

Lending Relationships, Banking Crises, and Optimal Monetary Policy

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Abstract

This paper develops a dynamic model of lending relationships and monetary policy. Entrepreneurs can finance idiosyncratic investment opportunities through external finance – by forming lending relationships with banks – or internal finance – by accumulating partially liquid assets. We study the dynamic response of lending rates, inflation, and investment to a banking crisis that severs lending relationships. We characterize optimal monetary policy in the aftermath of a crisis and show it involves a positive nominal interest rate that trades off the need to reduce the cost of self insurance by unbanked entrepreneurs and the need to promote the creation of lending relationships with banks. We calibrate the model to the U.S. economy and study quantitatively the optimal policy problem in and out of steady state, with and without commitment by the policymaker.

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1 Introduction

Most businesses (especially small ones) rely on dependable access to bank credit to finance investments and expand operations.¹ As illustrated by the Great Depression and the 2007-2008 Great Recession, financial and banking crises that disrupt the supply of credit and sever valuable relationships between banks and firms can have long-lasting effects on the real economy.² During the Great Recession, monetary policy responded by lowering interest rates in an attempt to reduce firms' costs of finance.³ Intuitively, a low interest rate reduces the opportunity cost of retaining earnings in the form of liquid assets (internal finance) and lowers lending rates (external finance) through the transmission mechanism of monetary policy. However, monetary policy faces a non-trivial trade off since low interest rates also have an adverse effect on banks' margins, which may slow down the recovery of the banking sector where the crisis originated.⁴

In this paper, we formalize these ideas by developing a model of lending relationships, corporate finance, and their relation with monetary policy. Incorporating these different elements in one framework proves challenging. For instance, it is not a trivial to explain the coexistence of bank credit and outside money – especially so if monetary policy is set optimally. It is also not obvious how to generate a pass through from the nominal interest rate set by the policymaker to the real lending rate. We develop a model that addresses these challenges and we use it to study the dynamics of lending rates, inflation, and investment following a banking crisis modeled as an unanticipated shock that severs lending relationships. We characterize the socially optimal monetary policy, in terms of an interest rate path, under alternative assumptions on the policymaker's commitment power.

We model an economy where entrepreneurs receive idiosyncratic investment opportunities, as in Kiyotaki and Moore (2005), which can be financed with bank credit or retained earnings in

¹Firms' use of bank lines of credit is documented by Agarwal et al. (2011), the Survey of Small Business Finance (2003), and the Joint Small Business Credit Survey (2014) and Sufi (2009) for U.S. public firms. Small businesses rely more on bank credit than large firms due to information problems and less access to spot loans or capital markets (Boot 2000).

²Bernanke (1983) was the first to attribute the severity of the Great Depression to the lending channel. Economists have since debated the extent to which this channel can explain real economic effects. A recent study by Cohen, Hachem, and Richardson (2016) finds small bank failures explains one third of the economic contraction during the Great Depression.

³For a discussion on the effects of banking crises on macroeconomic activity and policy responses, see, e.g., Garcia-Herrero (1997), Demirguc-Kunt and Detragiache (1998) and Dell'Ariccia, Detragiache, and Rajan (2004). There is a large empirical literature that establishes that monetary policy impacts the supply of credit, e.g. Bernanke and Blinder (1992) and Kashyap et al. (1993).

⁴Borio et al. (2015) and Claessens et al. (2016) find banks' net interest margins are low when short term interest rates are low, both for the U.S. and across countries.

liquid assets. Credit can be obtained through long-term credit lines negotiated between banks and entrepreneurs. Due to search and informational frictions, captured by a matching function in the credit market, these relationships take time to form. Liquid assets are modeled as fiat money, the rate of return of which is controlled by the Central Bank.

We first study the determinants of the lending rate and the dynamics of lending relationships in a nonmonetary economy. Here, firms do not have the possibility to accumulate liquid assets to self-insure against investment opportunities — there are no liquid assets — and rely exclusively on external finance provided by banks. The lending rate, which is negotiated in pairwise meetings between banks and firms, increases with banks' bargaining power, the return on investment opportunities, and the termination rate of lending relationships. Similarly, the rate at which relationships are formed increases with banks' bargaining power, firms' arrival rate of investment opportunities, and the expected duration of credit lines.

We then consider monetary equilibria where internal and external finance coexist. In equilibrium, banked entrepreneurs do not hold cash since they can finance investments with bank credit alone (because banks have the technology to monitor loans and enforce repayments). In contrast, unbanked entrepreneurs do not have access to credit and retain some of their earnings in cash in order to finance investment opportunities as they arise. Firms' holdings of cash increase with the frequency of investment opportunities but decrease with the opportunity cost of retaining earnings in cash as measured by the nominal interest rate. A key feature of our model is that it generates a pass through from the *nominal* interest rate set by the policymaker to the *real* lending rate negotiated between banks and firms (in the absence of any nominal rigidities). As the nominal rate increases, it becomes more costly for entrepreneurs to self finance investments, which makes lending relationships more valuable and allows banks to ask for higher interest payments on loans, which in turn leads to a higher rate of lending relationship creation.

To the equilibrium dynamics, we parametrize the model to match moments in the U.S. corporate credit market from the 2003 National Survey of Small Business Finances (SSBF). We use the calibrated model to describe the economy's response to a negative banking shock described as an exogenous destruction of lending relationships. Under a constant money growth rate, the banking shock creates an increase in the aggregate demand for real balances as entrepreneurs lose access to credit and turn to liquid assets. The initial deflation generated by this flight to liquidity is followed by a positive inflation that gradually falls over time as the banking sector recovers and

the firms' demand for cash shrinks. The positive inflation rates lead to higher real lending rates, which increases banks' interest margins and the creation rate of lending relationships. In contrast, if the monetary authority targets a constant nominal interest rate, then the creation rate of new lending relations remains constant throughout the transition to steady state.

In models without distributional considerations (most of the New Monetarist literature), the Friedman rule, which sets the nominal interest rate permanently to zero, is typically optimal and makes credit inessential. To generate a role for banks, even at the Friedman rule, we assume entrepreneurs can finance a larger number of investment opportunities in a relationship with a bank. We justify this assumption by the fact that cash is an imperfect store of value — it is subject to theft or counterfeiting — and banks not only provide financing means but also bring entrepreneurs new investment opportunities. To make the Friedman rule suboptimal, we assume that banks' bargaining power is low so there is inefficient entry of banks due to search externalities in the credit market. Positive interest rates raise banks' margins and promote more creation of lending relationships, which is socially beneficial.

We then determine the socially optimal sequence of interest rates in the aftermath of a banking crisis under different assumptions on the power of the policymaker to commit. If the policymaker can commit over an infinite time horizon, then we solve the Ramsey problem recursively. The optimal policy consists in setting low nominal interest rates, close to the zero lower bound, at the outset of the crisis to promote internal finance by the newly unbanked firms. In order to maintain banks' incentives to participate in the credit market despite low interest rates, the policymaker uses "forward guidance" by promising high inflation and high nominal interest rates in the future.

While "forward guidance" is effective in trading off the need to provide insurance to unbanked firms during the crisis and the need to preserve banks' incentives to supply credit, it is not time consistent. Indeed, once lending relationships have been rebuilt, the policymaker faces the temptation to set low interest rates to maximize investment by unbanked firms. Therefore, we relax the commitment assumption and let the policymaker set the interest rate period by period taking as given future policies. At the time of the crisis, the policymaker does not lower the initial interest rate as much as it would under commitment in order to maintain banks' incentives to create lending relationships. The interest rate falls over time but by a small amount. The recovery is considerably slower relative to the scenario under forward guidance – by about two years in the calibrated model.

We conclude the paper by investigating alternative shocks to the supply and demand of bank

credit. We consider shocks to the matching technology to form lending relationships. For instance, an increase in informational asymmetries during a banking crisis can lower the acceptance rate of loan applications. If the market for new lending relationship temporarily shuts down, it is optimal to set the interest rate at its lower bound, zero. As the matching process in the credit market recovers, the interest rate increases. We also describe the optimal response to a lower rate of investment opportunities, which can be interpreted as a negative productivity shock to the economy. We calibrate these shocks using the Senior Loan Officer Opinion Survey.

1.1 Literature

Our model is part of the New Monetarist literature on the coexistence of money and credit. A recent treatment and literature review appears in Gu, Mattesini, and Wright (2015).⁵ Our approach to make money an imperfect substitute to banks' liabilities is different but related to Sanches and Williamson (2010) who focus on theft. New Monetarist models of banks include He, Huang, and Wright (2005), in which banks provide safe-keeping services, and Gu, Mattesini, Monnet, and Wright (2014) in which banks are essential to issue circulating liabilities. Most of this literature, however, focuses on credit to households and does not formalize lending relationships. Exceptions include Corbae and Ritter (2004) in a model with indivisible money and Nosal and Rocheteau (2011, Ch.8) in a model with divisible money. The description of the credit market with search frictions is analogous to Wasmer and Weil (2004).

The focus of this paper is on corporate finance. The closest paper is Rocheteau, Wright, and Zhang (2016) that studies transaction lenders who provide one-time loans under pledgeability constraints and regulatory requirements. Our paper is also related to the literature on firms' liquidity management, e.g., Holmstrom and Tirole (1998), DeMarzo and Fishman (2007), and Acharya, Almeida, and Campello (2013). The role of cash in corporate financing decisions is documented by a large body of empirical research, e.g., Almeida et al. (2004), Opler et al. (1999), and Sufi (2009). A key difference relative to this literature is our focus on optimal monetary policy.

The literature on relationship lending is surveyed by Boot (2000) and Elyasiani and Goldberg (2004). Evidence on the costs and benefits of relationship lending is described in Petersen and Rajan (1994, 1995) and Berger (1999).⁶ Hachem (2011) studies relationship lending and the transmission

⁵Surveys on this approach and more discussion of the literature can be found in Nosal and Rocheteau (2011) and Lagos, Rocheteau, and Wright (2015).

⁶This research finds access to relationship lending by firms and the loan rate charged by banks depends on market structure and the other details in the credit market we make explicit in the model. For instance, Petersen (1999)

of monetary policy where monetary policy is introduced as an exogenous cost of funds for lenders. In contrast, in our analysis monetary policy sets the rate of return of liquid assets (fiat money) and, by the Fisher equation, the nominal interest on bonds. Boualam (2016) studies relationship lending in a dynamic contracting problem but without money or monetary policy.

An important contribution of our paper is to determine the optimal monetary policy, described as a sequence of nominal interest rates, in an environment where the Friedman rule is not optimal. Related models with search externalities where the constrained-efficient allocation requires both the Friedman rule and Hosios condition, include Berentsen, Rocheteau, and Shi (2007) and Rocheteau and Wright (2005, 2009). Those models do not have credit, banks, and lending relationships, and they do not characterize the optimal monetary policy. Our recursive formulation of the Ramsey problem is related to Chang (1998). Aruoba and Chugh (2010) study optimal monetary and fiscal policies in the Lagos-Wright model when the policymaker has commitment. The approach of the problem without commitment is similar to Klein, Krusell, and Rios-Rull (2008), i.e., we focus on Markov-perfect equilibria. Martin (2011, 2013) studies fiscal and monetary policy absent commitment in a New Monetarist model where the government finances the provision of a public good with money, nominal bonds, and distortionary taxes. See also Díaz-Giménez, Giovannetti, Marimon, and Teles (2008) on monetary policy with nominal bonds and Dominguez (2007) on public debt and optimal taxes. Cooley and Quadrini (2004) study the optimal monetary policy with and without commitment in a monetary model with cash-in-advance budget constraints where the labor market features search-and-matching frictions.

2 Environment

Time is denoted by $t \in \mathbb{N}_0$. Each period is divided in three stages. In the first stage, there is a competitive market for capital goods. The second stage is a credit market with search frictions where agents can form long-term lending relationships. The last stage is a frictionless centralized market where agents trade money and consumption goods and settle debts. See Figure 1. The capital good k is storable across stages but not across periods. The consumption good c is taken as the numéraire.

There are three types of agents: *entrepreneurs* who need capital, *suppliers* who can produce

finds a negative correlation between competitiveness of banks in local loan markets and access to relationship lending by private firms. Boot (2000) also reports the negotiation of relationship loan contracts depends on the bargaining position of the bank relative to the borrower.



Figure 1: Timing of a representative period

capital, and *banks* who can finance the acquisition of capital by entrepreneurs as explained below. The population of entrepreneurs is normalized to one. Given CRS for the production of capital goods (see below), the population size of suppliers is immaterial. The population of active banks is endogenous and will be determined through free entry. All agents have linear preferences, $c - h$, where c is consumption of numéraire and h is labor hours.⁷ They discount across periods according to $\beta = 1/(1 + \rho)$, $\rho > 0$.

At the start of each period, an entrepreneur gets an investment opportunity with probability λ , in which case he can transform k into $f(k)$ units of c , where $f(0) = 0$, $f'(0) = \infty$, $f'(\infty) = 0$ and $f'(k) > 0 > f''(k) \forall k > 0$. We define $k^* = \arg \max_k \{f(k) - k\}$. Capital is produced by suppliers in the first stage with a linear technology, $k = h$. Entrepreneurs can also produce c using their labor in the last stage with a linear technology, $c = h$. (Note that this linear technology generates no net gain.) Banks cannot produce neither c nor k .

Entrepreneurs lack commitment and their trading histories are private. As a result, suppliers would not accept IOUs issued by entrepreneurs since they understand entrepreneurs could renege without fear of retribution.⁸ In contrast, banks can issue one-period liabilities, called notes, and can commit to repay them in the last stage.⁹ Moreover, banks can accept entrepreneurs' IOUs because they can enforce repayment by inflicting arbitrarily large punishments in case of default.

A bank can only extend a loan to an entrepreneur it is matched with. In spirit with Pissarides' (2000) one-firm-one-job assumption, a bank manages at most one lending relationship. (One can think of actual banks as a large collection of such relationships.) At the beginning of the second

⁷Alternatively, we could follow the New-Monetarist literature and adopt quasi-linear preferences of the form, $U(c) - h$. We could also assume that banks and suppliers consume the numéraire but cannot produce it while entrepreneurs consume and produce it.

⁸Alternatively, we could relax this assumption and allow for direct finance or trade credit in a fraction of matches where the entrepreneur can obtain a loan directly from the supplier. In addition, trade credit can be subject to a pledgeability constraint where only a fraction of the entrepreneur's output is pledgeable due to a moral hazard problem. See Rocheteau, Wright, and Zhang (2016) for a formalization in a related set up.

⁹Banks' liabilities cannot circulate across periods because they can be counterfeited at no cost in future periods. For additional details on the counterfeiting interpretation, see Nosal and Wallace (2007) and Li, Rocheteau, and Weill (2012).

stage, banks without a lending relationship decide whether to participate in the credit market at a cost, $\zeta > 0$. There is then a bilateral matching process between unbanked entrepreneurs and unmatched banks. The number of new lending relationships formed in the second stage of period t is $\alpha_t = \alpha(\theta_t)$, which depends on credit market tightness, θ_t , defined as the ratio of unmatched banks to unbanked entrepreneurs. We assume $\alpha(\theta)$ is increasing and concave, $\alpha(0) = 0$, $\alpha'(0) = 1$, $\alpha(\infty) = 1$, and $\alpha'(\infty) = 0$. Because matches are formed at random, the probability an entrepreneur matches with a bank is α_t , and the probability a bank matches with an entrepreneur is $\alpha_t^b = \alpha(\theta_t)/\theta_t$. An existing match is terminated at the start of the last stage with probability $\delta \in (0, 1)$, but newly-formed matches in the second stage are not subject to the risk of termination.

In addition to banks' short-term liabilities, there is another liquid asset, fiat money. Fiat money is storable and its supply, M_t , grows at the gross growth rate $\gamma \equiv M_{t+1}/M_t \geq \beta$. Changes in the money supply are implemented through lump-sum transfers or taxes to entrepreneurs at the start of the last stage. The value of money in the last stage of period t in units of numéraire is ϑ_t . We define $\pi_{t+1} = \vartheta_t/\vartheta_{t+1} - 1$ as the inflation rate.

An unbanked entrepreneur with an investment opportunity can finance his purchase of k with money with probability $\nu < 1$. With probability $1 - \nu$ his money holdings are not accepted by suppliers, e.g., because of the difficulty to authenticate money in the presence of informational frictions.¹⁰ The parameter ν can also be a proxy for corporate theft and embezzlement that reduce the possibility to use money to finance random investment opportunities. Yet another interpretation is that $\lambda\nu$ is the probability of an investment opportunity for an unbanked entrepreneur, which captures the idea that banks bring both financing and profit opportunities to entrepreneurs.¹¹ As we will see, $\nu < 1$ makes banks essential and, under some condition on banks' bargaining power, the Friedman rule suboptimal.

3 Pure banking equilibria

We begin by studying nonmonetary equilibria where bank liabilities are the only means of payment. In the first stage of each period, suppliers choose the amount of k to produce at a linear cost taking

¹⁰We assume ν is exogenous but can make partial acceptability endogenous along the lines of Lester, Postlewaite, and Wright (2012) and Li, Rocheteau, and Weill (2012).

¹¹The idea that cash is subject to theft, which makes credit desirable, was first formalized by Sanches and Williamson (2010). The idea that it takes time for agents to reach their targeted real balances was formalized in Rocheteau, Weill, and Wong (2015). Graham and Leary (2015) estimate the speed of adjustment towards cash targets is 20% per year.

its price in terms of numéraire, q_t , as given. Formally, they solve $\max_{k \geq 0} \{-k + q_t k\}$. If $q_t > 1$, the problem has no solution, and if $q_t < 1$, $k_t = 0$. Hence, if the capital market is active, $q_t = 1$.

In the second stage, lending relationships are formed where newly matched entrepreneurs and banks negotiate the terms of the loan contract. In a lending relationship, all actions are observable and the bank can enforce the terms of the contract. The contract is a sequence, $\{(k_t, \phi_t)\}_{t=0}^{\infty}$, that specifies for all dates: (i) a loan of size k_t the entrepreneur obtains from the bank if he requests it and (ii) a repayment ϕ_t to the bank in units of numéraire. Each component, (k_t, ϕ_t) , is interpreted as a loan conditional on an investment opportunity.¹² The terms of the loan contract are determined through bargaining between the two parties.

To determine the bargaining outcome, we first compute the entrepreneur's surplus from a lending relationship. The lifetime expected utility of an unbanked entrepreneur at the beginning of period t is

$$U_t^e = \alpha_t \beta Z_{t+1}^e + (1 - \alpha_t) \beta U_{t+1}^e, \quad (1)$$

where Z_t^e is the lifetime utility of an entrepreneur in a lending relationship at the beginning of period t . An unbanked entrepreneur must pass on any investment opportunity in the first stage of period t . Hence, his profits are zero. In the second stage, he enters a lending relationship with probability α_t and remains unmatched with probability $1 - \alpha_t$.

The lifetime expected utility of an entrepreneur in a lending relationship solves

$$Z_t^e = \lambda [f(k_t) - k_t - \phi_t] + \delta \beta U_{t+1}^e + (1 - \delta) \beta Z_{t+1}^e. \quad (2)$$

The entrepreneur receives an investment opportunity with probability λ , in which case he can draw from his credit line to finance k_t in exchange for an interest payment ϕ_t . The lending relationship is terminated at the start of the last stage with probability δ , in which case his continuation value is βU_{t+1}^e . The relationship is maintained with probability $1 - \delta$, in which case his continuation value is βZ_{t+1}^e .

The entrepreneur's surplus from entering a lending relationship in period $t - 1$ is βS_t^e where $S_t^e \equiv Z_t^e - U_t^e$. From (2),

$$S_t^e = \lambda [f(k_t) - k_t - \phi_t] - (U_t^e - \beta U_{t+1}^e) + (1 - \delta) \beta S_{t+1}^e. \quad (3)$$

¹²There are many lending contracts that are payoff equivalent. Loan contracts in practice include a commitment fee where the bank charges the firm for setting up a credit line, including charges and penalties if the firm exceeds the line limit or violates contract terms (see e.g. Agarwal et al 2011). In our model, the absence of agency or incentive problems means all that matters for determining the loan contract is the discounted sum of payments to the bank.

The first term on the right side of (3) is the entrepreneur's expected profits from an investment opportunity. The second term is his outside option, $U_t^e - \beta U_{t+1}^e$, which is the flow value of being unbanked. The last term is the discounted surplus from being in a lending relationship in $t + 1$ if maintained with probability $1 - \delta$.

Similarly, U^b is defined as the bank's lifetime profits of being unmatched and Z^b as the bank's lifetime value of being in a lending relationship. They solve

$$U_t^b = -\zeta + \alpha_t^b \beta Z_{t+1}^b + (1 - \alpha_t^b) \beta \max \{U_{t+1}^b, 0\}, \quad (4)$$

$$Z_t^b = \lambda \phi_t + \delta \beta \max \{U_{t+1}^b, 0\} + (1 - \delta) \beta Z_{t+1}^b. \quad (5)$$

From (4), an unmatched bank incurs ζ at the start of the second stage to participate in the credit market; there, the bank is matched with an entrepreneur with probability α_t^b and remains unmatched with probability $1 - \alpha_t^b$. From (5), the bank's expected profits are composed of expected interest payments, $\lambda \phi_t$. At the beginning of the last stage, the lending relationship is terminated with probability δ and is maintained with probability $1 - \delta$.

The surplus of a bank from being in a lending relationship in period $t - 1$ is βS_t^b where $S_t^b \equiv Z_t^b - \max \{U_t^b, 0\}$. Free entry of banks in the credit market means $U_t^b = 0$. From (5),

$$S_t^b = \lambda \phi_t + \beta(1 - \delta) S_{t+1}^b. \quad (6)$$

From (2) and (6), the total surplus from a lending relationship, $S_t \equiv S_t^b + S_t^e$, solves

$$S_t = \lambda[f(k_t) - k_t] - (U_t^e - \beta U_{t+1}^e) + \beta(1 - \delta) S_{t+1}. \quad (7)$$

Since ϕ_t transfers utility perfectly between the entrepreneur and the bank, all Pareto-efficient contracts maximize S_t . From (7), this implies $k_t = k^*$ for all t . So the lending relationship implements the first-best production of capital goods.

Next we determine the interest payments to the bank, $\{\phi_{t+\tau}\}_{\tau=1}^\infty$, by assuming generalized Nash bargaining. In our context with transferable utility, this means $(1 - \eta) S_{t+1}^b = \eta S_{t+1}^e$, or equivalently, $S_{t+1}^b = \eta S_{t+1}$, where $\eta \in (0, 1)$ is the bank's share of the match surplus. Notice this solution only pins down $\sum_{\tau=1}^\infty \beta^\tau \phi_{t+\tau}$. To determine the full sequence of intermediation fees uniquely, we refine the bargaining solution by imposing $S_{t+\tau}^b = \eta S_{t+\tau}$ to hold throughout the lending relationship. Substituting S_t from (7) into $S_t^b = \eta S_t$ and solving for ϕ_t , we obtain

$$\phi_t = \eta [f(k^*) - k^*] - \frac{\eta}{\lambda} (U_t^e - \beta U_{t+1}^e). \quad (8)$$

From (8), the interest payment is composed of a share in the profits from an investment opportunity, $f(k^*) - k^*$, net of the entrepreneur's flow value of being unbanked, $U_t^e - \beta U_{t+1}^e$, which arises from the fact that the entrepreneur renounces on the option to look for another bank. (Recall that by assumption that there is no search while being matched.) From (1),

$$U_t^e - \beta U_{t+1}^e = \alpha_t \beta S_{t+1}^e = \frac{1 - \eta}{\eta} \theta_t \zeta.$$

Substituting in (8), we compute the real interest rate on a loan as $r_t = \phi_t / k_t$:

$$r_t = \frac{\eta [f(k^*) - k^*] - (1 - \eta) \theta_t \zeta / \lambda}{k^*}. \quad (9)$$

The lending rate depends negatively on θ_t . A tighter credit market increases the flow value of being unbanked, which allows the entrepreneur to negotiate a lower r_t . For a given θ_t , $\partial r_t / \partial \lambda > 0$ and $\partial r_t / \partial \eta > 0$.

Substituting $U_t^b = 0$ and ϕ_t from (8) into (5), and assuming positive entry, credit market tightness solves

$$\frac{\theta_{t-1}}{\alpha(\theta_{t-1})} = \frac{\beta \lambda \eta [f(k^*) - k^*]}{\zeta} - \beta (1 - \eta) \theta_t + \beta (1 - \delta) \frac{\theta_t}{\alpha(\theta_t)}. \quad (10)$$

The difference equation (10) gives the credit market tightness in period $t - 1$ taking into account the perfectly foreseen expected market tightness in period t .

The measure of lending relationships at the beginning of a period evolves according to

$$\ell_{t+1} = (1 - \delta) \ell_t + \alpha_t (1 - \ell_t). \quad (11)$$

The number of lending relationships at the beginning of $t + 1$ equals the measure of lending relationships at the beginning of t that have not been severed, $(1 - \delta) \ell_t$, plus newly created relationships, $\alpha_t (1 - \ell_t)$. An active non-monetary equilibrium is a bounded sequence, $\{\theta_t, \ell_t, r_t\}_{t=0}^\infty$, that solves (9), (10), and (11) for a given ℓ_0 .

A steady state equilibrium is a triple, (θ, ℓ, r) , that solves

$$(\rho + \delta) \frac{\theta}{\alpha(\theta)} + (1 - \eta) \theta = \frac{\lambda \eta [f(k^*) - k^*]}{\zeta}, \quad (12)$$

$$\ell = \frac{\alpha(\theta)}{\delta + \alpha(\theta)}, \quad (13)$$

$$r = \frac{\eta [f(k^*) - k^*] - (1 - \eta) \theta \zeta / \lambda}{k^*}. \quad (14)$$

Credit market tightness is obtained uniquely from (12). Given θ , ℓ is given by (13) and r is given by (14).

Proposition 1 (*Steady-state nonmonetary equilibrium*) *There exists a unique steady-state nonmonetary equilibrium. It features $\theta > 0$ if and only if*

$$(\rho + \delta)\zeta < \lambda\eta[f(k^*) - k^*]. \quad (15)$$

Comparative statics are summarized below:

	η	λ	ζ	δ
θ	+	+	-	-
ℓ	+	+	-	-
r	+	-	+	+

From the table above, ℓ and r covary negatively following changes in λ , ζ , and δ , and they covary positively following a change in η . An increase in λ raises the value of lending relationships, which generates more bank entry. Higher competition in the credit market drives the lending rate down. The opposite comparative statics hold for an increase in either ζ or δ since costlier search and a higher likelihood of a severed credit match lowers the value of being in a lending relationship.

Starting from any initial condition ℓ_0 , there exists an equilibrium where $\theta_t = \theta$ given by (12). Moreover, equilibrium is unique if $\partial\theta_{t-1}/\partial\theta_t \in (-1, 1)$, evaluated at the steady state. From (10),

$$\frac{\partial\theta_{t-1}}{\partial\theta_t} = \beta(1 - \delta) - \frac{\beta(1 - \eta)\alpha(\theta)}{1 - \varepsilon(\theta)},$$

where $\varepsilon(\theta) \equiv \theta\alpha'(\theta)/\alpha(\theta)$. When $\varepsilon(\theta) = \eta$, which is the Hosios condition, the condition for uniqueness holds. It also holds if the time period is small so that $\alpha(\theta)$ and δ are small. From (11), the measure of lending relationships solves

$$\ell_t = \ell + (\ell_0 - \ell)[1 - \delta - \alpha(\theta)]^t, \quad (16)$$

where ℓ and $\alpha(\theta)$ are steady state values. Suppose the economy begins in steady state and receives an unanticipated shock that reduces the measure of lending relationships to ℓ_0 . This shock has long lasting effects since rebuilding lending relationships is a time-consuming process. However, tightness in the credit market is unaffected.

4 Liquidity and lending relationships

We now turn to monetary equilibria with both internal and external finance. If fiat money is introduced and valued, then unbanked entrepreneurs can accumulate liquid assets with retained

earnings and finance k_t^m if an investment opportunity arises. Moreover, the loan contract in a lending relationship can now specify a down payment in real balances, d_t .

The lifetime expected value of an unbanked entrepreneur holding m real balances (units of money in terms of numéraire) in the last stage of period t is

$$W_t^e(m) = m + T_t + \max_{m_{t+1} \geq 0} \{-(1 + \pi_{t+1})m_{t+1} + \beta U_{t+1}^e(m_{t+1})\}, \quad (17)$$

where T_t is the lump-sum transfer in terms of numéraire and $U_t^e(m)$ is the value of an unbanked entrepreneur at the beginning of period t with m real balances. According to (17), the entrepreneur accumulates $(1 + \pi_{t+1})m_{t+1}$ real balances in order to hold m_{t+1} at the start of period $t+1$. Similarly, the lifetime utility of a banked entrepreneur with m real balances in the last stage of period t is

$$X_t^e(m) = m + T_t + \max_{m_{t+1} \geq 0} \{-(1 + \pi_{t+1})m_{t+1} + \beta Z_{t+1}^e(m_{t+1})\}. \quad (18)$$

Both $W_t^e(m)$ and $X_t^e(m)$ are linear in real balances.

Entrepreneur's choice of real balances The lifetime expected utility of an unbanked entrepreneur at the beginning of the second stage (credit market) solves:

$$\begin{aligned} V_t^e(m) &= \alpha_t X_t^e(m) + (1 - \alpha_t) W_t^e(m) \\ &= m + \alpha_t X_t^e(0) + (1 - \alpha_t) W_t^e(0). \end{aligned} \quad (19)$$

With probability α_t , an unmatched entrepreneur enters a lending relationship and, with probability $1 - \alpha_t$, he proceeds to the last stage unmatched. From (19), $V_t^e(m)$ is linear in m .

The problem of an unbanked entrepreneur in the capital market upon receiving an investment opportunity is

$$k_t^m \in \arg \max_{k_t \geq 0} \{f(k_t) + V_t^e(m_t - k_t)\} \quad \text{s.t.} \quad k_t \leq m. \quad (20)$$

The entrepreneur's purchase of k_t^m is bounded above by his real balances. Using the linearity of V_t^e , $f(k_t^m) + V_t^e(m_t - k_t^m) = f(k_t^m) - k_t^m + V_t^e(m_t)$. The solution to (20) is $k_t^m = k^*$ if $m \geq k^*$ and $k_t^m = m$ otherwise. As a result, the lifetime utility of an unbanked entrepreneur at the beginning of period t is

$$U_t^e(m_t) = \lambda \nu [f(k_t^m) - k_t^m] + V_t^e(m_t). \quad (21)$$

With probability λ , the entrepreneur receives an investment opportunity where he purchases k_t^m in the fraction ν of trades where money is accepted.

Substituting $U_t^e(m_t)$ from (21) into (17) and using the linearity of V_t^e , an unmatched entrepreneur's choice of real balances solves

$$\Delta_t \equiv \max_{m_t \geq 0} \{-i_t m_t + \lambda \nu [f(k_t^m) - k_t^m]\}, \quad (22)$$

where $i_t \equiv (1 + \pi_t - \beta)/\beta$ is the nominal interest rate on an illiquid bond. The FOC associated with (22) is

$$i_t = \lambda \nu [f'(k_t^m) - 1]. \quad (23)$$

Lending contract. We now turn to the terms of the contract in a lending relationship. The contract negotiated at time $t - 1$ specifies $\{k_{t+\tau}, \phi_{t+\tau}, d_{t+\tau}\}_{\tau=0}^{\infty}$ where $d_{t+\tau}$ is the down-payment in real balances and $k_{t+\tau} - d_{t+\tau}$ is interpreted as the loan size. We assume the entrepreneur can commit to the terms of the contract, which requires $m_{t+\tau} \geq d_{t+\tau}$. The entrepreneur's surplus from being in a lending relationship in the third stage of $t - 1$ is defined as the difference between the discounted value of being in a lending relationship, $X_{t-1}^e(m)$, and the value of not being in a lending relationship, $W_{t-1}^e(m)$. This surplus is denoted as βS_t^e where $S_t^e = [X_{t-1}^e(0) - W_{t-1}^e(0)] / \beta$.

The lifetime expected utility of a banked entrepreneur with $m \geq d_t$ real balances solves

$$Z_t^e(m) = \lambda [f(k_t) - k_t - \phi_t - d_t] + \delta W_t^e(m) + (1 - \delta) X_t^e(m). \quad (24)$$

With probability λ , the entrepreneur receives an investment opportunity. In accordance with the terms of the lending contract, the size of the investment is k_t , which is financed with d_t real balances and a loan with interest payment ϕ_t . The lending relationship is destroyed with probability δ , in which case the value of the entrepreneur in the last stage is $W_t^e(m)$. Otherwise, the continuation value of the entrepreneur is $X_t^e(m)$. For all $m \geq d_t$, $Z_t^e(m) = m + Z_t^e(0)$. From (18), it follows that $m = d_t$ if $1 + \pi_t > \beta$. In words, if money is costly to hold, a banked entrepreneur does not carry more money than what is needed to honor the terms of the lending contract. In the following we denote $Z_t^e = Z_t^e(d_t)$.

Following the same algebra as in the previous section the surplus of a banked entrepreneur solves

$$S_t^e = -i_t d_t + \lambda [f(k_t) - k_t - \phi_t - d_t] - [\Delta_t + V_t^e - W_t^e(0)] + (1 - \delta) \beta S_{t+1}^e. \quad (25)$$

The first two terms on the right side represent the expected profit of the entrepreneur from an investment opportunity net of the cost of holding the real balances required for the down payment.

The surplus of the bank solves (6), $S_t^b = \lambda\phi_t + \beta(1 - \delta)S_{t+1}^b$. Hence, the total surplus of a lending relationship, $S_t = S_t^e + S_t^b$, solves:

$$S_t = -i_t d_t + \lambda[f(k_t) - k_t] + W_t^e(0) - (\Delta_t + V_t^e) + (1 - \delta)\beta S_{t+1}.$$

Any Pareto-efficient contract maximizes the joint surplus, which implies $k_{t+\tau} = k^*$ and $d_{t+\tau} = 0$ for all τ . Given that banks can finance the first-best level of investment contracts specifying positive down payments are Pareto inefficient.¹³ An implication of our model is that banked entrepreneurs do not hold real balances while unbanked entrepreneur do hold real balances. As before, interest payments, $\{\phi_{t+\tau}\}_{\tau=1}^{\infty}$, are determined so that $(1 - \eta)S_{t+\tau}^b = \eta S_{t+\tau}^e$ holds at all dates during the relationship. After some calculation,

$$\phi_t = \eta[f(k^*) - k^*] - \frac{(1 - \eta)\theta_t \zeta + \eta\Delta_t}{\lambda}. \quad (26)$$

The real interest rate on a loan is

$$r_t = \frac{\eta\lambda[f(k^*) - k^*] - (1 - \eta)\theta_t \zeta - \eta\Delta_t}{\lambda k^*}. \quad (27)$$

Given θ_t , an increase in the nominal interest rate i_t raises r_t by reducing Δ_t . Intuitively, higher i_t lowers the entrepreneur's net profits from internal finance, which allows the bank to charge a higher r_t .

Credit market tightness. To determine credit market tightness, we substitute ϕ_t from (26) into (5). From free entry of banks, $Z_{t+1}^b = \zeta\theta_t/\beta\alpha_t$ and hence $\{\theta_t\}_{t=0}^{\infty}$ solves the first order difference equation:

$$\frac{\theta_t}{\alpha(\theta_t)} = \beta\eta \left\{ \frac{\lambda[f(k^*) - k^*] - \Delta_{t+1}}{\zeta} \right\} - \beta(1 - \eta)\theta_{t+1} + \beta(1 - \delta)\frac{\theta_{t+1}}{\alpha(\theta_{t+1})}. \quad (28)$$

An increase in Δ_t reduces ϕ_t and hence bank profits and their incentives to enter the credit market. This provides a channel for monetary policy to affect credit market outcomes.

Equilibrium. Market clearing implies the demand for real balances from the $1 - \ell_t$ unbanked entrepreneurs must equal the aggregate supply of real balances:

$$(1 - \ell_t)k_t^m = \vartheta_t M_t. \quad (29)$$

¹³It is key for our result that the terms of the lending contract are negotiated before investment opportunities occur and agents can commit to them. In Rocheteau, Wright, and Zhang (2016) banks and entrepreneurs cannot form lending relationships and therefore cannot commit on the terms of long-term lending contracts. In this case, entrepreneurs hold real balances to improve their outside option and to negotiate better terms for their loans.

From (29), the rate of return of money can be expressed as

$$\frac{\vartheta_{t+1}}{\vartheta_t} = \frac{M_t}{M_{t+1}} \frac{k_{t+1}^m}{k_t^m} \frac{1 - \ell_{t+1}}{1 - \ell_t}. \quad (30)$$

The rate of return of money decreases with the money growth rate, $\gamma = M_{t+1}/M_t$, and increases with the growth rate of the unbanked sector, $(1 - \ell_{t+1})/(1 - \ell_t)$. A monetary equilibrium is a bounded sequence, $\{\theta_t, \ell_t, k_t^m, r_t, \vartheta_t\}_{t=0}^\infty$, that solves (11), (23), (29), (27), and (28) for a given ℓ_0 .

In steady state, credit market tightness is the unique solution to

$$(\rho + \delta) \frac{\theta}{\alpha(\theta)} + (1 - \eta) \theta = \frac{\lambda \eta [f(k^*) - k^*] - \eta \Delta}{\zeta}. \quad (31)$$

Given θ , closed-form solutions for $(\ell, k^m, r, \vartheta)$ are:

$$\ell = \frac{\alpha(\theta)}{\delta + \alpha(\theta)} \quad (32)$$

$$k^m = f'^{-1} \left(1 + \frac{i}{\lambda \nu} \right) \quad (33)$$

$$r = \frac{\eta \lambda [f(k^*) - k^*] - (1 - \eta) \theta \zeta - \eta \Delta}{\lambda k^*} \quad (34)$$

$$\vartheta M = \frac{\delta}{\delta + \alpha(\theta)} f'^{-1} \left(1 + \frac{i}{\lambda \nu} \right). \quad (35)$$

Proposition 2 (*Transmission of monetary policy.*) *There exists a unique steady-state monetary equilibrium. It features an active credit market if and only if*

$$(\rho + \delta) \zeta < \lambda \eta [f(k^*) - k^*] - \eta \Delta(i). \quad (36)$$

If $\eta > 0$, an increase in i raises θ and r , but lowers k^m and ϑM . In the neighborhood of $i = 0$,

$$\frac{\partial \theta}{\partial i} = \frac{\eta k^*}{\zeta \{(\rho + \delta) [1 - \epsilon(\theta)] / \alpha(\theta) + 1 - \eta\}} \geq 0 \quad (37)$$

$$\frac{\partial r}{\partial i} = \frac{(\rho + \delta) \eta}{\lambda \{(\rho + \delta + (1 - \eta) \alpha(\theta) / [1 - \epsilon(\theta)])\}} \geq 0, \quad (38)$$

where $\epsilon(\theta) = \alpha'(\theta) \theta / \alpha(\theta)$.

The model delivers a pass through from the nominal policy rate to the real lending rate. An increase in i has two effects on r . First, higher i reduces Δ which tends to increase r . In words, an increase in i raises the opportunity cost of holding liquid assets. As a result, unbanked entrepreneurs reduce their money holdings. The outside option of entrepreneurs worsens, which allows banks to charge a higher r . Second, θ increases, thereby raising competition among banks, which tends to lower r . In the proof of Proposition 2, we show the first effect dominates.

Suppose that the matching function takes the form $\alpha(\theta) = \bar{\alpha}\theta/(1 + \theta)$. From (31), we can solve for market tightness in closed form. Assuming an interior solution,

$$\theta\zeta = \frac{\bar{\alpha}\lambda\eta[f(k^*) - k^*] - \bar{\alpha}\eta\Delta - (\rho + \delta)}{\bar{\alpha}(1 - \eta) + \rho + \delta}.$$

Substituting $\theta\zeta$ by its expression into (34),

$$r = \frac{\eta\lambda[f(k^*) - k^*] + (1 - \eta)\zeta - \eta\Delta}{\lambda k^* [\bar{\alpha}(1 - \eta)/(\rho + \delta) + 1]}.$$
 (39)

Hence, the pass-through rate is

$$\frac{\partial r}{\partial i} = \frac{\eta}{\lambda [\bar{\alpha}(1 - \eta)/(\rho + \delta) + 1]} \frac{k^m}{k^*}.$$
 (40)

The pass-through rate is strictly positive provided that $\eta > 0$. It increases with η and decreases with λ . It is higher for low interest rates since k^m is a decreasing function of i . Finally, the pass-through rate increases with δ . Hence, the more stable the lending relationships the lower the pass-through.¹⁴

5 Calibration

We now describe the dynamic response of the economy to a banking shock modeled as an exogenous destruction of lending relationships starting from a steady state. To illustrate these dynamics, we calibrate the model with data on small businesses in the U.S. economy.

The period length is a month and $\rho = 1.04^{1/12} - 1$. The production function is $f(k) = k^a$ with $a = 0.75$.¹⁵ Hence, $k^* = a^{\frac{1}{1-a}} = 0.316$. We adopt the matching function commonly used in the New Monetarist literature, $\alpha(\theta) = \bar{\alpha}\theta/(1 + \theta)$, where $\bar{\alpha} \in [0, 1]$. This matching function has the property that the contribution of agents to the matching process (as measured by elasticities) is equal to their shares in the overall population. The parameters to calibrate are $(\bar{\alpha}, \delta, \lambda, \nu, \eta, \zeta)$. We obtain these parameters by matching targets from the 2003 National Survey of Small Business Finances (SSBF).

We define a credit relationship broadly to include a credit line, business credit card, or owner credit card used for business purposes.¹⁶ A fraction 84.9% of small businesses report such a credit

¹⁴Cohen, Hachem, and Richardson (2016) use similar comparative statics to measure the intensity of relationship lending during the Great Depression.

¹⁵It turns out that the choice of a is important when calibrating the model to obtain plausible values for the acceptability of monetary assets, ν . For instance, for $a = 1/3$ we obtained $\nu = 0.23$.

¹⁶The SSBF defines a credit line as an arrangement with a financial institution that allows a firm to borrow funds during a specified period up to a specific credit limit. In a different sample of larger firms, Sufi (2009) finds 74.5% of U.S. public, non-financial firms have access to lines of credit provided by banks.

Table 1: Parameter values

Value	Target
$\rho = 1.04^{1/12} - 1$	Annual discount rate = 4%
$\bar{\alpha} = 0.0934$	Share of firms with access to credit = 84.9%, SSBF (2003)
$\delta = 0.009$	Duration of lending relationship = 111.5 months, SSBF (2003)
$\lambda = 0.037$	Credit utilization rate = 44.9%, SSBF (2003)
$\nu = 0.694$	Cash to credit ratio = 55.5%, SSBF (2003)
$\eta = 0.1016$	Annual pass through rate, 1994 to 2003 = 19.21%, FRED
$\zeta = 1.6641 \times 10^{-4}$	Annual lending rate, 1994 to 2003 = 9.78%, FRED

relationship. In our model, $\ell = \alpha/(\alpha + \delta) = 0.849$. The average duration of credit relationships is 9.3 years. This gives $\delta = 0.009$ and $\alpha = \delta\ell/(1 - \ell) = 0.0504$, i.e., it takes about 1.65 years to form a new lending relationship.

The average annual credit utilization rate, calculated as the amount drawn on current credit lines over the total available credit line, is 44.9%.¹⁷ To match this moment, we make two assumptions. First, the total credit line corresponds to the optimal investment size, k^* , since it corresponds to the amount that the bank and the entrepreneur negotiate in an optimal contract. Second, loans are repaid within a period.¹⁸ Hence, the total available credit line is ℓk^* and the amount drawn on credit lines is $\ell\lambda k^*$. It follows that the credit utilization rate, $\ell\lambda k^*/\ell k^*$, reduces to the arrival probability of investment opportunities, λ . Hence, 44.9% of businesses receive an investment opportunity within a year, which gives $\lambda = 0.449/12 = 0.037$.

We take the 3 month Treasury Bill secondary market rate as the interest rate targeted by monetary policy, which averages 4.9% per year from 1994 to 2003. Hence, $i = 0.049/12 = 0.0041$. The ratio of average cash holdings for unbanked small businesses to the average credit line for businesses in a lending relationship, k^m/k^* in our model, is 55.5%.¹⁹ Hence, $\nu = i/\lambda[(k^m/k^*)^{a-1} - 1] = 0.69$. So unbanked firms miss out approximately 30% of investment opportunities by not being

¹⁷Since firms in the SSBF are heterogenous in size, we scale the related credit variables with firm size proxied by the net book value.

¹⁸Suppose the aggregate stock of bank loans is L and firms repay a fraction ϱ of their outstanding balances each period. The law of motion for the stock of bank loans is $\Delta L_t = \lambda\ell_t k^* - \varrho L_t$. In steady state, $L = \lambda\ell k^*/\varrho$, and the credit utilization rate is $L/\ell k^* = \lambda/\varrho$. We assume $\varrho = 100\%$ over a year.

¹⁹An advantage of the SSBF is that we directly obtain average cash holdings conditional on being an unbanked firm and average credit line conditional on being a banked firm. The SSBF defines cash as the total amount of cash on hand (currency and coin used in operations, petty cash, etc.), balances in business checking accounts, and total balances of all savings accounts, money market accounts, time deposits, and certificates of deposit. In addition, some cash, like petty cash held in cashiers and minimal required balances in bank accounts, is needed to run a business no matter if an investment opportunity is available or not. Thus in our model, cash holdings for unbanked firms' investment is the cash holding per firm net of operational cash.

in a lending relationship.

The bank's bargaining power, η , is chosen to target the pass through rate. We match the lending rate in the model to the prime loan rate, which is the best rate offered by banks to those with minimal concern of default. In our model, bank loans are repaid within a period. If loans were repaid with one period lag, then the interest on the loan would be r^1 such that $\phi = \beta(1 + r^1)k^*$. This gives $r = (1 + r^1)/(1 + \rho) \approx r^1 - \rho$. So we interpret r as the difference between the nominal prime loan rate and nominal interest rate on government bonds. (Alternatively, if the government were to issue (illiquid) bonds at the start of a period and redeem then at the end of the period, then those bonds would no bear interest). Regressing the 3 month Treasury Bill secondary market rate on the prime loan rate we obtain $\partial r/\partial i = 0.1921$. See Figure 2 where we plot the monthly series of the 3-month Treasury Bill rates and the spread of bank prime loan rates in the data. From (40) $\eta = 0.1016$.

We set banks' entry cost, ζ , to match the average spread between the nominal prime loan rate and nominal interest rate is 4.87% per year from 1994 to 2003, $r = 0.0487/12 = 0.0041$. Using (39), $\zeta = 1.6641 \times 10^{-4}$. In order to illustrate the fit of our model we plot in Figure 2 the theoretical relationship between r and i for the calibrated values for η and ζ .

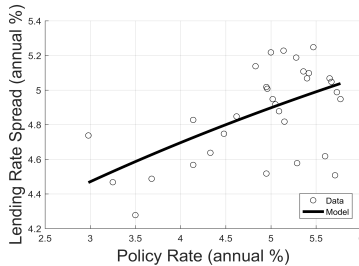


Figure 2: Pass-through estimation

We are now in position to use our calibrated model to study the response of the economy to a shock that destroys lending relationships. The size of the shock matches a measure of corporate credit contraction during the last financial crisis. Ivashina and Scharfstein (2010) report the number of bank loans decreased on average by 59% during the peak of the U.S. banking crisis of 2008. We attribute this credit crunch to the destruction of credit lines only. This gives $\ell_0 = 0.85 \times (1 - 0.59) = 0.35$, i.e., the measure of entrepreneurs with a line of credit falls from 85% to 35%.²⁰ In Figure 3,

²⁰Ivashina and Scharfstein (2010) provide evidence on syndicated loans during the 2008 financial crisis. While these loans are typically made to large firms, small and medium business were likely to be hit even harder than large firms during the crisis (Mills and McCarthy 2014).

we consider two policy rules: a constant money growth rate equal to its calibrated value, 0.9% per year (solid blue line); a constant nominal interest rate equal to its calibrated value, 4.9% per year (dashed black line).

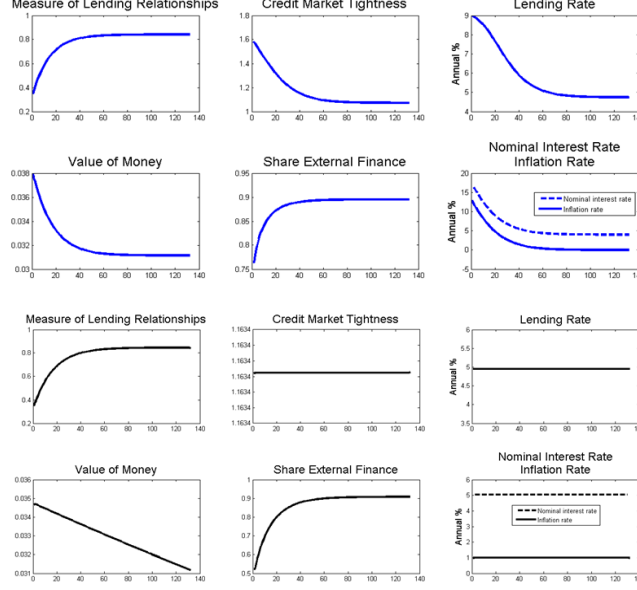


Figure 3: Effects of banking shock under two monetary policy rules

When the money growth rate is constant, the value of money ϑ_t jumps above its value in a stationary equilibrium since the aggregate demand for real balances by unbanked entrepreneurs increases. The initial deflation is followed by inflation at a higher rate than the money growth rate which subdues over time as the demand for money falls. The lending rate, r_t , jumps above its steady state value since inflation reduces the entrepreneur's profits from internally financed investment opportunities. Higher interest margins in turn promote bank entry. As lending relations recover, both r and θ fall gradually over time. When the nominal interest rate is constant, ϑ_t falls at a constant rate over time while market tightness and the real lending rate are constant. The share of external finance falls initially as lending relationships are destroyed then increases as credit recovers. The constant money growth rate implements a faster recovery than the constant interest rate.

6 Optimal monetary policy in the aftermath of a banking crisis

So far we have studied arbitrary policies that consist in keeping either the money growth rate or the nominal interest rate constant following the exogenous destruction of a fraction of the lending relationships from ℓ to $\ell_0 < \ell$. We now characterize the optimal monetary policy described as a sequence of nominal interest rates chosen by the policymaker. We will consider two polar cases regarding the policymaker's ability to commit. In the first case the policymaker can announce and commit to an infinite sequence of interest rates. We view this case as a benchmark. In the second case we take away the policymaker's power to commit to future interests. Instead, the policymaker reoptimizes its policy every period by setting a new interest rate taking as given the policies of future policymakers.

6.1 Commitment and forward guidance

Before the credit market opens (in the second stage) the policymaker announces a sequence of interest rates, $\{i_t\}_{t=1}^{\infty}$, and commits to it. We measure society's welfare, $\mathcal{W}$, starting in the second stage of $t = 0$ after the announcement is made,

$$\mathcal{W} = \sum_{t=0}^{\infty} \beta^t \left\{ -\zeta \theta_t (1 - \ell_t) + \beta (1 - \ell_{t+1}) \lambda \nu [f(k_{t+1}^m) - k_{t+1}^m] + \beta \ell_{t+1} \lambda [f(k^*) - k^*] \right\}. \quad (41)$$

It is the sum of profits arising from investment opportunities net of bank entry costs. The policymaker chooses a sequence of interest rates, $\{i_t\}$, to maximize $\mathcal{W}$, taking into account that the equilibrium allocation is a function of $\{i_t\}$, i.e.,

$$\widetilde{\mathcal{W}}(\ell_0) = \max_{\{\theta_t\}_{t=1}^{\infty}} \mathcal{W} \text{ s.t. } \{\theta_t, \ell_t, k_{t+1}^m\}_{t=0}^{\infty} \text{ being an equilibrium given } \ell_0. \quad (42)$$

Since there is a one-to-one relation between i_t and k_t^m , i.e., $i_t = \lambda \nu [f'(k_t^m) - 1]$, the policymaker effectively chooses a path for k_t^m . When making this choice, the policymaker faces a tradeoff between providing insurance to unbanked entrepreneurs at time t by setting low interest rates, i.e., high k^m , and promoting entry of banks by setting high interest rates, i.e., low k^m .

We first establish the conditions under which the Friedman rule, which consists of setting $i_t = 0$ for all t , is suboptimal. We denote $\underline{\theta}$ the credit market tightness at the Friedman rule and $\varepsilon(\underline{\theta}) \equiv \alpha'(\underline{\theta})\underline{\theta}/\alpha(\underline{\theta})$ the associated elasticity of the matching function.

Proposition 3 (*Suboptimality of the Friedman rule.*) *Suppose*

$$\zeta < \frac{\alpha'(0)\lambda\eta(1-\nu)[f(k^*) - k^*]}{\rho + \delta}. \quad (43)$$

If $\eta < \varepsilon(\underline{\theta})$, a deviation from the Friedman rule is optimal, i.e., the optimal monetary policy does not require $i_t = 0$ for all t .

The condition (43) guarantees the credit market at the Friedman rule is active, $\underline{\theta} > 0$. A necessary condition for (43) to hold is that money is partially acceptable. If $\nu = 1$, banks have no incentive to enter since entrepreneurs can self insure perfectly provided that $i_t = 0$ for all t . If $\nu < 1$, money only provides partial insurance so that banks have a role to play even when the Friedman rule is implemented. Provided that banks' bargaining power, η , is less than their contribution to the matching process in the credit market as measured by $\varepsilon(\underline{\theta})$, i.e., the Hosios condition is violated, it is socially beneficial to have positive nominal interest rates. Indeed, if $\eta < \varepsilon(\underline{\theta})$, banks' entry is inefficiently low as they fail to internalize the surplus they provide to entrepreneurs. In this case positive nominal rates raise bank profits and promote entry. From now on, we suppose the conditions in Proposition 3 hold.

We now describe a recursive formulation of the Ramsey problem in (42). To simplify the exposition, we adopt the functional forms $f(k) = Ak^a$, $\alpha(\theta) = \bar{\alpha}\theta/(1 + \theta)$, and the parametric condition $\delta + \bar{\alpha}(1 - \eta) < 1$. The policymaker takes as a constraint the relationship between current and future market tightness given by (28):

$$\theta_t = \frac{\bar{\alpha}\beta\eta\lambda(1-a)A}{\zeta} \left[(aA)^{\frac{a}{1-a}} - \nu(k_{t+1}^m)^a \right] + \beta[1 - \delta - \bar{\alpha}(1 - \eta)]\theta_{t+1} + \beta(1 - \delta) - 1. \quad (44)$$

By setting θ_t in period t , the policymaker is making a promise of future profits to banks which must be honored in period $t + 1$ by choosing k_{t+1}^m and θ_{t+1} consistent with the equilibrium condition. Since $k_t^m = f'^{-1}(1 + i_t/\lambda\nu)$ and $i_t \in \mathbb{R}_+$, the relevant state space is $k_t^m \in \mathbb{K} = [0, k^*]$ where $k^* = (aA)^{\frac{1}{1-a}}$. Values of θ_t consistent with an equilibrium are $\Omega = [\underline{\theta}, \bar{\theta}]$ where

$$\underline{\theta} = \frac{\bar{\alpha}\beta\eta\lambda(1-a)(1-\nu)A(aA)^{\frac{a}{1-a}}/\zeta + \beta(1-\delta) - 1}{1 - \beta[1 - \delta - \bar{\alpha}(1 - \eta)]} \quad (45)$$

$$\bar{\theta} = \frac{\bar{\alpha}\beta\eta\lambda(1-a)A(aA)^{\frac{a}{1-a}}/\zeta + \beta(1-\delta) - 1}{1 - \beta[1 - \delta - \bar{\alpha}(1 - \eta)]}, \quad (46)$$

The quantity $\underline{\theta}$ is the steady-state value of θ in a monetary equilibrium at the Friedman rule. We assume (43) holds so that $\underline{\theta} > 0$. The quantity $\bar{\theta}$ is the steady-state value of θ in a non-monetary equilibrium.

Proposition 4 (Recursive formulation of the optimal policy.) *The policymaker's value*

function solves:

$$\widetilde{\mathcal{W}}(\ell_0) = \max_{\theta_0 \in [\underline{\theta}, \bar{\theta}]} \mathcal{W}(\ell_0, \theta_0) \quad (47)$$

where $\mathcal{W}$ is the unique solution in $\mathcal{B}([0, 1] \times \Omega)$ to

$$\begin{aligned} \mathcal{W}(\ell_t, \theta_t) = & \max_{\theta_{t+1} \in \Gamma(\theta_t), \ell_{t+1}, k_{t+1}^m} \left\{ -\zeta \theta_t (1 - \ell_t) + \beta (1 - \ell_{t+1}) \lambda \nu [f(k_{t+1}^m) - k_{t+1}^m] \right. \\ & \left. + \beta \ell_{t+1} \lambda [f(k^*) - k^*] + \beta \mathcal{W}(\ell_{t+1}, \theta_{t+1}) \right\}, \end{aligned} \quad (48)$$

where

$$k_{t+1}^m = \nu^{\frac{-1}{a}} \left\{ (aA)^{\frac{a}{1-a}} - \frac{\zeta [\theta_t - \beta [1 - \delta - \bar{\alpha}(1 - \eta)] \theta_{t+1} - \beta(1 - \delta) + 1]}{\bar{\alpha} \beta \eta \lambda (1 - a) A} \right\}^{\frac{1}{a}} \quad (49)$$

$$\ell_{t+1} = (1 - \delta) \ell_t + \frac{\bar{\alpha} \theta_t}{1 + \theta_t} (1 - \ell_t), \quad (50)$$

and the feasibility correspondence $\Gamma(\theta_t)$ is defined as

$$\begin{aligned} \Gamma(\theta_t) = & \left[\frac{\theta_t + 1 - \beta(1 - \delta) - \bar{\alpha} \beta \eta \lambda (1 - a) A (aA)^{\frac{a}{1-a}} / \zeta}{\beta [1 - \delta - \bar{\alpha}(1 - \eta)]}, \right. \\ & \left. \frac{\theta_t + 1 - \beta(1 - \delta) - \bar{\alpha} \beta \eta \lambda (1 - a) A (1 - \nu) (aA)^{\frac{a}{1-a}} / \zeta}{\beta [1 - \delta - \bar{\alpha}(1 - \eta)]} \right] \cap \Omega. \end{aligned} \quad (51)$$

The recursive formulation of the optimal policy problem in (42) takes θ_t as a state variable and describes its law of motion by the promise-keeping constraint (44). Given the state, the Bellman equation (48) is obtained from the Principle of Optimality. The equilibrium allocation is then pinned down by choosing θ_0 to maximize (47). In the following proposition we prove that the equilibrium under the optimal policy is monetary, $\theta_t < \bar{\theta}$, since there are always unbanked entrepreneurs who need liquid assets to finance investments, and bank entry is larger than its level at the Friedman rule, i.e., $\theta_t > \underline{\theta}$.

Proposition 5 (Interior optimal policy.) *Assuming (43) holds and $\eta < \epsilon(\underline{\theta})$, the solution to (47) is such that $\theta_t \in (\underline{\theta}, \bar{\theta})$ for all t .*

We compute the optimal policy and the associated equilibrium by value function iteration (see the Appendix for a description of the algorithm). Figure 4 illustrates the optimal policy response following a banking shock that destroys lending relationships based on the parameterizations in Table 1. The top left panel plots the optimal path for the nominal interest rate. At the onset of the crisis, i_t is set at a low value close to the Friedman rule, about 0.25%, in order to reduce the

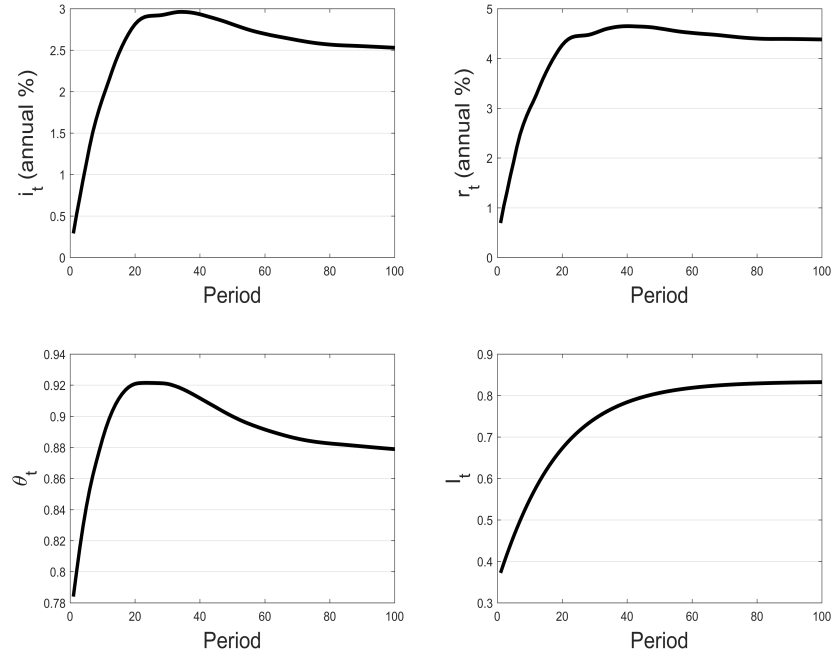


Figure 4: Optimal Ramsey policy with commitment

holding cost of liquidity for the 65% of unbanked entrepreneurs. In order to keep banks in the market despite such low interest rates, the policymaker promises a large increase in interest rates over the two years following the credit crunch. This "forward guidance" allows the policymaker to both provide liquidity at a low cost to entrepreneurs who lost access to a credit line and promote the creation of lending relationships by promising future profits to banks. The nominal interest rate peaks at about 3% and then falls to a steady-state level of 2.5%. The top right panel shows the real lending rate, r_t , which is set at a low value at the start of the banking crisis, less than 1%, but peaks at 4.6% after about two years before returning gradually to its steady state value of about 4.2%.

In the bottom left panel, credit market tightness follows a hump-shaped path similar to the one for i_t , but θ_t reaches its maximum before i_t . The rate of creation of lending relationships, $\alpha(\theta_t)$, overtakes its steady state value about a year after the start of the crisis. The overshooting occurs because the policymaker had to commit to high interest rates in order to maintain the creation of lending relationships early on.

Relative to the laissez-faire case with constant money growth, the optimal policy prescribes lower interest rates at all dates but also lower credit market tightness. So investment financed

internally is higher under the optimal policy but the return to the steady state is slower.

6.2 Markov policy without commitment

The policy described in Section 6 that consists in committing to an infinite sequence of interest rates is not time consistent. In our example the policymaker promises high nominal interest rates to induce banks to keep supplying loans but would like to renege later in order to reduce entrepreneurs' cost of holding real balances. We now relax the assumption of commitment and assume that the policymaker sets i_{t+1} in period t but cannot commit to $\{i_{t+j}\}_{j \geq 2}$. The timing of actions within period t is analogous to Klein, Krusell, and Rios-Rull (2008), i.e., the policymaker moves first by choosing i_{t+1} , and the private sector moves next by choosing θ_{t+1} and k_{t+1}^m .

Given the one-to-one relationship between i_{t+1} and $k_{t+1}^m = f'^{-1}(1 + i_{t+1}/\lambda\nu)$, a strategy of the policymaker is a mapping, $k_{t+1}^m = \mathcal{K}(h^t)$, that assigns an investment level for unbanked entrepreneurs to each public history up to period t , h^t . In each period each atomistic bank chooses whether to enter or not the credit market given $\mathcal{K}$. Because banks are small they do not take into account how their entry decisions affects other banks' decisions and the policymaker's choices. We aggregate these individual decisions to obtain market tightness, $\theta_t = \Theta(h^t)$, for all histories ending at the beginning of the second stage of period t . Both the strategy of the policymaker and the strategies of banks must be sequentially rational to form a subgame perfect equilibrium. Because the set of subgame perfect equilibria of infinitely-repeated games is vast, we restrict our attention to Markov Perfect Equilibria composed of strategies that are only functions of the aggregate state, i.e., $k_{t+1}^m = \mathcal{K}(\ell_t)$ and $\theta_t = \Theta(\ell_t, k_{t+1}^m)$. In other words, in a Markov perfect equilibrium the policymaker chooses the same interest rate, and hence the same k_{t+1}^m , after any h^t leading to the same measure of lending relationships ℓ_t .

Given $\mathcal{K}(\ell)$ the market-tightness function, $\Theta(\ell, k)$, solves the following functional equation:

$$\frac{\Theta(\ell, k)}{\alpha[\Theta(\ell, k)]} = \beta\eta \left\{ \frac{\lambda(1-a)A}{\zeta} \left[(aA)^{\frac{a}{1-a}} - \nu k^a \right] \right\} - \beta(1-\eta)\Theta(\ell', k') + \beta(1-\delta) \frac{\Theta(\ell', k')}{\alpha[\Theta(\ell', k')]}, \quad (52)$$

where $\ell' = (1-\delta)\ell + \alpha(\theta)(1-\ell)$ and $k' = \mathcal{K}(\ell)$. From (52), θ_t depends on the interest rate set prior to the opening of the credit market, which determines the investment unbanked entrepreneurs can finance internally, and it depends on the current state, ℓ_t , which determines the future state through the law of motion. When forming expectations about market tightness in $t+1$, banks anticipate that the policymaker in period $t+1$ will adhere to his policy rule, $k_{t+2}^m = \mathcal{K}(\ell_{t+1})$, and hence $\theta_{t+1} = \Theta(\ell_{t+1}, k_{t+2}^m)$.

Given Θ from (52), the problem of the policymaker can be written recursively as

$$\begin{aligned} \mathcal{W}(\ell_t) = & \max_{k_{t+1}^m \in [0, k^*], \ell_{t+1} \in [0, 1]} \{ -\zeta(1 - \ell_t)\Theta(\ell_t, k_{t+1}^m) + \beta\lambda\nu(1 - \ell_{t+1}) [f(k_{t+1}^m) - k_{t+1}^m] \\ & + \beta\ell_{t+1}\lambda [f(k^*) - k^*] + \beta\mathcal{W}(\ell_{t+1}) \}, \end{aligned} \quad (53)$$

subject to

$$\ell_{t+1} = (1 - \delta)\ell_t + \alpha [\Theta(\ell_t, k_{t+1}^m)] (1 - \ell_t). \quad (54)$$

A Markov perfect equilibrium can be computed by a two-dimensional iteration detailed in the Appendix. In a nutshell, in the n^{th} iteration of the algorithm, we first compute $\Theta(\ell, k)$ as a fixed point to the mapping implied by (52) taking as given some policy rule $\mathcal{K}_n(\ell)$. Second, given $\Theta(\ell, k)$, we update $\mathcal{K}_{n+1}(\ell)$ from (53) by standard value function iteration. We repeat these iterations until $\mathcal{K}_n(\ell)$ converges up to a sufficient accuracy. We take as our initial guess for the Markov strategy, $\mathcal{K}_0(\ell)$, the policy function under the full commitment.

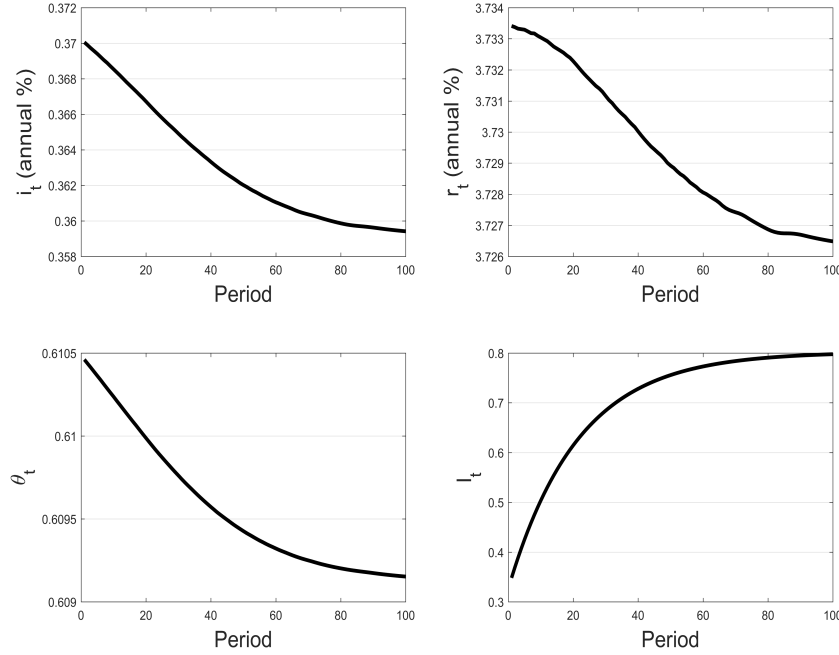


Figure 5: Optimal Markov policy without commitment

Figure 5 plots the optimal policy without commitment under the benchmark calibration. The top left panel shows that the time path for $\{i_t\}$ differs substantially from the one obtained with commitment. The policymaker who cannot credibly promise higher future interest rates sets the interest rate at the outset of the crisis at a higher value than the one under commitment, 0.37%

instead of 0.25%, in order to promote bank entry. It reduces i_t slightly over time as ℓ_t returns to its steady-state value. In contrast, under full commitment the nominal rate first increases and then gradually decreases. Quantitatively, changes in i_t are smaller in the absence of commitment, from 0.37% to 0.36%, and the level is closer to the Friedman rule. The recovery in terms of lending relationships is weaker than under full commitment, and credit market tightness is lower at all dates. After four years, the fraction of entrepreneurs in a lending relationship is 70% whereas under commitment such a fraction is reached in about two years. Hence, the recovery is considerably slower when the policymaker lacks commitment. The welfare loss associated with the lack of commitment is about 0.45% of total output.

7 Shocks to the supply and demand of credit

We now consider alternative shocks to the credit market temporarily that might involve different policy responses. First, we describe a shock that affect the efficiency of the matching process in the credit market, $\{\bar{\alpha}_t\}_{t=0}^T$ and $\bar{\alpha}_t = \bar{\alpha}$ for all $t \geq T+1$. This shock could represent tightening of lending standards and the difficulty for banks to screen applicants in times of large uncertainty. Second, we will describe a shock to the frequency of investment opportunities, $\{\lambda_t\}$. A recession could correspond to a period where such opportunities arrive less frequently. Third, we will describe a shock to the business cost, $\{\xi_t\}_{t=0}^T$ and $\xi_t = \xi$ for all $t \geq T+1$. This shock could represent the raise of various cost like screening and satisfying new regulation requirement like Dodd-Frank. Finally, we will describe a shock to the probability of separation, $\{\delta_t\}_{t=0}^T$ and $\delta_t = \delta$ for all $t \geq T+1$. Thus shock could capture the fact that the lending relationship becomes unstable and vulnerable to break down during the recession.

Before we go into the full analysis, to sharpen the intuition the next proposition focuses on the case where the credit market temporarily shuts down because no matches are formed, $\bar{\alpha}_t = 0$.

Proposition 6 *Suppose $\{\bar{\alpha}_t\}_{t=0}^{+\infty}$ is such that $\bar{\alpha}_t = 0$ for all $t = 0, \dots, T$. The optimal monetary policy with and without commitment consists of setting $i_{t+1} = 0$ for all $t = 0, \dots, T$.*

When the credit market shuts down, the optimal policy is the Friedman rule since the creation rate of lending relationships is zero irrespective of monetary policy, no matter the policymaker can make commitment or not. During the crisis the measure of lending relationships falls at a constant rate, which leaves the policymaker an economy with little lending relationship after the crisis. The

optimal policy after the crisis is analogous to the one in the previous section. Under commitment the policymaker sets low interest rates initially but promises future high interest rates in order to promote bank entry. If the policymaker lacks commitment, it will raise interest rates as soon as the economy recovers but interest rates will be kept at moderate levels.

We now describe the dynamics of the optimal policy responding to various shocks. Consider that the efficiency of the matching process falls by 30% and then gradually recovers to its steady state in a year. Figure 23 illustrates the optimal policy response to this shock with commitment. In this case the the policymaker reduces the interest rate at the onset of crisis and commit to raise it gradually back to its long-run level. During the crisis the measure of lending relationships falls, but at a diminishing rate, and then starts rising again before the end of the crisis. The decline in the lending relationship measure is mitigated by the bank's entry under the optimal Ramsey policy since the policymaker promises a rising interest rates, which improve bank's profit against the rising competition (and hence less favoring bargaining position) as the efficiency of the matching process improves. Figure 23 also illustrates the Ramsy policy after a similar shock to the frequency of investment opportunities (30% reduction initially and back to the steady state gradually in one year), and to the probability of separation and to the business cost (30% increase initially and back to the steady state gradually in one year). Unlike the shock to lending relationship, the zero lower bound can be binding initially after severe shocks to the parameters of the model.

Figure 7 illustrates the optimal policy response without commitment. It is different from the Ramsey policy given the same shock, and also different from the Markov policy after the shock to lending relationship. The nature of the shock matters. Unlike the Markov policy after the shock to lending relationship, promising rising interest rates becomes more credible, since the policymaker always wants to induce bank entry when the matching efficiency and investment probability are increasing. It is illustrated by the rising interest rates during period T . Nevertheless, after the crisis promising rising interest rates in the future is no longer credible, for the same reason in Section 7. Hence the Markov policy in Figure 7 becomes flat after T . For the shocks to the probability of separation and to the business cost, the policymaker cannot credibly raise the interest rate in the future since it is against the future policymaker's incentive. For example, the policymaker needs to raise the interest rate immediately after the crisis in order to compensate the rise in business cost. With commitment the policymaker can avoid raising the interest rate immediately by postponing it to the later periods when the economy is in better condition.

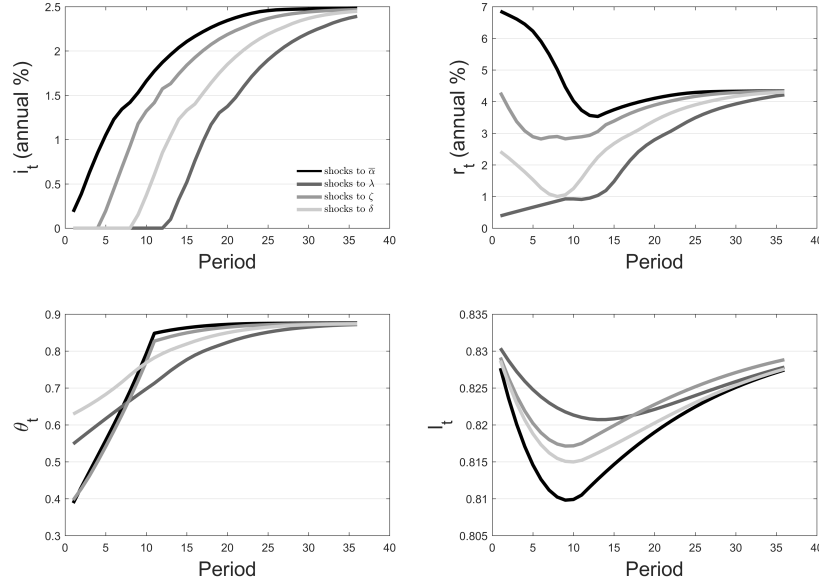


Figure 6: Optimal Ramsey policy with calibrated shocks to credit supply and demand

8 Conclusion

This paper achieved two objectives. First, we developed a corporate finance model of lending relationship and monetary policy. The formation of lending relationships between entrepreneurs and banks involves a time-consuming matching process and the terms of the loan contract are negotiated bilaterally. Because entrepreneurs can be unmatched for some period of time, there is a role for internal finance by retaining earnings in the form of cash. Our model delivers a transmission mechanism for monetary policy according to which the nominal interest rate set by the policymaker affects the real lending rate, banks' interest margins, and the supply of credit.

Second, we used our model to determine the optimal monetary policy following a banking crisis described as an exogenous destruction of a fraction of the existing lending relationships. We made two assumptions regarding the power of the policymaker to commit to future interest rates. If the policymaker can commit over an infinite time horizon the optimal policy involves some "forward guidance": the interest is set close to its lower bound at the outset of the crisis and it increases over time as the economy recovers. It is the promise of future high interest and inflation rates that gives banks incentives to keep creating lending relationships in a low interest rate environment. However, such promises are not time consistent. If the policymaker cannot commit more than one period ahead, then the interest rate increases when the crisis hits and it falls slightly over time

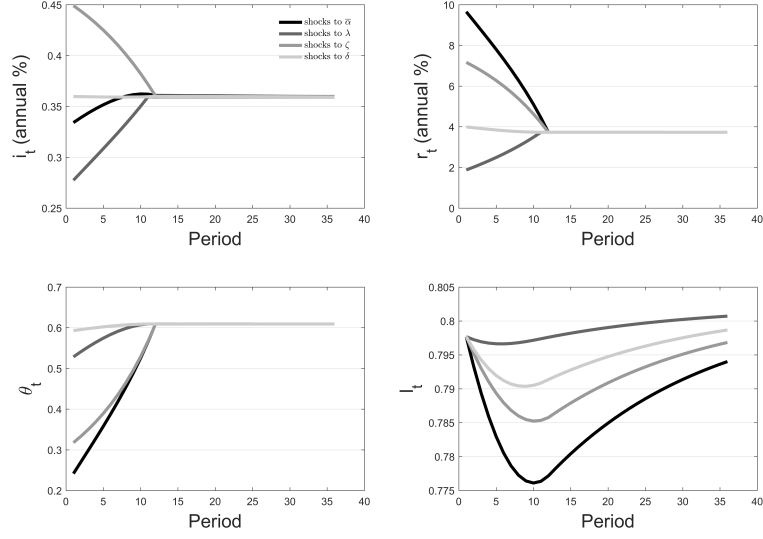


Figure 7: Optimal Markov policy with calibrated shocks to credit supply and demand

as the economy recovers. The inability to commit generates a more prolonged recession. We also studied alternative shocks on the supply and demand for credit and showed that the optimal policy response varies with the shock.

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Appendix A1: Proofs of Propositions and Lemmas

Proof of Proposition 2

The existence and uniqueness of the steady-state monetary equilibrium follow directly from (31)-(35). From (31) and $\partial\Delta/\partial i = -k^m < 0$, $\partial\theta/\partial i > 0$. To show $\partial r/\partial i > 0$, we can rewrite (31) as

$$(\rho + \delta) \frac{\theta \zeta}{\alpha(\theta)} = \lambda r k^*.$$

Hence, r is increasing with θ . From (33),

$$\frac{\partial k^m}{\partial i} = \frac{1}{\lambda \nu f''(k^m)} < 0.$$

From (35),

$$\frac{\partial(\vartheta M)}{\partial i} = \frac{-\delta \alpha'(\theta) k^m}{[\delta + \alpha(\theta)]^2} \frac{\partial \theta}{\partial i} + \frac{\delta}{\delta + \alpha(\theta)} \frac{\partial k^m}{\partial i} < 0.$$

To obtain (37), we differentiate (31) in the neighborhood of $i = 0$:

$$\frac{\partial \theta}{\partial i} = - \frac{\eta}{\zeta \{(\rho + \delta) [(1 - \epsilon(\theta))/\alpha(\theta) + 1 - \eta]\}} \frac{\partial \Delta}{\partial i}.$$

Using $\partial\Delta/\partial i = -k^*$ gives (37). Finally, to obtain (38), we differentiate (34) in the neighborhood of $i = 0$:

$$\frac{\partial r}{\partial i} = - \frac{(1 - \eta) \zeta}{\lambda k^*} \frac{\partial \theta}{\partial i} - \frac{\eta}{\lambda k^*} \frac{\partial \Delta}{\partial i}.$$

Using (37) and $\partial\Delta/\partial i = -k^*$ then gives (38).

Proof of Proposition 3

We consider an economy that starts with $\ell_0 = \underline{\ell} = \alpha(\underline{\theta})/[\delta + \alpha(\underline{\theta})]$ lending relationships, where $\underline{\theta}$ is credit market tightness at the Friedman rule. From (31), it solves:

$$(\rho + \delta) \zeta + \alpha(\underline{\theta}) \zeta = \frac{\alpha(\underline{\theta})}{\underline{\theta}} \eta \{ \lambda(1 - \nu) [f(k^*) - k^*] + \underline{\theta} \zeta \}. \quad (55)$$

According to (43), $\underline{\theta} > 0$. We measure social welfare in the second stage of $t = 0$ before banks make entry decisions and entrepreneurs make portfolio decisions:

$$\mathcal{W} = -\zeta \theta_0 (1 - \ell_0) + \sum_{t=1}^{\infty} \beta^t \{ (1 - \ell_t) \lambda \nu [f(k_t^m) - k_t^m] + \ell_t \lambda [f(k^*) - k^*] - \zeta \theta_t (1 - \ell_t) \}. \quad (56)$$

The first term on the RHS is the entry cost of banks in the initial period. The second term is the discounted sum of entrepreneurs' profits net of banks' entry cost in all subsequent periods. We consider a small deviation of the nominal interest rate from $i_1 = 0$. (The nominal interest rate is

known at the time banks make entry decisions.) For $t \geq 2$, $i_t = 0$. As a result, for all $t \geq 1$, $\theta_t = \underline{\theta}$ and for all $t \geq 2$, $k_t^m = k^*$. The measure of lending relationships solves:

$$\ell_1 = (1 - \delta)\underline{\ell} + \alpha(\theta_0)(1 - \underline{\ell}) \quad (57)$$

$$\ell_t = \underline{\ell} + (\ell_1 - \underline{\ell})[1 - \delta - \alpha(\underline{\theta})]^{t-1} \text{ for all } t \geq 2. \quad (58)$$

The welfare starting at time $t = 1$ if there is no deviation from the Friedman rule, $i_1 = 0$, is:

$$\begin{aligned} \mathcal{W}_1^{FR} &= \sum_{t=1}^{\infty} \beta^{t-1} \{[(1 - \ell_t)\nu + \ell_t] \lambda [f(k^*) - k^*] - \zeta \underline{\theta}(1 - \ell_t)\} \\ &= \frac{[(1 - \underline{\ell})\nu + \underline{\ell}] \lambda [f(k^*) - k^*] - (1 - \underline{\ell})\zeta \underline{\theta}}{1 - \beta} + (\ell_1 - \underline{\ell}) \frac{\lambda(1 - \nu) [f(k^*) - k^*] + \zeta \underline{\theta}}{1 - \beta[1 - \delta - \alpha(\underline{\theta})]}. \end{aligned}$$

Welfare from $t = 0$ is:

$$\begin{aligned} \mathcal{W} &= -\zeta \theta_0(1 - \underline{\ell}) + \beta(1 - \ell_1)\lambda\nu \{[f(k_1^m) - k_1^m] - [f(k^*) - k^*]\} + \beta \mathcal{W}_1^{FR} \\ &= -\zeta \theta_0(1 - \underline{\ell}) + \beta(1 - \ell_1)\lambda\nu \{[f(k_1^m) - k_1^m] - [f(k^*) - k^*]\} \\ &\quad + \beta \frac{\lambda[(1 - \underline{\ell})\nu + \underline{\ell}] [f(k^*) - k^*] - (1 - \underline{\ell})\zeta \underline{\theta}}{1 - \beta} \\ &\quad + \beta(\ell_1 - \underline{\ell}) \frac{\lambda(1 - \nu) [f(k^*) - k^*] + \zeta \underline{\theta}}{1 - \beta[1 - \delta - \alpha(\underline{\theta})]}. \end{aligned} \quad (59)$$

The second term on the RHS corresponds to the change in unbanked entrepreneurs' profits in $t = 1$ following a deviation from $i_1 = 0$. The relationship between θ_0 and k_1^m is given by (28) where we use that $\theta_1 = \underline{\theta}$, i.e.:

$$\begin{aligned} \frac{\theta_0}{\alpha(\theta_0)} &= \beta\eta \left\{ \frac{\lambda [f(k^*) - k^*] - \max_{k_1^m} \{-i_1 k_1^m + \lambda\nu [f(k_1^m) - k_1^m]\}}{\zeta} \right\} \\ &\quad - \beta(1 - \eta)\underline{\theta} + \beta(1 - \delta) \frac{\underline{\theta}}{\alpha(\underline{\theta})}. \end{aligned} \quad (60)$$

Differentiating (60) we obtain:

$$\begin{aligned} \left. \frac{\partial \theta_0}{\partial i_1} \right|_{i_1=0} &= \beta\eta \frac{\alpha(\theta_0)}{1 - \epsilon(\theta_0)} \frac{k^*}{\zeta} > 0 \\ \left. \frac{\partial k_1^m}{\partial i_1} \right|_{i_1=0} &= \frac{1}{\lambda\nu f''(k^*)} < 0. \end{aligned}$$

So a small increase of i_1 above 0 raises credit market tightness but reduces investment by unbanked entrepreneurs. From (59) the change in social welfare is

$$\left. \frac{\partial \mathcal{W}}{\partial i_1} \right|_{i_1=0} = \left\{ -\zeta + \beta\alpha'(\underline{\theta}) \frac{\lambda(1 - \nu) [f(k^*) - k^*] + \zeta \underline{\theta}}{1 - \beta[1 - \delta - \alpha(\underline{\theta})]} \right\} (1 - \ell) \frac{\partial \theta_0}{\partial i_1}, \quad (61)$$

where we have used that a small increase in i_1 above the Friedman rule only has a second-order effect on the profits of unbanked entrepreneurs. Using (55) to simplify the term between brackets in (61) we obtain:

$$\left. \frac{\partial \mathcal{W}}{\partial i_1} \right|_{i_1=0} = \left(\frac{\epsilon(\underline{\theta}) - \eta}{\eta} \right) \zeta (1 - \ell) \frac{\partial \theta_0}{\partial i_1}.$$

If $\epsilon(\underline{\theta}) > \eta$, then a deviation from the Friedman rule is optimal.

Proof of Proposition 4

We first restrict the policymaker's choice to bounded sequences $\{\theta_t\}$ that solve (44) given some initial condition, θ_{-1} . The policymaker solves:

$$\begin{aligned} \mathcal{W}(\ell_0, \theta_0) = \max_{\{k_{t+1}^m\}} \sum_{t=0}^{\infty} \beta^t \{ & -\zeta\theta_t(1-\ell_t) + \beta(1-\ell_{t+1})\lambda\nu [f(k_{t+1}^m) - k_{t+1}^m] + \beta\ell_{t+1}\lambda [f(k^*) - k^*] \} \\ \text{s.t. } & \{\theta_t, \ell_t\}_{t=0}^{\infty} \text{ being a solution to (44)-(50) given } \ell_0 \text{ and } \theta_0. \end{aligned}$$

The restriction $\theta_t \in \Omega$ is justified as follows. Suppose $\theta_t > \bar{\theta}$ for some t . Then,

$$\theta_{t+1} - \bar{\theta} = \frac{\theta_t - \bar{\theta} + \bar{\alpha}\beta\eta\lambda(1-a)A\nu(k_{t+1}^m)^a/\zeta}{\beta[1-\delta-\bar{\alpha}(1-\eta)]}.$$

Since $\beta[1-\delta-\bar{\alpha}(1-\eta)] \in (0, 1)$, the sequence $\{\theta_t - \bar{\theta}\}$ is unbounded, which is inconsistent with optimality as entry costs would be unbounded. Suppose next $\theta_t \in (0, \underline{\theta})$ for some t . With $\underline{\theta} > 0$,

$$\underline{\theta} - \theta_{t+1} = \frac{\underline{\theta} - \theta_t + \bar{\alpha}\beta\eta\lambda(1-a)A\nu[(aA)^{\frac{a}{1-a}} - (k_{t+1}^m)^a]/\zeta}{\beta[1-\delta-\bar{\alpha}(1-\eta)]}.$$

So θ_t becomes negative in finite time, which is inconsistent with an equilibrium. The feasibility condition $\theta_{t+1} \in \Gamma(\theta_t)$ is obtained from (44) by varying k_{t+1}^m from zero to k^* . By the Principle of Optimality, $\mathcal{W}(\ell_t, \theta_t)$ solves the Bellman Equation (48), i.e., it is the fixed point of a mapping from $\mathcal{B}([0, 1] \times [\underline{\theta}, \bar{\theta}])$ into itself. The mapping in (48) is a contraction by Blackwell's sufficient conditions (Theorem 3.3 in Stokey and Lucas, 1989), and by the contraction mapping theorem (Theorem 3.2 in Stokey and Lucas, 1989), the fixed point exists and is unique. The correspondence Γ is continuous and the policymaker's period utility is also continuous. So $\mathcal{W}(\ell, \theta)$ is continuous by the Contraction Mapping Theorem. Given there is no initial value for θ in the original sequence problem, (42), we choose $\theta_0 \in \Omega$ to maximize $\mathcal{W}(\ell_0, \theta_0)$. Such a solution exists by the continuity of $\mathcal{W}$ and the compactness of Ω . Given θ_0 , we use the policy function associated with $\mathcal{W}$ to pin down the entire trajectory for $\{\theta_t, k_t^m, \ell_t\}$.

Proof of Proposition 5

We now establish that $\theta_t \in (\underline{\theta}, \bar{\theta})$ for all t . Suppose $\theta_T = \underline{\theta}$. Since $\Gamma(\underline{\theta}) = \{\underline{\theta}\}$, it follows that $\theta_t = \underline{\theta}$ for all $t \geq T$. Such an equilibrium is implemented under the Friedman rule. Since we imposed a condition for the Friedman rule to be suboptimal, this contradicts $\theta_T = \underline{\theta}$. Suppose next $\theta_T = \bar{\theta}$. The equilibrium is a non-monetary equilibrium, i.e., $k_t^m = 0$ and $\theta_t = \bar{\theta}$ for all t . Consider an alternative equilibrium with a constant (k^m, θ) .

$$\mathcal{W} = -\zeta\theta_0(1-\ell_0) + \sum_{t=1}^{\infty} \beta^t \{ (1-\ell_t)\lambda\nu [f(k^m) - k^m] + \ell_t\lambda [f(k^*) - k^*] - \zeta\theta(1-\ell_t) \}.$$

Provided that $\nu > 0$, $\partial\mathcal{W}/\partial k^m = \infty$ when evaluated at $k^m = 0$. So $\theta_T = \bar{\theta}$ cannot be an equilibrium under an optimal policy.

Proof of Proposition 6

From (28) θ_t solves $-\zeta + \alpha_t^b \beta Z_{t+1}^b \leq 0$, with an equality if $\theta_t > 0$. Using that $\alpha_t(\theta) = \bar{\alpha}_t \theta / (1 + \theta)$ it can be reexpressed as

$$\frac{1 + \theta_t}{\bar{\alpha}_t} \geq \beta \eta \left\{ \frac{\lambda [f(k^*) - k^*] - \Delta_{t+1}}{\zeta} \right\} - \beta(1 - \eta)\theta_{t+1} + \beta(1 - \delta) \frac{1 + \theta_{t+1}}{\bar{\alpha}_{t+1}},$$

with an equality if $\theta_t > 0$. Rearranging this inequality we obtain:

$$\theta_t \geq \beta \eta \bar{\alpha}_t \left\{ \frac{\lambda [f(k^*) - k^*] - \Delta_{t+1}}{\zeta} \right\} + \beta \bar{\alpha}_t \left[\frac{1 - \delta}{\bar{\alpha}_{t+1}} - (1 - \eta) \right] \theta_{t+1} + \beta(1 - \delta) \frac{\bar{\alpha}_t}{\bar{\alpha}_{t+1}} - 1, \quad (62)$$

with an equality if $\theta_t > 0$. If $\bar{\alpha}_t = 0$, then $\theta_t = 0$. For all $t = 0, \dots, T$ the policymaker's problem simplifies to

$$\mathcal{W}_t(\ell_t) = \max_{k_{t+1}^m \in [0, k^*]} \{ \beta \lambda \nu (1 - \ell_{t+1}) [f(k^m) - k^m] + \beta \lambda \ell_{t+1} [f(k^*) - k^*] + \beta \mathcal{W}_{t+1}(\ell_{t+1}) \}$$

$$\text{s.t. } \ell_{t+1} = (1 - \delta)\ell_t,$$

where $\mathcal{W}_{T+1}$ which is welfare function with or without commitment. In either case, the optimal policy is always $k_{t+1}^m = k^*$, which is equivalent to $i_{t+1} = 0$ for $t = 0, \dots, T$.

Appendix A2: Derivations for the terms of the loan contract

Here we show how we obtain the expression for ϕ_t given by (26). First, we can write the value function of an unbanked entrepreneur with m real balances at the start of period t as

$$U_t^e(m) = \Delta_t + i_t m_t + V_t^e(m).$$

Substituting this into the expression for $W_t^e(m)$ gives

$$W_t^e(m) = m + T_t + \beta(\Delta_{t+1} + V_{t+1}^e).$$

Hence, $W_t^e(0) = T_t + \beta(\Delta_{t+1} + V_{t+1}^e)$. Assuming that $1 + \pi_t > \beta$, the value function of a banked entrepreneur with m real balances in the last stage is

$$X_t^e(m) = m + T_t - (1 + \pi_{t+1})d_{t+1} + \beta Z_{t+1}^e(d_{t+1}),$$

which uses the fact that $m_{t+1} = d_{t+1}$. This gives $X_t^e(0) = T_t - (1 + \pi_{t+1})d_{t+1} + \beta Z_{t+1}^e$.

To derive (25), we use $X_{t-1}^e(0) = T_{t-1} - (1 + \pi_t)d_t + \beta Z_t^e$ and $W_{t-1}(0) = T_{t-1} + \beta(\Delta_t + V_t^e)$, so that the surplus of a banked entrepreneur, $S_t^e = [X_{t-1}^e(0) - W_{t-1}^e(0)] / \beta$, solves

$$S_t^e = -(1 + i_t)d_t + Z_t^e - (\Delta_t + V_t^e).$$

Substituting Z_t^e by its expression in (24), we obtain

$$S_t^e = -i_t d_t + \lambda[f(k_t) - k_t - \phi_t] + T_t - (\Delta_t + V_t^e) + \beta(\Delta_{t+1} + V_{t+1}^e) + (1 - \delta)\beta S_{t+1}^e.$$

We now derive (26) as follows. Using (25) and solving for ϕ_t ,

$$\phi_t = [f(k_t) - k_t] + \frac{1}{\lambda} [W_t^e(0) - (\Delta_t + V_t^e) + (1 - \delta)\beta S_{t+1}^e - S_t^e].$$

Using $S_t^e = \frac{1-\eta}{\eta} S_t^b = \frac{1-\eta}{\eta} [\lambda \phi_t + \beta(1 - \delta)S_{t+1}^b]$ and substituting above, we obtain

$$\phi_t = \eta [f(k^*) - k^*] + \frac{\eta}{\lambda} [W_t^e(0) - (\Delta_t + V_t^e)]. \quad (63)$$

A novelty in (63) is the term Δ_t which depends on the rate of return on money and in turn affects the determination of r_t . The second term on the RHS of (63) arises from the fact that an unmatched entrepreneur has the option of purchasing k_t^m with his real balances, which reduces ϕ_t . From (19) and the fact that $\beta S_{t+1}^e = X_t^e(0) - W_t^e(0)$, this outside option can be expressed as

$$\begin{aligned} (\Delta_t + V_t^e) - W_t^e(0) &= (\Delta_t + V_t^e) + T_t - \beta(\Delta_{t+1} + V_{t+1}^e) \\ &= \Delta_t + \alpha_t \beta (T_t / \beta - W_t^e / \beta + Z_{t+1}^e) \\ &= \Delta_t + \alpha_t \beta S_{t+1}^e. \end{aligned}$$

Since $\eta S_{t+1}^e = (1 - \eta)Z_{t+1}^b$ and $Z_{t+1}^b = \zeta \theta_t / \beta \alpha_t$ from free entry of banks, the outside option reduces to

$$(\Delta_t + V_t^e) - W_t^e(0) = \frac{(1 - \eta)}{\eta} \theta_t \zeta + \Delta_t. \quad (64)$$

From the RHS of (64), the entrepreneur's reservation utility consists of the option of continuing to search for a bank and the option of using internal finance. Substituting (64) into (63) gives ϕ_t as expressed in (26).

Appendix A3. Numerical procedure for optimal policy problem

The numerical method is based on the observation that the following mapping is a contraction:

$$T\mathcal{W}_{n+1}(\ell, \theta) = \max_{\theta', \ell', k^m} \left\{ -\zeta\theta(1-\ell) + \beta(1-\ell')\lambda\nu [f(k^m) - k^m] + \beta\ell'\lambda [f(k^*) - k^*] + \beta\mathcal{W}_n(\ell', \theta') \right\},$$

where

$$\begin{aligned} k^m &= \nu^{\frac{-1}{a}} \left\{ (aA)^{\frac{a}{1-a}} - \frac{\zeta [\theta - \beta [1 - \delta - \bar{\alpha}(1 - \eta)] \theta' - \beta(1 - \delta) + 1]}{\bar{\alpha}\beta\eta\lambda(1-a)A} \right\}^{\frac{1}{a}} \\ \ell' &= (1 - \delta)\ell + \frac{\bar{\alpha}\theta}{1 + \theta}(1 - \ell). \end{aligned}$$

Therefore we will iterate this mapping to obtain a sequence of value functions, $\{T^n\mathcal{W}_0\}$. This sequence is Cauchy and converges to the unique fixed point of T .

1. Choose the precision of the grid for the state space: $N_\ell \in \mathbb{N}, N_\theta \in \mathbb{N}$.

This means there are $N_\ell + 1$ values for state ℓ and $N_\theta + 1$ values for state θ .

Denote $\varepsilon_\ell = 1/N_\ell$ and $\varepsilon_\theta = \bar{\theta}/N_\theta$. A state is a pair $\ell = \varepsilon_\ell i$ and $\theta = \varepsilon_\theta j$ where $(i, j) \in \{0, N_\ell\} \times \{0, N_\theta\}$.

2. Initialize the iterations by choosing $\mathcal{W}_0(\ell, \theta)$.

The initial guess for the value function is:

$$\begin{aligned} \mathcal{W}_0(\ell, \theta) &= -\zeta\theta + (1 - \ell) \frac{\lambda\nu [f(k^m) - k^m] - \zeta\theta}{\rho} + \ell^{ss} \frac{\lambda [f(k^*) - k^*]}{\rho} \\ &\quad + (\ell - \ell^{ss}) \frac{\lambda \{ [f(k^*) - k^*] - \nu [f(k^m) - k^m] \} + \zeta\theta}{\rho + \delta + \alpha(\theta)}. \end{aligned}$$

where

$$\begin{aligned} \ell^{ss} &= \frac{\bar{\alpha}\theta}{\delta + (\delta + \bar{\alpha})\theta} \\ k^m &= \nu^{\frac{-1}{a}} \left\{ (aA)^{\frac{a}{1-a}} - \frac{\{1 - \beta [1 - \delta - \bar{\alpha}(1 - \eta)]\} \theta - \beta(1 - \delta) + 1}{\bar{\alpha}\beta\eta\lambda(1-a)A} \zeta \right\}^{\frac{1}{a}}. \end{aligned}$$

3. Suppose $\mathcal{W}_n$ is known. To compute $\mathcal{W}_{n+1}(\ell, \theta)$, do loops over the entire state space: $i = 0 \dots N_\ell$ and $j = 0 \dots N_\theta$. Each loop corresponds a (θ, ℓ) . For each state find $\theta' \in \Gamma(\theta)$ that maximizes the right side of the Bellman equation, i.e.,

$$-\zeta\theta(1-\ell) + \beta(1-\ell')\lambda\nu [f(k^m) - k^m] + \beta\ell'\lambda [f(k^*) - k^*] + \beta\mathcal{W}_n(\ell', \theta').$$

If ℓ' does not belong to the grid, do a linear interpolation to approximate the value function.

4. Once $\mathcal{W}_{n+1}(\ell, \theta)$ has been computed, return to Step 3 to compute $\mathcal{W}_{n+2}(\ell, \theta)$. The criterion to stop is: $\left(\sum_{(\ell, \theta)} [\mathcal{W}_{n+1}(\ell, \theta) - \mathcal{W}_n(\ell, \theta)]^2 \right)^{\frac{1}{2}} < \epsilon$.

Appendix A4: Model with time-varying parameters

For all $t \geq T + 1$, the model is identical to the one presented in Section 4. For $t \leq T$, the probability an existing lending relationship is terminated is δ_t , the matching probability for unbanked entrepreneurs is $\bar{\alpha}_t\theta/(1+\theta)$, and the matching probability for unmatched banks is $\bar{\alpha}_t/(1+\theta)$. Now the utility of an unbanked entrepreneur at the start of the second stage is

$$V_t^e(m) = m + \frac{\bar{\alpha}_t\theta_t}{1+\theta_t}\beta Z_{t+1}^e + \left(1 - \frac{\bar{\alpha}_t\theta_t}{1+\theta_t}\right)W_t^e(0). \quad (65)$$

The only novelty is to allow the matching efficiency, $\bar{\alpha}_t$, to be a function of time. The value of the unbanked entrepreneur in the third stage is still given by (17). The value of being in a lending relationship at the start of the period for entrepreneurs and banks are respectively,

$$Z_t^e = \lambda[f(k_t) - k_t - \phi_t] + \delta_t W_t^e(0) + (1 - \delta_t)\beta Z_{t+1}^e \quad (66)$$

$$Z_t^b = \lambda\phi_t + \delta_t\beta U_t^b + (1 - \delta_t)\beta Z_{t+1}^b, \quad (67)$$

where the bank's lifetime value of not being in a lending relationship solves

$$U_t^b = -\zeta + \frac{\bar{\alpha}_t}{1+\theta_t}\beta Z_{t+1}^b + \left(1 - \frac{\bar{\alpha}_t}{1+\theta_t}\right)\beta U_{t+1}^b. \quad (68)$$

With free entry, $U_t^b = 0$. Hence, from (67), the surplus of bank from offering a lending relationship is $S_t^b = Z_t^b = \lambda\phi_t + (1 - \delta_t)\beta Z_{t+1}^b$. The entrepreneur's value of being in a lending relationship at the end of $t - 1$, $S_t^e \equiv (T_{t-1}/\beta + Z_t^e) - W_{t-1}^e(0)/\beta$, solves

$$S_t^e = \lambda[f(k_t) - k_t - \phi_t] - (\Delta_t + V_t^e) + T_t + \beta(\Delta_{t+1} + V_{t+1}^e) + (1 - \delta_t)\beta S_{t+1}^e, \quad (69)$$

where we used that $W_{t-1}^e(0)/\beta = T_{t-1} + \Delta_t + V_t^e(0)$. Using $(1 - \eta)S_{t+\tau}^b = \eta S_{t+\tau}^e$, $\{\phi_{t+\tau}\}_{\tau=1}^T$ solves

$$\phi_t = \eta[f(k^*) - k^*] - \frac{\eta}{\lambda}[(\Delta_t + V_t^e) - T_t - \beta(\Delta_{t+1} + V_{t+1}^e)].$$

Using the expression above for V_t^e ,

$$(\Delta_t + V_t^e) - T_t - \beta(\Delta_{t+1} + V_{t+1}^e) = \frac{\bar{\alpha}_t\theta_t}{1+\theta_t}\beta S_{t+1}^e + \Delta_t = \frac{1-\eta}{\eta}\theta_t\zeta + \Delta_t,$$

where we have used $\eta S_{t+1}^e = (1 - \eta)Z_{t+1}^b$ and $Z_{t+1}^b = \zeta(1 + \theta_t)/(\beta\bar{\alpha}_t)$. Hence, the interest payment solves (26). From free entry of banks, credit market tightness now solves

$$\frac{1 + \theta_{t-1}}{\bar{\alpha}_{t-1}} = \beta\eta \left\{ \frac{\lambda[f(k^*) - k^*] - \Delta_t}{\zeta} \right\} - \beta(1 - \eta)\theta_t + \beta(1 - \delta_t)\frac{1 + \theta_t}{\bar{\alpha}_t}.$$

We now describe the optimal policy problem with time-varying parameters. Starting in $T + 1$, the policymaker's problem is identical to the one in Proposition 4. For all $t \leq T$ the value function

of the policymaker solves:

$$\begin{aligned} \mathcal{W}_t(\ell_t, \theta_t) = & \max_{\theta_{t+1} \in \Gamma_{t+1}(\theta_t), \ell_{t+1}, k_{t+1}^m} \left\{ -\zeta \theta_t (1 - \ell_t) + \beta (1 - \ell_{t+1}) \lambda \nu [f(k_{t+1}^m) - k_{t+1}^m] \right. \\ & \left. + \beta \ell_{t+1} \lambda [f(k^*) - k^*] + \beta \mathcal{W}_{t+1}(\ell_{t+1}, \theta_{t+1}) \right\}. \end{aligned} \quad (70)$$

The novelty in (70) is that both $\mathcal{W}_t$ and Γ_{t+1} are indexed by time. The equilibrium condition for market tightness is:

$$\frac{1 + \theta_t}{\bar{\alpha}_t} = \beta \eta \left\{ \frac{\lambda [f(k^*) - k^*] - \Delta_{t+1}}{\zeta} \right\} - \beta (1 - \eta) \theta_{t+1} + \beta (1 - \delta_{t+1}) \frac{1 + \theta_{t+1}}{\bar{\alpha}_{t+1}}. \quad (71)$$

Following a similar reasoning as before, the state space, $\Omega_t = [\underline{\theta}_t, \bar{\theta}_t]$, is defined recursively as:

$$\frac{1 + \underline{\theta}_t}{\bar{\alpha}_t} = \beta \eta \frac{\lambda (1 - \nu) [f(k^*) - k^*]}{\zeta} - \beta (1 - \eta) \underline{\theta}_{t+1} + \beta (1 - \delta_{t+1}) \frac{1 + \underline{\theta}_{t+1}}{\bar{\alpha}_{t+1}} \quad (72)$$

$$\frac{1 + \bar{\theta}_t}{\bar{\alpha}_t} = \beta \eta \frac{\lambda [f(k^*) - k^*]}{\zeta} - \beta (1 - \eta) \bar{\theta}_{t+1} + \beta (1 - \delta_{t+1}) \frac{1 + \bar{\theta}_{t+1}}{\bar{\alpha}_{t+1}}. \quad (73)$$

From (72), the lower bound of Ω_t corresponds to market tightness at the Friedman rule when $k_{t+1}^m = k^*$, anticipating a market tightness in $t + 1$ associated at the Friedman rule, $\underline{\theta}_{t+1}$. From (73), the upper bound of Ω_t corresponds to market tightness when $k_{t+1}^m = 0$, anticipating a market tightness in $t + 1$ associated with a non-monetary equilibrium, $\bar{\theta}_{t+1}$. The feasible set, $\Gamma_{t+1}(\theta_t)$, is obtained from (71) by varying k_{t+1}^m from zero to k^* :

$$\Gamma_{t+1}(\theta_t) = \left[\frac{\frac{1 + \theta_t}{\bar{\alpha}_t} - \beta \eta \frac{\lambda [f(k^*) - k^*]}{\zeta} - \frac{\beta (1 - \delta_{t+1})}{\bar{\alpha}_{t+1}}}{\beta \left[\frac{1 - \delta_{t+1}}{\bar{\alpha}_{t+1}} - (1 - \eta) \right]}, \frac{\frac{1 + \theta_t}{\bar{\alpha}_t} - \beta \eta \frac{\lambda (1 - \nu) [f(k^*) - k^*]}{\zeta} - \frac{\beta (1 - \delta_{t+1})}{\bar{\alpha}_{t+1}}}{\beta \left[\frac{1 - \delta_{t+1}}{\bar{\alpha}_{t+1}} - (1 - \eta) \right]} \right] \cap \Omega_{t+1}. \quad (74)$$

From (71), investment by unbanked entrepreneurs solves:

$$k_{t+1}^m = \nu^{\frac{-1}{a}} \left\{ (aA)^{\frac{a}{1-a}} - \frac{\zeta \left[\frac{1 + \theta_t}{\bar{\alpha}_t} + \beta (1 - \eta) \theta_{t+1} - \beta (1 - \delta_{t+1}) \frac{1 + \theta_{t+1}}{\bar{\alpha}_{t+1}} \right]}{\beta \eta \lambda (1 - a) A} \right\}^{\frac{1}{a}}. \quad (75)$$

Finally, the law of motion for ℓ is:

$$\ell_{t+1} = (1 - \delta_t) \ell_t + \frac{\bar{\alpha}_t \theta_t}{1 + \theta_t} (1 - \ell_t). \quad (76)$$