The Macro Dynamics of the Wage Distribution

Rasmuz Lentz* Jeremy Lise† Jean-Marc Robin‡

December 2016

Abstract

In this note we develop a tractable model of the evolution of the wage distribution, extending Lise and Robin (2016, LR). LR develop a tractable model for allocations of heterogeneous workers to heterogeneous firms in a frictional labor market with aggregate productivity shocks. However, the assumption on wage determination made in LR mean that wages cannot be solved for exactly, indeed one needs to solve for a fixed point in worker values where the distribution of workers across jobs is a state variable. Our current model delivers identical allocations as LR and identical values to workers associated with moving to a new job. We adopt a slightly different assumption about how the firm delivers the value of the job to the worker that implies we can solve for the wage paid to a worker in any given match and any aggregate state of the economy exactly. We now have a way to calculate the exact dynamics of the wage distribution associated with the dynamics of worker firm matches.

Keywords: On-the-job search; Heterogeneity; Aggregate fluctuations; Wage distributions

JEL codes: E24; E32; J63; J64

*University of Wisconsin-Madison; NBER; LMDG; CAP. Email `rlentz@ssc.wisc.edu`

†University of Minnesota. Email `jeremy.lise@gmail.com`

‡Sciences-Po, Paris and University College of London. Email `jmarc.robin@gmail.com`

1 The Model

1.1 Heterogeneity and Aggregate Shocks

The economy is populated by a very large number of infinitely-lived workers indexed by ability x. The density function of the measure of types in the worker population is exogenous and denoted by $\ell(x)$. Jobs are characterized by a vector of two variables $y = (\psi, \varepsilon)$. The first component ψ is specific to the firm and is distributed across firms according to density $\nu(\psi)$. The second component is specific to the match and is drawn from a distribution $\mu(\varepsilon|\psi)$ when worker and firm meet for the first time. The aggregate numbers of workers–$\int \ell(x)\,\mathrm{d}x$–and firms–$\int \nu(\psi)\,\mathrm{d}\psi$–are arbitrary. However, μ is a proper PDF and $\int \mu(\varepsilon|\psi)\,\mathrm{d}\varepsilon = 1$. The distribution of ψ from which workers draw job offers is endogenous and will be specified later. Finally, the aggregate state of the economy is indexed by z_t. At the beginning of each period the aggregate state changes from z to z' according to the Markov transition probability $\pi(z, z')$, $\pi(z, z') \geq 0$, $\int \pi(z, z')\,\mathrm{d}z' = 1$.

1.2 The Meeting Technology

At the beginning of period t, a measure $u_{t-1}(x)$ of unemployed workers of type x and a measure $h_{t-1}(x, y)$ of workers of type x employed at jobs of type y are inherited from period $t-1$, with

$$u_{t-1}(x) + \int h_{t-1}(x, y)\,\mathrm{d}y = \ell(x).$$

Then, the aggregate state changes from z_{t-1} to z_t. For simplicity, we assume that separations and meetings occur sequentially after the realization of the aggregate productivity shock: separations first, then the unemployed and the surviving employees get a chance to draw a new offer.

Let $u_{t+}(x)$ denote the stock of unemployed workers of type x immediately after the realization of z_t (at time $t+$) and the ensuing job destructions, and let $h_{t+}(x, y)$ be the stock of matches of type (x, y) that survive the destruction shocks. Together they produce effective search effort

$$L_t = \int u_{t+}(x)\,\mathrm{d}x + s \iint h_{t+}(x, y)\,\mathrm{d}x\,\mathrm{d}y,$$

where the search effort of unemployed workers has been normalized to one and s is the relative search effort of employed workers.

Let $v_t(y)$ denote the density function of type-y job offers (see Section 1.5 for details). Let $V_t = \int v_t(y)\,\mathrm{d}y$ denote the aggregate number of job opportunities. The total measure of meetings at time t is given by $M_t = M(L_t, V_t)$. Then $\lambda_t = M_t/L_t$ is the probability an unemployed searcher contacts a vacancy, and $s\lambda_t$ is the probability an employed searcher contacts a vacancy in period t. Let $q_t = M_t/V_t$ be the probability per unit of recruiting effort $v_t(y)$ that a firm contacts any searching worker.

1.3 The Value of Unemployment

Let $B_t(x)$ be the value of unemployment to a type-x worker at t. The t subscript indicates that this value function depends on all the time varying states, including the aggregate productivity z_t, the distribution of workers $u_{t-1}(x)$ and $h_{t-1}(x, y)$, and the offer distribution $v_t(y)$ (possibly). Consider a worker of type x who is unemployed for the whole period t. During that period she earns $b(x, z_t)$, which depends on her own type and the current aggregate productivity of the economy. She anticipates that at the beginning of period $t + 1$, after revelation of the new aggregate state, she will meet a vacancy of type y with probability $\lambda_{t+1}\frac{v_{t+1}(y)}{V_{t+1}}$.

Let $P_t(x, y)$ denote the value of a match (x, y) at any calendar time t. The difference $S_t(x, y) = P_t(x, y) - B_t(x)$ is called the surplus of the match. We assume that only matches generating a value $P_t(x, y)$ greater than $B_t(x)$, or positive surplus, can be formed. We also assume that the surplus is entirely appropriated by the employer when a match is formed with an unemployed worker.

The value to this unemployed worker is therefore

$$\begin{aligned} B_t(x) &= b(x, z_t) + \frac{1}{1+r}\mathbb{E}_t\left[(1-\lambda_{t+1})B_{t+1}(x) + \lambda_{t+1}\int B_{t+1}(x)\frac{v_{t+1}(y)}{V_{t+1}}\,\mathrm{d}y\right] \\ &= b(x, z_t) + \frac{1}{1+r}\mathbb{E}_t B_{t+1}(x), \end{aligned} \tag{1}$$

where r is the discount rate and $\mathbb{E}_t$ is the expectation operator with respect to future aggregate states given the information set at time t.

It follows that there exists a solution $B_t = \mathrm{B}[z_t]$, where the operator B solves the fixed point equation

$$\mathrm{B}[z](x) = b(x,z) + \frac{1}{1+r}\int \mathrm{B}[z'](x)\,\pi(z,z')\,\mathrm{d}z'. \tag{2}$$

1.4 The Value and Surplus of a Match

Firms have access to a production technology, defined at the match level, that combines the skills of a worker and the technology of a firm with aggregate productivity to create value added $p(x,y,z)$.

A match (x,y) thus produces $p(x,y,z_t)$ in period t. At the beginning of period $t+1$, after revelation of the new aggregate shock z_{t+1}, the worker and the firm are better off separated than staying together if and only if $P_{t+1}(x,y) < B_{t+1}(x)$. In addition, we allow for a source of idiosyncratic job destruction shocks δ. The match is therefore destroyed with probability

$$\mathbf{1}\left\{P_{t+1}(x,y) < B_{t+1}(x)\right\} + \delta \times \mathbf{1}\left\{P_{t+1}(x,y) \geq B_{t+1}(x)\right\},$$

and if the job is destroyed the continuation value of the match is the value of unemployment $B_{t+1}(x)$.

The current match continues in period $t+1$ with probability $(1-\delta)\mathbf{1}\{P_{t+1}(x,y) \geq B_{t+1}(x)\}$. Then the worker draws an alternative offer of type y' with probability $s\lambda_{t+1}\frac{v_{t+1}(y')}{V_{t+1}}$. We adopt the sequential auction framework of Postel-Vinay and Robin (2002). Incumbent and poaching firms engage in Bertrand competition which grants the worker a value equal to the second highest bid. Specifically, either $P_{t+1}(x,y') > P_{t+1}(x,y)$ and the worker moves to firm y' and receives the incumbent employer's reservation value $P_{t+1}(x,y)$ as continuation value; or $P_{t+1}(x,y') \leq P_{t+1}(x,y)$ and the worker stays with her current employer with continuation value the minimum of $P_{t+1}(x,y')$ and the worker's current contract. Hence, Bertrand competition makes the continuation value of the match independent of whether the employee is poached or not:

$$\begin{aligned} P_t(x,y) &= p(x,y,z_t) \\ &\quad + \frac{1}{1+r}\mathbb{E}_t\left\{B_{t+1}(x) + (1-\delta)\mathbf{1}\left\{P_{t+1}(x,y) \geq B_{t+1}(x)\right\}\left[P_{t+1}(x,y) - B_{t+1}(x)\right]\right\}. \end{aligned}$$

Finally, making use of equation (1), the preceding equation simplifies to

$$S_t(x,y) = p(x,y,z_t) - b(x,z_t) + \frac{1-\delta}{1+r}\mathbb{E}_t S_{t+1}(x,y)^+, \tag{3}$$

where we denote $X^+ = \max\{X, 0\}$, for any scalar X. It follows that there exists a solution $S_t = \mathrm{S}[z_t]$, where $\mathrm{S}[z]$ solves the equation

$$\mathrm{S}[z](x,y) = p(x,y,z) - b(x,z) + \frac{1-\delta}{1+r}\int \mathrm{S}[z'](x,y)^+\,\pi(z,z')\,\mathrm{d}z'. \tag{4}$$

Notice that matches with $p(x,y,z)-b(x,z) < 0$ may be formed or survive in recessions if they are sufficiently profitable in better times.

We can also explicitly express the stocks $h_{t+}(x,y)$ as

$$h_{t+}(x,y) = (1-\delta)\mathbf{1}\left\{S_t(x,y) \geq 0\right\} h_t(x,y). \tag{5}$$

and $u_{t+}(x) = \ell(x) - \int h_{t+}(x,y)\,\mathrm{d}y$.

1.5 Vacancy Creation

Each period firms can advertise n job opportunities at a cost $c(n) \geq 0$ that is assumed independent of the firm's type, increasing and convex. In equilibrium, the number of advertised job opportunities is determined by equating the marginal cost to the expected value of a job opening,

$$c'\left[n_t(\psi)\right] = q_t J_t(\psi), \tag{6}$$

where $J_t(\psi)$ denotes the expected value of a contact by a vacancy of type ψ, and q_t is the probability, per unit of recruiting effort, that a firm contacts a searching worker. The assumption that $c(\cdot)$ is increasing and convex guarantees a non-degenerate distribution of vacancies $n_t(\psi)$.

Any job opportunity that does not deliver a contact with a worker in the period is lost and generates no continuation value. Any contact that does not end up in an employment contract is lost and has zero value.

The expected value of a contact is calculated as

$$J_t(\psi) = \int \frac{u_{t+}(x)}{L_t} S_t(x,y)^+ \mu(\varepsilon|\psi)\,\mathrm{d}\varepsilon\,\mathrm{d}x + \iint \frac{sh_{t+}(x,y')}{L_t}\left[S_t(x,y) - S_t(x,y')\right]^+ \mu(\varepsilon|\psi)\,\mathrm{d}\varepsilon\,\mathrm{d}x\,\mathrm{d}y'. \quad (7)$$

The contact is with an unemployed worker of type x with probability $\frac{u_{t+}(x)}{L_t}$ and a match is formed if the net match surplus $S_t(x,y)$, for $y = (\psi, \varepsilon)$, is positive, in which case it is entirely appropriated by the employer. The contact is with a worker of type x that is currently employed at a firm of type y' with complementary probability $\frac{sh_{t+}(x,y')}{L_t}$. Poaching is successful if $S_t(x,y) > S_t(x,y')$ and Bertrand competition grants the poacher a value $S_t(x,y) - S_t(x,y') = P_t(x,y) - P_t(x,y')$.

Note that $J_t = \mathrm{J}[h_{t-1}, z_t]$ is explicitly defined by equation (7) given S.

Finally, the offer distribution of $y = (\psi, \varepsilon)$ has density

$$v_t(y) = \mu(\varepsilon|\psi) n_t(\psi) \nu(\psi),$$

where $\nu(\psi)$ is the number of firms of type ψ. Note that $q_t = M(L_t, V_t)/V_t$ depends on V_t. So, V_t must first be solved as a solution to equation

$$V_t = \int n_t(\psi)\nu(\psi)\,\mathrm{d}\psi = \int (c')^{-1}\left[\frac{M(L_t,V_t)}{V_t} J_t(\psi)\right] \nu(\psi)\,\mathrm{d}\psi. \quad (8)$$

The distribution of vacancies created in period t is a deterministic functional of the distribution of worker job matches in period $t-1$ and the aggregate productivity shock in period t: $v_t = \mathrm{v}[h_{t-1}, z_t]$.

1.6 Contractual Environment

We consider employment contracts with limited commitment stipulating a fixed share of the match surplus that the employer commits to.[1] A contract can be renegotiated

[1] Alternative assumptions about how to deliver a given surplus to the worker would be a piece rate on output or a fixed wage until both parties agree to renegotiate. All assumptions deliver identical allocations of workers to jobs. The assumption of working with a share of the surplus, as opposed to a share of flow output or a fixed wage simplifies calculations to the point of delivering a closed form expression for the wage. Lise and Postel-Vinay (2015) adopt this specification to simplify the computations in a model with multi dimensional human capital accumulation and production that is non-additive in worker skills, firm productivity and skill requirements. Bagger et al. (2014) use a

only if both parties agree. Employers can fire workers and workers can quit at will. There is no severance payment or experience rating, and unemployment benefit is not contingent on previous work history.

Let $W_t(x, y, \sigma)$ denote the present value for a worker of type x employed by a firm of type y, with a contract that delivers a share σ of the match surplus to the worker. By definition of σ,

$$W_t(x, y, \sigma) = B_t(x) + \sigma S_t(x, y).$$

Hiring from unemployment requires setting $\sigma = 0$, and competition between two firms y and y' at time t_0, with $S_{t_0}(x, y) \leq S_{t_0}(x, y')$, yields $W_{t_0}(x, y', \sigma) = B_{t_0}(x) + S_{t_0}(x, y)$ or $\sigma = S_{t_0}(x, y)/S_{t_0}(x, y')$. The employer commits to yield σ share of the rent at any future date $t > t_0$, i.e.

$$W_t(x, y', \sigma) = B_t(x) + \sigma S_t(x, y') = B_t(x) + \frac{S_{t_0}(x, y)}{S_{t_0}(x, y')} S_t(x, y'),$$

unless some change in the environment forces partners to renegotiate.

Now, consider a match (σ, x, y). The firm pays some wage $w_t(\sigma, x, y)$ in period t. The optimal decision of the firm is to terminate any match which produces negative surplus, and to yield surplus in order to retain workers in positive surplus matches who have outside offers. Thus, in period $t + 1$, with probability $(1 - \delta)\mathbf{1}\{S_{t+1}(x, y) \geq 0\}\, s\lambda_{t+1} \frac{v_{t+1}(y')}{V_{t+1}}$, the worker draws an alternative offer y' generating the following renegotiation of σ to

$$\sigma' = \begin{cases} S_{t+1}(x, y)/S_{t+1}(x, y') & \text{if } S_{t+1}(x, y') > S_{t+1}(x, y), \\ S_{t+1}(x, y')/S_{t+1}(x, y) & \text{if } \sigma S_{t+1}(x, y) < S_{t+1}(x, y') \leq S_{t+1}(x, y), \\ \sigma & \text{if } S_{t+1}(x, y') \leq \sigma S_{t+1}(x, y). \end{cases} \tag{9}$$

Working with contracts that specify a surplus share have the convenient property that while aggregate shocks will generally lead to a wage change, they do not lead to a contract negotiation, except in the extreme case where $S_t(x, y) < 0$ and both parties agree to terminate the relationship.

---

piece rate rather than fixed wage to obtain tractability in a model with human capital accumulation. Note that, in the absence of human capital accumulation or aggregate shocks, a constant share of the surplus or a constant share of output deliver the same constant wage as in Postel-Vinay and Robin (2002).

In any period, a contract σ induces a wage $w_t(\sigma, x, y)$ that is such that

$$\begin{aligned} W_t(\sigma, x, y) &= B_t(x) + \sigma S_t(x, y) \\ &= w_t(\sigma, x, y) + \frac{1}{1+r} \mathbb{E}_t B_{t+1}(x) \\ &\quad + \frac{1-\delta}{1+r} \mathbb{E}_t \left[\mathbf{1}\left\{ S_{t+1}(x, y) \geq 0 \right\} \left(s\lambda_{t+1} \int I_{t+1}(\sigma, x, y, y') \frac{v_{t+1}(y')}{V_{t+1}} \, \mathrm{d}y' \right.\right. \\ &\quad \left.\left. + (1 - s\lambda_{t+1}) \, \sigma S_{t+1}(x, y) \right) \right], \end{aligned}$$

where $I_{t+1}(\sigma, x, y, y')$ is the second best of the three values: $S_{t+1}(x, y')$, $S_{t+1}(x, y)$, $\sigma S_{t+1}(x, y)$. That is,

$$I_{t+1}(\sigma, x, y, y') = \begin{cases} S_{t+1}(x, y) & \text{if } S_{t+1}(x, y') > S_{t+1}(x, y), \\ S_{t+1}(x, y') & \text{if } \sigma S_{t+1}(x, y) < S_{t+1}(x, y') \leq S_{t+1}(x, y), \\ \sigma S_{t+1}(x, y) & \text{if } S_{t+1}(x, y') \leq \sigma S_{t+1}(x, y). \end{cases}$$

After eliminating unemployment values using equation (1),

$$\begin{aligned} \sigma S_t(x, y) &= w_t(\sigma, x, y) - b(x, z_t) \\ &\quad + \frac{1-\delta}{1+r} \mathbb{E}_t \left[\mathbf{1}\left\{ S_{t+1}(x, y) \geq 0 \right\} \left(s\lambda_{t+1} \int I_{t+1}(\sigma, x, y, y') \frac{v_{t+1}(y')}{V_{t+1}} \, \mathrm{d}y' \right.\right. \\ &\quad \left.\left. + (1 - s\lambda_{t+1}) \, \sigma S_{t+1}(x, y) \right) \right]. \end{aligned}$$

And finally, using equation (3),

$$\begin{aligned} & w_t(\sigma, x, y) = \sigma p(x, y, z_t) + (1 - \sigma) b(x, z_t) \\ & - \frac{1-\delta}{1+r} \mathbb{E}_t \left[\mathbf{1}\left\{ S_{t+1}(x, y) \geq 0 \right\} s\lambda_{t+1} \int \left[I_{t+1}(\sigma, x, y, y') - \sigma S_{t+1}(x, y) \right] \frac{v_{t+1}(y')}{V_{t+1}} \, \mathrm{d}y' \right]. \end{aligned} \tag{10}$$

The expectation on the right hand side can be computed exactly given the results above. The wage is the piece rate wage $\sigma p(x, y, z_t) + (1 - \sigma) b(x, z_t)$ diminished of future renegotiation opportunities ($I_{t+1}(\sigma, x, y, y') - \sigma S_{t+1}(x, y) \geq 0$).

The model also has potentially interesting implications for the cyclicality of wages.

- Workers recently hired from unemployment (who will generally have a low σ) will have counter cyclical wages (at $\sigma = 0$ they will be positively correlated with $b(x, z)$ and negatively correlated with the expected future surplus gains).
- Experienced workers (those with high σ) will have pro-cyclical wages (for $\sigma = 1$ they are perfectly correlated with output).
- It is not obvious a priori whether the average wage is pro-, counter-, or a-cyclical.

1.7 Labor Market Flows

The law of motion for employment is therefore

$$h_t(x,y) = h_{t+}(x,y) \times \left[1 - s\lambda_t + \int s\lambda_t \frac{v_t(y')}{V_t} \mathbf{1}\{S_t(x,y') \leq S_t(x,y)\}\, \mathrm{d}y'\right] + \int h_{t+}(x,y') s\lambda_t \frac{v_t(y)}{V_t} \mathbf{1}\{S_t(x,y) > S_t(x,y')\}\, \mathrm{d}y' + u_{t+}(x)\lambda_t \frac{v_t(y)}{V_t} \mathbf{1}\{S_t(x,y) \geq 0\}, \quad (11)$$

subtracting those lost to more productive poachers, and adding the (x, y)-jobs created by poaching from less productive firms and hiring from unemployment. Unemployment follows as $u_t(x) = \ell(x) - \int h_t(x, y)\, \mathrm{d}y$.

Note that h_t is a deterministic functional of h_{t-1} and z_t, $h_t = \mathrm{h}[h_{t-1}, z_t]$, which is implicitly given by equation (11).

Finally we can determine the law of motion of the cross-sectional distribution function of contracts:

$$G_t(\sigma,x,y) = G_{t+}(\sigma,x,y) \times \left[1 - s\lambda_t + \int s\lambda_t \frac{v_t(y')}{V_t} \mathbf{1}\{S_t(x,y') \leq \sigma S_t(x,y)\}\, \mathrm{d}y'\right] + \int h_{t+}(x,y') \mathbf{1}\{\sigma S_t(x,y) > S_t(x,y')\}\, \mathrm{d}y' \times s\lambda_t \frac{v_t(y)}{V_t} + u_{t+}(x)\lambda_t \frac{v_t(y)}{V_t} \mathbf{1}\{S_t(x,y) \geq 0\}, \quad (12)$$

where

$$G_{t+}(\sigma, x, y) = (1 - \delta)\mathbf{1}\left\{S_t(x, y) \geq 0\right\} G_{t-1}(\sigma, x, y).$$

Notice also that, by definition, $G_t(1, x, y) = h_t(x, y)$. The first row of the RHS of equation (12) is the stock of matches (x, y) with contract less than σ which remain unchanged. The second row counts all poaching occurrences (a match (x, y') draws an offer y such that $S(x, y) > S(x, y')$) delivering a contract $S(x, y')/S(x, y)$ that is less than σ. The last row counts all hires from unemployment.

References

Bagger, J., F. Fontaine, F. Postel-Vinay, and J.-M. Robin (2014): "A Tractable Equilibrium Search Model of Individual Wage Dynamics with Experience Accumulation," *American Economic Review*, 104, 1551–96.

Lise, J. and F. Postel-Vinay (2015): "Multidimensional Skills, Sorting, and Human Capital Accumulation," Unpublished manuscript.

Lise, J. and J.-M. Robin (2016): "The Macro-dynamics of Sorting between Workers and Firms," Forthcoming *American Economic Review*.

Postel-Vinay, F. and J.-M. Robin (2002): "Equilibrium wage dispersion with worker and employer heterogeneity," *Econometrica*, 70, 2295–2350.