Precautionary Savings and Pecuniary Externalities: Analytical Results for Optimal Capital Income Taxation

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Abstract

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1 Introduction

[TBC]

Literature on precautionary saving in partial equilibrium: Sandmo, Barsky, Mankiw and Zeldes

Literature on overaccumulation of capital in general equilibrium and capital income taxation: Aiyagari (1995), Davila, Hong, Krusell and Rios-Rull (20xx)

OLG with idiosyncratic risk and fiscal policy.

2 Model

2.1 Time and Demographics

Time is discrete and extends from $t = 0$ to $t = \infty$. In each period a new generation (indexed by the time of birth t) is born that lives for two periods. Thus at any point in time there is a $y$oung and an $o$ld generation. We normalize household size to 1 for each age cohort. In addition there is an initial old generation that has one remaining year of life.

2.2 Household Preferences and Endowments

2.2.1 Endowments

Each household has one unit of time in both periods, supplied inelastically to the market. Labor productivity when young is equal to 1, in the second period labor productivity is given by

$$\lambda\eta_{t+1}$$

where $\lambda \geq 0$ is a parameter and η_{t+1} is an idiosyncratic labor productivity shock. We assume that the cdf of η_{t+1} is given by $\Psi(\eta_{t+1})$ in every period and deonote the corresponding pdf by $\psi(\eta_{t+1})$. We assume that Ψ is both the population distribution of η_{t+1} as well as the cdf of the productivity shock for any given individual (that is, we assume a Law of Large Numbers, LLN henceforth). Whenever there is no scope for confusion we suppress the time subscript of the productivity shock η_{t+1}. We make the following

Assumption 1.

$$\int \eta_{t+1} d\Psi = 1.$$

Each member of the initial old generation is addtitionally endowed with assets equal a_0, equal to the initial capital stock K_0 in the economy. The asset endowment is independent of the household's realization of the shock η.

2.2.2 Preferences

A household of generation $t \geq 0$ has preferences over consumption allocations $c_t^y, c_{t+1}^o(\eta_{t+1})$ given by

$$V_t = u(c_t^y) + \beta \int u(c_{t+1}^o(\eta_{t+1}))d\Psi \tag{1}$$

Lifetime utility of the initial old generation is determined as

$$V_{-1} = \int u(c_0^o(\eta_0))d\Psi$$

2.3 Technology

The representative firm operates a standard Cobb-Douglas production technology of the form

$$F(K_t, L_t) = K_t^{\alpha} (L_t)^{1-\alpha}.$$

Capital fully depreciates between two (30 year) periods.

2.4 Government

The government levies a (potentially time varying) capital tax τ_t on capital (including interest) and rebates the proceeds in a lump-sum fashion to all members of the current old generation as a transfer T_t. For the optimal tax policiy analysis we assume the government has the following social welfare function

$$SWF = \sum_{t=-1}^{\infty} \omega_t V_t$$

where $\{\omega_t\}_{t=-1}^{\infty}$ are the Pareto weights on different generations and satisfy $\omega_t \geq 0$. Since lifetime utilities of each generation will be bounded, so will be the social welfare function as long as $\sum_{t=-1}^{\infty} \omega_t < \infty$. We will also consider the case $\omega_t = 1$ for all t, in which case we will take the social welfare function to be defined as

$$SWF = \lim_{T \to \infty} \frac{\sum_{t=-1}^{T} V_t}{T}$$

which is equivalent to maximizing steady state welfare (as long as a steady state exists).

2.5 Competitive Equilibrium

2.5.1 Household Budget Set and Optimization Problem

The budget constraints in both periods read as

$$c_t^y + a_{t+1} = w_t \tag{2}$$

$$c_{t+1}^o = a_{t+1} R_{t+1}(1 - \tau_{t+1}) + \lambda \eta_{t+1} w_{t+1} + T_{t+1} \tag{3}$$

where w_t, w_{t+1} are the aggregate wages in period t and $t+1$, $R_{t+1} = 1 + r_{t+1}$ is the gross interest rate between period t and $t+1$, and T_{t+1} are lump-sum transfers to the old generation. η_{t+1} is the age-2 period-$t+1$ idiosyncratic shock to wages.[1]

[1] Notice that instead of working with a tax on capital τ_t, one could work, completely equivalently, with a (standard) capital income tax τ_t^k given by

$$1 + r_t(1 - \tau_t^k) = (1 + r_t)(1 - \tau_t)$$

and thus

$$\tau_t^k = 1 - \frac{R_t(1 - \tau_t) - 1}{R_t - 1}.$$

2.5.2 Firm Optimization

From the firms first order conditions

$$R_t = \alpha k_t^{\alpha-1} \tag{4a}$$

$$w_t = (1-\alpha)k_t^{\alpha} \tag{4b}$$

where

$$k_t = \frac{K_t}{L_t}$$

is the capital-labor ratio.

2.5.3 Equilibrium Definition

Definition 1. *Given the initial condition* $a_0 = K_0$ *an allocation is a sequence* $\{c_t^y, c_t^o(\eta_t), L_t, a_{t+1}, K_{t+1}\}_{t=0}^{\infty}$.

Definition 2. *Given the initial condition* $a_0 = K_0$ *and a sequence of tax policies* $\tau = \{\tau_t\}_{t=0}^{\infty}$*, a competitive equilibrium is an allocation* $\{c_t^y, c_t^o, L_t, a_{t+1}, K_{t+1}\}_{t=0}^{\infty}$*, prices* $\{R_t, w_t\}_{t=0}^{\infty}$ *and transfers* $\{T_t\}_{t=0}^{\infty}$ *such that*

1. *Given prices* $\{R_t, w_t\}_{t=0}^{\infty}$ *and policies* $\{\tau_t, T_t\}_{t=0}^{\infty}$ *for each* $t \geq 0$, $(c_t^y, c_{t+1}^o(\eta_{t+1}), a_{t+1})$ *maximizes (1) subject to (2) and (3) (for each realization of* η_{t+1}*).*

2. *Consumption* $c_0^o(\eta_0)$ *of the initial old satisfies (3) (for each realization of* η_0*):*
$$c_0^o = a_0 R_0(1-\tau_0) + \lambda\eta_0 w_0 + T_0$$

3. *Prices satisfy equations (4a) and (4b).*

4. *The government budget constraint is satisfied in every period: for all* $t \geq 0$
$$T_t = \tau_t R_t K_t$$

5. *Markets clear*

$$\begin{aligned} L_t &= L = 1 + \lambda \\ a_{t+1} &= K_{t+1}. \\ c_t^y + \int c_t^o(\eta_t) d\Psi + K_{t+1} &= K_t^\alpha (L_t)^{1-\alpha} \end{aligned}$$

Denote by $SWF(\tau)$ social welfare associated with an equilibrium for given tax policy τ. As we will show below, for a given tax policy τ the associated competitive equilibrium in our economy exists and is unique and thus the function $SWF(\tau)$ is well-defined.[2]

Definition 3. *Given the initial condition $a_0 = K_0$, a Ramsey equilibrium is a sequence of tax policies $\hat{\tau} = \{\hat{\tau}_t\}_{t=0}^{\infty}$ and equilibrium allocations, prices and transfers associated with $\hat{\tau}$ (in the sense of the previous definition) such that*

$$\hat{\tau} \in \arg\max_{\tau} SWF(\tau)$$

3 Analysis of Equilibrium for a Given Tax Policy

3.1 Partial Equilibrium

We first proceed to analyze the household problem for given prices and policies. We will first proceed under the assumption that a unique solution characterized by the Euler equation exists, and then make sufficient parametric assumptions to insure that this is indeed the case.

The optimal asset choice a_{t+1} satisfies

$$1 = \beta(1 - \tau_{t+1}) \int \frac{R_{t+1} \left[u'(a_{t+1}R_{t+1}(1 - \tau_{t+1}) + \lambda \eta_{t+1} w_{t+1} + T_{t+1})\right]}{u'(w_t - a_{t+1})} d\Psi(\eta_{t+1}).$$

[2]This requires some restrictions on the τ_t that we need to spell out. I believe we need $\tau_t \leq 1$ and $\tau_t \geq -\bar{\tau}$.

Defining the saving rate as

$$s_t = \frac{a_{t+1}}{w_t}$$

we can rewrite the above equation as

$$1 = \beta(1-\tau_{t+1}) \int \frac{R_{t+1}\left[u'(s_t R_{t+1}(1-\tau_{t+1})w_t + \lambda\eta_{t+1}w_{t+1} + T_{t+1})\right]}{u'\left[w_t(1-s_t)\right]} d\Psi(\eta_{t+1}) \tag{5}$$

which defines the solution

$$s_t = s_t(w_t, w_{t+1}, R_{t+1}, \tau_{t+1}, T_{t+1}; \beta, \lambda, \theta, \Psi)$$

Here θ is the parameter vector characterizing the utility function u. Of course without further assumptions on the fundamentals we can't make progress.

3.1.1 Special Case 1: Constant Relative Risk Aversion

In this case we need to make sure that consumption in the second period is positive η-almost surely. It should be sufficient to assume that $\eta > 0$ almost surely, although weaker conditions might do. With CRRA utility the Euler equation becomes

$$1 = \beta(1-\tau_{t+1})R_{t+1} \int \left(\frac{s_t R_{t+1}(1-\tau_{t+1}) + \lambda\eta_{t+1}\frac{w_{t+1}}{w_t} + \frac{T_{t+1}}{w_t}}{(1-s_t)} \right)^{-\sigma} d\Psi(\eta_{t+1})$$

or

$$1 = \beta(1-\tau_{t+1}) \left(R_{t+1}\right)^{1-\sigma} \int \left(\frac{1-s_t}{s_t(1-\tau_{t+1}) + \frac{\lambda w_{t+1}}{w_t R_{t+1}}\eta_{t+1} + \frac{T_{t+1}}{w_t R_{t+1}}} \right)^{\sigma} d\Psi(\eta_{t+1}) \tag{6}$$

If $\sigma = 1$, this equation becomes:

$$1 = \beta(1-\tau_{t+1}) \int \frac{1-s_t}{s_t(1-\tau_{t+1}) + \frac{\lambda w_{t+1}}{w_t R_{t+1}}\eta_{t+1} + \frac{T_{t+1}}{w_t R_{t+1}}} d\Psi(\eta_{t+1}) \tag{7}$$

Equations (6) and (7) implicitly define the optimal savings rate $s_t =$

$s_t(w_t, w_{t+1}, R_{t+1}, \tau_{t+1}, T_{t+1}; \beta, \lambda, \sigma, \Psi)$.

3.1.2 Special Case 2: Constant Absolute Risk Aversion

CARA with appropriate assumptions on the distribution of η (see Willen, 200x and Wang 200x). With CARA utility the Euler equation becomes

$$\exp\left[-\gamma w_t(1-s_t)\right] = \beta(1-\tau_{t+1})R_{t+1}\int \exp\left[-\gamma(s_t R_{t+1}(1-\tau_{t+1})w_t + \lambda\eta_{t+1}w_{t+1} + T_{t+1})\right] d\Psi(\eta_{t+1}).$$

Note that this can be rewritten as

$$\begin{aligned}\exp\left[-\gamma w_t(1-s_t)\right] = \beta(1-\tau_{t+1})R_{t+1}\exp\left(-\gamma s_t R_{t+1}(1-\tau_{t+1})w_t\right) \\ \exp(-\gamma T_{t+1})\int \exp(-\gamma\lambda\eta_{t+1}w_{t+1})d\Psi(\eta_{t+1}).\end{aligned}$$

and taking logs yields

$$\begin{aligned}&\gamma s_t R_{t+1}(1-\tau_{t+1})w_t + \gamma s_t w_t = \\ &\gamma\left(w_t - T_{t+1}\right) + \log\left[\beta(1-\tau_{t+1})R_{t+1}\right] + \log\left(\int \exp(-\gamma\lambda\eta_{t+1}w_{t+1})d\Psi(\eta_{t+1})\right)\end{aligned}$$

and thus the closed form solution for the saving rate is given by:

$$\begin{aligned}s_t = \frac{w_t - T_{t+1} + \frac{1}{\gamma}\left[\log\left[\beta(1-\tau_{t+1})R_{t+1}\right] + \log\left(\int \exp(-\lambda\eta_{t+1}w_{t+1})d\Psi(\eta_{t+1})\right)\right]}{w_t\left[R_{t+1}(1-\tau_{t+1})+1\right]} = \\ s_t(w_t, w_{t+1}, R_{t+1}, \tau_{t+1}, T_{t+1}; \beta, \lambda, \gamma, \Psi).\end{aligned}$$

Note that this result is a special case of Wang (2006). The term $\log\left(\int \exp(-\lambda\eta_{t+1}w_{t+1})d\Psi(\eta_{t+1})\right)$ captures the additional saving due to the presence of idiosyncratic risk. Note that this term is zero if $\lambda = 0$.

3.2 General Equilibrium

Now we exploit the remaining equilibrium conditions. For this it is useful to define the equilibrium capital-labor ratio as $k_t = \frac{K_t}{L_t} = \frac{K_t}{1+\lambda}$. Since labor supply is exogenous we note that there is a one-to one mapping between K_t and k_t. In equilibrium factor prices and transfers are given by

$$w_t = (1-\alpha)k_t^{\alpha} \tag{8a}$$
$$w_{t+1} = (1-\alpha)k_{t+1}^{\alpha} \tag{8b}$$
$$R_{t+1} = \alpha k_{t+1}^{\alpha-1} \tag{8c}$$
$$T_{t+1} = \tau_{t+1}R_{t+1}K_{t+1} = (1+\lambda)\tau_{t+1}R_{t+1}k_{t+1} \tag{8d}$$

From the definition of the saving rate $s_t = \frac{a_{t+1}}{w_t}$ and market clearing in the asset market, which implies $a_{t+1} = K_{t+1} = (1+\lambda)k_{t+1}$, we find that

$$k_{t+1} = \frac{a_{t+1}}{1+\lambda} = \frac{s_t w_t}{1+\lambda}$$

and thus

$$k_{t+1} = s_t \frac{1-\alpha}{1+\lambda} k_t^{\alpha} \tag{9}$$

Sometimes it will be useful to express the saving rate as a function of the capital stocks and write

$$s_t = \frac{(1+\lambda)\, k_{t+1}}{(1-\alpha)\, k_t^{\alpha}} \tag{10}$$

In general, for a given sequence of capital income taxes $\{\tau_t\}_{t=0}^{\infty}$ the competitive equilibrium is a sequence of capital stocks $\{k_{t+1}\}_{t=0}^{\infty}$ that solves, for a given initial condition $k_0 = \frac{K_0}{1+\lambda}$, the first order difference equation (equation (5))

$$1 = \beta(1-\tau_{t+1}) \int \frac{R_{t+1}\left[u'(s_t R_{t+1}(1-\tau_{t+1})w_t + \lambda\eta_{t+1}w_{t+1} + T_{t+1})\right]}{u'\left[w_t(1-s_t)\right]} d\Psi(\eta_{t+1})$$

or, using (8) and (10) in the above

$$1 = \beta(1-\tau_{t+1})\alpha k_{t+1}^{\alpha-1} \int \frac{u'\left\{\left[(1+\lambda)\,\alpha + \lambda\eta_{t+1}(1-\alpha)\right] k_{t+1}^{\alpha}\right\}}{u'\left\{(1-\alpha)\, k_t^{\alpha} - (1+\lambda)\, k_{t+1}\right\}} d\Psi(\eta_{t+1}) \tag{11}$$

which implicitly defines the function $k_{t+1} = \Omega(k_t, \tau_{t+1})$.

Again note that instead of expressing the solution as $k_{t+1} = \Omega(k_t, \tau_{t+1})$ we could also express it as

$$s_t = \frac{(1+\lambda)\,\Omega(k_t, \tau_{t+1})}{(1-\alpha)\,k_t^{\alpha}} = \Lambda(k_t, \tau_{t+1}) \tag{12}$$

Finally we can calculate lifetime utility of a household as

$$\begin{aligned} V_t &= V(k_t, \tau_{t+1}) = u((1-\alpha)\,k_t^{\alpha} - (1+\lambda)\,\Omega(k_t, \tau_{t+1})) + \beta \int u\left([(1+\lambda)\,\alpha + \lambda\eta_{t+1}(1-\alpha)]\,[\Omega(k_t, \cdots\right. \\ V_{-1} &= V(k_0, \tau_0) = \int u\left([(1+\lambda)\,\alpha + \lambda\eta_0(1-\alpha)]\,k_0^{\alpha}\right) d\Psi(\eta_0) \end{aligned}$$

or, in terms of savings rates

$$V_t = V(k_t, \tau_{t+1}) = u((1-s_t)\,(1-\alpha)\,k_t^{\alpha}) + \beta \int u\left([(1+\lambda)\,\alpha + \lambda\eta_{t+1}(1-\alpha)]\left[\frac{(1-\alpha)\,k_t^{\alpha} s_t}{(1+\lambda)}\right]^{\alpha}\right) d\Psi(\eta_{t+\cdots}$$

Note that the last result implies that τ_0 is irrelevant for welfare of the initial old generation (and all future generations) and can be set arbitrarily. This is due to the fact that τ_0 is non-distortionary, is lump-sum rebated and (most crucially) that the government is assumed to have a period-by-period budget balance. Thus there is no effect of τ_0 on anything and the standard result (tax existing capital at highest possible rate) does not emerge here. In fact, the expression shows that with the set of policies we consider here lifetime utility of the initial old cannot be affected at all, which is useful since we therefore do not need to include it in the social welfare function.

3.2.1 Constant Relative Risk Aversion

For the case of constant relative risk aversion equation (11) becomes

$$1 = \beta(1-\tau_{t+1})\alpha k_{t+1}^{\alpha-1} \int \left(\frac{[(1+\lambda)\,\alpha + \lambda\eta_{t+1}(1-\alpha)]\,k_{t+1}^{\alpha}}{(1-\alpha)\,k_t^{\alpha} - (1+\lambda)\,k_{t+1}}\right)^{-\sigma} d\Psi(\eta_{t+1})$$

and thus

$$
\begin{aligned}
1 &= \beta(1-\tau_{t+1})\left(\alpha k_{t+1}^{\alpha-1}\right)^{1-\sigma}\left(\frac{(1-\alpha)}{(1+\lambda)}\frac{k_t^{\alpha}}{k_{t+1}}-1\right)^{\sigma}\int\left[1+\frac{\lambda(1-\alpha)\eta_{t+1}}{(1+\lambda)\,\alpha}\right]^{-\sigma}d\Psi(\eta_{t+1}) \\
1 &= \beta(1-\tau_{t+1})\left(\alpha k_{t+1}^{\alpha-1}\right)^{1-\sigma}\left(\frac{(1-\alpha)}{(1+\lambda)}\frac{k_t^{\alpha}}{k_{t+1}}-1\right)^{\sigma}\Gamma(\alpha,\lambda,\sigma;\Psi)
\end{aligned}
$$

which implicitly determines $k_{t+1}=\Omega(k_t,\tau_{t+1})$ as a function of the capital stock today and the tax rate as well as model parameters.

Alternatively, we can again rewrite this problem in terms of saving rates and write

$$
1=\beta(1-\tau_{t+1})\left(\alpha\left[\frac{(1-\alpha)\,k_t^{\alpha}}{(1+\lambda)}\right]^{\alpha-1}\right)^{1-\sigma}(s_t)^{(\alpha-1)(1-\sigma)}\left(\frac{1}{s_t}-1\right)^{\sigma}\Gamma(\alpha,\lambda,\sigma;\Psi)
$$

or compactly

$$
1=(1-\tau_{t+1})\,(k_t)^{\alpha(\alpha-1)(1-\sigma)}\,(s_t)^{(\alpha-1)(1-\sigma)}\left(\frac{1}{s_t}-1\right)^{\sigma}\Gamma(\alpha,\lambda,\sigma;\Psi)\gamma(\alpha,\beta,\lambda,\sigma) \tag{15}
$$

which determines the savings rate, in general equilibrium $s_t=\Lambda(k_t,\tau_{t+1})$ as a function of the current capital stock and the future capital income tax rate (as well as parameters).

Here

$$
\gamma(\alpha,\beta,\lambda,\sigma)=\beta\left(\alpha\left[\frac{(1-\alpha)}{(1+\lambda)}\right]^{\alpha-1}\right)^{1-\sigma}>0
$$

is a constant and

$$
\Gamma=\Gamma(\alpha,\lambda,\sigma;\Psi)=\int\left(1+\frac{\lambda(1-\alpha)}{(1+\lambda)\alpha}\eta_{t+1}\right)^{-\sigma}d\Psi(\eta_{t+1})
$$

is a constant that determines the impact of idiosyncratic risk on saving and the dynamics of the capital stock. Also define

$$
\bar{\Gamma}=\bar{\Gamma}(\alpha,\lambda,\sigma)=\left(1+\frac{\lambda(1-\alpha)}{(1+\lambda)\alpha}\right)^{-\sigma}=\left(\frac{\alpha+\alpha\lambda}{\alpha+\lambda}\right)^{\sigma}<1
$$

Now we make

Assumption 2. $\eta_{t+1} > 0$ *almost surely.*

Note that under this assumption $\Gamma(\alpha, \lambda, \sigma; \Psi) \in (0, 1)$ and by Jensen's inequality $\bar{\Gamma}(\alpha, \lambda, \sigma) < \Gamma(\alpha, \lambda, \sigma; \Psi) < 1$.

Log-Utility In the case of log-utility we can solve for the optimal savings rate in closed form

$$\frac{s_t}{1 - s_t} = \beta(1 - \tau_{t+1})\Gamma(\alpha, \lambda, \sigma = 1; \Psi)$$

or

$$s_t = \frac{1}{1 + [\beta(1 - \tau_{t+1})\Gamma]^{-1}} = s_t(\tau_{t+1})$$

Comparative Statics We can rewrite (15) as

$$\frac{(s_t)^{1-\alpha+\alpha\sigma}}{(1 - s_t)^{\sigma}} = \gamma(\alpha, \beta, \lambda, \sigma)\frac{(1 - \tau_{t+1})\Gamma(\alpha, \lambda, \sigma; \Psi)}{(k_t)^{\alpha(1-\alpha)(1-\sigma)}} > 0 \tag{16}$$

How the savings rate changes with the tax rate, with income risk and with the existing capital stock depends on parameters. Define the function

$$f(s) = \frac{(s)^{1-\alpha+\alpha\sigma}}{(1 - s)^{\sigma}}$$

and note that $f(s) \geq 0$ for all s,

$$\begin{aligned} f(0) &= 0 \\ \lim_{s \to 1} f(s) &= \infty \\ f'(s) &> 0 \end{aligned}$$

and thus there is a unique solution $s_t = \Lambda(k_t, \tau_{t+1})$ which is strictly increasing in γ, Γ, strictly decreasing in τ_{t+1} and the impact of k_t on the saving rate depends on the size of σ. Thus the government can implement uniquely, for a given current capital stock k_t, any savings rate s_t by appropriate choice of the capital income tax rate $\tau_{t+1} \in (-\infty, 1]$.

Lifetime Utility: Towards the Primal Approach Note that for any given k_t, equation defines, uniquely, the tax rate τ_{t+1} that implements a given savings rate s_t. Or, in terms of the function $\Lambda(k_t, \tau_{t+1})$, this tax rate $\tau_{t+1}(s_t; k_t)$ is implicitly given by

$$s_t = \Lambda(k_t, \tau_{t+1}(s_t; k_t))$$

and for the log-case has the explicit solution

$$\tau_{t+1}(s_t; k_t) = \tau_{t+1}(s_t) = 1 - \frac{s_t}{(1 - s_t)\beta\Gamma(\alpha, \lambda, \sigma = 1; \Psi)}$$

which is independent of the level of capital. Thus, rather than choosing tax rates the Ramsey government can be thought of as choosing savings rates of households directly. This, of course, is the essence of the primal approach to optimal taxation.

In terms of savings rates, lifetime utility of generation t is given by

$$V(k_t, s_t) = u((1-s_t)(1-\alpha)k_t^{\alpha}) + \beta \int u\left([(1+\lambda)\alpha + \lambda\eta_{t+1}(1-\alpha)]\left[\frac{(1-\alpha)k_t^{\alpha}s_t}{(1+\lambda)}\right]^{\alpha}\right) d\Psi(\eta_{t+1})$$

with the implied law of motion for capital

$$k_{t+1} = \frac{(1-\alpha)k_t^{\alpha}s_t}{(1+\lambda)} \tag{17}$$

and the initial condition $k_0 = \frac{K_0}{1+\lambda} > 0$ given.

For CRRA utility with $\sigma \neq 1$ we have

$$V(k_t, s_t) = \frac{((1-s_t)(1-\alpha)k_t^{\alpha})^{1-\sigma} + \beta\left[k_t^{\alpha}s_t\right]^{\alpha(1-\sigma)}\tilde{\Gamma}}{1-\sigma} \tag{18}$$

where

$$\tilde{\Gamma} = \int \left([(1+\lambda)\alpha + \lambda\eta(1-\alpha)]\left[\frac{(1-\alpha)}{(1+\lambda)}\right]^{\alpha}\right)^{1-\sigma} d\Psi(\eta) > 0 \tag{19}$$

whereas for log-utility we find

$$\begin{aligned}
V(k_t, s_t) &= \log((1-s_t)(1-\alpha)k_t^{\alpha}) + \beta \int \log\left([(1+\lambda)\alpha + \lambda\eta_{t+1}(1-\alpha)] \left[\frac{(1-\alpha)k_t^{\alpha} s_t}{(1+\lambda)} \right]^{\alpha} \right) d\Psi(\eta_t \\
&= \log(1-s_t) + \alpha\beta\log(s_t) + \alpha(1+\alpha\beta)\log(k_t) + \log(1-\alpha) + \beta\hat{\Gamma}
\end{aligned}$$

where

$$\hat{\Gamma} = \int \log\left([(1+\lambda)\alpha + \lambda\eta(1-\alpha)] \left[\frac{(1-\alpha)}{(1+\lambda)} \right]^{\alpha} \right) d\Psi(\eta)$$

4 Ramsey Tax Problem

The objective of the Ramsey government, stated in its primal form, is then to maximize, by choice of $\{s_t\}_{t=0}^{\infty}$

$$SWF(\{s_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \omega_t V(k_t, s_t)$$

subject to the law of motion for capital (17) and the initial condition k_0.

4.1 Special Case 1: Log-Utility

For the log-utility case we first recognize that

$$\begin{aligned}
\log(k_{t+1}) &= \log(1-\alpha) - \log(1+\lambda) + \alpha\log(k_t) + \log(s_t) \\
&= \varkappa + \log(s_t) + \alpha\left[\alpha\log(k_{t-1}) + \log(s_{t-1})\right] \\
&= \varkappa_{t+1} + \sum_{\tau=0}^{t} \alpha^{\tau}\log(s_{t-\tau})
\end{aligned}$$

and thus the objective of the social planner problem is given by

$$\begin{aligned}
\sum_{t=0}^{\infty}\omega_t V(k_t,s_t) &= \sum_{t=0}^{\infty}\omega_t\left[\log(1-s_t)+\alpha\beta\log(s_t)+\alpha\left(1+\alpha\beta\right)\log(k_t)+\log(1-\alpha)+\beta\hat{\Gamma}\right] \\
&= \varkappa+\sum_{t=0}^{\infty}\left[\omega_t\log(1-s_t)+\log(s_t)\left(\alpha\beta\omega_t+\alpha\left(1+\alpha\beta\right)\sum_{\tau=t+1}^{\infty}\omega_\tau\alpha^{\tau-(t+1)}\right)\right] \\
V(\{s_t\}_{t=0}^{\infty}) &= \varkappa+\sum_{t=0}^{\infty}\omega_t\left[\log(1-s_t)+\log(s_t)\left(\alpha\beta+\alpha\left(1+\alpha\beta\right)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)\right]
\end{aligned}$$

where $\varkappa$ is a messy constant that depends on the initial capital stock k_0, but is irrelevant for maximization.

The problem of the Ramsey government is to maximize $SWF(\{s_t\}_{t=0}^{\infty})$ by choice of the savings rates $\{s_t\}_{t=0}^{\infty}$. These savings rates in turn imply, given k_0, sequences of capital determined by

$$k_{t+1}=\frac{(1-\alpha)\,k_t^{\alpha}}{(1+\lambda)}s_t$$

and capital tax rates defined by

$$s_t=\frac{1}{1+\left[\beta(1-\tau_{t+1})\Gamma\right]^{-1}}$$

or

$$(1-\tau_{t+1})=\frac{s_t}{\beta\Gamma\left(1-s_t\right)} \tag{20}$$

In the next theorem we completely and analytically characterize the solution to the Ramsey tax problem, not just in the steady state, but along the entire transition path!

Theorem 1. *Suppose the utility function is logarithmic and the idiosyncratic shock process satisfies assumptions 1 and 2. Then the solution to the Ramsey problem is given by savings rates*

$$s_t=\frac{1}{1+\left(\alpha\beta+\alpha\left(1+\alpha\beta\right)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)^{-1}}\in(0,1)$$

and can be implemented by the capital taxes

$$1 - \tau_{t+1} = \frac{\left(\alpha\beta + \alpha\left(1+\alpha\beta\right)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)}{\beta\Gamma} > 0$$

Proof. The first order conditions read as

$$\frac{s_t}{1-s_t} = \left(\alpha\beta + \alpha\left(1+\alpha\beta\right)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)$$

and thus immediately give the optimal savings rates as stated in the theorem. The tax rates immediately follow from equation (20). □

Corollary 1. *The optimal savings rates are independent of the extent of income risk in the economy. The optimal capital tax rates are strictly increasing in the extent of income risk (as measured by* Γ*).*

Corollary 2. *Suppose the ratio of social welfare weights* $\frac{\omega_{t+j}}{\omega_t}$ *do not depend on time* t*. Then optimal savings rates and capital tax rates are constant over time.*

Example 1. *Consider the case of constant welfare weights* $\frac{\omega_{t+j}}{\omega_t} = 1$*, corresponding to a steady state analysis. Then the optimal savings rate is given by*

$$s_t = \frac{\alpha(1+\beta)}{1+\alpha\beta}$$

and can be implemented by the capital taxes

$$1 - \tau_{t+1} = \frac{\alpha(1+\beta)}{(1-\alpha)\beta\Gamma}$$

Example 2. *Suppose the weights on future generations is geometrically de-*

clining such that $\frac{\omega_{t+1}}{\omega_t} = \gamma < 1$. *Then*

$$\begin{aligned} s_t &= \frac{1}{1 + \left(\alpha\beta + \frac{\alpha(1+\alpha\beta)}{1-\gamma\alpha}\right)^{-1}} \\ &= \frac{1}{1 + \left(\frac{\alpha(\beta+\gamma)}{1-\gamma\alpha}\right)^{-1}} \\ &= \frac{\alpha(\gamma+\beta)}{1+\alpha\beta} \end{aligned}$$

and can be implemented by the capital taxes

$$\begin{aligned} 1 - \tau_{t+1} &= \frac{\alpha\beta(1-\gamma\alpha) + \alpha\gamma\,(1+\alpha\beta)}{(1-\gamma\alpha)\,\beta\Gamma} \\ &= \frac{\alpha\,(\gamma+\beta)}{(1-\alpha\gamma)\,\beta\Gamma} \end{aligned}$$

4.2 Special Case II; CRRA Utility with $\sigma \neq 1$

4.2.1 Steady States

Note that a steady state analysis corresponds to Pareto weights $\omega_t = 1$, at least as t gets large. To approach this problem, consider maximizing steady state utility. From (17) we have that the steady state capital stock is given by

$$k = \left(\frac{(1-\alpha)s}{1+\lambda}\right)^{\frac{1}{1-\alpha}}. \tag{21}$$

and (18) writes in steady state as

$$V(k,s) = \frac{((1-s)\,(1-\alpha)\,k^{\alpha})^{1-\sigma} + \beta\,[k^{\alpha}s]^{\alpha(1-\sigma)}\,\tilde{\Gamma}}{1-\sigma}$$

Using (21) in the above we get

$$
\begin{aligned}
V(s) &= \frac{\left((1-s)\,(1-\alpha)\left(\frac{(1-\alpha)s}{1+\lambda}\right)^{\frac{\alpha}{1-\alpha}}\right)^{1-\sigma} + \beta\left[\left(\frac{(1-\alpha)s}{1+\lambda}\right)^{\frac{\alpha}{1-\alpha}} s\right]^{\alpha(1-\sigma)}\tilde{\Gamma}}{1-\sigma} \\
&= \left(\phi_1(1-s)^{1-\sigma} + \beta\phi_2\tilde{\Gamma}\right) s^{\frac{\alpha(1-\sigma)}{1-\alpha}}
\end{aligned}
$$

where $\phi_1 = \frac{\left((1-\alpha)\left(\frac{1-\alpha}{1+\lambda}\right)^{\frac{\alpha}{1-\alpha}}\right)^{1-\sigma}}{1-\sigma}$ and $\phi_2 = \frac{\left(\left(\frac{1-\alpha}{1+\lambda}\right)^{\frac{\alpha^2(1-\sigma)}{1-\alpha}}\right)}{1-\sigma}$. It is convenient to rewrite this further as

$$
V(s) = \phi_1\left((1-s)^{1-\sigma} + \beta\tilde{\phi}\tilde{\Gamma}\right) s^{\frac{\alpha(1-\sigma)}{1-\alpha}}
$$

where $\tilde{\phi} = \frac{\phi_2}{\phi_1} = \frac{\left(\frac{1+\lambda}{1-\alpha}\right)^{\alpha(1-\sigma)}}{(1-\alpha)^{1-\sigma}} > 0$.

Now, take the FOC w.r.t. s to get

$$
\begin{aligned}
V_s = \phi_1 \Bigg(&-(1-\sigma)(1-s)^{-\sigma} s^{\frac{\alpha(1-\sigma)}{1-\alpha}} + \\
&+ \left((1-s)^{1-\sigma} + \beta\tilde{\phi}\tilde{\Gamma}\right)\frac{\alpha(1-\sigma)}{1-\alpha} s^{\frac{\alpha(1-\sigma)}{1-\alpha}-1}\Bigg) = 0
\end{aligned}
$$

From this we get

$$
f(s) \equiv \frac{s}{(1-s)^{\sigma}} - \frac{\alpha}{1-\alpha}(1-s)^{1-\sigma} - \beta\frac{\alpha}{1-\alpha}\tilde{\phi}\tilde{\Gamma} = 0 \tag{22}
$$

Observation 1. *Observe that for $\sigma = 1$ we get—noticing that $\tilde{\phi} = \tilde{\Gamma} = 1$—*

$$
\begin{aligned}
\frac{s}{(1-s)} - \frac{\alpha}{1-\alpha} &= \beta\frac{\alpha}{1-\alpha} \\
\Leftrightarrow \qquad s &= \frac{\alpha(1+\beta)}{1+\alpha\beta}
\end{aligned}
$$

which is just the solution derived above for the case $\frac{\omega_{t+1}}{\omega_1} = 1$.

Observation 2. *Because*

$$\lim_{\sigma\to\infty} \tilde{\phi} = 0$$

(22) converges to

$$\lim_{\sigma\to\infty} f(s) = \frac{s}{(1-s)^{\sigma}} - \frac{\alpha}{1-\alpha}(1-s)^{1-\sigma}$$

Solving $\lim_{\sigma\to\infty} f(s) = 0$ *for* $s^\star$ *gives* $s^\star = \alpha$.

Proposition 1. *The steady state exists and is unique.*

Proof. Define from (22)

$$f(s) \equiv \frac{s}{(1-s)^{\sigma}} - \frac{\alpha}{1-\alpha}(1-s)^{1-\sigma} - \beta\frac{\alpha}{1-\alpha}\tilde{\phi}\tilde{\Gamma}. \tag{23}$$

and observe that

$$f(s=0) = -\frac{\alpha}{1-\alpha}\left(1+\beta\tilde{\phi}\tilde{\Gamma}\right) < 0$$

and $\lim_{s\to1} f(s) = \infty$.

It remains to establish that there is only a single root of $f(s)$. Observe that

$$\frac{\partial f(s)}{\partial s} = \frac{(1-s)^{\sigma} + s\sigma(1-s)^{\sigma-1}}{(1-s)^{2\sigma}} + (1-\sigma)\frac{\alpha}{1-\alpha}(1-s)^{-\sigma}. \tag{24}$$

For $\sigma \leq 1$ (24) is positive for all s. Hence there is a single root.

For $\sigma > 1$ the minimum of $f(s)$ is characterized by the condition

$$\begin{aligned}
\Leftrightarrow \qquad & (1-s)^{\sigma} + s\sigma(1-s)^{\sigma-1} = (\sigma-1)\frac{\alpha}{1-\alpha}(1-s)^{\sigma} \\
\Leftrightarrow \qquad & (1-s)^{\sigma}\left(1 + s\sigma(1-s)^{-1}\right) = (\sigma-1)\frac{\alpha}{1-\alpha}(1-s)^{\sigma} \\
\Leftrightarrow \qquad & \frac{s}{1-s} = \frac{(\sigma-1)\frac{\alpha}{1-\alpha} - 1}{\sigma} \equiv \phi \\
\Leftrightarrow \qquad & s = \frac{\phi}{1+\phi}.
\end{aligned}$$

To characterize the minimum further, observe that for $\sigma > 1$ we have $\phi > -1$. To see this, notice that

$$\begin{aligned}
& \phi > -1 \\
\Leftrightarrow \qquad & \frac{1 - (\sigma-1)\frac{\alpha}{1-\alpha}}{\sigma} < 1 \\
\Leftrightarrow \qquad & 1 - (\sigma-1)\frac{\alpha}{1-\alpha} < \sigma \\
\Leftrightarrow \qquad & 1 + \frac{\alpha}{1-\alpha} < \sigma\left(1 + \frac{\alpha}{1-\alpha}\right)
\end{aligned}$$

which always holds for $\sigma > 1$.

Suppose that $\phi \in (-1, 0]$, for which we need

$$\phi \equiv \frac{(\sigma-1)\frac{\alpha}{1-\alpha} - 1}{\sigma} \leq 0 \qquad \Leftrightarrow \qquad \sigma \leq \frac{1}{\alpha}.$$

Hence, for $\sigma \leq \frac{1}{\alpha}$ we have $f(s)$ upward sloping on the entire interval $s \in [0, 1)$. For $\sigma > \frac{1}{\alpha}$, we have $f(s)$ downward sloping on the interval $s \in [0, \frac{\phi}{1+\phi})$ and upward sloping on $s \in (\frac{\phi}{1+\phi}, 1)$.

Therefore, the steady state is unique. $\square$

Now, analyze (22) by $\sigma <> 1$

1. $\sigma > 1$: Observe from (19) that a mean preserving spread of η ($MPS(\eta)$) increases $\tilde{\Gamma}$. Hence a $MPS(\eta)$ increases the RHS in (22). Rewrite the

LHS of (22) as

$$\frac{s}{(1-s)^{\sigma}} - \frac{\alpha}{1-\alpha}\frac{1}{(1-s)^{\sigma-1}} = \frac{1}{(1-s)^{\sigma-1}}\left(\frac{s}{1-s} - \frac{\alpha}{1-\alpha}\right)$$

to see that the LHS is increasing in s. Hence a $MPS(\eta)$ increases the optimal savings rate s.

2. $\sigma < 1$: Observe from (19) that a $MPS(\eta)$ decreases $\tilde{\Gamma}$. From (22) we observe that $\frac{s}{(1-s)^{\sigma}}$ is increasing in s whereas $(1-s)^{1-\sigma}$ is decreasing in s. Therefore the LHS of (22) is increasing in s. Therefore, a $MPS(\eta)$ leads to a reduction of s.

To understand this result (which is a generalization of our earlier findings for the log case) notice that there are two opposing effects: As a reaction to a $MPS(\eta)$ it is optimal to

1. **Precautionary Savings Effect:** increase savings out of precautionary motives,

2. **General Equilibrium Effect:** decrease savings because a $MPS(\eta)$ increases risk exposure in general equilibrium as second period income is given by $w_{t+1}\eta_{t+1} = (1-\alpha)k_{t+1}(s_t)^{1-\alpha}\eta_{t+1}$.

For the case of log utility ($\sigma = 1$) these two effects turn out to offset each other. For low risk aversion (or, better, low prudence) the first effect dominates, for high prudence the second effect dominates.

4.2.2 General Ramsey Problem

Concentrate on the case where $\frac{\omega_t+1}{\omega_t} = \gamma$. Let the value function of the government be given by

$$W(k) = \max_{s}\{V(k,s) + \gamma W(k')\} \tag{25}$$

subject to

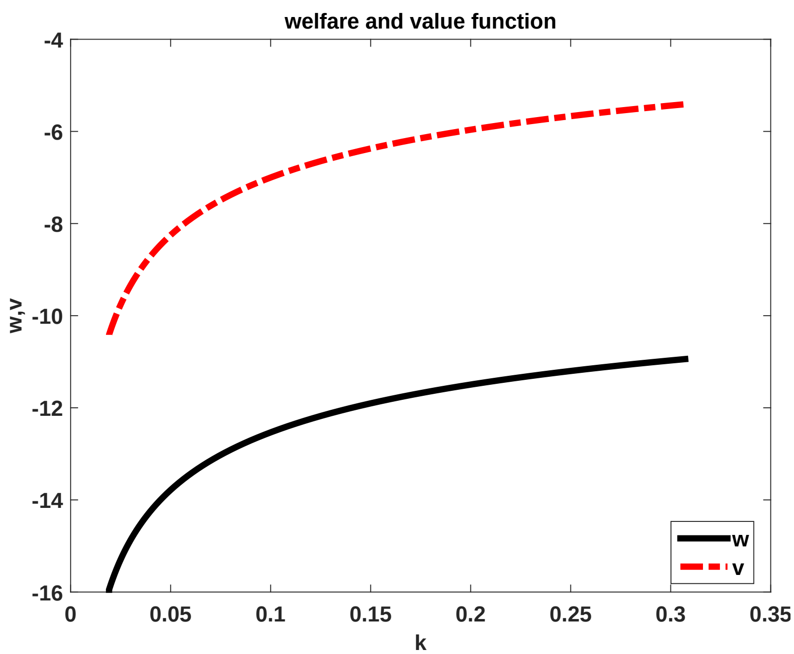
$$k' = g(k, s) = \frac{1-\alpha}{1+\lambda} s k^{\alpha}$$

5 Numerical Results for CRRA Utility

We solve the dynamic program (25). We next simulate the economy along the transition setting the initial capital stock equal to the lowest gridpoint in the capital grid and compute the saving rate such that this initial capital stock constitutes and initial steady state of the economy. We thereby implicitly some initial tax rates (which we do not solve for below). Next, we characterize the transitional dynamics of the economy under two scenarios. One is that the government implements in period $t = 1$ the optimal steady state policy ($\gamma = 1$). The second it that the government implements the optimal policy along a transition for which we choose $\gamma = 0.7$.

Below, we show results for $\sigma = 2$ and $\sigma = 0.5$. From Figure 1 we observe that both the utility function $V(s, k)$ as well as the social welfare function, $W(k)$, are well behaved (globally concave in k). The policy function for the saving rate, shown in Figure 3, behaves according to the predictions we made earlier for the steady state comparison (there for a MPS(η): for high σ the **precautionary savings** effect dominates so that when the capital stock increases the optimal saving rate increases (because risk exposure increases); for low σ, the optimal saving rate decreases in k in order to reduce capital accumulation so that risk exposure in the next period decreases, the **general equilibrium effect**. Figures 7 and 9 look at the transitional dynamics when, from period 1 onwards, the optimal steady state ($\gamma = 1$), respectively the optimal transition policy $\gamma = 0.7$) is implemented. Also for the optimal transition policy, the simulations converge nicely to a steady state. Finally, Figure 11 displays the tax rates τ_t which we derive from the allocations, using equation (15) and Figure 13 the associated capital income tax rate τ_t^k which we back out from the solution, cf. Footnote 1.

Figure 1: Value and Welfare Function: $\sigma = 2$



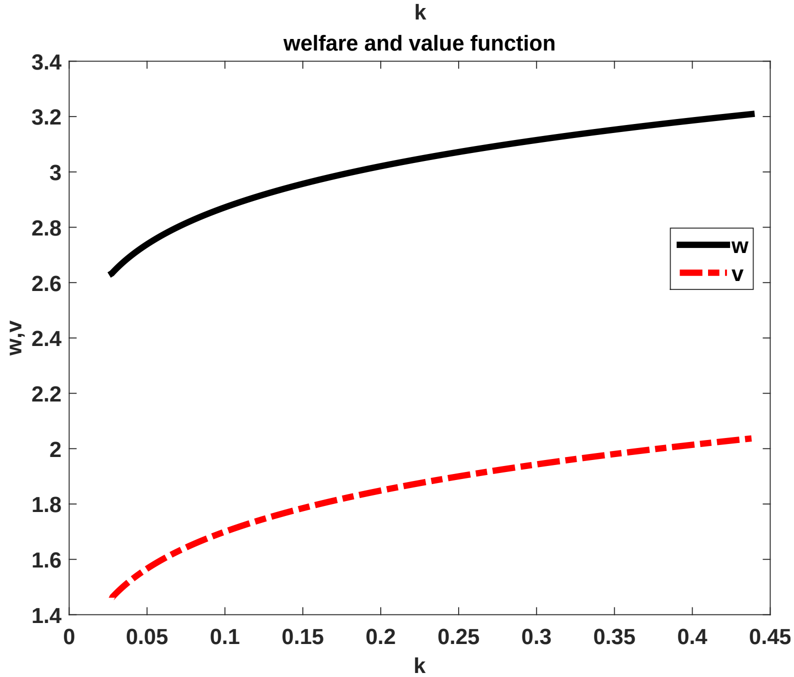
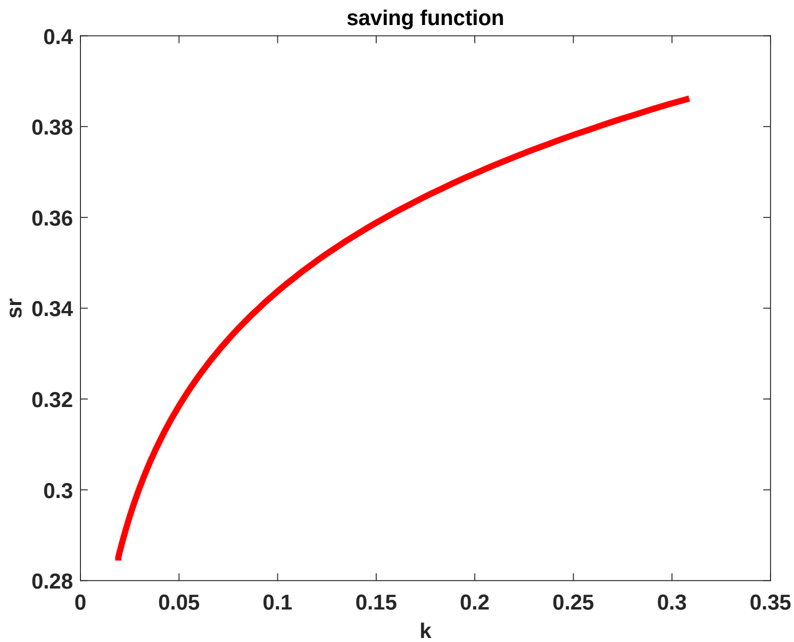


Figure 2: Value and Welfare Function: $\sigma = 0.5$

Figure 3: Policy Function: Saving rate $s(k)$, $\sigma = 2$



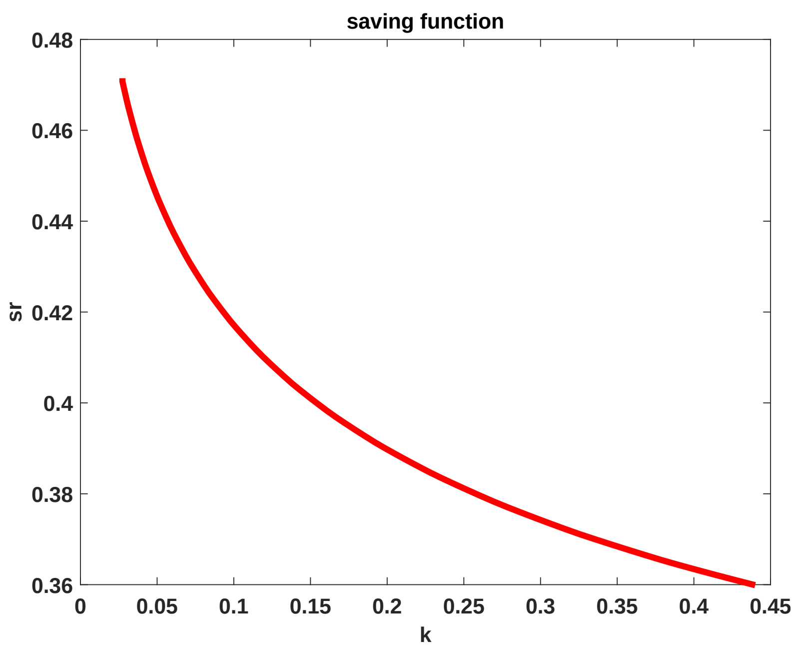
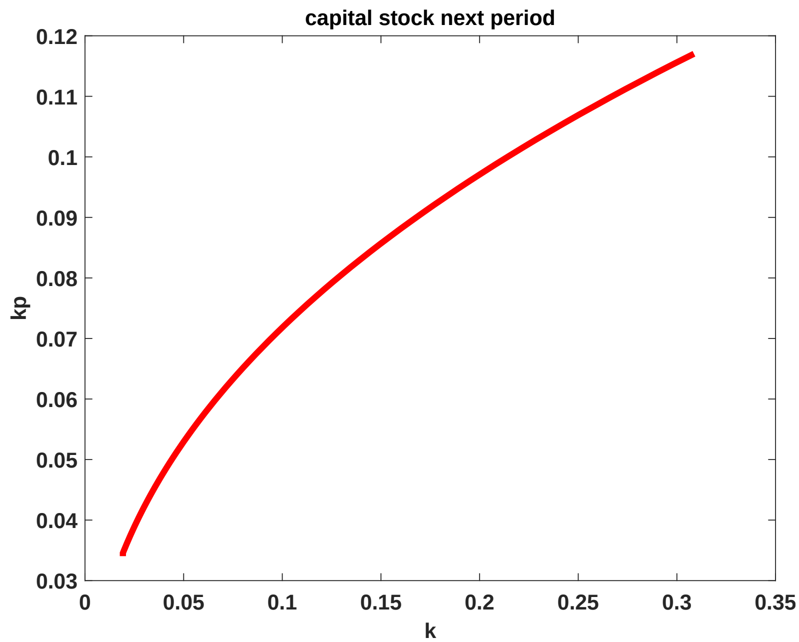


Figure 4: Policy Function: Saving rate $s(k)$, $\sigma = 0.5$

Figure 5: Policy Function: Next period's capital $k'(k)$, $\sigma = 2$



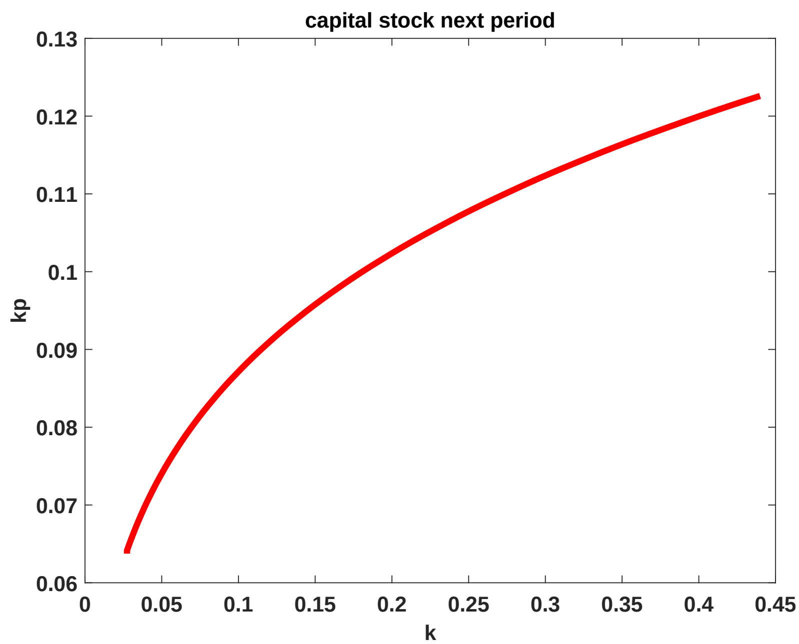
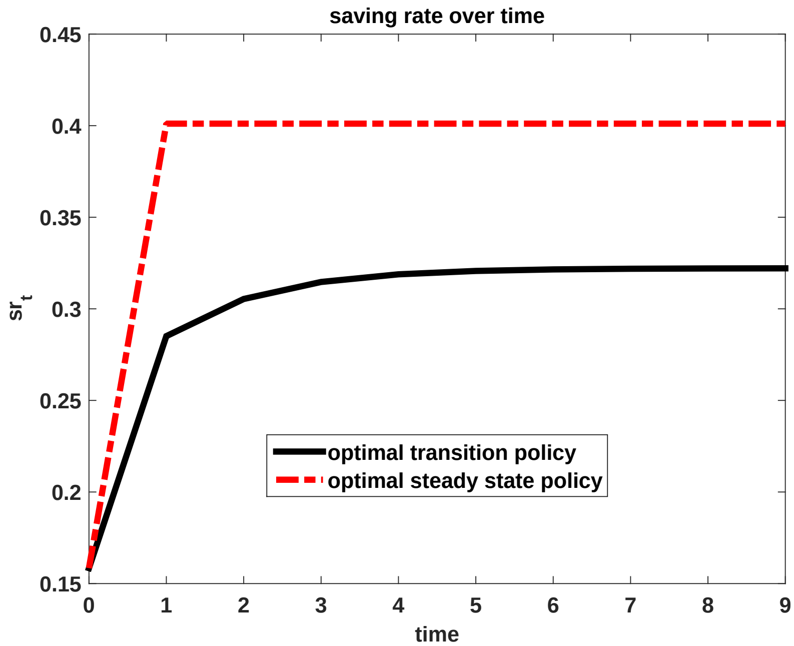


Figure 6: Policy Function: Next period's capital $k'(k)$, $\sigma = 0.5$

Figure 7: Simulation: Saving rate $s(t)$, $\sigma = 2$



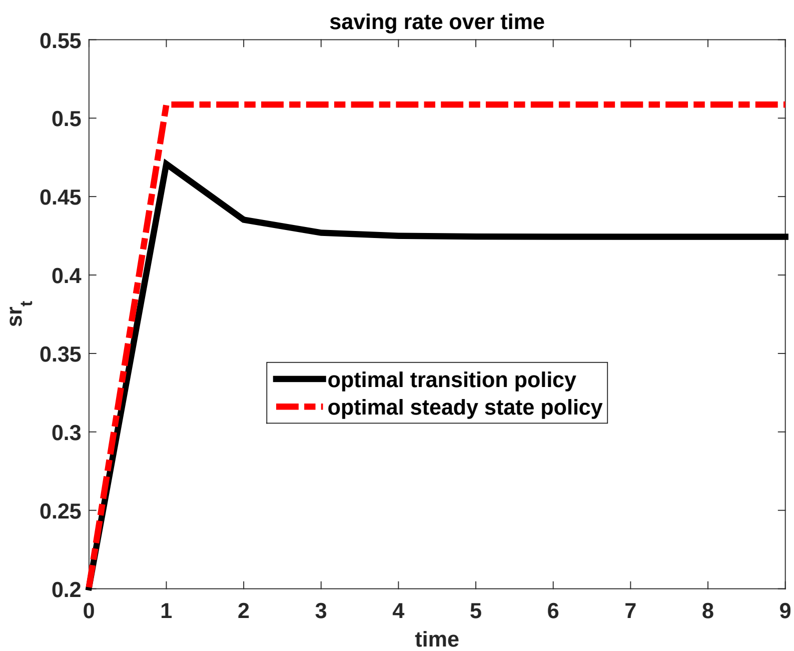
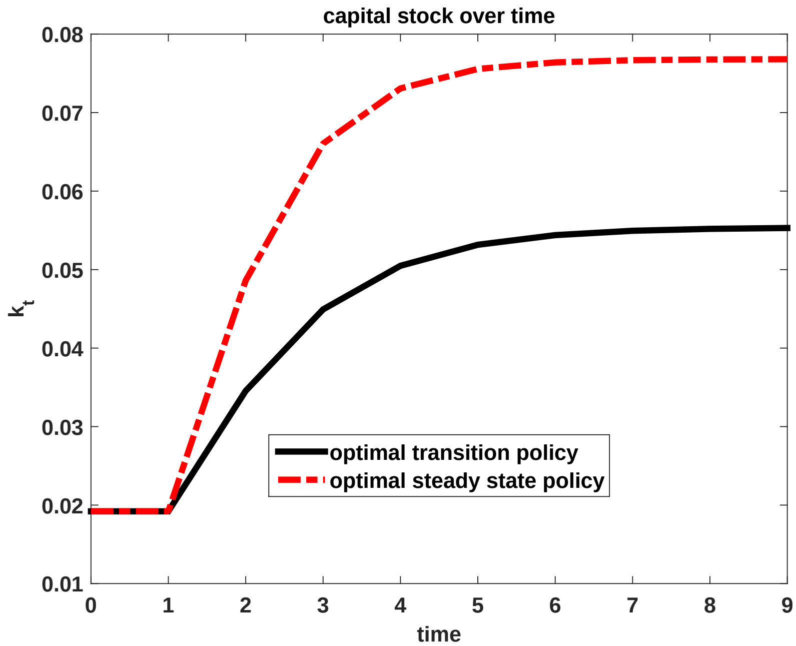


Figure 8: Simulation: Saving rate $s(t)$, $\sigma = 0.5$

Figure 9: Simulation: Capital stock $k(t)$, $\sigma = 2$



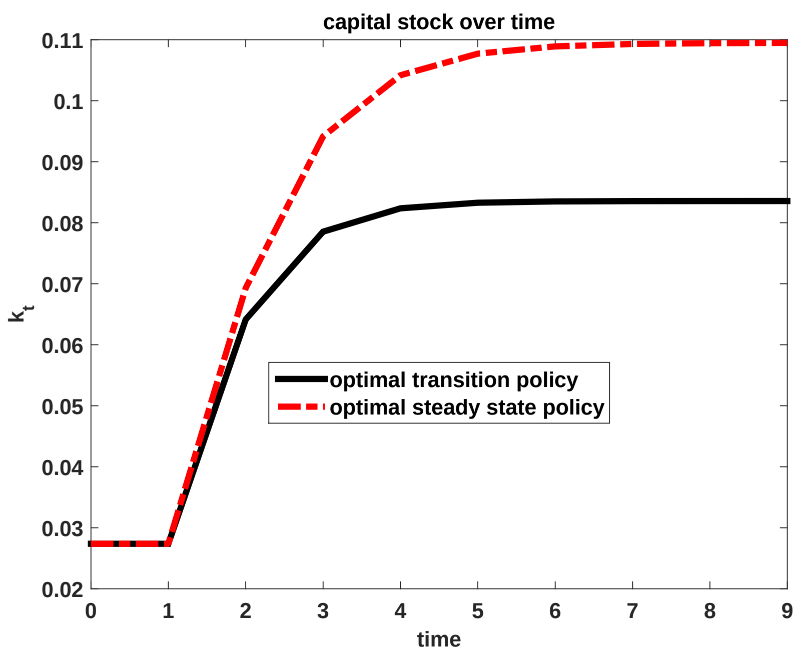
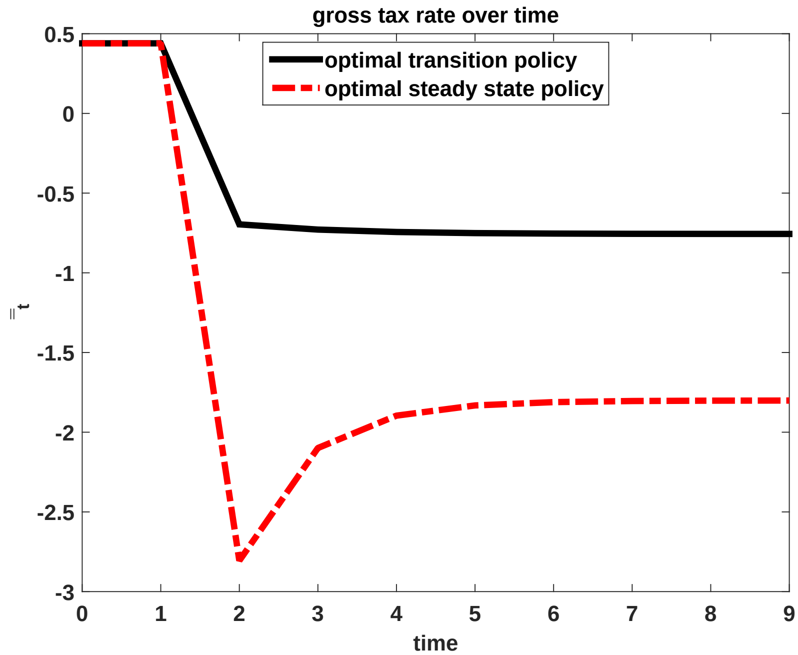


Figure 10: Simulation: Capital stock $k(t)$, $\sigma = 0.5$

Figure 11: Simulation: Tax Rate $\tau(t)$, $\sigma = 2$



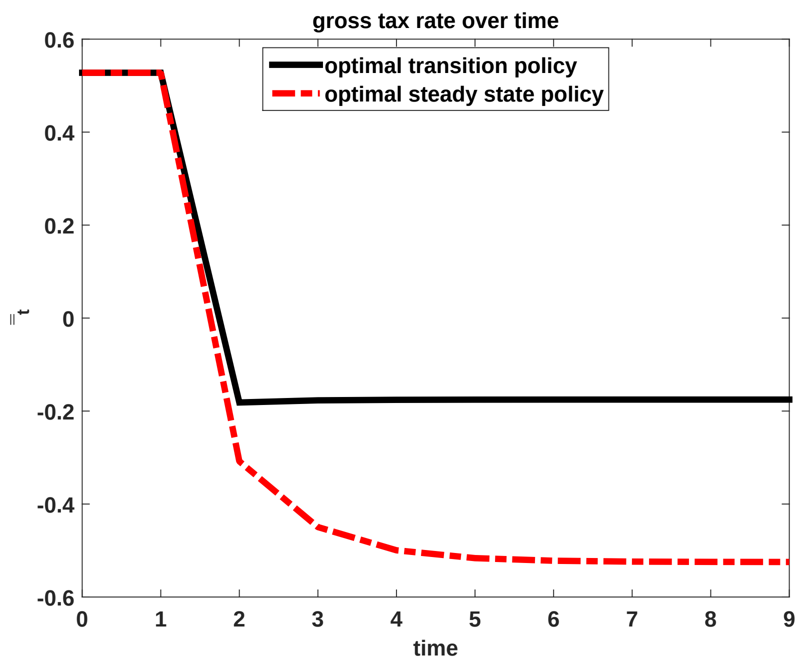
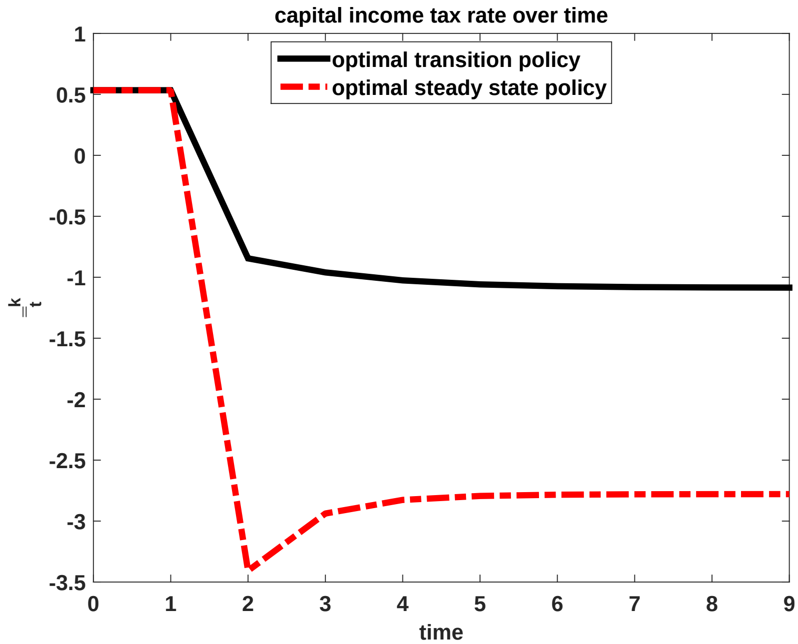


Figure 12: Simulation: Tax Rate $\tau(t)$, $\sigma = 0.5$

Figure 13: Simulation: Capital Income Tax Rate $\tau^k(t)$, $\sigma = 2$



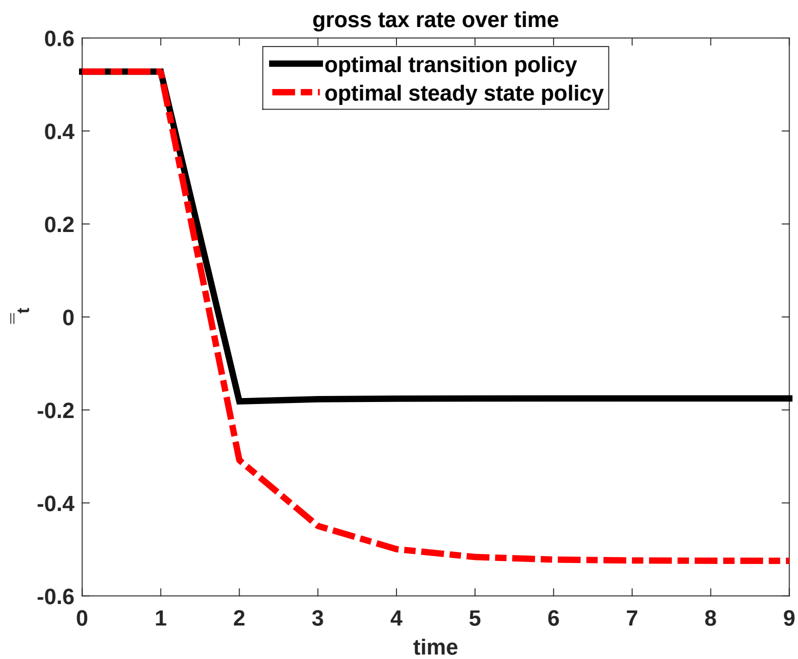


Figure 14: Simulation: Capital Income Tax Rate $\tau^k(t)$, $\sigma = 0.5$

6 Conclusion

[TBC]