Risk and monetary policy in a New Keynesian model*

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Abstract

Asset pricing data indicates that shocks to the nominal interest rates are primarily reflected as changes in risk premium. In this paper we build a New Keynesian model in which the behavior of interest rates and risk premium is consistent with this observation. We show that monetary shocks are transmitted via a novel *risk channel* – when nominal price adjustment is costly, firms need to balance needs to maximize current period profit and minimize future cost of price adjustments. Increase in risk, which follows a decrease in interest rates, increases firm's weight of future costs in inflationary environments, and leads to an increase in inflation, nominal and real marginal costs and output. Effectiveness of monetary policy varies with the state of the economy and generally is low in low inflationary environments. We also show that a number of well-known predictions of New Keynesian models reverses when interest rate shocks affect risk premium.

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1 Introduction

Most monetary theories are fundamentally at odds with the data. The common assumption in conventional monetary models is that changes in nominal interest rates directly affect conditional growth rates of inflation and aggregate output through the stochastic discount factor and consumer's Euler equation. Yet the empirical asset pricing literature establishes that movements in interest rates are substantially reflected in conditional variances and other higher moments of those variables; to the extent that the growth rates are affected, they move in the opposite direction from what conventional monetary models would predict.

If there is no arbitrage on the market for federal funds, the logarithm of the nominal one period risk-free interest rate ι_t can be written as

$$-\iota_t = \mathbb{E}_t\left[m_{t+1}\right] + \left(\frac{1}{2}var_t\left[m_{t+1}\right] + \sum_{j=3}^{\infty}\frac{1}{j!}\left[\kappa_{jt}\right]\right)$$

where m_{t+1} is a (log) nominal stochastic discount factor and $\mathbb{E}_t$, var_t and κ_j are the conditional mean, variance and j-th order conditional cumulants of m_{t+1}. Since most consumption capital asset pricing models predict that the first term of this equation is primarily related to the expected growth rates of the aggregate price level and output, we call it the *growth component* of the interest rates. We call the remaining terms the *risk component.* The empirical evidence strongly suggests that fluctuations in interest rates over the business cycle frequencies affect primarily the risk component. Well-documented phenomena such as the failure of the expectations hypothesis in the yield curve data, near unpredictability of nominal exchange rates changes, failure of the uncovered interest rate parity, time-varying Sharpe ratios all imply that the risk component must account for a significant part of movements in interest rates. For example, the unpredictability of the nominal exchange rate changes implies the idiosyncratic shocks to nominal interest rates of a small country should be fully reflected in the changes in the risk component and not affect the growth component at all.

In this paper we incorporate the findings of the empirical asset pricing literature into the workhorse model of modern monetary economics – a New Keynesian model of costly nominal price adjustments. We identify a novel transmission mechanism of interest rates shocks to nominal and real variables. In particular, we show that a price setting firm in such a model balances two objectives when setting its price in the current period. The first objective is to maximize its current period profit. The second objective is to minimize costs of future price changes. Shocks to the risk component of the nominal interest rates affect how the firm weights these two objectives. In particular, if firms expect future aggregate price level to be higher than current price level, an increase in risk induces such a firm to weight future costs more heavily and increase its price in the current period. This leads in general equilibrium to an increase in the aggregate price level, nominal and real marginal costs and the output. Since risk and nominal interest rates are negatively related, this mechanism implies that lower interest rates leads to higher output and inflation. The effect of the interest rates is state-contingent. It is the highest when the expected future inflation is high and becomes small and even negative when expected rates of inflation is low or negative.

We show that this mechanism implies that shocks to interest rates have persistent but mean-reverting effect on output and inflation and that in equilibrium risk has a U-shaped relationship with inflation with highest risk associated with high inflation and high deflation. We also show that a number of implications of the NK models do not hold when the interest shocks operate through the risk rather than growth component. For example, in our model all policy functions are continuous in the cost of price changes and converge to the flexible price level as this cost converges to 0. This property implies that output responds to productivity shocks when interest rates is pegged (e.g. due to the zero lower bound constraint). Furthermore, in our model, the effectiveness of forward guidance declines with the horizon at which it operates. Standard NK models that shut down the risk channel have output neutral productivity shocks and suffer from the forward guidance puzzle.

Our work is inspired and builds on the observations made by Alvarez, Atkeson, and Kehoe (2007) and Atkeson and Kehoe (2009) about the implications of the asset pricing data to monetary shocks. Similarly to Alvarez, Atkeson, and Kehoe (2009), interest rates in our model operates through changing the risk premium. Alvarez, Atkeson, and Kehoe (2009) focus on the endowment economy while study a standard NK production economy. Campbell, Pflueger, and Viceira (2014) develop an NK model with endogenous risk premium. However, in their approach they simply linearize the firms' optimality condition which de-facto eliminates the monetary transmission channel that emerges in our model.

The rest of the paper is organized as follows. In Section 2 we overview the empirical evidence about the relationship between movements in interest rates and their growth and risk components. In Section 3 we develop a benchmark model NK closed economy model in which there are no real shocks, interest rate shocks translate one to one to its risk component, preferences are isoelastic and all endogenous variables follow a log-normal stochastic process. We show that equilibrium can be characterized in such settings without any approximations, prove its existence, uniqueness in a certain sense and describe its properties. In Section 4 we show to how extend this analysis to economies with Epstein-Zin preferences, real shocks, open economies, and economies in which shocks to interest rates affect both the growth and the risk components.

2 Empirical motivation

In this section we discuss empirical evidence about the relationship between monetary policy, changes in nominal interest rates and a notion of "risk" measured using higher moments of the pricing kernel. Most of the discussion here is standard and we build on Backus, Foresi, and Telmer (2001) and Alvarez, Atkeson, and Kehoe (2007).

Let e_t be the logarithm of the nominal exchange rate between the domestic country and a foreign country in period t, and ι_t and ι_t^* be the logarithms of the nominal risk-free interest rates in the two countries. If there is no arbitrage in bond or currency markets, then there exist strictly positive stochastic processes $\{M_t, M_t^*\}_t$ such that

$$e_{t+1} - e_t = \ln M_{t+1}^* - \ln M_{t+1}, \tag{1}$$

$$-\iota_t = \ln \mathbb{E}_t M_{t+1},\ -\iota_t^* = \ln \mathbb{E}_t M_{t+1}^*. \tag{2}$$

We will refer to M_t^* and M_t process as the nominal stochastic discount factors for the home and the foreign country. The derivation of these equations uses only absence of arbitrage and no other assumptions on market structure.[1] We use lower case letters to denote logarithms of upper case letters, e.g. $m_t \equiv \ln M_t$.

[1]If we impose complete markets, the processes M_t^* and M_t are unique. For detailed arguments see Duffie (2010) and Backus, Foresi, and Telmer (2001).

Our organizing equation is a simple decomposition that we obtain from rewriting (2)

$$-\iota_t = \mathbb{E}_t\left[m_{t+1}\right] + \sum_{j=2}^{\infty}\frac{1}{j!}\left[\kappa_{jt}\right], \tag{3}$$

where κ_{jt} are the j-th cumulants of random variable m_{t+1}. As mentioned in the introduction, the first term, $\mathbb{E}_t\left[m_{t+1}\right]$ is the expected growth of marginal utilities and for several choices of preferences would be a function of the growth rate of nominal consumption. The remaining terms, $\sum_{j=2}^{\infty}\frac{1}{j!}\left[\kappa_{jt}\right]$ are a function of the higher moments of m_t and capture the risk adjustment. In particular, if the stochastic discount factor is conditionally log-normal, then all the higher moments are compactly summarized by conditional variance of m_{t+1}. For brevity we refer to the first term as "growth" and the second term as "risk". Our goal is to infer from the asset market data what fractions of changes in nominal interest rates, on the left hand side of (3) are associated with changes in the growth and risk component, respectively, on the right hand side of (3).

We start by using data from exchange rate markets. Equation (1) and (3) imply that

$$\iota_t - \iota_t^* = \mathbb{E}_t\left[e_{t+1} - e_t\right] + \sum_{j=2}^{\infty}\frac{1}{j!}\left[\kappa_{jt}^* - \kappa_{jt}\right], \tag{4}$$

or for the conditionally lognormal case, the last equation becomes

$$\iota_t - \iota_t^* = \mathbb{E}_t\left[e_{t+1} - e_t\right] + \frac{1}{2}\left[var_t\left(m_{t+1}^*\right) - var_t\left(m_{t+1}\right)\right]. \tag{5}$$

Equation (4) or (5) shows that data on the exchange rates allow to separate movements in interest rates associated into the growth and risk components.

To see the implications of these equations for the relationship between domestic interest rates and domestic stochastic discount factor, suppose that economy is stationary and recursive in some vector of state variables $\mathbf{Z}$. With a slight abuse of notation we use $\mathbb{E}_{\mathbf{Z}}$ to denote the conditional expectation $\mathbb{E}_t$ when the state is $\mathbf{Z}$. In addition, suppose that domestic and foreign countries are "small", in the sense that pure idiosyncratic shocks to domestic country have negligible effect on the foreign country and vice versa. Then we can partition $\mathbf{Z}$ as $\mathbf{Z} = \left(\bar{Z}, Z, Z^*\right)$ where $\bar{Z}$ are common shocks to both countries and Z, Z^* are pure idiosyncratic shocks to the domestic and the foreign country. Since both countries are small, we can write $\mathbb{E}_{\mathbf{Z}} m_{t+1} = \mathbb{E}_{\left(\bar{Z},Z\right)} m_{t+1}$, $\mathbb{E}_{\mathbf{Z}} m_{t+1}^* = \mathbb{E}_{\left(\bar{Z},Z^*\right)} m_{t+1}^*$.

Consider first the case in which the nominal exchange rate follows a random walk. This assumption is a common benchmark in the literature. Since Meese and Rogoff (1983), it has been shown that this assumption approximates well the behavior of exchange rates in the data for a variety of OECD countries and sample periods from 1973 to 2010; see Cheung, Chinn, and Pascual (2005) and Rossi (2013) for recent surveys of the literature. If the exchange rate is a random walk then $\mathbb{E}_{\mathbf{Z}}\left[m_{t+1}^* - m_{t+1}\right] = 0$ for all $\mathbf{Z}$ and any changes in domestic interest rates are entirely related to changes in risk. Assuming for simplicity that M_{t+1} is lognormal and equation (5) then implies

$$\mathbb{E}_{\left(\bar{Z},Z\right)} m_{t+1} = f\left(\bar{Z}\right), \tag{6}$$

$$-\iota_{\left(\bar{Z},Z\right)} = f\left(\bar{Z}\right) + \frac{1}{2} var_{\left(\bar{Z},Z\right)}\left(m_{t+1}\right), \tag{7}$$

where $f\left(\bar{Z}\right)$ is some functions of common shocks $\bar{Z}$ but not idiosyncratic shocks Z. As one can see from this equation, the exogenous shocks to the nominal interest rates translate one for one to changes in risk and do not affect the first moment of the log stochastic discount factor at all.

This analysis can be extended if exchange rates are predictable.[2] There is some evidence that over longer time horizon exchange rates of the countries with higher interest rates tend to appreciate. In the regression of the type

$$e_{i,t+1} - e_t = b_0 + b_1\left[\iota_t - \iota_t^i\right] + \epsilon_{t+1}$$

the researchers have found that the coefficient b_1 is almost always negative and often statistical significant. This finding is called the *forward premium anomaly* and indicates the failure of uncovered interest rate parity.[3]

More generally if exchange rates are predictable and

$$\mathbb{E}_{\mathbf{Z}} e_{i,t+1} - e_t = b^T\left[Z - Z^i\right] \tag{8}$$

for some vector b with $b \neq \mathbf{0}$. Substituting this back into (5) we get

$$\begin{aligned} \mathbb{E}_{(\bar{Z},Z)} m_{t+1} &= const_1\left(\bar{Z}\right) - b^T Z \\ -\iota_{(\bar{Z},Z)} &= const_2\left(\bar{Z}\right) + b^T Z + \frac{1}{2} var_{(\bar{Z},Z)}\left(m_{t+1}\right). \end{aligned}$$

In particular, if interest rates differential is the only predictor of exchange rates, this equation becomes

$$\begin{aligned} \mathbb{E}_{(\bar{Z},Z)} m_{t+1} &= f\left(\bar{Z}\right) - b_1 \iota_{(\bar{Z},Z)} \\ -\left(1 - b_1\right)\iota_{(\bar{Z},Z)} &= f\left(\bar{Z}\right) + \frac{1}{2} var_{(\bar{Z},Z)}\left(m_{t+1}\right). \end{aligned} \tag{9}$$

We make two observations about these equations. First, as long as $b_1 \neq 1$, any idiosyncratic movements in nominal interest rates must be associated with changes in risk. Since the asset pricing literature typically finds $b_1 \leq 0$, changes in risk move more than one for one with changes in interest rates and the first moment of the log stochastic discount factor co-moves *positively* with nominal interest rates. To the extent that the same statements can be made about exogenous shocks to the nominal interest rates, as suggested by the work of Eichenbaum and Evans (1995), the same conclusions hold about transmission of monetary policy shocks.

In the last section we illustrate these findings by estimating (8) using interest rate differentials, productivity differentials. We use OECD Main Economic Indicator database to obtain quarterly series of exchange rates, unit labor costs, and short-term interest rates for U.S., Canada, Australia, UK, Sweden and a basket of 19 Euro Area (EA19).[4] We set the base country to be U.S. and estimate the following three specifications using OLS

[2] Evidence for predictability is not systematic and depends on the time-period, set of countries, choice of predictors and other details of the empirical implementation. Some examples are Clark and West (2006) using interest rate differentials, Molodtsova and Papell (2009) using Taylor rule fundamentals, Gourinchas and Rey (2007) using current account balances and Chen and Rogoff (2003) using commodity prices.

[3] See Fama (1984) for early evidence, Cochrane (2009) and Rossi (2013) for recent summaries. Eichenbaum and Evans (1995) used a variety of identification techniques to estimate the effect of exogenous changes in interest rate on exchange rates and found those coefficients to also be negative and often statistically significant.

[4] Most of the data is post 1975 and individual time periods differ slightly across countries. We use all available data in the MEI database.

(I)	Only int. rates	$e_{i,t+1} - e_t = b_0 + b_\iota(\iota_t^{US} - \iota_t^i) + \epsilon$
(II)	Only productivity	$e_{i,t+1} - e_t = b_0 + b_a(a_t^{US} - a_t^i) + \epsilon$
(III)	Both int. rate and productivity	$e_{i,t+1} - e_t = b_0 + b_\iota(\iota_t^{US} - \iota_t^i) + b_a(a_t^{US} - a_t^i) + \epsilon$

The results are listed in Table 1. For the quarterly data we see that except for UK and EA19, exchange rates are not predicted by either differences in interest rates or differences in labor productivities. This is consistent findings in Cheung, Chinn, and Pascual (2005) and other related papers cited in Rossi (2013). Even for UK and EA19, the coefficient b_ι is negative; again indicating the failure of uncovered interest rate parity. To see if exogenous movements in interest rates predict exchange rates, we add the Romer and Romer (2004) index for monetary policy. We find that the coefficients are either negative or statistically zero, in part confirming the findings in Eichenbaum and Evans (1995) that the forward premium anomaly surfaces even in causal-type analysis of effects of monetary policy on exchange rates.

As a final robustness exercise we estimate (I)-(III) using annual data. In Table 1 we see that for longer horizons, in some cases, interest rate differences do predict exchange rates, albeit with a statistically negative b_ι.[5] The coefficient on productivity differences, b_a are statistically insignificant.

3 A New Keynesian model of pure risk

The previous section reviewed evidence that indicates that a substantial part of monetary shocks is transmitted in the economy through changes in risk. In this section we investigate the implication of this observation in a workhorse model of monetary economics – a New Keynesian model. We show that changes in risk provide a novel transmission mechanism of monetary shocks to real variables that differs from the one typically considered in NK models that shut off the risk channel. To illustrate our results as transparently as possible we take a particular convenient functional form for preferences, abstract from real shocks and assume that changes in interest rates are related one to one to changes in risk, as consistent with equation (6). We relax all these assumptions in Section 4.

We consider a textbook New Keynesian model adopted from Chapter 3 in Gali (2015). We assume a representative infinitely-lived household with preferences

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t\left[\ln C_t - \frac{1}{2}N_t^2\right], \tag{10}$$

where C_t is a consumption index, N_t is labor supply and $\beta \in (0,1)$ is a discount factor. Consumption goods are produced by a continuum of firms $j \in [0,1]$. Let $C_t(j)$ be consumption of good produced by firm j. The consumption index is then defined by

$$C_t \equiv \left(\int C_t(j)^{\frac{1-\varepsilon}{\varepsilon}}dj\right)^{\frac{\varepsilon}{1-\varepsilon}}$$

for $\varepsilon > 1$. Markets are complete and consumer's budget constraint is

$$\int P_t(j)C_t(j)dj + \mathbb{E}_t Q_{t+1}B_{t+1} \leq B_t + W_t N_t + T_t,$$

[5] Evidence for predictability at longer horizons also documented in several studies, for example see Mark (1995), Chinn and Meese (1995) and Cochrane (2009).

Country		Baseline specifications (I)	(II)	(III)	Robustness (III) + RomerRomer	(III) with annual data
Canada		b_0, b_ι	b_0, b_a	b_0, b_ι, b_a	$b_0, b_\iota, b_a, b_{rom}$	b_0, b_ι, b_a
	coeff	0.00, -0.01	-0.01, 0.02	-0.01,-0.08,0.02	-0.00,-0.04,0.01,-0.01	-0.03,0.01,0.07
	std. err.	0.00, 0.09	0.00, 0.01	0.01,0.11,0.011	0.01, 0.13,0.014,0.01	0.02,0.54,0.04
	R^2	0.00	0.02	0.02	0.01	0.07
	nobs	187	187	187	151	46
Australia						
	coeff	-0.00, 0.01	-0.01, 0.03	-0.01, -0.00, 0.03	-0.02, -0.00, 0.03, 0.00	-0.06, -0.4, 0.10
	std. err.	0.00,0.11	0.01,0.01	0.00,0.11,0.014	0.00,0.10,0.015,0.014	0.024, 0.39, 0.05
	R^2	0.00	0.02	0.02	0.03	0.09
	nobs	187	187	187	151	46
UK						
	coeff	-0.01,-0.51	-0.01,0.04	-0.01,-0.47, 0.04	-0.01,-0.52,0.03,-0.02	-0.03,-0.62,0.16
	std. err.	0.00,0.25	0.00,0.03	0.00,0.25,0.03	0.01,0.26,0.03,0.01	0.02,0.74,0.10
	R^2	0.05	0.02	0.07	0.09	0.10
	nobs	155	155	155	119	38
Sweden						
	coeff	0.00,-0.03	-0.01,0.04	-0.01,-0.15,0.05	-0.01,-0.15,0.07,0.08	-0.03,-0.70,0.26
	std. err.	0.00,0.23	0.00,0.03	0.00,0.22,0.03	0.01,0.21,0.03,0.05	0.02,0.6,0.11
	R^2	0.00	0.02	0.02	0.05	0.11
	nobs	139	139	139	103	34
EA19						
	coeff	0.00,-0.53	0.00,0.03	0.00,-0.72,0.05	0.00,-1.11,0.06,0.07	-0.03,-2.8,0.24
	std. err.	0.00,0.34	0.00,0.03	0.00,0.33,0.03	0.01,0.28,0.04,0.05	0.02,1.55,0.10
	R^2	0.03	0.01	0.06	0.21	0.23
	nobs	91	91	91	55	22

Table 1: Exchange rate regressions

where $P_t(j)$ is the price of good j, W_t is wage, B_{t+1} and Q_{t+1} are the quantities and prices of Arrow securities that pay off 1 dollar in a given state in period $t+1$, and T_t are various payments such as dividends. The key optimality conditions are consumer's demand for good j

$$C_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon} C_t$$

and consumption-labor choice

$$C_t N_t = \frac{W_t}{P_t}. \tag{11}$$

The (nominal) SDF is given by

$$M_{t+1} = \beta \frac{C_t}{C_{t+1}} \frac{P_t}{P_{t+1}}. \tag{12}$$

A firm j that employs $N_t(j)$ workers produces

$$Y_t(j) = N_t(j)$$

units of output. All firms are monopolistically competitive and set their nominal prices. We assume, in the spirit of Rotemberg (1982), that changing nominal prices is costly. In particular, if a firm j that had price $P_{t-1}(j)$ in period t and sets price $P_t(j)$ in period t, it pays a cost

$$\frac{\theta}{2}\left(\frac{P_t(j) - P_{t-1}(j)}{P_{t-1}(j)}\right)^2 P_t C_t$$

where $\theta \geq 0$ is the parameter that captures the cost of changing prices. We assume that this cost is reimbursed lump-sum to households so that feasibility condition in good j is given by

$$C_t(j) = Y_t(j).$$

Firm j optimal pricing decision can be characterized recursively. Let $V_t(P_-(j))$ be the present discounted value of profits of a firm in period t if its nominal price was $P_-(j)$ in period $t-1$. We can write V_t recursively as

$$\begin{aligned} V_t(P_-(j)) \quad = \quad & \max_{P(j)} P(j)\left(\frac{P(j)}{P_t}\right)^{-\varepsilon} C_t - W_t\left(\frac{P(j)}{P_t}\right)^{-\varepsilon} C_t - \frac{\theta}{2}\left(\frac{P(j) - P_-(j)}{P_-(j)}\right)^2 P_t C_t \\ & + \mathbb{E}_t M_{t+1} V_{t+1}(P(j)). \end{aligned} \tag{13}$$

The optimality condition to this problem, using the envelope theorem and equations (11) and (12), can we written as

$$\begin{aligned} & (1-\varepsilon)\left(\frac{P(j)}{P_t}\right)^{-\varepsilon} + \varepsilon C_t^2 \left(\frac{P(j)}{P_t}\right)^{-\varepsilon-1} - \theta\left(\frac{P(j) - P_-}{P_-}\right)\frac{P_t}{P_-} \\ & + \beta\theta\mathbb{E}_t\left[\left(\frac{P^*_{t+1}(j) - P(j)}{P(j)}\right)\frac{P^*_{t+1}(j)}{P(j)}\frac{P_t}{P(j)}\right] = 0, \end{aligned} \tag{14}$$

where $P^*_{t+1}(j)$ is the optimal price of firm j in period $t+1$.

Let $\Pi_t \equiv P_t/P_{t-1}$ be (gross) inflation and Y_t the total output in the economy. In a symmetric equilibrium $P_t(j) = P_t$ for all j and equation (14) becomes

$$(1-\varepsilon) + \varepsilon Y_t^2 - \theta(\Pi_t - 1)\Pi_t + \theta\beta\mathbb{E}_t(\Pi_{t+1} - 1)\Pi_{t+1} = 0. \tag{15}$$

Equation (15) together with the definition of the nominal interest rate,

$$-\iota_t \equiv \ln \mathbb{E}_t M_{t+1} = \ln \mathbb{E}_t \left\{ \frac{\beta Y_t}{Y_{t+1}} \frac{1}{\Pi_{t+1}} \right\} \tag{16}$$

are the two key equations on the New Keynesian models. The standard approach to study these models is to linearize (15) and (16) around the no inflation steady state to obtain the so-called the New Keynesian Phillips Curve and the dynamic IS equations:

$$\begin{aligned} \pi_t &= \beta \mathbb{E}_t \pi_{t+1} + const \cdot y_t, & (17) \\ 0 &= -\mathbb{E}_t m_{t+1} - \iota_t & (18) \\ &= \mathbb{E}_t y_{t+1} - (\iota_t - \mathbb{E}_t \pi_{t+1} + \ln \beta) - y_t. \end{aligned}$$

As before, we use the lower case letters denote the logarithms of upper case letters, e.g. $\pi_t = \ln \Pi_t$. Equations (17) and (18) are the two central equations of New Keynesian models (see, e.g., Woodford (2011), Gali (2015)). Equation (18), obtained by linearizing the Euler equation (16), describes the monetary transmission mechanism: how shocks to nominal interest rates feed into output and inflation. Usual representations of the dynamic IS curve do not show explicitly the relationship between the nominal interest rate ι_t and log SDF m_{t+1} but we show it here to emphasize the relationship with the empirical evidence discussed in Section 2.

A fundamental problem is that if central banks have some ability to set nominal interest rates then the equation (18) is inconsistent with the data. For example, if the exchange rate follows a random walk then equation (7) would imply that $\mathbb{E}_t m_{t+1}$ should be independent of exogenous changes in nominal interest rates by a central bank. But then equation (18) says that it is impossible for the central bank to change its nominal rates exogenously. The same conclusion holds even if exchange rates are predictable by nominal interest rates. In this case equations (9) and (18) imply, as long as $b_1 \neq 1$, that

$$\iota_t = -\frac{1}{1 - b_1} f_t$$

where f_t is orthogonal to idiosyncratic shocks. There is some disagreement in the literature whether exchange rates are approximately a random walk (which implies $b_1 = 0$) or positive shocks to interest rates lead to appreciation of the domestic currency (which implies $b_1 < 0$) but it is nearly impossible to find any evidence in support of $b_1 = 1$.[6]

This problem arises not because of any deficiency of the New Keynesian models per se but rather from an extraneous assumption that monetary shocks operate solely through the growth channel. Often this assumption is implicit. For example, a common approach to the analysis of such model is to take their first order Taylor expansion. This solution technique make fluctuations in the risk component impossible.

In this section we construct a tractable class of equilibria in a New Keynesian model that is consistent with the evidence presented in Section 2 and do not require any approximations to characterize its solution. For now we assume that random walk is a good approximation to the exchange rates $b_1 = 0$. In this case equation (6) implies

$$\mathbb{E}_t \ln \left[\frac{Y_t}{Y_{t+1}} \frac{1}{\Pi_{t+1}} \right] = -x. \tag{19}$$

for some constant $x \geq 0$. This constraint is equivalent to $\mathbb{E}_t m_{t+1} = -x + \ln \beta$.

[6]Another way to make the same point is to observe that $b_1 = 1$ only if the uncovered interest rate parity (UIRP) holds. It is well know that the UIRP fails to hold in the data.

Definition 1 *A log-normal equilibrium is an equilibrium is a stochastic process* $\{\Pi_t, Y_t, \iota_t\}_t$ *that satisfies (15), (16) and (19) and have the properties that (i)* M_{t+1} *is a conditionally log-normal r.v., (ii)* $Y_t = D\Pi_t^\delta$ *for some* D, δ*, (iii)* $var_t(\pi_{t+1}) = 0$ *if* $\pi_t = 0$.

A few words about this definition is in order. Requirements (i) is a common assumptions in the asset pricing literature. Requirement (ii) then ensures that Y_t and Π_t are log-normal random variables that allows us to characterize the equilibrium without resorting to any approximations. Finally, (iii) is made for the ease of comparison with commonly used linear NK equilibria that implicitly assume that the variance of π_{t+1} is approximately zero around the no inflation steady state. As we show below, a log-normal equilibrium exists, is unique and generically follows a non-degenerate stochastic process.

In our simplified model, where "fundamental" shocks such as TFP are absent, there are at least two ways to interpret the source of shocks. One source is simply exogenous changes in nominal interest rates by the central bank unrelated to fundamentals, in the spirit of Romer and Romer (1989). The other is changes in the expectations about variability of future inflation to which the central bank reacts by adjusting its interest rate. When exchange rates are random walk, these "exogenous" and "endogenous" changes in interest rates are indisinguishable in our model and have the same effect.

Since M_{t+1} is log-normal r.v. and $M_{t+1} = \beta\Pi_t^\delta/\Pi_{t+1}^{1+\delta}$, then Π_{t+1} is also log-normal. Let (μ_t, ω_t) be its mean and variance of Π_{t+1}. Using equation (19) we have

Lemma 1 *In any log-normal equilibrium*

$$\mu_t - x = \frac{\delta}{1+\delta}\left[\pi_t - x\right]. \tag{20}$$

Proof. In a log-normal equilibrium we have $y_t = \ln D + \delta\pi_t$. Substitute this into (19) and rearrange to get the result. ■

Corollary 1 *Let* $\mathbb{E}$ *be an expectation operator in an invariant distribution. Then* $\mathbb{E}\pi_t = x$.

Proof. In an invariant distribution $\mathbb{E}\pi_t = \mathbb{E}\pi_{t+1} = \mathbb{E}\left[\mathbb{E}_t\pi_{t+1}\right] = \mathbb{E}\mu_t$. The previous lemma then implies $\mathbb{E}\pi_t = x$. ■

Equation (15) simplifies in a log-normal equilibrium to

$$(1-\varepsilon) + \varepsilon D^2\Pi_t^{2\delta} - \theta\left(\Pi_t - 1\right)\Pi_t + \theta\beta\left[\exp\left(2\mu_t + 2\omega_t\right) - \exp\left(\mu_t + \frac{1}{2}\omega_t\right)\right] = 0, \tag{21}$$

where μ_t is given by (20). This equation shows the relationship between the level of inflation Π_t and the conditional variance of the next period inflation, ω_t. As we show below, this relationship is unique as long as θ is not too high. With a slight abuse of notation we use ω to denote this relationship, i.e. $\omega_t = \omega(\Pi_t)$. Substituting this into the definition of interest rates (16) and using properties of the log-normal distribution, we also obtain that there is a unique equilibrium relationship between interest rate ι_t and inflation Π_t given by

$$\iota(\Pi) \equiv x - \ln\beta - \frac{(1+\delta)^2}{2}\omega(\Pi). \tag{22}$$

Thus, the interest rates are inversely related to the uncertainty about inflation in the next period.

Given these observations, a log-normal equilibrium is fully characterized by a pair (D, δ) and a function ω. The next proposition, that we prove in the appendix, characterizes other properties of the equilibrium.

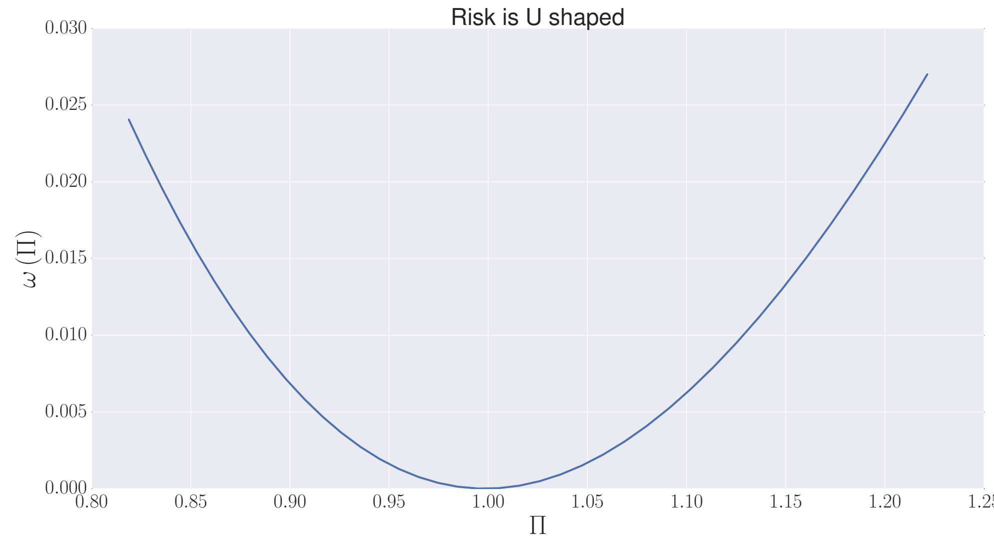

Figure 1: Equilibrium relationship $\omega(\Pi)$. Parameters used for calculation: $\varepsilon = 6$, $\beta = 0.96$, $\theta = 8$, $x = 0.02$.

Proposition 1 *There exists $\bar{\theta} > 0$ such that for all $\theta \in (0, \bar{\theta})$ the log-normal equilibrium exists, is unique and risk $\omega(\Pi)$ is differentiable and U-shaped in the sense that $\lim_{\Pi \to 0,\infty} \omega(\Pi) = \infty$, $\delta \in (0, 1]$ and parameters (D, δ) are continuous in θ and converge to $\left(\sqrt{\frac{\varepsilon-1}{\varepsilon}}, 0\right)$ as $\theta \to 0$, coinciding with their values in the flexible price equilibrium.*

As long as the cost of price setting is not too high,[7] the equilibrium exists and the relationship between inflation and output is uniquely pinned down. These parameters are continuous in θ and the relationship between output and inflation converges to their flexible price analogue as $\theta \to 0$. As we show in the next section, the same property is preserved with real shocks are added, in contrast to the equilibria considered by studying linearized equations (17) and (18).

Proposition 1 shows that output and inflation must be positively. This comes from the fact that monopolistically competitive firms in a-la Dixit and Stiglitz (1977) set their price as some mark up over their nominal marginal costs. In the absence of productivity shocks real marginal costs are proportional to real output. Thus in equilibrium real output and nominal price level (and therefore inflation) co-move positively. Since $\delta < 1$, real output moves less than inflation.

Since $\delta \in (0, 1]$, inflation is a persistent but mean reverting processes. Risk is U-shaped in inflation and also persistent and mean reverting. To illustrate this, we plot $\omega(\cdot)$ in Figure 1 for a common choice of parameter values. The shape of function $\omega(\cdot)$ is driven by the monetary transmission mechanism, to which we turn next.

3.1 Monetary transmission mechanism

In this section we study how shocks to the nominal interest rates affect real and nominal variables in our economy. Let $\hat{x}$ denote changes in variable x. We have

[7] The requirement that θ is not too high is a sufficient condition to establish the existence of the equilibrium. All other statements hold for any θ, provided that an equilibrium exists.

$$\hat{\imath}_t = -\frac{1}{2}\widehat{var}_t\left(m_{t+1}\right) = -\frac{1}{2}\widehat{var}_t\left(y_{t+1}+\pi_{t+1}\right) \propto -\widehat{var}_t\left(\pi_{t+1}\right).$$

The first equality is our restriction that interest rates effect the SDF through the risk channel, the second equation is the implication of C-CAPM and our equation (12), the third relationship comes from the fact that in NK models inflation and output has positive comovement as discussed above. Thus, changes of interest rates in period t in our model directly change the conditional variance of inflation in period $t+1$, with a decrease in interest rates leading to an increase in variance of future inflation and therefore future price levels. The main channel in our model is how the increase in uncertainty about prices in period $t+1$ affect prices and quantities in period t.

Consider the price optimization problem of an individual firm. If an individual firm chooses price Q, its per-unit profit in the current period is given by

$$\Phi_t\left(Q\right) \equiv Q\left(\frac{Q}{P_t}\right)^{-\varepsilon} - W_t\left(\frac{Q}{P_t}\right)^{-\varepsilon} - \frac{\theta}{2}\left(\frac{Q}{Q_{t-1}}-1\right)^2 P_t,$$

where Q_{t-1} is its price in period $t-1$. The price that maximizes current period profit, P^{cur}, solves $\Phi_t'\left(P^{cur}\right)=0$. Function Φ_t is convex in the neighborhood of P^{cur}.

The firm's dynamic optimality condition (14) can be re-written more simply as

$$\Phi_t'\left(P\left(j\right)\right) + \beta\theta\mathbb{E}_t\left[\left(\frac{P_{t+1}-P\left(j\right)}{P\left(j\right)}\right)\frac{P_{t+1}}{P\left(j\right)}\frac{P_t}{P\left(j\right)}\right] = 0. \tag{23}$$

This equation shows that a firm pursues two objectives when it sets its price in period t. The first objective is to maximize its current period profit Φ_t. This objective is captured by the first term in (23). The second objective is to minimize costs of future price changes and is captured by the second term. In equilibrium, firm's price balances these two objectives. For example, if firms expect positive inflation in the future, in the sense that $\mathbb{E}_t P_{t+1} \geq P^{cur}$, equation (23) shows that firms optimally forgo some of the current profits to decrease future costs of price increases and choose price $P_t(j)$ that is above P^{cur} (with $p^{cur} \equiv \ln P^{cur}$).

We want to understand how changes in uncertainty about future inflation π_{t+1}, or equivalently, about future log price level p_{t+1} affect the optimal pricing decision of an individual firm. To do that, we fix all the period t aggregate variables in problem (23), assume that p_{t+1} is normally distributed with mean and variance (μ,ω) and study how the solution to (23), $Q^*\left(\mu,\omega\right)$, is affected by changes in ω. As we show in the appendix, as long as θ is not too high, Q^* is unique and the current period profit function Φ_t is convex in the neighborhood of Q^*, the assumption that we maintain throughout. In the appendix we prove the following proposition.

Proposition 2 *There exists $\bar{\theta}>0$ such that for all $\theta \in \left(0,\bar{\theta}\right)$*

(a). If $\mu > \log P^{cur} - \ln 4$ then $\frac{\partial}{\partial\omega}Q^*\left(\mu,\omega\right) > 0$, $\frac{\partial^2}{\partial\omega\partial\mu}Q^*\left(\mu,\omega\right) > 0$ for all ω

(b). If $\mu < \log P^{cur} - \ln 4$ then $\frac{\partial}{\partial\omega}Q^*\left(\mu,\omega\right) \leq 0$ for all ω sufficiently low and $\frac{\partial}{\partial\omega}Q^*\left(\mu,\omega\right) \geq 0$ for all ω sufficiently high

To get an intuition for this result, observe that an increase in $var_t\left(\ln P_{t+1}\right)$ raises both the mean and the variance of P_{t+1}. The effect of an increase in $var_t\left(P_{t+1}\right)$ on the marginal cost of future mispricing,

$$\mathbb{E}_t\left[\left(\frac{P_{t+1}}{Q}-1\right)\frac{P_{t+1}}{Q^2}\right],$$

is unambiguously positive as this function is convex in P_{t+1}. The effect of an increase in $\mathbb{E}_t(P_{t+1})$ on the marginal cost of mispricing is ambiguous. It is positive if $\mathbb{E}_t(P_{t+1})$ is sufficiently high relative to Q and negative if it is sufficiently low. If firms expect inflation in the future (or at least not too severe deflation) than both the increase in the mean and in the variance of P_{t+1} increases the marginal cost of mispricing and induces the firm to increase its current price Q^*. This can be seen from part (a) of Proposition 2. Moreover, the higher the expected inflation μ is, the greater is the marginal effect of any given increase in uncertainty, $\frac{\partial^2}{\partial\omega\partial\mu}Q^* > 0$. The same conclusion holds for mild deflations, but is reversed if the expected deflation is sufficiently severe (part b).[8]

This discussion shows that in inflationary environments individual firms on average want to increase their prices in the response to an increase in uncertainty about future inflation, but the magnitude of this increase depends on the state of economy. This effect may become negative in the states in which deflation is expected. Since the aggregate price level is the sum of individual prices and the aggregate output is proportional to the aggregate price level, the same conclusion can be reached about the response of P_t and Y_t to an increase in risk. In our general equilibrium, rational expectations model, the expectations about inflation in period $t+1$ follow a persistent Markov process that depends on the realized inflation in period $t-1$. One state-contingent measure of effectiveness of monetary policy in period t is conditional covariance $cov_{t-1}(\iota_t, y_t)$, that captures how much any given increase in interest rates affect the output. Figure 2 illustrates these relationships graphically. Consistent with our discussion above, this covariance is negative except for most deflationary states. It is also monotonically decreasing in the past inflation rate, so that the effectiveness of monetary policy is higher in high- rather than low-inflation environments. Finally, since we assume in the simulations that $x > 0$, the average inflation in the ergodic distribution is positive and ergodic covariance $cov(\iota, y) = -0.001 < 0$.

Note that our model the monetary transmission mechanism differs from the one traditionally studied in the New Keynesian models. Both ours and the traditional approaches start with the Euler equation (16) that can be decomposed into the growth and risk components. The traditional approach, by using first order approximation, implicitly *assumes* that any changes in interest rates affect one for one the growth component, directly changing growth rates of aggregate output and prices. In contrast, in our settings shocks to the interest rates feed into risk, which affect how firms balance their static and dynamic objectives in price setting. Thus, in our model all the real effects comes from firm's optimality condition, equation (15). The main driving force of our mechanism is that an increase in risk associated with a reduction in the nominal interest rates endogenously requires firm to put higher weight on its dynamic objectives. In inflationary environments that leads to an increase in both current prices and output, but the magnitude and even the sign of this response is state-contingent.

[8]In NK models firms face strategic complementarities in price setting. As a result, firms can use fluctuations in risk level as a coordinating device in setting their prices. The effect of this channel is, in general, ambiguous. Requirement (iii) in the definition of the log-normal equilibrium uniquely pins it down. If we drop it, then one can show that any equilibrium has the U-shaped relationship between inflaiton and risk, and that the relationship between interest rates and output will be state-contingent as we discussed in this section. Thus, the effect discussed in this section holds in all equilibria. The inflation level at which risk is minimized is not determined. If we impose a natural refinement that the sign of the response of the aggregate price level to an increase in risk is the same as the sign of the response of individual price levels to an increase in risk holding aggregate price level fixed, then the risk-minimizing level of inflaiton is unique.

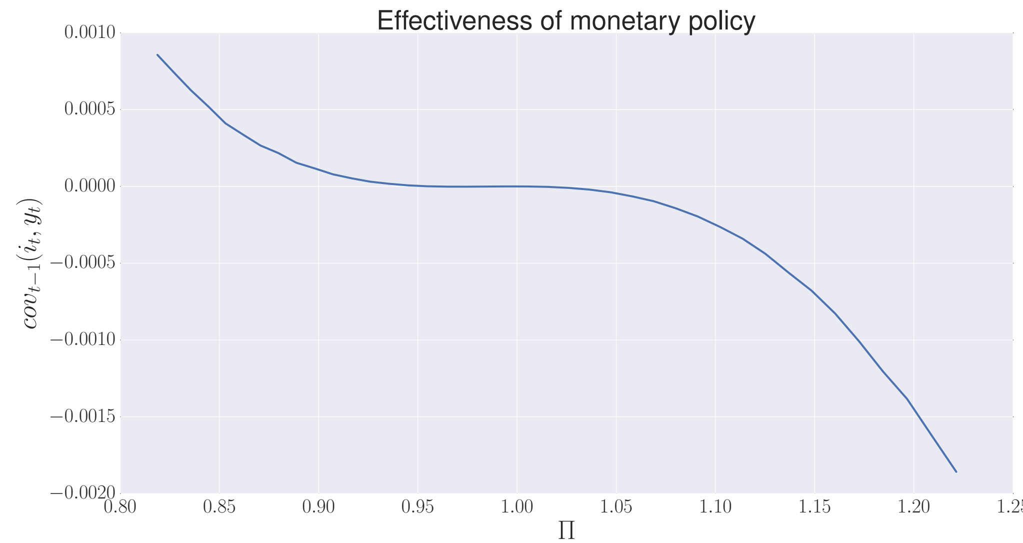

Figure 2: Effectiveness of monetary policy. Same parameters as in Figure 1.

4 Extensions

In this section we show that the mechanism discussed in the previous section carries over to economies with more general preferences, real shocks, multicounty economies, and economies in which interest rates affect both the growth and the risk channels. We also used these extensions to derive additional insights about the effect of monetary shocks.

4.1 Preferences

In our baseline economy we assume log-quadratic preferences (10). Our analysis of the previous section would go literally unchanged if we used a very commonly used isoelastic specification of preferences

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t\left[\frac{C_t^{1-\rho}}{1-\rho}-\frac{N_t^{1+\phi}}{1+\phi}\right] \tag{24}$$

instead. This new specification simply imply that the risk component of the interest rates is proportional to $var_t(\pi_{t+1}+\rho y_{t+1})$ rather than $var_t(\pi_{t+1}+y_{t+1})$, and the real marginal costs in general equilibrium are equal to $Y_t^{\sigma+\phi}$ rather than Y_t^2. We get an immediate corollary to Proposition 1.

Corollary 2 *Proposition 1 holds if preferences are given by (24).*

One limitation of time-separable preferences such as (24) is that models in which consumers have such preferences fail to match quantitatively a number of empirical regularities about asset pricing behavior. A common approach in the asset pricing literature is to instead adopt recursive preferences, such as those developed by Epstein and Zin (1989). Such preferences are defined recursively by

$$v_t = \left[(1-\beta)\, u\left(C_t, N_t\right)^{1-\rho} + \beta \left(\mathbb{E}_t v_{t+1}^{1-\gamma} \right)^{\frac{1-\rho}{1-\gamma}} \right]^{\frac{1}{1-\rho}},$$

where $1/\rho$ captures the intertemporal elasticity of substitution, γ represents risk aversion and $u\left(C_t, N_t\right)$ is the utility from consumption and leisure and assumed to be positive. We assume that

$$u\left(C, N\right) = \left[C^{1-\rho} - (1-\rho) \frac{N^{1+\phi}}{1+\phi} \right]^{\frac{1}{1-\rho}},$$

so that we nest specification (24) when $\rho = \gamma$. For our purposes it is useful to use a monotonic transformation $v_t \equiv \frac{v_t^{1-\rho}-1}{(1-\beta)(1-\rho)}$ which allows us to use a more convenient representation (see Karantounias (2013) for details)

$$v_t = \frac{C_t^{1-\rho}}{1-\rho} - \frac{N_t^{1+\phi}}{1+\phi} + \beta \frac{\left[\mathbb{E}_t \left[1 + (1-\beta)(1-\rho)\, v_{t+1} \right]^{\frac{1-\gamma}{1-\rho}} \right]^{\frac{1-\rho}{1-\gamma}} - 1}{(1-\beta)(1-\rho)}. \tag{25}$$

The stochastic discount factor is now given by

$$M_{t+1} = \beta S_{t+1}^{\frac{\rho-\gamma}{1-\gamma}} \left(\frac{C_t}{C_{t+1}} \right)^{\rho} \frac{P_t}{P_{t+1}},$$

where $S_{t+1} \equiv \frac{v_{t+1}^{1-\gamma}}{\mathbb{E}_t v_{t+1}^{1-\gamma}}$ is the risk-adjustment.

To simplify the exposition and make parallels to our analysis in Section 3 transparent, we assume that $\rho = \phi = 1$. In this case the intra-temporal consumption-leisure decision is still given by equation (11) and the firm's value function is given by (13). The equilibrium relationship (15) now becomes

$$(1-\varepsilon) + \varepsilon Y_t^2 - \theta \left(\Pi_t - 1 \right) \Pi_t + \theta \beta \mathbb{E}_t S_{t+1} \left(\Pi_{t+1} - 1 \right) \Pi_{t+1} = 0, \tag{26}$$

where

$$S_{t+1} = \frac{\exp\left[(1-\beta)(1-\gamma)\, v_{t+1} \right]}{\mathbb{E}_t \exp\left[(1-\beta)(1-\gamma)\, v_{t+1} \right]}. \tag{27}$$

Thus the only difference with our analysis in Section 3 is that the marginal cost of re-pricing $\left(\Pi_{t+1} - 1 \right) \Pi_{t+1}$ are risk-adjusted with S_{t+1}.

The explicit closed form characterization of equilibrium that we constructed in Section 3 is difficult because even if $\left(Y_{t+1}, P_{t+1} \right)$ follow a log-normal process, the risk-adjustment S_{t+1} does not, and therefore neither does M_{t+1}. Therefore the simple analytical expression we used to obtain (21) no longer apply. To make progress, we use a class of tractable second order approximations of policy functions around small uncertainty that was developed in Bhandari, Evans, Golosov, and Sargent (2017). Here we sketch the main idea.

We study equilibria in which π_t follows a stochastic process

$$\pi_t = \bar{\pi} + \sigma z_t, \tag{28}$$

where z_t is a first order Markov process on a compact set $[-\bar{z}, \bar{z}]$ and $\sigma \geq 0$ is a parameter. We assume that $\bar{z}$ is large but finite. In the spirit of Definition 1, we assume that log output is a function of log inflation and can be represented as $y_t = y(\pi_t;\sigma)$ and that $var_t(\pi_{t+1}) = 0$ if $\pi_t = 0$. The equilibrium relationship between y_t and π_t depends in general on σ. The equilibrium is characterized by functions $\pi(\sigma z;\sigma), y(\sigma z;\sigma), s(\sigma z;\sigma), v(\sigma z;\sigma),\ n(\sigma z;\sigma)$ that satisfy (in exponents, when necessary) $n(\sigma z;\sigma) = y(\sigma z;\sigma)$ and equation (25), (27), (26), (28) for all z, σ. In line with our analysis in Section 3 we assume that interest rates affect the economy only through the risk channel and impose

$$\mathbb{E}_t \ln\left[S_{t+1}\frac{Y_t}{Y_{t+1}}\frac{1}{\Pi_{t+1}}\right] = -x, \tag{29}$$

where $x \geq 0$. Our approach consists of taking second order Taylor expansions of policy functions with respect to σ around $\sigma = 0$. This expansion characterizes properties of the stochastic process z, and hence π and all other variables in the competitive equilibrium.

Proposition 3 *There exists $\bar{\theta} > 0$ such that for all $\theta \in (0, \bar{\theta})$, up to the $O(\sigma^3)$ terms: the equilibrium is unique; inflation satisfies $\mathbb{E}_t\pi_{t+1} = \alpha\pi_t$ for some $\alpha \in (0,1)$; y_t is increasing in π_t; ι_t is U-shaped in π_t with sensitivity of ι_t to π_t increasing in γ; the ratio $\frac{\pi_t}{var_t(\pi_{t+1})}$ is increasing in γ; y_t converges to the flexible price output for all σ, z as $\theta \to 0$.*

Proposition 3 shows that all the insights of our model in Section 3 carry over to the economy considered in this section. Note, the relative to the economy considered in that section, not only we allowed for the more general Epstein-Zin preferences but also we dropped the assumption that shocks are log-normal and that the output is proportional to inflation. The monetary transmission mechanism discussed in Section 3.1 remains operational even if we relax those assumptions. As a result, a reduction of interest rates is expansionary in the inflationary environments but its effect is state dependent and low in the low inflation states. Inflation follows a persistent process and risk is U-shaped in inflation.

The Epstein-Zin preferences do not change the qualitative nature of these results but affect quantitative magnitude. Holding the elasticity of the intertemporal substitution fixed, higher risk aversion implies that the interest rates are more sensitive to the volatility of inflation, and that changes in uncertainty about period $t+1$ have a bigger impact on prices and output in period t. This occurs since higher degree of the risk-aversion amplifies the effect of risk.

Our approach allows to derive the generalization of standard two equations of the NK model, (17) and (18), when interest rates operate through the risk channel. We show in the appendix that they take the form

$$2\hat{y}_t + 2\hat{y}_t^2 - \left[\zeta_1\hat{\pi}_t + \frac{1}{2}\zeta_2\hat{\pi}_{t+1}\right] + \beta\mathbb{E}_t\left[\zeta_1\hat{\pi}_t + \frac{1}{2}\zeta_2\hat{\pi}_{t+1}\right] + \frac{\xi^2\zeta_0}{2}var_t(\hat{y}_{t+1}) - \zeta_1 cov_t(\hat{\pi}_{t+1}, \xi(\gamma)\hat{y}_{t+1}) = 0$$

and

$$\iota_t = x + \frac{1}{2}\left[var_t(\hat{y}_{t+1} + \hat{\pi}_{t+1}) + 2cov_t(\tilde{y}_{t+1} + \tilde{\pi}_{t+1}, \xi(\gamma)\hat{y}_{t+1}) + var_t(\xi(\gamma)\hat{y}_{t+1})\right],$$

where $\zeta_0, \zeta_1, \zeta_2$ are constants that depend on the value of x, $\xi(\gamma)$ is an increasing function of γ with $\xi(1) = 0$, and hats denote deviations from the means. These equations make it clear that interest rates in period t operate through changing conditional variances of inflation and output in period $t+1$, which in term feeds into period t inflation and output through firms' optimality condition.

4.2 Productivity shocks

So far we considered a simplified economy with no real shocks. Adding such shocks is straightforward. Consider settings in Section 3 but suppose that production is given by

$$C_t(j) = A_t Y_t(j),$$

where A_t is the aggregate productivity process that follows some stochastic process. In this case equation (15) becomes

$$(1-\varepsilon) + \varepsilon\left(\frac{Y_t}{A_t}\right)^2 - \theta(\Pi_t - 1)\Pi_t + \theta\beta\mathbb{E}_t(\Pi_{t+1}-1)\Pi_{t+1} = 0.$$

We assume that in the log-normal equilibrium $Y_t = D(A_t)\Pi_t^{\delta}$ and $D(A_t)$ is a function of productivity A_t. For simplicity, we assume that A_t are i.i.d with mean 1 and extend (iii) in Definition 1 to have $var_t(\pi_{t+1}) = 0$ if $\pi_t = 0$ and $A_t = 1$.

Proposition 4 *In a log-normal equilibrium shocks are* $\mathbb{E}_t\pi_{t+1} = \frac{\delta}{1+\delta}\pi_t + const(A_t)$*; volatility of inflation satisfies* $\omega_t = \omega(\Pi_t, A_t)$ *for some function* ω *with* $\omega(\cdot, A)$ *being U-shaped for each* A *as in Proposition 11; interest rates given by* $\iota_t = \iota(\Pi_t, A_t)$ *where* $\iota(\cdot,\cdot) = const - \frac{(1+\delta)^2}{2}\omega(\cdot,\cdot)$*;* $\delta \in (0,1]$ *and parameters* $(D(A), \delta)$ *are continuous in* θ *and converge to* $\left(A\sqrt{\frac{\varepsilon-1}{\varepsilon}}, 0\right)$ *as* $\theta \to 0$*, coinciding with their values in the flexible price equilibrium.*

Thus, all the insights of our model of pure risk extent to the economy with productivity shocks. Interest rate shocks operate through the risk channel, but their effect now depends on the current period productivity A. Policy functions are continuous in θ and converge to their flexible price analogue as $\theta \to 0$. This implies that all impulse responses to shocks also converge to their flexible price analogues.

4.3 Multi-country world

In the previous analysis we focused on an individual country abstracting from the possibility of trade. We extent our analysis here to a standard open economy NK model. We follow very closely Gali and Monacelli (2005) and Chapter 7 in Gali (2015) with the main difference is that we use Rotemberg rather than Calvo pricing assumption. Following them, our world consists of a continuum of small open economies. Each country is of measure zero. For now we abstract from real shocks.

Preferences of household in the domestic country are given by 10 but now

$$C_t = \frac{1}{(1-\alpha)^{1-\alpha}\alpha^{\alpha}} C_{H,t}^{1-\alpha} C_{F,t}^{\alpha}$$

and $C_{H,t}$ and $C_{F,t}$ are indexed of goods produced in the home country and in foreign county and $1-\alpha \in [0,1]$ measure the degree of home bias. Each country is denoted by $i \in [0,1]$ and firms in country i are denoted by j. We have

$$C_{H,t} = \left(\int C_{H,t}(j)^{\frac{1-\varepsilon}{\varepsilon}} dj\right)^{\frac{\varepsilon}{1-\varepsilon}}, \; C_{i,t} = \left(\int C_{i,t}(j)^{\frac{1-\varepsilon}{\varepsilon}} dj\right)^{\frac{\varepsilon}{1-\varepsilon}}, \; C_{F,t} = \exp\left(\int c_{i,t} di\right).$$

This implies that expenditure shares are given by

$$\begin{aligned} P_{H,t}C_{H,t} &= (1-\alpha)\,P_tC_t \\ P_{F,t}C_{F,t} &= \alpha P_tC_t \\ P_{i,t}C_{i,t} &= P_{F,t}C_{F,t} \end{aligned}$$

and that individual demand of individual goods produced by firm j, at home and at country i is, respectively

$$C_{H,t}(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon} C_{H,t},\; C_{i,t}(j) = \left(\frac{P_{i,t}(j)}{P_{i,t}}\right)^{-\varepsilon} C_{i,t}$$

Let $E_{i,t}$ be the nominal exchange rate. Then $Q_{i,t}$ is real exchange rate:

$$Q_{i,t} \equiv \frac{E_{i,t}P_t^i}{P_t} \tag{30}$$

where P_t^i is the price level in country i, in currency of country i, and S_t is terms of trade:

$$S_t \equiv \frac{P_{F,t}}{P_{H,t}}.$$

Note that CPI can be written as

$$P_t = P_{H,t}S_t^{\alpha}. \tag{31}$$

Following Gali (2015) we assume that trade is frictionless, markets are complete, and all countries are symmetric. These assumptions imply that the law of one price holds

$$P_{i,t}(j) = E_{i,t}P_{i,t}^i(j) \text{ for all } i, j$$

and that

$$C_t = C_t^i Q_{i,t} \text{ for all } i. \tag{32}$$

We assume that producers are as in Section 3 with the cost of price changes given by $\frac{\theta}{2}\left(\frac{P(j)-P_{\text{-}}}{P_{\text{-}}}\right)^2 P_{H,t}C_t$, so that their value function can be written as

$$\begin{aligned} V_t\left(P_{\text{-}}(j)\right) \;=\; & \max_{P(j)} P(j)\left(\frac{P(j)}{P_{H,t}}\right)^{-\varepsilon} C_t - W_t\left(\frac{P(j)}{P_{H,t}}\right)^{-\varepsilon} C_t - \frac{\theta}{2}\left(\frac{P(j)-P_{\text{-}}(j)}{P_{\text{-}}(j)}\right)^2 P_{H,t}C_t \\ & +\mathbb{E}_t M_{t+1}V_{t+1}\left(P(j)\right). \end{aligned}$$

The main result of this section can be stated as follows

Proposition 5 *Let Y^* be the average would output. There is some constant κ such that in equilibrium*

$$(1-\varepsilon) + \varepsilon\left(\frac{\exp(\kappa)}{Y^*}\right)^{\alpha} Y_t^{2+\alpha} - \theta\left(\Pi_t - 1\right)\Pi_t + \beta\theta\mathbb{E}_t\left(\Pi_{t+1}-1\right)\Pi_{t+1} = 0. \tag{33}$$

Equation (33) together with (19) fully describe the equilibrium in the domestic economy. The analysis of the log-normal equilibrium in this economy is identical to the analysis in Section 3. Note that the output in each country fluctuates but those fluctuations are uncorrelated across countries and so that aggregate output remains constant by the law of the large numbers. It is straightforward to add idiosyncratic productivity shocks to this economy along the lines of Section 4.2. As long as they remain uncorrelated across countries, it is immediate to obtain an extension of Proposition 4.

Adding a correlated component to the productivity shocks across countries yields additional insight. In this case the aggregate allocations are no longer constant, and even if the exchange rate follow a random walk, equation (19) would read

$$\mathbb{E}_t \ln \left[\frac{Y_t}{Y_{t+1}} \frac{1}{\Pi_{t+1}} \right] = -x\left(\bar{A}_t\right),$$

where $\bar{A}_t$ is the aggregate productivity component. Thus, the world-wide shocks may operate through the growth component of the interest rates. Since average interest and inflation rates are increasing in x in Section 3 and inflation and output are positively related, one can construct equilibria with unconditional positive correlation between the interest rates and output along the equilibrium path, while having negative correlation between the interest rates and output conditional on an unexpected idiosyncratic shock to the interest rates.

4.4 Incorporating the predictability of exchange rates

In the analysis above we assume that the nominal exchange rates are unpredictable and follow a random walk. As we discussed in Section 2, while this assumption is a good approximation over the business cycle frequencies, there is some evidence that exchange rate movements are predictable, in particularly using the interest rate differentiable. In this section we discuss how to extend our analysis to incorporate such predictability.

Suppose that interest rates predict the exchange rate movements. Abstracting from the world aggregate shocks, equation (9) then implies

$$\mathbb{E}_t m_{t+1} = -x - b_1 \iota_t,$$

where empirical evidence implies that $b_1 \leq 0$. In a log-normal equilibrium the interest rate satisfies

$$-\iota_t = \mathbb{E}_t m_{t+1} + \frac{1}{2} var_t\left(m_{t+1}\right),$$

which reduces the previous equation to

$$\mathbb{E}_t m_{t+1} = -\frac{1}{1-b_1} x + \frac{1}{2} \frac{b_1}{1-b_1} var_t\left(m_{t+1}\right). \tag{2**}$$

Note that $b_1 < 0$ implies that risk channel moves more than one for one with movements in the interest rates, and that the risk channel moves in the *opposite* direction from the one usually assumed in NK literature.

It is straightforward to characterize the log-normal equilibrium in this economy. In particular, assuming the inflation follows a conditionally log-normal process, equation (20) becomes

$$\mu_t - \left(x - \frac{1}{2} \frac{b_1}{1-b_1} (1+\delta)^2 \omega_t\right) = \frac{\delta}{1+\delta} \left[\pi_t - \left(x - \frac{1}{2} \frac{b_1}{1-b_1} (1+\delta)^2 \omega_t\right)\right],$$

so that inflation is persistent and mean-reverting, but the process for the mean-reversion is more involved. Equation (21) then pin down the relationship between inflation and risk $\omega(\Pi)$ just as it did in Section 3. As the result, most of the qualitative properties of the equilibrium remain unchanged.

5 Conclusion

Interest rates can be decomposed into what we refer as growth and risk components. Empirical evidence suggest that over the business cycle frequencies movements in interest rates mainly, if not entirely, affect the risk component. However, much of the conventional New Keynesian literature assumes that all shocks to nominal interest rates affect only the growth component.

This assumption has significant implications for the monetary transmission mechanism and many of well-known predictions of the literature rely heavily on this assumption. For example, the NK literature shows that forward guidance (i.e. a promise to change interest rates by a given amount at a certain date in the future) can be arbitrarily effective if done at sufficiently distant date; that effects of the real shocks can be arbitrarily big with an interest rate peg (for example, due to the ZLB constraint); and that behavior of real variables in the economy with negligible costs of price changes can be arbitrarily far away from their behavior in the economy with flexible prices.

In this paper we study the implications of New Keynesian models when the risk channel is active. and highlight a novel channel of transmission of shocks to nominal interest rates to real economy. Consistent with the conventional wisdom, our channel indicates the unexpected reduction of nominal interest rates general stimulates the economy by increasing output. However, the magnitude and even the sign of this effect varies with past inflation rates and expectations about the level of future inflation. In low inflationary environments a reduction of interest rate has small effect on the output and inflation.

This finding has some intriguing policy implications. In the aftermath of the 2007 financial crises several central banks made an effort to introduce negative interest rates in the hopes of increasing inflation and output. Since such efforts were undertaken during the historically low inflation rates, our model suggests that such efforts not result in a sizable improvement and may even backfire.

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6 Appendix

6.1 Proof for Section 3

Let

$$F(\mu,\omega) \equiv \exp(2\mu + 2\omega) - \exp\left(\mu + \frac{1}{2}\omega\right).$$

We first show the following useful intermediate result

Lemma 2 *If $\mu \geq -\ln 4$ then $F(\mu,\cdot)$ is increasing; if $\mu < -\ln 4$, then $F(\mu,\omega)$ is decreasing in ω for $\omega \in \left[0, \frac{2}{3}(-\mu - \ln 4)\right]$ and increasing for $\omega \geq \frac{2}{3}(-\mu - \ln 4)$. $F(\mu,\omega)$ is uniformly bounded below by a finite number. The range of $F(\mu,\cdot)$ is $[f(\mu),\infty)$ for some continuous function $f(\mu)$.*

Proof. Differentiate F w.r.t. ω :

$$F_\omega(\mu,\omega) = 2\exp(2\mu + 2\omega) - \frac{1}{2}\exp\left(\mu + \frac{1}{2}\omega\right).$$

We have $F_\omega(\mu,\omega) \leq 0$ if $\frac{3}{2}\omega \leq -\mu - \ln 4$ and $F_\omega(\mu,\omega) \geq 0$ if $\frac{3}{2}\omega \geq -\mu - \ln 4$. Therefore function $F(\mu,\cdot)$ attains its minimum at

$$\bar{\omega}(\mu) = \max\left\{0, \frac{2}{3}(-\mu - \ln 4)\right\},$$

which establishes the first part of the lemma. Let $f(\mu) \equiv F(\mu, \bar{\omega}(\mu))$. We have

$$f'(\mu) = \exp(2\mu + 2\bar{\omega}(\mu))\left[2 + 2\bar{\omega}'(\mu)\right] - \exp\left(\mu + \frac{1}{2}\omega\right)\left[1 + \frac{1}{2}\bar{\omega}'(\mu)\right] \text{ for } \mu \neq -\ln 4.$$

If $\mu \geq -\ln 4$ then $\bar{\omega}'(\mu) = 0$ and this expression becomes

$$f'(\mu) = 4\exp(2\mu) - \exp(\mu) \geq 0.$$

If $\mu < -\ln 4$ then $\bar{\omega}'(\mu) = -\frac{2}{3}$ and this expression becomes

$$f'(\mu) = \frac{2}{3}\left[\exp\left(\frac{2}{3}\mu - \frac{4}{3}\ln 4\right) - \exp\left(\frac{2}{3}\mu - \frac{1}{3}\ln 4\right)\right] < 0.$$

Therefore f attains its minimum at $\mu = -\ln 4$. Since $F(\mu,\cdot)$ is a continuous function with $\lim_{\omega\to\infty} F(\mu,\omega) = \infty$ for all μ, the range of F is $[f(\mu),\infty)$. ■

Lemma 3 *In any equilibrium, $\delta \in [0,1]$.*

Proof. Let $G(\Pi) \equiv \frac{(1-\varepsilon)}{\theta} + \frac{\varepsilon}{\theta} D\Pi^{2\delta} - (\Pi - 1)\Pi$ so that we can write (21) as

$$G(\Pi) + \beta F(\mu(\Pi), \omega(\Pi)) = 0. \tag{34}$$

If $\delta > 1$ then $\lim_{\Pi\to\infty} G(\Pi) = \infty$; if $\delta < 0$ then $\lim_{\Pi\to 0} G(\Pi) = \infty$. In either case (34) is violated as F is bounded below by Lemma 2. This establishes that $\delta \in [0,1]$. ■

Proof. [Proof of Proposition 1] First, observe that if an equilibrium exists then D satisfies $D^2 = \left[\frac{\varepsilon-1}{\varepsilon} - \frac{\theta}{\varepsilon}\beta F\left(\mu\left(1\right),0\right)\right]$ and $\lim_{\theta\to 0} D_\theta = \sqrt{\frac{\varepsilon-1}{\varepsilon}}$.

An equilibrium exists if for any $\Pi \in (0,\infty)$ we can find $\omega\left(\Pi\right)$ that satisfies (34). Note that for all Π s.t. $\mu\left(\Pi\right) \geq -\ln 4$, $F\left(\mu\left(\Pi\right),\cdot\right)$ is an increasing function by Lemma 2 and therefore the solution $\omega\left(\Pi\right)$, if its exists, is unique and differentiable in this range. Since $\mu\left(1\right) = \frac{\delta}{1+\delta}\ln 1 + \frac{1}{1+\delta}x \geq 0$ by Lemma 1 and $\omega\left(1\right) = 0$, this implies that $G'\left(1\right) + \beta F_\mu\left(\mu\left(1\right),0\right)\mu'\left(1\right) = 0$ or

$$\frac{\varepsilon}{\theta}D2\delta - 1 + \left[2\exp\left(\frac{2x}{1+\delta}\right) - \exp\left(\frac{x}{1+\delta}\right)\right]\frac{\delta}{1+\delta} = 0. \tag{35}$$

This equation implies that $\lim_{\theta\to 0}\delta_\theta = 0$.

Given the range of F function shown in Lemma 2, it is sufficient to show that for all Π

$$\begin{aligned} K\left(\Pi\right) &\equiv G\left(\Pi\right) + \beta F\left(\mu\left(\Pi\right),0\right) \\ &= \frac{\varepsilon-1}{\theta}\left[\frac{\varepsilon}{\varepsilon-1}D\Pi^{2\delta} - 1\right] - \left[\Pi^2 - \Pi\right] + \beta\left[\Pi^{\frac{2\delta}{1+\delta}}\exp\left(\frac{2x}{1+\delta}\right) - \Pi^{\frac{\delta}{1+\delta}}\exp\left(\frac{x}{1+\delta}\right)\right] \leq 0. \end{aligned}$$

We show that this is the case for all θ sufficiently small. Choose any $\epsilon \in (0,1)$.

Case 1. Suppose $\Pi \leq 1-\epsilon$. Then $\Pi^{2\delta} \leq (1-\epsilon)^2$ and $\Pi^{\frac{2\delta}{1+\delta}} \leq 1$ and therefore

$$K\left(\Pi\right) \leq \frac{\varepsilon-1}{\theta}\left[(1-\epsilon)^2\left(1 - \frac{\theta}{\varepsilon-1}\beta F\left(\mu\left(1\right),0\right)\right) - 1\right] + 1 + \beta\exp\left(2x\right).$$

The first term diverges to $-\infty$ as $\theta \to 0$ and therefore there exists $\bar{\theta}'$ s.t. $K\left(\Pi\right) \leq 0$ for all $\theta \leq \bar{\theta}', \Pi \leq (1-\epsilon)$.

Case 2. Suppose $\Pi > 1-\epsilon$. Then $\Pi^{2\delta-2}, \Pi^{\frac{2\delta}{1+\delta}-2}, \Pi^{\frac{\delta}{1+\delta}-2} \leq (1-\epsilon)^{-2}$. We know by construction of δ that $K'\left(1\right) = 0$. Therefore it is sufficient to show that $K''\left(\Pi\right) \leq 0$ for all $\Pi > 1-\epsilon$. Differentiate to get

$$\begin{aligned} K''\left(\Pi\right) &= \frac{\varepsilon}{\theta}D2\delta\left(2\delta-1\right)\Pi^{2\delta-2} - 2 + \beta\frac{\partial^2}{\partial\Pi^2}\left[\Pi^{\frac{2\delta}{1+\delta}}\exp\left(\frac{2x}{1+\delta}\right) - \Pi^{\frac{\delta}{1+\delta}}\exp\left(\frac{x}{1+\delta}\right)\right] \\ &= -2 - \left(\frac{\varepsilon}{\theta}D2\delta\right)\Pi^{2\delta-2} + O\left(\theta\right) \leq -2 + O\left(\theta\right) \end{aligned}$$

uniformly for all $\Pi > 1-\epsilon$, which follows from the fact that all powers of Π that appear in the expression are abounded by $(1-\epsilon)^{-2}$ and $\delta_\theta = O\left(\theta\right)$. That we can choose $\bar{\theta}'' > 0$ s.t. $K\left(\Pi\right) \leq 0$ for all $\theta \leq \bar{\theta}'', \Pi > (1-\epsilon)$. This completes the proof of existence.

Uniqueness. We already shows that $\omega\left(\Pi\right)$ that solves (34) is unique for all Π satisfying $\mu\left(\Pi\right) \geq -\ln 4$. Suppose $\mu\left(\Pi\right) < -\ln 4$. Since it follows from the previous arguments that $K\left(\Pi\right) < 0$ for all such Π, it must be true that in equilibrium $F\left(\mu\left(\Pi\right),\omega\left(\Pi\right)\right) > F\left(\mu\left(\Pi\right),0\right)$. Since $F\left(\mu\left(\Pi\right),\cdot\right)$ is U-shaped by Lemma 2, there can be only one such ω. This also implies that $\omega\left(\Pi\right)$ is differentiable for all Π.

It remains to show that $\lim_{\Pi\to 0,\infty}\omega\left(\Pi\right) = \infty$. Observe that equation (35) implies that $\delta > 0$ for all $\theta > 0$ and that for fixed ω we have

$$\lim_{\Pi\to 0,\infty}\frac{F\left(\mu\left(\Pi\right),\omega\right)}{\Pi^2} = 0 \text{ for any } \delta \in (0,1). \tag{36}$$

Since $\lim_{\Pi\to\infty}\frac{G(\Pi)}{\Pi^2}=-1$ equation (36) implies that (34) must be violated if $\omega(\Pi)$ is bounded above for high Π. Therefore $\lim_{\Pi\to\infty}\omega(\Pi)=\infty$. Similarly, $\lim_{\Pi\to 0}G(\Pi)=\frac{(1-\varepsilon)}{\theta}<0$ and (36) implies that equation (34) must be violated if $\omega(\Pi)$ is bounded above for low Π. Therefore $\lim_{\Pi\to 0}\omega(\Pi)=\infty$. ■

Proof. [Proof of Proposition 2] Since Φ_t is convex in the neighborhood of P^{cur} the solution to (23) is unique for all θ sufficiently small and satisfies $\lim_{\theta\to 0}Q^*_\theta=P^{cur}$. Direct differentiation yields evaluated at $Q=Q^*=P_t$ yields

$$\frac{\partial}{\partial\omega}Q^*_\theta(\mu,\omega)=-\theta\frac{\beta\left[\frac{2}{\left(Q^*_\theta\right)^2}\exp(2\mu+2\omega)-\frac{1}{2}\frac{1}{Q^*_\theta}\exp\left(\mu+\frac{1}{2}\omega\right)\right]}{\Phi''_t\left(Q^*_\theta\right)+\beta\theta\left[-\frac{3}{\left(Q^*_\theta\right)^3}\exp(2\mu+2\omega)+\frac{2}{\left(Q^*_\theta\right)^2}\exp\left(\mu+\frac{1}{2}\omega\right)\right]}.$$

The denominator of this expression converges to $\Phi''_t(P^{cur})<0$ as $\theta\to 0$. The numerator is positive iff $\ln 4+\mu+\frac{3}{2}\omega\geq q^*_\theta$.

Suppose $\ln 4+\mu>p^{cur}$. Since $\lim_{\theta\to 0}q^*_\theta=p^{cur}$ then $\ln 4+\mu\geq q^*_\theta$ for all θ sufficiently small and therefore $\frac{\partial}{\partial\omega}Q^*_\theta>0$. Similar arguments show properties of $\frac{\partial}{\partial\omega}Q^*_\theta$ when $\ln 4+\mu<p^{cur}$.

To show properties of $\frac{\partial^2}{\partial\omega\partial\mu}Q^*_\theta$ observe that for any μ',μ'' we have

$$\begin{aligned}\frac{\frac{\partial}{\partial\omega}Q^*_\theta(\mu'',\omega)}{\frac{\partial}{\partial\omega}Q^*_\theta(\mu',\omega)} &= \frac{\frac{2}{\left(Q^*_\theta(\mu'')\right)^2}\exp(2\mu''+2\omega)-\frac{1}{2}\frac{1}{Q^*_\theta(\mu'')}\exp\left(\mu''+\frac{1}{2}\omega\right)}{\frac{2}{\left(Q^*_\theta(\mu')\right)^2}\exp(2\mu'+2\omega)-\frac{1}{2}\frac{1}{Q^*_\theta(\mu')}\exp\left(\mu'+\frac{1}{2}\omega\right)}\\ &\times\frac{\Phi''_t\left(Q^*_\theta(\mu')\right)+\beta\theta\left[-\frac{3}{\left(Q^*_\theta(\mu')\right)^3}\exp(2\mu'+2\omega)+\frac{2}{\left(Q^*_\theta(\mu')\right)^2}\exp\left(\mu'+\frac{1}{2}\omega\right)\right]}{\Phi''_t\left(Q^*_\theta(\mu'')\right)+\beta\theta\left[-\frac{3}{\left(Q^*_\theta(\mu'')\right)^3}\exp(2\mu''+2\omega)+\frac{2}{\left(Q^*_\theta(\mu'')\right)^2}\exp\left(\mu''+\frac{1}{2}\omega\right)\right]}.\end{aligned}$$

The second terms on the right hand side converges to 1 as $\theta\to 0$. The first term converges to

$$\frac{\frac{2}{(P^{cur})^2}\exp(2\mu''+2\omega)-\frac{1}{2}\frac{1}{P^{cur}}\exp\left(\mu''+\frac{1}{2}\omega\right)}{\frac{2}{(P^{cur})^2}\exp(2\mu'+2\omega)-\frac{1}{2}\frac{1}{P^{cur}}\exp\left(\mu'+\frac{1}{2}\omega\right)}.$$

If $\mu''>\mu'>p^{cur}_t-\ln 4$, then this term is strictly greater than 1. Since $\frac{\partial}{\partial\omega}Q^*_\theta(\mu,\omega)>0$ for such μ, this establishes that $\frac{\partial}{\partial\omega}Q^*_\theta(\mu'',\omega)>\frac{\partial}{\partial\omega}Q^*_\theta(\mu',\omega)$ for all θ sufficiently low. If $p^{cur}_t-\ln 4>\mu''>\mu'$, then there exists $(\mu',\omega),(\mu'',\omega)$ such that $\frac{\partial}{\partial\omega}Q^*_\theta(\mu'',\omega)>\frac{\partial}{\partial\omega}Q^*_\theta(\mu',\omega)$ and such that $\frac{\partial}{\partial\omega}Q^*_\theta(\mu'',\omega)<\frac{\partial}{\partial\omega}Q^*_\theta(\mu',\omega)$. ■