beta(1-\rho)} \right] \frac{\partial x_L}{\partial \rho} - \frac{\partial \hat{x}_F}{\partial \rho} \left[u'(\hat{x}_F) - \frac{\alpha\beta(1-\rho)+1-\beta+\beta\rho}{\alpha\beta(1-\rho)} \right] < 0$$

Proof.

From (30) it follows that if $1 < u'(\hat{x}_F) < \frac{\alpha\beta(1-\rho)+1-\beta+\beta\rho}{\alpha\beta(1-\rho)}$ then $\frac{\partial \{\}}{\partial \rho} < 0$. Within the space ρ, β this is the case when $u'\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'}-\rho}{1-\rho}\right)\right) < 1 + \frac{1-\beta+\beta\rho}{\alpha\beta(1-\rho)}$, which can be easily rearranged as $\beta(1-\alpha-\rho) < 1-\alpha\gamma$. Thus, under the assumption that $\gamma\alpha \leq 1$ made above, if $1-\alpha \leq \rho$ then the inequality is always satisfied for all β , so that $\frac{\partial \{\}}{\partial \rho} < 0$. If instead $1-\alpha > \rho$ then the inequality is satisfied, implying that $\frac{\partial \{\}}{\partial \rho} < 0$, for all $\beta \leq \bar{\beta} = \frac{1-\alpha\gamma}{(1-\alpha-\rho)}$. Alternatively, the result follows from requiring that (30) is negative.

■

Thus, if $\frac{\partial \{\}}{\partial \rho} < 0$, for any given $\beta \in (0, 1)$ there exists a $\bar{\rho}_\beta \in (\frac{1}{2}, 1)$ such that (29) is satisfied for all $\rho \leq \bar{\rho}_\beta$.

Evaluate (29) at $\beta = 1$ to get:

$$\begin{aligned} &\frac{\rho}{\alpha(1-\rho)} \left[u'^{-1}\left(\frac{\gamma-\rho}{1-\rho}\right) - u'^{-1}\left(\frac{\gamma-(1-\rho)}{\rho}\right) \right] \\ &+(u(x^*)-x^*)-\left[u\left(u'^{-1}\left(\frac{\gamma-\rho}{1-\rho}\right)\right)-u'^{-1}\left(\frac{\gamma-\rho}{1-\rho}\right)\right] \geq 0 \end{aligned}$$

We can solve this for ρ to obtain $\bar{\rho}_1$. We know that such $\bar{\rho}_1$ exists and is unique because for any β , thus also $\beta = 1$, (29) is satisfied for $\rho = \frac{1}{2}$ and is violated for $\rho = 1$. Because (29) is continuous and monotonically decreasing in $\rho \in (\frac{1}{2}, 1)$ (under the assumptions above for $\frac{\partial \{\}}{\partial \rho} < 0$) then the intermediate value theorem implies that there exists $\bar{\rho}_1 \in (\frac{1}{2}, 1)$ where the above inequality holds at equality.

Lemma 13 If $\frac{1-\beta+\beta\rho}{\alpha} + \beta < \gamma$ then (30) is decreasing in β .

Proof.

Differentiating (29) with respect to β yields:

$$\frac{\partial \{\}}{\partial \beta} = \frac{\rho(2-\rho)-\beta(1-\rho)}{\alpha\beta(1-\rho)^2} [\hat{x}_F - x_L] + \left[\frac{1-\beta+\beta\rho}{\alpha\beta(1-\rho)} \right] \left[\frac{\partial \hat{x}_F}{\partial \beta} - \frac{\partial x_L}{\partial \beta} \right] - [u'(\hat{x}_F) - 1] \frac{\partial \hat{x}_F}{\partial \beta}$$

that is, substituting out for the endogenous variables:

$$\begin{aligned} \frac{\partial \{\}}{\partial \beta} = & \frac{(1-\beta+\beta\rho-(1-\rho)^2)}{\alpha\beta(1-\rho)^2} \left[u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) - u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right] \\ & + \frac{1-\beta+\beta\rho}{\alpha\beta(1-\rho)} \frac{\gamma}{\beta^2 \rho u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) \right)} - \left[\frac{1-\beta+\beta\rho+\alpha\beta(1-\rho)}{\alpha\beta(1-\rho)} - u'(\hat{x}_F) \right] \frac{\gamma}{\beta^2 (1-\rho) u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) \right)} \end{aligned}$$

If $\frac{1-\beta+\beta\rho+\alpha\beta(1-\rho)}{\alpha\beta(1-\rho)} < u'(\hat{x}_F)$ then it follows easily that $\frac{\partial \{\}}{\partial \beta} < 0$. We can rearrange $\frac{1-\beta+\beta\rho+\alpha\beta(1-\rho)}{\alpha\beta(1-\rho)} < u'(\hat{x}_F)$ as $\frac{1-\beta+\beta\rho}{\alpha} + \beta < \gamma$.

Thus we are left with characterizing (29) for $\rho > \frac{1}{2}$. We know that there exists a $\bar{\rho}_\beta \in (\frac{1}{2}, 1)$ such that (29) is satisfied for all $\rho \leq \bar{\rho}_\beta$, and we know that for $\beta = 1$ we still have $\bar{\rho}_1 \in (\frac{1}{2}, 1)$. Because we know that (29) is also decreasing in β (under the assumption $\frac{1-\beta+\beta\rho}{\alpha} + \beta < \gamma$) we expect that for small values of β it will be that $\bar{\rho}_\beta \geq \bar{\rho}_1$. Because (29) is not defined at $\beta = 0$, then we use l' Hospital theorem to characterize (29) for $\beta \rightarrow 0$. Rearranging (29) as

$$\begin{aligned} \left[\frac{1-\beta+\beta\rho}{\alpha\beta(1-\rho)} \right] \left[u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) - u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right] & \geq \\ u \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) \right) - u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) - (u(x^*) - x^*) & \end{aligned}$$

notice that $\lim_{\beta \rightarrow 0} u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) = 0$ under the assumption of Inada conditions on u such that $\lim_{x \rightarrow 0} u'(x) = \infty$. Similarly $\lim_{\beta \rightarrow 0} u \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) \right) = 0$. Thus the right hand side is such that:

$$\lim_{\beta \rightarrow 0} \left\{ u \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) \right) - u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) - (u(x^*) - x^*) \right\} = -(u(x^*) - x^*) < 0$$

For the left hand side, which is always negative because $\hat{x}_F < x_L$, we have:

$$\begin{aligned} \lim_{\beta \rightarrow 0} \left\{ \frac{1-\beta+\beta\rho}{\alpha\beta(1-\rho)} \right\} & = +\infty \\ \lim_{\beta \rightarrow 0} \left\{ u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) - u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right\} & = 0 \end{aligned}$$

Applying the L'Hospital theorem we have that $\lim_{\beta \rightarrow 0} \left[\frac{1-\beta+\beta\rho}{\alpha\beta(1-\rho)} \right] \left[u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) - u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right] =$

$$\lim_{\beta \rightarrow 0} \left\{ \frac{\frac{\partial}{\partial \beta} \left[u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) - u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right]}{\frac{\partial}{\partial \beta} \left(\frac{\alpha\beta(1-\rho)}{1-\beta+\beta\rho} \right)} \right\} \text{ if the latter exists. Thus:}$$

$$\begin{aligned} \frac{\frac{\partial}{\partial \beta} \left[u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) - u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right]}{\frac{\partial}{\partial \beta} \left(\frac{\alpha\beta(1-\rho)}{1-\beta+\beta\rho} \right)} &= \frac{-\frac{\gamma}{(1-\rho)\beta^2}}{u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) \right)} - \frac{-\frac{\gamma}{\rho\beta^2}}{u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right)} \\ &= \frac{\gamma}{\beta^2} \frac{(1-\rho) u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) \right) - \rho u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right)}{\rho(1-\rho) u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) \right) u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right)} \end{aligned}$$

and

$$\begin{aligned} \frac{\frac{\partial}{\partial \beta} \left(\frac{\alpha\beta(1-\rho)}{1-\beta+\beta\rho} \right)}{\frac{\partial}{\partial \beta} \left(\frac{\alpha\beta(1-\rho)}{1-\beta+\beta\rho} \right)} &= \frac{\alpha(1-\rho)}{(1-\beta+\beta\rho)^2} [1-\beta+\beta\rho+\beta(1-\rho)] \\ &= \frac{\alpha(1-\rho)}{(1-\beta+\beta\rho)^2} > 0 \end{aligned}$$

Thus

$$\lim_{\beta \rightarrow 0} \left\{ \frac{\frac{\partial}{\partial \beta} \left[u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) - u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right]}{\frac{\partial}{\partial \beta} \left(\frac{\alpha\beta(1-\rho)}{1-\beta+\beta\rho} \right)} \right\} = \lim_{\beta \rightarrow 0} \frac{\gamma(1-\beta+\beta\rho)^2}{\beta^2 \alpha(1-\rho)} \frac{\left[(1-\rho) u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) \right) - \rho u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right) \right]}{\rho(1-\rho) u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) \right) u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right)}$$

Thus the sign of the limit above is determined by the sign of $\left[(1-\rho) u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) \right) - \rho u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right) \right]$,

since all other terms are always strictly positive for all $\rho > \frac{1}{2}$. For values of ρ arbitrarily close to $\frac{1}{2}$, under the assumption that $u''' < 0$ it is always the case that $\left[(1-\rho) u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) \right) - \rho u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right) \right] > 0$. Therefore, for values of ρ arbitrarily close to $\frac{1}{2}$, we have that

$$\lim_{\beta \rightarrow 0} \left[\frac{1-\beta+\beta\rho}{\alpha\beta(1-\rho)} \right] \left[u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) - u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right] > 0$$

which implies that (29) is satisfied for $\beta \rightarrow 0$. For values of ρ sufficiently larger than

$\frac{1}{2}$ despite $u''' < 0$ implies $u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) \right) > u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right)$, because $u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) > u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right)$, we cannot easily sign the above limit because we also have $1-\rho < \rho$.

Notice, however, that the term $\left[(1-\rho) u'' \left(u'^{-1} \left(\frac{\gamma}{\beta} - \rho \right) \right) - \rho u'' \left(u'^{-1} \left(\frac{\gamma}{\beta} - (1-\rho) \right) \right) \right]$ is increasing in ρ :

$$\begin{aligned} & \frac{\partial}{\partial \rho} \left[(1-\rho) u'' \left(u'^{-1} \left(\frac{\gamma}{\beta} - \rho \right) \right) - \rho u'' \left(u'^{-1} \left(\frac{\gamma}{\beta} - (1-\rho) \right) \right) \right] = \\ & -u'' \left(u'^{-1} \left(\frac{\gamma}{\beta} - \rho \right) \right) + (1-\rho) u''' \left(u'^{-1} \left(\frac{\gamma}{\beta} - \rho \right) \right) \frac{\partial u'^{-1} \left(\frac{\gamma}{\beta} - \rho \right)}{\partial \rho} \\ & -u'' \left(u'^{-1} \left(\frac{\gamma}{\beta} - (1-\rho) \right) \right) - \rho u''' \left(u'^{-1} \left(\frac{\gamma}{\beta} - (1-\rho) \right) \right) \frac{\partial u'^{-1} \left(\frac{\gamma}{\beta} - (1-\rho) \right)}{\partial \rho} \end{aligned}$$

where $\frac{\partial \hat{x}_F}{\partial \rho} = \frac{\partial u'^{-1} \left(\frac{\gamma}{\beta} - \rho \right)}{\partial \rho} = \frac{\frac{1}{(1-\rho)^2} \left(\frac{\gamma}{\beta} - 1 \right)}{u'' \left(u'^{-1} \left(\frac{\gamma}{\beta} - \rho \right) \right)} < 0$ and $\frac{\partial x_L}{\partial \rho} = \frac{\partial u'^{-1} \left(\frac{\gamma}{\beta} - (1-\rho) \right)}{\partial \rho} = \frac{\frac{1}{\rho^2} \left(1 - \frac{\gamma}{\beta} \right)}{u'' \left(u'^{-1} \left(\frac{\gamma}{\beta} - (1-\rho) \right) \right)} >$

0. Thus, we can rearrange the above equation as

$$\begin{aligned} & \frac{\partial}{\partial \rho} \left[(1-\rho) u''(\hat{x}_F) - \rho u''(x_L) \right] = \\ & -u''(\hat{x}_F) - u''(x_L) + (1-\rho) u'''(\hat{x}_F) \frac{\partial \hat{x}_F}{\partial \rho} - \rho u'''(x_L) \frac{\partial x_L}{\partial \rho} \end{aligned}$$

Because $u''' < 0$, and $u'' < 0$, it then follows that $\frac{\partial}{\partial \rho} \left[(1-\rho) u''(\hat{x}_F) - \rho u''(x_L) \right] > 0$. Therefore, for any $\rho > \frac{1}{2}$ we have

$$\lim_{\beta \rightarrow 0} \left[\frac{1-\beta+\beta\rho}{\alpha\beta(1-\rho)} \right] \left[u'^{-1} \left(\frac{\gamma}{\beta} - \rho \right) - u'^{-1} \left(\frac{\gamma}{\beta} - (1-\rho) \right) \right] > 0$$

implying that (29) is satisfied for $\beta \rightarrow 0$ even for large enough values of ρ .

It would be now useful to characterize the curvature of the level curves associated with inequality (29) in the space ρ, β . To do that differentiate (30) with respect to β

$$\begin{aligned} \frac{\partial^2 \{ \}}{\partial \rho \partial \beta} &= \frac{\left(\frac{\partial \hat{x}_F}{\partial \beta} - \frac{\partial x_L}{\partial \beta} \right) \beta - (\hat{x}_F - x_L)}{\alpha \beta^2 (1-\rho)^2} \\ & - \frac{\partial \left[\frac{1-\beta+\beta\rho}{\alpha\beta(1-\rho)} \right]}{\partial \beta} \frac{\partial x_L}{\partial \rho} - \left[\frac{1-\beta+\beta\rho}{\alpha\beta(1-\rho)} \right] \frac{\partial^2 x_L}{\partial \rho \partial \beta} \\ & - \frac{\partial^2 \hat{x}_F}{\partial \rho \partial \beta} \left[u'(\hat{x}_F) - \frac{\alpha\beta(1-\rho) + 1 - \beta + \beta\rho}{\alpha\beta(1-\rho)} \right] \\ & - \frac{\partial \hat{x}_F}{\partial \rho} \left[u''(\hat{x}_F) \frac{\partial \hat{x}_F}{\partial \beta} + \frac{1}{\alpha\beta^2(1-\rho)} \right] \end{aligned}$$

but signing this is not manageable.

Thus, what we can tell is that (29) is satisfied in the shaded region in the following figure.

3.1.5 Necessary and sufficient conditions in the space β, γ

Notice that in this case we are restricting attention to $\gamma > \beta$.

Notice that:

- as a function of γ we have that $\frac{\partial}{\partial \gamma} \left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho} \right) = \frac{1}{\beta(1 - \rho)}$ thus $\frac{\partial \hat{x}_F}{\partial \gamma} = \frac{1}{\beta(1 - \rho)u''(\hat{x}_F)} < 0$
- as a function of γ we have that $\frac{\partial}{\partial \gamma} \left(\frac{\frac{\gamma}{\beta} - 1 + \rho}{\rho} \right) = \frac{1}{\beta\rho}$. Thus $\frac{\partial x_L}{\partial \gamma} = \frac{1}{\beta\rho u''(x_L)} < 0$.
Thus, if $u''' < 0$, because $x_L \geq \hat{x}_F$, for $\rho \leq \frac{1}{2}$ we have that $\frac{\partial x_L}{\partial \gamma} < \frac{\partial \hat{x}_F}{\partial \gamma}$, meaning x_L decreases faster in γ than $\hat{x}_F$ does.

ASSUMPTIONS: i) $u''' < 0$

Condition 1 1-Consider $1 - \beta + \beta\rho + \alpha\beta\left(1 - \frac{\gamma}{\beta}\right) > 0$

Lemma 14 At $\beta = 0$ this is always satisfied, at $\beta = 1$ it is satisfied if and only if $\left(\frac{\rho}{\alpha} + 1\right) > \gamma$. If $\alpha > 1 - \rho$ then the set $\Gamma_\beta = \left\{ \gamma > \beta : \frac{1 - \beta(1 - \rho)}{\alpha} + \beta > \gamma \right\}$ gets larger as β increases, alternatively, If $\alpha < 1 - \rho$ then the set $\Gamma_\beta = \left\{ \gamma > \beta : \frac{1 - \beta(1 - \rho)}{\alpha} + \beta > \gamma \right\}$ gets smaller as β increases.

Proof. Rearrange the inequality as $f(\beta) = \frac{1 - \beta(1 - \rho)}{\alpha} + \beta > \gamma$. Evaluate it at $\beta = 0$ and $\beta = 1$ and the result is straightforward. Notice that $f(\beta)$ is increasing in β if and only if $\alpha > 1 - \rho$ (in fact $\frac{\partial \left[\frac{1 - \beta(1 - \rho)}{\alpha} + \beta \right]}{\partial \beta} = 1 - \frac{(1 - \rho)}{\alpha}$). Also, $f(1) > f(0)$ if and only if $\alpha > 1 - \rho$. Thus the result follows. ■

The results of the above lemma are summarized in the figure 2, where condition $1 - \beta + \beta\rho + \alpha\beta\left(1 - \frac{\gamma}{\beta}\right) > 0$ is satisfied in the area between all colored dashed lines and the vertical axes.

Condition 2 2-Consider inequality (28)

after substituting out for the endogenous variables $x_L = u'^{-1} \left(\frac{\frac{\phi}{\beta\phi'} - (1 - \rho)}{\rho} \right)$, $\beta\phi'\hat{m} = u'^{-1} \left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1 - \rho} \right)$, $\beta\phi'\hat{m} = u'^{-1} \left(\frac{\phi}{\beta\phi'} \right)$, where $\hat{m} > \hat{m}$ and $x_L \geq \beta\phi'\hat{m}$ (from the DM IC), we have

$$\begin{aligned} & \left[\frac{1 - \beta + \beta\rho}{\alpha\beta(1 - \rho)} \right] \left[u'^{-1} \left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho} \right) - u'^{-1} \left(\frac{\frac{\gamma}{\beta} - (1 - \rho)}{\rho} \right) \right] \\ & + (u(x^*) - x^*) - \left[u \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho} \right) \right) - u'^{-1} \left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho} \right) \right] \geq 0 \end{aligned}$$

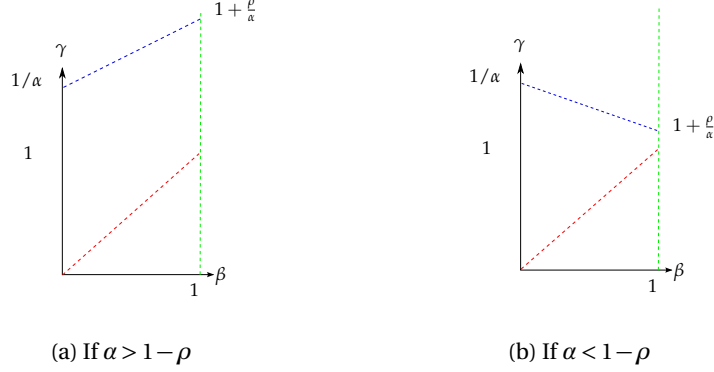


Figure 2: Necessary and sufficient condition $1 - \beta + \beta\rho + \alpha\beta\left(1 - \frac{\gamma}{\beta}\right) > 0$

Lemma 15 Condition (28) is satisfied for all $\gamma > \beta$ and $\rho \in (0, \frac{1}{2}]$.

Proof. Notice that at $\rho = \frac{1}{2}$ the above inequality is always satisfied for all $\gamma > \beta$, at $\rho = 0$ it must be that $u'^{-1}\left(\frac{\frac{\gamma}{\beta} - (1-\rho)}{\rho}\right) = 0$, so the inequality is still satisfied for all $\gamma > \beta$. ■

Consider now the general case when $\rho > \frac{1}{2}$.

Lemma 16 If either i) $1 - \alpha \leq \rho$, or if ii) $1 - \alpha > \rho$ and $\gamma \leq \frac{1 - \beta(1 - \alpha - \rho)}{\alpha}$ then condition (28) is satisfied for all $\gamma > \beta$ and $\rho \in (\frac{1}{2}, 1)$.

Proof. Notice that $\lim_{\rho \rightarrow 1} u'^{-1}\left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}\right) = 0$, thus, if

$$\lim_{\rho \rightarrow 1} \left[\frac{1 - \beta + \beta\rho}{\alpha\beta(1 - \rho)} \right] \left[u'^{-1}\left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}\right) - u'^{-1}\left(\frac{\frac{\gamma}{\beta} - (1 - \rho)}{\rho}\right) \right] > -(u(x^*) - x^*)$$

then (28) is satisfied, regardless of whether it is increasing or decreasing in ρ , as long as it is monotonically so. Then, applying l'Hospital rule we have

$$\lim_{\rho \rightarrow 1} \left[\frac{1 - \beta + \beta\rho}{\alpha\beta(1 - \rho)} \right] \left[u'^{-1}\left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}\right) - u'^{-1}\left(\frac{\frac{\gamma}{\beta} - (1 - \rho)}{\rho}\right) \right] = \lim_{\rho \rightarrow 1} \frac{\frac{\partial \left[u'^{-1}\left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}\right) - u'^{-1}\left(\frac{\frac{\gamma}{\beta} - (1 - \rho)}{\rho}\right) \right]}{\partial \rho}}{\frac{\partial \left(\frac{\alpha\beta(1 - \rho)}{1 - \beta + \beta\rho} \right)}{\partial \rho}}$$

if the latter exists. Thus

$$\frac{\frac{\partial \left[u'^{-1}\left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}\right) - u'^{-1}\left(\frac{\frac{\gamma}{\beta} - (1 - \rho)}{\rho}\right) \right]}{\partial \rho}}{\frac{\partial \left(\frac{\alpha\beta(1 - \rho)}{1 - \beta + \beta\rho} \right)}{\partial \rho}} = \left(\frac{\gamma}{\beta} - 1 \right) \left(\frac{\rho^2 u''\left(u'^{-1}\left(\frac{\frac{\gamma}{\beta} - (1 - \rho)}{\rho}\right)\right) + (1 - \rho)^2 u''\left(u'^{-1}\left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}\right)\right)}{(1 - \rho)^2 u''\left(u'^{-1}\left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho}\right)\right) \rho^2 u''\left(u'^{-1}\left(\frac{\frac{\gamma}{\beta} - (1 - \rho)}{\rho}\right)\right)} \right)$$

$$\frac{\partial \left(\frac{\alpha\beta(1-\rho)}{1-\beta+\beta\rho} \right)}{\partial \rho} = \frac{-\alpha\beta}{(1-\beta+\beta\rho)^2}$$

Thus

$$\begin{aligned} & \lim_{\rho \rightarrow 1} \left[\frac{1-\beta+\beta\rho}{\alpha\beta(1-\rho)} \right] \left[u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) - u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right] = \\ & \lim_{\rho \rightarrow 1} \frac{-(1-\beta+\beta\rho)^2}{\alpha\beta} \left(\frac{\gamma}{\beta} - 1 \right) \left(\frac{\rho^2 u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right) + (1-\rho)^2 u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) \right)}{(1-\rho)^2 u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) \right) \rho^2 u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right)} \right) > 0 > \\ & \quad \quad \quad -(u(x^*) - x^*) \end{aligned}$$

Then, the previous lemma (on the space ρ, β) implies that condition (28) is monotonically decreasing in ρ if either i) $1-\alpha \leq \rho$, or if ii) $1-\alpha > \rho$ and $\gamma \leq \frac{1-\beta(1-\alpha-\rho)}{\alpha}$. Combining this result with the previous lemma, showing that (28) is always satisfied for all $\rho \leq \frac{1}{2}$, the result follows. ■

Also notice that $\frac{\partial \left(\frac{1-\beta+\beta\rho}{\alpha\beta(1-\rho)} \right)}{\partial \beta} > 0$ if and only if $\beta < \frac{\rho(2-\rho)}{(1-\rho)}$, from results in 3- above. We also know that the term $\left[u \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) \right) - u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) \right]$ is increasing in $u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right)$ because $u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) < x^*$. Because u' is decreasing, then $\left[u \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) \right) - u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) \right]$ is decreasing in $\frac{\gamma}{\beta}$. On the left hand side, the term $\left[u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) - u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right]$ is increasing (decreasing) in $\frac{\gamma}{\beta}$ (in β) if and only if $\rho u'' \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) - (1-\rho) u'' \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) > 0$. If $u''' < 0$ then $u'' \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) < u'' \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right)$, thus $\left[u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) - u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right]$ is always decreasing in $\frac{\gamma}{\beta}$ if $\rho \geq \frac{1}{2}$.

Given that (30) is increasing in ρ if and only if

$$\frac{(\hat{x}_F - x_L)}{\alpha\beta(1-\rho)^2} - \left[\frac{1-\beta+\beta\rho}{\alpha\beta(1-\rho)} \right] \frac{\partial x_L}{\partial \rho} - \frac{\partial \hat{x}_F}{\partial \rho} \left[u'(\hat{x}_F) - \frac{\alpha\beta(1-\rho)+1-\beta+\beta\rho}{\alpha\beta(1-\rho)} \right] > 0$$

if we can find a large enough γ for any $\rho > \frac{1}{2}$ such that this inequality is satisfied, then condition (28) is satisfied. Let us rewrite this inequality as

$$\frac{u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) - u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right)}{\alpha\beta(1-\rho)^2} - \left[\frac{1-\beta+\beta\rho}{\alpha\beta(1-\rho)} \right] \frac{\partial x_L}{\partial \rho} - \frac{\partial \hat{x}_F}{\partial \rho} \frac{1}{(1-\rho)} \left[\frac{\gamma}{\beta} - \rho - \frac{\alpha\beta(1-\rho)+1-\beta+\beta\rho}{\alpha\beta} \right] > 0$$

where $\frac{\partial \hat{x}_F}{\partial \rho} = \frac{\frac{\gamma}{\beta}-1}{(1-\rho)^2 u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-\rho}{1-\rho} \right) \right)} < 0$ and $\frac{\partial x_L}{\partial \rho} = -\frac{\frac{\gamma}{\beta}-1}{\rho^2 u'' \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta}-(1-\rho)}{\rho} \right) \right)} > 0$. The last term

in square brackets can be rearranged as

$$\frac{\gamma}{\beta} - \rho - \frac{\alpha\beta(1-\rho) + 1 - \beta + \beta\rho}{\alpha\beta} = \frac{1}{\alpha\beta} [\alpha(\gamma - \beta) - (1 - \beta + \beta\rho)]$$

Thus, because the first two terms in the inequality are negative, for it to be satisfied it is necessary that $\gamma > \beta + \frac{(1-\beta+\beta\rho)}{\alpha}$

3.2 Case with LC_F is slack and LC_L binds and $m = 0$.

Intuitively this will be an equilibrium only for $v - \hat{v} > 0$ and sufficiently large to support enough consumption for the good buyer in LI meetings to be consistent with no money holdings. Thus, with $x_F = x^* > x_L$ and $d_F = d_L = m = 0$ the good buyer's problem becomes

$$\begin{aligned} v = \max \quad & \left\{ (1-\rho)[u(x^*) - l_F] + \rho \left[u(x_L) - \frac{l_L}{\alpha} \right] + \beta v' \right\} \\ \text{s.t.} \quad & x_L = l_L \\ & l_F \leq \beta(v - \hat{v}) \\ & l_L = \alpha\beta(v - \hat{v}) \end{aligned}$$

and for $m = 0$ it is necessary that

$$-\phi + \beta\phi'[(1-\rho) + \rho u'(x_L)] < 0$$

This simply becomes

$$v = (1-\rho)[u(x^*) - x^*] + \rho \left[u(\alpha\beta(v - \hat{v})) - \frac{\alpha\beta(v - \hat{v})}{\alpha} \right] + \beta v'$$

Now, in order to solve for $\hat{v}$, we solve the problem of the bad buyer

$$\hat{v} = \max_{\hat{m}} \{ -\phi\hat{m} + (1-\rho)u(\beta\phi'\hat{m}) + \rho[u(x_L) + \beta\phi'\hat{m}] + \beta\hat{v}' \}$$

Their choice of $\hat{m}$ satisfies $-\phi + \beta\phi'[(1-\rho)u'(\beta\phi'\hat{m}) + \rho] = 0$, implying $\hat{m} =$

$$\frac{u'^{-1}\left(\frac{\phi}{\beta\phi' - \rho}\right)}{\beta\phi'} > 0, \text{ as in the previous pooling equilibrium with } m > 0.$$

Thus

$$(v - \hat{v}) = \frac{[(1-\rho)[u(x^*) - x^*] + \phi\hat{m} - (1-\rho)u(\beta\phi'\hat{m}) - \rho\beta\phi'\hat{m}]}{(1-\beta + \beta\rho)} \quad (31)$$

Notice that $v - \hat{v} > 0$ if and only if

$$(1-\rho)[u(x^*) - x^*] + (\phi - \rho\beta\phi')\hat{m} - (1-\rho)u(\beta\phi'\hat{m}) > 0 \quad (32)$$

That can be rearranged as

$$(1-\rho)[u(x^*) - x^* - u(\beta\phi'\hat{m})] + \beta\phi'\left(\frac{\gamma}{\beta} - \rho\right)\hat{m} > 0 \quad (33)$$

Also, as in the previous equilibrium, we have two incentive constraints, one in the DM and one in the CM.

3.2.1 DM incentive constraint

If the DM constraint was violated, the bad buyer would strictly prefer to separate, conditional on being in a LI meeting with money holdings $\hat{m}$, by spending all his money instead of imitating the good buyer. The DM incentive constraint is $u(x_L) + \beta\phi'\hat{m} \geq u(\beta\phi'\hat{m})$ which can be rewritten as $u(\alpha\beta(v - \hat{v})) + \beta\phi'\hat{m} \geq u(\beta\phi'\hat{m})$. Substituting out from (31) yields

$$u\left(\frac{\alpha\beta}{(1-\beta+\beta\rho)}\{(1-\rho)[u(x^*)-x^*-u(\beta\phi'\hat{m})]+\hat{m}(\phi-\rho\beta\phi')\}\right)+\beta\phi'\hat{m} \geq u(\beta\phi'\hat{m}) \quad (34)$$

3.2.2 CM incentive constraint

As in the previous equilibrium, the CM incentive constraint for the bad buyer to choose money holdings $\hat{m}$ in CM and play pooling in DM. Differently from the DM incentive constraint, the bad buyer could choose money holdings $\hat{m}$ if the consumption stemming from transferring $\hat{m}$ to the seller, namely $\beta\phi'\hat{m}$ by the seller's participation constraint, is sufficiently larger than x_L . The choice of $\hat{m}$, if the bad buyer planned on deviating to a separating strategy in a LI meeting, satisfies $-\phi + \beta\phi'u'(\beta\phi'\hat{m}) = 0$, yielding $\hat{m} = \frac{u'^{-1}(\frac{\phi}{\beta\phi'})}{\beta\phi'}$. Notice that, as in the previous case, if $\frac{\gamma}{\beta} > 1$ then $\hat{m} > \hat{m}$. The CM incentive constraint that rules out the deviation by an individual bad buyer (thus not affecting prices in the CM) is

$$-\phi\hat{m} + (1-\rho)u(\beta\phi'\hat{m}) + \rho[u(\alpha\beta(v - \hat{v})) + \beta\phi'\hat{m}] \geq -\phi\hat{m} + u(\beta\phi'\hat{m}) \quad (35)$$

Substituting out from (31) yields

$$\begin{aligned} & \phi(\hat{m} - \hat{m}) + (1-\rho)u(\beta\phi'\hat{m}) + \rho\beta\phi'\hat{m} - u(\beta\phi'\hat{m}) \\ & + \rho\left\{u\left(\frac{\alpha\beta}{(1-\beta+\beta\rho)}\{(1-\rho)[u(x^*)-x^*-u(\beta\phi'\hat{m})]+\hat{m}(\phi-\rho\beta\phi')\}\right)\right\} \geq 0 \end{aligned} \quad (36)$$

where, if (33) is satisfied, then the term in curly brackets is strictly positive, but $(1-\rho)u(\beta\phi'\hat{m}) + \rho\beta\phi'\hat{m} - u(\beta\phi'\hat{m}) < 0$ since $\hat{m} > \hat{m}$. Therefore both (33) and (36) are necessary and sufficient conditions for this pooling equilibrium to exist.

3.2.3 Equilibrium characterization

The allocation $m = 0, x^*, x_L, \hat{m}, \hat{x}_F, x_L$ is an equilibrium if and only if (33), (34) and (36) are satisfied.

Lemma 17 *If $\gamma > \beta$ then (34) implies (36) if and only if*

$$\phi(\hat{m} - \hat{m}) \geq u(\beta\phi'\hat{m}) - u(\beta\phi'\hat{m}) \quad (37)$$

Proof. Combining (34) with (36) yields

$$\begin{aligned} & u\left(\frac{\alpha\beta}{(1-\beta+\beta\rho)}\{(1-\rho)[u(x^*)-x^*-u(\beta\phi'\hat{m})]+\hat{m}(\phi-\rho\beta\phi')\}\right) \geq \\ & \max\left(u(\beta\phi'\hat{m}), -\frac{(\phi(\hat{m}-\hat{m})+(1-\rho)u(\beta\phi'\hat{m})-u(\beta\phi'\hat{m}))}{\rho}\right) - \beta\phi'\hat{m} \end{aligned} \quad (38)$$

which implies that (34) implies (36) if and only if . Otherwise the opposite is true.

³ ■

Lemma 18 *If $\gamma > \beta$ then (38) implies that $v - \hat{v} > 0$.*

Proof. The right hand side of (38) is always non negative because if $u(\beta\phi'\hat{m}) \geq -\frac{(\phi(\hat{m}-\hat{m})+(1-\rho)u(\beta\phi'\hat{m})-u(\beta\phi'\hat{m}))}{\rho}$ then $\beta\phi'\hat{m} \leq x^*$ implies that $u(\beta\phi'\hat{m}) - \beta\phi'\hat{m} > 0$.

Similarly, if $u(\beta\phi'\hat{m}) < -\frac{(\phi(\hat{m}-\hat{m})+(1-\rho)u(\beta\phi'\hat{m})-u(\beta\phi'\hat{m}))}{\rho}$ then

$$\begin{aligned} & -\frac{(\phi(\hat{m}-\hat{m})+(1-\rho)u(\beta\phi'\hat{m})-u(\beta\phi'\hat{m}))}{\rho} - \beta\phi'\hat{m} \geq \\ & u(\beta\phi'\hat{m}) - \beta\phi'\hat{m} > 0 \end{aligned}$$

Thus the right hand side of (38) is always strictly positive. Notice that the left hand side of (38) is strictly positive if and only if $(1-\rho)[u(x^*)-x^*-u(\beta\phi'\hat{m})]+\hat{m}(\phi-\rho\beta\phi') > 0$, which is equivalent to $v - \hat{v} > 0$. It follows that if (38) is satisfied then $u\left(\frac{\alpha\beta}{(1-\beta+\beta\rho)}(v - \hat{v})\right) > 0$ and therefore $v - \hat{v} > 0$. ■

Proposition 19 *If $\gamma > \beta$ then the allocation $m = 0, x^*, x_L, \hat{m}, \hat{x}_F, x_L$ is an equilibrium if and only if (38) is satisfied.*

Proof. It follows from the previous lemmas. ■

3.3 Case with both LC_F and LC_L binding.

This case is very similar to the case characterized in section 3.1, with the difference being that in this section the equilibrium is such that the good buyer cannot achieve x^* in FI meetings. Intuitively, a necessary condition for this to be true must be that he spends all his money in FI meetings as well, not only in LI meetings when the limited commitment constraint is tighter due to pooling with bad buyers. In fact, as long as $x_F < x^*$ then the good buyer's marginal utility from consuming an additional unit of DM good exceeds the marginal utility from saving on labor in the

³Notice that it is not always the case that $\phi(\hat{m}-\hat{m}) \geq u(\beta\phi'\hat{m}) - u(\beta\phi'\hat{m})$ is satisfied because we only know that a sufficient condition for it is always violated: $\frac{\gamma}{\beta} > 1$ implies that $\phi(\hat{m}-\hat{m}) > \beta\phi'(\hat{m}-\hat{m})$. Thus (37) is satisfied if $\beta\phi'(\hat{m}-\hat{m}) \geq u(\beta\phi'\hat{m}) - u(\beta\phi'\hat{m})$ is, which is always true violated because $\hat{m} > \hat{m}$ implies that $u(\beta\phi'\hat{m}) - \beta\phi'\hat{m} - [u(\beta\phi'\hat{m}) - \beta\phi'\hat{m}] > 0$.

next CM (because he carries over some money and needs to work less to acquire new money).

In this case then, the good buyer's problem is

$$\begin{aligned}
v = \max \quad & \left\{ -\phi m + (1-\rho)[u(x_F) + \beta\phi'(m-d_F) - l_F] + \rho \left[u(x_L) + \beta\phi'(m-d_L) - \frac{l_L}{\alpha} \right] + \beta v' \right\} \\
s.t. \quad & x_F = \beta\phi'd_F + l_F \\
& x_L = \beta\phi'd_L + l_L \\
& l_F = \beta(v - \hat{v}) \\
& l_L = \alpha\beta(v - \hat{v})
\end{aligned}$$

Lemma 20 Assume $\gamma > \beta$. If $d_i < m$ then $d_{-i} = m > 0$.

Proof. Otherwise the buyer is better off acquiring less money in the CM in the first place. ■

Thus at least in one type of meetings the buyer transfers his entire money holdings to the seller. In the following lemma we characterize money transfers in each type of meeting in more detail.

Lemma 21 Assume $\gamma > \beta$. If $d_F \leq m$ then $d_L = m > 0$.

Proof. The case $d_F < m$ and $d_L = m > 0$ follows from the previous lemma. For the case $d_F = m$ suppose, by way of contradiction, that $d_L < m$. Then there exists an alternative feasible allocation such that $m \geq d'_L = d_L + \varepsilon$ for $\varepsilon > 0$ arbitrarily small, and $x'_L = \beta\phi'(d_L + \varepsilon) + \alpha\beta(v - \hat{v})$, and such that the allocation in FI meetings is unchanged. The implied change in the value of the objective function is

$$\beta\phi'[u'(\beta\phi'(d_L + \varepsilon) + \alpha\beta(v - \hat{v})) - 1] > 0$$

because $x_L < x_F \leq x^*$, otherwise the LC_F would be slack. ■

Combining the results in the above lemmas, it follows that $d_L = m$, so that

$$\begin{aligned}
v = \max \quad & -\phi m + (1-\rho)[u(\beta\phi'd_F + \beta(v - \hat{v})) + \beta\phi'(m - d_F) - \beta(v - \hat{v})] + \\
& \rho[u(\beta\phi'm + \alpha\beta(v - \hat{v})) - \beta(v - \hat{v})] + \beta v'
\end{aligned} \tag{39}$$

and first order conditions for m and d_F respectively:

$$\begin{aligned}
-\phi + \beta\phi'[(1-\rho) + \rho u'(\beta\phi'm + \alpha\beta(v - \hat{v}))] &= 0 \\
(1-\rho)\beta\phi'[u'(\beta\phi'd_F + \beta(v - \hat{v})) - 1] &\geq 0
\end{aligned}$$

Then

$$m = \frac{u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right) - \alpha\beta(v - \hat{v})}{\beta\phi'} \tag{40}$$

as in (16).

Lemma 22 Assume $\gamma > \beta$. Then $d_F = m$.

Proof. It follows from the first order condition for d_F and LC_F binding, which imply $u'(x_F) = u'(\beta\phi'd_F + \beta(v - \hat{v})) > 1$. ■

This results differs from the case in which LC_F was slack, where $d_F < m$ was a possibility, given that sufficient credit was allowing the good buyer to purchase the first best consumption x^* in FI meetings, thus making it profitable for him to carry the money that wasn't necessary to achieve x^* into the next CM.

The good buyer value is then simply

$$\begin{aligned}(1-\beta)v &= -\phi m + (1-\rho)u(\beta\phi'm + \beta(v - \hat{v})) + \rho u(\beta\phi'm + \alpha\beta(v - \hat{v})) - \beta(v - \hat{v}) \\ v &= -\phi m + (1-\rho)u(\beta\phi'm + \beta(v - \hat{v})) + \rho u(\beta\phi'm + \alpha\beta(v - \hat{v})) + \beta\hat{v}\end{aligned}$$

Analogously, bad buyers' problem is

$$\hat{v} = \max_{\hat{m}} \{-\phi\hat{m} + (1-\rho)u(\beta\phi'\hat{m}) + \rho[u(\beta\phi'm + \alpha\beta(v - \hat{v})) + \beta\phi'(\hat{m} - m)] + \beta\hat{v}'\} \quad (41)$$

Their choice of $\hat{m}$ satisfies $-\phi + \beta\phi'[(1-\rho)u'(\beta\phi'\hat{m}) + \rho] = 0$, implying $\hat{m} =$

$\frac{u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right)}{\beta\phi'}$, as defined also in (18). Thus the value of a bad buyer is

$$(1-\beta)\hat{v} = -\phi\hat{m} + (1-\rho)u(\beta\phi'\hat{m}) + \rho[u(\beta\phi'm + \alpha\beta(v - \hat{v})) + \beta\phi'(\hat{m} - m)] \quad (42)$$

3.3.1 Equilibrium characterization

Combining (39) with (42) yields

$$\begin{aligned}(v - \hat{v}) &= -\phi m + (1-\rho)u(\beta\phi'm + \beta(v - \hat{v})) + \rho u(\beta\phi'm + \alpha\beta(v - \hat{v})) - (1-\beta)\hat{v} \\ &= \phi(\hat{m} - m) + (1-\rho)u(\beta\phi'm + \beta(v - \hat{v})) \\ &\quad - (1-\rho)u(\beta\phi'\hat{m}) - \rho\beta\phi'(\hat{m} - m)\end{aligned}$$

Similarly

$$\beta\phi'(\hat{m} - m) = u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right) - u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right) + \alpha\beta(v - \hat{v})$$

which, substituted back into the previous equation yields

$$\begin{aligned}(v - \hat{v}) &= \phi \left(\frac{u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right) - u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right) + \alpha\beta(v - \hat{v})}{\beta\phi'} \right) + (1-\rho)u(\beta\phi'm + \beta(v - \hat{v})) \\ &\quad - (1-\rho)u(\beta\phi'\hat{m}) - \rho \left[u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right) - u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right) + \alpha\beta(v - \hat{v}) \right]\end{aligned}$$

which can be rearranged as

$$\begin{aligned}
(v - \hat{v}) = & \frac{\left(\frac{\phi}{\beta\phi'} - \rho\right)}{\left[1 - \left(\frac{\phi}{\beta\phi'} - \rho\right)\alpha\beta\right]} \left[u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1 - \rho}\right) - u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1 - \rho)}{\rho}\right) \right] \\
& + \frac{(1 - \rho)}{\left[1 - \left(\frac{\phi}{\beta\phi'} - \rho\right)\alpha\beta\right]} \left[u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1 - \rho)}{\rho}\right)\right) - u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1 - \rho}\right)\right) \right] \quad (43)
\end{aligned}$$

Thus, solving for $\hat{m} - m$ yields

$$\begin{aligned}
\beta\phi'(\hat{m} - m) = & \frac{\left[u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1 - \rho}\right) - u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1 - \rho)}{\rho}\right) \right]}{\left[1 - \left(\frac{\phi}{\beta\phi'} - \rho\right)\alpha\beta\right]} \\
& + \frac{\alpha\beta(1 - \rho) \left[u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1 - \rho)}{\rho}\right)\right) - u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1 - \rho}\right)\right) \right]}{\left[1 - \left(\frac{\phi}{\beta\phi'} - \rho\right)\alpha\beta\right]} \quad (44)
\end{aligned}$$

Necessary and sufficient conditions for $m, x_F, x_L, \hat{m}, \hat{x}_F, x_L$ to be an equilibrium are i) $(\hat{m} - m) \geq 0$, ii) $(v - \hat{v}) > 0$ iii) DM and CM incentive constraints.

3.3.2 DM incentive constraint

The incentive constraint in the DM is $u(x_L) + \beta\phi'(\hat{m} - m) \geq u(\beta\phi'\hat{m})$ which can be rewritten as

$$u(\beta\phi'm + \alpha\beta(v - \hat{v})) + \beta\phi'(\hat{m} - m) \geq u(\beta\phi'\hat{m}) \quad (45)$$

Substituting out form (40) yields

$$\begin{aligned}
u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1 - \rho)}{\rho}\right)\right) \geq & u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1 - \rho}\right)\right) - \frac{\left[u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1 - \rho}\right) - u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1 - \rho)}{\rho}\right) \right]}{\left[1 - \left(\frac{\phi}{\beta\phi'} - \rho\right)\alpha\beta\right]} \\
& - \frac{\alpha\beta(1 - \rho) \left[u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1 - \rho)}{\rho}\right)\right) - u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1 - \rho}\right)\right) \right]}{\left[1 - \left(\frac{\phi}{\beta\phi'} - \rho\right)\alpha\beta\right]}
\end{aligned}$$

Assuming $\left[1 - \left(\frac{\phi}{\beta\phi'} - \rho\right)\alpha\beta\right] > 0$ yields:⁴

$$\begin{aligned}
\left[1 + \left(1 - \frac{\phi}{\beta\phi'}\right)\alpha\beta\right] \left[u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1 - \rho)}{\rho}\right)\right) - u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1 - \rho}\right)\right) \right] \geq \\
\left[u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1 - \rho)}{\rho}\right) - u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1 - \rho}\right) \right] \quad (46)
\end{aligned}$$

⁴This is equivalent to $1 > \alpha(\gamma - \beta\rho)$, so it is either an upper bound on α, γ or a lower bound on β, ρ .

3.3.3 CM incentive constraint

Differently from the DM incentive constraint, the bad buyer could choose money holdings $\hat{m}$ if the consumption stemming from transferring $\hat{m}$ to the seller, namely $\beta\phi'\hat{m}$ by the seller's participation constraint, is sufficiently larger than x_L . The choice of $\hat{m}$, if the bad buyer planned on deviating to a separating strategy in a LI meeting, satisfies $-\phi + \beta\phi' u'(\beta\phi'\hat{m}) = 0$, yielding $\hat{m} = \frac{u'^{-1}(\frac{\phi}{\beta\phi'})}{\beta\phi'}$. Notice that, as in the previous case, if $\frac{\gamma}{\beta} > 1$ then $\hat{m} > \hat{m}$. The CM incentive constraint that rules out the deviation by an individual bad buyer (thus not affecting prices in the CM) is

$$-\phi\hat{m} + (1-\rho)u(\beta\phi'\hat{m}) + \rho[u(\beta\phi'm + \alpha\beta(v-\hat{v})) + \beta\phi'(\hat{m}-m)] \geq -\phi\hat{m} + u(\beta\phi'\hat{m})$$

This can be rewritten as

$$\begin{aligned} & u(\beta\phi'm + \alpha\beta(v-\hat{v})) + \beta\phi'(\hat{m}-m) \geq \\ & \frac{1}{\rho}[-\phi(\hat{m}-\hat{m}) + u(\beta\phi'\hat{m}) - (1-\rho)u(\beta\phi'\hat{m})] \end{aligned} \quad (47)$$

Substituting out from the equilibrium allocation yields

$$\begin{aligned} & u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right)\right) - u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right) \\ & + \left(\frac{1}{[1 + (\rho - \frac{\phi}{\beta\phi'})\alpha\beta]} - \frac{1}{\rho} \frac{\phi}{\beta\phi'}\right) u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right) \\ & + \frac{(1-\rho)(1 - \frac{\phi}{\beta\phi'})\alpha\beta}{\rho[1 + (\rho - \frac{\phi}{\beta\phi'})\alpha\beta]} u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right)\right) \geq \\ & + \frac{1}{\rho} \left[u\left(u'^{-1}\left(\frac{\phi}{\beta\phi'}\right)\right) - \frac{\phi}{\beta\phi'} u'^{-1}\left(\frac{\phi}{\beta\phi'}\right) \right] \end{aligned}$$

Lemma 23 *If $\gamma > \beta$ then (47) implies (45) if and only if*

$$u\left(u'^{-1}\left(\frac{\phi}{\beta\phi'}\right)\right) - u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right)\right) \geq \frac{\phi}{\beta\phi'} \left(u'^{-1}\left(\frac{\phi}{\beta\phi'}\right) - u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right) \right)$$

Proof. Combining (47) with (45) yields

$$\begin{aligned} & u(\beta\phi'm + \alpha\beta(v-\hat{v})) + \beta\phi'(\hat{m}-m) \geq \\ & \max\left(u(\beta\phi'\hat{m}), \frac{1}{\rho}[-\phi(\hat{m}-\hat{m}) + u(\beta\phi'\hat{m}) - (1-\rho)u(\beta\phi'\hat{m})]\right) \end{aligned}$$

Then (47) implies (45) if and only if

$$\frac{1}{\rho}[-\phi(\hat{m}-\hat{m}) + u(\beta\phi'\hat{m}) - (1-\rho)u(\beta\phi'\hat{m})] \geq u(\beta\phi'\hat{m})$$

which, after some rearranging, is simply $u(\beta\phi'\hat{m}) - u(\beta\phi'\hat{m}) \geq \phi(\hat{m} - \hat{m})$, and

using $\hat{m} = \frac{u'^{-1}\left(\frac{\phi}{\beta\phi'}\right)}{\beta\phi'}$ and $\hat{m} = \frac{u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right)}{\beta\phi'}$, becomes

$$u\left(u'^{-1}\left(\frac{\phi}{\beta\phi'}\right)\right) - u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right)\right) \geq \frac{\phi}{\beta\phi'} \left(u'^{-1}\left(\frac{\phi}{\beta\phi'}\right) - u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right)\right)$$

■

Thus necessary and sufficient conditions for this pooling equilibrium are 1) $\hat{m} - m \geq 0$, 2) $v - \hat{v} > 0$, 3) DM and CM IC:

1) $\hat{m} - m \geq 0$ if and only if

$$\begin{aligned} & \frac{\left[u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right) - u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right)\right]}{\left[1 - \left(\frac{\phi}{\beta\phi'} - \rho\right)\alpha\beta\right]} \\ & + \frac{\alpha\beta(1-\rho)\left[u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right)\right) - u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right)\right)\right]}{\left[1 - \left(\frac{\phi}{\beta\phi'} - \rho\right)\alpha\beta\right]} \geq 0 \end{aligned}$$

which is also, in terms of endogenous variables:

$$\frac{(\hat{x}_F - x_L) + \alpha\beta(1-\rho)(u(x_L) - u(\hat{x}_F))}{\left[1 - \left(\frac{\phi}{\beta\phi'} - \rho\right)\alpha\beta\right]} \geq 0$$

and, assuming $\left[1 - \left(\frac{\phi}{\beta\phi'} - \rho\right)\alpha\beta\right] > 0$, is satisfied if and only if

$$\begin{aligned} (\hat{x}_F - x_L) + \alpha\beta(1-\rho)(u(x_L) - u(\hat{x}_F)) & \geq 0 \\ u(x_L) - u(\hat{x}_F) & \geq \frac{1}{\alpha\beta(1-\rho)}(x_L - \hat{x}_F) \end{aligned} \quad (48)$$

2) $v - \hat{v} > 0$ if and only if

$$\begin{aligned} & \frac{\left(\frac{\phi}{\beta\phi'} - \rho\right)}{\left[1 - \left(\frac{\phi}{\beta\phi'} - \rho\right)\alpha\beta\right]} \left[u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right) - u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right)\right] \\ & + \frac{(1-\rho)}{\left[1 - \left(\frac{\phi}{\beta\phi'} - \rho\right)\alpha\beta\right]} \left[u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right)\right) - u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right)\right)\right] > 0 \end{aligned}$$

which is also, in terms of endogenous variables:

$$\frac{\left(\frac{\gamma}{\beta} - \rho\right)(\hat{x}_F - x_L) + (1-\rho)(u(x_L) - u(\hat{x}_F))}{\left[1 - \left(\frac{\phi}{\beta\phi'} - \rho\right)\alpha\beta\right]} > 0$$

and, assuming $\left[1 - \left(\frac{\phi}{\beta\phi'} - \rho\right)\alpha\beta\right] > 0$, is satisfied if and only if

$$\begin{aligned} \left(\frac{\gamma}{\beta} - \rho\right)(\hat{x}_F - x_L) + (1 - \rho)(u(x_L) - u(\hat{x}_F)) &> 0 \\ u(x_L) - u(\hat{x}_F) &> \frac{\left(\frac{\gamma}{\beta} - \rho\right)}{(1 - \rho)}(x_L - \hat{x}_F) \end{aligned} \quad (49)$$

Notice that this is always violated if $\rho = \frac{1}{2}$.

3) DM and CM IC, which we can rewrite as simply CM IC and the necessary and sufficient condition for it to imply DM IC:

$$\begin{aligned} u(\beta\phi'\hat{m}) - u(\beta\phi'\hat{m}) &\geq \phi(\hat{m} - \hat{m}) \\ u(\beta\phi'm + \alpha\beta(v - \hat{v})) + \beta\phi'(\hat{m} - m) &\geq \\ \frac{1}{\rho}[-\phi(\hat{m} - \hat{m}) + u(\beta\phi'\hat{m}) - (1 - \rho)u(\beta\phi'\hat{m})] \end{aligned}$$

which can also be rewritten in terms of endogenous variables as

$$\begin{aligned} u(\beta\phi'\hat{m}) - \phi(\hat{m} - \hat{m}) &\geq u(\hat{x}_F) \\ u(x_L) - u(\hat{x}_F) &\geq \\ \frac{[-\phi(\hat{m} - \hat{m}) - \rho\beta\phi'(\hat{m} - m) + u(\beta\phi'\hat{m}) - u(\hat{x}_F)]}{\rho} \end{aligned}$$

or the opposite, that is DM IC and the necessary and sufficient condition for it to imply CM IC

$$\begin{aligned} u(\beta\phi'\hat{m}) - u(\beta\phi'\hat{m}) &< \phi(\hat{m} - \hat{m}) \\ u(\beta\phi'm + \alpha\beta(v - \hat{v})) + \beta\phi'(\hat{m} - m) &\geq u(\beta\phi'\hat{m}) \end{aligned}$$

which can also be rewritten in terms of endogenous variables as:

$$\begin{aligned} u(\beta\phi'\hat{m}) - \phi(\hat{m} - \hat{m}) &< u(\hat{x}_F) \\ u(\beta\phi'm + \alpha\beta(v - \hat{v})) + \beta\phi'(\hat{m} - m) &\geq u(\hat{x}_F) \end{aligned}$$

Notice that we can rewrite (48) as a lower bound on α if $x_L \geq \hat{x}_F$ or as an upper bound on α if $x_L < \hat{x}_F$.

Also, the following lemma helps us in characterizing (49) more closely.

Lemma 24 *If $x_L \geq \hat{x}_F$ then a sufficient condition for (49) to be satisfied is*

$$u'(x_L) \geq \frac{\left(\frac{\gamma}{\beta} - \rho\right)}{(1 - \rho)}.$$

If instead $x_L < \hat{x}_F$ then a sufficient condition for (49) to be satisfied is

$$u'(x_L) < \frac{\left(\frac{\gamma}{\beta} - \rho\right)}{(1 - \rho)}.$$

Where $\gamma > \beta$ by assumption.

Proof. Using concavity of u , if $x_L \geq \hat{x}_F$ then $u(x_L) - u(\hat{x}_F) > u'(x_L)(x_L - \hat{x}_F)$, and if $x_L < \hat{x}_F$ then $u(x_L) - u(\hat{x}_F) < u'(x_L)(x_L - \hat{x}_F)$, which imply the result. ■

As for the CM incentive constraint, maintaining the assumption $\left[1 + \left(1 - \frac{\phi}{\beta\phi'}\right)\alpha\beta\right] > 0$, we can rewrite (47) as

$$\begin{aligned} & \left[1 + \left(\rho - \frac{\phi}{\beta\phi'}\right)\alpha\beta\right] \left\{ u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right)\right) - u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right) \right\} \\ & \left[1 + \left(\rho - \frac{\phi}{\beta\phi'}\right)\alpha\beta\right] \left\{ -\frac{1}{\rho} \left[u\left(u'^{-1}\left(\frac{\phi}{\beta\phi'}\right)\right) - \frac{\phi}{\beta\phi'} u'^{-1}\left(\frac{\phi}{\beta\phi'}\right) \right] \right\} \geq \\ & -\frac{(1-\rho)}{\rho} \left(1 - \frac{\phi}{\beta\phi'}\alpha\beta\right) u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right)\right) \\ & \left(\frac{1}{\rho} \frac{\phi}{\beta\phi'} \left[1 + \left(\rho - \frac{\phi}{\beta\phi'}\right)\alpha\beta\right] - 1\right) u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right) \end{aligned}$$

and rearrange as a lower bound on α :

$$\begin{aligned} & \alpha\beta \left(\rho - \frac{\phi}{\beta\phi'}\right) \left\{ u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right)\right) - u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right) \right\} \\ & -\frac{1}{\rho} \alpha\beta \left(\rho - \frac{\phi}{\beta\phi'}\right) \left[u\left(u'^{-1}\left(\frac{\phi}{\beta\phi'}\right)\right) - \frac{\phi}{\beta\phi'} u'^{-1}\left(\frac{\phi}{\beta\phi'}\right) \right] \\ & -\alpha\beta \left\{ \frac{(1-\rho)}{\rho} \frac{\phi}{\beta\phi'} u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right)\right) + \frac{1}{\rho} \frac{\phi}{\beta\phi'} \left(\rho - \frac{\phi}{\beta\phi'}\right) u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right) \right\} \geq \\ & \left[\frac{1}{\rho} \frac{\phi}{\beta\phi'} - 1 \right] u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right) - \frac{(1-\rho)}{\rho} u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right)\right) \\ & - \left\{ u\left(u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right)\right) - u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - (1-\rho)}{\rho}\right) \right\} \\ & + \frac{1}{\rho} \left[u\left(u'^{-1}\left(\frac{\phi}{\beta\phi'}\right)\right) - \frac{\phi}{\beta\phi'} u'^{-1}\left(\frac{\phi}{\beta\phi'}\right) \right] \end{aligned}$$

4 Separating equilibria

When playing separating strategies the bad buyer always chooses $\hat{m} = \frac{u'^{-1}\left(\frac{\phi}{\beta\phi'}\right)}{\beta\phi'}$ as defined in the previous section, since it is the solution to

$$\hat{v} = \max_{\hat{m}} \{ -\phi \hat{m} + u(\beta\phi' \hat{m}) + \beta \hat{v}' \}$$

However, the value of $\hat{m}$ from the pooling equilibrium may be different because it depends on the the actual value of x_L , so it will be characterized case by case.

The decision problem of the good buyer is

$$\begin{aligned}
v = \max \quad & \{-\phi m + (1-\rho)u(x_F) + \rho u(x_L) + \beta[(1-\rho)(\phi'(m-d_F) - \tau_F) + \rho(\phi'(m-d_L) - \tau_L)] + \beta v\} \\
s.t. \quad & x_F \leq \beta \phi' d_F + l_F \\
& x_L \leq \beta \phi' d_L + l_L \\
& l_F \leq \beta(v - \hat{v}), l_L \leq \beta(v - \hat{v}) \\
& \tau_F = \frac{l_F}{\beta}, \tau_L = \frac{l_L}{\beta} \\
& u(\beta \phi' \hat{m}) \geq u(x_L) + \beta \phi'(\hat{m} - d_L) \\
& -\phi \hat{m} + u(\beta \phi' \hat{m}) \geq -\phi \hat{m} + \rho[u(x_L) + \beta \phi'(\hat{m} - d_L)] + (1-\rho)u(\beta \phi' \hat{m})
\end{aligned} \tag{51}$$

where (51) and (52) and the DM and CM incentive constraints respectively.

Lemma 25 *If $\gamma > \beta$ then $d_i = m$ for at least one i*

Proof.

Otherwise $d_i < m$ for all i , which implies that there exists a feasible allocation in which no consumption allocation changes but the buyer carries over $m' < m$, and which yields higher utility because it saves on money purchases in the current CM.

■

Lemma 26 *If $\gamma > \beta$ and if the incentive constraints (51) and (52) are satisfied, then $d_i = m \geq 0$ for all i .*

Proof.

A solution to (50) is by definition such that the incentive constraints (51) and (52) are satisfied. Then a FI meeting is equivalent to a LI meeting, implying that if $d_L = m$ then also $d_F = m$, which by feasibility on money holdings are non negative.

■

From the above lemmas we can conclude that if $d_L < d_F = m$ it is because it is necessary for the good buyer to separate from the bad buyer by making it unappealing to pool in LI meeting. This is achieved by lowering consumption in LI meetings, thus satisfying the incentive constraints. Clearly, the good buyer prefers to use all of the credit he can use a LI meeting, and save on money which he can always bring into the next CM.

Then we can rewrite the good buyer's problem as

$$\begin{aligned}
v = \max \quad & \{-\phi m + (1-\rho)u(\beta \phi' m + l_F) + \rho u(\beta \phi' d_L + l_L) - (1-\rho)l_F + \rho(\beta \phi'(m-d_L) - l_L) + \beta v'\} \\
s.t. \quad & l_F \leq \beta(v - \hat{v}), l_L \leq \beta(v - \hat{v}) \\
& u(\beta \phi' \hat{m}) \geq u(\beta \phi' d_L + l_L) + \beta \phi'(\hat{m} - d_L) \\
& -\phi \hat{m} + u(\beta \phi' \hat{m}) \geq -\phi \hat{m} + \rho[u(\beta \phi' d_L + l_L) + \beta \phi'(\hat{m} - d_L)] + (1-\rho)u(\beta \phi' \hat{m})
\end{aligned}$$

where $\hat{m}$ solves

$$\hat{v} = \max_{\hat{m}} \{-\phi \hat{m} + (1-\rho)u(\beta \phi' \hat{m}) + \rho[u(x_L) + \beta \phi' \hat{m}] + \beta \hat{v}'\}$$

with $x_L = \beta \phi' d_L + l_L$ which may be different case by case.

Let η_{DM}, η_{CM} denote the Lagrange multipliers to (51) and (52) respectively.

Optimality conditions for m and $d_L \leq m$ are, respectively:

$$-\phi + \beta \phi' (1 - \rho) u'(\beta \phi' m + l_F) + \rho \beta \phi' = 0 \quad (53)$$

$$\beta \phi' \rho u'(\beta \phi' d_L + l_L) - \rho \beta \phi' - \eta_{DM} [\beta \phi' u'(\beta \phi' d_L + l_L) - \beta \phi'] - \eta_{CM} \rho [\beta \phi' u'(\beta \phi' d_L + l_L) - \beta \phi'] \geq 0 \quad (54)$$

Lemma 27 *There is no solution to (50) where both incentive constraints bind. The CM incentive constraint, (52), implies the DM incentive constraint, (51), if and only if $\phi(\hat{m} - \hat{m}) + (1 - \rho)[u(\beta \phi' \hat{m}) - u(\beta \phi' \hat{m})] \leq 0$.*

Proof.

Constraints (51) and (52) can be rewritten respectively as:

$$u(\beta \phi' \hat{m}) \geq u(\beta \phi' d_L + l_L) + \beta \phi' (\hat{m} - d_L)$$

$$\frac{1}{\rho} [\phi(\hat{m} - \hat{m}) + u(\beta \phi' \hat{m}) - (1 - \rho) u(\beta \phi' \hat{m})] \geq u(\beta \phi' d_L + l_L) + \beta \phi' (\hat{m} - d_L)$$

Therefore, either

$$\phi(\hat{m} - \hat{m}) + u(\beta \phi' \hat{m}) - (1 - \rho) u(\beta \phi' \hat{m}) \leq \rho u(\beta \phi' \hat{m})$$

and the CM IC (52) implies that the DM IC (51) is satisfied. Or the opposite is true. The above inequality can be rewritten as

$$\phi(\hat{m} - \hat{m}) + (1 - \rho)[u(\beta \phi' \hat{m}) - u(\beta \phi' \hat{m})] \leq 0$$

■

Thus there can be several types of separating equilibrium: one with both incentive constraints slack, one where only the DM IC binds, one where only the CM IC binds. Moreover there can be equilibria with the limited commitment constraint slack or binding. Once we characterize the solutions to the decision problem of a good buyer using separating strategies we need to check that his utility is not larger under pooling strategies.

4.0.1 Case with both DM and CM incentive constraints slack

We consider two cases: first the limited commitment constraint is slack, second it is binding.

4.0.2 Case with limited commitment constraint slack.

With all constraints slack – apart from feasibility constraints – the good buyer's problem is simply

$$\begin{aligned} v = \max \quad & \{-\phi m + (1 - \rho) u(x_F) + \rho u(x_L) - (1 - \rho) l_F + \rho(\beta \phi'(m - d_L) - l_L) + \beta v'\} \\ \text{s.t.} \quad & l_F \leq \beta(v - \hat{v}), l_L \leq \beta(v - \hat{v}) \\ & x_F = \beta \phi' m + l_F \\ & x_L = \beta \phi' d_L + l_L \end{aligned}$$

Lemma 28 *If $\gamma > \beta$ then $d_F = d_L = m = 0$.*

Proof. Idea: It follows from limited commitment constraint slack: if $m > 0$ was a solution to the good buyer's problem then he could always marginally reduce his money holdings and maintain his former consumption level by marginally increasing his credit, which is feasible because $l_i < \beta(v - \hat{v})$ for $i = F, L$. ■

Then, from the first order conditions to the good buyer's problem, (53) and (54), we have

$$\begin{aligned} -\phi + \beta\phi'[(1-\rho)u'(x^*) + \rho] &\leq 0 \\ \beta\phi'\rho[u'(x^*) - 1] &\leq 0 \end{aligned}$$

where we have substituted out $\eta_{DM} = \eta_{CM} = 0$ because both incentive constraints are slack in this case. Because $u'(x^*) = 1$ we then simply have

$$-\phi + \beta\phi' \leq 0$$

And the value of a good buyer is

$$(1-\beta)v = u(x^*) - x^*$$

And a necessary condition for this equilibrium is that $\beta(v - \hat{v}) \geq x^*$.

The value of a bad buyer in a separating equilibrium is simply

$$(1-\beta)\hat{v} = -\phi\hat{m} + u(\beta\phi'\hat{m})$$

Thus

$$(1-\beta)(v - \hat{v}) = u(x^*) - x^* + \phi\hat{m} - u(\beta\phi'\hat{m})$$

Necessary and sufficient conditions for this to be an equilibrium are 1) $\beta(v - \hat{v}) \geq x^*$, 2) DM and CM incentive constraint satisfied and 3) $-\phi + \beta\phi' \leq 0$.

1) $\beta(v - \hat{v}) \geq x^*$:

$$\begin{aligned} \frac{\beta}{(1-\beta)}[u(x^*) - x^* + \phi\hat{m} - u(\beta\phi'\hat{m})] &\geq x^* \\ \beta[u(x^*) + \phi\hat{m} - u(\beta\phi'\hat{m})] &\geq x^* \end{aligned}$$

2') DM and CM incentive constraints respectively

$$u(\beta\phi'\hat{m}) \geq u(x^*) + \beta\phi'\hat{m}$$

$$\frac{1}{\rho}[\phi(\hat{m} - \hat{m}) + u(\beta\phi'\hat{m}) - (1-\rho)u(\beta\phi'\hat{m})] \geq u(x^*) + \beta\phi'\hat{m}$$

It is easy to see, however, that the first one is never satisfied if $\gamma > \beta$.

Proposition 29 *If $\gamma > \beta$ there is no equilibrium with both incentive constraints slack and the limited commitment constraints slack.*

Proof. It follows from the DM incentive constraint where $x_L = \beta\phi'd_L + l_L$ is set to x^* with $d_L = 0$. ■

4.0.3 Case with limited commitment constraint binding.

With the incentive constraints slack and the limited commitment constraint binding the good buyer's problem is simply

$$\begin{aligned} v = \max \quad & \{-\phi m + (1-\rho)u(x_F) + \rho u(x_L) - (1-\rho)l_F + \rho(\beta\phi'(m-d_L) - l_L) + \beta v'\} \\ \text{s.t.} \quad & l_F = l_L = \beta(v - \hat{v}) \\ & x_F = \beta\phi' m + l_F \\ & x_L = \beta\phi' d_L + l_L \end{aligned}$$

There can be two types of equilibrium of this type (DM and CM IC slack and LC binding): one with $m = 0$ (for example when γ is sufficiently larger than β , or analogously, when $v - \hat{v}$ is large enough) and one with $m > 0$.

4.0.4 Case with limited commitment constraint binding and $m = 0$.

In this case first order condition for m implies $-\phi + \beta\phi' u'(\beta(v - \hat{v})) \leq 0$, which in turn implies $\beta\phi' < \phi$ because with the limited commitment constraint binding $u'(\beta(v - \hat{v})) > 1$. Thus a necessary and sufficient condition for this equilibrium to exist is $u'(\beta(v - \hat{v})) \leq \frac{\gamma}{\beta}$.

The good buyer's value is $(1-\beta)v = u(\beta(v - \hat{v})) - \beta(v - \hat{v})$ while the bad buyer's value is $(1-\beta)\hat{v} = -\phi\hat{m} + u(\beta\phi'\hat{m})$. Thus $(v - \hat{v})$ is the solution to:

$$(v - \hat{v}) = u(\beta(v - \hat{v})) + \phi\hat{m} - u(\beta\phi'\hat{m}) \quad (55)$$

A necessary condition for this equilibrium to exist is that $(v - \hat{v}) > 0$, which is

$$u(\beta(v - \hat{v})) + \phi\hat{m} > u(\beta\phi'\hat{m})$$

The DM incentive constraint in this case is

$$u(\beta\phi'\hat{m}) \geq u(\beta(v - \hat{v})) + \beta\phi'\hat{m}$$

which doesn't necessarily contradict $(v - \hat{v}) > 0$ because $\phi > \beta\phi'$ by assumption.

The CM incentive constraint in this case is

$$u(\beta\phi'\hat{m}) \geq \phi(\hat{m} - \hat{m}) + \rho u(\beta(v - \hat{v})) + (1-\rho)u(\beta\phi'\hat{m})$$

where $\hat{m}$ in this case solves

$$\hat{v} = \max_{\hat{m}} \{-\phi\hat{m} + (1-\rho)u(\beta\phi'\hat{m}) + \rho[u(\beta(v - \hat{v})) + \beta\phi'\hat{m}] + \beta\hat{v}'\}$$

and therefore satisfies

$$-\phi + \beta\phi'(1-\rho)u'(\beta\phi'\hat{m}) + \rho\beta\phi' = 0$$

Thus

$$\hat{m} = \frac{1}{\beta\phi'} u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{(1-\rho)}\right)$$

which is the same as in the previous section for the pooling equilibrium because the bad buyer's choice of $\hat{m}$ does not affect x_L . Thus, as in the previous section, $\hat{m} > \hat{m}$ if $\gamma > \beta$.

Proposition 30 *A separating equilibrium with $m = 0$, DM and CM IC slack, LC binding exists if and only if*

$$\begin{aligned} \max((\beta\phi' - \phi)\hat{m}, (1-\rho)[u(\beta\phi'\hat{m}) - u(\beta(v-\hat{v}))] - \phi\hat{m}) &\leq \\ u(\beta\phi'\hat{m}) - u(\beta(v-\hat{v})) - \phi\hat{m} &< 0 \end{aligned} \quad (56)$$

and $\frac{1}{\beta} < u'(\beta(v-\hat{v})) \leq \frac{\gamma}{\beta}$, where $(v-\hat{v})$ is defined in (55).

Proof. Combining both DM and CM incentive constraints with the condition $(v-\hat{v}) > 0$ we have (56). Then from the first order condition for m to the good buyer's problem a necessary and sufficient condition for $m = 0$ is $u'(\beta(v-\hat{v})) \leq \frac{\gamma}{\beta}$. Under this condition, however, a solution to (55) exists if and only if $\frac{1}{\beta} < u'(\beta(v-\hat{v}))$: if fact, the left hand side of (55) is increasing in $v-\hat{v}$ if and only if $\frac{1}{\beta} < u'(\beta(v-\hat{v}))$. Therefore, rearranging $u'(\beta(v-\hat{v})) \leq \frac{\gamma}{\beta}$ as simply $(v-\hat{v}) \leq \frac{1}{\beta} u'^{-1}(\frac{\gamma}{\beta})$, if $\frac{1}{\beta} \geq u'(\beta(v-\hat{v}))$ then $u(\beta(v-\hat{v})) - (v-\hat{v})$ is decreasing in $(v-\hat{v})$, so that (55) implies

$$u\left(u'^{-1}\left(\frac{\phi}{\beta\phi'}\right)\right) - \frac{1}{\beta} u'^{-1}\left(\frac{\phi}{\beta\phi'}\right) < u\left(u'^{-1}\left(\frac{\phi}{\beta\phi'}\right)\right) - \frac{\gamma}{\beta} u'^{-1}\left(\frac{\phi}{\beta\phi'}\right)$$

where $\beta\phi'\hat{m} = u'^{-1}(\frac{\phi}{\beta\phi'})$ has been substituted out. This is a contradiction because $\gamma > 1$ by assumption. ■

Intuitively, this is a solution to the good buyer's problem (not necessarily an equilibrium, as a good buyer may be better off pooling with the bad buyer) if $x_L = x_F$ is sufficiently low not to induce the bad buyer to pool in LI meetings. It is clear from the DM incentive constraint that $\hat{x}_F = \beta\phi'\hat{m} > x_F = x_L = \beta(v-\hat{v})$. At the same time γ must be sufficiently larger than β to induce good buyers not to carry money into the DM. The cost for the bad buyer is having to hold money in order to consume in the DM, and as long as this cost is large enough then $v-\hat{v}$ is also sufficiently large to not induce the good buyer to carry money. The cost of holding money, however, cannot be too large either, otherwise the bad buyer prefers to pool in LI meetings.

4.0.5 Case with limited commitment constraint binding and $m > 0$.

In this case

$$v = \max \left\{ -\phi m + (1-\rho)u(\beta\phi' m + \beta(v-\hat{v})) + \rho u(\beta\phi' d_L + \beta(v-\hat{v})) - \beta(v-\hat{v}) + \rho\beta\phi'(m-d_L) + \beta\phi' m \right\} \quad (57)$$

then the first order conditions for m and d_L respectively are

$$\begin{aligned} -\phi + (1-\rho)\beta\phi' u'(\beta\phi' m + \beta(v-\hat{v})) + \rho\beta\phi' &= 0 \\ \rho\beta\phi' u'(\beta\phi' d_L + \beta(v-\hat{v})) - \rho\beta\phi' &\geq 0 \text{ with } = \text{ if } d_L < m \end{aligned}$$

Thus yielding $m = \frac{u'^{-1}\left(\frac{\phi}{\beta\phi'} - \rho\right) - \beta(v - \hat{v})}{\beta\phi'}$ and, if $d_L < m$, then $u'(\beta\phi'd_L + \beta(v - \hat{v})) = 1$. This, however, implies that $x_L = \beta\phi'd_L + \beta(v - \hat{v}) = x^*$ which in turn implies that $x_F = \beta\phi'm + \beta(v - \hat{v}) > x^*$ which can never be a solution to the good buyer's problem because $\gamma > \beta$. Therefore there is no solution to the good buyer's problem with both DM and CM IC slack, the LC constraint binding, $m > 0$ but $d_L < d_F = m$.

With $d_L = m$ instead we have $x_L = x_F = \beta\phi'm + \beta(v - \hat{v})$ and $u'(x_L) = u'(x_F) > 1$.

Thus

$$(1 - \beta)v = -\phi m + u(\beta\phi'm + \beta(v - \hat{v})) - \beta(v - \hat{v})$$

where the choice of m satisfies

$$-\phi + \beta\phi'u'(\beta\phi'm + \beta(v - \hat{v})) = 0$$

yielding

$$m = \frac{u'^{-1}\left(\frac{\phi}{\beta\phi'}\right) - \beta(v - \hat{v})}{\beta\phi'}$$

where, with $(v - \hat{v}) \geq 0$ then $m \leq \hat{m} = \frac{u'^{-1}\left(\frac{\phi}{\beta\phi'}\right)}{\beta\phi'}$.

The bad buyer's value, when separating from the good buyer in LI meetings, is:

$$(1 - \beta)\hat{v} = -\phi\hat{m} + u(\beta\phi'\hat{m})$$

Thus $(v - \hat{v})$ is the solution to:

$$(v - \hat{v}) = u(\beta\phi'm + \beta(v - \hat{v})) + \phi(\hat{m} - m) - u(\beta\phi'\hat{m}) \quad (58)$$

The DM and CM incentive constraints are, respectively:

$$\begin{aligned} u(\beta\phi'\hat{m}) &\geq u(\beta\phi'm + \beta(v - \hat{v})) + \beta\phi'(\hat{m} - m) \\ \frac{1}{\rho}[\phi(\hat{m} - \hat{m}) + u(\beta\phi'\hat{m}) - (1 - \rho)u(\beta\phi'\hat{m})] &\geq u(\beta\phi'm + \beta(v - \hat{v})) + \beta\phi'(\hat{m} - m) \end{aligned}$$

With $\hat{m} - m > 0$, the DM incentive constraint implies that $\hat{x}_F = \beta\phi'\hat{m} > x_L = x_F = \beta\phi'm + \beta(v - \hat{v})$.

Proposition 31 *If $\gamma > \beta$ there exists a solution to (57) with $m > 0$ if and only if*

$$\begin{aligned} -\beta\phi'(\hat{m} - m) + \min\left(0, \left(\beta\phi' - \frac{\phi}{\rho}\right)(\hat{m} - \hat{m}) + \left(\frac{1 - \rho}{\rho}\right)[u(\beta\phi'\hat{m}) - u(\beta\phi'\hat{m})]\right) &\geq \\ u(\beta\phi'm + \beta(v - \hat{v})) - u(\beta\phi'\hat{m}) &> \\ -\phi(\hat{m} - m) &\quad (59) \end{aligned}$$

Proof. We can rearrange the DM, CM incentive constraints and the condition $v - \hat{v} > 0$ respectively as

$$\begin{aligned}
u(\beta\phi' m + \beta(v - \hat{v})) - u(\beta\phi' \hat{m}) &\leq -\beta\phi'(\hat{m} - m) \\
u(\beta\phi' m + \beta(v - \hat{v})) - u(\beta\phi' \hat{m}) &\leq \\
\left(\beta\phi' - \frac{\phi}{\rho}\right)(\hat{m} - \hat{m}) + \left(\frac{1-\rho}{\rho}\right)[u(\beta\phi' \hat{m}) - u(\beta\phi' \hat{m})] - \beta\phi'(\hat{m} - m) & \\
u(\beta\phi' m + \beta(v - \hat{v})) - u(\beta\phi' \hat{m}) &> -\phi(\hat{m} - m)
\end{aligned}$$

where the DM incentive constraint and the condition $v - \hat{v} > 0$ are consistent with each other because $\gamma > \beta$ by assumption. Combining them yields (59). ■

4.0.6 Case with one incentive constraint binding

Regardless of which incentive constraint binds, as in the previous section the following cases are possible:

1. LC slack. This implies $m = 0$. Otherwise the good buyer would be better off by reducing his money holdings and use more credit, since it is feasible.
2. LC binding and $d_L \leq d_F$. Then two subcases are possible: 2.1) $m = 0$, and 2.2) $m > 0$.

We start from the case of DM IC binding and case 1.

4.0.7 Case 1.a with DM incentive constraint binding and LC slack

From the arguments above we know that $m = 0$. Also, from the lemma in the previous section we know that if the DM IC is binding then the CM IC is slack and $\phi(\hat{m} - \hat{m}) + (1-\rho)[u(\beta\phi' \hat{m}) - u(\beta\phi' \hat{m})] > 0$ is a necessary and sufficient condition for this solution to exist.

The good buyer's problem is simply

$$\begin{aligned}
v = \max_{\{l_F, l_L\}} & \{ (1-\rho)u(l_F) + \rho u(l_L) - (1-\rho)l_F - \rho l_L + \beta v' \} \\
s.t. & \quad u(\beta\phi' \hat{m}) = u(l_L) + \beta\phi' \hat{m}
\end{aligned}$$

As in the previous cases, the DM IC implies that $\hat{x}_F > x_L$. The good buyer's choice of l_L is pinned down by the DM IC, while the choice of l_F satisfies $u'(l_F) = 1$ implying that $l_F = x^*$. Thus, the good buyer's value is

$$(1-\beta)v = (1-\rho)u(x^*) + \rho[u(\beta\phi' \hat{m}) - \beta\phi' \hat{m}] - (1-\rho)x^* - \rho u^{-1}[u(\beta\phi' \hat{m}) - \beta\phi' \hat{m}]$$

As in the previous cases the bad buyer's value is simply $(1-\beta)\hat{v} = -\phi\hat{m} + u(\beta\phi' \hat{m})$. Thus

$$(v - \hat{v}) = \frac{\phi\hat{m} + (1-\rho)[u(x^*) - x^* - u(\beta\phi' \hat{m})] - \rho\{\beta\phi' \hat{m} + u^{-1}[u(\beta\phi' \hat{m}) - \beta\phi' \hat{m}]\}}{(1-\beta)} \quad (60)$$

A necessary condition for the LC to be slack is that $\beta(v - \hat{v}) \geq x^*$ that is

$$\left[\frac{(1-\beta)}{\beta} + 1 - \rho \right] x^* \leq (\phi - \rho\beta\phi')\hat{m} + (1-\rho)[u(x^*) - u(\beta\phi'\hat{m})] - \rho u^{-1}[u(\beta\phi'\hat{m}) - \beta\phi'\hat{m}]$$

And, similarly, that $\beta(v - \hat{v}) \geq l_L$ that is

$$(1 - \beta + \beta\rho)u^{-1}[u(\beta\phi'\hat{m}) - \beta\phi'\hat{m}] \leq \beta\{(\phi - \rho\beta\phi')\hat{m} + (1-\rho)[u(x^*) - x^* - u(\beta\phi'\hat{m})]\}$$

The monotonicity of u implies $u^{-1}[u(\beta\phi'\hat{m}) - \beta\phi'\hat{m}] < \hat{x}_F$. Thus the above inequality is always satisfied if

$$0 \leq (\gamma - 1 - \beta\rho)\hat{x}_F + \beta(1-\rho)[u(x^*) - x^* - (u(\hat{x}_F) - \hat{x}_F)]$$

Easily, if $(\gamma - 1 - \beta\rho) \geq 0$ then it is always satisfied (thus γ sufficiently large). However, because $l_L \leq l_F = x^*$ it is necessary and sufficient that LC_F is satisfied.

The following proposition characterizes more in detail the necessary and sufficient condition for these separating strategies to solve both buyers' problems.

Proposition 32 *If $\gamma > \beta$ then there exists a separating solution to the good buyer's problem if and only if*

$$\left[\frac{(1-\beta)}{\beta} + 1 - \rho \right] x^* \leq (\phi - \rho\beta\phi')\hat{m} + (1-\rho)[u(x^*) - u(\beta\phi'\hat{m})] - \rho u^{-1}[u(\beta\phi'\hat{m}) - \beta\phi'\hat{m}]$$

and

$$\phi(\hat{m} - \hat{m}) + (1-\rho)[u(\beta\phi'\hat{m}) - u(\beta\phi'\hat{m})] > 0 \quad (62)$$

with $\beta\phi'\hat{m} = u'^{-1}\left(\frac{\phi}{\beta\phi'}\right)$ and $\beta\phi'\hat{m} = u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{1-\rho}\right)$.

Proof. Using (60) the necessary and sufficient condition $\beta(v - \hat{v}) \geq l_L$ can be rewritten as (61). Clearly, because $x^* \geq l_L$ then $\beta(v - \hat{v}) \geq x^* > 0$ implies $\beta(v - \hat{v}) \geq l_L$, thus the only necessary and sufficient conditions we do not need to consider also $\beta(v - \hat{v}) \geq l_L$. Condition (62) is necessary and sufficient for the DM IC to imply the CM IC. Because we are analysing the case where the DM IC is binding and the CM IC is slack then this is a necessary and sufficient condition for the separating strategies to solve both buyers' problems. ■

We can rearrange (61) and (62) as follows

$$\begin{aligned} \left[\frac{(1-\beta)}{\beta} + 1 - \rho \right] x^* &\leq \phi\hat{m} - (1-\rho)u(\beta\phi'\hat{m}) - \rho\beta\phi'\hat{m} + (1-\rho)u(x^*) - \rho u^{-1}[u(\beta\phi'\hat{m}) - \beta\phi'\hat{m}] \\ 0 &< -[\phi\hat{m} - (1-\rho)u(\beta\phi'\hat{m})] - (1-\rho)u(\beta\phi'\hat{m}) + \phi\hat{m} \end{aligned}$$

So a necessary condition possibly easier to check for (61) and (62) to be satisfied is

$$\begin{aligned} \left[\frac{(1-\beta)}{\beta} + 1 - \rho \right] x^* &< \phi\hat{m} + (1-\rho)[u(x^*) - u(\beta\phi'\hat{m})] \\ &\quad - \rho[\beta\phi'\hat{m} + u^{-1}[u(\beta\phi'\hat{m}) - \beta\phi'\hat{m}]] \end{aligned}$$

Notice that from the optimality conditions to the good buyer's problem $m = 0$ is always optimal, consistent with the observations in the previous subsection (case 1.): if the good buyer was purchasing money in the CM and using it in FI meetings, while spending potentially only part of it in LI meetings since the DM IC binds l_L , then his choice of m should satisfy

$$-\phi + \beta \phi' [(1-\rho) u'(x^*) + \rho] \leq 0$$

Because $\frac{\phi}{\beta \phi'} = \frac{\gamma}{\beta} > 1$ and $u'(x^*) = 1$ then the first order condition for m is always satisfied with strict inequality.

Intuitively, this is an equilibrium because $v - \hat{v}$ is large enough to allow good buyers to not carry money into the DM, and on the other hand, the reason why $v - \hat{v}$ is large enough is that good buyers can consume the first best allocation in FI meetings, despite they must reduce their consumption in LI meetings in order to separate from bad buyers.

4.0.8 Case 2.1a with DM incentive constraint binding, LC binding and $m = 0$

From the lemma in the previous section we know that if the DM IC is binding then the CM IC is slack and $\phi(\hat{m} - \hat{m}) + (1-\rho)[u(\beta \phi' \hat{m}) - u(\beta \phi' \hat{m})] > 0$ is a necessary and sufficient condition for this solution to exist, where $\hat{m}$ solves

$$\hat{v} = \max_{\hat{m}} \{-\phi \hat{m} + (1-\rho) u(\beta \phi' \hat{m}) + \rho [u(x_L) + \beta \phi' \hat{m}] + \beta \hat{v}'\}$$

The good buyer's problem is simply

$$\begin{aligned} v = \max_{\{m, d_L\}} & \{-\phi m + (1-\rho)[u(\beta \phi' m + l_F) - l_F] + \rho [u(\beta \phi' d_L + l_L) + (\beta \phi' (m - d_L) - l_L)] + \beta v'\} \\ \text{s.t.} & \quad l_F = l_L = \beta(v - \hat{v}) \\ & \quad u(\beta \phi' \hat{m}) = u(\beta \phi' d_L + l_L) + \beta \phi' (\hat{m} - d_L) \end{aligned}$$

with $m = d_L = 0$ if and only if, evaluated at $m = 0$, the first order condition for m satisfies:

$$(1-\rho) \beta \phi' u'(\beta \phi' m + l_F) + \rho \beta \phi' < \phi$$

Thus, the DM incentive constraint is simply $u(\beta \phi' \hat{m}) - \beta \phi' \hat{m} = u(\beta(v - \hat{v}))$, and the value of a good buyer is:

$$(1-\beta)v = (1-\rho)[u(\beta(v - \hat{v})) - \beta(v - \hat{v})] + \rho[u(\beta(v - \hat{v})) - \beta(v - \hat{v})]$$

As in the previous cases then the DM IC implies that $\hat{x}_F > x_L$. As in the previous cases the bad buyer's value is simply $(1-\beta)\hat{v} = -\phi \hat{m} + u(\beta \phi' \hat{m})$. Thus $(v - \hat{v})$ is the solution to:

$$(v - \hat{v}) = u(\beta(v - \hat{v})) + \phi \hat{m} - u(\beta \phi' \hat{m}) \quad (63)$$

combined with the DM incentive constraint, $u(\beta \phi' \hat{m}) - \beta \phi' \hat{m} = u(\beta(v - \hat{v}))$, we have

$$(v - \hat{v}) = (\phi - \beta \phi') \hat{m}$$

But from the DM incentive constraint it must also be that

$$(v - \hat{v}) = \frac{u^{-1}(u(\beta\phi'\hat{m}) - \beta\phi'\hat{m})}{\beta} \quad (64)$$

Proposition 33 *If $\gamma > \beta$ then there exists a separating solution to the good buyer's problem with the DM IC and the LC binding and $m = 0$ if and only if $\phi(\hat{m} - \hat{m}) + (1 - \rho)[u(\beta\phi'\hat{m}) - u(\beta\phi'\hat{m})] > 0$ and*

$$(1 - \rho)\beta\phi' u'(\beta(v - \hat{v})) + \rho\beta\phi' < \phi \quad (65)$$

$$\frac{u^{-1}(u(\beta\phi'\hat{m}) - \beta\phi'\hat{m})}{\beta} = (\phi - \beta\phi')\hat{m} \quad (66)$$

Proof. The inequality $\phi(\hat{m} - \hat{m}) + (1 - \rho)[u(\beta\phi'\hat{m}) - u(\beta\phi'\hat{m})] > 0$ is necessary and sufficient for the DM IC to be binding (and implying the CM IC is satisfied). Inequality (65) is necessary and sufficient for $m = 0$ while equation (66) follows from (63) combined with (64). ■

Notice that necessary and sufficient condition for DM IC to imply DM IC, $\phi(\hat{m} - \hat{m}) + (1 - \rho)[u(\beta\phi'\hat{m}) - u(\beta\phi'\hat{m})] > 0$, can be rewritten as

$$\frac{\gamma}{\beta} \left[u'^{-1} \left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho} \right) - u'^{-1} \left(\frac{\gamma}{\beta} \right) \right] + (1 - \rho) \left[u \left(u'^{-1} \left(\frac{\gamma}{\beta} \right) \right) - u \left(u'^{-1} \left(\frac{\frac{\gamma}{\beta} - \rho}{1 - \rho} \right) \right) \right] > 0$$

and that (65) can be rewritten as $(1 - \rho)u'(l_F) + \rho < \frac{\gamma}{\beta}$.

An example of when the assumptions of the above proposition are violated is when $\gamma - \beta > 1$: in fact we can rewrite the (66) as:

$$\begin{aligned} u^{-1}(u(\beta\phi'\hat{m}) - \beta\phi'\hat{m}) &= \left(\frac{\phi}{\phi'} - \beta \right) \beta\phi'\hat{m} \\ u^{-1}(u(\hat{x}_F) - \hat{x}_F) &= (\gamma - \beta)\hat{x}_F \end{aligned}$$

Notice that because $\hat{x}_F > 0$ then $u^{-1}(u(\hat{x}_F) - \hat{x}_F) < \hat{x}_F$. Therefore it must be that $(\gamma - \beta)\hat{x}_F < \hat{x}_F$, which is clearly violated if $\gamma - \beta > 1$.

Also, (66) can be rearranged using the definition of $\hat{m}$ as a restriction on exogenous parameters:

$$u^{-1} \left[u \left(u'^{-1} \left(\frac{\gamma}{\beta} \right) \right) - u'^{-1} \left(\frac{\gamma}{\beta} \right) \right] = (\gamma - \beta) u'^{-1} \left(\frac{\gamma}{\beta} \right)$$

which is equivalent to $l_F = (\gamma - \beta)\hat{x}_F$. Thus, in terms of endogenous variables the three necessary and sufficient condition in the above propositions are

$$\begin{aligned} l_F &= (\gamma - \beta)\hat{x}_F \\ \frac{\gamma}{\beta} &> (1 - \rho)u'(l_F) + \rho \\ -\frac{\gamma}{\beta}\hat{x}_F + (1 - \rho)u(\hat{x}_F) &> -\phi\hat{m} + (1 - \rho)u(\beta\phi'\hat{m}) \end{aligned}$$

4.0.9 Case 2.2a with DM incentive constraint binding, LC binding and $m \geq d_L > 0$

From the lemma in the previous section we know that if the DM IC is binding then the CM IC is slack and $\phi(\hat{m} - \hat{m}) + (1 - \rho)[u(\beta\phi'\hat{m}) - u(\beta\phi'\hat{m})] > 0$ is a necessary and sufficient condition for this solution to exist where $\hat{m}$ solves

$$\hat{v} = \max_{\hat{m}} \{-\phi\hat{m} + (1 - \rho)u(\beta\phi'\hat{m}) + \rho[u(x_L) + \beta\phi'\hat{m}] + \beta\hat{v}'\}$$

with $x_L = \beta\phi'd_L + l_L$ which may be different case by case. The good buyer's problem is

$$\begin{aligned} v = \max \quad & \{-\phi m + (1 - \rho)u(\beta\phi'm + l_F) + \rho u(\beta\phi'd_L + l_L) - (1 - \rho)l_F + \rho(\beta\phi'(m - d_L) - l_L) + \beta v'\} \\ \text{s.t.} \quad & l_F = l_L = \beta(v - \hat{v}) \\ & u(\beta\phi'\hat{m}) = u(\beta\phi'd_L + l_L) + \beta\phi'(\hat{m} - d_L) \end{aligned}$$

As in the previous cases, if $\hat{m} > d_L$ the DM IC implies that $\hat{x}_F > x_L$. The choice of m , if $m > d_L$, satisfies

$$\beta\phi'(1 - \rho)u'(\beta\phi'm + \beta(v - \hat{v})) + \rho\beta\phi' = \phi$$

yielding, as in the previous case:

$$m = \frac{u'^{-1}\left[\frac{\left(\frac{\phi}{\beta\phi'} - \rho\right)}{(1 - \rho)}\right] - \beta(v - \hat{v})}{\beta\phi'}$$

The DM IC constraint pins down d_L :

$$u(\beta\phi'\hat{m}) - \beta\phi'\hat{m} = u(\beta\phi'd_L + \beta(v - \hat{v})) - \beta\phi'd_L$$

Notice that because the LC is binding, then $x_L < x^*$ and $u'(x_L) > 1$, implying that the right hand side of the DM IC as rewritten above is strictly increasing in d_L . Then, a necessary and sufficient condition for $d_L < m$ is:

$$u(\beta\phi'\hat{m}) - \beta\phi'\hat{m} < u(\beta\phi'm + \beta(v - \hat{v})) - \beta\phi'm$$

The value of a good buyer is then

$$(1 - \beta)v = -\phi m + (1 - \rho)u(\beta\phi'm + \beta(v - \hat{v})) + \rho[u(\beta\phi'\hat{m}) - \beta\phi'(\hat{m} - m)] - \beta(v - \hat{v})$$

Combining it with the bad buyer's value we have that $(v - \hat{v})$ solves:

$$(v - \hat{v}) = (\phi - \rho\beta\phi')(\hat{m} - m) + (1 - \rho)[u(\beta\phi'm + \beta(v - \hat{v})) - u(\beta\phi'\hat{m})] \quad (67)$$

Notice that in this case, because $\hat{m} > d_L$ then the DM IC implies that $\hat{x}_F > x_L = \beta\phi'm + \beta(v - \hat{v})$, thus $(v - \hat{v}) > 0$ only if $(\phi - \rho\beta\phi')$ is sufficiently large.

Proposition 34 *If $\gamma > \beta$ then there is no separating solution to the good buyer's problem with the DM IC and the LC binding and $m > 0$.*

Proof. The necessary and sufficient conditions for such separating strategies to be a solution to the buyers' problems are i) $\phi(\hat{m} - \hat{m}) + (1 - \rho)[u(\beta\phi'\hat{m}) - u(\beta\phi'\hat{m})] > 0$ and, using the DM IC, ii):

$$u(\beta\phi'\hat{m}) - \beta\phi'\hat{m} < u(\beta\phi'm + \beta(v - \hat{v})) - \beta\phi'm$$

for $m > d_L$, and using (67). Inequality i) is necessary and sufficient for the DM IC to imply the CM IC, as implied by lemma DM-DM IC in the previous section. Inequality ii) follows from $m > d_L$ and the DM IC, which is a necessary and sufficient condition for the conjectured allocation to be a solution to the buyers' problems. Also the definition of $v - \hat{v}$ in (67) requires the necessary and sufficient condition

$$(\phi - \rho\beta\phi')(\hat{m} - m) + (1 - \rho)[u(\beta\phi'm + \beta(v - \hat{v})) - u(\beta\phi'\hat{m})] > 0$$

for $(v - \hat{v}) > 0$. However, combining this with ii) yields:

$$\begin{aligned} u(\beta\phi'm + \beta(v - \hat{v})) - u(\beta\phi'\hat{m}) &> -\beta\phi'(\hat{m} - m) \\ u(\beta\phi'm + \beta(v - \hat{v})) - u(\beta\phi'\hat{m}) &> -\frac{(\phi - \rho\beta\phi')}{(1 - \rho)}(\hat{m} - m) \end{aligned}$$

Notice that $\gamma > \beta$ implies that

$$\frac{(\phi - \rho\beta\phi')}{(1 - \rho)} > \beta\phi'$$

Thus assumption ii) guarantees $(v - \hat{v}) > 0$. Moreover, notice that the necessary and sufficient condition i) for the DM IC to imply the CM IC can be rewritten as

$$u(\beta\phi'\hat{m}) - u(\beta\phi'\hat{m}) > \frac{\phi}{(1 - \rho)}(\hat{m} - \hat{m})$$

which, combined with ii) yields

$$u(\beta\phi'm + \beta(v - \hat{v})) + \beta\phi'(\hat{m} - m) > u(\beta\phi'\hat{m}) > u(\beta\phi'\hat{m}) + \frac{\phi}{(1 - \rho)}(\hat{m} - \hat{m})$$

Because $\frac{\phi}{(1 - \rho)} > \beta\phi'$ then it is necessary that $u(\beta\phi'm + \beta(v - \hat{v})) > u(\beta\phi'\hat{m})$ for the above sequence of inequality to hold, where

$$\begin{aligned} \beta\phi'\hat{m} &= u'^{-1}\left(\frac{\frac{\phi}{\beta\phi'} - \rho}{(1 - \rho)}\right) \\ x_L = \beta\phi'm + \beta(v - \hat{v}) &= u'^{-1}\left[\frac{\left(\frac{\phi}{\beta\phi'} - \rho\right)}{(1 - \rho)}\right] \end{aligned}$$

Then, necessary condition $u(\beta\phi'm + \beta(v - \hat{v})) > u(\beta\phi'\hat{m})$, and therefore necessary and sufficient condition ii), are never satisfied. ■

4.0.10 Case 2.3a with DM incentive constraint binding, LC binding and $m = d_L > 0$

From the lemma in the previous section we know that if the DM IC is binding then the CM IC is slack and $\phi(\hat{m} - \hat{m}) + (1 - \rho)[u(\beta\phi'\hat{m}) - u(\beta\phi'\hat{m})] > 0$ is a necessary and sufficient condition for this solution to exist where $\hat{m}$ solves

$$\hat{v} = \max_{\hat{m}} \{-\phi\hat{m} + (1 - \rho)u(\beta\phi'\hat{m}) + \rho[u(x_L) + \beta\phi'\hat{m}] + \beta\hat{v}'\}$$

with $x_L = \beta\phi'd_L + l_L$ which may be different case by case. The good buyer's problem is

$$\begin{aligned} v = \max_m \quad & \{-\phi m + u(\beta\phi'm + \beta(v - \hat{v})) - \beta(v - \hat{v}) + \beta v'\} \\ \text{s.t.} \quad & u(\beta\phi'\hat{m}) = u(\beta\phi'm + \beta(v - \hat{v})) + \beta\phi'(\hat{m} - m) \end{aligned}$$

Proposition 35 *If $\gamma > \beta$ then there exists a separating solution to the good buyer's problem with the DM IC and the LC binding and $m = d_L > 0$ if and only if: i) $\phi(\hat{m} - \hat{m}) + (1 - \rho)[u(\beta\phi'\hat{m}) - u(\beta\phi'\hat{m})] > 0$, ii) m solving $u(\beta\phi'\hat{m}) = u((\phi - \beta\phi')\beta\hat{m} - (\phi - \phi'(1 + \beta))\beta m) + \beta\phi'(\hat{m} - m)$ is such that $m < \hat{m}$, and iii) $u'(\beta\phi'm + \beta(v - \hat{v})) - \frac{\gamma}{\beta}$, which guarantees that the Lagrange multiplier on the DM IC is positive.*

Proof. Condition i) is necessary and sufficient for the CM IC to be satisfied. The DM IC then pins down m . Notice that the DM IC can be rearranged, assuming $(v - \hat{v}) > 0$, as:

$$\begin{aligned} u(\beta\phi'\hat{m}) - \beta\phi'\hat{m} &= u(\beta\phi'm + \beta(v - \hat{v})) - \beta\phi'm \\ u(\beta\phi'\hat{m}) - \beta\phi'\hat{m} &= u(x_L) - x_L + \beta(v - \hat{v}) \end{aligned}$$

Since both $x_L < x^*$ and $\hat{x}_F < x^*$ then the last inequality implies that $\hat{x}_F > x_L = x_F$, because $u(x) - x$ is increasing in x for all $x < x^*$. Combining the above results with the bad buyer's value, $(1 - \beta)\hat{v} = -\phi\hat{m} + u(\beta\phi'\hat{m})$, we have that $(v - \hat{v})$ solves:

$$(v - \hat{v}) = \phi(\hat{m} - m) + u(\beta\phi'm + \beta(v - \hat{v})) - u(\beta\phi'\hat{m})$$

We can then substitute from the DM IC, $(v - \hat{v})$ solves:

$$(v - \hat{v}) = \phi(\hat{m} - m) - \beta\phi'(\hat{m} - m)$$

Substituting this back into the DM IC yields

$$u(\beta\phi'\hat{m}) = u((\phi - \beta\phi')\beta\hat{m} - (\phi - \phi'(1 + \beta))\beta m) + \beta\phi'(\hat{m} - m)$$

Thus $v - \hat{v} > 0$ if and only if $\hat{m} > m$, which is required by condition ii). Optimality conditions to the good buyer's problem imply that the choice of m satisfies

$$u'(\beta\phi'm + \beta(v - \hat{v})) - \eta[u'(\beta\phi'm + \beta(v - \hat{v})) - 1] = \frac{\phi}{\beta\phi'}$$

yielding, because $u'(\beta \phi' m + \beta (v - \hat{v})) > 1$:

$$\frac{u'(\beta \phi' m + \beta (v - \hat{v})) - \frac{\gamma}{\beta}}{u'(\beta \phi' m + \beta (v - \hat{v})) - 1} = \eta$$

Thus $\eta > 0$ if and only if $u'(\beta \phi' m + \beta (v - \hat{v})) - \frac{\gamma}{\beta}$, which is necessary and sufficient condition iii). ■

4.0.11 Case 1.b with CM incentive constraint binding and LC slack

From the arguments above we know that $m = 0$. Also, from the lemma in the previous section we know that if the CM IC is binding then the DM IC is slack and $\phi(\hat{m} - \hat{m}) + (1 - \rho)[u(\beta \phi' \hat{m}) - u(\beta \phi' \hat{m})] \leq 0$ is a necessary and sufficient condition for the DM IC this solution to exist.

The good buyer's problem is simply

$$\begin{aligned} v = \max_{\{l_F, l_L\}} & \{ (1 - \rho) u(l_F) + \rho u(l_L) - (1 - \rho) l_F - \rho l_L + \beta v' \} \\ \text{s.t.} & \quad u(\beta \phi' \hat{m}) = u(l_L) + \beta \phi' \hat{m} \end{aligned}$$

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