Gilded Bubbles*
Preliminary

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Abstract

Excessive credit growth and high asset prices increase the probability of a crisis. Because these two variables are determined in equilibrium, the analysis of systemic risk and the cost-benefit analysis of macroprudential regulation requires a specific framework consistent with the empirical observation. We argue that an overlapping generation model of rational bubbles can explain some of the main features of banking crises and, therefore, provide a microfounded framework for the rigorous analysis of macroprudential policy. We find that credit financed bubbles may have a role as a buffer in reducing excessive investiment at the firms' level and, thus, increasing efficiency. Still, when banks have a risk of going bankrupt a trade-off appears between financial stability and efficiency. When this is the case, macroprudential policy has a key role in improving efficiency while preserving financial stability.

JEL Classification: Banks, bubbles, macroprudential regulation
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> "Beautiful credit! The foundation of modern society. Who shall say that this is not the golden age of mutual trust, of unlimited reliance upon human promises? "

–Mark Twain

1 Introduction

The recent empirical evidence has confirmed, once more, the existing connections between credit growth, leverage, bubbles and the likelihood of a banking crisis. Intuitively, it is clear that once a bubble starts, it explains both the credit growth and leverage because the bubble creates a demand for credit that is backed by the bubble itself as collateral. Still, to our knowledge, a rigorous modeling of the link between rational bubbles, credit expansion and systemic risk has not been developed yet, in spite of the need to understand these issues as a basis for macroprudential policy.

This article builds a simple model that emphasizes the connections between the supply of credit, the equilibrium price of bubbles and systemic risk in the presence of banking. The explicit introduction of financial intermediaries (banks for short) is of interest for three reasons: first, banks allow for levered bubbles; second, banks' solvency depends upon the behavior of the bubble, and third, a systemic crisis when bubbles are financed by banks may have much more devastating consequences than if bubbles are directly financed by investors, as the anecdotal evidence of the comparison between the dot.com bubble and the housing price bubble of 2007 shows and the empirical analysis of Jordà et al. (2015) rigorously establishes.

There is now a large consensus on the impact of excessive credit growth on financial stability (Jordà et al., 2015; Laeven and Valencia, 2012; Schularick and Taylor, 2012) to the point that Schularick and Taylor (2012) conclude that "Credit growth is a powerful predictor of financial crises". Still, even if the empirical literature points at excessive credit growth as one important correlate to financial crises, it is important to keep in mind that the amount of credit in an economy is endogenously determined, and so the question that arises is simply: how come there is a sufficient amount of positive net present value projects expected to be financed? One answer is that credit rationing was limiting credit, so that deregulation and access to wholesale funds suddenly allow to unleash credit and finance these projects. The alternative is that the existence of credit may be a necessary condition for asset bubbles to emerge, so that, provided the bubbles expected return is sufficiently high, the demand for the bubble will create a demand for credit. This is an important point, as the supply of credit fuels bubble prices while larger bubbles fuels the demand for credit. So, in this market, the supply of credit creates its own demand.

From an empirical point of view, attributing the credit growth to the existence of bubbles is a complex exercise, as it requires defining a fundamental value and identifying the (positive) residual as the bubble. This will be difficult for real estate prices and may prove impossible for stock markets. It is no wonder that, prior to the crisis, the dominant view was that bubbles, if they ever exist could not be identified. Still, a number of contributions point out the importance of bubbles bursting as a cause of banking crises.[1]

It is not surprising that the impact of bubble bursting on banks' solvency is of paramount importance when these bubbles have been financed by banks' credit. Using a measure of bubbles that combines the trend of real estate prices with an ulterior price correction, (Jordà et al., 2015, p. S1) show that "when fueled by credit booms, asset price bubbles increase financial crisis risks". Anundsen et al. (2014) recursively test whether credit and house prices are in a regime characterized by explosive behavior or not, and establish a positive and highly significant effect of exuberant behavior in the housing and credit market on the likelihood of a crisis.

From a theoretical perspective, the analysis of bubbles is particularly appealing because, first, it justifies simultaneously the bubble and the credit boom; second, because it relates it to the supply of credit, and, as a consequence, to saving gluts and capital mobility; third, it makes risk perception endogenous and, last but not least it frames the macroprudential policy in a set up where efficiency is well defined, so that the trade-off between growth and financial stability is tractable.

Our research will focus on rational bubbles because this framework allows us to define the welfare properties of the equilibrium in a straightforward way. In our model, contrarily to Farhi and Tirole (2012) and Martin and Ventura (2015), bubbles are not owned by entrepreneurs but by consumers, as bubbles constitute their best investment opportunity. Consequently, their positive role in resource allocation does not stem from their value in providing additional collateral to credit rationed firms but from preventing inefficient over-

[1] A first method to identify a bubble consists in detecting housing price deviations of real house prices above some specified threshold relative to trend (Borio and Lowe, 2002; Detken and Smets, 2004; Goodhart and Hofmann, 2008). A second approach focuses on the rate of growth of prices and diagnoses the existence of a bubble by the rate of growth being consistently above some threshold (Bordo and Jeanne, 2002). A third view identifies the bubble by the extent of the peak-trough (Helbling, 2005; Helbling and Terrones, 2003; Claessens et al., 2008). The different approaches can be combined to filter non-bubbles. This is why Jordà et al. (2015), while using the first approach of real increase relative to a trend require also a subsequent price correction larger than 15%. A simpler, completely different strategy in identifying bubbles consists in analyzing the ratio of wealth to GDP, as the numerator, but not the denominator, might reflect the existence of a bubble. This is considered by Carvalho et al. (2012) who motivate their analysis by observing that the ratio of wealth to GDP grows before a crisis.

investment; that is, from preserving Solow's golden rule whereby the efficiency allocation requires the equality between the interest rate and the growth rate.

This paper is not the first to address the issue of rational bubbles in their connection to systemic risk. Indeed, Aoki and Nikolov (2015) consider, as we do, bubbles in a banking economy and their role in generating banking crises. Still, their objective is to compare the impact of household held and bank-held bubbles, and their results show that bank-held bubbles imply a higher systemic risk. As Reinhart and Rogoff (2009) and Jordà et al. (2015) argue, credit-boom-fueled housing price spirals are particularly pernicious, and therefore this is also one of the objectives of our research. Still, our aim is broader as we would like to explore the mechanism of bubble creation and bursting in connection with credit booms.

The analysis of bubbles require to be specific about the alternative investment vehicles available to savers, as in any overlapping generations (OLG) model the transfer of goods from one generation to the next will be done through these vehicles. Our framework considers three different investment opportunities: acquiring the bubble, depositing in the bank and using the storage technology. Because the storage technology is always available, in equilibrium, the return on deposits and on bubble acquisition should be larger than the return on the storage technology. We show that the existence of equilibria will critically depend on whether the return on the storage technology is positive or negative. In particular, if this return is positive, there will be no bubbles in stationary equilibria. Interestingly, in a fiat money economy, it seems natural to interpret the return on the storage technology as the return on holding cash, which constitutes a second bubble. Our results could then be viewed as the effect of the competition between two bubbles, with the possibility that one is relegated to its transactional role while the other provides a store of value role.

Our concern for bank stability will lead us to consider a set up where the equilibrium allocation depends upon a stochastic element, that could be endowments, productivity or capital inflows, that will affect both the current allocation as well as future bubble prices. As we will see, some of the intuition for our results can be obtained from the allocation in a riskless steady state economy, but the analysis of bubble bursting and banks' bankruptcies requires the additional complexity of random shocks. For expositional reasons we take this stochastic shocks to stem from the liability side of banks and this will affect the supply of credit. This is a strong simplification, as we would expect the amount of funding a bank is able to attract depends upon the equilibrium interest rate in the economy. Still, we show that endowment and productivity related stochastic shocks should have the same impact, as they would also affect interest rates and the overall allocation. Focusing on the liability side has the benefit to simplify the analysis of the impact of capital regulation on the growth rate and riskiness of the resulting allocation. It allows also to consider the role of foreign capital inflows (or sudden stops) as well as Central Banks liquidity injections or withdrawals.

We obtain three regimes, one where households borrow from the bank to buy the bubble, a second one where household distribute their savings between deposits and their investment in the bubble and a third one where the bubble is zero.

We show that, when the storage technology is costly, that is, when its net return is negative, the efficient equilibrium may be the one characterized by leveraged bubbles. This is the case because, first, when the return on the storage technology is negative, in the absence of a bubble, too low interest rates will lead to excessive capital accumulation. Second, the equilibrium characterized by unlevered bubbles, where the return of the bubble is the same as the return on a deposit is also inefficient, because the spread between deposit and loan rates implies high interest rate and low levels of capital, the opposite of what happened in the bubbleless economy.

Thus, our results are in line with Jordà et al. (2015) empirical finding and with the conjectures put forward by Mishkin (2008), Mishkin (2009) and other policy- makers after the crisis: bubbles that threaten financial stability are those that are fueled by credit and leverage. More precisely, the finding that they obtain that the coefficient the interaction between the credit variable and the bubble indicators is highly significant in determining the probability of a crisis, which fits our theoretical model.

2 The model

We will consider an overlapping generations economy with households and entrepreneurs that live for two periods. Entrepreneurs and households need financing and banks have a role as providing funds, because of their ability to screen, monitor and enforce contracts (Diamond, 1984). Absent banks, households may provide funding to firms but at a much higher cost that reduces the productivity of the economy. There is a cost of monitoring that implies that, in equilibrium, there is a spread between the bank offered deposit rate, r^d_{t+1} and its lending rate to firms r^f_{t+1} or the households r^h_{t+1}.

2.1 Households

We assume that risk neutral households supply one unit of labor when young, and derive utility from their consumption only when old. We denote $c_{i,t}$ the consumption of generation i at time t. We make the usual assumption that households receive a labor endowment N_t when young, which will allow them to obtain an equilibrium wage, and receive no endowment when old when they consume. For simplicity we take the labor supply to be inelastic and equal to 1. Consequently, young households at time t will save the equilibrium wage W_t in order to consume when old. This they will be able to do either by depositing in the bank, by buying the bubble or by investing in a storage technology. In contrast to Farhi and Tirole (2012) or Martin and Ventura (2015) it is households, not firms, that may invest in the asset that is susceptible of trading at a price above its fundamental value, denoted by $\underline{B}$, which we will refer to as the bubble. Consequently, any effect that we may capture is completely unrelated to the role of collateral in providing better access to financial market.

The banking system allows households to either borrow an amount L_t at a rate r^h_{t+1} to invest in the bubble or to deposit D_t at the bank with a return of r^d_{t+1}. In addition, households, as any other agent, can also save by investing in a riskless asset, which offers a return that we denote by $\underline{r}$.

Consequently, household will have to determine how to allocate their savings, between purchasing q_t units of the bubble at a price B_t or depositing in the bank or investing in the riskless asset. The following maximization program (1) corresponds to the problem a household born at t solves.

$$
\begin{aligned}
& \max_{c_{t,t+1}, q_t, D_t, L_t} \mathbb{E}_t c_{t,t+1} \qquad (1) \\
\text{s. t. } & q_t B_t + D_t + O_t \leq W_t + L_t \\
& c_{t,t+1} \leq \max(0, q_t B_{t+1} - (1 + r^h_{t+1}) L_t + (1 + r^d_{t+1}) D_t) \\
& c_{t,t+1}, q_t, D_t, L_t \geq 0.
\end{aligned}
$$

Notice that households are protected by limited liability that ensures $c_{t,t+1} \geq 0$.

2.2 Entrepreneurs

Each generation of entrepreneurs has a production technology and no other endowment. Similar to households, generation t of entrepreneurs lives at t and $t+1$ and consumes only in $t+1$.

The production process takes one period and allows to produce an output Y_{t+1} out of its inputs in labor and capital (N_{t+1}, K_t). K_t is borrowed at t, then labor is hired from generation $t+1$ and is paid out of the product Y_{t+1}, without requiring additional borrowing. The production process is simplified as we assume it is riskless. We assume an exogenous level of labor productivity, $\gamma > 1$; i.e. $Y_{t+1} = F(\gamma^t N_{t+1}, K_t)$ and, consequently, the standard result whereby the marginal product of an input equals its cost applies. The equilibrium in the labor market will therefore determine the wage W_{t+1} at time $t+1$ as a function of the previous period capital K_t. Because entrepreneurs live only two periods they will fully consume their profits.

The problem entrepreneurs solve is the following:

$$\max_{c^E_{t,t+1}, N_{t+1}, K_t} \mathbb{E}_t c^E_{t,t+1} \tag{2}$$
$$c^E_{t,t+1} \leq F(\gamma^t N_{t+1}, K_t) - W_{t+1}N_{t+1} - (1 + r^f_{t+1})K_t$$
$$K_t \geq 0, N_t \geq 0, c^E_{t,t+1} \geq 0.$$

where we assume, for the sake of simplification, that capital fully depreciates in production, that $F(\cdot,\cdot)$ is homogeneous of degree one and that the productivity of labor grows at a constant rate of γ. The homogeneity property allows to rewrite the firms profits as:

$$\pi^F_{t+1} = \gamma^t N_{t+1} F\left(1, \frac{K_t}{\gamma^t N_{t+1}}\right) - \frac{W_{t+1}}{\gamma^t}(\gamma^t N_{t+1}) - (1 + r^f_{t+1})(\gamma^t N_{t+1})\frac{K_t}{\gamma^t N_{t+1}}.$$

In a Solow economy, with a supply of labor equal to 1, the output will grow at the rate γ, and the capital to labor ratio will decrease at the rate γ, so that it will be constant in term of efficient units, $\frac{K_t}{\gamma^t}$.

In what follows we will assume $\gamma = 1$ for the sake of exposition and analyze the stationary steady state as our reference, but the reinterpretation of the different variables as being measured in efficient units imply that our results stand for any $\gamma-$homothetic economy with a rate of growth γ.

2.3 Bankers

The role of banks in the economy is to screen monitor and enforce payment on loans. This entails a cost that implies a spread between deposit and loan rates. We simplify the analysis and assume there exists a representative bank that lives for as long as it is solvent. It starts with equity E_0 and has equity E_t at time t. It lends to households that buy the bubble and to entrepreneurs to borrow to invest in their firm. It obtains an amount of deposits S_t that could be interpreted as exogenous foreign investment in the country or central bank injection. These funds are remunerated at the same rate as the riskless asset, $\underline{r}$.

The equality of assets and liabilities implies

$$K_t + L_t + O^B_t = E_t + S_t + D_t \tag{3}$$

where O^B_t is the bank's investment in the riskless asset and E_t is the bank's equity before the t period profits Π_t are realized.

For simplicity we take the bank intermediation costs as a proportion $1-\varphi$ of the banks assets

The bank's profits are be determined by:

$$\Pi_{t+1} = \min\left\{B_{t+1}, (1+r^h_{t+1})L_t\right\} - L_t + r^f_{t+1}K_t + \underline{r}(O^B_t - S_t) + r^d_{t+1}D_t) - (1-\varphi)(L_t + K_t). \quad (4)$$

and the banks participation constraint will be that the expected return on equity is at least equal to the deposit rate.

We will assume banks are price takes and, as a consequence, they will offer any quantity of loans or deposits at the market rates. For the sake of simplicity we also assume depositors are insured at a zero insurance premium and that the random supply of liquidity S_t is inelastic.

The three interest rates the bank faces are related. First, the bank will lend both to households and firms provided the expected return on the two types of loans the bank offers is the same. This implies that in any equilibrium where credit to households and firms is non zero, we have

$$\begin{aligned}(1+r^f_{t+1})B_t = {} & \Pr\left(B_{t+1} \geq L_t(1+r^h_{t+1}) \mid L_t > 0\right)(1+r^h_{t+1})B_t \\ & + \Pr\left(B_{t+1} < L_t(1+r^h_{t+1}) \mid L_t > 0\right)\mathbb{E}\left(B_{t+1} \mid B_{t+1} < L_t(1+r^h_{t+1}) \text{ and } L_t > 0\right). \end{aligned} \quad (5)$$

Equality (5) relates r^f_{t+1} to r^h_{t+1} and, as we will see, will allow to use only r^f_{t+1}. Notice that probabilities are conditional on $L_t > 0$, since otherwise r^h_{t+1} is not defined.

Second, the relationship between r^f_{t+1} and r^d_{t+1} is given by the cost of financial intermediation, and, for the sake of simplicity, we will assume it is linear:

$$\varphi\left(1+r^f_{t+1}\right) = 1 + r^d_{t+1}, \quad \varphi \in (0,1) \quad (6)$$

This implies that $r^d_{t+1} < r^h_{t+1}$, so households will never borrow and deposit at the same time.

The process for equity will determine whether the bank is solvent or bankrupt. It will depend upon the previous period's profits, that are only realized once the price of the bubble B_t is obtained:

$$E_{t+1} = \max\left\{0, E_t + \Pi_t\right\}. \quad (7)$$

This means that the bank will lend on the basis of the equity it has before profits, E_t and will only then incorporate its profits.[2]

2.4 Equilibrium

An equilibrium is characterized by a path

$$\{r^f_{t+1}, r^b_{t+1}, r^d_{t+1}, q_t, N_t, W_t, K_t, L_t, D_t, B_t, E_t, q_t, c_{t-1,t,} c^E_{t-1,t,}, c^B_{t-1,t,}\}.$$

This is simplified as, at each period t, the supply of bubbles and labor is fixed, so that $q_t = 1$ and $N_t = 1$ and the variables $E_t, c_{t-1,t,} c^E_{t-1,t,}, c^B_{t-1,t,}$ and W_t, are determined by the decisions from the previous period, so we are left with six key variables, $(r^f_{t+1}, K_t, L_t, D_t, B_t)$, and r^h_{t+1} and r^d_{t+1} deriving from r^f_{t+1} through the cost of financial intermediation and credit risk.

Because of the existence of three different regimes, the equilibrium should also specify under what circumstances a regime switch occurs. In our context it seems natural to hypothesize the hysteresis of decisions so that a regime switch will only occur when driven by changes in parameters that forces it out of its trajectory. We will therefore assume that the probability of staying in the same equilibrium is either 1 if the equilibrium is feasible or 0 otherwise. This means that the determination of the equilibrium and the expectations have to be solved simultaneously, which implies an additional complexity.

3 Equilibrium Properties

We will consider here the laws of motion of the different variables and abstract from the issue of banks' dividend distribution and capital accumulation that will be discussed later on. This allow us to consider the bank's supply of funds, which equals the sum of external funds deposited, S_t, and the bank accumulated equity, E_t, as the realization of a random variable whose distribution is known. We will denote this liquidity by $\mathcal{L}_t$.

To begin with, in equilibrium, the demand for the bubble is derived from the first order condition of (1) with respect to q_t, which implies that the expected return from buying the bubble equals the interest rate (a condition sometimes referred to as "no arbitrage", a term that does not apply stricto sensu in our framework).

[2]Interestingly, if we consider the alternative formulation where banks lend before knowing their own capital and then find the required liquidity in the market, the supply of bank credit will determine the price of the bubble and make banks' profits indeterminate. This is in line with the joint determination of bank credit and bubbles' prices. Still, it means pushing the hypothesis of a representative bank too far, as the bank will then determine its own solvency.

In our framework, because households may be on the borrowing or on the deposit side, the expected return on the bubble may equal either the lending rate or the deposit rate. In addition, as there is always the option of investing in the riskless asset, in some cases the deposit rate will equal the riskless rate. As a consequence, three different regimes appear: a levered $L-$regime, where households borrow to buy the bubble, and an unlevered $U-$regime, where investors are indifferent between buying the bubble or depositing and an $\underline{r}$ regime, where the interest rate on deposits is determined by the return on the riskless asset and depositors are indifferent between depositing in the bank or investing in the riskless asset. Denoting the expected future price of the bubble by $\mathcal{E}^L_{t+1} = E(B_{t+1} \mid L)$, by $\mathcal{E}^U_{t+1} = E(B_{t+1} \mid U)$ and by $\mathcal{E}^0_{t+1} = E(B_{t+1} \mid \underline{r})$, the "no-arbitrage" condition requires that $\frac{\mathcal{E}^0_{t+1}}{B_t} = 1 + \underline{r}$ and $\frac{\mathcal{E}^U_{t+1}}{B_t} = 1 + r^d_{t+1}$, in the $\underline{r}$ and U regimes respectively and an equivalent expression, though complicated by the household limited liability,

$$\mathbb{E}\left(B_{t+1} \mid B_{t+1} \geq L_t(1 + r^h_{t+1})\right) = (1 + r^h_{t+1})B_t$$

for the $L-$regime. It is easy to show that, in equilibrium, the above expression simplifies to $\mathcal{E}^L_{t+1} = (1 + r^f_{t+1})B_t$, simply by multiplying by $\Pr(B_{t+1} \mid B_{t+1} \geq L_t(1 + r^h_{t+1}))$ and using (5).

Consequently we can establish the characterization of the three different regimes:

In the $L-$regime the laws of motion are defined by the following equations:

$$\begin{aligned}
\mathcal{E}^L_{t+1} &= B_t(1 + r^f_t) \\
K_t + L_t &= \mathcal{L}_t \\
B_t &= W_t + L_t \\
D_t &= 0 \\
\frac{\partial F(N_{t+1}, K_t)}{\partial K_t} &= 1 + r^f_{t+1} \\
\frac{\partial F(N_t, K_{t-1})}{\partial N_t} &= W_t \\
N_t &= N_{t+1} = 1,
\end{aligned} \tag{8}$$

provided $L_t \geq 0$, $r^d_{t+1} \geq \underline{r}$ and $B_t \geq \underline{B}$.

In the $U-$regime the laws of motion are similar to the previous case, but now the no arbitrage condition refers to the deposit rate and L_t becomes $-D_t$. The equilibrium is characterized by the following equations:

$$
\begin{aligned}
\mathcal{E}^U_{t+1} &= B_t(1+r^d_t) \\
K_t - D_t &= \mathcal{L}_t \\
B_t &= W_t - D_t \geq \underline{B} \\
L_t &= 0 \\
\frac{\partial F(N_{t+1}, K_t)}{\partial K_t} &= 1 + r^f_{t+1} \\
\frac{\partial F(N_t, K_{t-1})}{\partial N_t} &= W_t \\
N_t &= N_{t+1} = 1,
\end{aligned}
\tag{9}
$$

provided $D_t \geq 0$, $r^d_{t+1} \geq \underline{r}$ and $B_t \geq \underline{B}$.

In the $\underline{r}-$regime the laws of motion are similar but the minimum interest rate accepted on the banks' liabilities is $\underline{r}$, so that $r^d_{t+1} = \underline{r}$ and $1 + r^f_{t+1} = \frac{1+r}{\varphi}$ immediately determines now the no arbitrage condition refers to the deposit rate and L_t becomes $-D_t$. The equilibrium is characterized by the following equations:

$$
\begin{aligned}
\mathcal{E}^{\underline{r}}_{t+1} &= B(\underline{r})(1+\underline{r}) \\
K_t - D_t &= \mathcal{L}_t \\
B(\underline{r}) &= W_t - D_t \geq \underline{B} \\
L_t &= 0 \\
\frac{\partial F(N_{t+1}, K_t)}{\partial K_t} &= 1 + \underline{r} \\
\frac{\partial F(N_t, K_{t-1})}{\partial N_t} &= W_t \\
N_t &= N_{t+1} = 1,
\end{aligned}
\tag{10}
$$

provided $D_t \geq 0$, and $B_t \geq \underline{B}$.

We now prove that an equilibrium exists. To do this we first show properties that B and r^f satisfy in equilibrium (Lemma 1). Then we derive thresholds on $\mathcal{L}$ for which the different regimes exist (Lemma 2). We use the results in these lemmas to prove the existence of an equilibrium in Proposition 1.

Lemma 1. *Let $B^L(\mathcal{E}^L, \mathcal{L})$ and $B^U(\mathcal{E}^U, \mathcal{L})$ denote be the equilibrium price of the bubble for the L and U regimes for given values of $\mathcal{E}^L$ and $\mathcal{E}^U$ and a given realization of $\mathcal{L}$. Similarly let $r^L(\mathcal{E}^L, \mathcal{L})$ and $r^U(\mathcal{E}^U, \mathcal{L})$ denote the equilibrium interest rates that the entrepreneurs pay* *to borrow in these regimes. Then*

$$\frac{\partial r^L}{\partial \mathcal{E}^L} > 0; \frac{\partial r^L}{\partial \mathcal{L}} < 0; \frac{\partial B^L}{\partial \mathcal{E}^L} > 0; \frac{\partial B^L}{\partial \mathcal{L}} > 0$$
$$\frac{\partial r^U}{\partial \mathcal{E}^U} > 0; \frac{\partial r^U}{\partial \mathcal{L}} < 0; , \frac{\partial B^U}{\partial \mathcal{E}^U} > 0; \frac{\partial B^U}{\partial \mathcal{L}} > 0$$

Proof. See Appendix A. □

Lemma 2. *(i)For $\mathcal{L}\epsilon(\min(\mathcal{L}^L(\mathcal{E}^L), \mathcal{L}^{L,\underline{B}}(\mathcal{E}^L)), \mathcal{L}^{L,\underline{r}}(\mathcal{E}^L))$ an L-regime equilibrium exists; (ii) For $\mathcal{L} \leq \mathcal{L}^{\mathcal{U}}(\mathcal{E}^{\mathcal{U}})$ a U-regime equilibrium exists; (iii) For $\mathcal{L} > \mathcal{L}^{L,\underline{r}}(\mathcal{E}^L)$ an $\underline{r}$ -regime equilibrium exists*

Furthermore,

$$\frac{\partial \mathcal{L}^L}{\partial \mathcal{E}^L}, \frac{\partial \mathcal{L}^U}{\partial \mathcal{E}^U}, \frac{\partial \mathcal{L}^{L,\underline{B}}}{\partial \mathcal{E}^L}, \frac{\partial \mathcal{L}^{U,\underline{B}}}{\partial \mathcal{E}^U} < 0.$$

Proof. See Appendix A. □

Conjecture: For φ sufficiently high (low intermediation cost), $\mathcal{L}^L(\mathcal{E}^L) < \mathcal{L}^U(\mathcal{E}^U)$

Intuition: $\mathcal{E}_{t+1}^L$ corresponds to the integral for $L > W$ while $\mathcal{E}_{t+1}^U$corresponds to the integral for $L < W$, so that $\mathcal{E}_{t+1}^L > \mathcal{E}_{t+1}^U$.By definition, the thresholds $\mathcal{L}^L(\mathcal{E}^L)$ and $\mathcal{L}^U(\mathcal{E}^U)$ satisfy respectively :$\frac{\partial F}{\partial K_l}(1, \mathcal{L}^L(\mathcal{E}^L)) = \frac{\mathcal{E}_{t+1}^L}{W_t}$ and $\frac{\partial F}{\partial K_l}(1, \mathcal{L}^U(\mathcal{E}^U)) = \frac{1}{\varphi}\frac{\mathcal{E}_{t+1}^U}{W_t}$. For $\varphi = 1$, $\frac{\mathcal{E}_{t+1}^L}{W_t} > \frac{1}{\varphi}\frac{\mathcal{E}_{t+1}^U}{W_t}$ and, because of the concavity of $F(1, K)$, this implies $\mathcal{L}^L(\mathcal{E}^L) < \mathcal{L}^U(\mathcal{E}^U)$

Proposition 1. *Let $\mathcal{G}$ denote the probability distribution of $\mathcal{L}$. Assume that the support of $\mathcal{G}$ is $[0, \bar{\mathcal{L}}]$ for $\bar{\mathcal{L}} < \infty$. Then an equilibrium exists in this economy.*

Proof. See Appendix A. □

3.1 The steady state riskless benchmark

The steady state certainty case provides an interesting benchmark to examine some of the characteristics of the equilibrium, even if in the certainty steady state the economy will stay indefinitely in one of the three regimes. These properties will match the welfare properties of the equilibrium in the stochastic OLG equilibrium.

Because the price of the bubble is constant, this implies that, in a bubbly regime, the return on holding the bubble is zero. Still in an $L-$regime, the interest rate on the bubble is the lending interest rate, so that $r^f = 0$, and the golden rule of equality between interest and growth rates holds. On the other hand, because in the $U-$regime the no-arbitrage condition holds for a deposit interest rate, $r^d = 0$, implying firms face a lending rate of $r^f = \frac{1}{\varphi} - 1 > 0$

that is higher than the growth rate. Finally in the $\underline{r}$ case, interest rates on deposits are reduced to $\underline{r}$ as the bank has excess liquidity invests itself in the riskless asset.

3.2 Generalization

So far our equilibrium depends exclusively upon the realization of the capital inflow/outflow shock. Still, it is easy to see that productivity shocks could play the same role, as stated and proven in the following proposition.

Proposition 2. *Consider a production function that is subject to a productivity shock, A_t. Then for every $\mathcal{L}$ there is an A such that the economy attains the same equilibrium.*

Proof. See Appendix A. □

Remark 1. *We are going to consider shocks that occur before K is chosen. If the shock occurs after K is chosen, then capital allocation would not be affected by the shock, and production will. This will incorporate an extra source of risk in the economy, i. e. the firm might not be able to pay its loans, which will only complicate the setup, without changing considerably our main results.*

3.3 Welfare Properties

In defining welfare we have to take into account the welfare of the current as well as all future generations and, in particular, the risks they inherit of a bubble bursting. Because each generation consumes only when old, we cannot discount the different utilities, so we assume, as, for instance Allen and Gale (1997), a planner that wants to maximize the long run average of the expected utilities of the different generations, given the starting capital K_0. This corresponds to the ex ante preferences of agents under the veil of ignorance whereby agents neither know ex ante whether they will be consumers, entrepreneurs or bankers and at what time they will be born. Assume $\mathcal{G}_t$ is the probability distribution of $\mathcal{L}_t$ taking into account all information relative to $t-1$.[3]

$$V(K_0) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \left[\mathbb{E}_t c_{t,t+1} + \mathbb{E}_t c^E_{t,t+1} + \mathbb{E}^B_t c_{t,t+1}\right] \mid K_0, \mathcal{G}_t).$$

The utilitarian solution combined with risk neutrality allows to simplify the problem. This is the case because the welfare of each generation is the sum of the consumption of

[3] We maximize the limit of the expecte value, while Allen and Gale (1997) maximize the expected value of the limit. The reason is that in our case risk neutrality might cause the limit to not exist.

households, entrepreneurs and bankers, allowing us to disregard redistribution between the three classes of agents: $\mathbb{E}_t c_{t,t+1} + \mathbb{E}_t c^E_{t,t+1} + \mathbb{E}^B_t c_{t,t+1}$.

At any time $t+1$, because repayment of loans and deposits are transfers between the bank and the agents, the aggregate goods available, $Y_{t+1} + S_{t+1}$ are either consumed $(c^E_{t,t+1} + c_{t,t+1} + c^B_{t,t+1})$, invested in the production function, K_{t+1} as capital or used to repay capital and interest to foreign investors, $(1 + r^d_{t+1})S_t$.[4] Consequently, we have:

$$c_{t,t+1} + c^E_{t,t+1} + c^B_{t,t+1} = Y_{t+1} - ((1 + r^d_{t+1})S_t - S_{t+1}) - K_{t+1}. \tag{11}$$

Denote by δ the probability that banks go bankrupt. As default can only occur when the bubble burst and the bank equity is not sufficient to absorb the losses, $\delta = \Pr(E_{t+1} < 0)$. We assume that, when a banking crisis occurs, banks' assets are put to a different use and the banks' balance sheet shrinks. In order to model the cost of a banking crises we make the simplifying assumption that a crisis implies a loss in output, denoted ΔY, during the period where it occurs.[5]

Consequently, the ex ante welfare associated to a random consumption $(c_{t,t+1}, c^E_{t,t+1}, c_{t,t+1})$ is equivalent, under our utilitarian assumptions, to the maximization of

$$\mathbb{E}\left[Y_{t+1} - ((1 + r^d_{t+1})S_t - S_{t+1}) - K_{t+1}\right],$$

with $Y_{t+1} = F(N_{t+1,}K_t)$ when banks are solvent and $Y_{t+1} = F(N_{t+1,}K_t) - \Delta Y$ in a banking crisis. Denote by $\widetilde{\Delta Y}$ the random variable that takes the value 0 in the absence of a crisis and ΔY when there is a crisis, then the equilibrium in each perido will be characterized by the realization of a random variable $\widetilde{S}_t$, which, in turn will determine a realization for the equilibrium value of capital, $\widetilde{K}_t$.

Consequently, if we define $W_T(K_0, K_T)$ as

$$\begin{aligned} V(K_0) &= \lim_{T\to\infty} W_T(K_0, K_T) \\ W_T(K_0, K_T) &= \frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}\left[F(N_{t+1,}\widetilde{K}_t) - \widetilde{K}_t - \widetilde{\Delta Y} - ((1 + r^d_{t+1})S_t - S_{t+1})\right]. \end{aligned}$$

Reordering the summations, we obtain

[4] If instead of foreign investors it is the central bank that injects liquidity, then the central bank remuneration is redistributed and is part of the aggregate consumption.

[5] The alternative of considering explicitly the cost of credit misallocation, with a reduction of the aggregate capital would imply a lower level of wages in the next period with a more complex dynamics as the absorption of the shock will only take place progressively

$$W_T(K_0, K_T) = T^{-1}(F(N_{t+1,}K_0) - ((1+r_1^d)S_0 + \mathbb{E}(\widetilde{S_T} - K_T) + \\ + \sum_{t=1}^{T-1}\left[\mathbb{E}(F(N_{t+1,}\widetilde{K_t}) - \widetilde{K_t} - \widetilde{\Delta Y} + r_{t+1}^d \widetilde{S}_t)\right].$$

This formulation allows to see three main partial equilibrium properties of the welfare function that may be helpful later on in providing some intuition on the efficiency of macro-prudential policies is analyzed. Unfortunately, the general equilibrium properties are much more involved.

First, at any time t, a decrease in the probability of a banking crisis, is welfare increasing.

Second, at any time t, a mean preserve spread for $\widetilde{K_t}$ is welfare decreasing (Rothschild and Stiglitz, 1976).

Third, a change in the distribution of the capital equilibrium from $\widetilde{K_t}$ to $\widetilde{K_t} + \widetilde{\Delta}(\widetilde{K_t})$ will increase welfare provided it increases $\mathbb{E}(F(N_{t+1,}\widetilde{K_t} + \widetilde{\Delta}) - \widetilde{\Delta}(\widetilde{K_t}))$.

When $\delta = 0$, which occurs for the U-regime, the condition states that the expected marginal productivity of capital should be equal to one, which is corresponds to the golden rule. This unsurprising result constitutes a natural extension of the steady state case.

4 Equilibrium in a Cobb-Douglas Economy

For the sake of tractability, we will now assume that $Y_{t+1} = F(N_{t+1}, K_t) = N_{t+1}^{1-\alpha}K_t^{\alpha}$ and let's set $\alpha = \frac{1}{2}$. The first order condition, for K_t is given by $\frac{1}{2}K_t^{-\frac{1}{2}}N_{t+1}^{\frac{1}{2}} = 1 + r_{t+1}^f$, so that in equilibrium $K_t = \frac{1}{4\left(1+r_{t+1}^f\right)^2}$, while the first order condition for W_{t+1} is $\frac{1}{2}N_t^{-\alpha}K_{t-1}^{\alpha} = W_{t+1}$, implying $W_{t+1} = \frac{1}{4\left(1+r_t^f\right)}$.

These specifications allow to provide an explicit solution to the system of equations (8,9) and obtain necessary conditions for the existence of the three different regimes. The $L-$regime is a function of $\mathcal{L}_t$ and the conditional expectation $E(B\ |\mathrm{L}) = \mathcal{E}_{t+1}^L$. So, as $B_t = \frac{\mathcal{E}_{t+1}^L}{1+r^L(\mathcal{L}))}$and $K^L(\mathcal{L}) = \frac{1}{4(1+r^L(\mathcal{L}))^2}$, after replacing in $B_t = W_t + \mathcal{L}_t - \frac{1}{4(1+r^L(\mathcal{L}))^2}$ it is possible to solve for the equilibrium values of $r^L(\mathcal{L})$, $B(\mathcal{L})$ and $K(\mathcal{L})$.

$$
\begin{aligned}
r^L(\mathcal{L}) &= \frac{1}{2}\frac{1}{W+\mathcal{L}}\left[\mathcal{E}^L_{t+1} + \sqrt{\left(\mathcal{E}^L_{t+1}\right)^2 + W + \mathcal{L}}\right] - 1 \\
K^L(\mathcal{L}) &= \frac{1}{4(1+r^L(\mathcal{L}))^2} \\
B^L(\mathcal{L}) &= W + \mathcal{L} - K^L(\mathcal{L}) \\
&= \frac{2(W+\mathcal{L})\mathcal{E}^L_{t+1}}{\mathcal{E}^L_{t+1} + \sqrt{\left(\mathcal{E}^L_{t+1}\right)^2 + W + \mathcal{L}}} \\
&= 2\mathcal{E}^L_{t+1}\left[\sqrt{\left(\mathcal{E}^L_{t+1}\right)^2 + W + \mathcal{L}} - \mathcal{E}^L_{t+1}\right] \\
B^L(\mathcal{L}) &\geq W.
\end{aligned}
$$

Two conditions are required for the equilibrium to exist, first that $B^L(\mathcal{L}) > W$, and second that the price of the bubble is greater than its intrinsic value, $B^L(\mathcal{L}) \geq \underline{B}$. From Lemma 1 and its corollary we can establish

$$
\begin{aligned}
\mathcal{L}^L &= \left(\frac{W}{2\mathcal{E}^L_{t+1}}\right)^2 \\
\mathcal{L}^{L,\underline{B}} &= \left(\frac{\underline{B}}{2\mathcal{E}^L_{t+1}}\right)^2 + \underline{B} - W.
\end{aligned}
$$

Symmetrically, the $U-$regime solution as a function of the conditional expectation $E(B\,|\mathrm{U}) = \mathcal{E}^U_{t+1}$ is obtained as

$$
\begin{aligned}
r^U(\mathcal{L}) &= \frac{1}{2\varphi}\frac{1}{W+\mathcal{L}}\left[\mathcal{E}^U_{t+1} + \sqrt{\left(\mathcal{E}^U_{t+1}\right)^2 + \varphi^2\left(W+\mathcal{L}\right)}\right] - 1 \\
K^U(\mathcal{L}) &= \frac{1}{4(1+r^U(\mathcal{L}))^2} \\
B^U(\mathcal{L}) &= W + \mathcal{L} - K^U(\mathcal{L}) \\
&= \frac{2(W+\mathcal{L})\mathcal{E}^U_{t+1}}{\mathcal{E}^U_{t+1} + \sqrt{\left(\mathcal{E}^U_{t+1}\right)^2 + \varphi^2\left(W+\mathcal{L}\right)}} \\
&= \frac{2\mathcal{E}^U_{t+1}}{\varphi^2}\left[\sqrt{\left(\mathcal{E}^U_{t+1}\right)^2 + \varphi^2\left(W+\mathcal{L}\right)} - \mathcal{E}^U_{t+1}\right] \\
B^U(\mathcal{L}) &< W.
\end{aligned}
$$

where $r^U(\mathcal{L})$ is the borrowing rate in this regime.

Similarly, from Lemma 1 and its corollary we can establish

$$\mathcal{L}^U = \left(\frac{\varphi W}{2\mathcal{E}^U_{t+1}}\right)^2$$
$$\mathcal{L}^{U,\underline{B}} = \left(\frac{\varphi \underline{B}}{2\mathcal{E}^U_{t+1}}\right)^2 + \underline{B} - W.$$

Consequently, the U-regime exists for $\mathcal{L} \in \left[\mathcal{L}^{U,\underline{B}}, \mathcal{L}^U\right]$, so a necessary condition for the existence of this regime is that $\mathcal{L}^{U,\underline{B}} \leq \mathcal{L}^U$, which occurs if $W \geq \underline{B}$.

We also assume that there is an outside option that offers a return $\underline{r}$ and banks can also invest in this option. Therefore, banks will only lend to firms as long as $r^L(\mathcal{L}) \geq \underline{r}$. This occurs as long as $\mathcal{L} \leq \mathcal{L}^{L,\underline{r}}$, where

$$\mathcal{L}^{L,\underline{r}} = \frac{1 + 4(1+\underline{r})\mathcal{E}^{\underline{r}}_{t+1}}{4(1+\underline{r})^2} - W.$$

Additionally, as long as $r^L(\mathcal{L}) \geq \underline{r}$ banks will not be find it optimal to invest in the outside option, since they would rather lend to firms.

Consider a $\mathcal{L} > \mathcal{L}^{L,\underline{r}}$. Then the we can't be in the L-regime, because banks would not be willing to lend at a rate r^f, $r^f \leq \underline{r}$. So only equivalent of the no arbitrage equation holds for a deposit rate $r^d = \underline{r}$, that is:

$$B(\underline{r}) = \frac{\mathcal{E}^{\underline{r}}_{t+1}}{1+\underline{r}}.$$

From the household's budget constraint, we know

$$D + O = W - B(\underline{r}),$$

so

$$-D \geq \frac{\mathcal{E}^{\underline{r}}_{t+1}}{1+\underline{r}} - W,$$

where O is the amount invested in the outside option. Similarly, if O^B denotes the amount invested by banks in the outside option, the balance sheet of the bank will be

$$K(\underline{r}) + O^B = \mathcal{L} + D.$$

Conditions for this to be an equilibrium are the following

$$\begin{aligned}
W &> \frac{\mathcal{E}^{r}_{t+1}}{1+\underline{r}} \\
\mathcal{L} &\geq K(\underline{r}) - D \\
&= \frac{1}{4(1+\underline{r})^2} - D \\
&\geq \frac{1}{4(1+\underline{r})^2} + \frac{\mathcal{E}^{r}_{t+1}}{1+\underline{r}} - W \\
&= \mathcal{L}^{L,\underline{r}}.
\end{aligned}$$

4.1 Random walk

We consider an economy where $\mathcal{L}$ follows a random walk. As before, in this economy a bubbleless equilibrium will always exist. Now, if $B_0 > 0$ a bubbly equilibrium will exist, provided that the probability of the bubble bursting is not too high. The interesting point is that the probability of a bubble bursting changes at every point t, so that the cost benefit of a macroprudential policy as well as its effectiveness will change at each date.

The change in $\mathcal{L}$ will then affect in different ways the equilibrium and the effectiveness of the policy. Thus, for instance, the impact of a random walk can be seen with a binomial random walk on $\mathcal{L}$, and this allows to establish the impact on the bubble and on the expectations.

To illustrate this point, consider a T horizon geometric binomial random walk on $\mathcal{L}$, with a initial price for the bubble of B_t at time t, for a realization of the liquidity shock $\mathcal{L}_t$.

Assume that at each period $\mathcal{L}_{t+1} = u\mathcal{L}_t$, which occurs with probability p or $\mathcal{L}_{t+1} = d\mathcal{L}_t$, which occurs with probability $1-p$, with $u > d$. . Let $\mathcal{L}^L$ and $\mathcal{L}^{L,\underline{B}}$ as defined before and consider $\mathcal{L}^L < u\mathcal{L}$ for some $\mathcal{L}$, so that the $L-$regime exists. Then there are two cases:

1. If $\mathcal{L}^L \leq d\mathcal{L} < u\mathcal{L}$

$$\begin{aligned}
\mathcal{E}^{L}_{t+1} &= p(W_t + u\mathcal{L}_t - K^L(u\mathcal{L}_t)) + (1-p)(W_t + d\mathcal{L}_t - K^L(d\mathcal{L}_t)) \\
\mathcal{E}^{L}_{t+1} &= \frac{1}{4(1+r^L(\mathcal{L}_t))} + (pu + (1-p)d)\mathcal{L}_t - p\left(\frac{1}{4(1+r^L(u\mathcal{L}_t))^2}\right) \\
&\quad -(1-p)\left(\frac{1}{4(1+r^L(d\mathcal{L}_t))^2}\right),
\end{aligned}$$

where $r^L(\mathcal{L}) = \frac{1}{2}\frac{1}{W_t+\mathcal{L}}\left[\mathcal{E}^{L}_{t+1} + \sqrt{\left(\mathcal{E}^{L}_{t+1}\right)^2 + W_t + \mathcal{L}}\right] - 1.$

2. If $d\mathcal{L} < \mathcal{L}^L < u\mathcal{L}$

$$\begin{aligned}
\mathcal{E}^L_{t+1} &= p(W_t + u\mathcal{L}_t - K^L(u\mathcal{L}_t)) \\
\mathcal{E}^L_{t+1} &= p\left(\frac{1}{4(1+r^L(\mathcal{L}_t))} + u\mathcal{L}_t - \frac{1}{4(1+r^L(u\mathcal{L}_t))^2}\right).
\end{aligned}$$

Similarly, in the U-regime, assume $\mathcal{L}^{U,\underline{B}} < d\mathcal{L} < \mathcal{L}^U$ for some $\mathcal{L}$ so that the U-regime exists. Two cases are possible:

1. If $\mathcal{L}^{U,\underline{B}} < d\mathcal{L} < u\mathcal{L} \leq \mathcal{L}^U$ then

$$\begin{aligned}
\mathcal{E}^U_{t+1} &= p(W_t + u\mathcal{L}_t - K^U(u\mathcal{L}_t)) + (1-p)(W_t + d\mathcal{L}_t - K^U(d\mathcal{L}_t)) \\
\mathcal{E}^U_{t+1} &= \frac{1}{4(1+r^U(\mathcal{L}_t))} + (pu + (1-p)d)\mathcal{L}_t - p\left(\frac{1}{4(1+r^U(u\mathcal{L}_t))^2}\right) \\
&\quad -(1-p)\left(\frac{1}{4(1+r^U(d\mathcal{L}_t))^2}\right),
\end{aligned}$$

where $r^U(\mathcal{L}) = \frac{1}{2\varphi}\frac{1}{W_t+\mathcal{L}}\left[\mathcal{E}^U_{t+1} + \sqrt{\left(\mathcal{E}^U_{t+1}\right)^2 + \varphi(W_t+\mathcal{L})}\right] - 1.$

2. If $\mathcal{L}^{U,\underline{B}} < d\mathcal{L} < \mathcal{L}^U < u\mathcal{L}$ then

$$\begin{aligned}
\mathcal{E}^U_{t+1} &= (1-p)(W_t + d\mathcal{L}_t - K^U(d\mathcal{L}_t)) \\
\mathcal{E}^U_{t+1} &= (1-p)\left(\frac{1}{4(1+r^U(\mathcal{L}_t))} + d\mathcal{L}_t - \frac{1}{4(1+r^U(d\mathcal{L}_t))^2}\right).
\end{aligned}$$

4.2 Productivity shocks

With the previous specifications, assume now the shock comes from the productivity side, so that the production function depends upon a random parameter A, namely, $Y_{t+1} = F(N_{t+1}, K_t) = AN_{t+1}^{1-\alpha}K_t^{\alpha}$ and let's set $\alpha = \frac{1}{2}$. The first order condition, for K_t is given by $\frac{A}{2}K_t^{-\frac{1}{2}}N_{t+1}^{\frac{1}{2}} = 1 + r^f_{t+1}$, so that in equilibrium $K_t = \left(\frac{A}{2\left(1+r^f_{t+1}\right)}\right)^2$,while the first order condition for W_t is $\frac{A}{2}N_t^{-\alpha}K_{t-1}^{\alpha} = W_t$, implying $W_t = \frac{A^2}{4\left(1+r^f_t\right)}$

Again, these specifications allow to provide an explicit solution to the system of equations (8,9) and obtain necessary conditions for the existence of the three different regimes.

The $L-$regime is a function of A_t and the conditional expectation $E(B\ |\text{L}) = \mathcal{E}^L_{t+1}$. So, as $B_t = \frac{\mathcal{E}^L_{t+1}}{1+r^L(A))}$and $K^L(A) = \frac{A^2}{4(1+r^L(A))^2}$, after replacing in $B_t = W_t + \mathcal{L}_t - \frac{A^2}{4(1+r^L(A))^2}$ we solve for the equilibrium values of $r^L(A), B(A)$ and $K(A)$.

$$
\begin{aligned}
r^L(A) &= \frac{1}{2}\frac{1}{W+\mathcal{L}}\left[\mathcal{E}^L_{t+1} + \sqrt{\left(\mathcal{E}^L_{t+1}\right)^2 + A^2(W+\mathcal{L})}\right] - 1 \\
K^L(A) &= \frac{A^2}{4(1+r^L(A))^2} \\
B^L(A) &= \frac{2(W+\mathcal{L})\mathcal{E}^L_{t+1}}{\mathcal{E}^L_{t+1} + \sqrt{\left(\mathcal{E}^L_{t+1}\right)^2 + A^2\left(W+\mathcal{L}\right)}} \\
&= \frac{2}{A^2}\mathcal{E}^L_{t+1}\left[\sqrt{\left(\mathcal{E}^L_{t+1}\right)^2 + A^2\left(W+\mathcal{L}\right)} - \mathcal{E}^L_{t+1}\right]
\end{aligned}
$$

Similarly, as before we can establish a lower bound for A^L so that for $A \leq A^L$ the bubble satisfies $B^L(A) \geq \max(W, \underline{B})$, which is given by

$$
A^L = \frac{2\mathcal{E}^L_{t+1}}{\max(W,\underline{B})}\sqrt{W+\mathcal{L}-\max(W,\underline{B})}.
$$

To satisfy the $r^L(A) \geq \underline{r}$ condition, the equivalent condition is

$$
A \geq A^{L,\underline{B}} = \sqrt{4(1+\underline{r})\left[\left(W+\mathcal{L}\right)\left(1+\underline{r}\right) - \mathcal{E}^L_{t+1}\right]}
$$

required for the equilibrium to exist, imposes a limit on the values of the productivity shock

Symmetrically, the $U-$regime solution as a function of the conditional expectation $E(B\,|\mathrm{U}) = \mathcal{E}^U_{t+1}$ is obtained as

$$
\begin{aligned}
r^U(A) &= \frac{1}{2\varphi}\frac{1}{W+\mathcal{L}}\left[\mathcal{E}^U_{t+1} + \sqrt{\left(\mathcal{E}^U_{t+1}\right)^2 + \varphi^2 A^2\left(W+\mathcal{L}\right)}\right] - 1 \\
K^U(A) &= \frac{A^2}{4(1+r^U(A))^2} \\
B^U(A) &= \frac{2(W+\mathcal{L})\mathcal{E}^U_{t+1}}{\mathcal{E}^U_{t+1} + \sqrt{\left(\mathcal{E}^U_{t+1}\right)^2 + \varphi^2 A^2\left(W+\mathcal{L}\right)}} \\
&= \frac{2\mathcal{E}^U_{t+1}}{A^2\varphi^2}\left[\sqrt{\left(\mathcal{E}^U_{t+1}\right)^2 + \varphi^2 A^2\left(W+\mathcal{L}\right)} - \mathcal{E}^U_{t+1}\right] \\
B^U(\mathcal{L}) &< W.
\end{aligned}
$$

4.2.1 Welfare Properties

As before, the efficient allocation requires that the expected marginal productivity of capital equals one. Contrarily to the random liquidity case, it implies that the amount of capital is contingent on the productivity shock, with higher productivity necessitating a higher capital.

5 Macroprudential policy

As it is well known, macroprudential policy has a time and a cross section dimension, and while the time dimension is fully present, the structural dimension is absent from our set up. Still, the cost of a systemic crisis may related to the cross section dimension is partially present in the cost of a bubble bursting. Indeed, lower connections between banks will imply lower systemic banks, lower contagion and less complexity (Allen and Babus, Shin 2010),

In our framework, a banks' bankruptcy can only occur in the $L-$regime. This will be the case only when the bubble burst and the equilibrium switches to the zero bubble price, although, if the bank is sufficiently capitalized the losses will be coped by the capital buffer.

A macroprudential policy $m(\cdot)$ will be a function of a vector of observable variables $\mathbf{X}(A, \mathcal{L})$ that affects the equilibrium outcome, so that the equilibrium Θ is a function $\Theta(A, \mathcal{L}, m(\mathbf{X}(A, \mathcal{L})))$. We take $m(\cdot)$ to be a real function. Thus, for example a loan to value rule could be defined by $m(\mathbf{X}(A, \mathcal{L})) = 0$ if the threshold is not reached and $m(\mathbf{X}(A, \mathcal{L})) = 1$ if the threshold is reached, in which case it affects the equilibrium. Our benchmark $m(\cdot) = 0$ will refer to the absence of binding macroprudential regulation.

In an $L-$regime, a sufficient condition for a macroprudential policy to be welfare improving, is that it is equivalent to a reduction in the riskiness of $\mathcal{L}$. Our analysis, will have to spell out what are the observable variables, $\mathbf{X}(A, \mathcal{L})$, on which the macroprudential policy is based, which constitutes a realistic limitation of the scope of the macroprudential policy before analyzing its welfare properties. It is important to remark that $m(\mathbf{X}(A, \mathcal{L}))$ affects the current outcome also through the change in expectations, and in particular through the change in the expected future value of the bubble.

We assume the policy $m(\mathbf{X}(A, \mathcal{L}))$ once in place is kept forever and that agents know the impact of the policy on the equilibrium.

A number of particular cases will be helpful in illustrating the impact of a macroprudential policy.

Definition: A full information unconstrained macroprudential policy is characterized by $\mathbf{X}(A, \mathcal{L}) = (A, \mathcal{L})$ and no constraint on $m()$.

Such a macroprudential policy is based on full information and will allow to sterilize the negative impact of any shock, so that the first best will be obtained.

Definition: An asymmetric full information constrained macroprudential policy is characterized by $\mathbf{X}(A,\mathcal{L}) = (A,\mathcal{L})$ and $m(\mathcal{L}) \leq \mathcal{L}$

The constraint seems in line with a number of macroprudential policies, that allow to limit bank lending in a boom, but, contrary to monetary policy, do not provide a mechanism to increase bank lending in a bust.

A simple way to visualize $m(\mathcal{L})$ is to consider it as a limit to loanable funds. In this case, it implies the bank has resources $\mathcal{L}$ but can only lend $m(\mathcal{L})$ to finance entrepreneurs and investment in the bubble, and has to invest the difference $\mathcal{L} - m(\mathcal{L})$ in liquid, less profitable investments, such as sovereign debt.

- As an initial benchmark, assume the macroprudential policy is full information and unconstrained.Then it is obvious that the efficient macroprudential policy will be something close to a perfect sterilization policy. By injecting or withdrawing $\Delta\mathcal{L}$, the regulator will accomodate any shock and set $r = 0$, a constant level B, and values for K, W and L such that the efficient allocation is reached. If we assume that the production function is a Cobb-Douglas with $\alpha = \frac{1}{2}$, this corresponds to $B > \frac{1}{4}$ for the bubble, $K = \frac{1}{4}$, and $W = \frac{1}{4}, L = B - \frac{1}{4}$.

It is not surprising that by sterilizing the shocks it is possible to reach the efficient allocation. What is interesting is that this could be reached for any level of B, provided B is larger than W. It is therefore intuitive that efficient macroprudential policies will try to accomodate shocks and will thus reduce the volatility of shocks as well as the probability of a bank bankruptcy. But this may be possible with a very small levered bubble B.

- A second interesting characterization is obtained with the full information constrained policy.

To define the optimal macroprudential policy, the first step is to determine the new equilibrium it defines. This is here simplified by the fact that it suffices to replace $\mathcal{L}$ by $m(\mathcal{L})$ in the equilibrium conditions (8,9).

The next step will be to characterize the optimal macroprudential policy:

$$\begin{aligned} W(K_0, K_{T+1}) = \max_{K_1} &\left[F(N_{t+1,}K_0) - ((1 + r^d_{t+1})S_t - S_{t+1}) - K_1\right] \\ &+ (1-\delta)(W(K_1, K_{T+1}) - \Delta Y) + \delta W(K_1, K_{T+1}). \end{aligned}$$

under the constraints

$$
\begin{aligned}
r^L(\mathcal{L}) &= \frac{1}{2}\frac{1}{W+m(\mathcal{L})}\left[\mathcal{E}_{t+1}^L + \sqrt{\left(\mathcal{E}_{t+1}^L\right)^2 + W + m(\mathcal{L})}\right] - 1 \\
K^L(m(\mathcal{L})) &= \frac{1}{4(1+r^L(m(\mathcal{L})))^2} \\
B^L(m(\mathcal{L})) &= 2\mathcal{E}_{t+1}^L\left[\sqrt{\left(\mathcal{E}_{t+1}^L\right)^2 + W + m(\mathcal{L})} - \mathcal{E}_{t+1}^L\right]
\end{aligned}
$$

$$
\begin{aligned}
r^U(m(\mathcal{L})) &= \frac{1}{2\varphi}\frac{1}{W+m(\mathcal{L})}\left[\mathcal{E}_{t+1}^U + \sqrt{\left(\mathcal{E}_{t+1}^U\right)^2 + \varphi^2\left(W + m(\mathcal{L})\right)}\right] - 1 \\
K^U(m(\mathcal{L})) &= \frac{1}{4(1+r^U(m(\mathcal{L})))^2} \\
B^U(m(\mathcal{L})) &= \frac{2\mathcal{E}_{t+1}^U}{\varphi^2}\left[\sqrt{\left(\mathcal{E}_{t+1}^U\right)^2 + \varphi^2\left(W + m(\mathcal{L})\right)} - \mathcal{E}_{t+1}^U\right]
\end{aligned}
$$

and

$$
m(\mathcal{L}) \leq \mathcal{L}
$$

5.1 Relationship to current macroprudential policies

When considering macroprudential policies, notice that in our model this can be written as $L_t \leq \eta(\mathbf{X}(A, \mathcal{L}))$. Indeed, this is the case for a risk weighted capital requirement, for a credit to GDP ratio, where $L_t \leq \gamma_0 Y_t - K_t$, a loan to value ratio, $L_t \leq \gamma_1 B_t$, with $B_t = W_t + \mathcal{L}_t - K_t$, or a loan to income where $L_t \leq \gamma_1 W_t$.

Consequently, for this type of macroprudential interventions, the macroprudential policy can be thought of as substituting the initial distribution of $\mathcal{L}_t$ with a distribution on $\mathcal{L}_t$ truncated on the upper side by a random upper bound. Thus, for instance, a loan to value ratio implies $\mathcal{L}_t - K_t \leq \gamma_1(W_t + \mathcal{L}_t - K_t)$, so that $\mathcal{L}_t(1-\gamma_1) \leq \gamma_1 W_t + K_t(1-\gamma_1)$ truncates the distribution of $\mathcal{L}_t$ at $\frac{\gamma_1 W_t + K_t(\mathcal{L}_t)(1-\gamma_1)}{(1-\gamma_1)}$ provided that $\gamma_1 < 1$ and has no impact if $\gamma_1 = 0$, as a one dollar increase in the bubble allows for an one dollar increase in credit.

We assume the observables are $B_t, Y_t, K_t, r_{t+1}^f, L_t, E_t$. This allow to define macroprudential policies that include:

- Credit to GDP ratio
- Asset price growth (includes real estate)
- Impose loan to value limits
- Impose loan to income limits

Whether this causes a bank bankruptcy will depend upon the repayment of the loan by consumers, which could be zero under the non-recourse contract or W in the recourse contract.

Credit to GDP ratio

The credit to GDP ratio will impose the total credit, $K_t + L_t \leq \theta Y_t$ for some positive θ. Because $K_t + L_t = \mathcal{L}$, this implies the equilibrium using the credit to GDP ratio, $\Theta(A, \mathcal{L}, m(\mathbf{X}(A, \mathcal{L})))$, will be equivalent to an equilibrium where the distribution of $\mathcal{L}$ is truncated at θY_t, with $Y_t = F(1, K_t)$ which is ex ante random.

Conjecture: Because the expected interest rate is zero, the steady state capital distribution is close to the initial one. At the same time, because future lending is restricted, with the same level of capital L_t is lower and so is $\mathcal{E}_t^L$. The cost of the macroprudential regulation is the increase in the probability of a switch to the $U-$regime. Also, the bank is less profitable, as the fraction of resources $\mathcal{L}-\theta Y_t$ is to be invested in the storage technology.

5.2 Productivity shocks

For the sake of symmetry, assume that macroprudential authorities have perfect information and are unconstrained, which, led to a sterilization policy in the case of liquidity shocks.

It is then easy to define the efficient regime as one where L is constant and the liquidity is adjusted so that marginal product of capital equals zero. In the Cobb-Douglas case, this will imply:

$K^L(A) = \frac{A^2}{4}; B^L(A) = W + \mathcal{L} - \frac{A^2}{4}, \mathbb{E}(B^L(A)) = \mathcal{E}^L$, which can be obtained with $B^L(A) = \widehat{B}$ by setting $\mathcal{L}(A) = \widehat{B} - W + \frac{A^2}{4}$, so that the liquidity policy should fine tune changes in the productivity.

Appendix A Mathematical Appendix

Proof of Lemma 1:

To avoid cumbersome notation, in this proof we abstract from time subscripts in this proof and we refer to r^f as r. Since F is strictly concave and twice differentiable, then demand for credit, $K(r) = \left(\frac{\partial F(1,K)}{\partial K}\right)^{-1}(r)$, is strictly decreasing and differentiable in r.

Consider first the L-regime. To make notation simpler, let $x \equiv \mathcal{E}^L$, and define B^L as the value of the bubble in this regime. Using the second and third equations in (8) we have

$$B^L = W + L = W + \mathcal{L} - K(r) \geq \underline{B}.$$

Additionally, from the first equation in (8) we can derive

$$x = (1 + r)(W + \mathcal{L} - K(r)),$$

which implicitly determines $r^L(x, \mathcal{L})$ that satisfies

$$\frac{\partial r^L}{\partial x} = \frac{1}{W + \mathcal{L} - K(r^L) - (1 + r^L)K'(r^L)} > 0$$
$$\frac{\partial r^L}{\partial \mathcal{L}} = -\frac{1 + r^L}{W + \mathcal{L} - K(r^L) - (1 + r^L)K'(r^L)} < 0.$$

We can now derive $B^L(x, \mathcal{L}) = W + \mathcal{L} - K(r^L(x, \mathcal{L}))$ which satisfies

$$\frac{\partial B^L}{\partial x} = -K'(r^L)\frac{\partial r^L}{\partial x} > 0$$
$$\frac{\partial B^L}{\partial \mathcal{L}} = 1 - K'(r^L)\frac{\partial r^L}{\partial \mathcal{L}} > 0.$$

To see why the second inequality holds, notice that

$$K'(r^L)\frac{\partial r^L}{\partial \mathcal{L}} = \frac{1}{1 - \frac{W+\mathcal{L}-K(r^L)}{(1+r^L)K'(r^L)}} < 1.$$

Now consider the U-regime. To make notation simpler, let $y \equiv \mathcal{E}^U$, and define B^U as the value of the bubble in this regime. Using the second and third equations in (9) we have

$$B^U = W - D = W + \mathcal{L} - K(r) \geq \underline{B}.$$

Additionally, from the first equation in (9) and the fact that $1 + r^d = \varphi(1 + r)$ we can derive

$$y = \varphi(1 + r)(W + \mathcal{L} - K(r)),$$

which implicitly determines $r^U(y, \mathcal{L})$ that satisfies

$$\frac{\partial r^U}{\partial y} = \frac{1}{\varphi(W + \mathcal{L} - K(r^U) - (1 + r^U)K'(r^U))} > 0$$
$$\frac{\partial r^U}{\partial \mathcal{L}} = -\frac{1 + r^U}{W + \mathcal{L} - K(r^U) - (1 + r^U)K'(r^U)} < 0.$$

We can now derive $B^U(y, \mathcal{L}) = W + \mathcal{L} - K(r^U(y, \mathcal{L}))$ which satisfies

$$\frac{\partial B^U}{\partial y} = -K'(r^U)\frac{\partial r^U}{\partial y} > 0$$
$$\frac{\partial B^U}{\partial \mathcal{L}} = 1 - K'(r^U)\frac{\partial r^U}{\partial \mathcal{L}} > 0.$$

To see why the second inequality holds, notice that

$$K'(r^U)\frac{\partial r^U}{\partial \mathcal{L}} = \frac{1}{1 - \frac{W+\mathcal{L}-K(r^U)}{(1+r^U)K'(r^U)}} < 1.$$

Proof of Lemma 2:

Two conditions are required for the equilibrium in the L-regime to exist: $L_t \geq 0$ and $B_t \geq \underline{B}$, and the corresponding conditions for the $U-$regime equilibrium are: $D_t \geq 0$ and $B_t \geq \underline{B}$.

The four thresholds will be given by these conditions holding with equaltiy.

Consider, first, the threshold $\mathcal{L}^L(\mathcal{E}^L)$, which occurs for $L_t = 0$, i.e., $K_t = \mathcal{L}_t$ for $\frac{\partial F}{\partial K_l}(N_{t+1}, K_t) = 1 + r^f_{t+1}$, $\mathcal{E}^L_{t+1} = B_t(1 + r^f_t)$,and $B_t = W_t$. Such a threshold exists as $\frac{\mathcal{E}^L_{t+1}}{W_t} > 0$, so that $r^f_t > -1$ and, under the Inada conditions there $\mathcal{L}^L(\mathcal{E}^L)$ that satisfies $\frac{\partial F}{\partial K_l}(1, \mathcal{L}^L(\mathcal{E}^L)) = \frac{\mathcal{E}^L_{t+1}}{W_t}$. Now, because of lemma 1, we know that $\mathcal{L}_t > \mathcal{L}^L(\mathcal{E}^L)$, we have $L_t > 0$. Regarding the second constraint, $B_t \geq \underline{B}$., notice that it never holds if $\frac{\mathcal{E}^L_{t+1}}{\underline{B}} \geq \frac{\partial F}{\partial K_t}(1, \mathcal{L}_t + W_t - \underline{B})$.

If $\frac{\mathcal{E}^L_{t+1}}{\underline{B}} \geq \frac{\partial F}{\partial K_t}(1, \mathcal{L}_t + W_t - \underline{B})$ were to hold, define the auxiliary function $h_L(B) = \frac{\mathcal{E}^L_{t+1}}{B} - \frac{\partial F}{\partial K_t}(1, \mathcal{L}_t + W_t - B)$, and let B_t be the equilibrium prevailing in the absence of the $B_t \geq \underline{B}$ constraint. Assume the constraint is not fulfilled, that is $B_t < \underline{B}$. Because B_t is an equilibrium, we have $h_L(B_t) = \frac{\mathcal{E}^L_{t+1}}{B_t} - \frac{\partial F}{\partial Kt}(1, \mathcal{L}_t + W_t - B_t) = 0$. As $h'_L(B) = -\frac{\mathcal{E}^L_{t+1}}{B^2} + \frac{\partial^2 F}{\partial K^2_t}(1, \mathcal{L}_t + W_t - B) < 0$, $\varphi(\underline{B}) < 0$, that is, $\frac{\mathcal{E}^L_{t+1}}{\underline{B}} < \frac{\partial F}{\partial K_t}(1, \mathcal{L}_t + W_t - \underline{B})$, a contradiction.

Therefore, if $\frac{\mathcal{E}^L_{t+1}}{\underline{B}} < \frac{\partial F}{\partial K_t}(1, \mathcal{L}_t + W_t - \underline{B})$ then the $B_t \geq \underline{B}$. constraint implies a second constraint on $\mathcal{L}_t$, that is, $\mathcal{L}_t \geq \mathcal{L}^{L,\underline{B}}(\mathcal{E}^L)$, where $\mathcal{L}^{L,\underline{B}}(\mathcal{E}^L)$ satisfies $\frac{\mathcal{E}^L_{t+1}}{\underline{B}} = \frac{\partial F}{\partial K_t}(1, \mathcal{L}^{L,\underline{B}}(\mathcal{E}^L_{t+1}) + W_t - \underline{B})$. This implies, that the equilibrium in the $L-$regime exists provided $\mathcal{L}_t \geq \max(\mathcal{L}^L(\mathcal{E}^L), \mathcal{L}^{L,\underline{B}}(\mathcal{E}^L))$

Second, consider, the threshold $\mathcal{L}^U(\mathcal{E}^U)$, which occurs for $D_t = 0$, i.e., $K_t = \mathcal{L}_t$ for $\frac{\partial F}{\partial K_l}(N_{t+1}, K_t) = 1 + r^f_{t+1}$, $\mathcal{E}^U_{t+1} = B_t(1 + r^d_t)$,and $B_t = W_t$. As before, such a threshold exists as $\frac{\mathcal{E}^U_{t+1}}{W_t} > 0$, so that $r^d_t > -1$ and, under the Inada conditions there exists $\mathcal{L}^U(\mathcal{E}^U)$ that satisfies $\frac{\partial F}{\partial K_l}(1, \mathcal{L}^U(\mathcal{E}^U)) = \frac{1}{\varphi}\frac{\mathcal{E}^U_{t+1}}{W_t}$. Now, because of lemma 1, $\frac{\partial B^U}{\partial \mathcal{L}} > 0$, and, consequently, for any $\mathcal{L}_t < \mathcal{L}^U(\mathcal{E}^L)$, we have $B_t < W_t$, so that $D_t > 0$.

Regarding the second constraint, $B_t \geq \underline{B}$., as before, it never holds if $\frac{1}{\varphi}\frac{\mathcal{E}^L_{t+1}}{\underline{B}} \geq \frac{\partial F}{\partial K_t}(1, \mathcal{L}_t + W_t - \underline{B})$.

Therefore, If $\frac{1}{\varphi}\frac{\mathcal{E}^U_{t+1}}{\underline{B}} < \frac{\partial F}{\partial K_t}(1, \mathcal{L}_t + W_t - \underline{B})$ the $B_t \geq \underline{B}$ constraint is not satisfied. This implies a second constraint on $\mathcal{L}_t$, $\mathcal{L}_t \geq \mathcal{L}^{U,\underline{B}}(\mathcal{E}^U)$, where $\mathcal{L}^{U,\underline{B}}(\mathcal{E}^U)$ satisfies $\frac{1}{\varphi}\frac{\mathcal{E}^U_{t+1}}{\underline{B}} = \frac{\partial F}{\partial K_t}(1, \mathcal{L}^{U,\underline{B}}(\mathcal{E}^L_{t+1}) + W_t - \underline{B})$. This implies, that the equilibrium in the $U-$regime exists provided $\mathcal{L}^U(\mathcal{E}^U) \geq \mathcal{L}_t \geq, \mathcal{L}^{U,\underline{B}}(\mathcal{E}^U)$.

Furthermore, from $\frac{\partial F}{\partial K_l}(1, \mathcal{L}_t) = \frac{\mathcal{E}^L_{t+1}}{W_t}$, we get $\frac{\partial \mathcal{L}^L}{\partial \mathcal{E}^L} < 0$ and the equivalent condition yields $\frac{\partial \mathcal{L}^U}{\partial \mathcal{E}^U} < 0$, $\frac{\partial \mathcal{L}^{L,\underline{B}}}{\partial \mathcal{E}^L} < 0$ and $\frac{\partial \mathcal{L}^{U,\underline{B}}}{\partial \mathcal{E}^U} < 0$.

Proof of Proposition 1:

Notice that for all $\mathcal{E}^L$ and $\mathcal{E}^U$ and $\mathcal{G}(\mathcal{L})$ we can solve the problems of the agents in this economy. The proof will therefore consist in finding a fixed point in the mapping from $\mathcal{E}^L$ and $\mathcal{E}^U$ into the equilibrium values $\mathcal{E}^L$ and $\mathcal{E}^U$ that are generated by $\mathcal{G}(\mathcal{L})$. This amounts to proving that there exists $\mathcal{E}^L$ and $\mathcal{E}^U$ such that $\mathcal{E}^L = \phi_L(\mathcal{E}^L, \mathcal{E}^U)$ and $\mathcal{E}^U = \phi_U(\mathcal{E}^L, \mathcal{E}^U)$, where

$$\phi_L(x,y) = \int_{\mathcal{L}^L<\mathcal{L}} B^L(x,\mathcal{L})d\mathcal{G}(\mathcal{L}) + \int_{\mathcal{L}^{L,\underline{B}}\leq\mathcal{L}\leq\mathcal{L}^L} B^U(y,\mathcal{L})d\mathcal{G}(\mathcal{L})$$
$$\phi_U(x,y) = \int_{\mathcal{L}^U<\mathcal{L}} B^L(x,\mathcal{L})d\mathcal{G}(\mathcal{L}) + \int_{\mathcal{L}^{U,\underline{B}}\leq\mathcal{L}\leq\mathcal{L}^U} B^U(y,\mathcal{L})d\mathcal{G}(\mathcal{L})$$

and $B^L(x,\mathcal{L})$ and $B^U(x,\mathcal{L})$ are the expressions for the bubbles that we derive in the proof of Lemma 1.

Notice that the function $\phi(x,y) = (\phi_L(x,y), \phi_U(x,y))$ is a continuous mapping from $\mathbb{R}^2$ into $\mathbb{R}^2$. We will prove that there exists $(x,y) = \phi(x,y)$ by proving that there exists $0 < \varepsilon < b < \infty$ such that $\phi : [\varepsilon, b] \times [\varepsilon, b] \to [\varepsilon, b] \times [\varepsilon, b]$. The proof of existence of an equilibrium then follows from Brouwer's fixed point theorem.

The existence of ε follows from the fact that $B^{L'}(x,\mathcal{L}), B^{U'}(y,\mathcal{L}) \to \infty$ as $x, y \to 0$.

Proof of Proposition 2:

It is sufficient to prove that the resulting equilibrium r is monotonic in the shock A. The result then follows from the proof of Lemma 1. We'll restrict the proof to the L-regime, since an equivalent result holds for the U-regime. Similar to before, to avoid cumbersome notation, in this proof we'll abstract from time subscripts in this proof and we'll refer to r^f as r. Since F is strictly concave in K and strictly increasing in A, then demand for credit, $K(r,A) = \left(\frac{\partial F(A,1,K)}{\partial K}\right)^{-1}(r)$, is strictly decreasing in r and strictly increasing in A.

In the L-regime we have

$$B^L = W + L = W + \mathcal{L} - K(r,A) \geq \underline{B}.$$

Additionally, from the first equation in (8) we can derive

$$x = (1+r)(W + \mathcal{L} - K(r,A)),$$

which implicitly determines $r^L(x,A)$ that satisfies

$$\frac{\partial r^L}{\partial x} = \frac{1}{W + \mathcal{L} - K(r^L,A) - (1+r^L)\frac{\partial K}{\partial r^L}} > 0$$
$$\frac{\partial r^L}{\partial A} = \frac{(1+r^L)\frac{\partial K}{\partial A}}{W + \mathcal{L} - K(r^L,A)} > 0.$$

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