

Debt Runs and the Value of Liquidity Reserves*

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Abstract

This article analyzes a firm prone to debt runs, and the effect of its portfolio liquidity composition on the run behavior of its creditors. The firm holds cash and an illiquid cash flow generating asset, and is financed with debt held by a continuum of creditors. At each point in time, a constant fraction of the firm's outstanding liabilities matures, leading the maturing creditors to decide whether to roll-over or ask for their funds back. When the firm's portfolio value deteriorates, creditors are inclined to run, but their propensity to run decreases with the amount of available liquidity resources. The theory has policy implications for micro-prudential bank liquidity regulation: for any leverage ratio, it characterizes the quantity of liquidity reserves a firm should hold in order to deter a run. I solve the model numerically and perform comparative statics, varying the firm's illiquid asset characteristics and the firm's debt maturity profile. I discuss the influence of the firm's portfolio choice and dividend policy on the run behavior of creditors. The model can also be transported into an international macroeconomic context: the firm can be reinterpreted as a central bank/government, having issued foreign-currency denominated sovereign debt that is regularly rolled over. A high debt-to-GDP ratio combined with low levels of foreign currency reserves will prompt foreign creditors to run. The theory can therefore provide guidance on the appropriate sizing of central banks' foreign currency reserves for countries issuing large amounts of short term foreign exchange debt.

Key Words: Bank Runs, Cash Holdings

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1 Introduction

Corporates die of cancer, but financial firms die of heart attacks. This quote from [Valukas et al. \(2010\)](#) emphasizes an important characteristic of firms prone to debt runs: they rarely fail due to a lack of equity capital, but rather due to a lack of liquidity. The last-minute rescue of Bear Stearns in March 2008 and the bankruptcy of Lehman Brothers in September 2008 illustrate well this phenomenon: days before their collapse, both U.S. broker-dealers were in compliance with the SEC’s net capital rules, and had investment grade ratings by all major credit rating agencies. Instead, it is the shortage of available funds and liquid unencumbered assets that precipitated those firms’ demise. Of course, liquidity and solvency should be viewed as tightly interconnected concepts: a deteriorating capital situation at a run-prone institution will cause its creditors to be reluctant to roll-over their maturing debt, which in turn leads to increasing liquidity needs for the institution, and potentially to its default. It seems that, at the time, institutions prone to run risk held insufficient liquidity reserves.

The funding stress suffered by financial institutions during the 2008-2009 financial crisis attracted the attention of both academia and policy makers. [Brunnermeier, Krishnamurthy, and Gorton \(2013\)](#) focus on the U.S. banking sector and measure the mismatch between the market liquidity of bank assets and the funding liquidity of their liabilities. Their “liquidity mismatch index” worsens dramatically in the two years leading to the crisis. This liquidity mismatch also prompted the Basel Committee on Banking Supervision to develop several regulations that would force banks to hold minimum amounts of cash and liquid unencumbered assets. The Liquidity Coverage Ratio test, for example, ensures that banks hold sufficient high quality liquid assets to survive a significant stress scenario lasting 30 calendar days, whereas the Net Stable Funding Ratio test requires that at least 100% of a bank’s long term asset portfolio is financed with stable funding.

But those rules seem ad-hoc and fail to capture certain key characteristics of runs on run-prone institutions. Why is 30 day the right time horizon for the Liquidity Coverage Ratio test? Should rules driving a bank’s liquidity pool also be linked to its solvency – in other words, should a bank with a lower capital ratio be forced to hold more liquidity reserves than a healthier bank, since it is potentially more prone to runs? How do these rules capture the strategic behavior of creditors who determine their actions not only by looking at the bank’s asset composition, but also by reacting to the assumed behavior of other creditors?

My paper will provide some answers to these micro-prudential regulation design issues. Leveraging the canonical framework of [He and Xiong \(2012\)](#), I develop a model in which a financial firm holds a portfolio consisting of cash and liquid reserves on the one hand,

and illiquid assets on the other. Cash and liquid reserves can be sold immediately at their fundamental price, whereas illiquid assets can only be disposed of at a discount to their intrinsic value. The firm finances itself with debt that is purchased by a continuum of creditors. Each year, a constant fraction of the firm’s outstanding liabilities matures, a feature first introduced by [Leland \(1994\)](#). This feature guarantees that the firm’s average debt maturity is constant. When a creditor’s debt claim matures, the creditor decides whether to continue financing the firm, or to stop rolling over its debt claim. The creditor behaves strategically by taking into account the firm’s asset composition and the assumed behavior of the firm’s other creditors.

I look for a symmetric Markov perfect equilibrium in cutoff strategies for creditors: they run when the firm’s illiquid asset value falls below an endogenous threshold that depends on the amount of liquid resources available at the firm. Otherwise they roll-over their maturing claims. The runs in my model are thus directly linked to the solvency and liquidity position of the firm. A deteriorating solvency position combined with a weak liquidity situation leads creditors to start running on the firm.

For the model parameters considered, I find only one equilibrium in cutoff strategies. This result stems from two key ingredients first introduced by [Frankel and Pauzner \(2000\)](#): creditors make asynchronous roll/run decisions, and the state variables (the fundamental value of the illiquid asset and the liquid resources of the firm) are time varying. This proves fundamental in looking at the model from a policy recommendation’s perspective. I use this feature to compute probabilities of runs and probabilities of bankruptcy for any balance-sheet composition. This can provide clear quantitative guidance to policy makers interested in designing bank micro-prudential regulations.

In the model, cash is a key stabilizing force against runs. When the firm’s portfolio value deteriorates, creditors will be inclined to run, but their propensity to run will decrease with the amount of liquid resources available to the firm. [Valukas et al. \(2010\)](#) mentions this phenomenon in the context of Lehman Brothers’ bankruptcy: *[...] the size of Lehman’s liquidity pool provided comfort to market participants and observers, including rating agencies. The size of Lehman’s liquidity pool encouraged counterparties to continue providing essential short-term financing and intraday credit to Lehman. In addition, the size of Lehman’s liquidity pool provided assurance to investors that if certain sources of short-term financing were to disappear, Lehman could still survive.* My model provides a theoretical justification to this assertion by characterizing the set of firm’s portfolios that leave creditors indifferent between rolling over their debt claims and running. This boundary can again provide transparent guidance to policy makers designing bank liquidity regulations: for any leverage ratio, it characterizes clearly the quantity of liquidity reserves a firm should hold in order to deter a

run.

My paper contributes to a growing literature on dynamic debt runs that includes [He and Xiong \(2012\)](#), [Schroth, Suarez, and Taylor \(2014\)](#) or [Cheng and Milbradt \(2011\)](#). I build on previous work by [He and Xiong \(2012\)](#). They assume that a firm subject to a run can rely on an emergency credit line that might fail. I, on the other hand, make the assumption that the firm maintains a cash buffer that can be used to pay off maturing creditors. This added dimensionality enriches the model in two important dimensions. First, the extended model replicates some of the stylized facts characteristic of debt runs. Second, it also allows me to focus on aspects that have not been studied before, such as quantifying the value of cash as run deterrent for run-prone firms.

On the empirical side, I know from the data that debt runs are not instantaneous events, but rather can be prolonged before an institution runs out of cash and defaults. In my model, when creditors start running, the firm uses its available cash to meet debt redemptions. Only after all liquidity resources have been exhausted does the firm sell its illiquid assets to repay remaining creditors and potentially defaults. Moreover, debt runs do not always lead to the failure of the firm being the target of such run. The experience of Goldman Sachs in the fall of 2008 is well suited. In response to a request from the Financial Crisis Inquiry Committee (“FCIC”) related to its liquidity pool at the time, the firm indicated that its cash buffer, which averaged \$113bn in the third quarter 2008, declined to a low of \$66bn on September 18th, following *both anticipated contractual obligations and other flows of cash and collateral that were driven by counterparty confidence and market volatility*. In other words, Goldman Sachs did suffer the equivalent of a debt run in September 2008. However, the firm’s liquidity position, at the time, was strong enough to deter the run, as its response to the FCIC highlights: *Our liquidity policies and position gave us enough time to make these tactical and strategic decisions in an appropriate manner that preserved the markets confidence in our institution. This confidence led to the reduction of customer-driven outflows of liquidity and allowed us to return to our pre-crisis liquidity buffer levels, with our buffer at an average of \$111bn in the fourth quarter of 2008.*¹ In my model, during a run, creditors’ strategy switches from running to rolling over once the firm’s solvency or its liquidity position improves sufficiently.

In addition, by adding the cash dimension to the firm’s problem, I am able to calculate the marginal value of cash for an institution that is subject to run risk. Although there is a large and growing literature discussing the value of cash holdings for firms, nobody has yet studied the value of cash as run deterrent. [Décamps et al. \(2011\)](#) analyze the optimal

¹source: <http://www.goldmansachs.com/media-relations/in-the-news/archive/response-to-fcic-folder/gs-liquidity.pdf>

dividend policy of a firm holding a cash-flow generating asset and facing external financing costs. In their model, the firm balances the cost of holding cash (which earns an interest lower than the firm’s discount rate due to agency costs) with the savings realized due to less frequent equity issuances upon the occurrence of operating losses. Bolton, Chen, and Wang (2011) and Bolton, Chen, and Wang (2013) enhance the model by studying optimal investment and cash retention under similar capital markets frictions. However, none of the firms studied in those papers issue any debt that might lead to roll-over risk². Hugonnier and Morellec (2014) analyze the effects of liquidity and leverage requirements on banks’ solvency risk. The authors build from the model developed by Décamps et al. (2011), but assume an asset dividend process that makes the bank debt claims risky. The bank’s motive for holding liquidity reserves is however identical to the previously cited papers: the bank faces flotation costs, and thus stores cash as buffer mechanism to save on future issuance costs. Hugonnier, Malamud, and Morellec (2011) study a firm’s investment, payout and financing policies when capital markets are imperfect due to search frictions: the firm has to look for investors when in need of capital, leading to the need to store cash. In my paper, internal cash is valuable to the firm as a run deterrent, since creditors’ incentive to pull back their funding will decrease when the cash internal to the firm increases. It is to my knowledge a novel role for cash within a firm.

The model’s added dimensionality also opens the number of issues that I hope to study in subsequent research. What is the optimal dynamic portfolio choice (cash and liquid low-yielding securities vs. illiquid higher yielding long term investments) for an institution subject to run risk? How does a run-prone firm’s dividend policy influence creditors’ run behavior, and what is the firm’s optimal dividend policy? I am hoping to shed some light on these questions in future work thanks to the model developed in this paper.

Finally, my model can be reinterpreted in an international macroeconomic context and can provide answers to questions related to the optimal sizing of central banks’ foreign currency reserves for countries where monetary policy is not totally independent of the government³. As an example, the 1997 Asian financial crisis featured countries (Thailand, Indonesia, South-Korea and the Philippines) with large amounts of foreign-currency denominated debt and high debt-to-GDP ratios experiencing sudden withdrawals of dollar funding. Central banks in those crisis countries reacted to the collapse of their currencies by raising interest rates and intervening in the foreign exchange markets. In my model, the analog of the firm is the government/central bank of a country such as Thailand. Its illiquid cash flow generating asset is now the country’s fiscal revenues (converted into USD), and its liquid reserves are the USD

²Bolton, Chen, and Wang (2011) does analyze a firm with a credit line, but there is no strategic interactions amongst creditors and the credit line is perpetual – i.e. does not mature.

³I want to thank Fernando Alvarez for pointing this out to me.

and other foreign currencies held at its central bank. The country has financed itself with foreign-currency denominated sovereign debt held by a continuum of creditors. Creditors then face a decision to run or roll-over their debt claims, and their decision depends on the debt-to-GDP ratio of the country, as well as the amount of foreign currency reserves held by its central bank.

My paper thus contributes to the literature on currency attacks and international reserve holdings – a large literature that has gone through multiple phases over the past 40 years. The seminal papers of [Krugman \(1979\)](#) and [Flood and Garber \(1984\)](#) focus on currency crisis in a country with a pegged exchange rate regime and domestic credit expansion, which result in a depletion of the central bank’s foreign exchange reserves and a currency attack by agents with perfect foresight. Within that class of models, [Obstfeld \(1986\)](#) emphasizes the role of expectations about future government policies and highlights the possibility for self-fulfilling crisis and multiple equilibria in economies where a fixed exchange rate would otherwise be sustainable. In order to move away from equilibrium indeterminacy and self-fulfilling beliefs, [Morris and Shin \(1998\)](#) leverage the global games framework. By assuming heterogeneous information across agents, their model features a unique equilibrium in cutoff strategies, which facilitates comparative statics and policy analysis.

[Chang and Velasco \(2001\)](#) adapts the celebrated framework of [Diamond and Dybvig \(1983\)](#) to an international context, by assuming that domestic banks can borrow from international lenders in addition to obtaining funds from domestic depositors. They illustrate the connection between the behavior of international lenders and domestic creditor runs, and show that the presence of international reserves can prevent equilibria where international creditors refuse to roll-over their debt. [Hur and Kondo \(2013\)](#) extends the concept to a multi-period setting and analyze a country that is financed by foreign creditors who are hit by exogenous random liquidity shocks. The country has access to an illiquid production technology, giving rise to maturity mismatch and the need to store international reserves. They study the optimal reserve holdings and sudden stop probability as a function of liquidity risk, but assume away the “bad” [Diamond and Dybvig \(1983\)](#) run equilibrium. The authors replicate some stylized facts about sudden stops: when rollover risk increases, sudden stops occur and countries optimally increase their stock of foreign currency reserves. This class of models takes as exogenous the demand for liquidity by some creditors, whereas my model generates runs purely due to concerns over liquidity and solvency.

Finally, [Bianchi, Hatchondo, and Martinez \(2012\)](#) build upon the model of [Eaton and Gersovitz \(1981\)](#) to study a country that issues long-term defaultable debt to facilitate consumption smoothing, and that is exposed to exogenous sudden stop shocks. Since a sovereign default leads to financial autarky, the accumulation of international reserves enables the

country to survive (at least in the short run) to a sudden stop shock without having to immediately default, which is welfare-improving. Similarly, [Jeanne and Ranciere \(2011\)](#) derive a useful close form expression for the optimal international reserve holding for a country that might be locked out of international capital markets with some exogenous probability. In both articles, the elasticity of intertemporal substitution crucially influences the demand for international reserves, whereas my model operates in an environment where all agents are risk-neutral. Those last two papers also assume that sudden stop events are entirely exogenous, while my sudden stops occur following a deterioration of macroeconomic fundamentals ⁴.

My article is organized as follows. I first discuss some empirical facts that motivate this research. I then describe the basic model. I establish a few properties that will be useful in solving numerically for the threshold equilibrium. I derive the Hamilton-Jacobi-Bellman equations that the key equilibrium value functions of my model satisfy. I solve the model numerically and perform comparative statics, varying the firm's illiquid asset characteristics and the firm's debt maturity profile. Finally, I analyze the influence of the firm's dividend policy and portfolio choice on the run behavior of creditors.

2 Empirical Facts

2.1 Bank Runs and Liquidity Reserves

The failure of several banks and broker dealers during the 2008-2009 crisis illustrates the fact that financial firms, while in compliance with their minimum capital requirements, can collapse due to a shortage of available funds.

In 2008, bank failures occurred in the U.S. even with the ability to access the Federal Reserve discount window. Washington Mutual Bank for example was placed into the receivership of the FDIC on September 25, 2008. Its demise resulted from a deteriorating mortgage portfolio, beginning in 2007, combined with a sudden outflow of deposits⁵⁶. [He](#)

⁴Note that the source of sudden stops is still a hotly debated topic in the empirical literature on international capital flows and the forces driving such flows. While [Forbes and Warnock \(2012\)](#) concludes that global factors are significantly associated with extreme capital flow episodes, [Gourinchas and Obstfeld \(2011\)](#) shows that currency crisis for emerging market economies tend to be preceded by high external debt, a deteriorating current account, and low levels of foreign currency reserves.

⁵John Reich, director of the federal Office of Thrift Supervision, is quoted by the Huffington Post on September 25, 2008, as saying the following: "Pressure on WaMu intensified in the last three months as market conditions worsened. An outflow of deposits, that began on Sept. 15, reached \$16.7bn, and without sufficient cash to meet its obligations, WaMu was in an unsafe and unsound condition to transact business".

⁶It is worthwhile noting that Washington Mutual Bank had lost primary-credit access to the Federal Reserve discount window in September 2008 due to its CAMELS rating downgrade by the OTS – an institution with a CAMELS rating of 4 or below is not eligible to receive primary credit at the Federal Reserve discount window. At the time of its bankruptcy filing however, Washington Mutual Bank still had secondary-credit

and Manela (2014) illustrate the dynamic nature of the run on Washington Mutual Bank by plotting the daily deposit balances at the institution in the months leading to its failure; while Washington Mutual Bank survived a first run in July 2008 (through which it lost \$9bn worth of deposits), the second run in September was fatal for the bank. In the United Kingdom, Northern Rock received emergency funding from the Bank of England in September 2007, following a deterioration of its mortgage portfolio and an inability to roll over its short and medium term wholesale funding, as documented extensively by Shin (2009). In each of these examples, the creditors' run is preceded by a deterioration of the fundamentals of the firm subject to the run – in stark contrast with the original model of Diamond and Dybvig (1983), in which runs are merely the result of a coordination failure amongst creditors of an otherwise healthy firm.

While non-bank U.S. financial firms did not have access to the discount window, the Federal Reserve had created, in March 2008, two facilities to enable primary dealers to borrow funds against eligible collateral either overnight (through the Primary Dealer Credit Facility) or for a period of 28 days (for the Term Securities Lending Facility). This did not prevent Lehman Brothers to file for bankruptcy in September 2008⁷. Valukas et al. (2010) discusses extensively the liquidity position of the firm in the months leading to its collapse. According to the authors, *by the second week of September 2008, Lehman found itself in a liquidity crisis; it no longer had sufficient liquidity to fund its survival*. The report describes a variety of collateral and other margin calls during August and early September 2008 that depleted Lehman Brothers' liquidity resources. Both Morgan Stanley and Goldman Sachs suffered runs in September 2008. Morgan Stanley's liquidity pool shrunk from \$177bn (i.e. 26% of total assets as of year-end 2008) the week-end prior to Lehman Brothers' Chapter 11 filing to \$101bn (i.e. 15% of total assets as of year-end 2008) two weeks later, before rebounding⁸. Goldman Sachs' liquidity levels, which averaged \$113bn in the third quarter 2008, reached a low of \$66bn on September 18, 2008. Both firms managed to survive, highlighting the fact that runs are not always fatal to a financial institution⁹.

Figure 1 gives a picture of aggregate liquidity of the U.S. banking sector between 2005

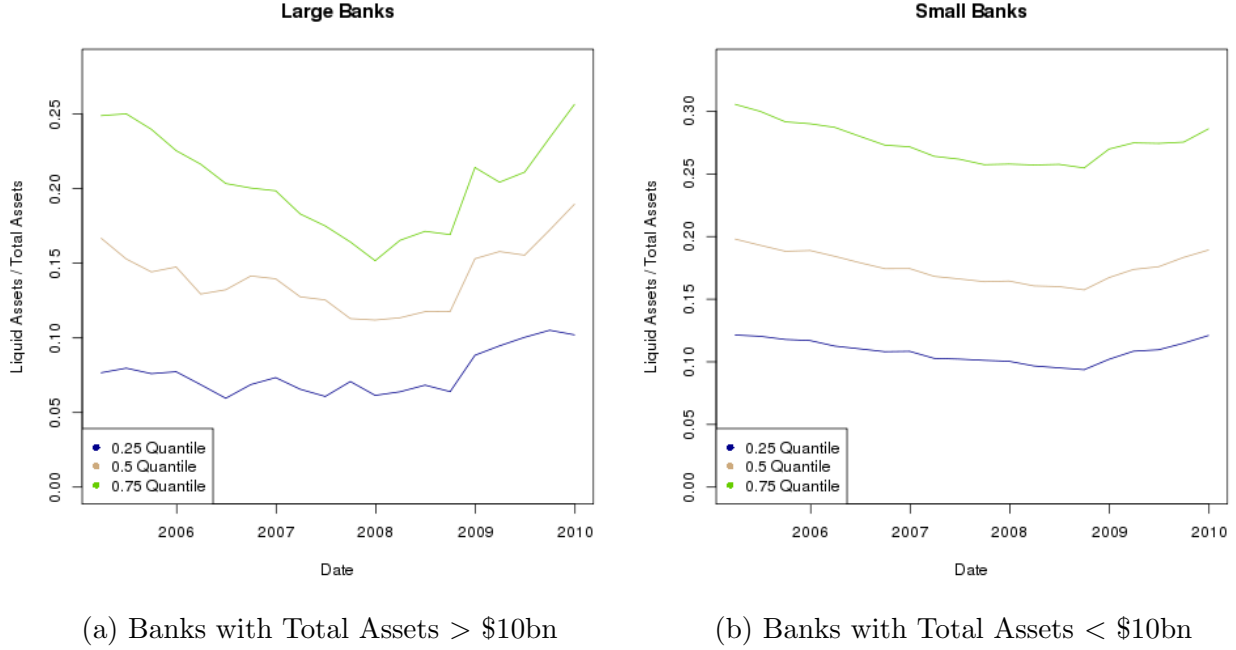
discount window access, with the Federal Reserve Bank of San Francisco.

⁷Note that Bear Stearns, at the time of its collapse in March 2008, had access neither to the PDCF, nor to the TSLF.

⁸Source: New York Fed.

⁹The financial section of Goldman Sachs' 2008 annual report mentions the following: *In the latter half of 2008, we were unable to raise significant amounts of long-term unsecured debt in the public markets, other than as a result of the issuance of securities guaranteed by the FDIC under the TLGP. It is unclear when we will regain access to the public long-term unsecured debt markets on customary terms or whether any similar program will be available after the TLGPs scheduled June 2009 expiration. However, we continue to have access to short-term funding and to a number of sources of secured funding, both in the private markets and through various government and central bank sponsored initiatives.*

Figure 1: Liquid Assets - Total Assets Ratio



and 2010. For that time period, I plot the ratio of (a) cash, U.S. treasury securities and U.S. agency mortgage backed securities divided by (b) total assets for different quantiles of the distribution of banks regulated by the FDIC¹⁰. The figure also separates large banks (defined as banks with assets over \$10bn) from small banks, as those two groups differ by several important factors¹¹. Figure 1 indicates that U.S. banks' liquidity reserves (as a percentage of total assets) trended downwards in the 3 years preceding the financial crisis. Those liquidity reserves then rebounded rapidly starting in the fourth quarter 2008, most likely due to (a) a more conservative portfolio liquidity choice by banks' management in a period of funding stress, as well as (b) anticipation of future liquidity requirements enacted by regulators worldwide¹².

The events of 2008 illustrated another key assumption of the model developed in this paper – the “illiquidity” of a large fraction of financial institutions' balance-sheet at such time. In the model, illiquidity is characterized by an inability for the firm to sell its risky asset without incurring a loss, when compared to the “fundamental” value of such asset. In his

¹⁰Source: FDIC call reports.

¹¹As Figure 30 and Figure 31 in the appendix show, large U.S. banks have (i) tier 1 risk based capital ratios and (ii) retail deposits to total asset ratios significantly lower than small banks.

¹²It is worth noting that the multiple rounds of quantitative easing implemented by the Federal Reserve, starting in December 2008, did not impact the liquidity reserves of U.S. banks, since these operations merely transformed U.S. treasuries and agency MBS into reserves, therefore not impacting banks' total liquidity pools.

report, Valukas et al. (2010) highlights the lack of liquidity of Lehman Brothers’ assets during the crucial weeks preceding its failure: *Its ability to deleverage by selling assets was severely limited by the illiquidity and depressed prices of the assets it had accumulated*¹³. Thus, the combination of (a) a low liquidity buffer combined with (b) an inability to monetize its assets in order to meet its coming liabilities ultimately precipitated Lehman Brothers’ demise.

The 2008-2009 financial crisis prompted the Basel Committee on Banking Supervision to develop new liquidity regulations. The most important of these rules, to be implemented in 2015, constrains banks to maintain their Liquidity Coverage Ratio (“LCR”) at a level greater than 100%¹⁴. The LCR is defined as (i) High Quality Liquid Assets (“HQLA”) divided by (ii) total 30-days Net Cash Outflows (“NCO”):

$$LCR = \frac{HQLA}{NCO}$$

Eligible HQLA are defined as unencumbered assets that can be readily converted to cash at little or no loss of value. Some of these assets are only contributing a fraction of their market value, due to regulatory haircuts used to account for a potentially lower market liquidity in times of stress. Table 2 in the appendix lists eligible HQLA and their related regulatory haircuts.

The NCO is calculated as the difference between (i) total expected cash outflows and (ii) total expected cash inflows over the next 30 calendar days during a period of market stress. Each category of liability that is projected to come due within 30 days, weighted by its probability of not being rolled over, contributes to the total expected cash outflows. Similarly, cash inflows projected to be received within the next 30 days are discounted at various haircut rates driven by the probability of those cash inflows being actually received in a period of market stress. Table 3 in the appendix shows the different liability runoff rates assumed by the Basel Committee on Banking Supervision.

While international bank regulators are just starting to discuss implementation of minimum liquidity requirements, certain countries have been imposing such requirements for some time now. Regulators in the Netherlands for example impose a floor on domestic banks’ “Liquidity Balance” – a ratio that resembles the LCR ratio designed by the Basel Committee on Banking Supervision. The availability of high-frequency data on banks’ liquid asset holdings

¹³Volume 1, page 18 in the report. The author also refers to an interview of Tim Geithner, dated November 24, 2009: *Starting in mid 2007, many of Lehman’s inventory positions had grown increasingly “sticky” – i.e., difficult to sell without incurring substantial losses. Moreover, selling sticky inventory at reduced prices could also have led to loss of market confidence in Lehman’s valuations for inventory remaining on the firms balance sheet).*

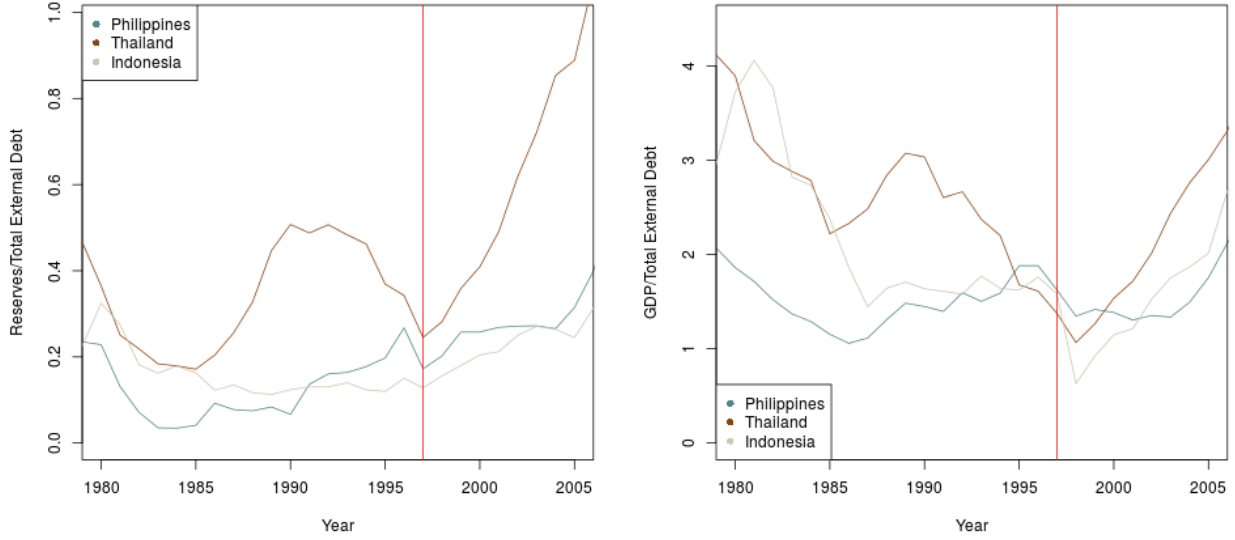
¹⁴In 2015, banks only need to maintain their LCR at a level above 60%, and this minimum level increases each year, reaching 100% in 2019.

in the Netherlands enables De Haan and van den End (2013) to analyze the determinants of such liquidity holdings for Dutch banks from 2004 to 2010. By regressing banks' liquid asset holdings vs. measures of short term liabilities and short term cash inflows and outflows, the authors find that Dutch banks store liquid assets as a buffer against short term liabilities, as well as longer term liabilities and other projected net cash outflows, up to one year ahead. By including measures of solvency on the right-handside of their regressions, the authors also find that better capitalized banks hold less liquid assets against their stock of short term liabilities, an empirical result that will resonate with the model developed in this paper.

At this point, it should be clear to the reader that regulators' approach to liquidity risk completely abstracts from any solvency considerations. In other words, two banks that have the same short term debt and short term net cash outflows will have the same minimum liquidity holdings requirements, irrespective of the fact that one bank might be significantly better capitalized than the other. Kowalik et al. (2013) makes a similar observation when discussing Basel III's revised liquidity rules published in January 2013: *estimates of both run-off and return rates are set without accounting for the fact that, in the event of a liquidity crisis, the actual rates will be determined by the individual characteristics of a given institution and the particular market environment in which it functions.*¹⁵ Unfortunately, liquidity and solvency are highly interconnected concepts. A bank whose solvency deteriorates will find it more difficult to find creditors willing to purchase its debt claims. A reduction of the bank's ability to refinance itself will force the bank to use its available liquid resources to repay maturing creditors that are not rolling over. This downward pressure on the liquidity resources of the bank might eventually deplete the cash buffer of such bank, forcing the bank to sell less liquid assets to service its debt. Eventually, such bank might end up totally illiquid and having to default. This reasoning motivates the model presented in the next section: a regulatory framework for bank minimum liquidity holdings must take into account the bank's solvency level.

¹⁵Kowalik et al. (2013) adds a concrete example focused on portfolio diversification as opposed to solvency ratios: *[...] the Basel III approach requires that, all else equal, an institution with diversified assets must maintain the same size of liquidity buffer as an institution with assets concentrated only in one area. Such an approach does not account for the fact that the well-diversified institution will likely have higher cash inflows during a crisis and thus potentially will be less susceptible to severe liquidity problems. Hence under the Basel III approach, a well-diversified institution bears the same cost of insuring against liquidity shocks as a less-diversified institution, even though the less-diversified institution is more vulnerable to liquidity shocks.* The model developed thereafter will be able to deal with portfolio diversification, via focusing on asset volatility and the sensitivity of creditors' propensity to run w.r.t. asset volatility.

Figure 2: The Asian Tiger Crisis



(a) Foreign Currency Reserves / Total Ext. Debt

(b) Real GDP / Total Ext. Debt

2.2 Sudden Stops and Currency Runs

As discussed in the introduction, the model developed in this paper can be reinterpreted in an international macroeconomic context. In that reinterpretation, the consolidated government includes its central bank, which holds a stock of foreign currency reserves. The government finances some of its activities through the issuance of foreign currency denominated liabilities, and relies on fiscal receipts to repay its creditors. The fiscal receipts can be viewed as an “illiquid asset” on the government’s balance-sheet. Foreign creditors financing this government will focus on both the fiscal revenues of the government, as well as its stock of foreign currency reserves when making roll-over decisions. A “sudden stop” happens when both the fiscal revenues and the foreign currency reserves of the country have declined below certain threshold levels. It is thus informative to look empirically at the occurrence of sudden stops and understand the extent to which those sudden stops are related to a country’s external debt-to-GDP ratio and its foreign-currency reserves-to-GDP ratio.

The 1997 Asian Tiger crisis provides anecdotal evidence of the mechanism described in my paper. For different reasons, government and corporate foreign currency debt (as a percentage of GDP) in Thailand and the Philippines had increased significantly during the 90s’. At the time, these countries were operating under a fixed exchange rate regime, with the Thai baht and the Indonesian rupiah pegged to a basket of currencies, in which the US dollar had the largest weight. The mid-90s’ saw a substantial appreciation of the US dollar, in part due to

the U.S. Federal Reserve increase in short term rates. The reduced competitiveness of Thai and Indonesian exports led to a deterioration of those countries' current account balance which, combined with the existing vulnerability of those countries' financial systems, led to rapid US dollar withdrawals in May 1997 for Thailand, and in August 1997 for Indonesia. Both countries initially tried to defend their US dollar pegs by using large amounts of central bank foreign currency reserves, but eventually had to devalue and float the Thai baht and the Indonesian rupiah. **Figure 2** shows the evolution of foreign currency reserves and external debt (both variables normalized by GDP) for those countries, highlighting the link between “liquidity” (the ratio of foreign currency reserves over GDP), “solvency” (the ratio of external debt over GDP), and the occurrence of debt runs.

Gourinchas and Obstfeld (2011) studies a similar question, using data on 57 emerging market economies over the period 1973 – 2010. For different types of crisis episodes (sovereign, banking, and currency crisis), using a fixed-effect panel specification, the authors analyze the evolution of different macroeconomic variables pre- and post- crisis, relative to “tranquil times”. For emerging market countries, in the years preceding either a currency crisis or a sovereign default, the authors find that (a) external leverage is high relative to “tranquil times”, (b) the current account is low relative to “tranquil times”, (c) foreign exchange reserves are below their value in “tranquil times”, and (d) short term external debt is high relative to “tranquil times”. As will become clear in the next few sections, some of these empirical facts will be featured in the model developed in this article.

3 The Model

3.1 Agents

Time is continuous, indexed by t , and the horizon is infinite. $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space, which I will call the “physical” probability space, and $(\mathcal{F}_t)_{t \geq 0}$ is a family of sub- σ -fields of $\mathcal{F}$ such that $\mathcal{F}_s \leq \mathcal{F}_t$ for $s \leq t$ and such that $\mathcal{F}_t$ contains the null sets of $\mathbb{P}$. $B^\mathbb{P}(t)$ will be a standard Brownian motion w.r.t. $\mathcal{F}_t$. I consider a firm that owns two types of assets: illiquid assets that can only be sold at a discount to “fundamental” value (to be defined in the next section), and liquid reserves (which I will refer to as cash). The firm is controlled by its shareholders and finances itself by issuing debt that is sold to a continuum of creditors. I assume that both creditors and shareholders' stochastic discount factor $\Xi(t)$ evolves as follows:

$$\frac{d\Xi(t)}{\Xi(t)} = -\rho dt - \nu dB^\mathbb{P}(t) \quad (1)$$

ρ is the risk free rate, while ν is the market price of risk. Using Girsanov's theorem, [equation \(1\)](#) enables me to define a new probability measure $\mathbb{Q}$, such that $B^{\mathbb{Q}}(t) := B^{\mathbb{P}}(t) + \nu dt$ is a standard Brownian motion under $\mathbb{Q}$. I will use $\mathbb{E}^{\mathbb{P}}$ for the expectation operator under the physical measure $\mathbb{P}$, and will use $\mathbb{E}$ for the expectation operator under the risk-neutral measure $\mathbb{Q}$.

3.2 The Firm's Assets

The firm's portfolio consists of two different assets. First, the firm holds cash and short term liquid assets, which can be sold at any point in time at a price of 1, and which pay a flow interest at rate r_c , where $0 < r_c < \rho$. The assumption that the cash yield is lower than the time preference rate of agents creates a friction that can be motivated as follows. First, consistent with [Jensen \(1986\)](#), the firm's managers could be engaging in wasteful activities – such as expensing the available cash to derive private benefits. Those wasteful activities lead to agency costs that can be modeled in a reduced form fashion by assuming that cash yields a rate r_c that is lower than the risk free rate ρ . Second, cash in my model is valuable as it can be liquidated immediately without incurring transaction costs. But this benefit comes at a shadow cost, in the form of a lower cash yield. I will note $C(t)$ the cash value of those short term liquid assets at the firm at time t .

Second, the firm holds $N(t)$ units of a long term illiquid investment, which generates cash-flows $Y(t)$ per unit of time and per unit of investment. $Y(t)$ is assumed to follow geometric Brownian motion dynamics under the physical measure $\mathbb{P}$, in other words $\frac{dY(t)}{Y(t)} = \mu^{\mathbb{P}} dt + \sigma dB^{\mathbb{P}}(t)$. Using $\mu := \mu^{\mathbb{P}} - \nu\sigma$ for the risk-neutral drift of the cash-flow growth rate, I assume that $\mu < \rho$. The fundamental value $Q(t)$ of one unit of the illiquid investment satisfies the risk-neutral pricing equation:

$$Q(t) = \mathbb{E}^Y \left[\int_t^{+\infty} e^{-\rho(s-t)} Y(s) ds \right] = \frac{Y(t)}{\rho - \mu}$$

The notation $\mathbb{E}^Y$ represents the risk neutral expectation operator conditioned on $Y(t) = Y$. The excess return of the illiquid asset satisfies $\frac{dQ(t)+Y(t)dt}{Q(t)} - \rho dt = \sigma \nu dt + \sigma dB^{\mathbb{P}}(t)$, in other words $\sigma \nu$ is the risk premium of the illiquid asset the firm is holding. I will note $P(t) = N(t)Q(t)$ the aggregate fundamental value of the illiquid investments held by the firm. The firm's long term illiquid investment “matures” according to a Poisson process with arrival intensity ϕ . At such maturity date, the illiquid investment is sold for its fundamental value, and the proceeds, in addition to any cash available at the firm, are distributed to creditors and shareholders according to a standard priority of payments. Outside the maturity date,

the long term illiquid investments can only be sold at a discount α to the fundamental value $P(t)$. I assume that the maturity jump process is not “priced” by investors, meaning that the maturity jump process follows the same dynamics under $\mathbb{Q}$ and under $\mathbb{P}$.

In the first part of my analysis, firm’s portfolio cash flows received in excess of funding costs and potential debt redemptions are entirely reinvested into the cash reserve, meaning that $N(t) = N$ is constant (normalized to 1). I then discuss a modified environment where the firm reinvests a portion of these net cash inflows into the illiquid asset by solving a portfolio choice problem. I do not allow fractional sales of the illiquid investment: if sold before its maturity date, the illiquid investment is sold in its entirety.

3.3 The Firm’s Liability Structure

I assume that the firm cannot raise additional outside equity or debt financing in addition to what it has already raised: it has to finance its operations and costs using cash generated from its asset portfolio. While this assumption is in stark contrast with models that assume that firm’s negative net cashflows are replenished via additional share issuances (Leland (1994) and all following articles), one can instead assume that equity issuance costs are so high that shareholders would prefer defaulting, if and when they have to, instead of injecting additional equity capital into the firm. In addition, the firm in my framework will default when its asset portfolio performs poorly and when its cash reserve is depleted – the lack of share issuance in such situation is corroborated by the overwhelming evidence that financial institutions whose balance-sheets deteriorate and who are suffering runs rarely manage to raise additional private capital¹⁶.

The debt that has been raised by the firm is held by a continuum of creditors with initial measure D . Each creditor has invested 1 unit of cash into the firm, in the form of short term debt. This short term debt matures according to a Poisson arrival process with intensity λ . Thus, at each point in time, a constant fraction of the firm’s outstanding liabilities matures, guaranteeing that the firm’s debt average life remains constant. At the maturity of a given debt instrument, the relevant creditor decides whether or not to roll-over into a new short term debt instrument. While I remain silent on the underlying reasons for the firm to issue debt, one could imagine that tax considerations – for example, the tax rate on interest income being lower than the corporate tax rate – provide the firm with an incentive to use nominal debt contracts for financing purposes. The short term constant average maturity debt structure can also be justified by the presence of asymmetric information, leading the firm’s management to favor debt over equity issuances (Myers and Majluf (1984)). Finally,

¹⁶The vast majority of capital raised by financial institutions following Lehman Brothers’ collapse in the Fall 2008 was coming from the Troubled Asset Relief Program, in other words from the public sector.

while the modeling choice of a creditor’s debt contract’s maturity time as a Poisson arrival process might seem unusual, I can reinterpret this maturity structure as if each atomistic creditor was holding a debt instrument that is amortizing exponentially.

Creditors receive a flow interest rate r_d (with $r_d > \rho$) on their debt claims. The parameter restriction $r_d > \rho$ ensures that creditors have some incentive to be invested in the debt issued by the firm. The interest rate r_d at which the firm finances itself is not set by the market, in contrast to classic firms’ contingent claims’ models in which debt is issued at “fair market value” (meaning that the risk-neutral expected value of the cash-flows of the debt claim, at issuance, equals the total proceeds raised by the firm). In those latter models, so long as the firm performs on its contractual obligations, debt can be issued, which means that those models cannot generate the type of debt runs I am interested in studying in the present paper¹⁷.

Since the firm’s creditors have an initial measure equal to D , the aggregate short term debt at time $t = 0$ is equal to $D(0) = D$, and generally, the aggregate short term debt at the firm at time t is equal to $D(t)$. In my model, $D(t)$ is a weakly decreasing function of time: it is constant when creditors continuously roll over their maturing claims, but it decreases when maturing creditors refuse to roll and receive their principal balance back. If the firm needs to repay a creditor before the maturity date of the firm’s illiquid investment, the firm will first use its cash reserves before selling a single unit of the illiquid investment, since such illiquid investment can only be sold at a discount to fundamental value. Before the maturity date of the illiquid asset, if the firm’s cash reserves are depleted, the firm has to sell its illiquid assets and distribute the proceeds to creditors. If any money is left once creditors have been repaid, the balance is distributed to the firm’s shareholders. Going forward, I will assume the following parameter restriction.

ASSUMPTION 1. The liquidation fraction α is such that $1 > \alpha > \frac{\rho - \mu}{r_d + \lambda}$.

The inequality above is easily satisfied for reasonable parameter configurations, once I emphasize the fact that I will be considering firms that are involved in maturity transformation, and for which the average debt maturity is less than one year. In the first part of the paper, I assume that no dividends are paid to shareholders. The only cash-flows received by shareholders come either (a) at the maturity date of the illiquid asset, when the firm’s

¹⁷Several contingent claims’ models of the firm, such as Della Seta, Morellec, and Zucchi (2015), generate rollover losses when the firm issues finite maturity debt in bad times, leading to a downward pressure on cash holdings; those roll-over losses are however not sufficient to generate the speed of runs observed in the data. In addition, there is no empirical evidence that distressed financial institutions issue debt at discounts; instead, unsecured bonds are issued at par with higher coupons, and financial institutions rarely issue any bonds with spreads over the risk free rate significantly over [5] %.

cash and illiquid assets are liquidated at their fundamental value, or (b) when the firm's cash reserves are entirely depleted and the firm's illiquid asset is sold for a fraction α of its fundamental value. In each of these events, shareholders are entitled to the residual proceeds (if any) after creditors have been repaid the funds they are owed. I will then discuss a modified environment where the firm has the ability to pay dividends to shareholders. I will do so by first assuming an exogenous dividend payment rule, and will then discuss the firm's optimal dividend payment policy. I will note by $U(t)$ the cumulative payouts made to shareholders.

4 Model Solution

4.1 Strategy Space

In the next few sections and unless otherwise specified, I assume that no dividend is payable to shareholders before the maturity date of the illiquid asset, and that all the firm's net cash-flows above debt service and potential debt redemptions are reinvested in cash – in other words, $N(t) = N = 1$ at all times. I will study the problem faced by a specific creditor i , and will focus on symmetric Markov perfect equilibria, as defined in more details below. A given creditor only makes decisions when his debt claim matures. The state variables that are payoff-relevant for creditor i 's decision at such time t are (i) the value of the illiquid portfolio $P(t)$ (since the cash flows related to such portfolio are simply proportional to $P(t)$), (ii) the amount of cash $C(t)$ in the liquidity reserve, and (iii) the amount of debt outstanding $D(t)$. A pure Markov strategy for a given creditor can then be defined as a mapping $s : \mathbb{R}_+^3 \rightarrow \{0, 1\}$, where action 0 corresponds to the decision to roll over, and action 1 corresponds to the decision to run. Under strategy s , at each time t at which creditor i 's debt claim matures, creditor i will roll into a new debt claim if $s(C(t), P(t), D(t)) = 0$, and creditor i will instead request his principal balance back if $s(C(t), P(t), D(t)) = 1$. In what follows, I will adopt the following convention: s will denote the strategy assumed to be followed by a given creditor i , while S is the strategy assumed to be followed by all other creditors. Note that a pure Markov strategy for creditors can be also characterized by the subset of $\mathbb{R}_+^3$ for which creditors elect to run. In other words, for any strategy S , the set $\mathcal{R}_S := \{(C, P, D) : S(C, P, D) = 1\}$ (or “run region”) uniquely characterizes the creditors' strategy. I will use similarly the creditors' roll region $\mathcal{NR}_S := \{(C, P, D) : S(C, P, D) = 0\}$.

4.2 State Space

Since I am first assuming that the firm's illiquid asset holding $N(t)$ is constant and normalized to 1, the aggregate value of the firm's illiquid investments $P(t) = N(t)Q(t)$ under the measure

$\mathbb{Q}$ follows the same dynamics as the price of one unit of the illiquid investment:

$$dP(t) = \mu P(t)dt + \sigma P(t)dB^{\mathbb{Q}}(t) \quad (2)$$

For a given arbitrary strategy $S : \mathbb{R}_+^3 \rightarrow \{0, 1\}$ followed by the firm's creditors, the endogenous state variable C follows the dynamics:

$$dC(t) = (\rho - \mu)P(t)dt + r_c C(t)dt - r_d D(t)dt - 1_{\{S(C(t), P(t), D(t))=1\}} \lambda D(t)dt \quad (3)$$

$C(t)$ increases with cash flows $Y(t)dt = (\rho - \mu)P(t)dt$ received on the illiquid investment and with interest $r_c C(t)dt$ collected on the liquid assets kept by the firm, and $C(t)$ decreases with interest $r_d D(t)dt$ paid to creditors and with redeeming creditors $\lambda D(t)dt$, whenever the state $(C(t), P(t), D(t))$ is in the “run” region. Similarly, the endogenous state variable $D(t)$ follows the dynamics:

$$dD(t) = -1_{\{S(C(t), P(t), D(t))=1\}} \lambda D(t)dt \quad (4)$$

Thus, the outstanding debt balance of the firm is constant in the roll region, but declines exponentially in the run region.

4.3 Payoff Functions

Let $\tau_b := \inf\{t : C(t) = 0, (\rho - \mu)P(t) < (r_d + \lambda 1_{\{S(C(t), P(t), D(t))=1\}})D(t)\}$ be the firm's default time. For the firm to default, its cash reserve needs to hit zero, and the drift of its cash reserve needs to be strictly negative¹⁸. Note that the firm's shareholders do not have any incentive to default anytime sooner than τ_b ; indeed, they are protected by limited liability and never contribute additional equity to the firm, in stark contrast with models of endogenous defaults in the tradition of [Leland \(1994\)](#). I focus on a specific creditor, following a strategy s . Let $\{\tau_\lambda^n\}_{n \geq 1}$ be a sequence of independent exponentially distributed stopping times, with arrival intensity λ . Let τ_r be defined as follows:

$$\tau_r := \inf_k \left\{ \sum_{i=1}^k \tau_\lambda^i \text{ s.t. } s \left(C \left(\sum_{i=1}^k \tau_\lambda^i \right), P \left(\sum_{i=1}^k \tau_\lambda^i \right), D \left(\sum_{i=1}^k \tau_\lambda^i \right) \right) = 1 \right\}$$

τ_λ^n are times at which creditor i has the opportunity to roll over its debt claim, and τ_r is the first time at which creditor i elects to run. For a given strategy s followed by creditor i and a

¹⁸If the cash reserve is equal to zero but the drift of cash is positive, the firm would simply start accumulating liquidity reserves and would thus not be forced to sell its illiquid investment.

given strategy S followed by all other creditors, the risk-neutral payoff function V of creditor i is equal to the following:

$$V(C, P, D; s, S) = \mathbb{E}^{C, P, D} \left[\int_0^\tau e^{-\rho t} r_d dt + e^{-\rho \tau} 1_{\{\tau = \tau_r\}} + e^{-\rho \tau} 1_{\{\tau = \tau_\phi\}} \min \left(1, \frac{P(\tau) + C(\tau)}{D(\tau)} \right) + e^{-\rho \tau} 1_{\{\tau = \tau_b\}} \min \left(1, \alpha \frac{P(\tau)}{D(\tau)} \right) \right] \quad (5)$$

In the above, $\tau := \tau_r \wedge \tau_b \wedge \tau_\phi$ is the earliest of (a) the creditor running, (b) the firm filing for bankruptcy, and (c) the illiquid asset maturing. τ is potentially infinite. Equation (5) says that creditor i derives value (a) from flow interest payments at the rate r_d per unit of time, (b) at the first creditor's debt maturity τ_r for which the creditor elects not to roll (if such stopping time comes first) from the collection of his principal balance, (c) at the illiquid asset maturity date τ_ϕ (if such stopping time comes first) from the value $\min(1, \frac{P(\tau) + C(\tau)}{D(\tau)})$ of collecting his principal balance to the extent of funds available at the firm, and (d) at the firm's bankruptcy date τ_b (if such stopping time comes first) from the value $\min(1, \frac{\alpha P(\tau)}{D(\tau)})$ of collecting his principal balance to the extent of funds available at the firm. The notation above emphasizes that the payoff function V depends on creditor i 's strategy s , as well as the strategy of all other creditors S (via the dynamics of the state variables). Note that I can similarly define shareholders' risk-neutral payoff function E as follows:

$$E(C, P, D; S) = \mathbb{E}^{C, P, D} \left[e^{-\rho(\tau_\phi \wedge \tau_b)} 1_{\{\tau_\phi < \tau_b\}} \max(0, P(\tau_\phi) + C(\tau_\phi) - D(\tau_\phi)) \right] + \mathbb{E}^{P, D, C} \left[e^{-\rho(\tau_\phi \wedge \tau_b)} 1_{\{\tau_b < \tau_\phi\}} \max(0, \alpha P(\tau_b) - D(\tau_b)) \right] \quad (6)$$

Shareholders derive value (a) at the illiquid asset maturity date τ_ϕ (if such stopping time comes first) from the excess value $\max(0, P(\tau_\phi) + C(\tau_\phi) - D(\tau_\phi))$ of the firm's asset portfolio over the aggregate debt outstanding, and (b) at the firm's bankruptcy date τ_b (if such stopping time comes first) from the excess value $\max(0, \alpha P(\tau_b) - D(\tau_b))$ of the firm's illiquid asset liquidation proceeds over the aggregate debt outstanding. I will be studying strategies that have a particular homogeneity property.

ASSUMPTION 2. Creditors' strategies S are homogeneous of degree zero in (C, P, D) .

I should point out that Assumption 2 is merely a restriction on the set of strategies I will be looking into. This assumption makes intuitive sense: since creditors are infinitesimally small, a given creditor should be indifferent between a credit exposure to the risk of a firm with illiquid assets worth P , cash worth C , and debt worth D , or a credit exposure to the risk of a firm with illiquid assets worth aP , cash worth aC , and debt worth aD , for any

$a > 0$. One question that arises naturally is whether a creditor's best response $s^*(S)$ to a common homogeneous strategy S followed by all other creditors is also homogeneous. The following lemma helps shed light on this issue.

LEMMA 1. For any homogeneous of degree zero strategy $S : \mathbb{R}_+^3 \rightarrow \{0, 1\}$ followed by all other firm's creditors, creditor i 's best response $s^*(S) := \arg \max_s V(\cdot, \cdot, \cdot; s, S)$ is homogeneous of degree zero.

Thus, homogeneous strategies are “stable”, in the sense that a creditor, taking into account the fact that all other creditors follow a common homogeneous strategy, will respond by also following a homogeneous strategy. All the results that follow are thus derived under **Assumption 2**, satisfied for (i) all other creditors (strategy S), and (ii) for the particular creditor of interest (strategy s). The homogeneity property discussed above simplifies considerably the problem to be studied, as **Lemma 2** demonstrates.

LEMMA 2. For any strategy $S : \mathbb{R}_+^3 \rightarrow \{0, 1\}$ followed by the firm's creditors and strategy $s : \mathbb{R}_+^3 \rightarrow \{0, 1\}$ followed by creditor i , the payoff function V is homogeneous of degree zero, and the value function E is homogeneous of degree one.

Lemma 2 enables me to summarize the state of the system by two state variable only:

$$c(t) := \frac{C(t)}{D(t)} \tag{7}$$

$$p(t) := \frac{P(t)}{D(t)} \tag{8}$$

Note that p and c are appropriately defined: since creditors' claims mature with Poisson arrival rate λ , I know that $D(t) > 0$ for all t almost surely, irrespective of the decisions made by creditors or the firm. With a slight abuse of notation, a strategy will now be a mapping $S : \mathbb{R}_+^2 \rightarrow \{0, 1\}$. I now focus on the dynamics of the state variables (c, p) under $\mathbb{Q}$, for a given arbitrary strategy $S : \mathbb{R}_+^2 \rightarrow \{0, 1\}$ followed by creditors. Using Ito's lemma, I have;

$$dp(t) = (\mu + \lambda 1_{\{S(c(t), p(t))=1\}}) p(t) dt + \sigma p(t) dB^{\mathbb{Q}}(t) \tag{9}$$

$$dc(t) = ((\rho - \mu)p(t) + (r_c + \lambda 1_{\{S(c(t), p(t))=1\}})c(t) - (r_d + \lambda 1_{\{S(c(t), p(t))=1\}})) dt \tag{10}$$

The drift of the illiquid asset price (per unit of outstanding debt) $p(t)$ is greater (by the term $+\lambda p(t)$) when a run is occurring, since the outstanding debt of the firm decreases, meaning that the remaining creditors can rely on a greater “share of the pie”. Similarly, when a run occurs, there are two effects on the drift of the cash per unit of outstanding debt $c(t)$:

cash is used to pay down maturing creditors (the drift term $-\lambda dt$), but since the number of remaining creditors is decreasing, those remaining creditors are entitled to a greater “share of the pie” (the drift term $+\lambda c(t)dt$). Note that in the region of the state space where the firm holds less than 1 unit of cash per unit of debt outstanding (in other words, where $c < 1$ – arguably the empirically relevant region), a run leads to a decrease of the cash drift. Given the homogeneity properties described above, I can write $V(C, P, D; s, S) = v(c, p; s, S)$, and $E(C, P, D; S) = De(c, p; S)$. Using the state variables $(c(t), p(t))$, the creditor’s payoff function V can be re-written as follows:

$$v(c, p; s, S) = \mathbb{E}^{c,p} \left[\int_0^\tau e^{-\rho t} r_d dt + e^{-\rho \tau} 1_{\{\tau=\tau_r\}} + e^{-\rho \tau} 1_{\{\tau=\tau_\phi\}} \min(1, p(\tau) + c(\tau)) + e^{-\rho \tau} 1_{\{\tau=\tau_b\}} \min(1, \alpha p(\tau)) \right] \quad (11)$$

In the above, $\tau = \tau_r \wedge \tau_\phi \wedge \tau_b$. Similarly, the shareholders’ payoff function, per unit of debt outstanding, can be re-written:

$$e(c, p; S) = \mathbb{E}^{c,p} \left[e^{-\rho(\tau_b \wedge \tau_\phi)} \left[1_{\{\tau_\phi < \tau_b\}} \max(0, p(\tau_\phi) + c(\tau_\phi) - 1) + 1_{\{\tau_b < \tau_\phi\}} \max(0, \alpha p(\tau_b) - 1) \right] \right] \quad (12)$$

4.4 Symmetric Markov Perfect Equilibrium

For a given strategy $S : \mathbb{R}_+^2 \rightarrow \{0, 1\}$, creditor i finds the strategy $s : \mathbb{R}_+^2 \rightarrow \{0, 1\}$ that maximizes his payoff function. In other words, creditor i solves for any (c, p) :

$$v^*(c, p; S) = \sup_s v(c, p; s, S)$$

DEFINITION 1. A symmetric Markov perfect equilibrium of the game is a mapping $S : \mathbb{R}_+^2 \rightarrow \{0, 1\}$ such that for any $(c, p) \in \mathbb{R}^2$:

$$v(c, p; S, S) = \sup_s v(c, p; s, S)$$

4.5 Strategic Complementarity and Supermodularity

One question that naturally arises is whether creditors’ payoff functions exhibit supermodularity. The answer to this question will guide the solution method adopted. Indeed, if the

game that creditors play is supermodular, I can rely on lattice theory and iterated deletions of interim dominated strategies, as illustrated by [Vives \(1990\)](#) or [Milgrom and Roberts \(1990\)](#), in order to establish the existence of equilibria of the game described in this paper. Such a strategy has been used in the bank run literature, for example by [Rochet and Vives \(2004\)](#) or more recently [Vives \(2014\)](#).

Intuitively, one might want to postulate that the game’s payoffs indeed has the supermodularity property. When creditors run, they reduce the cash balance available for the firm to service the remaining creditors’ debt, making the length of time needed to hit the zero-cash boundary shorter and prompting the remaining creditors to be tempted to run at the first opportunity. Proving that the game considered in this paper is supermodular would require showing that for an arbitrary strategy S_1 followed by all creditors, and an arbitrary strategy $S_2 \geq S_1$ (i.e. creditors run in more states under S_2 than under S_1), a specific creditor’s incentive to run is higher when responding to S_2 than when responding to S_1 :

$$1 - v(c, p; s, S_2) \geq 1 - v(c, p; s, S_1) \quad \text{for any strategy } s \text{ and } (c, p) \in \mathbb{R}_+^2$$

Instead, I will show that this condition does not hold for the game considered in this paper.

PROPOSITION 1. [The payoff function \$v\(\cdot, \cdot; s, S\)\$ does not exhibit supermodularity.](#)

I develop the proof in the appendix, by focusing on the region of the state space $p = 0$. The failure of the game to be supermodular stems from the fact that conflicting forces are at play upon the occurrence of a run. On one side, the firm’s cash balance decreases due to debt redemptions, putting the firm closer to an illiquid situation, at which point the firm might have to sell its illiquid asset and default on its debts. This contributes to increasing the incentives for a given creditor to run. However, a run has several beneficial effects for the remaining creditors. As [equations \(9\) and \(10\)](#) indicate, upon the occurrence of a run, the drift of the illiquid asset price (per unit of debt outstanding) is higher (than in the no-run regime) by the term $\lambda p(t)$. As discussed previously, this means that the remaining creditors can claim a bigger “share of the pie” upon a firm’s default. In addition, upon the occurrence of a run, the firm’s expensive debt (which yields r_d) is being paid down, helping the firm reduce its future debt interest expense. This is another source of “good news” for a remaining creditor, leading to a potentially counterintuitive result that the run incentive might decrease when the run region expands. This feature of the model is one of the striking differences between this paper and [He and Xiong \(2012\)](#), whose payoff function does exhibit strategic complementarity in terminal payoffs and could therefore be solved using iterated deletions of

interim dominated strategies¹⁹.

This result reminds of a similar observation made by Goldstein and Pauzner (2005) for their model: their bank run payoffs do not exhibit global, but rather local strategic complementarity. This prompts the authors to use a solution technique that differs from the traditional tools available in the presence of supermodularity.

4.6 Dominance Regions

In what follows, I will want to restrict the equilibria of the game to cutoff Markov perfect equilibria.

DEFINITION 2. A strategy $S : \mathbb{R}_+^2 \rightarrow \{0, 1\}$ followed by the firm's creditors is a cutoff strategy if the sets $\mathcal{R}_S := \{(c, p) \in \mathbb{R}_+^2 : S(c, p) = 1\}$ and $\mathcal{NR}_S := \{(c, p) \in \mathbb{R}_+^2 : S(c, p) = 0\}$ are disjoint connected sets that form a partition of $\mathbb{R}_+^2$.

In words, under a cutoff strategy S , the northeast quadrant is divided into two disjoint areas: one area in which creditors run, and one area in which creditors roll over their debt contracts. I will show that there exists a Markov perfect equilibrium in cutoff strategies, as defined above. In order to do this, I first establish some preliminary results.

PROPOSITION 2. For any strategy $S : \mathbb{R}_+^2 \rightarrow \{0, 1\}$ followed by the firm's creditors, the payoff function v is non-negative and bounded above by $\frac{r_d + \phi}{\rho + \phi}$.

The upper bound established in Proposition 2 corresponds to the value for a creditor rolling his debt claim into a new debt claim forever while facing no credit risk. Such creditor earns an interest rate r_d greater than his discount rate ρ until the time the illiquid asset matures, yielding a present value $\frac{r_d + \phi}{\rho + \phi} > 1$.

PROPOSITION 3. There exists non-empty lower and upper dominance regions, in other words there exists $\mathcal{D}_l \subset \mathbb{R}_+^2$ and $\mathcal{D}_u \subset \mathbb{R}_+^2$ such that for any strategy $S : \mathbb{R}_+^2 \rightarrow \{0, 1\}$

¹⁹Note that He and Xiong (2012) establish the existence and uniqueness of the equilibrium of their game by leveraging closed-form solutions for the payoff function of creditors; assuming the constant value of debt is $D = 1$, the authors could instead have analyzed the properties of the value function $V(P; s, S) = \mathbb{E}^P \left[\int_0^\tau e^{-\rho t} r_d dt + e^{-\rho \tau} 1_{\{\tau = \tau_r\}} + e^{-\rho \tau} 1_{\{\tau = \tau_\phi\}} \max(1, P(\tau)) + e^{-\rho \tau} 1_{\{\tau = \tau_b\}} \max(1, \alpha P(\tau)) \right]$. τ_r is as usual the first (stopping) time at which the state is such that $s(P_{\tau_r}) = 1$, and τ_b is the default time, which is the stopping time at which $S(P_{\tau_b}) = 1$ and the credit line fails. The value function V as defined is supermodular, as Doh (2015) shows: $1 - V(P, s, S) \leq 1 - V(P, s, S')$, whenever $S < S'$, for any strategy s and $P > 0$.

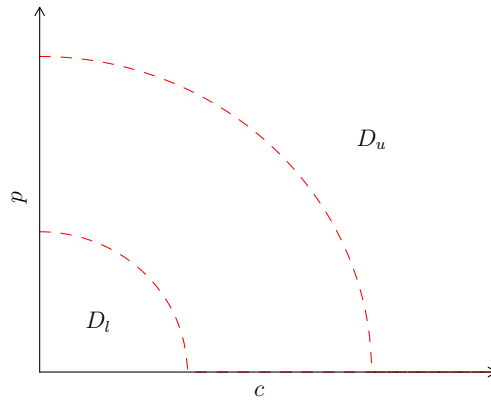
followed by creditors, any strategy $s : \mathbb{R}_+^2 \rightarrow \{0, 1\}$ followed by creditor i , I have:

$$(c, p) \in \mathcal{D}_l \Rightarrow v(c, p; s, S) < 1$$

$$(c, p) \in \mathcal{D}_u \Rightarrow v(c, p; s, S) > 1$$

The proof of the propositions above can be found in the appendix. The lower dominance region is a region of the state space near the origin $(c, p) = (0, 0)$: for small values of p and c , it is a dominant strategy for a creditor to run when he has the opportunity to do so, irrespective of the strategy S employed by all other creditors. The upper dominance region is a region of the state space characterized by high values of p , c , or both. In that region, it is a dominant strategy for a creditor to roll over his debt claim upon the maturity of his existing debt claim, irrespective of the strategy S employed by all other creditors. The lower and upper dominance regions are illustrated in [Figure 3](#). Given that these regions are non-empty, I can define $\mathcal{D}_l$ as the largest connected set containing $(0, 0)$ such that running is a dominant strategy, and similarly I can define $\mathcal{D}_u$ as the largest connected set containing $(+\infty, +\infty)$ such that rolling is a dominant strategy. The existence of dominance regions suggests that I should be looking for symmetric Markov perfect Equilibria in cutoff strategies.

Figure 3: Dominance Regions



4.7 Creditor's Problem

For a given strategy $S : \mathbb{R}_+^2 \rightarrow \{0, 1\}$ followed by all other creditors, I now study the optimization problem that creditor i solves:

$$v^*(c, p; S) = \sup_s v(c, p; s, S)$$

The following proposition establishes existence and uniqueness of v^* as the solution to a standard functional equation.

PROPOSITION 4. For any cutoff strategy $S : \mathbb{R}_+^2 \rightarrow \{0, 1\}$ followed by the firm's creditors, creditor i 's optimal value function is the unique continuous bounded function that is solution to the following fixed point problem:

$$\begin{aligned} v^*(c, p; S) = \mathbb{E}^{p,c} & \left[\int_0^\tau e^{-\rho t} r_d dt + e^{-\rho \tau} 1_{\{\tau=\tau_\lambda\}} \max(1, v^*(p(\tau), c(\tau); S)) \right] \\ & + \mathbb{E}^{p,c} \left[e^{-\rho \tau} 1_{\{\tau=\tau_\phi\}} \min(1, p(\tau) + c(\tau)) + e^{-\rho \tau} 1_{\{\tau=\tau_b\}} \min(1, \alpha p(\tau)) \right] \end{aligned} \quad (13)$$

Where τ_λ is exponentially distributed (with parameter λ), and $\tau = \tau_\lambda \wedge \tau_\phi \wedge \tau_b$.

The proof of **Proposition 4** relies on an appropriately constructed contraction map. The functional equation for $v^*(\cdot, \cdot; S)$ reflects the fact that for $\tau = \tau_\lambda$ (in other words, when the first stopping time to occur is the maturity date of creditor i 's debt claim), the creditor has the option to either roll over into a new debt claim (with payoff $v^*(p(\tau_\lambda), c(\tau_\lambda); S)$), or to take his money out (with payoff 1).

4.8 Cutoff Markov Perfect Equilibrium when $p = 0$

In order to make progress on the characterization of any cutoff Markov perfect equilibrium, I now derive the value function v^* in the special case where the illiquid asset cash flow – and thus the illiquid asset fundamental value – is zero. When p approaches zero, the firm can only rely on its cash resources to pay creditors' interest and principal (for those creditors who are seeking repayment). At the extreme, when $p = 0$, only the cash balance c is relevant for creditors' roll-over decisions. Note that this is a deterministic problem with perfect foresight for creditors: creditors can predict perfectly the evolution of the state variable $c(t)$. Since c is the only state variable, I can postulate a threshold c^* above which it will be optimal for creditors to continue rolling their debt claims, and below which it will be optimal for creditors to run when the opportunity arises. In what follows, I prove that the threshold

c^* is unique. I will note $v_0(c) := v^*(c, 0)$ the *equilibrium* value function of a creditor of a game with no illiquid asset – in other words, the value function of a given creditor following the cutoff strategy c^* , when all other creditors follow the same strategy. Similarly, I will use $e_0(c) := e^*(c, 0)$ for the shareholder value.

PROPOSITION 5. In an economy without illiquid asset, there exists a unique equilibrium in cutoff strategies, characterized by the cutoff $c^* \in \left(1, \frac{r_d + \lambda}{r_c + \lambda}\right)$. The equilibrium value function v_0 of creditors is equal to:

$$v_0(c) = \begin{cases} \left(\frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \lambda + \phi)(\phi + \rho - r_c)} \right) \left[1 - \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c \right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} \right] + \frac{\phi}{\phi + \rho - r_c} c & 0 \leq c < 1 \\ H_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c \right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} + \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} & 1 \leq c < c^* \\ H_2 \left(1 - \frac{r_c}{r_d} c \right)^{\frac{\rho + \phi}{r_c}} + \frac{r_d + \phi}{\rho + \phi} & c^* \leq c < \frac{r_d}{r_c} \\ \frac{r_d + \phi}{\rho + \phi} & c \geq \frac{r_d}{r_c} \end{cases} \quad (14)$$

The equilibrium cutoff c^* satisfies $v_0(c^*) = 1$. The formula for c^* is derived in the appendix. The value function v_0 is strictly increasing for $c < \frac{r_d}{r_c}$, and constant for $c > \frac{r_d}{r_c}$. The equilibrium value function for shareholders is equal to:

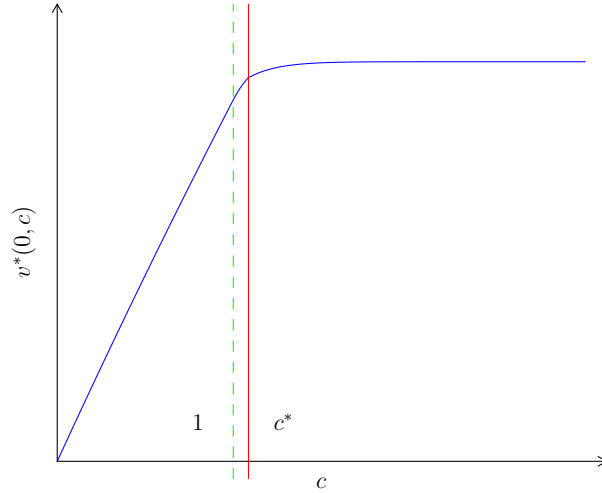
$$e_0(c) = \begin{cases} 0 & 0 \leq c < 1 \\ K_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c \right)^{\frac{\rho + \phi}{r_c + \lambda}} + \frac{\phi}{\phi + \rho - (r_c + \lambda)} c - \frac{\phi}{\rho + \phi} \left(1 + \frac{r_d + \lambda}{\phi + \rho - (r_c + \lambda)} \right) & 1 \leq c < c^* \\ K_2 \left(1 - \frac{r_c}{r_d} c \right)^{\frac{\rho + \phi}{r_c}} + \frac{\phi}{\phi + \rho - r_c} c - \frac{\phi}{\rho + \phi} \left(1 + \frac{r_d}{\phi + \rho - r_c} \right) & c^* \leq c < \frac{r_d}{r_c} \\ \frac{\phi}{\phi + \rho - r_c} c - \frac{\phi}{\rho + \phi} \left(1 + \frac{r_d}{\phi + \rho - r_c} \right) & c \geq \frac{r_d}{r_c} \end{cases} \quad (15)$$

H_1, H_2, K_1 and K_2 are constants described in the appendix.

Under the assumptions of **Proposition 5**, there is no “aggregate” uncertainty (in other words creditors have perfect foresight w.r.t. the evolution of the state variable $c(t)$), but each atomistic creditor faces idiosyncratic uncertainty (since each creditor’s maturity date is an exponentially distributed random variable). In this setting, a unique cutoff Markov perfect equilibrium emerges – due to a combination of (i) a time-varying state variable, (ii) asynchronous decisions by creditors, and (iii) the existence of dominance regions. In the model of **He and Xiong (2012)**, shutting down the volatility of the risky asset leads to the existence of multiple cutoff Markov perfect equilibria; while creditors make their decisions asynchronously, and while dominance regions exist, the state variable is no longer time-varying, leading to the multiplicity result.

Proposition 5 is also useful for the following reason: when looking for a symmetric cutoff Markov perfect equilibrium strategy, I now know that the cutoff boundary must intersect the axis $p = 0$ at $c = c^*$. **Figure 4** illustrates the shape of the value function when the firm does not hold any illiquid asset²⁰. Notice that the cutoff c^* is always strictly greater than 1. This means that when the firm's cash reserve is exactly equal to the aggregate outstanding debt, creditors are already running. This makes sense – when $C(t) = D(t)$, the firm owns an asset yielding r_c per unit of time, while it has outstanding debt yielding $r_d > r_c$. This is an irreversible situation for such firm, whose cash reserves will be depleted in finite time, and creditors choose to run before the asset to debt ratio of the firm is unity.

Figure 4: Value function $v_0(\cdot)$

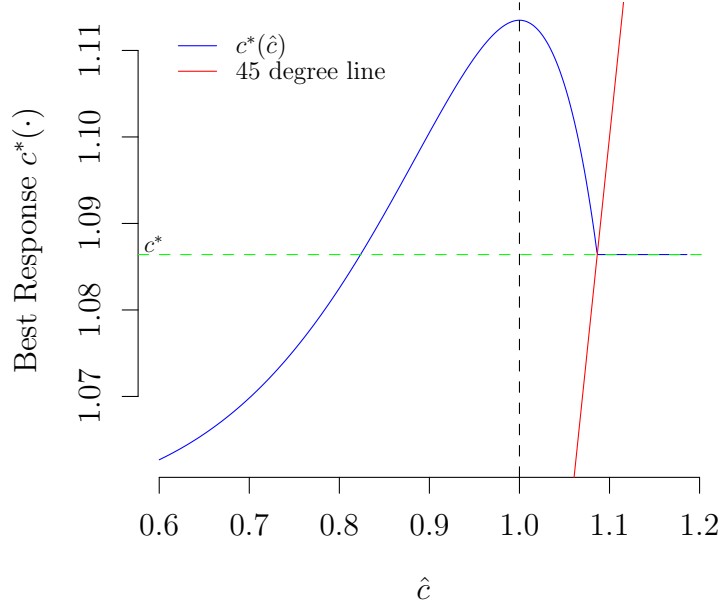


It is also instructive to look at the best response function $c^*(\hat{c})$, defined as the cutoff that a particular creditor i finds optimal to use, when all other creditors use a cutoff $\hat{c}$. **Figure 5** illustrates such best response function for the same parameters used to plot **Figure 4**. The figure illustrates a property already discussed in **Section 4.5** – namely, the lack of strategic complementarity in the payoff structure of this game. **Figure 5** shows that when the cutoff $\hat{c}$ played by other creditors is below 1, an increase in such cutoff increases the best response $c^*(\hat{c})$ of agent i – in other words $c \in [0, 1]$ is a region of strategic complementarity. However,

²⁰The value function is plotted using the following model parameters: $\rho = 0.04$, $\lambda = 0.2$, $\phi = 0.2$, $r_d = 0.05$, $r_c = 0.01$.

when the cutoff $\hat{c}$ played by other creditors is above 1, the best response function becomes downward sloping – thus highlighting strategic substitutability in creditors’ payoffs in the region $c \geq 1$.

Figure 5: Best Response $c^*(\hat{c})$ when $p = 0$



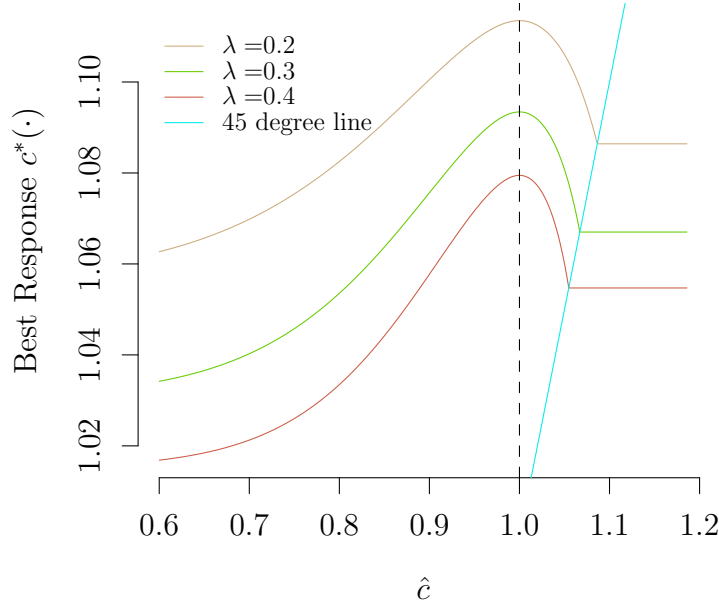
My next proposition looks at the comparative statics for the cutoff c^* .

PROPOSITION 6. In an economy without illiquid asset, the unique equilibrium cutoff strategy c^* is decreasing in λ .

Proposition 6 might seem surprising at first. It says that the longer the average firm’s debt maturity $1/\lambda$, the more conservative the equilibrium strategy of creditors, in other words the “earlier” they run. This result is due to the lack of strategic complementarity highlighted by **Proposition 1** in the neighborhood of the equilibrium. Specifically, for the game with no illiquid asset, the region of the state space where the equilibrium cutoff c^* is located exhibits strategic substitutability: the earlier other creditors run, the better off a given creditor i is. As the proof of **Proposition 6** (in the appendix) shows, this substitutability leads to the seemingly strange result that a longer debt average maturity leads to a more run-prone firm. **Figure 6** shows the best response function $c^*(\cdot)$ for different values of the parameter λ and provides a graphic illustration of **Proposition 6**. I will show in the numerical section of the

paper that this result also obtains in the presence of illiquid assets.

Figure 6: Best Response $c^*(\hat{c})$ when $p = 0$ – Sensitivity to λ



4.9 Boundary $c = 0$

I now discuss the boundary of the state space $c = 0$. When this boundary is reached (or when the system is started at $c = 0$), two events can occur. Either the drift of the cash reserve is positive, in which case the game continues, or the drift is negative, in which case the firm is forced to sell its illiquid asset and distribute the proceeds to its creditors and its shareholders. The following proposition characterizes the run and roll regions on the subset $\{(c, p) : c = 0\}$, for any symmetric cutoff Markov perfect equilibrium.

PROPOSITION 7. Given any symmetric cutoff Markov perfect equilibrium, the set $\{(0, p) : p < \frac{1}{\alpha}\}$ is in the run region and the set $\{(p, 0) : p \geq \frac{1}{\alpha}\}$ is in the roll region.

The proof of **Proposition 7** is developed in the appendix. It shows that for any symmetric cutoff Markov perfect equilibrium, its boundary must be anchored on the axis $c = 0$, which will prove useful for my numerical implementation of this model.

4.10 Existence of Symmetric Cutoff Markov Perfect Equilibrium

The previous sections facilitated a better understanding of the economics of the debt run model developed in this paper in different regions of the state space. The following theorem establishes the existence of a symmetric cutoff Markov perfect equilibrium of the game.

PROPOSITION 8. *There exists a symmetric cutoff Markov perfect equilibrium of the game.*

The proof of **Proposition 8** is detailed in the appendix. It leverages a fixed point theorem applicable for an appropriate space of functions, and bypasses the difficult issue of whether the best response to a cutoff strategy played by other creditors is indeed a cutoff strategy.

5 Hamilton-Jacobi-Bellman Equations

In the section below, I derive Hamilton-Jacobi-Bellman equations that the creditors' optimal value function v^* satisfies in different regions of the state space, under the assumption that S is a symmetric Markov perfect equilibrium strategy. The value function v^* implicitly depends on the strategy S followed by all other creditors, but I omit this dependence in this section in order to simplify notation. I assume that the value function v^* is twice differentiable except at specific boundaries of the state space. Finally, I introduce the following differential operators, indexed explicitly by S as they depend on the strategy followed by all creditors:

$$\begin{aligned}\mathcal{L}_S^p &:= (\mu + \lambda 1_{\{S(c,p)=1\}}) p \frac{\partial}{\partial p} \\ \mathcal{L}^{pp} &:= \frac{\sigma^2}{2} \frac{\partial^2}{\partial p^2} \\ \mathcal{L}_S^c &:= ((\rho - \mu)p + (r_c + \lambda 1_{\{S(c,p)=1\}})c - (r_d + \lambda 1_{\{S(c,p)=1\}})) \frac{\partial}{\partial c}\end{aligned}$$

A standard verification theorem (see for example **Fleming and Soner (2006)**) can show that the optimal value function $v^*(\cdot, \cdot)$ is solution to the following Hamilton-Jacobi-Bellman equation, for $c > 0$:

$$(\rho + \phi + \lambda) v^* = r_d + \phi \min(1, p + c) + \lambda \max(1, v^*) + (\mathcal{L}_S^p + \mathcal{L}^{pp} + \mathcal{L}_S^c) v^* \quad (16)$$

Similarly, the value function e^* satisfies the following equation:

$$(\rho + \phi) e^* = \phi \max(0, 1 - (p + c)) + (\mathcal{L}_S^p + \mathcal{L}^{pp} + \mathcal{L}_S^c) e^* \quad (17)$$

Since the firm cannot pay dividends before the maturity date of the illiquid asset, if $c > \frac{r_d}{r_c}$, the cash available (per unit of outstanding debt) is strictly increasing with time, irrespective of the value of the illiquid asset p and irrespective of creditor's decisions. Thus, a dominant strategy for creditors in this case is to roll over their maturing debt. Creditor's debt is thus risk-free, with a value equal to the discounted stream of interest payments at rate r_d , up to the stopping time τ_ϕ . In other words, I must have for any (c, p) such that $c > \frac{r_d}{r_c}$:

$$v^*(c, p) = \frac{r_d + \phi}{\rho + \phi}$$

Additionally, for p large, the illiquid asset cash flows per unit of time are large as well. For p large enough, the available cash per unit of debt outstanding exceeds the bound $\frac{r_d}{r_c}$ in a short amount of time, at which point the debt is risk-free, as established above. This means that I have:

$$\lim_{p \rightarrow +\infty} v^*(c, p) = \frac{r_d + \phi}{\rho + \phi}$$

Finally, I establish the following lemma for shareholder value when p or c are large.

LEMMA 3. When $p \rightarrow +\infty$ or $c \rightarrow +\infty$, and when no dividends are payable to shareholders, the function e^* verifies:

$$e^*(c, p) = \frac{\phi}{\rho + \phi - r_c} c + \frac{\phi}{\rho + \phi - \mu} \left(1 + \frac{\rho - \mu}{\rho + \phi - r_c} \right) p - \frac{\phi}{\rho + \phi} \left(\frac{r_d}{\rho + \phi - r_c} + 1 \right) + o(1)$$

When $c = 0$, **Proposition 7** shows that the locus of points $\{(c, p) : c = 0, p < \frac{1}{\alpha}\}$ is in the run region $\mathcal{R}_S$. On such boundary, the value of the debt and equity claims is equal to:

$$v^*(p, 0) = \min(\alpha p, 1) \tag{18}$$

$$e^*(p, 0) = \max((\alpha p - 1), 0) \tag{19}$$

Symmetry and optimality of creditors' cutoff strategy means that for points on the boundary $\partial\mathcal{R}_S$ which are not on the vertical axis $c = 0$ and which are not on the horizontal axis $p = 0$ must satisfy:

$$v^*(c, p) = 1$$

In order to confirm that the cutoff strategy s is optimal, I must verify that:

$$\begin{aligned} v^*(c, p) &\geq 1 \text{ for all } (c, p) \in \mathcal{NR}_S \\ v^*(c, p) &< 1 \text{ when } (c, p) \in \mathcal{R}_S \end{aligned}$$

Finally, given the Brownian shocks that the state variable p is exposed to, I will require the functions v^* and e^* to be continuously differentiable with respect to p on the locus of points $\partial\mathcal{R}_S$ that are in the interior of the state space.

5.1 The Particular Case $\sigma = 0$

The case $\sigma = 0$ helps getting a better intuitive understanding of the mechanics of the model and the role of the different parameters. When I shut down the shocks to the illiquid asset price, the state dynamics are deterministic, and agents with perfect foresight can predict the evolution of the state space $(c(t), p(t))$. Creditors are still subject to the uncertainty related to the realization of the maturity date of their debt contract, in addition to the uncertain realization of the maturity date τ_ϕ of the illiquid asset.

PROPOSITION 9. In an economy with $\sigma = 0$, if $\frac{r_d - \mu}{\rho - \mu} > \frac{1}{\alpha}$ there exists a continuum of symmetric cutoff Markov perfect equilibria. One such symmetric cutoff Markov perfect equilibrium is characterized by a strictly decreasing and differentiable function $\Psi : [0, c^*] \rightarrow \mathbb{R}_+$ such that creditors run in the region $\{(c, p) \in \mathbb{R}_+^2 : p \leq \psi(c), c \leq c^*\}$, and roll otherwise. This is the unique equilibrium whose cutoff boundary intersects the axis $c = 0$ at $p = 1/\alpha$. All other symmetric cutoff Markov perfect equilibria are characterized by some $\tilde{c} < c^*$, such that creditors' cutoff boundary will be (a) for $c < \tilde{c}$, the state trajectory conditioned on a run and which goes through the point $(\Psi(\tilde{c}), \tilde{c})$, and (b) for $c \geq \tilde{c}$, the function Ψ . If $\frac{r_d - \mu}{\rho - \mu} < \frac{1}{\alpha}$, a unique equilibrium exists, characterized by the function Ψ .

The proof of **Proposition 9** is developped in the appendix, and is broken into multiple lemmas. It relies on solving a set of first order linear partial differential equations in closed form. When the volatility $\sigma \neq 0$, I established in **Proposition 7** that any symmetric cutoff Markov perfect equilibrium of the game must intersect the axis $c = 0$ at $p = 1/\alpha$. My proof uses the continuity of the best response function $v^*(\cdot, \cdot; S)$, and thus the continuity of any equilibrium value function v^* . Such continuity relies on the presence of Brownian shocks hitting the state variables $(c(t), p(t))$; the absence of those Brownian shocks leads to the possibility of discontinuities in the value function, and thus the existence of multiple cutoff Markov perfect equilibria as **Proposition 9** shows. One particular equilibrium however

(and it turns out, the only such equilibrium) is characterized by a cutoff boundary that is a strictly decreasing function $\Psi : [0, c^*] \rightarrow \mathbb{R}_+$. **Proposition 9** also highlights the importance of the Brownian shocks in reducing the number of equilibria of my model. Without such shocks, a potentially infinite number of equilibria arise, all of them characterized by creditor value functions that are not continuous functions of the state space. The introduction of Brownian shocks “smooths” the value function on the state space, and restricts the number of equilibria obtained in this model.

6 Numerical Implementation

6.1 Algorithm

I compute the value function v numerically over the compact set $[0, \bar{p}] \times [0, \bar{c}]$, by determining the value of v^* on a grid G_h , where $h > 0$ is my scalar approximation parameter. I choose $\bar{c} = \frac{r_d}{r_c}$, and $\bar{p}$ large enough to ensure that v^* is close to its maximum at $p = \bar{p}$. I will use a Markov Chain approximation method, as explained in **Kushner and Dupuis (2001)**, and solve the model assuming no dividends are payable. I start with a guess equilibrium map $S^{(1)}$, and a guess value function $v^{(1,1)}$. My guess functions will take the following form:

$$S^{(1)}(c, p) = 1_{\{p < \frac{1}{\alpha}(1-c/c^*)\}}$$

$$v^{(1,1)}(c, p) = \min \left(v_0(c) + \alpha p, \frac{r_d + \phi}{\rho + \phi} \right)$$

The initial guess equilibrium map corresponds to a run boundary that is a linear in the (c, p) space, intersecting the axis $p = 0$ at $c = c^*$, and intersecting the axis $c = 0$ at $p = 1/\alpha$. My algorithm has an outer-loop, which updates the equilibrium map $S^{(i)}$, and an inner loop, which, for a given $S^{(i)}$, updates the function $v^{(i,j)}$. In the inner loop, I calculate the function $v^*(\cdot, \cdot; S^{(i)})$ as follows. Given the map $S^{(i)}$, the state space (c, p) evolves according to the following:

$$dp(t) = \left(\mu + \lambda 1_{\{S^{(i)}(c(t), p(t))=1\}} \right) p(t) dt + \sigma p(t) dB^{\mathbb{Q}}(t)$$

$$dc(t) = \left((\rho - \mu)p(t) + (r_c + \lambda 1_{\{S^{(i)}(c(t), p(t))=1\}})c(t) - (r_d + \lambda 1_{\{S^{(i)}(c(t), p(t))=1\}}) \right) dt$$

I will use the following notation:

$$\begin{aligned} a_p(c, p) &= \sigma p \\ b_p(c, p) &= (\mu + \lambda 1_{\{S^{(i)}(p, c)=1\}}) p \\ b_c(c, p) &= (\rho - \mu)p + (r_c + \lambda 1_{\{S^{(i)}(p, c)=1\}})c - (r_d + \lambda 1_{\{S^{(i)}(p, c)=1\}}) \end{aligned}$$

In the inner loop, I create a Markov Chain $\{(p_n^h, c_n^h), n < \infty\}$ that approximates the process $\{(c(t), p(t))\}_{t \geq 0}$. Let $\gamma > 0$ be an arbitrary constant. I introduce $Q^h(c, p)$ and $\Delta t^h(c, p)$ as follows:

$$\begin{aligned} Q^h(c, p) &:= a_p(c, p)^2 + hb_p(c, p) + h|b_c(c, p)| + h\gamma \\ \Delta t^h(c, p) &:= \frac{h^2}{Q^h(c, p)} \end{aligned}$$

Note that $\inf_{p, c} Q^h(c, p) > 0$, which means that $\Delta t^h(c, p)$ is well defined. Note also that I have for all (c, p) :

$$\lim_{h \rightarrow 0} \Delta t^h(c, p) = 0$$

I then define the following transition probabilities:

$$\begin{aligned} \Pr((p_{n+1}^h, c_{n+1}^h) = (p + h, c) | (p_n^h, c_n^h) = (c, p)) &= \frac{a_p(c, p)^2/2 + hb_p(c, p)}{Q^h(c, p)} \\ \Pr((p_{n+1}^h, c_{n+1}^h) = (p - h, c) | (p_n^h, c_n^h) = (c, p)) &= \frac{a_p(c, p)^2/2}{Q^h(c, p)} \\ \Pr((p_{n+1}^h, c_{n+1}^h) = (c, p) | (p_n^h, c_n^h) = (c, p)) &= \frac{h\gamma}{Q^h(c, p)} \\ \Pr((p_{n+1}^h, c_{n+1}^h) = (p, c + h) | (p_n^h, c_n^h) = (c, p)) &= \frac{h \max(0, b_c(c, p))}{Q^h(c, p)} \\ \Pr((p_{n+1}^h, c_{n+1}^h) = (p, c - h) | (p_n^h, c_n^h) = (c, p)) &= \frac{h \max(0, -b_c(c, p))}{Q^h(c, p)} \end{aligned}$$

Notice that these transition probabilities are all greater than zero, less than 1, and they add up to 1. Noting $\Delta(p_n^h, c_n^h) := (p_{n+1}^h, c_{n+1}^h) - (p_n^h, c_n^h)$, the Markov chain created satisfies the

local consistency condition:

$$\begin{aligned}\mathbb{E}^{p,c} \left[\Delta \begin{pmatrix} p_n^h \\ c_n^h \end{pmatrix} \right] &= \begin{pmatrix} b_p(c,p) \\ b_c(c,p) \end{pmatrix} \Delta t^h(c,p) \\ \text{var}^{p,c} \left[\Delta \begin{pmatrix} p_n^h \\ c_n^h \end{pmatrix} \right] &= \begin{pmatrix} a_p(c,p)^2 & 0 \\ 0 & 0 \end{pmatrix} \Delta t^h(c,p) + o(\Delta t^h(c,p))\end{aligned}$$

For $\bar{p} > p > 0$ and $\bar{c} > c > 0$, and given a function $v^{(i,j)}$, I compute $v^{(i,j+1)}$ on the grid G_h as follows:

$$\begin{aligned}v^{(i,j+1)}(c,p) &= r_d \Delta t^h(c,p) \\ &+ e^{-\rho \Delta t^h(c,p)} \times \left\{ \begin{aligned} &\phi \Delta t^h(c,p) \min(1, p+c) + \lambda \Delta t^h(c,p) \max(1, v^{(i,j)}(c,p)) \\ &+ (1 - (\lambda + \phi) \Delta t^h(c,p)) \sum_{(p',c')} \Pr((p',c')|(c,p)) \times v^{(i,j+1)}(p',c') \end{aligned} \right\}\end{aligned}$$

For $c > 0$, when the Markov chain is in a state with $p = h$, the algorithm puts a non-zero probability onto the next state being such that $p = 0$. When that happens, I will assume that such next state is absorbing, with a terminal value $v_0(c)$. Similarly, for $p > 0$, when the Markov chain is in a state with $c = h$, our algorithm puts a non-zero probability onto the next state being such that $c = 0$ and $p \leq \frac{1}{\alpha}$. When that happens, I will assume that such next state is absorbing, with a terminal value $\alpha p \wedge 1$. Finally, when $c = \frac{r_d}{r_c} - h$ and the Markov chain transitions to a state where $c = \frac{r_d}{r_c}$, or when $p = \bar{p} - h$ and the Markov chain transitions to a state where $p = \bar{p}$, I assume that such state is absorbing, with value $\frac{r_d + \phi}{\rho + \phi}$.

So long as $\|v^{(i,j+1)} - v^{(i,j)}\|_\infty > \epsilon$, for ϵ small taken arbitrarily, I continue on the inner loop. When $\|v^{(i,j+1)} - v^{(i,j)}\|_\infty \leq \epsilon$, I have obtained $v^*(\cdot, \cdot; S^{(i)})$ as the limit of the shooting algorithm. I then set $S^{(i+1)}$ by solving for each (c, p) on the grid:

$$S^{(i+1)}(c, p) = 1_{\{v^*(c, p; S^{(i)}) < 1\}}$$

So long as $\|S^{(i+1)} - S^{(i)}\|_\infty > \hat{\epsilon}$, for $\hat{\epsilon}$ small taken arbitrarily, I continue on the outer loop, and stop when $\|S^{(i+1)} - S^{(i)}\|_\infty < \hat{\epsilon}$.

6.2 Numerical Results

6.2.1 Base Case Parameters

I first study the behavior of the model in the context of pre-financial crisis US broker dealers. The model parameters selected (displayed in [Table 1](#)) thus reflect some of the key characteristics of those financial intermediaries.

Parameter	Value	Description
ρ	0.06	Risk free rate
λ	2.00	1/Average debt maturity
ϕ	0.10	1/Average illiquid asset maturity
r_d	0.07	Interest rate on debt
r_c	0.05	Interest rate on internal cash
μ	0.02	Risk neutral illiquid asset growth rate
σ	0.15	Illiquid asset volatility
α	0.40	Recovery upon illiquid asset sale

Table 1: Calibration Parameters

Constrained by data availability on recovery rates, I focus for all relevant parameters²¹ on the time period 1982 – 2008. I set the risk free rate ρ to 6%, which is slightly above the average federal funds effective rate between January 1982 and December 2008²². r_d is the interest rate received by creditors. Since creditors need to have an incentive to roll-over their debt, I need to impose $r_d > \rho$. I calibrate $r_d - \rho$ to be equal to the historical 5-year unsecured broker dealer CDS spread²³. Since the illiquid asset value follows the Gordon growth formula $P(t) = Y(t)/(\rho - \mu)$, I need to set $\rho > \mu$ to guarantee that the illiquid asset value is finite; I thus choose $\mu = 2\%$, leading to an asset payout ratio of $\rho - \mu = 4\%$. r_c is the yield on the cash internal to the firm, and I have assumed $r_c < \rho$ in order to introduce an agency cost for the firm to hold liquid reserves (this will be useful when I discuss portfolio choice and dividend policy). Following [Bolton, Chen, and Wang \(2011\)](#), I assume an agency cost of $\rho - r_c = 1\%$, leading to a rate of return on cash of $r_c = 5\%$. The parameter α is calibrated using Moody’s historical recovery rate data²⁴. In order to calibrate σ , I look at equity returns of publicly traded US banks and broker dealers²⁵ between January 1982 and December 2008 and use as

²¹As discussed further, the only notable exception is the calibration of the spread $r_d - \rho$, which relies on historical bank CDS spreads during the time period 2001 – 2014.

²²Using historical data from the Federal Reserve website, I calculate an average effective federal funds rate of 5.60% over that time period.

²³Source: Markit. CDS levels downloaded for the 5 main broker dealers (namely Goldman Sachs, Morgan Stanley, Merrill Lynch, Lehman Brothers and Bear Stearns) between 2001 – the earliest year available – and the end of 2014 average 96bps over that time period.

²⁴See [Emery et al. \(2009\)](#). According to the report, the issuer-weighted historical recovery rate on senior unsecured bonds is 36.4% for the time period 1982-2008 when using post-default unsecured bond trading levels, while such historical recovery rate is 46.2% for the time period 1987-2008 when using ultimately realized recoveries. I thus select a recovery level in-between for my base case calibration. The report also computes an average senior unsecured bond recovery rate of 35.4% for financial institution defaults in 2008, but once again uses 30-day post default trading levels, which has a tendency to bias downwards the realized recovery levels – media reports in 2014 for example were calculating an estimated payout for Lehman Brothers’ unsecured claims of 26.9%, which needs to be compared to the 9.3% market value of such claims shortly after Lehman Brothers’ bankruptcy filing.

²⁵The list of banks and broker dealers used for this analysis is available upon request.

a proxy for the asset volatility $\sigma^2 \approx \text{Lev}^2 \sigma_{\text{eq}}^2$, where “Lev” is a firm’s market leverage (i.e. the ratio of (i) the firm’s equity market value over (ii) the firm’s market value), and σ_{eq} is the firm’s stock price volatility. This calibration method leads to a floor on asset volatility – as [Schaefer and Strebulaev \(2008\)](#) points out, a more realistic estimate of asset volatilities would be:

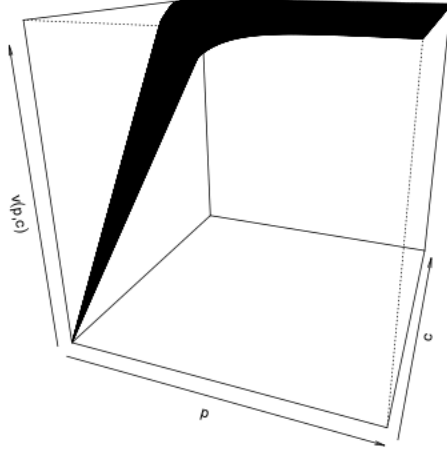
$$\hat{\sigma}^2 = \text{Lev}^2 \sigma_{\text{eq}}^2 + (1 - \text{Lev})^2 \sigma_{\text{debt}}^2 + 2\text{Lev}(1 - \text{Lev})\sigma_{\text{eq,debt}}$$

The parameter ϕ influences the duration of the illiquid asset – its inverse $1/\phi$ represents the expected maturity of the asset. I choose $\phi = 0.1$, which means that I am assuming an illiquid asset average life of 10 years. This assumption differs from [He and Xiong \(2012\)](#): they focus on a firm that holds a mortgage portfolio, and thus set the expected life of their asset at 13 years. I have in mind a broker-dealer or investment bank, whose portfolio includes not only mortgages, but corporate bonds, asset-backed securities, non-agency mortgage-backed securities, and assets whose average lives are typically shorter than a standard agency mortgage expected life. The parameter λ drives the debt maturity structure – its inverse $1/\lambda$ represents the average debt maturity of the firm. The base case parameter $\lambda = 2$ thus means that the firm’s average debt maturity is 6 months. This is consistent with the liability structure of many broker-dealers: firms such as Goldman Sachs or Morgan Stanley have a large tri-party repo book with a duration of approximately 3 months, combined with some long term debt with a duration of 5 to 10 years. The base case parameters lead to a value $c^* = 1.013909$. As expected, $1 < c^* < \frac{r_d + \lambda}{r_c + \lambda} = 1.0199$.

[Figure 7](#) is the 3-D representation of the value function v^* . [Figure 8a](#) represents the value function $v^*(\cdot, p)$ for several price levels p arbitrarily chosen. $v^*(c, 0) = v_0(c)$ is directly drawn from the expression obtained in [Proposition 5](#). It exhibits kinks at $c = 1$ and $c = c^*$. For the selected values of p , $v^*(\cdot, p)$ is an increasing function of c , bounded above by $\frac{r_d + \phi}{\rho + \phi}$. [Figure 8b](#) represents the value function $v^*(\cdot, c)$ for several levels of cash c arbitrarily chosen. I focus on $v^*(\cdot, 0)$ to start with – in other words the debt value function when the cash (per unit of debt outstanding) is zero. The function is linear for $p < 1/\alpha$, since for those values of p , the firm runs out of cash, sells its illiquid assets, with creditors realizing $\min(1, \alpha p)$. For $p \geq 1/\alpha$, the illiquid asset value is high enough that maturing creditors elect to roll-over their maturing debt claims, even if the bank has no cash resources available to it.

[Figure 9](#) shows the endogenous equilibrium run boundary $\partial\mathcal{R}$ in the (c, p) space. For this specific parameter environment, the equilibrium run boundary is a decreasing function of the cash level. I have also included the locus of points (c, p) such that the cash drift is

Figure 7: $v^*(\cdot, \cdot)$



zero when a run occurs, and when creditors roll over²⁶. Points of the state space below these lines are points where the cash drift is negative. As expected, the slope of equilibrium run boundary $\partial\mathcal{R}$ is steeper than -1 , since cash has value as run-deterrent. Indeed, remember that leverage, which can be measured as total assets divided by total outstanding debt, is simply equal to $p + c$ in my model. For a given leverage value $p + c$, creditors will be more inclined to run when the firm's liquid resources are low. When $c = 0$, unless the price of the illiquid asset (per unit of debt outstanding) is greater than $1/\alpha$, creditors start running. For this parameter environment, as the amount of cash (per unit of debt outstanding) increases, the minimum illiquid asset price level required to deter a run decreases. At the limit $p = 0$, a run is deterred when $c > c^*$.

I then consider the following question: given a firm's asset liquidity composition, measured via $\Lambda := \frac{c}{p+c}$ (the "Liquidity Ratio", i.e. the fraction of the firm's assets that is liquid), what leverage (as measured by the "Solvency Ratio" $\Sigma := p + c$, i.e. the ratio of total assets to total liabilities) does the firm need to maintain in order to deter a run. **Figure 10** answers this question specifically: it plots the threshold leverage $c + p$ as a function of the percentage of the firm invested in liquid reserves $\frac{c}{c+p}$, for points $(c, p) \in \partial\mathcal{R}$. I can then relate those theoretical predictions to the empirical facts discussed in **Section 2**. For example, a firm that

²⁶When creditors are rolling over their debt claims, the cash drift is equal to $(\rho - \mu)p + r_c c - r_d$. Thus, the locus of points (c, p) for which the cash drift is zero is characterized by $p = \frac{1}{\rho - \mu} (r_d - r_c c)$. Similarly, when a run is occurring, the locus of points (c, p) for which the cash drift is zero is characterized by $p = \frac{1}{\rho - \mu} (r_d + \lambda - (r_c + \lambda)c)$.

Figure 8: Value function v^*

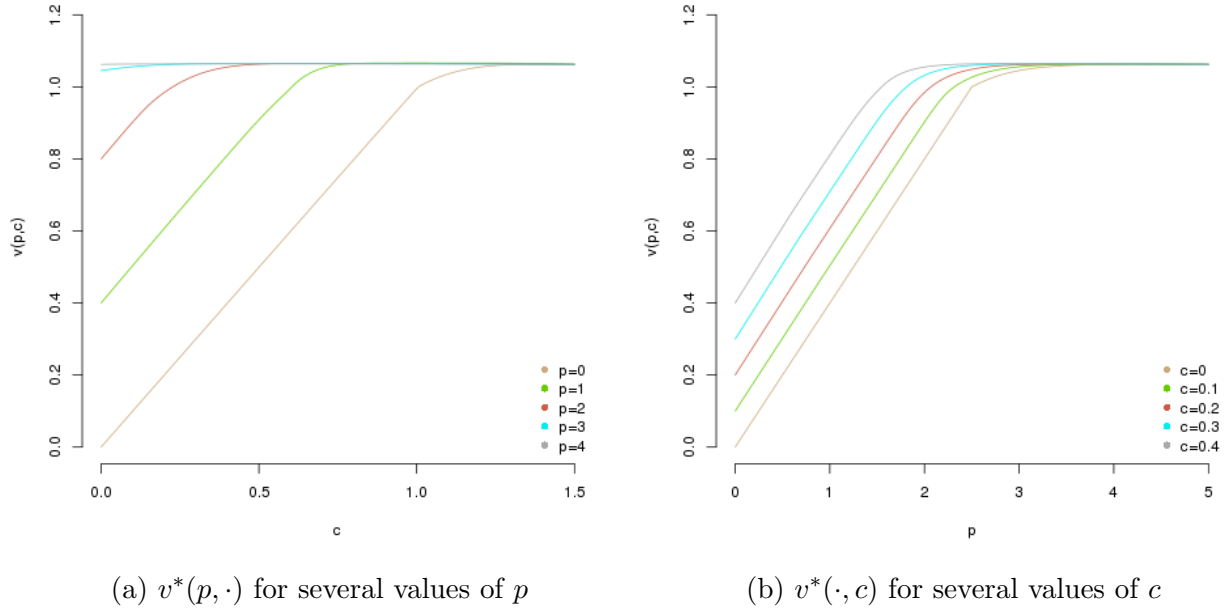
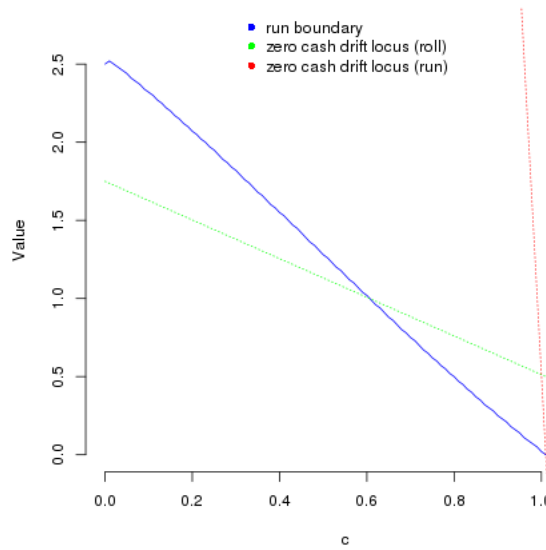
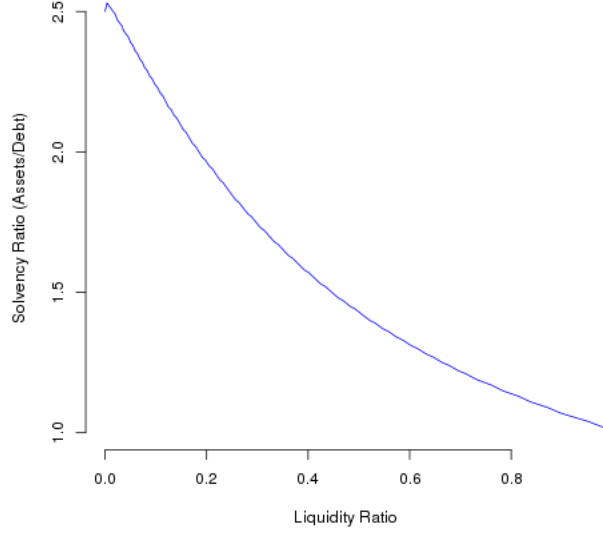


Figure 9: Equilibrium threshold $\Psi(\cdot)$



holds 20% of its balance-sheet in liquid resources needs to maintain an asset-to-debt leverage above 2 in order to deter a run.

Figure 10: Liquidity vs. Solvency



6.2.2 Connection to Recent Regulatory Proposals

As discussed in [Section 1](#), one of the key changes in the micro-prudential bank regulatory framework relates to the Liquidity Coverage Ratio (“LCR”) requirement. In a nutshell, the proposed regulation constrains banks to maintain sufficient liquidity reserves in order to withstand a one-month run. I can look at this proposed regulation through the lense of my model. Noting Δ the amount of time that regulators want a given bank to survive upon the occurrence of a run (Δ would be equal to one month under the Basel Committee proposal), the model-implied minimum liquidity holdings $C_{\min}(t)$ need to satisfy:

$$C(t) \geq \mathbb{E}^{P,C,D} \left[\int_t^{t+\Delta} ((r_d + \lambda)D(s) - (\rho - \mu)P(s) - r_c C(s)) ds \right]$$

The model-implied worst case one-month outflow is equal to $\int_t^{t+\Delta} (r_d + \lambda)D(s)ds$, while the model-implied one-month inflow is equal to $\int_t^{t+\Delta} ((\rho - \mu)P(s) + r_c C(s)) ds$. [Lemma 4](#) derives a closed form expression for this model-implied minimum liquidity requirement under the new Basel rules.

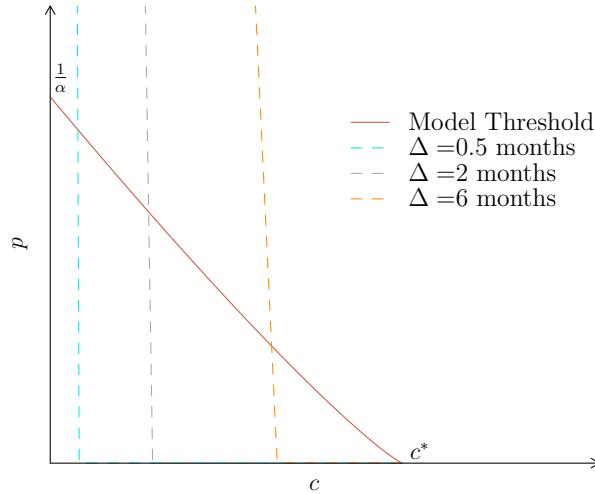
LEMMA 4. The model-implied LCR requirement $C_{\min}$ satisfies:

$$C_{\min} = \frac{1}{1 + r_c \Delta} \left[\frac{r_d + \lambda}{r_c + \lambda} D (e^{r_c \Delta} - e^{-\lambda \Delta}) - \frac{\rho - \mu}{r_c - \mu} P (e^{r_c \Delta} - e^{\mu \Delta}) \right]$$

At the limit, when the time period Δ under consideration is small, the model-implied LCR requirement per unit of debt outstanding satisfies $c_{\min} \approx (r_d + \lambda)\Delta - (\rho - \mu)\Delta p$.

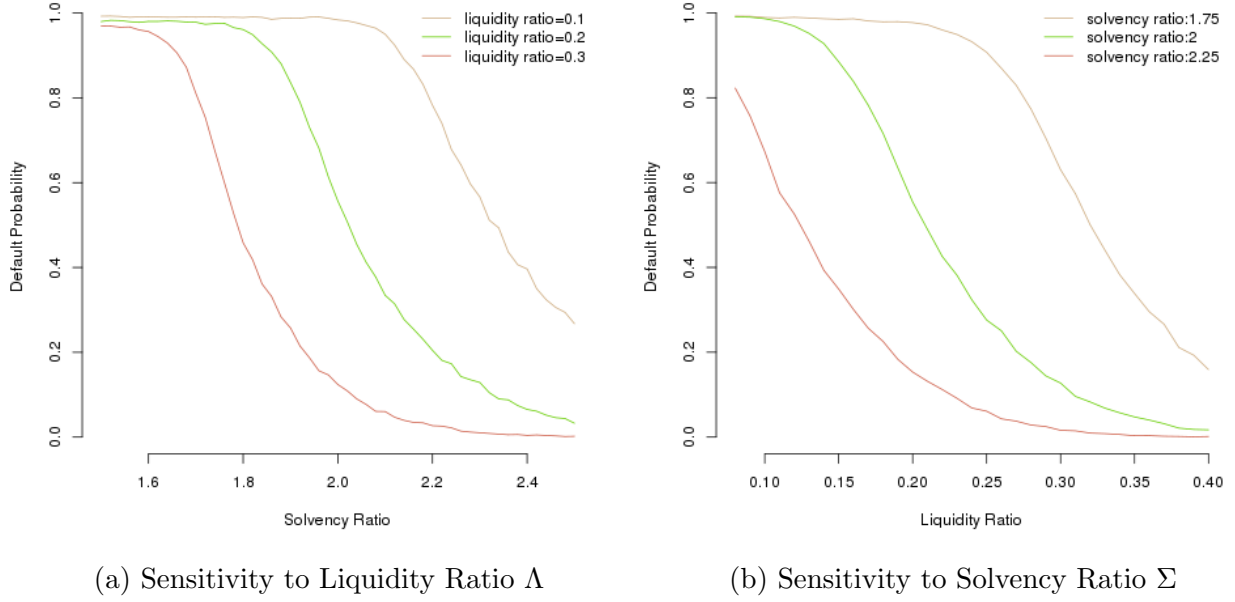
The derivation of the model-implied LCR requirement when Δ is small has a natural interpretation: for a small time interval Δ , the firm's cash outflows (per unit of debt outstanding) correspond to $(r_d + \lambda)\Delta$, the cost of interest and principal payments over such time period, while the firm's cash inflows (per unit of debt outstanding) correspond to $(\rho - \mu)\Delta p$, the income received on the illiquid asset. Given my base case parameter values, and most importantly given the fact that the firm I am considering finances itself with short term debt, I have $r_d + \lambda \gg \rho - \mu$, which means that for reasonable illiquid asset values (per unit of debt outstanding), the model-implied LCR requirement is approximately equal to the projected amount of debt maturing over the time period Δ .

Figure 11: Model-Implied LCR Requirement



For the base case parameter values under consideration, [Figure 11](#) plots such requirement in the state space (c, p) for different values of the time horizon Δ , as well as the equilibrium run threshold. The figure makes it clear that the LCR requirement does not take into account the liquidation value of the bank's illiquid asset: the required cash holding is substantially larger than necessary when the firm's illiquid asset liquidation value covers the aggregate debt outstanding (in [Figure 11](#), this is region of the state space with low values of c and

Figure 12: Default Probability



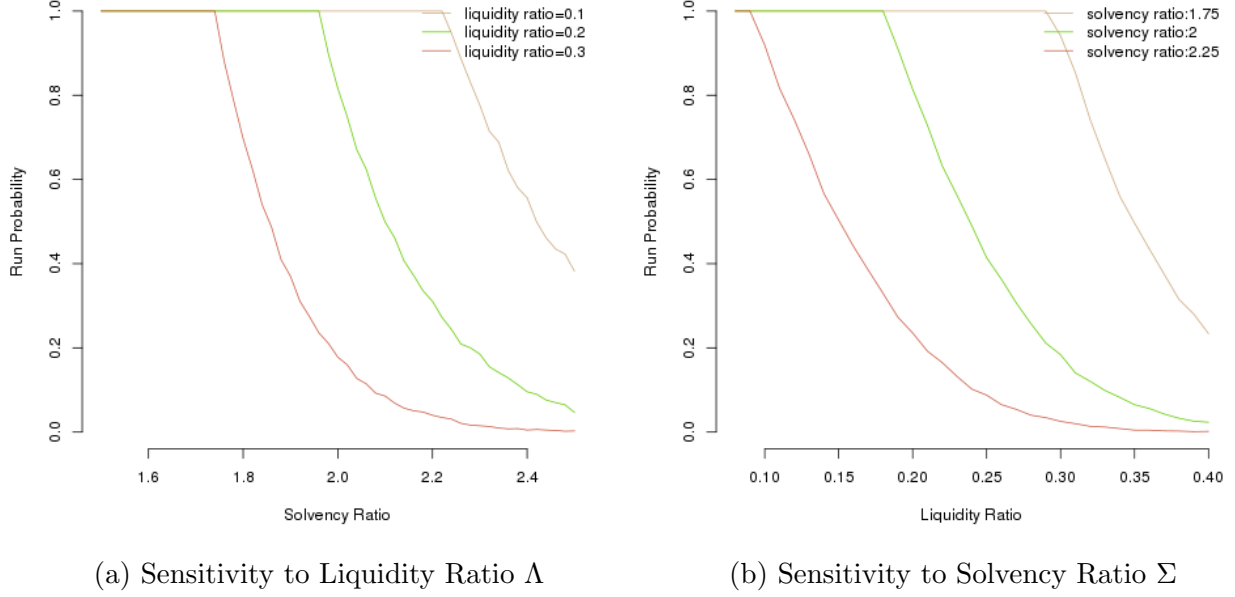
high values of p). At the other extreme, when the illiquid asset liquidation value is not high enough to cover the aggregate debt outstanding, the firm might be experiencing a run on its liabilities while at the same time satisfying its LCR requirement (in Figure 11, this is region of the state space with low values of p and values of c on the right side of the model-implied LCR requirement).

6.2.3 Default Probabilities, Run Probabilities and Credit Spreads

The unique equilibrium obtained for the parameters considered enables me to calculate default and run probabilities for any initial level of cash and illiquid asset price (per unit of debt outstanding). If I note $\pi_d(c, p; T) := \mathbb{E}^{c,p} [1_{\{\tau_b < T\}}]$ the probability that a default occurs before time T given initial states (c, p) , I can compute $\pi_d(c, p; T)$ either via solving a system of partial differential equations similar to Section 5 (except for the fact that partial derivatives with respect to time would also appear), or via monte-carlo simulation. I choose the second approach, and plot in Figure 12 the default probability $\pi_d(c, p; T)$ for large T (since the model is non-stationary, I know that $\pi_d(c, p; T)$ has a limit when $T \rightarrow +\infty$), and plotted for various initial liquidity and solvency ratios.

Using a similar reasoning, I can also look at the probability that the firm, starting at initial cash and illiquid asset price levels (c, p) , suffers a run on the time interval $[0, T]$ – this corresponds to the probability of hitting the boundary $\partial\mathcal{R}_S$ within T years: $\pi_r(c, p; T) :=$

Figure 13: Run Probability



$\mathbb{E}^{c,p} [1_{\{\inf\{t:p(t) \leq \Psi(c(t))\} < T\}}]$. I plot such probabilities in [Figure 13](#), for $T \rightarrow +\infty$, in the liquidity solvency space. As the figure indicates, the run probability is 1 whenever the firm starts with a solvency ratio below the threshold level Ψ , which is the case for example when the firm has a liquidity ratio of 30% and a solvency ratio below 1.75, or when the firm has a liquidity ratio of 20% and a solvency ratio below 2.

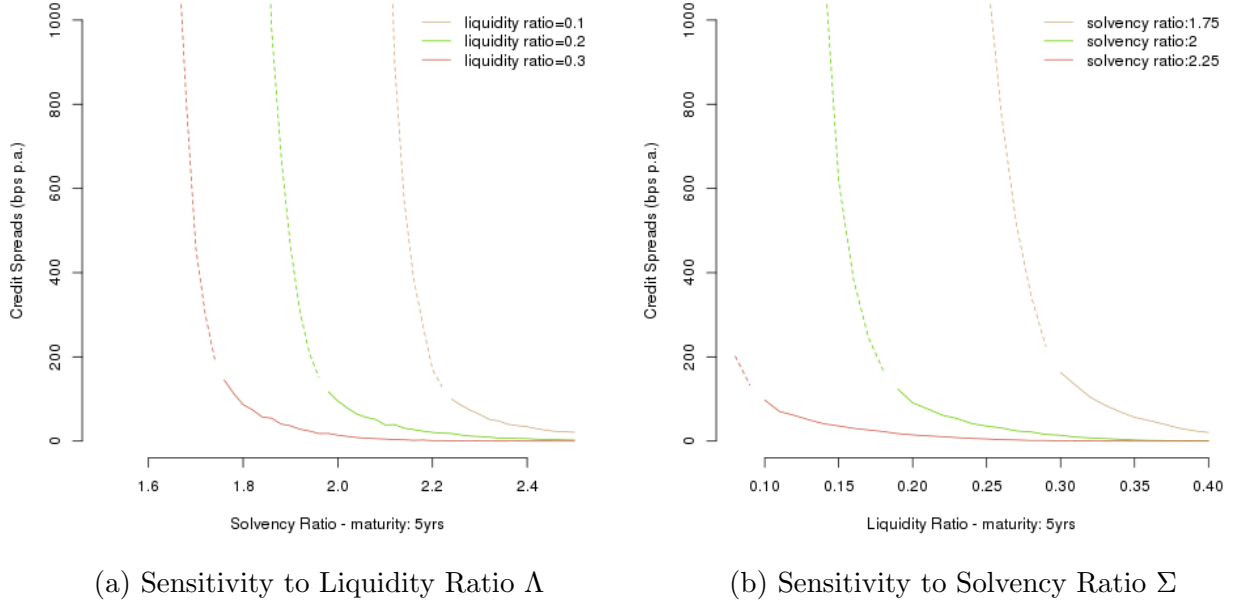
Finally, I compute par CDS spreads at different maturities; for a given maturity T , those par CDS spread correspond to the running premium a counterparty A would need to receive during T years in order to accept to compensate counterparty B for credit losses suffered at or before T in connection with a debt investment in the firm. Such par CDS spread $CDS(c, p; T)$ thus satisfies:

$$CDS(c, p; T) := \frac{\mathbb{E}^{c,p} [1_{\{\tau_b < T \wedge \tau_\phi\}} e^{-\rho \tau_b} \max(0, 1 - \alpha p(\tau_b)))]}{\mathbb{E}^{c,p} \left[\int_0^{\tau_b \wedge \tau_\phi \wedge T} e^{-\rho t} dt \right]}$$

The dotted lines in [Figure 14](#) are par CDS spreads computed in states where the firm is suffering a run. Those figures highlight two features of broker dealer credit default swap levels: in “normal times” (i.e. whenever creditors roll over), broker dealer CDS trade below 200bps per annum, while such CDS levels increase rapidly in times of funding stress, such as those experienced during the financial crisis.

As an illustration of this phenomenon, I plotted in [Figure 15](#) the time series of Lehman

Figure 14: 5-year Par CDS Spreads



Brothers' and Bear Stearns' 5-year CDS levels from January 2007 to their demises (the red vertical line in the figure), illustrating the very quick rise of their CDS levels as institutional counterparties were refusing to roll over their repo contracts. The run and default mechanisms illustrated in this paper are thus radically different from the corresponding default mechanism of industrial firms, whose fundamental deterioration leads to slow increase in credit spreads, during which equity holders might be recapitalizing the firm before an eventual default – a mechanism studied extensively in the literature started by [Leland \(1994\)](#).

The benchmark model of default following [Leland \(1994\)](#) takes the followin form.

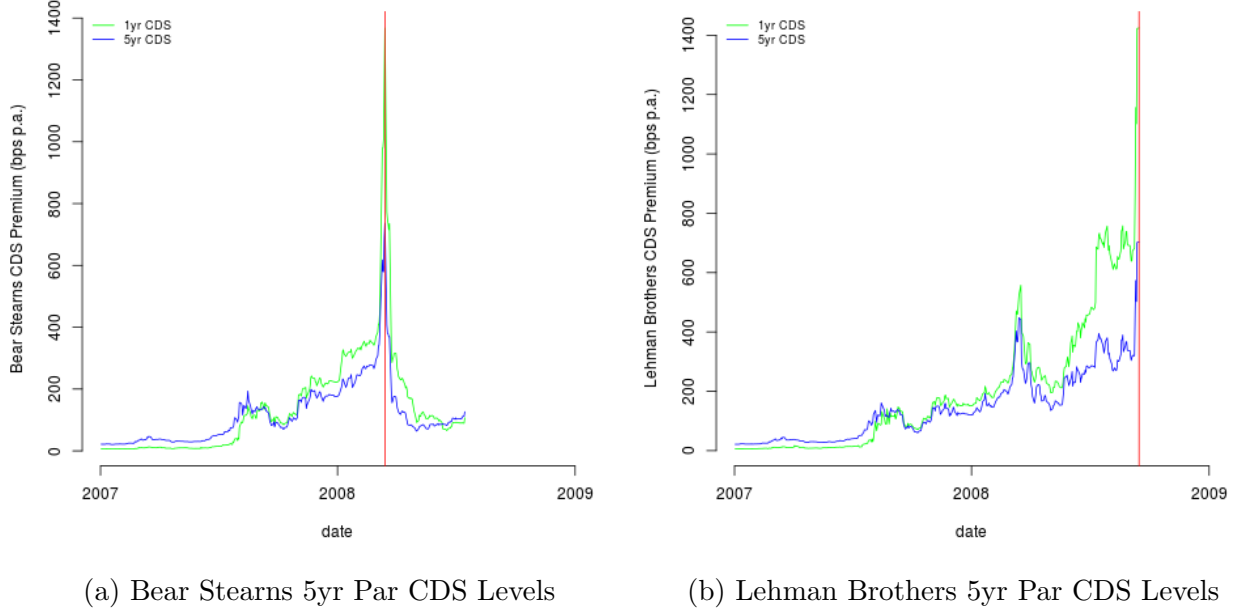
PROPOSITION 10. In a standard model of endogeneous default, the debt price d is equal to:

$$d(p) = \begin{cases} \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} + k_1 p^{x_1} & p \geq 1 \\ \frac{\phi p}{\rho + \lambda + \phi - \mu} + \frac{r_d + \lambda}{\rho + \lambda + \phi} + l_1 p^{x_1} + l_2 p^{x_2} & 1 > p \geq p^* \end{cases} \quad (20)$$

6.2.4 Sensitivity to the Asset Volatility σ

I then study the sensitivity of the threshold strategy to the illiquid asset volatility σ . For parameter values $\sigma = 0.1\%$, $\sigma = 5\%$ and $\sigma = 40\%$, [Figure 16a](#) shows the endogeneous run boundary $\partial\mathcal{R}$, while [Figure 16b](#) shows the trade-off between firm's leverage and the

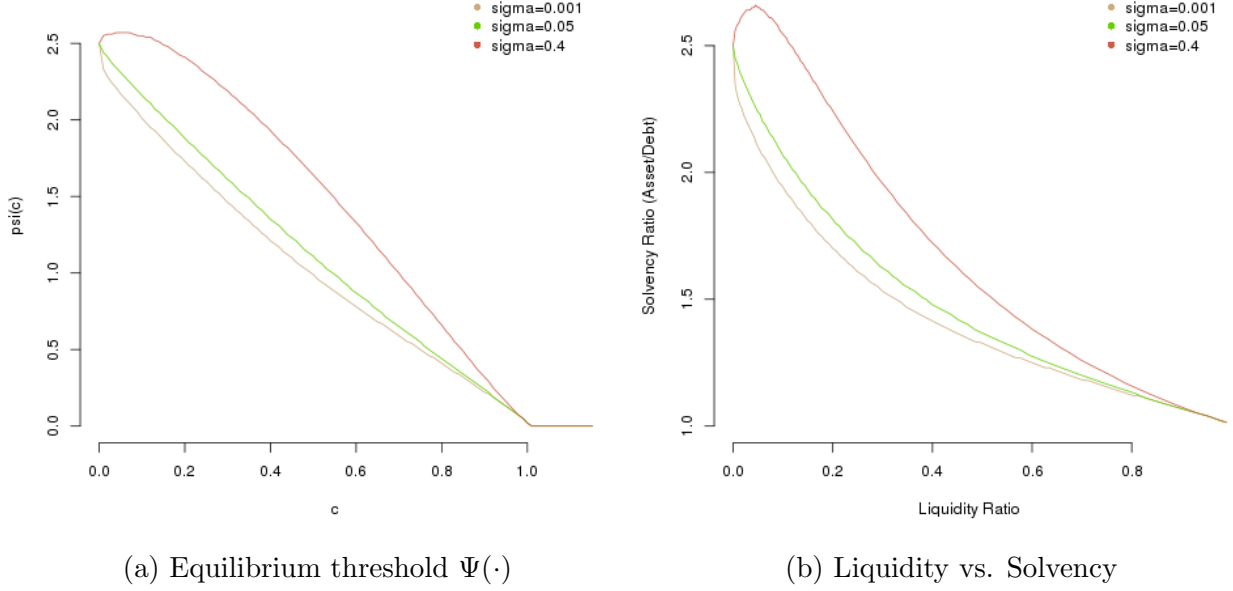
Figure 15: 5yr CDS during Crisis Times



percentage of the balance-sheet needed to be invested in cash in order to deter a run. At the base case parameter values selected, the equilibrium run boundary $\partial\mathcal{R}$ increases as the illiquid asset volatility increases. This result is intuitive: for a given creditor, with higher illiquid asset volatility, there is a higher chance that a sequence of bad shocks occurs between two debt claim maturities, making such creditor more conservative in his run/roll strategy.

I do note however that the equilibrium run boundary $\partial\mathcal{R}$ is no longer monotone in c . For high volatility values σ and small values of the liquidity reserve c , the equilibrium run boundary $\partial\mathcal{R}$ first increases with c , before decreasing. This suggests that the value function v^* is not monotone in c for these parameter configurations. The intuition behind this surprising result is as follows. On the vertical axis $c = 0$, the debt value function is a concave function of p . At the point $(c, p) = (1/\alpha, 0)$, the debt value is exactly equal to 1. However, if at time t the state is $(c, p) = (1/\alpha, \epsilon)$, the game continues at least for a small time period $dt = \epsilon/|\text{drift}(c)|$. If the drift at that point of the state space is negative (which would be the case if creditors are running), then the state at time $t + dt$ will be $(1/\alpha + dp, 0)$, where dp is normally distributed with mean $\frac{\mu}{\alpha}dt$ and variance $\frac{\sigma^2}{\alpha^2}dt$. Given the concavity of $v^*(\cdot, 0)$, it is clear that for sufficiently high values of σ , Jensen's inequality will be such that $v^*(1/\alpha, \epsilon) < v^*(1/\alpha, 0)$, leading to a value function v^* that can be decreasing in c for values of p close to $1/\alpha$ and values of c close to zero.

Figure 16: Sensitivity to σ



6.2.5 Sensitivity to the Average Debt Maturity $1/\lambda$

I then analyze the sensitivity of the threshold strategy to the firm's debt maturity structure. For parameter values $\lambda = 0.25$, $\lambda = 1$ and $\lambda = 4$, corresponding to firm's debt average lives of 4 years, 1 year and 3 months respectively, Figure 17a shows the endogenous run boundary $\partial\mathcal{R}$, while Figure 17b shows the trade-off between firm's leverage and the percentage of the balance-sheet needed to be invested in cash in order to deter a run. At the base case parameter values selected, the run boundary $\partial\mathcal{R}$ is surprisingly insensitive to the term structure of the firm's liabilities. To gain some more intuition behind this result, I increase the base case volatility level from 15% to 30% and recompute the sensitivity analysis w.r.t. λ .

Figure 18 shows the result of this new analysis. It now appears that the longer the firm's average life, the sooner creditors run. This result seems to contradict the traditional belief that a firm with longer term debt is less run-prone than a firm with shorter term debt. This counterintuitive result, first highlighted by He and Xiong (2012), can be explained as follows. A smaller value for λ means a longer debt average life. Since the drift of the cash reserve depends on the term $-\lambda 1_{\{S(c,p)=1\}}(1-c)$, and since the values of interest for c are values $c < 1$, a lower value for λ leads to a lower downward pressure on the cash reserve, and therefore a longer period of time needed for the firm to go bankrupt. But a lower value of λ also means that a given creditor will need to wait a longer period of time between two roll-over decisions. If the volatility of the illiquid asset is high enough, a longer time period

Figure 17: Sensitivity to λ when $\sigma = 15\%$

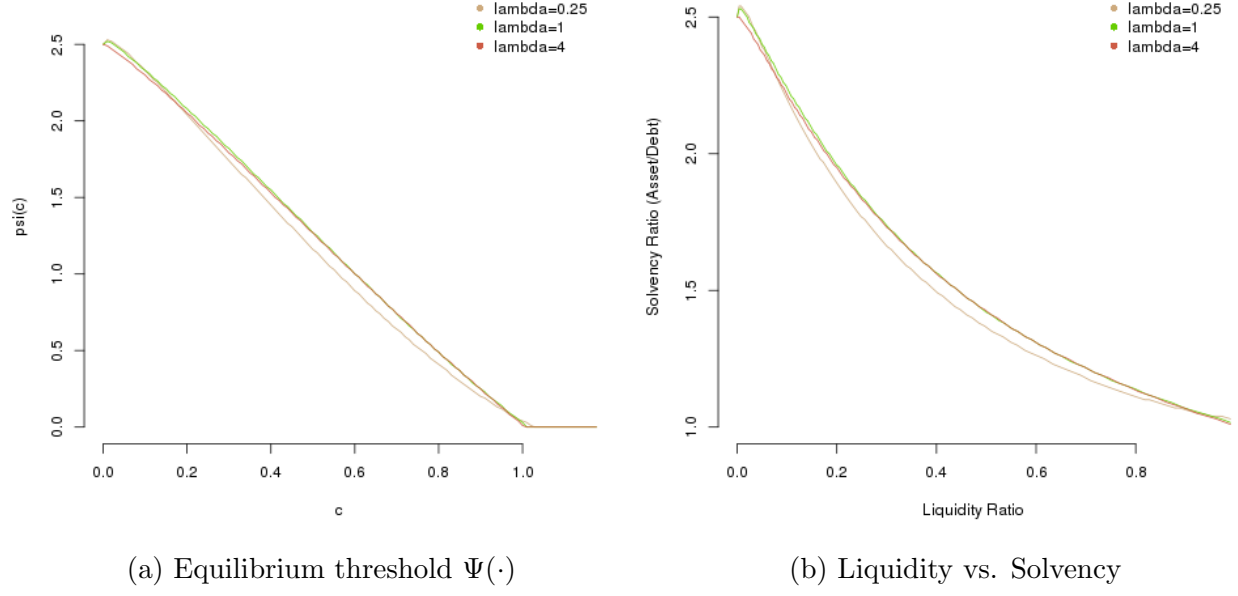
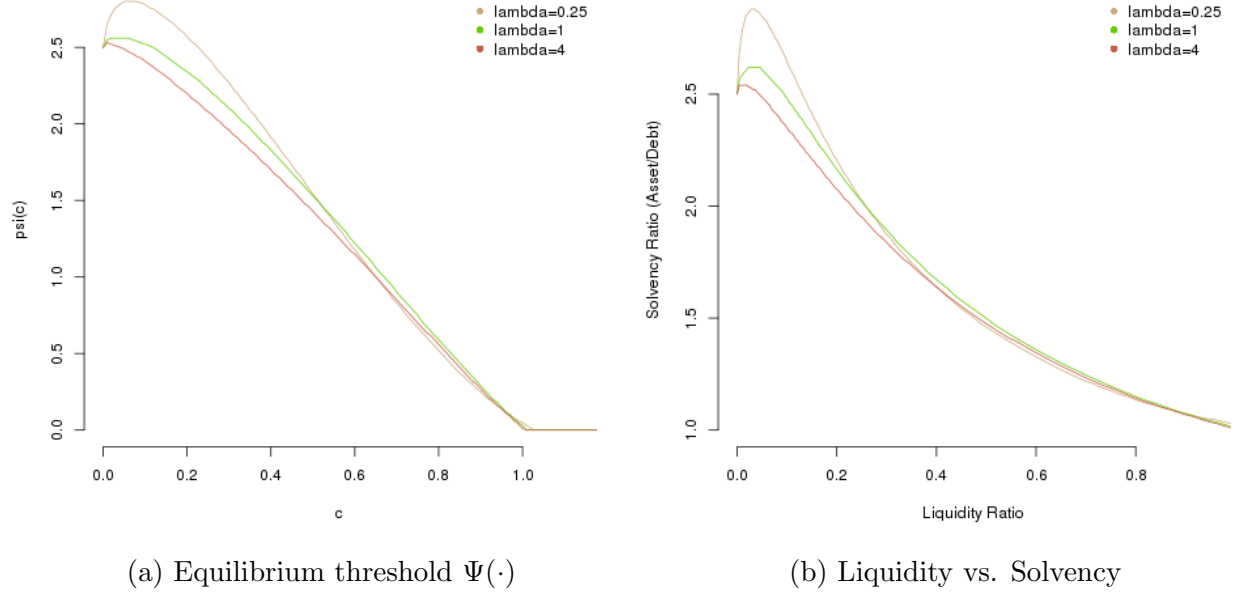


Figure 18: Sensitivity to λ when $\sigma = 30\%$



between two roll-over decisions means a greater probability that a bad sequence of shocks occur, making creditors potentially more conservative in their decision to roll-over or run. In the example of [Figure 18](#), the volatility of the illiquid asset is sufficiently high for the second effect to dominate the first, leading to the counterintuitive result that creditors run sooner when the firm's debt average life is longer.

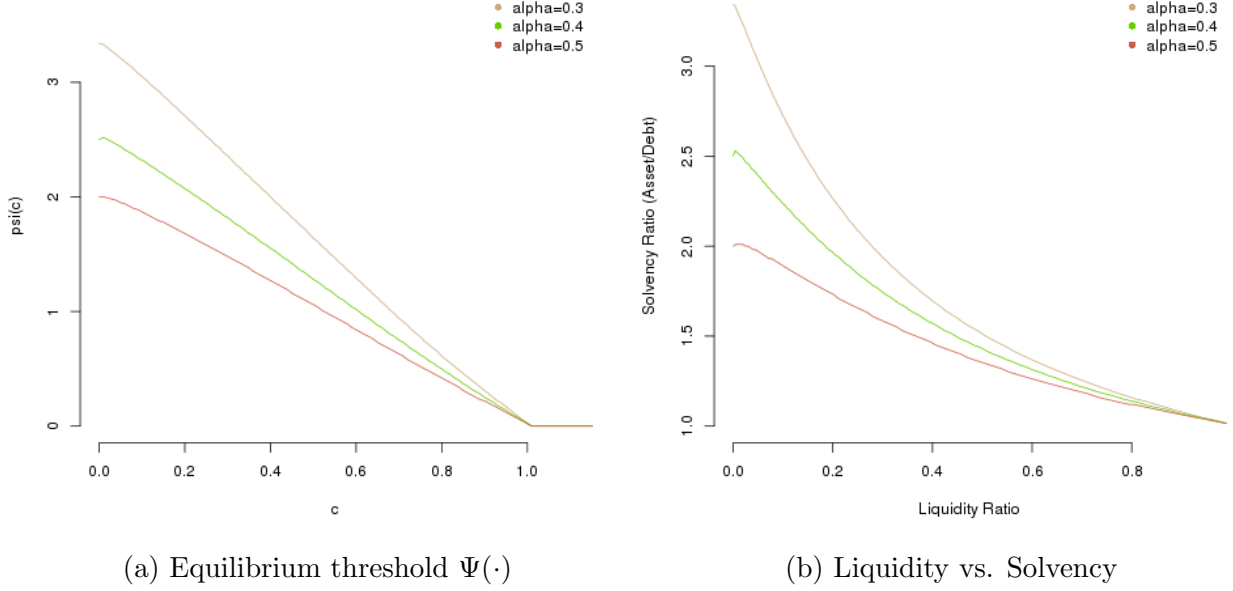
The trade-off uncovered above is relatively different from the traditional trade-off in the corporate finance literature discussing costs and benefits of longer vs. shorter debt maturity choices (see for example [He and Milbradt \(2014\)](#)): in this latter literature, short term debt is less sensitive to the firm's asset value than long term debt, but short term debt needs to be rolled-over more frequently. Those two effects lead to roll-over losses in bad times, hurting shareholder current cashflows, and making the ex-ante choice of optimal debt maturity non-trivial. Instead, in the model developed in this paper, longer term debt reduces the rate of cash depletion conditioned on the occurrence of a run, but makes creditors roll-over decisions more temporally distant, leading to a different ex-ante trade-off.

Finally, instead of modeling the maturity of a creditor's debt contract as a Poisson arrival process, I could have elected to model the debt contract as a bullet bond with initial maturity T , as in [Leland and Toft \(1996\)](#). This choice would have led to an additional state variable for the creditor's problem: the time to maturity of the creditor's debt contract. However, I suspect that this alternative modeling strategy might attenuate if not entirely remove the counterintuitive result described above: a shorter average debt maturity T leads to greater liquidity pressure on the firm once it suffers a run, while a given creditor with a remaining maturity t will not benefit from greater arrival intensity of his debt maturity.

6.2.6 Sensitivity to the Illiquid Asset Liquidation Value α

I continue my sensitivity analysis by looking at the role of the firm's illiquid asset recovery rate α . For parameter values $\alpha = 0.15$, $\alpha = 0.3$ and $\alpha = 0.45$, [Figure 19a](#) shows the endogenous run boundary $\partial\mathcal{R}$, while [Figure 19b](#) shows the trade-off between firm's leverage and the percentage of the balance-sheet needed to be invested in cash in order to deter a run. The recovery rate α plays a crucial role in the determination of the run boundary $\partial\mathcal{R}$: as α increases, the propensity of creditors to run decreases. This phenomenon was to be expected, in light of [Proposition 7](#): the run boundary $\partial\mathcal{R}$ cuts the axis $c = 0$ at $p = \frac{1}{\alpha}$. Therefore, the parameter α turns out to be a critical driver of the value function close to the boundary $c = 0$.

Figure 19: Sensitivity to α



7 Externality

As pointed in [He and Xiong \(2012\)](#), maturing creditors who are electing to withdraw their funds from the firm increase the probability that the firm runs out of liquid resources and defaults, thus imposing an externality on remaining creditors. In this section, I discuss how to quantify the loss in value, for both creditors and shareholders, due to this externality. In other words, I discuss how to compute the debt value and the shareholder value of a firm that is still subject to run risk, but for which the externality imposed by running creditors on remaining creditors is absent. In order to achieve this, I change the debt ownership assumption of the base case model: instead of having a continuum of creditors invested in the debt issued by the firm, I now assume that a unique large creditor owns the entire debt stack of the firm. This large creditor controls whether to reinvest the maturing debt into new debt issued by the firm, or whether to withdraw funding. More specifically, in the infinitesimal time interval $[t, t + dt]$, an amount $\lambda D(t)dt$ of debt is maturing. The large creditor controls the fraction $\eta(t) \in [0, 1]$ of maturing debt that he decides not to roll over; $1 - \eta(t)$ thus represents the fraction of maturing debt reinvested into the firm's debt.

I first study the problem faced by the large creditor. As usual, $\tau_b = \inf\{t : C(t) = 0, (\rho - \mu)P(t) < (r_d + \lambda\eta(t))D(t)\}$ is the firm default time (potentially infinite), and τ_ϕ is the maturity date of the illiquid investment. Let $\tau := \tau_\phi \wedge \tau_b$ be the earlier of (a) the illiquid

asset maturity date, and (b) the firm's default. For a given adapted process $\{\eta(t)\}_{t \geq 0}$ taking values in the interval $[0, 1]$, the large creditor's payoff function is equal to:

$$W(P, D, C; \eta) = \mathbb{E}^{P, D, C} \left[\int_0^\tau e^{-\rho t} r_d D(t) dt + \int_0^\tau e^{-\rho t} \lambda \eta(t) D(t) dt \right. \\ \left. + e^{-\rho \tau} 1_{\{\tau = \tau_\phi\}} \min(D(\tau), P(\tau) + C(\tau)) + e^{-\rho \tau} 1_{\{\tau = \tau_b\}} \min(D(\tau), \alpha P(\tau)) \right]$$

The notation $W(\cdot, \cdot, \cdot; \eta)$ highlights the fact that the payoff function depends on the roll-over strategy η employed by the large creditor. In the expression above, the first integral represents interest collections on outstanding debt and the second integral represents principal proceeds received on maturing debt that is not rolled. Upon the stopping time τ_ϕ , the large creditor receives the greater of (a) his outstanding debt holding $D(\tau_\phi)$ and (b) the fundamental value of the firm's assets $P(\tau_\phi) + C(\tau_\phi)$. Finally, upon the stopping time τ_b , the large creditor receives the greater of (a) his outstanding debt holding $D(\tau_b)$ and (b) the liquidation value of the firm's illiquid investment $\alpha P(\tau_b)$. The debt $D(t)$ evolves according to $dD(t) = -\lambda \eta(t) D(t) dt$, which can also be written:

$$D(t) = D e^{-\lambda \int_0^t \eta(s) ds}$$

Using this expression, I can re-write the payoff function $W(\cdot, \cdot, \cdot; \eta)$ as follows:

$$W(P, D, C; \eta) = D \mathbb{E}^{P, D, C} \left[\int_0^\tau e^{-\rho t} r_d e^{-\lambda \int_0^t \eta(s) ds} dt + \int_0^\tau e^{-\rho t} \lambda \eta(t) e^{-\lambda \int_0^t \eta(s) ds} dt \right. \\ \left. + e^{-\rho \tau} 1_{\{\tau = \tau_\phi\}} e^{-\lambda \int_0^\tau \eta(s) ds} \min \left(1, \frac{P(\tau) + C(\tau)}{D(\tau)} \right) \right. \\ \left. + e^{-\rho \tau} 1_{\{\tau = \tau_b\}} e^{-\lambda \int_0^\tau \eta(s) ds} \min \left(1, \alpha \frac{P(\tau)}{D(\tau)} \right) \right]$$

Given the laws of motion for the state variables $\{D(t)\}_{t \geq 0}$, $\{P(t)\}_{t \geq 0}$ and $\{C(t)\}_{t \geq 0}$, it is thus clear that the value function $W(\cdot, \cdot, \cdot; \eta)$ is homogeneous of degree 1 in (P, D, C) , and can be expressed as:

$$W(P, D, C; \eta) = D w(c, p; \eta)$$

For a given adapted process $\{\eta(t)\}_{t \geq 0}$ in $[0, 1]$, the function $w(\cdot, \cdot; \eta)$ is equal to:

$$w(c, p; \eta) = \mathbb{E}^{p, c} \left[\int_0^\tau (r_d + \lambda \eta(t)) e^{-\int_0^t (\rho + \lambda \eta(s)) ds} dt \right. \\ \left. + 1_{\{\tau = \tau_\phi\}} e^{-\int_0^\tau (\rho + \lambda \eta(s)) ds} \min(1, p(\tau) + c(\tau)) + 1_{\{\tau = \tau_b\}} e^{-\int_0^\tau (\rho + \lambda \eta(s)) ds} \min(1, \alpha p(\tau)) \right]$$

The state variables $\{p(t)\}_{t \geq 0}$ and $\{c(t)\}_{t \geq 0}$ evolve according to:

$$dp(t) = (\mu + \lambda \eta(t)) p(t) dt + \sigma p(t) dB(t) \\ dc(t) = ((\rho - \mu)p(t) + (r_c + \lambda \eta(t))c(t) - (r_d + \lambda \eta(t))) dt$$

The firm's large creditor maximizes the function $w(\cdot, \cdot; \eta)$ over all possible adapted processes $\{\eta(t)\}_{t \geq 0}$ in $[0, 1]$. A standard verification theorem (see for example [Fleming and Soner \(2006\)](#)) can show that the optimal value function $w(c, p) := \arg \max_\eta w(c, p; \eta)$ is solution to the following Hamilton-Jacobi-Bellman equation, for $c > 0$:

$$0 = \max_{\eta \in [0, 1]} \left[-(\rho + \lambda \eta) w + r_d + \lambda \eta + (\mu + \lambda \eta) p \frac{\partial w}{\partial p} + \frac{1}{2} \sigma^2 p^2 \frac{\partial^2 w}{\partial p^2} \right. \\ \left. + [(\rho - \mu)p + (r_c + \lambda \eta)c - (r_d + \lambda \eta)] \frac{\partial w}{\partial c} + \phi(\min(1, p + c) - w) \right]$$

This is a bang-bang control problem, with an optimal roll-over policy as follows:

$$\eta(c, p) = \begin{cases} 0 & \text{if } w(c, p) - 1 > p \frac{\partial w}{\partial p}(c, p) + (c - 1) \frac{\partial w}{\partial c}(c, p) \\ 1 & \text{if } w(c, p) - 1 < p \frac{\partial w}{\partial p}(c, p) + (c - 1) \frac{\partial w}{\partial c}(c, p) \end{cases} \quad (21)$$

The boundary $\{(c, p) : w(c, p) - 1 = p \frac{\partial w}{\partial p}(c, p) + (c - 1) \frac{\partial w}{\partial c}(c, p)\}$ is the set of points in the state space such that the single large creditor is indifferent between rolling over his debt claim or running. The large creditor's optimal decision can then be interpreted as follows. At each time t , a quantity of debt λdt comes due. The large creditor's flow benefit of reinvesting into the firm's debt is equal to $(w(c, p) - 1) \lambda dt$. Reinvesting into the firm's debt has a marginal cost, equal to the difference between the flow capital gains when running and the flow capital gains when rolling, which is equal to $\left(p \frac{\partial w}{\partial p}(c, p) + (c - 1) \frac{\partial w}{\partial c}(c, p) \right) \lambda dt$. The reinvestment condition uncovered in [equation \(21\)](#) illustrates the fact that the large creditor internalizes the effect of his decision to run or roll onto the dynamics of the state variables. This is the key difference between the model without creditor externality and the previously-studied model, in which creditor's roll-over decisions are driven by $v(c, p) - 1 \leq 0$.

7.1 Large Creditor's Behavior when $p = 0$

I now derive the value function w in the special case where the illiquid asset cash flow – and thus the illiquid asset fundamental value – is zero. Similar to [Section 4.8](#), this problem is deterministic with perfect foresight for the large creditor, who can predict perfectly the evolution of the state variable $c(t)$. Since c is the only state variable, I postulate a threshold $\hat{c}$ above which it will be optimal for the large creditor to continue rolling its debt claim, and below which it will be optimal to run. In what follows, I prove that the threshold $\hat{c}$ is unique, and that it verifies $\hat{c} > c^*$, where c^* is the corresponding threshold in the equilibrium with the run externality. I will note $w_0(c) := w(0, c)$ the *optimal* value function of the large creditor of a game with no illiquid asset – in other words, the value function of the large creditor following the optimal cutoff strategy.

PROPOSITION 11. In an economy without illiquid asset and with a unique large creditor, the creditor value function w_0 is equal to:

$$w_0(c) = \begin{cases} \left(\frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \lambda + \phi)(\phi + \rho - r_c)} \right) \left[1 - \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c \right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} \right] + \frac{\phi}{\phi + \rho - r_c} c & 0 \leq c < 1 \\ H_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c \right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} + \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} & 1 \leq c < \hat{c} \\ H'_2 \left(1 - \frac{r_c}{r_d} c \right)^{\frac{\rho + \phi}{r_c}} + \frac{r_d + \phi}{\rho + \phi} & \hat{c} \leq c < \frac{r_d}{r_c} \\ \frac{r_d + \phi}{\rho + \phi} & c \geq \frac{r_d}{r_c} \end{cases} \quad (22)$$

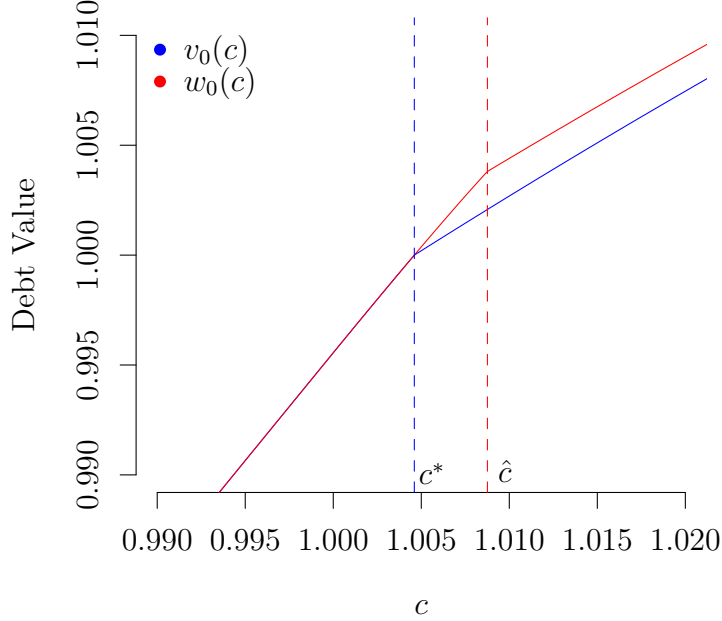
$\hat{c}$ defines the unique threshold equilibrium of this economy, and it satisfies:

$$\begin{aligned} w_0(c) - 1 &< (c - 1)w'_0(c) \quad \forall c < \hat{c} \\ w_0(c) - 1 &> (c - 1)w'_0(c) \quad \forall c > \hat{c} \end{aligned}$$

The implicit equation satisfied by $\hat{c}$ is derived in the appendix, and the following inequality holds: $\hat{c} > c^*$. The value function w_0 is strictly increasing for $c < \frac{r_d}{r_c}$, and constant for $c > \frac{r_d}{r_c}$. H_1 and H'_2 are constants described in the appendix.

I prove [Proposition 11](#) in the appendix. In [Figure 20](#), I plot v_0 and w_0 , the debt values for $p = 0$ with and without the run externality. The figure illustrates the fact that the no-externality debt value w_0 is weakly greater than the debt value v_0 , and that the large creditor runs “earlier” (i.e. when the cash level per unit of debt outstanding is greater) than the continuum of atomistic creditors. This latter fact might be surprising at first – one might have expected that a setup without externality leads to a more “stable” firm – i.e. a firm where creditors have a lower propensity to run. The result $\hat{c} > c^*$ stems from the fact that the

Figure 20: Debt Values $v_0(\cdot)$ and $w_0(\cdot)$



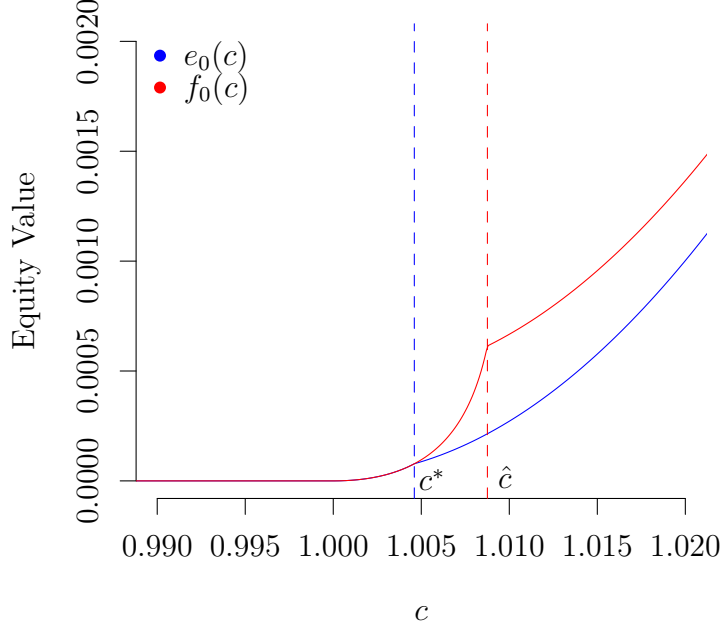
region of the state space where c^* is located is a region that exhibits strategic substitutability: the “later” other creditors run, the “earlier” a given creditor will want to run. Assuming a unique large creditor removes entirely the effect of strategic substitutability, leading to the inequality $\hat{c} > c^*$. Another way to see this result is to remember that when $p = 0$, the firm is keeping cash yielding r_c , while creditors and shareholders discount cashflows at rate $\rho > r_c$. Thus both creditors and shareholders would rather liquidate the firm and distribute the available cash to its different claimants immediately, as opposed to storing such cash inside the firm.

Figure 21 shows the equity value e_0 with externality, as well as the equity value f_0 without. Once again, shareholders of the firm with the large creditor are better off than shareholders of the firm financed by a continuum of atomistic creditors, for reasons similar to those mentioned previously.

8 Debt Maturity Choice

In this section, I discuss an extension of the model that is applicable to sovereign debt. Several articles [to be cited] within the sovereign debt literature have documented a shortening of

Figure 21: Equity Values $e_0(\cdot)$ and $f_0(\cdot)$



the duration of sovereign bonds when fundamentals of the country deteriorate. In a sense, when a country's performance worsens, creditors act conservatively, giving up spread income for the safety of a shorter investment. In my model, I introduce creditors' maturity choice as follows: creditors will choose the intensity λ at which their debt claim matures, in exchange for giving up some spread income, modeled as an exogenous function $r_d(\lambda)$ that is decreasing in the intensity controlled by the creditor. I am assuming that λ can be chosen on a compact set $[\underline{\lambda}, \bar{\lambda}]$, and that over the interval, the function $r_d(\cdot)$ is differentiable and strictly decreasing. A strategy for a given creditor is now a mapping $s : \mathbb{R}_+^2 \rightarrow [\underline{\lambda}, \bar{\lambda}] \times \{0, 1\}$ which specifies, at each point (c, p) of the state space, (i) the maturity intensity λ of the creditor's debt claim, and (ii) whether or not such creditor decides to roll-over if he gets the opportunity to do so. Similar to what I prove in [Proposition 4](#), for a given strategy S followed by all other creditors, there exists a unique function v^* that solves the functional equation:

$$v^*(c, p; S) = \max_{\{\lambda(t)\}_{t \leq \tau}} \left\{ \mathbb{E}^{p, c} \left[\int_0^\tau e^{-\rho t} r_d dt + e^{-\rho \tau} 1_{\{\tau = \tau_\lambda\}} \max(1, v^*(p(\tau), c(\tau); S)) \right] \right. \\ \left. + \mathbb{E}^{p, c} \left[e^{-\rho \tau} 1_{\{\tau = \tau_\phi\}} \min(1, p(\tau) + c(\tau)) + e^{-\rho \tau} 1_{\{\tau = \tau_b\}} \min(1, \alpha p(\tau)) \right] \right\}$$

Where τ_λ has arrival intensity $\lambda(t)$ and where $\tau = \tau_\lambda \wedge \tau_\phi \wedge \tau_b$. The Hamilton-Jacobi-Bellman equation corresponding to the creditor's problem is as follows:

$$0 = \max_{\lambda \in [\underline{\lambda}, \bar{\lambda}]} [-(\rho + \phi) v^* + r_d(\lambda) + \phi \min(1, p + c) + \lambda \max(0, 1 - v^*) + (\mathcal{L}_S^p + \mathcal{L}^{pp} + \mathcal{L}_S^c) v^*] \quad (23)$$

Note that the equation above assumes that a given creditor takes as given the strategy S followed by all other creditors – in other words, it takes as given the maturity intensity chosen by all other creditors, which drives the dynamics of the aggregate state variables $c(t)$ and $p(t)$. Thus, a given creditor does not internalize its maturity intensity choice on the dynamics of the state variables. This leads to the following optimal behavior for a given creditor:

$$\lambda(c, p) = \begin{cases} \underline{\lambda} & \text{if } v^*(c, p; S) \geq 1 \\ \max(\bar{\lambda}, (r'_d)^{-1} (1 - v^*(c, p; S))) & \text{if } v^*(c, p; S) < 1 \end{cases} \quad (24)$$

9 Dividends and Portfolio Choice

While a full analysis of the optimal dividend and portfolio choice is beyond the scope of this paper, I discuss in this section the incentives for the firm's management (assumed to be acting on behalf of shareholders) to pay dividends or reinvest the available cash into illiquid assets.

9.1 The Incentive to Pay Dividends

A dividend payment policy for shareholders can be defined as a non-decreasing adapted process $U(t)$, satisfying $U(0) = 0$, representing the cumulative dollar amount of dividends paid, per unit of debt outstanding at the time of payment. For a given run strategy S employed by all creditors, and a given dividend payment strategy $U(t)$ employed by the firm's management (acting on behalf of shareholders), it can be showed that the creditor's payoff function and the equity value function continue to exhibit the homogeneity property established in [Lemma 2](#). The normalized state variables then evolve according to:

$$dp(t) = (\mu + \lambda 1_{\{S(c(t), p(t))=1\}}) p(t) dt + \sigma p(t) dB^\mathbb{Q}(t) \quad (25)$$

$$dc(t) = ((\rho - \mu)p(t) + (r_c + \lambda 1_{\{S(c(t), p(t))=1\}})c(t) - (r_d + \lambda 1_{\{S(c(t), p(t))=1\}})) dt - dU(t) \quad (26)$$

A dividend policy $U(t)$ is admissible if $\Pr((p(t), c(t)) \in \mathbb{R}_+^2 \forall t \geq 0) = 1$. The optimization problem solved by equity holders takes as given creditors' behavior, as encoded by the strategy

S :

$$e(c, p; S) = \sup_U \mathbb{E}^{p,c} \left[\int_0^{\tau_b \wedge \tau_\phi} e^{-\rho t} dU(t) + e^{-\rho(\tau_b \wedge \tau_\phi)} \left[1_{\{\tau_\phi < \tau_b\}} \max(0, p(\tau_\phi) + c(\tau_\phi) - 1) \right. \right. \\ \left. \left. + 1_{\{\tau_b < \tau_\phi\}} \max(0, \alpha p(\tau_b) - 1) \right] \right] \quad (27)$$

Noting $\lambda(c, p; S) = \lambda 1_{\{S(c,p)=1\}}$, standard arguments (see for example [Fleming and Soner \(2006\)](#)) can show that the equity value function is solution to the following Hamilton-Jacobi-Bellman equation:

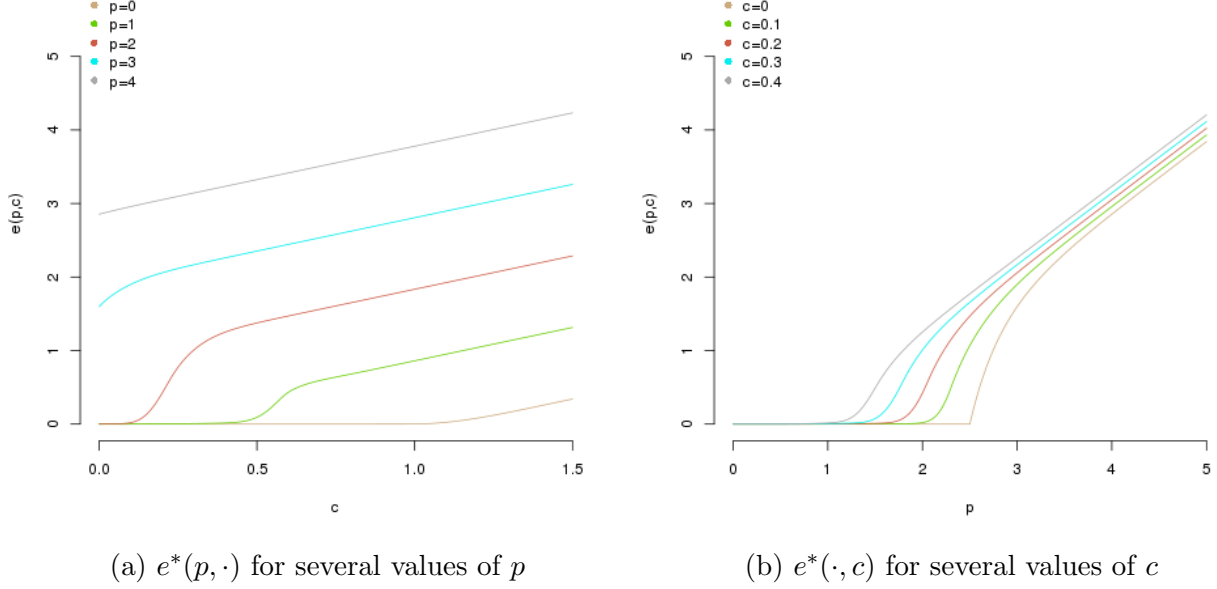
$$0 = \max \left[-(\rho + \phi)e(c, p; S) + \phi \max(0, p + c - 1) + (\mathcal{L}_S^p + \mathcal{L}^{pp} + \mathcal{L}_S^c) e(c, p; S), 1 - \frac{\partial e}{\partial c}(c, p; S) \right]$$

In the region of the state space where the marginal value of cash is strictly above unity, the firm's management finds it optimal to store such cash internally. Instead, at the points of the state space where the marginal value of cash is less than or equal to unity, the firm pays dividends to shareholders. As an example, taking as fixed a strategy S followed by the firm's creditors and which respects the dominance regions established in [Proposition 3](#), [Lemma 3](#) suggests that the equity marginal value of cash converges to $\frac{\phi}{\rho + \phi - r_c} < 1$ as p or c gets large. In other words for large values of p or c the firm has an incentive to pay dividends. This discussion is not completely rigorous, since it does not take into account the fact that creditors will change their behavior, once the firm adopts a dividend payment rule. However, it supports the idea that any equilibrium of the new economy will feature the standard endogeneous run boundary characterized by the mapping S , as well as a new endogeneous reflecting boundary $\{(c, p) : \frac{\partial e}{\partial c}(c, p) = 1\}$ beyond which the firm pays dividends to its shareholders. When defining an equilibrium for this new game, I will need to take into account the fact that (a) creditors are choosing their optimal run behavior taking as given the firm's dividend policy, and (b) the firm is choosing its optimal dividend policy taking into account the run behavior of creditors. An equilibrium in the economy where the firm can pay dividends can be defined as follows:

DEFINITION 3. A symmetric Markov perfect equilibrium of the game with dividend choice is (a) a mapping $S : \mathbb{R}_+^2 \rightarrow \{0, 1\}$ representing the run behavior of creditors, and (b) a dividend payment region $\mathcal{D} \subset \mathbb{R}_+^2$ representing the region of the state space where the firm is using excess cash to pay dividends, and such that:

- i. for any $(c, p) \in \mathbb{R}_+^2$, $v(c, p; S, S, \mathcal{D}) = \sup_s v(c, p; s, S, \mathcal{D})$;
- ii. for any $(c, p) \notin \mathcal{D}$, $\frac{\partial e}{\partial c}(c, p; S, \mathcal{D}) > 1$;

Figure 22: Value function e^*



iii. for any $(c, p) \in \partial \mathcal{D}$, $\frac{\partial e}{\partial c}(c, p; S, \mathcal{D}) = 1$.

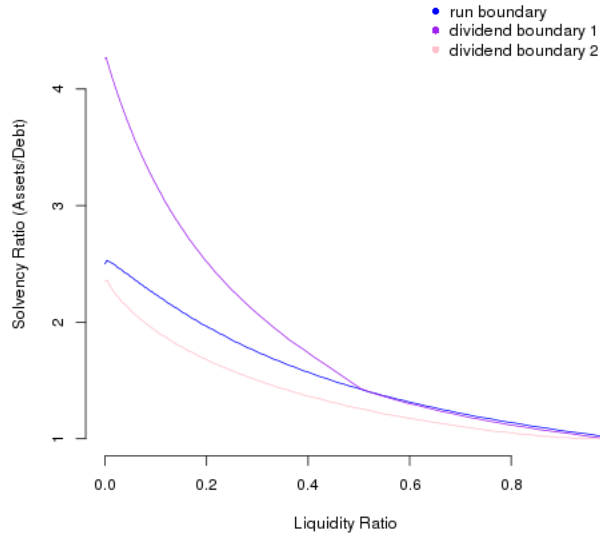
iv. for any $(c, p) \in \mathcal{D}$, there exists $h > 0$ defined via $(c - h, p) \in \partial \mathcal{D}$ and such that $e(c, p; S, \mathcal{D}) = e(c - h, p; S, \mathcal{D}) + h$ and $v(c, p; S, \mathcal{D}) = v(c - h, p; S, \mathcal{D})$.

Studying such equilibria (if any of them exists) is beyond the scope of this paper. However, fixing the creditors' behavior, I can study the firm's incentive to pay dividends, as given by the condition $\frac{\partial e}{\partial p}(c, p) \leq 1$, where the equity value is taken to be the equity value in the game without dividend payments, and where creditors' run behavior is taken to be the equilibrium run behavior for a game without dividend payment. To begin with, I plot in [Figure 22](#) the equity value function for different levels of price p and cash c (per unit of debt outstanding), for the base case parameters in [Table 1](#).

I then plot in the liquidity/solvency space (Λ, Σ) the run boundary as well as two new boundaries, representing points at which the equity marginal value of cash is equal to 1 ([Figure 23](#)). The region of the state space above the purple dividend boundary ("dividend boundary 1" in [Figure 23](#)) is a region where $\frac{\partial e}{\partial c} < 1$, meaning that management's incentive is to make dividend payments. In such region, the firm is highly liquid and highly solvent, such that the contribution of one incremental dollar to the cash reserve is strictly less than 1. Similarly, the region of the state space below the pink dividend boundary ("dividend boundary 2" in [Figure 23](#)) is a region where $\frac{\partial e}{\partial c} < 1$. In that region of the state space, the firm

is suffering a run, and there is almost no hope for shareholders to see the illiquid asset value recover fast enough to improve solvency in order to exit the run region. In such case, the firm's management has an incentive to pay out as dividends the entire firm's cash holdings before such cash ends up paid out to running creditors. Those dividend payment incentive regions are of course plotted taking as given the equilibrium behavior of creditors. In any equilibrium satisfying [Definition 3](#), creditors' behavior would then adjust to such dividend payment policy by running "sooner", i.e. not rolling over in regions of the state space where they would have rolled over, had dividend payments not been allowed. Given this feedback effect, it is entirely possible that under certain parameter configurations, no equilibrium might exist for the game studied above. Under parameter configurations where an equilibrium satisfying [Definition 3](#) exists, the creditor's run behavior must be more conservative than under a framework where no dividend payment is allowed. In such case, a properly designed regulation preventing dividend payments when liquidity and/or solvency ratios are too low would reduce creditors' incentives to run.

Figure 23: Dividend Boundaries



9.2 The Incentive to Reinvest

In this section, I analyze the portfolio choice problem facing the management of the firm, who is acting on behalf of the firm's shareholders. A portfolio policy for shareholders is now a non-decreasing adapted process $L(t)$, satisfying $L(0) = 0$, representing the cumulative dollar amount of additional illiquid assets purchased, per unit of debt outstanding at the time of

purchase. For a given run strategy S employed by all creditors, and a given portfolio strategy $L(t)$ employed by shareholders, the state variables evolve according to:

$$dp(t) = (\mu + \lambda 1_{\{S(c(t), p(t))=1\}}) p(t) dt + \sigma p(t) dB(t) + dL(t) \quad (28)$$

$$dc(t) = ((\rho - \mu)p(t) + (r_c + \lambda 1_{\{S(c(t), p(t))=1\}})c(t) - (r_d + \lambda 1_{\{S(c(t), p(t))=1\}})) dt - dL(t) \quad (29)$$

A portfolio policy $L(t)$ is admissible if $\Pr((p(t), c(t)) \in \mathbb{R}_+^2 \forall t \geq 0) = 1$. The optimization problem solved by equity holders takes as given creditors' behavior, as encoded by the strategy S :

$$e(c, p; S) = \sup_L \mathbb{E}^{p, c} \left[e^{-\rho(\tau_b \wedge \tau_\phi)} \left[1_{\{\tau_\phi < \tau_b\}} \max(0, p(\tau_\phi) + c(\tau_\phi) - 1) + 1_{\{\tau_b < \tau_\phi\}} \max(0, \alpha p(\tau_b) - 1) \right] \right] \quad (30)$$

Noting $\lambda(c, p) = \lambda 1_{\{S(c, p)=1\}}$, the corresponding HJB equation is:

$$0 = \max \left[-(\rho + \phi)e(c, p; S) + \phi \max(0, p + c - 1) + (\mathcal{L}_S^p + \mathcal{L}^{pp} + \mathcal{L}_S^c) e(c, p; S), \frac{\partial e}{\partial p}(c, p; S) - \frac{\partial e}{\partial c}(c, p; S) \right]$$

As expected, the optimal portfolio choice problem is a bang bang control problem, featuring two regions of the state space: a region where the firm accumulates cash reserves (and characterized by $\partial_p e(c, p; S) < \partial_c e(c, p; S)$), and a region where the firm uses its cash reserve to purchase the illiquid asset. At the boundary of the two regions, one extra unit of cash yields the same additional value to shareholders than one extra unit of the illiquid asset: $\frac{\partial e}{\partial p}(c, p; S) = \frac{\partial e}{\partial c}(c, p; S)$.

I note also that the equity value $e(c, p; S)$ for high values of p or c calculated when the firm does not solve any portfolio choice problem indicates that the marginal condition for the firm to want to invest into the illiquid asset is satisfied. Indeed, when p or c is large, I established in [Lemma 3](#) that:

$$e(c, p; S) = \frac{\phi}{\rho + \phi - r_c} c + \frac{\phi}{\rho + \phi - \mu} \left(1 + \frac{\rho - \mu}{\rho + \phi - r_c} \right) p - \frac{\phi}{\rho + \phi} \left(\frac{r_d}{\rho + \phi - r_c} + 1 \right) + o(1)$$

Thus, when p or c is large, I have:

$$\begin{aligned}\frac{\partial e}{\partial c}(c, p; S) &= \frac{\phi}{\rho + \phi - r_c} + o(1) \\ \frac{\partial e}{\partial p}(c, p; S) &= \frac{\phi}{\rho + \phi - \mu} \left(1 + \frac{\rho - \mu}{\rho + \phi - r_c} \right) + o(1) \\ &= \frac{\phi}{\rho + \phi - r_c} + \frac{\phi}{\rho + \phi - \mu} \left(1 - \frac{\phi}{\rho + \phi - r_c} \right) + o(1) > \frac{\phi}{\rho + \phi - r_c}\end{aligned}$$

Thus, for large values of p or c , $\frac{\partial e}{\partial p}(c, p; S) > \frac{\partial e}{\partial c}(c, p; S)$. The discussion above supports the idea that any equilibrium of our new economy will feature the standard endogenous run boundary characterized by the mapping S , as well as a new endogenous reflecting boundary $\{(c, p) : \frac{\partial e}{\partial p}(c, p; S) = \frac{\partial e}{\partial c}(c, p; S)\}$ beyond which the firm invests its cash into the illiquid asset. I am now ready to define an equilibrium in the economy where the firm can invest into the illiquid asset.

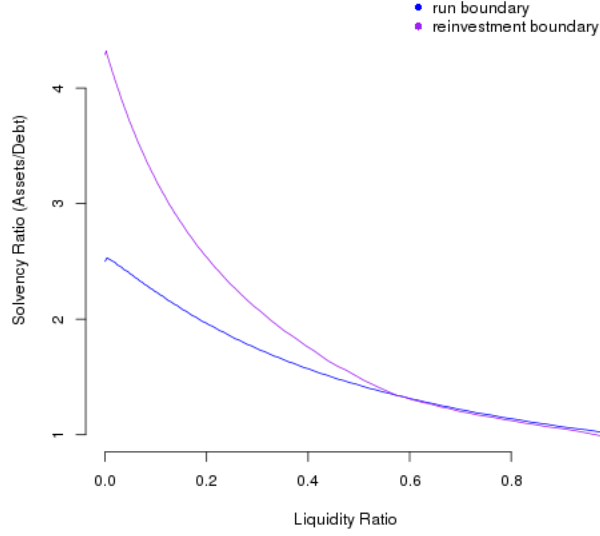
DEFINITION 4. A symmetric Markov perfect equilibrium of the game with portfolio choice is (a) a mapping $S : \mathbb{R}_+^2 \rightarrow \{0, 1\}$ representing the run behavior of creditors, and (b) an investment region $\mathcal{I} \subset \mathbb{R}_+^2$ representing the region of the state space where the firm is using excess cash to purchase the illiquid asset, and such that:

- i. for any $(c, p) \in \mathbb{R}_+^2$, $v(c, p; S, S, \mathcal{I}) = \sup_s v(c, p; s, S, \mathcal{I})$;
- ii. for any $(c, p) \notin \mathcal{I}$, $\frac{\partial e}{\partial p}(c, p; S, \mathcal{I}) < \frac{\partial e}{\partial c}(c, p; S, \mathcal{I})$;
- iii. for any $(c, p) \in \partial \mathcal{I}$, $\frac{\partial e}{\partial p}(c, p; S, \mathcal{I}) = \frac{\partial e}{\partial c}(c, p; S, \mathcal{I})$.
- iv. for any $(c, p) \in \mathcal{I}$, there exists $h > 0$ defined via $(c - h, p + h) \in \partial \mathcal{I}$ and such that $e(c, p; S, \mathcal{I}) = e(c - h, p + h; S, \mathcal{I})$ and $v(c, p; S, \mathcal{I}) = v(c - h, p + h; S, \mathcal{I})$.

The first condition above is identical to the condition used to define the equilibrium without reinvestments. The second condition says that in the region of the state space where the firm is not investing its cash into the illiquid asset, it must be the case that the equity marginal value of cash is strictly greater than the equity marginal value of illiquid asset. The third condition describes the condition that needs to be satisfied at the reflecting boundary – namely, that both equity marginal values of cash and illiquid assets are equal. Finally, the last condition stipulates that in the investment region, the firm uses a lump sum of cash h to instantaneously purchase an incremental amount of illiquid assets.

Studying such equilibria (if any of them exists) is beyond the scope of this paper. However, fixing the creditors' behavior, I can study the firm's incentive to reinvest into illiquid assets

Figure 24: Portfolio Reinvestment Boundary



vs. increasing the cash reserve, as given by the condition $\frac{\partial e}{\partial p}(c, p) \leq \frac{\partial e}{\partial c}(c, p)$, where the equity value is taken to be the equity value in the game without reinvestments, and where creditors' run behavior is taken to be the equilibrium run behavior for a game without reinvestment. I plot in the liquidity/solvency space (Λ, Σ) the run boundary as well as a new boundary, representing points at which the equity marginal value of cash is equal to the equity marginal value of illiquid assets (Figure 24). The region of the state space above the purple reinvestment boundary is a region where $\frac{\partial e}{\partial p} > \frac{\partial e}{\partial c}$, meaning that management's incentive is to reinvest into the illiquid asset. [Incomplete]

10 Conclusion

[To be completed]

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A Proofs

Proof of Proposition 1: Supermodularity is typically defined for static games. I adapt the definition of supermodularity as follows: the payoff function $v(\cdot, \cdot; s, S)$ is supermodular if for any strategies S_1, S_2 such that $S_2(c, p) \geq S_1(c, p)$ for all (c, p) , and for any strategy s , the incentive for a creditor to run are greater when responding to S_2 than when responding to S_1 , in other words:

$$1 - v(c, p; s, S_2) \geq 1 - v(c, p; s, S_1)$$

In order to prove that the payoff function of the game studied in this paper does not exhibit supermodularity, I choose to focus on the region of the state space $p = 0$. Since the illiquid asset price is a geometric Brownian motion, when $p = 0$, $p(t) = 0$ for all $t \geq 0$. In the restricted domain $p = 0$, the firm holds cash yielding r_c on the asset side of its balance-sheet, and has debt yielding r_d on the liability side of its balance-sheet. The cash balance per unit of debt outstanding c is thus the only relevant state variable for creditors' roll-over decisions.

Consider an initial cash level $c(0) = c_0 < 1$. Let $c_f \in [0, c_0)$ be an arbitrary point on the state space, and define $\tau_{\mathcal{R}}(c_f; c_0)$ to be the c_f -hitting time assuming a run from $t = 0$ onwards and assuming the initial cash level is c_0 , and define $\tau_{\mathcal{NR}}(c_f; c_0)$ to be the c_f -hitting time assuming creditors roll from $t = 0$ onwards and assuming the initial cash level is c_0 . When $p = 0$ and when a run occurs, the cash balance evolves according to the ordinary differential equation $dc(t) = ((r_c + \lambda)c(t) - (r_d + \lambda))dt$. Instead, when creditors roll their debt claims into new debt claims, the cash balance evolves according to the ordinary differential equation $dc(t) = (r_c c(t) - r_d)dt$. In either regime, since $c_0 < 1$, the cash balance is decreasing as a function of time. Integrating the ordinary differential equations satisfied by $c(t)$ given the initial condition $c(0) = c_0$ leads to the following expressions for the stopping times $\tau_{\mathcal{R}}(c_f; c_0)$ and $\tau_{\mathcal{NR}}(c_f; c_0)$:

$$\begin{aligned}\tau_{\mathcal{R}}(c_f; c_0) &= \frac{1}{r_c + \lambda} \ln \left(\frac{\frac{r_d + \lambda}{r_c + \lambda} - c_f}{\frac{r_d + \lambda}{r_c + \lambda} - c_0} \right) \\ \tau_{\mathcal{NR}}(c_f; c_0) &= \frac{1}{r_c} \ln \left(\frac{\frac{r_d}{r_c} - c_f}{\frac{r_d}{r_c} - c_0} \right)\end{aligned}$$

When $\tau_{\mathcal{R}}(c_f; c_0)$ is viewed as a function of λ , it can be showed that it is decreasing in λ , so long as $0 \leq c_f < c_0 < 1$. This is true since:

$$\frac{\partial \tau_{\mathcal{R}}(c_f; c_0)}{\partial \lambda} = \frac{1}{(r_c + \lambda)^2} \left[\frac{\frac{r_d + \lambda}{r_c + \lambda} - 1}{\frac{r_d + \lambda}{r_c + \lambda} - c_0} - \frac{\frac{r_d + \lambda}{r_c + \lambda} - 1}{\frac{r_d + \lambda}{r_c + \lambda} - c_f} - \ln \frac{\frac{r_d + \lambda}{r_c + \lambda} - c_f}{\frac{r_d + \lambda}{r_c + \lambda} - c_0} \right] < 0$$

This means that the cash balance converges to the state $c = c_f$ faster upon the occurrence of a run than when all creditors are rolling.

Consider the family $\{S_{\hat{c}}\}$ of cutoff strategies followed by all other creditors – in other words, strategies such that creditors run when $c \in [0, \hat{c})$ and creditors roll over when $c \in [\hat{c}, +\infty)$, for some $\hat{c} \in \mathbb{R}_+$. First, take $1 > \hat{c}_2 > \hat{c}_1 \geq 0$, which means that $S_{\hat{c}_2}(0, c) \geq S_{\hat{c}_1}(0, c)$

for all c . Given my previous observation, the cash per unit of debt outstanding verifies:

$$c^{c_0}(t; S_{\hat{c}_1}) \geq c^{c_0}(t; S_{\hat{c}_2})$$

In the above, $c^{c_0}(t; S_{\hat{c}})$ represents the cash level at time t , assuming that the cash level at $t = 0$ is c_0 , and assuming that all creditors play a cutoff strategy $S_{\hat{c}}$. The inequality above is an equality when $\hat{c}_1 > c_0$. The inequality is strict for $t > 0$ when $\hat{c}_2 > c_0 > \hat{c}_1$, and for $t \geq \tau_{\mathcal{NR}}(\hat{c}_2; c_0)$ when $c_0 > \hat{c}_2$. Given the fact that a specific agent i 's flow payoff does not depend on other creditors' strategy but that such specific creditor's terminal payoff is increasing in the cash balance $c(\tau)$ (where τ is the earlier of (a) the default stopping time τ_b or (b) the maturity stopping time τ_ϕ), I must have, for any strategy s followed by a specific creditor i :

$$v(0, c; s, S_{\hat{c}_2}) \leq v(0, c; s, S_{\hat{c}_1})$$

Thus, when considering creditors' strategies $S_{\hat{c}}$ for $\hat{c} < 1$, the game's payoff function exhibits supermodularity. Unfortunately, the uncovered supermodularity property is local instead of global. Consider now (c_0, c_f) with $\frac{r_d + \lambda}{r_c + \lambda} > c_0 > c_f > 1$. A reasoning similar to the previous one shows that:

$$\tau_{\mathcal{R}}(c_f; c_0) \geq \tau_{\mathcal{NR}}(c_f; c_0)$$

This is due to the fact that on that interval,

$$\frac{\partial \tau_{\mathcal{R}}(c_f; c_0)}{\partial \lambda} > 0$$

Consider then two cutoff strategies $S_{\hat{c}_1}$ and $S_{\hat{c}_2}$, with $\frac{r_d + \lambda}{r_c + \lambda} > \hat{c}_2 > \hat{c}_1 > 1$. The cash per unit of debt outstanding verifies:

$$c^{c_0}(t; S_{\hat{c}_1}) \leq c^{c_0}(t; S_{\hat{c}_2})$$

The inequality above is an equality when $\hat{c}_1 > c_0$. The inequality is strict for $t > 0$ when $\hat{c}_2 > c_0 > \hat{c}_1$, and for $t \geq \tau_{\mathcal{NR}}(\hat{c}_2; c_0)$ when $\frac{r_d + \lambda}{r_c + \lambda} > c_0 > \hat{c}_2$. When the firm starts with an amount of cash $\hat{c}_2 > c_0 > \hat{c}_1$, the cash balance decreases more rapidly when all creditors are rolling. Using a reasoning similar to the one in the previous paragraph, it means that for any strategy s followed by a specific creditor:

$$v(0, c; s, S_{\hat{c}_2}) \geq v(0, c; s, S_{\hat{c}_1})$$

In other words, the game's payoff exhibits strategic substitutability for cutoff strategies $\hat{c} \in \left(1, \frac{r_d + \lambda}{r_c + \lambda}\right)$. Intuitively, when $\frac{r_d + \lambda}{r_c + \lambda} > c > 1$, a creditor run decreases the speed at which the cash reserve *per unit of debt outstanding* is depleted. Upon a run, expensive debt yielding r_d per unit of time ends up being paid down, which helps improve the financial health of the firm (since it can only rely on an asset yielding $r_c < r_d$). Thus, the earlier creditors run (in other words the higher the threshold $\hat{c}$), the better off a single creditor is. $\square$

Proof of Proposition 2: For a given strategy $S : \mathbb{R}_+^2 \rightarrow \{0, 1\}$ followed by creditors, the best achievable payoff for a given creditor is to be rolling over all the time, until the time τ_ϕ at which the illiquid asset matures and creditors are fully paid off. In other words, I have:

$$\begin{aligned} v(c, p; s, S) &\leq \mathbb{E}^{p, c} \left[\int_0^{\tau_\phi} e^{-\rho t} r_d dt + e^{-\rho \tau_\phi} \right] \\ &\leq \frac{r_d + \phi}{\rho + \phi} \end{aligned}$$

□

Proof of Proposition 3: I want to establish that for (c, p) small enough, it is optimal for a given creditor to run when he gets the chance to do so, irrespective of the strategy S followed by other creditors. In order to achieve that, take $S : \mathbb{R}_+^2 \rightarrow \{0, 1\}$ and $s : \mathbb{R}_+^2 \rightarrow \{0, 1\}$ arbitrary. When I integrate the stochastic differential equation for $c(t)$, I obtain:

$$e^{-\int_0^t (r_c + \lambda(s)) ds} c(t) = c + p \int_0^t (\rho - \mu) e^{(\mu - \frac{1}{2}\sigma^2 - r_c)s + \sigma B(s)} ds - \int_0^t (r_d + \lambda(s)) e^{-\int_0^s (r_c + \lambda(u)) du} ds$$

In the above, I have noted $\lambda(t) := \lambda 1_{\{S(c(t), p(t))=1\}}$. For a given realization $\{B(t, \omega)\}_{t \geq 0}$ of the Brownian motion, $\tau_b(\omega)$ is the smallest time that solves:

$$c + p \int_0^{\tau_b(\omega)} (\rho - \mu) e^{(\mu - \frac{1}{2}\sigma^2 - r_c)s + \sigma B(s, \omega)} ds = \int_0^{\tau_b(\omega)} (r_d + \lambda(s, \omega)) e^{-\int_0^s (r_c + \lambda(u, \omega)) du} ds \quad (31)$$

Note that the equation above might not have a solution, in which case $\tau_b(\omega) = \infty$. Let τ_λ be an exponentially distributed time, with arrival intensity λ , and $\tau := \tau_\lambda \wedge \tau_b \wedge \tau_\phi$. The value function can be re-written:

$$\begin{aligned} v(c, p; s, S) &= \mathbb{E}^{p, c} \left[\int_0^\tau e^{-\rho t} r_d dt + e^{-\rho \tau} 1_{\{\tau = \tau_\lambda\}} [(1 - s(p(\tau), c(\tau))) v(p(\tau), c(\tau); s, S) + s(p(\tau), c(\tau))] \right. \\ &\quad \left. + e^{-\rho \tau} 1_{\{\tau = \tau_\phi\}} \min(1, p(\tau) + c(\tau)) + e^{-\rho \tau} 1_{\{\tau = \tau_b\}} \min(1, \alpha p(\tau)) \right] \end{aligned}$$

Using the law of iterated expectations, and making explicit the dependence on the Brownian motion realization (via ω), the value function for creditor i can then be simplified as follows:

$$\begin{aligned} v(c, p; s, S) &= \mathbb{E}^{p, c} \left[e^{-(\lambda + \phi)\tau_b(\omega)} \frac{r_d}{\rho} (1 - e^{-\rho \tau_b(\omega)}) + e^{-(\lambda + \phi)\tau_b(\omega)} e^{-\rho \tau_b(\omega)} \min(1, \alpha p(\tau_b(\omega))) \right. \\ &\quad + \Pr(\tau_\lambda \leq \tau_b(\omega) \wedge \tau_\phi) \times \mathbb{E} \left[\frac{r_d}{\rho} (1 - e^{-\rho \tau_\lambda}) + e^{-\rho \tau_\lambda} [(1 - s(p(\tau), c(\tau))) v(p(\tau), c(\tau); s, S) \right. \\ &\quad \left. \left. + s(p(\tau), c(\tau))] | \tau_\lambda \leq \tau_b(\omega) \wedge \tau_\phi \right] \right. \\ &\quad \left. + \Pr(\tau_\phi \leq \tau_b(\omega) \wedge \tau_\lambda) \times \mathbb{E} \left[\frac{r_d}{\rho} (1 - e^{-\rho \tau_\phi}) + e^{-\rho \tau_\phi} \min(1, p(\tau_\phi) + c(\tau_\phi)) | \tau_\phi \leq \tau_b(\omega) \wedge \tau_\lambda \right] \right] \end{aligned}$$

For a fixed and given $\tau_b(\omega)$, simple algebra gives me the following:

$$\begin{aligned}\Pr(\tau_\lambda \leq \tau_b(\omega) \wedge \tau_\phi) &= \frac{\lambda}{\lambda + \phi} (1 - e^{-(\lambda+\phi)\tau_b(\omega)}) \\ \Pr(\tau_\lambda \leq \tau_b(\omega) \wedge \tau_\phi) \times \mathbb{E}[e^{-\rho\tau_\lambda} | \tau_\lambda \leq \tau_b(\omega) \wedge \tau_\phi] &= \frac{\lambda}{\rho + \lambda + \phi} (1 - e^{-(\rho+\lambda+\phi)\tau_b(\omega)}) \\ \Pr(\tau_\phi \leq \tau_b(\omega) \wedge \tau_\lambda) &= \frac{\phi}{\lambda + \phi} (1 - e^{-(\lambda+\phi)\tau_b(\omega)}) \\ \Pr(\tau_\phi \leq \tau_b(\omega) \wedge \tau_\lambda) \times \mathbb{E}[e^{-\rho\tau_\phi} | \tau_\phi \leq \tau_b(\omega) \wedge \tau_\lambda] &= \frac{\phi}{\rho + \lambda + \phi} (1 - e^{-(\rho+\lambda+\phi)\tau_b(\omega)})\end{aligned}$$

Note that given $p(0) = p$, I have $p(t) \leq pe^{(\mu - \frac{1}{2}\sigma^2 + \lambda)t + \sigma B(t)}$ irrespective of creditors' strategy. Given the upper bound computed for v and the previous comment, I have the following inequality:

$$\begin{aligned}v(c, p; s, S) &\leq \mathbb{E}^{p,c} \left[e^{-(\lambda+\phi)\tau_b(\omega)} \left(\frac{r_d}{\rho} (1 - e^{-\rho\tau_b(\omega)}) + e^{-\rho\tau_b(\omega)} \min \left(1, \alpha pe^{(\mu+\lambda-\frac{1}{2}\sigma^2)\tau_b(\omega)+\sigma B(\tau_b(\omega))} \right) \right) \right. \\ &\quad + \frac{r_d}{\rho} \frac{\lambda}{\lambda + \phi} (1 - e^{-(\lambda+\phi)\tau_b(\omega)}) + \left(\frac{r_d + \phi}{\rho + \phi} - \frac{r_d}{\rho} \right) \frac{\lambda}{\rho + \lambda + \phi} (1 - e^{-(\rho+\lambda+\phi)\tau_b(\omega)}) \\ &\quad \left. + \frac{r_d}{\rho} \frac{\phi}{\lambda + \phi} (1 - e^{-(\lambda+\phi)\tau_b(\omega)}) + \left(1 - \frac{r_d}{\rho} \right) \frac{\phi}{\rho + \lambda + \phi} (1 - e^{-(\rho+\lambda+\phi)\tau_b(\omega)}) \right]\end{aligned}$$

The inequality can be simplified further:

$$v(c, p; s, S) \leq \mathbb{E}^{p,c} \left[\frac{r_d + \phi}{\rho + \phi} (1 - e^{-(\rho+\lambda+\phi)\tau_b(\omega)}) + e^{-(\rho+\lambda+\phi)\tau_b(\omega)} \min \left(1, \alpha pe^{(\mu+\lambda-\frac{1}{2}\sigma^2)\tau_b(\omega)+\sigma B(\tau_b(\omega))} \right) \right]$$

The expectation above is taken over the random variable $\tau_b(\omega)$, defined implicitly by [equation \(31\)](#). The key idea is that by choosing (c, p) “small enough”, I can make $\tau_b(\omega)$ “small”, which means that the term $\frac{r_d + \phi}{\rho + \phi} (1 - e^{-(\rho+\lambda+\phi)\tau_b(\omega)})$ in the expectation above is dominated by the term $e^{-(\rho+\lambda+\phi)\tau_b(\omega)} \min \left(1, \alpha pe^{(\mu+\lambda-\frac{1}{2}\sigma^2)\tau_b(\omega)+\sigma B(\tau_b(\omega))} \right)$. Using Doob's optional sampling theorem and Jensen's inequality (on the concave function $x \rightarrow \min(1, x)$), for any finite T , I have:

$$\begin{aligned}\mathbb{E}^{p,c} \left[\frac{r_d + \phi}{\rho + \phi} (1 - e^{-(\rho+\lambda+\phi)(\tau_b(\omega) \wedge T)}) + e^{-(\rho+\lambda+\phi)(\tau_b(\omega) \wedge T)} \min \left(1, \alpha pe^{(\mu+\lambda-\frac{1}{2}\sigma^2)(\tau_b(\omega) \wedge T) + \sigma B((\tau_b(\omega) \wedge T))} \right) \right] &\leq \\ \mathbb{E}^{p,c} \left[\frac{r_d + \phi}{\rho + \phi} (1 - e^{-(\rho+\lambda+\phi)(\tau_b(\omega) \wedge T)}) + \min \left(e^{-(\rho+\lambda+\phi)(\tau_b(\omega) \wedge T)}, \alpha pe^{-(\rho+\phi-\mu)(\tau_b(\omega) \wedge T)} \right) \right]\end{aligned}$$

Taking $T \rightarrow +\infty$, I obtain the inequality:

$$v(c, p; s, S) \leq \mathbb{E}^{p,c} \left[\frac{r_d + \phi}{\rho + \phi} (1 - e^{-(\rho+\lambda+\phi)\tau_b(\omega)}) + e^{-(\rho+\lambda+\phi)\tau_b(\omega)} \min \left(1, \alpha pe^{(\mu+\lambda)\tau_b(\omega)} \right) \right]$$

Note that the right handside of [equation \(31\)](#) verifies the following inequality, for any t , and

any strategy S :

$$\begin{aligned} \int_0^t (r_d + \lambda(s, \omega)) e^{-\int_0^s (r_c + \lambda(u, \omega)) du} ds &\geq \int_0^t r_d e^{-(r_c + \lambda)s} ds \\ &\geq \frac{r_d}{r_c + \lambda} (1 - e^{-(r_c + \lambda)t}) \end{aligned}$$

Thus, the stopping time $\tau_b(\omega)$ is almost surely less than or equal to the stopping time $\bar{\tau}_b(\omega)$, defined implicitly as the smallest time solving:

$$c + p \int_0^{\bar{\tau}_b(\omega)} (\rho - \mu) e^{(\mu - \frac{1}{2}\sigma^2 - r_c)s + \sigma B(s, \omega)} ds = \frac{r_d}{r_c + \lambda} (1 - e^{-(r_c + \lambda)\bar{\tau}_b(\omega)}) \quad (32)$$

Note that the stopping time $\bar{\tau}_b$ does not depend on creditors' strategies. When the equation above does not have a solution, I set $\bar{\tau}_b(\omega) = +\infty$. Since the function $t \rightarrow \frac{r_d + \phi}{\rho + \phi} (1 - e^{-(\rho + \lambda + \phi)t}) + e^{-(\rho + \lambda + \phi)t} \min(1, \alpha p e^{(\mu + \lambda)t})$ is increasing in t for positive values of t , I can write:

$$v(c, p; s, S) \leq \mathbb{E}^{p, c} \left[\frac{r_d + \phi}{\rho + \phi} (1 - e^{-(\rho + \lambda + \phi)\bar{\tau}_b(\omega)}) + e^{-(\rho + \lambda + \phi)\bar{\tau}_b(\omega)} \min(1, \alpha p e^{(\mu + \lambda)\bar{\tau}_b(\omega)}) \right] := g(c, p)$$

$g(c, p)$ is thus an upper bound of $v(c, p; s, S)$ that does not depend on the strategies (s, S) . It remains to be shown that for (c, p) small enough, g is strictly less than 1. Notice that $g(0, 0) = 0$, and that g is continuous in a neighborhood of $(0, 0)$. The continuity of g means that there exists a neighborhood $\mathcal{D}_l \subset \mathbb{R}_+^2$ such that for $(c, p) \in \mathcal{D}_l$, $g(c, p) < 1$, which is the desired result. As an aside, the stopping time $\bar{\tau}_b$ is increasing in p and in c . Since $t \rightarrow \frac{r_d + \phi}{\rho + \phi} (1 - e^{-(\rho + \lambda + \phi)t}) + e^{-(\rho + \lambda + \phi)t} \min(1, \alpha p e^{(\mu + \lambda)t})$ is increasing in t , this means that the function g defined above is increasing in p and c , with value zero at $(0, 0)$. In other words, the region $\{(c, p) : g(c, p) < 1\}$ includes the point $(0, 0)$ and is path-connected, implying that the largest possible dominance region (which can be defined as $\{(c, p) : v(c, p; s, S) < 1 \forall (s, S)\}$) is path-connected and contains $(0, 0)$. $\square$

I now want to establish that for (c, p) large enough, it is optimal for a given creditor to roll over his maturing debt claim into a new debt claim when he gets the chance to do so, irrespective of the strategy S followed by other creditors. Note that for $c > \frac{r_d}{r_c}$, the cash reserve $c(t)$ is strictly increasing, irrespective of S . Thus the default time τ_b is infinite almost surely, which means that it is dominant for creditors to always roll over. I can also show that for p high enough, it is dominant for a given creditor to roll, irrespective of other creditors' strategy. The idea is that for an arbitrarily small $\epsilon > 0$ given, I can find a p high enough such that:

$$((\rho - \mu)p + r_c c - (r_d + \lambda))\epsilon > \frac{r_d}{r_c}$$

In other words, irrespective of creditors' strategies, I can find a p large enough such that after an arbitrarily small time interval ϵ , the state variable c ends up above r_d/r_c , value at which I know it becomes dominant for creditors to roll. $\square$

Proof of Proposition 4: Let me first introduce some notation:

$$\begin{aligned}\mu_p(c, p) &= (\mu + \lambda 1_{\{(c, p) \in \mathcal{R}_S\}}) p \\ \sigma_p(c, p) &= \sigma p \\ \mu_c(c, p) &= (\rho - \mu) p + (r_c + \lambda 1_{\{(c, p) \in \mathcal{R}_S\}}) c - (r_c + \lambda 1_{\{(c, p) \in \mathcal{R}_S\}})\end{aligned}$$

Given the drift coefficients μ_p, μ_c are sublinear and the volatility coefficient σ_p is Lipschitz, Karatzas (1991) (page 303) provides for the existence of a weak solution to the stochastic differential equation governing the evolution of the state variables.

Let $\mathcal{C}(\mathbb{R}_+^2)$ be the set of functions that are continuous and bounded on $\mathbb{R}_+^2$. For the cutoff strategy $S : \mathbb{R}_+^2 \rightarrow \{0, 1\}$ followed by other creditors, I define the operator $\mathbb{T}_S$ that maps any arbitrary continuous bounded function $f : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ into a function $\mathbb{T}_S(f)$ defined as follows:

$$\begin{aligned}\mathbb{T}_S(f)(c, p) &= \mathbb{E}^{p, c} \left\{ \int_0^\tau e^{-\rho t} r_d dt + e^{-\rho \tau} 1_{\{\tau = \tau_\lambda\}} \max(1, f(p(\tau), c(\tau))) \right\} \\ &\quad + \mathbb{E}^{p, c} \left\{ e^{-\rho \tau} 1_{\{\tau = \tau_\phi\}} \min(1, p(\tau) + c(\tau)) + e^{-\rho \tau} 1_{\{\tau = \tau_b\}} \min(1, \alpha p(\tau)) \right\}\end{aligned}$$

Where p and c evolve according to:

$$\begin{aligned}dp(t) &= \mu_p(p(t), c(t)) dt + \sigma_p(p(t), c(t)) dB(t) \\ dc(t) &= \mu_c(p(t), c(t)) dt\end{aligned}$$

I want to establish that $\mathbb{T}_S$ maps the set of continuous and bounded functions on $\mathbb{R}^2$ into itself. Let $f \in \mathcal{C}(\mathbb{R}_+^2)$ be given, and let M be its upper bound. For any $(c, p) \in \mathbb{R}_+^2$, I have:

$$\begin{aligned}\mathbb{T}_S(f)(c, p) &\leq \mathbb{E}^{p, c} \left\{ \int_0^\tau e^{-\rho t} r_d dt + e^{-\rho \tau} 1_{\{\tau = \tau_\lambda\}} M + e^{-\rho \tau} 1_{\{\tau = \tau_\phi\}} + e^{-\rho \tau} 1_{\{\tau = \tau_b\}} \right\} \\ &\leq \max\left(\frac{r_d}{\rho}, M\right)\end{aligned}$$

Thus $\mathbb{T}_S$ maps bounded functions into bounded functions. Let me pick $(c, p) \in \mathbb{R}_+^2$ and a sequence $\{(p_\epsilon, c_\epsilon)\}_\epsilon \in \mathbb{R}_+^2$, where $\lim_{\epsilon \rightarrow 0} (p_\epsilon, c_\epsilon) = (c, p)$. I want to prove that $\lim_{\epsilon \rightarrow 0} \mathbb{T}_S(f)(p_\epsilon, c_\epsilon) = \mathbb{T}_S(f)(p, c)$. Let $\tau_{b, \epsilon}$ be the firm default time conditioned on the starting values (p_ϵ, c_ϵ) . Similarly, let $(p_\epsilon(t), c_\epsilon(t))$ be the random variables associated with the values of the state variables (c, p) at time t , conditioned on initial values (p_ϵ, c_ϵ) . Let $\tau = \tau_{b, \epsilon} \wedge \tau_b \wedge \tau_\lambda \wedge \tau_\phi$. To simplify

notation, I define the following events:

$$\begin{aligned}
E_1 &:= \{\tau_\lambda < \tau_b \wedge \tau_{b,\epsilon} \wedge \tau_\phi\} \\
E_2 &:= \{\tau_\phi < \tau_b \wedge \tau_{b,\epsilon} \wedge \tau_\lambda\} \\
E_3 &:= \{\tau_{b,\epsilon} < \tau_b < \tau_\lambda \wedge \tau_\phi\} \\
E_4 &:= \{\tau_b < \tau_{b,\epsilon} < \tau_\lambda \wedge \tau_\phi\} \\
E_5 &:= \{\tau_{b,\epsilon} < \tau_\phi < \tau_\lambda \wedge \tau_b\} \\
E_6 &:= \{\tau_b < \tau_\phi < \tau_\lambda \wedge \tau_{b,\epsilon}\} \\
E_7 &:= \{\tau_{b,\epsilon} < \tau_\lambda < \tau_\phi \wedge \tau_b\} \\
E_8 &:= \{\tau_b < \tau_\lambda < \tau_\phi \wedge \tau_{b,\epsilon}\}
\end{aligned}$$

Note that those events do not intersect, and $\Pr(\bigcup_{i=1}^8 E_i) = 1$. I have the following:

$$\begin{aligned}
\mathbb{T}_S(f)(x_\epsilon, y_\epsilon) - \mathbb{T}_S(f)(x, y) = & \\
& \mathbb{E} \left[e^{-\rho\tau_\lambda} (\max(1, f(p_\epsilon(\tau_\lambda), c_\epsilon(\tau_\lambda))) - \max(1, f(p(\tau_\lambda), c(\tau_\lambda)))) | E_1 \right] \Pr(E_1) \\
& + \mathbb{E} \left[e^{-\rho\tau_\phi} (\min(1, p_\epsilon(\tau_\phi) + c_\epsilon(\tau_\phi)) - \min(1, p(\tau_\phi) + c(\tau_\phi))) | E_2 \right] \Pr(E_2) \\
& + \mathbb{E} \left[\int_{\tau_b}^{\tau_{b,\epsilon}} e^{-\rho t} r_d dt + e^{-\rho\tau_{b,\epsilon}} \min(1, \alpha p_\epsilon(\tau_{b,\epsilon})) - e^{-\rho\tau_b} \min(1, \alpha p(\tau_b)) | E_3 \right] \Pr(E_3) \\
& + \mathbb{E} \left[\int_{\tau_{b,\epsilon}}^{\tau_b} e^{-\rho t} r_d dt + e^{-\rho\tau_{b,\epsilon}} \min(1, \alpha p_\epsilon(\tau_{b,\epsilon})) - e^{-\rho\tau_b} \min(1, \alpha p(\tau_b)) | E_4 \right] \Pr(E_4) \\
& + \mathbb{E} \left[\int_{\tau_\phi}^{\tau_{b,\epsilon}} e^{-\rho t} r_d dt + e^{-\rho\tau_{b,\epsilon}} \min(1, \alpha p_\epsilon(\tau_{b,\epsilon})) - e^{-\rho\tau_\phi} \min(1, p(\tau_\phi) + c(\tau_\phi)) | E_5 \right] \Pr(E_5) \\
& + \mathbb{E} \left[\int_{\tau_b}^{\tau_\phi} e^{-\rho t} r_d dt + e^{-\rho\tau_\phi} \min(1, p_\epsilon(\tau_\phi) + c_\epsilon(\tau_\phi)) - e^{-\rho\tau_b} \min(1, \alpha p(\tau_b)) | E_6 \right] \Pr(E_6) \\
& + \mathbb{E} \left[\int_{\tau_\lambda}^{\tau_{b,\epsilon}} e^{-\rho t} r_d dt + e^{-\rho\tau_{b,\epsilon}} \min(1, \alpha p_\epsilon(\tau_{b,\epsilon})) - e^{-\rho\tau_\lambda} \max(1, f(p(\tau_\lambda), c(\tau_\lambda))) | E_7 \right] \Pr(E_7) \\
& + \mathbb{E} \left[\int_{\tau_b}^{\tau_\lambda} e^{-\rho t} r_d dt + e^{-\rho\tau_\lambda} \max(1, f(p_\epsilon(\tau_\phi), c_\epsilon(\tau_\phi))) - e^{-\rho\tau_b} \min(1, \alpha p(\tau_b)) | E_8 \right] \Pr(E_8)
\end{aligned}$$

From this expression, I have the following inequality:

$$\begin{aligned}
|\mathbb{T}_S(f)(x_\epsilon, y_\epsilon) - \mathbb{T}_S(f)(x, y)| \leq & \\
& \mathbb{E} \left[e^{-\rho\tau_\lambda} |f(p_\epsilon(\tau_\lambda), c_\epsilon(\tau_\lambda)) - f(p(\tau_\lambda), c(\tau_\lambda))| |E_1 \right] \Pr(E_1) \\
& + \mathbb{E} \left[e^{-\rho\tau_\phi} |(p_\epsilon(\tau_\phi) + c_\epsilon(\tau_\phi)) - (p(\tau_\phi) + c(\tau_\phi))| |E_2 \right] \Pr(E_2) \\
& + \mathbb{E} \left[\left(\frac{r_d}{\rho} + \min(1, \alpha p(\tau_b)) \right) |e^{-\rho\tau_{b,\epsilon}} - e^{\tau_b}| + \alpha e^{-\rho\tau_{b,\epsilon}} |p_\epsilon(\tau_{b,\epsilon}) - p(\tau_b)| |E_3 \right] \Pr(E_3) \\
& + \mathbb{E} \left[\left(\frac{r_d}{\rho} + \min(1, \alpha p(\tau_{b,\epsilon})) \right) |e^{-\rho\tau_{b,\epsilon}} - e^{\tau_b}| + \alpha e^{-\rho\tau_b} |p_\epsilon(\tau_{b,\epsilon}) - p(\tau_b)| |E_4 \right] \Pr(E_4) \\
& + \mathbb{E} \left[\int_{\tau_\phi}^{\tau_{b,\epsilon}} e^{-\rho t} r_d dt + e^{-\rho\tau_{b,\epsilon}} |E_5 \right] \Pr(E_5) \\
& + \mathbb{E} \left[\int_{\tau_b}^{\tau_\phi} e^{-\rho t} r_d dt + e^{-\rho\tau_b} |E_6 \right] \Pr(E_6) \\
& + \mathbb{E} \left[\int_{\tau_\lambda}^{\tau_{b,\epsilon}} e^{-\rho t} r_d dt + e^{-\rho\tau_{b,\epsilon}} |E_7 \right] \Pr(E_7) \\
& + \mathbb{E} \left[\int_{\tau_b}^{\tau_\lambda} e^{-\rho t} r_d dt + e^{-\rho\tau_b} |E_8 \right] \Pr(E_8)
\end{aligned}$$

Given that S is cutoff and given the drift and diffusion coefficients of $(p(t), c(t))$, [Nilssen \(2012\)](#) provides for the continuous differentiability of the process $(p_\epsilon(t), c_\epsilon(t))$ – in other words, the vector $(p_\epsilon(t), c_\epsilon(t))$ converges pathwise to the vector $(c(t), p(t))$. This means that:

$$\lim_{\epsilon \rightarrow 0} \mathbb{E} \left[e^{-\rho\tau_\phi} |(p_\epsilon(\tau_\phi) + c_\epsilon(\tau_\phi)) - (p(\tau_\phi) + c(\tau_\phi))| |E_2 \right] = 0$$

Given the pathwise convergence of $(p_\epsilon(t), c_\epsilon(t))$ to $(p(t), c(t))$, the stopping time $\tau_{b,\epsilon}$ converges in probability to τ_b . Thus I have:

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} \mathbb{E} \left[\left(\frac{r_d}{\rho} + \alpha p(\tau_b) \right) |e^{-\rho\tau_{b,\epsilon}} - e^{\tau_b}| + \alpha e^{-\rho\tau_{b,\epsilon}} |p_\epsilon(\tau_{b,\epsilon}) - p(\tau_b)| |E_3 \right] &= 0 \\
\lim_{\epsilon \rightarrow 0} \mathbb{E} \left[\left(\frac{r_d}{\rho} + \alpha p(\tau_{b,\epsilon}) \right) |e^{-\rho\tau_{b,\epsilon}} - e^{\tau_b}| + \alpha e^{-\rho\tau_b} |p_\epsilon(\tau_{b,\epsilon}) - p(\tau_b)| |E_4 \right] &= 0 \\
\lim_{\epsilon \rightarrow 0} \Pr(E_5) &= 0 \\
\lim_{\epsilon \rightarrow 0} \Pr(E_6) &= 0 \\
\lim_{\epsilon \rightarrow 0} \Pr(E_7) &= 0 \\
\lim_{\epsilon \rightarrow 0} \Pr(E_8) &= 0
\end{aligned}$$

Finally, the continuity of the function f provides for:

$$\lim_{\epsilon \rightarrow 0} \mathbb{E} \left[e^{-\rho\tau_\lambda} |f(p_\epsilon(\tau_\lambda), c_\epsilon(\tau_\lambda)) - f(p(\tau_\lambda), c(\tau_\lambda))| |E_1 \right] = 0$$

In other words, I have just established that:

$$\lim_{\epsilon \rightarrow 0} |\mathbb{T}_S(f)(x_\epsilon, y_\epsilon) - \mathbb{T}_S(f)(x, y)| = 0$$

This enables me to conclude that $\mathbb{T}_S$ maps the set of continuous and bounded functions on $\mathbb{R}^2$ into itself. Finally, I want to show that $\mathbb{T}_S$ is a contraction. For any pair of functions (f, g) in $\mathcal{C}(\mathbb{R}_+^2)$, and for any $(c, p) \in \mathbb{R}_+^2$, I have:

$$\begin{aligned} |\mathbb{T}_S(f)(c, p) - \mathbb{T}_S(g)(c, p)| &= |\mathbb{E}^{p,c} [e^{-\rho\tau} 1_{\{\tau=\tau_\lambda\}} (\max(1, f(p(\tau), c(\tau))) - \max(1, g(p(\tau), c(\tau))))] | \\ &\leq \mathbb{E}^{p,c} [e^{-\rho\tau} 1_{\{\tau=\tau_\lambda\}} |f(p(\tau), c(\tau)) - g(p(\tau), c(\tau))|] \end{aligned}$$

Noting with a slight abuse of notation $f(t) := f(c(t), p(t))$ and $g(t) := g(c(t), p(t))$, I can condition on the stopping time τ_λ being less than or greater than ϵ , for some fixed $\epsilon > 0$:

$$\begin{aligned} &\mathbb{E}^{p,c} [e^{-\rho\tau} 1_{\{\tau=\tau_\lambda\}} |f(p(\tau), c(\tau)) - g(p(\tau), c(\tau))|] \\ &= \mathbb{E}^{p,c} \left[e^{-\rho\tau} 1_{\{\tau=\tau_\lambda\}} |f(\tau) - g(\tau)| \middle| \tau_\lambda \leq \epsilon \right] \Pr(\tau_\lambda \leq \epsilon) + \mathbb{E}^{p,c} \left[e^{-\rho\tau} 1_{\{\tau=\tau_\lambda\}} |f(\tau) - g(\tau)| \middle| \tau_\lambda > \epsilon \right] \Pr(\tau_\lambda > \epsilon) \\ &\leq \|f - g\|_\infty \times (1 - e^{-\lambda\epsilon}) + e^{-\rho\epsilon} \|f - g\|_\infty \times e^{-\lambda\epsilon} \\ &\leq (1 - e^{-\lambda\epsilon} + e^{-(\rho+\lambda)\epsilon}) \|f - g\|_\infty \end{aligned}$$

Since $0 < 1 - e^{-\lambda\epsilon} + e^{-(\rho+\lambda)\epsilon} < 1$ for any strictly positive ϵ , $\mathbb{T}_S$ is a contraction map. Thus, for a given cutoff strategy S , a solution $v^*(\cdot, \cdot; S)$, fixed point of the mapping $\mathbb{T}_S$ defined above, exists and is unique. $\square$

Proof of Proposition 5: I will assume that there is a threshold c^* such that for $c \leq c^*$, it is optimal for creditors to run, while for $c \geq c^*$, it is optimal for creditors to continue rolling. The threshold c^* will need to verify $v_0(c^*) = 1$. I will establish that $c^* \in (1, \frac{r_d + \lambda}{r_c + \lambda})$, but for the time being, no specific assumption is made on the value of c^* . Finally, I will assume that the value functions v and e are continuous and continuously differentiable at $p = 0$.

1. $c \in (0, 1 \wedge c^*)$

On this interval, none of the maturing creditors are rolling over their debt. The value functions v_0 and e_0 must satisfy:

$$\begin{aligned} \rho v_0(c) &= r_d + ((r_c + \lambda)c - (r_d + \lambda)) v'_0(c) + \lambda(1 - v_0(c)) + \phi(\min(1, c) - v_0(c)) \\ \rho e_0(c) &= ((r_c + \lambda)c - (r_d + \lambda)) e'_0(c) + \phi(\max(0, c - 1) - e_0(c)) \end{aligned}$$

Since on this interval, $c < 1$, I have:

$$\begin{aligned} v'_0(c) &= \frac{\rho + \lambda + \phi}{(r_c + \lambda)c - (r_d + \lambda)} v_0(c) - \frac{r_d + \lambda + \phi c}{(r_c + \lambda)c - (r_d + \lambda)} \\ e'_0(c) &= \frac{\rho + \phi}{(r_c + \lambda)c - (r_d + \lambda)} e_0(c) \end{aligned}$$

Given the boundary $e_0(0) = v_0(0) = 0$, these ODEs admit the following solutions:

$$v_0(c) = \left(\frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \lambda + \phi)(\phi + \rho - r_c)} \right) \left[1 - \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c \right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} \right] + \frac{\phi}{\phi + \rho - r_c} c$$

$$e_0(c) = 0$$

The expression for $v_0(c)$ on this interval admits a natural interpretation. When $c < (1 \wedge c^*)$, the optimal strategy for creditors is to stop rolling their debt claims. Thus, the evolution of $c(t)$ is the following:

$$c'(t) = (r_c + \lambda)c(t) - (r_d + \lambda)$$

This means that $c(t)$ evolves as follows:

$$c(t) = \left(c(0) - \frac{r_d + \lambda}{r_c + \lambda} \right) e^{(r_c + \lambda)t} + \frac{r_d + \lambda}{r_c + \lambda}$$

Taking $c(0) = c < 1 < \frac{r_d + \lambda}{r_c + \lambda}$, the cash (per unit of debt) is a strictly decreasing function of time, which hits zero at time τ_0 :

$$\tau_0 = \frac{-1}{r_c + \lambda} \ln \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c \right)$$

Thus, for a given creditor i , if within a period of time of length τ_0 neither τ_λ (the creditor's claim maturing), nor τ_ϕ (the asset maturing) occurs, the creditor is guaranteed to only receive interest payment on its debt and lose its principal balance. This means that I can think of the creditor's value as the sum of interest collections until a stopping time $\tau = \tau_0 \wedge \tau_\lambda \wedge \tau_\phi$, plus principal collections that depend on whether τ_0, τ_λ or τ_ϕ occurs first:

$$v_0(c) = \Pr(\tau_0 < \tau_\lambda \wedge \tau_\phi) \times \int_0^{\tau_0} e^{-\rho t} r_d dt$$

$$+ [1 - \Pr(\tau_0 < \tau_\lambda \wedge \tau_\phi)] \times \mathbb{E}^c \left[\int_0^{\tau_\lambda \wedge \tau_\phi} e^{-\rho t} r_d dt \mid \tau_\lambda \wedge \tau_\phi < \tau_0 \right]$$

$$+ [1 - \Pr(\tau_0 < \tau_\lambda \wedge \tau_\phi)] \times \Pr(\tau_\lambda < \tau_\phi \mid \tau_0 < \tau_\lambda \wedge \tau_\phi) \times \mathbb{E}^c [e^{-\rho \tau_\lambda} \mid \tau_\lambda < \tau_\phi < \tau_0]$$

$$+ [1 - \Pr(\tau_0 < \tau_\lambda \wedge \tau_\phi)] \times \Pr(\tau_\phi < \tau_\lambda \mid \tau_0 < \tau_\lambda \wedge \tau_\phi) \times \mathbb{E}^c [e^{-\rho \tau_\phi} c(\tau_\phi) \mid \tau_\phi < \tau_\lambda < \tau_0]$$

A separate calculation of all the components above, taking into account the fact that $\tau_\lambda \wedge \tau_\phi$ is exponentially distributed with parameter $\lambda + \phi$, and taking into account the deterministic value of τ_0 calculated above, enables me to verify that the expression for $v_0(\cdot)$ previously obtained corresponds to the decomposition above. Given the expression for v_0 , I can immediately conclude that v_0 is strictly increasing on $(0, 1 \wedge c^*)$. The

derivative v'_0 on the interval $[0, 1]$ takes the following form:

$$v'_0(c) = \left(1 - \frac{\phi}{\phi + \rho - r_c}\right) \times \underbrace{\left(1 - \frac{r_c + \lambda}{r_d + \lambda}c\right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda} - 1}}_{\in (0,1)} + \left(\frac{\phi}{\phi + \rho - r_c}\right) \times 1$$

Since this is a weighted average of $\left(1 - \frac{r_c + \lambda}{r_d + \lambda}c\right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda} - 1}$ (which is strictly less than 1 for $c > 0$ and equal to 1 for $c = 0$) and 1, I have $v'_0(c) < 1$ for all $c \in (0, 1)$. Since $v_0(0) = 0$ and $v'_0(c) < 1$ for all $c < 1$, it must be the case that $v_0(1) < 1$. Since c^* must satisfy $v_0(c^*) = 1$, I must have $c^* > 1$: when the cash (per unit of debt outstanding) falls below unity, it is already “too late”: creditors have started running for cash levels greater than 1. This is intuitive: debt interest r_d is strictly greater than the discount rate ρ , while the cash only earns $r_c < \rho$, which means that the cash available plus its interest collections are insufficient to cover interest and principal repayments on the debt.

2. $c \in (1, c^* \wedge \frac{r_d + \lambda}{r_c + \lambda})$

On this interval, the value functions v_0 and e_0 satisfy:

$$\begin{aligned}\rho v_0(c) &= r_d + ((r_c + \lambda)c - (r_d + \lambda))v'_0(c) + \lambda(1 - v_0(c)) + \phi(\min(1, c) - v_0(c)) \\ \rho e_0(c) &= ((r_c + \lambda)c - (r_d + \lambda))e'_0(c) + \phi(\max(0, c - 1) - e_0(c))\end{aligned}$$

Since I am now focused on $c^* > c > 1$, I have:

$$\begin{aligned}v'_0(c) &= \frac{\rho + \lambda + \phi}{(r_c + \lambda)c - (r_d + \lambda)}v_0(c) - \frac{r_d + \lambda + \phi}{(r_c + \lambda)c - (r_d + \lambda)} \\ e'_0(c) &= \frac{\rho + \phi}{(r_c + \lambda)c - (r_d + \lambda)}e_0(c) - \frac{\phi(c - 1)}{(r_c + \lambda)c - (r_d + \lambda)}\end{aligned}$$

These ODEs admit the following solutions:

$$\begin{aligned}v_0(c) &= H_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda}c\right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} + \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} \\ e_0(c) &= K_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda}c\right)^{\frac{\rho + \phi}{r_c + \lambda}} + \frac{\phi}{\phi + \rho - (r_c + \lambda)}c - \frac{\phi}{\rho + \phi} \left(1 + \frac{r_d + \lambda}{\phi + \rho - (r_c + \lambda)}\right)\end{aligned}$$

K_1, H_1 are constants to be determined. Let me first focus on the expression for v_0 on this interval. Value matching at $c = 1$ gives us:

$$v_0(1) = H_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda}\right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} + \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi}$$

I know from the previous section that $v_0(1) < 1$. Since $\frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} > 1$, it must be the case that $H_1 < 0$, meaning that $v_0(\cdot)$ is strictly increasing on $(1, \frac{r_d + \lambda}{r_c + \lambda})$. I also know

that I must have $v_0(c^*) = 1$. Since $v_0\left(\frac{r_d+\lambda}{r_c+\lambda}\right) = \frac{r_d+\lambda+\phi}{\rho+\lambda+\phi} > 1$, and since $v_0(\cdot)$ is strictly increasing for $c < \frac{r_d+\lambda}{r_c+\lambda}$, it must be the case that $c^* < \frac{r_d+\lambda}{r_c+\lambda}$. Using value matching at $c = 1$, I can conclude that H_1 is equal to:

$$H_1 = -\frac{\phi(r_d - r_c)}{(\rho + \lambda + \phi)(\phi + \rho - r_c)} \left(1 - \frac{r_c + \lambda}{r_d + \lambda}\right)^{-\frac{\rho+\lambda+\phi}{r_c+\lambda}} - \frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \lambda + \phi)(\phi + \rho - r_c)}$$

I thus obtain the following functional form for v_0 on $(1, c^*)$:

$$v_0(c) = H_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c\right)^{\frac{\rho+\lambda+\phi}{r_c+\lambda}} + \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi}$$

The threshold c^* is the unique value that satisfies:

$$1 = H_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c^*\right)^{\frac{\rho+\lambda+\phi}{r_c+\lambda}} + \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi}$$

In other words, I have:

$$c^* = \frac{r_d + \lambda}{r_c + \lambda} \left(1 - \left(\frac{1}{H_1} \frac{\rho - r_d}{\rho + \phi + \lambda}\right)^{\frac{r_c+\lambda}{\rho+\lambda+\phi}}\right)$$

Similarly, value matching at $c = 1$ means that I must have $e_0(1) = 0$, implying that the constant K_1 is equal to:

$$K_1 = \frac{\phi(r_d - r_c)}{(\rho + \phi)(\phi + \rho - (r_c + \lambda))} \left(1 - \frac{r_c + \lambda}{r_d + \lambda}\right)^{-\frac{\rho+\phi}{r_c+\lambda}}$$

Some algebra enables me to compute $e'_0(c)$, for $c \in (1, c^*)$:

$$e'_0(c) = \frac{\phi}{\rho + \phi - (r_c + \lambda)} \left[1 - \left(\frac{1 - \frac{r_c+\lambda}{r_d+\lambda}c}{1 - \frac{r_c+\lambda}{r_d+\lambda}}\right)^{\frac{\rho+\phi-(r_c+\lambda)}{r_c+\lambda}}\right] > 0$$

The value function $e_0(\cdot)$ is strictly increasing on the interval $(1, c^*)$ and is equal to:

$$e_0(c) = K_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c\right)^{\frac{\rho+\phi}{r_c+\lambda}} + \frac{\phi}{\phi + \rho - (r_c + \lambda)} c - \frac{\phi}{\rho + \phi} \left(1 + \frac{r_d + \lambda}{\phi + \rho - (r_c + \lambda)}\right)$$

Once again, the equation above has a natural interpretation. Indeed, during a run (i.e. when $c < c^*$), I know the cash level (per unit of outstanding debt) is strictly decreasing. The equity value is only positive due to the fact that when $c \in (1, c^*)$, there is a chance that the stopping time τ_ϕ occurs, at which point cash is distributed to creditors and the remainder $c - 1$ is distributed to the equity holders. The cash $c(t)$ evolves according

to:

$$c(t) = \left(c(0) - \frac{r_d + \lambda}{r_c + \lambda} \right) e^{(r_c + \lambda)t} + \frac{r_d + \lambda}{r_c + \lambda}$$

When $c(0) = c$, I know that $c(t)$ is above 1 as long as $t \leq \tau$, where τ satisfies $c(\tau) = 1$:

$$\tau = \frac{-1}{r_c + \lambda} \ln \left(\frac{1 - \frac{r_c + \lambda}{r_d + \lambda} c}{1 - \frac{r_c + \lambda}{r_d + \lambda}} \right)$$

This means that I can think of the shareholder's value as the probability of τ_ϕ occurring before τ , multiplied by the expected discounted value of $c(\tau_\phi) - 1$ conditioned on $\tau_\phi < \tau$:

$$\begin{aligned} e_0(c) &= \Pr(\tau_\phi < \tau) \times \mathbb{E}^c [e^{-\rho\tau_\phi} (c(\tau_\phi) - 1) | \tau_\phi < \tau] \\ &= \int_0^\tau e^{-\rho x} (c(x) - 1) \phi e^{-\phi x} dx \end{aligned}$$

Further algebra enables me to recover the expression derived previously using the HJB equation for e_0 .

3. $c \in (c^*, \frac{r_d}{r_c})$

On this interval, the value functions v_0 and e_0 satisfy:

$$\begin{aligned} \rho v_0(c) &= r_d + (r_c c - r_d) v_0'(c) + \phi (\min(1, c) - v_0(c)) \\ \rho e_0(c) &= (r_c c - r_d) e_0'(c) + \phi (\max(0, c - 1) - e_0(c)) \end{aligned}$$

Since $c^* > 1$, I have for $c \in (c^*, \frac{r_d}{r_c})$:

$$\begin{aligned} v_0'(c) &= \frac{\rho + \phi}{r_c c - r_d} v_0(c) - \frac{r_d + \phi}{r_c c - r_d} \\ e_0'(c) &= \frac{\rho + \phi}{r_c c - r_d} e_0(c) - \frac{\phi(c - 1)}{r_c c - r_d} \end{aligned}$$

These ODEs admit the following solutions:

$$\begin{aligned} v_0(c) &= H_2 \left(1 - \frac{r_c}{r_d} c \right)^{\frac{\rho + \phi}{r_c}} + \frac{r_d + \phi}{\rho + \phi} \\ e_0(c) &= K_2 \left(1 - \frac{r_c}{r_d} c \right)^{\frac{\rho + \phi}{r_c}} + \frac{\phi}{\phi + \rho - r_c} c - \frac{\phi}{\rho + \phi} \left(1 + \frac{r_d}{\phi + \rho - r_c} \right) \end{aligned}$$

K_2, H_2 are constants to be determined. Value matching at $c = c^*$ gives me the following

equation for H_2 :

$$1 = H_2 \left(1 - \frac{r_c}{r_d} c^*\right)^{\frac{\rho+\phi}{r_c}} + \frac{r_d + \phi}{\rho + \phi}$$

$$\Rightarrow H_2 = \frac{\rho - r_d}{\rho + \phi} \left(1 - \frac{r_c}{r_d} c^*\right)^{-\frac{\rho+\phi}{r_c}}$$

Since $r_d > \rho$, it is clear that $H_2 < 0$, meaning that $v(\cdot)$ is increasing on $(c^*, \frac{r_d}{r_c})$. Let me now look at the function e_0 . Value matching at $c = c^*$ gives the following:

$$K_2 \left(1 - \frac{r_c}{r_d} c^*\right)^{\frac{\rho+\phi}{r_c}} + \frac{\phi}{\phi + \rho - r_c} c^* - \frac{\phi}{\rho + \phi} \left(1 + \frac{r_d}{\phi + \rho - r_c}\right)$$

$$= K_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c^*\right)^{\frac{\rho+\phi}{r_c + \lambda}} + \frac{\phi}{\phi + \rho - (r_c + \lambda)} c^* - \frac{\phi}{\rho + \phi} \left(1 + \frac{r_d + \lambda}{\phi + \rho - (r_c + \lambda)}\right)$$

I conclude this section by highlighting that on this interval $(c^*, \frac{r_d}{r_c})$, creditors are not running but the cash available at the firm is strictly decreasing with time. When noting $c(0) = c$, the evolution of $c(t)$ is as follows:

$$c(t) = \left(c - \frac{r_d}{r_c}\right) e^{r_c t} + \frac{r_d}{r_c}$$

Which is strictly decreasing since $c < \frac{r_d}{r_c}$. Thus, while creditors are not running, the cash reserves are decreasing, and will for sure reach the level c^* at which point it starts becoming optimal for creditors to stop rolling over.

4. $c \in (\frac{r_d}{r_c}, +\infty)$

Since shareholders do not receive dividends, $c(t)$ is a strictly increasing function of time since the initial cash reserve is above $\frac{r_d}{r_c}$. Creditors constantly roll over their debt, and their value function is constant, equal to:

$$v_0(c) = \frac{r_d + \phi}{\rho + \phi}$$

The equity value is then equal to the expected discounted value of $c(\tau_\phi) - 1$, in other words:

$$e_0(c) = \mathbb{E}^{p,c} [e^{-\rho\tau_\phi} (c(\tau_\phi) - 1)]$$

$$= \int_0^{+\infty} \phi e^{-\rho x} \left(\left(c - \frac{r_d}{r_c}\right) e^{r_c x} + \frac{r_d}{r_c} - 1 \right) e^{-\phi x} dx$$

$$= \frac{\phi}{\phi + \rho - r_c} c - \frac{\phi}{\rho + \phi} \left(1 + \frac{r_d}{\phi + \rho - r_c}\right)$$

To summarize, the value function of creditors is equal to:

$$v_0(c) = \begin{cases} \left(\frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \lambda + \phi)(\phi + \rho - r_c)} \right) \left[1 - \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c \right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} \right] + \frac{\phi}{\phi + \rho - r_c} c & \text{for } 0 < c < 1 \\ H_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c \right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} + \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} & \text{for } 1 < c < c^* \\ H_2 \left(1 - \frac{r_c}{r_d} c \right)^{\frac{\rho + \phi}{r_c}} + \frac{r_d + \phi}{\rho + \phi} & \text{for } c^* < c < \frac{r_d}{r_c} \\ \frac{r_d + \phi}{\rho + \phi} & \text{for } c > \frac{r_d}{r_c} \end{cases}$$

Similarly, the value function for shareholders is equal to:

$$e_0(c) = \begin{cases} 0 & \text{for } 0 < c < 1 \\ K_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c \right)^{\frac{\rho + \phi}{r_c + \lambda}} + \frac{\phi}{\phi + \rho - (r_c + \lambda)} c - \frac{\phi}{\rho + \phi} \left(1 + \frac{r_d + \lambda}{\phi + \rho - (r_c + \lambda)} \right) & \text{for } 1 < c < c^* \\ K_2 \left(1 - \frac{r_c}{r_d} c \right)^{\frac{\rho + \phi}{r_c}} + \frac{\phi}{\phi + \rho - r_c} c - \frac{\phi}{\rho + \phi} \left(1 + \frac{r_d}{\phi + \rho - r_c} \right) & \text{for } c^* < c < \frac{r_d}{r_c} \\ \frac{\phi}{\phi + \rho - r_c} c - \frac{\phi}{\rho + \phi} \left(1 + \frac{r_d}{\phi + \rho - r_c} \right) & \text{for } c > \frac{r_d}{r_c} \end{cases}$$

□

Proof of Proposition 6: I have established in Proposition 5 that the threshold c^* must be between 1 and $\frac{r_d + \lambda}{r_c + \lambda}$. I now establish that this cutoff is decreasing as λ increases. In order to do this, I will leverage the strategic substitutability property of the value function $v(0, \cdot; s, S)$ on the interval $[1, \frac{r_d + \lambda}{r_c + \lambda}]$. I first establish a preliminary result.

LEMMA 5. Consider the economy without illiquid asset. For any cutoff strategy $S_{\hat{c}}$ played by all other creditors, creditor i 's optimal value function $v^*(0, \cdot; S_{\hat{c}})$ is continuous, bounded and monotone in c .

The proof of Lemma 5 is straightforward. Take the cutoff strategy $S_{\hat{c}}$, and define the operator $\mathbb{T}_{S_{\hat{c}}}$ that maps any arbitrary continuous bounded function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ into a function $\mathbb{T}_{S_{\hat{c}}}(f)$ defined as follows:

$$\mathbb{T}_{S_{\hat{c}}}(f)(c) = \mathbb{E}^c \left[\int_0^\tau e^{-\rho t} r_d dt + e^{-\rho \tau} 1_{\{\tau = \tau_\lambda\}} \max(1, f(c(\tau))) + e^{-\rho \tau} 1_{\{\tau = \tau_\phi\}} \min(1, c(\tau)) \right],$$

with τ_λ, τ_ϕ independent and exponentially distributed times (with intensity λ and ϕ respectively), $\tau = \tau_b \wedge \tau_\phi \wedge \tau_b$, and with c evolving according to:

$$dc(t) = \left[(r_c + \lambda 1_{\{c(t) < \hat{c}\}}) c(t) - (r_d + \lambda 1_{\{c(t) < \hat{c}\}}) \right] dt$$

The operator $\mathbb{T}_{S_{\hat{c}}}$ maps the space of continuous bounded functions into itself. In addition, it is straightforward to show that the $\mathbb{T}_{S_{\hat{c}}}$ is a contraction. Since the space of continuous bounded functions is complete, $\mathbb{T}_{S_{\hat{c}}}$ admits a unique fixed point – the function $v^*(0, \cdot; S_{\hat{c}})$. Then note that the operator $\mathbb{T}_{S_{\hat{c}}}$ maps weakly increasing functions into weakly increasing

functions. Using a corollary of the contraction mapping theorem, I conclude that $v^*(0, \cdot; S_{\hat{c}})$ is weakly increasing. $\square$

Lemma 5 then enables me to conclude that creditor i 's best response to a cutoff strategy $S_{\hat{c}}$ employed by other creditors is also cutoff. Let $c^*(\hat{c})$ be the best response of a creditor i , when all other creditors use a cutoff strategy $S_{\hat{c}}$, with $\hat{c} \in [1, \frac{r_d + \lambda}{r_c + \lambda}]$. $c^*(\hat{c})$ is a well defined function – indeed, if there were several points $c_1 < \dots < c_n$ that satisfy $v^*(0, c_k; S_{\hat{c}}) = 1$ for all k , then $v^*(0, c; S_{\hat{c}}) = 1$ on the full non-empty interval $[c_1, c_n]$, since I established that $v^*(0, \cdot; S_{\hat{c}})$ is monotone. But that would mean that $\partial_c v^*(0, c; S_{\hat{c}}) = 0$ on such interval, and using the HJB equation satisfied by v^* , would suggest that $v^*(0, c; S_{\hat{c}}) = \frac{r_d + \phi}{\rho + \phi} > 1$ for $c \in [c_1, c_n]$, which is a contradiction.

Now that the best response function $c^*(\hat{c})$ is properly defined, consider $\hat{c} = \frac{r_d + \lambda}{r_c + \lambda}$. When $c(0) = \hat{c} = \frac{r_d + \lambda}{r_c + \lambda}$, the cash per unit of debt outstanding $c(t)$ is constant and equal to $c(0)$. In other words, $v^*(0, \frac{r_d + \lambda}{r_c + \lambda}; S_{\frac{r_d + \lambda}{r_c + \lambda}}) = \frac{r_d + \phi}{\rho + \phi} > 1$. This means that the best response $c^*(\frac{r_d + \lambda}{r_c + \lambda}) < \frac{r_d + \lambda}{r_c + \lambda}$. Consider then $\hat{c} = 1$. The proof of **Proposition 5** shows that $v^*(0, 1; S_1) < 1$, which means that $c^*(1) > 1$. Standard arguments can show that the function $c^*(\cdot)$ is continuous on $[1, \frac{r_d + \lambda}{r_c + \lambda}]$. A straight application of the intermediate value theorem, using the continuity of $c^*(\cdot)$ with the fact that $c^*(1) > 1$ and $c^*(\frac{r_d + \lambda}{r_c + \lambda}) < \frac{r_d + \lambda}{r_c + \lambda}$, delivers the existence of at least one fixed point of the function $c^*(\cdot)$.

Finally, the strategic substitutability property of the payoff function on $[1, \frac{r_d + \lambda}{r_c + \lambda}]$ leads to the result that the function $c^*(\cdot)$ is decreasing on the interval $[1, \frac{r_d + \lambda}{r_c + \lambda}]$, which means that the fixed point of the best response function is unique and on this interval. So far, I have established (using different arguments) the exact result that was proven analytically in the proof of **Lemma 5**. But the strategic substitutability property can also be leveraged further. Indeed, as I established in the proof of **Proposition 1**, when $\frac{r_d + \lambda}{r_c + \lambda} > c_0 > c_f \geq 1$, the stopping time function $\tau_{\mathcal{R}}(c_f; c_0)$ is increasing in the parameter λ . In other words, when the cutoff strategy $S_{\hat{c}}$ is such that $\hat{c} \in [1, \frac{r_d + \lambda}{r_c + \lambda}]$, the cash per unit of debt outstanding $c(t)$ decreases (for $c(0) < [1, \frac{r_d + \lambda}{r_c + \lambda}]$) faster, the smaller λ is. This leads to the conclusion that if $\lambda' > \lambda$, for any $x \in [1, \frac{r_d + \lambda}{r_c + \lambda}]$:

$$c^*(x; \lambda') \leq c^*(x; \lambda),$$

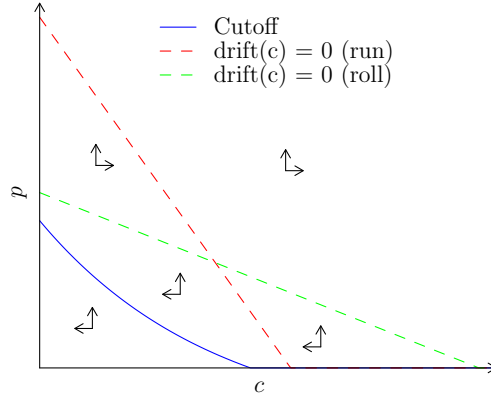
where $c^*(\cdot; \lambda)$ is the best response function for parameter λ . This leads to the conclusion that the fixed point c^* is decreasing in λ . $\square$

Proof of Proposition 7: The analysis of the boundary $c = 0$ needs to be split between two cases. I assume the existence of a symmetric cutoff Markov perfect equilibrium, with equilibrium strategy S , and related run and roll regions $\mathcal{R}_S$ and $\mathcal{N}_{\mathcal{R}_S}$. As a reminder, the locus of points where the cash drift is zero when creditors are rolling satisfies $p = \frac{r_d}{\rho - \mu}(1 - \frac{r_c}{r_d}c)$. This locus of points is a straight line in the (c, p) space, intersecting $c = 0$ at $p = \frac{r_d}{\rho - \mu}$. The locus of points where the cash drift is zero when creditors are running satisfies $p = \frac{r_d + \lambda}{\rho - \mu}(1 - \frac{r_c + \lambda}{r_d + \lambda}c)$. This locus of points is also a straight line in the (c, p) space, intersecting $c = 0$ at $p = \frac{r_d + \lambda}{\rho - \mu}$.

1. Case $\frac{r_d}{\rho-\mu} > \frac{1}{\alpha}$

This case corresponds to a relatively high recovery rate. **Figure 25** illustrates the parameter configuration studied. For any $p < \frac{1}{\alpha} < \frac{r_d}{\rho-\mu}$, the drift of the cash reserve is negative at $c = 0$, meaning that the firm has to sell its illiquid asset and distribute the proceeds to creditors. Since $p < 1/\alpha$, creditors take a loss, and the value function at any such point $(0, p)$ is strictly less than 1. Thus I must have $\{(p, 0) : p < 1/\alpha\} \subset \mathcal{R}_S$. In other words, the segment of the vertical axis that is below $p = 1/\alpha$ must be part of the run region.

Figure 25: Case $\frac{r_d}{\rho-\mu} > \frac{1}{\alpha}$



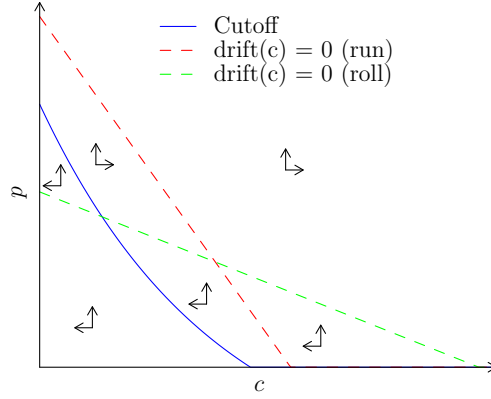
For any $\frac{r_d}{\rho-\mu} > p > \frac{1}{\alpha}$, the drift of the cash reserve is also negative at $c = 0$, meaning that the firm has to sell its illiquid asset and distribute the proceeds to creditors. But at those points of the state space, the recovery rate realized upon the asset sale is greater than 1, meaning that creditors' value must be exactly equal to 1 (since liquidation proceeds in excess of the outstanding debt are paid to shareholders). Since I am assuming that agents indifferent between running and rolling will chose to roll, I must have $\{(p, 0) : p \geq 1/\alpha\} \subset \mathcal{NR}_S$.

2. Case $\frac{r_d}{\rho-\mu} < \frac{1}{\alpha} < \frac{r_d+\lambda}{\rho-\mu}$

This case corresponds to intermediate and low recovery values. **Figure 26** illustrates the parameter configuration studied. I will prove (by contradiction) that I must have $\{(0, p) : p < 1/\alpha\} \subset \mathcal{R}_S$, and $\{(0, p) : p \geq 1/\alpha\} \subset \mathcal{NR}_S$. First, assume that there exists $\bar{p} > 1/\alpha$ such that $\{(0, p) : p < \bar{p}\} \subset \mathcal{R}_S$. If that was the case, for any point of the state space $(0, p)$ where $p \in (1/\alpha, \bar{p})$, since $p < \frac{r_d+\lambda}{\rho-\mu}$ the drift of cash is negative and the firm is forced to sell its illiquid asset. The liquidation proceeds are sufficient for creditors to be fully paid back, meaning that creditors' value function at that point

of the state space has to be equal to 1. In other words, creditors must be rolling over their debt at those points of the state space, leading to a contradiction.

Figure 26: Case $\frac{r_d}{\rho-\mu} < \frac{1}{\alpha} < \frac{r_d+\lambda}{\rho-\mu}$



Now assume that there exists $\bar{p} < 1/\alpha$ such that $\{(0, p) : p \geq \bar{p}\} \subset \mathcal{NR}_S$. Without loss of generality, assume that $\bar{p} = \inf\{\tilde{p} : \{(0, p) : p \geq \tilde{p}\} \subset \mathcal{NR}_S\}$. Take any arbitrary $\epsilon > 0$, since $\bar{p} - \epsilon < \frac{r_d+\lambda}{\rho-\mu}$, at the point $(0, \bar{p} - \epsilon)$ the cash drift is negative, meaning that the firm sells its illiquid assets and distributes the proceeds to creditors. Since $\bar{p} - \epsilon < 1/\alpha$, creditors realize a loss, and their value function at that point is equal to $\alpha(\bar{p} - \epsilon)$. Take $\epsilon \rightarrow 0$, since v^* is continuous on $\mathbb{R}_+^2$, I must have $\lim_{\epsilon \rightarrow 0} v^*(0, \bar{p} - \epsilon) = v^*(0, \bar{p}) = \alpha\bar{p} < 1$. But by construction, $(0, \bar{p}) \in \mathcal{NR}_S$, which means that $v(0\bar{p}) \geq 1$. This is the contradiction I was looking for.

□

Proof of Lemma 3: When p or c are very large, the probability that a run occurs before the illiquid asset matures converges to zero. It is thus clear that when p or c tends to $+\infty$, the value function e verifies:

$$e(c, p) = \mathbb{E}^{p,c} [e^{-\rho\tau_\phi} (p(\tau_\phi) + c(\tau_\phi) - 1)] + o(1)$$

Where the expectation is taken under the following dynamics for p and c :

$$\begin{aligned} dp(t) &= \mu p(t)dt + \sigma p(t)dB(t) \\ dc(t) &= ((\rho - \mu)p(t) + r_c c(t) - r_d) dt \end{aligned}$$

Given $p(0) = p$, since $e^{-\rho t}p(t) = pe^{(\mu-\rho-\frac{1}{2}\sigma^2)t+\sigma B(t)}$, I have:

$$\mathbb{E}^{p,c} [e^{-\rho\tau_\phi}p(\tau_\phi)] = p \int_0^{+\infty} \phi e^{-\phi x} e^{(\mu-\rho)x} dx = \frac{\phi}{\rho + \phi - \mu} p$$

Note also that for any fixed time t , I have:

$$\begin{aligned} \mathbb{E}^{p,c} [e^{-\rho t}c(t)] &= c + \mathbb{E}^{p,c} \left[\int_0^t ((\rho - \mu)p(s) + (r_c - \rho)c(s) - r_d) e^{-\rho s} ds \right] \\ &= c + (1 - e^{-(\rho-\mu)t}) p + (r_c - \rho) \int_0^t \mathbb{E}^{p,c} [e^{-\rho s}c(s)] ds - \frac{r_d}{\rho} (1 - e^{-\rho t}) \end{aligned}$$

Thus, I have:

$$\frac{d}{dt} (\mathbb{E}^{p,c} [e^{-\rho t}c(t)]) = (\rho - \mu)p e^{-(\rho-\mu)t} + (r_c - \rho)\mathbb{E}^{p,c} [e^{-\rho t}c(t)] - r_d e^{-\rho t}$$

This enables me to conclude that:

$$\mathbb{E}^{p,c} [e^{-\rho t}c(t)] = \left(c + \frac{\rho - \mu}{\mu - r_c} (e^{(\mu-r_c)t} - 1) p - \frac{r_d}{r_c} (1 - e^{-r_c t}) \right) e^{-(\rho-r_c)t}$$

And finally:

$$\begin{aligned} \mathbb{E}^{p,c} [e^{-\rho\tau_\phi}c(\tau_\phi)] &= \int_0^{+\infty} \phi \left[\left(c + \frac{\rho - \mu}{\mu - r_c} (e^{(\mu-r_c)t} - 1) p - \frac{r_d}{r_c} (1 - e^{-r_c t}) \right) e^{-(\rho-r_c)t} \right] e^{-\phi t} dt \\ &= \frac{\phi}{\rho + \phi - r_c} \left(c + \frac{\rho - \mu}{r_c - \mu} p - \frac{r_d}{r_c} \right) + \frac{\phi}{\rho + \phi - \mu} \frac{\rho - \mu}{\mu - r_c} p + \frac{\phi}{\rho + \phi} \frac{r_d}{r_c} \end{aligned}$$

I can then conclude that when p, c are large, I have:

$$e(c, p) = \frac{\phi}{\rho + \phi - r_c} c + \frac{\phi}{\rho + \phi - \mu} \left(1 + \frac{\rho - \mu}{\rho + \phi - r_c} \right) p - \frac{\phi}{\rho + \phi} \left(\frac{r_d}{\rho + \phi - r_c} + 1 \right) + o(1)$$

Note finally that when $c > \frac{r_d}{r_c}$, since c is monotone and increasing irrespective of the strategy followed by creditors, I know that creditors never run thereafter, which means that the approximation above is actually an equality. $\square$

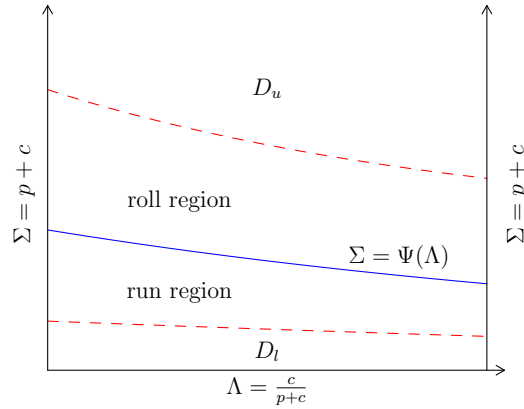
Proof of Proposition 8[Incomplete]: For the purpose of this proof, I find it appropriate to do a change in variable. Instead of working with (c, p) , I will work with (Λ, Σ) defined as follows:

$$\begin{aligned} \Lambda &= \frac{c}{p + c} \\ \Sigma &= p + c \end{aligned}$$

In other words, Σ represents the total asset-to-debt ratio of the firm (or solvency ratio), while Λ represents the liquidity ratio of the firm – in other words, the fraction of the firm's total

assets invested in cash. Note that $\Lambda \in [0, 1]$ while $\Sigma \in \mathbb{R}_+$. The proof will be divided in several components. First, I will prove that there exists a function $\Psi : [0, 1] \rightarrow \mathbb{R}_+$ that satisfies $v^*(\Lambda, \Psi(\Lambda); \Psi) = 1$, where I note (with a slight abuse of notation) $v^*(\cdot, \cdot; \Psi)$ the optimal value function of a given creditor, given that other creditors play a cutoff strategy encoded by the function Ψ (i.e. creditors run when $\Sigma < \Psi(\Lambda)$ and roll otherwise). In other words, points (Λ, Σ) of the state space satisfying $\Sigma = \Psi(\Lambda)$ are indifference points for creditors. Since I will be looking for a fixed point Ψ in the space of functions, I will want to use Schauder's fixed point theorem. Second, I will prove that points (Λ, Σ) of the state space satisfying $\Sigma \geq \Psi(\Lambda)$ are such that $v^*(\Lambda, \Sigma; \Psi) \geq 1$, establishing that above the cutoff boundary $\Sigma = \Psi(\Lambda)$, creditors roll over their debt claims when they have the opportunity to do so. Finally, I will prove that points (Λ, Σ) of the state space satisfying $\Sigma < \Psi(\Lambda)$ are such that $v^*(\Lambda, \Sigma; \Psi) < 1$, establishing that below the cutoff boundary $\Sigma = \Psi(\Lambda)$, creditors run.

Figure 27: Reparametrized State Space (Λ, Σ)



Proof – Part A[Incomplete]: In this section I prove that there exists a function $\Psi : [0, 1] \rightarrow \mathbb{R}_+$ that satisfies $v^*(\Lambda, \Psi(\Lambda); \Psi) = 1$. The dynamics of the state variables (Σ, Λ) can be computed using Ito's lemma, for any strategy S . Indeed, noting $\lambda 1_{\{S(\Sigma(t), \Lambda(t))=1\}} = \lambda(t)$, I have:

$$d\Lambda(t) = (1 - \Lambda(t)) \left((\rho - \mu) - (\rho - r_c)\Lambda(t) - \frac{r_d + \lambda(t)}{\Sigma(t)} + \Lambda(t)(1 - \Lambda(t))\sigma^2 \right) dt - \Lambda(t)(1 - \Lambda(t))\sigma dB_t$$

$$d\Sigma(t) = ((\lambda(t) + \rho)\Sigma(t) - (\rho - r_c)\Lambda(t)\Sigma(t) - (r_d + \lambda(t))) dt + (1 - \Lambda(t))\Sigma(t)\sigma dB_t$$

In the reparametrized state space, the dominance regions are now located as indicated in [Figure 27](#). I now focus on functions $\Psi : [0, 1] \rightarrow \mathbb{R}_+$ such that creditors run whenever $\Sigma < \Psi(\Lambda)$, and roll over otherwise. Going forward, I will therefore refer to creditor i 's

strategy as the function ψ , and all other creditors' strategy as the function Ψ .

The existence of dominance regions imply that there exists $\underline{\Sigma}, \bar{\Sigma}$, both strictly positive, such that for any (ψ, Ψ) , and for any Λ , I have:

$$\begin{aligned} v(\Lambda, \Sigma; \psi, \Psi) &< 1 \text{ if } \Sigma < \underline{\Sigma} \\ v(\Lambda, \Sigma; \psi, \Psi) &> 1 \text{ if } \Sigma > \bar{\Sigma} \end{aligned}$$

Let $A > \bar{\Sigma}$. Note $\mathbb{C}$ the space of continuous functions $\psi : [0, 1] \rightarrow [0, A]$. This space of functions is not compact when equipped with the sup norm, which means that I will not be able to use it when applying Schauder's fixed point theorem. A more restrictive space of functions would be any subset of $\tilde{\mathbb{C}} \subset \mathbb{C}$ that is closed, bounded and equicontinuous. Indeed, since $[0, 1]$ is compact, Arzela-Ascoli's theorem guarantees that any such subspace is compact. I am considering functions that are bounded (i.e. with images in $[0, A]$). I thus need to design a subspace $\tilde{\mathbb{C}} \subset \mathbb{C}$ that is closed and equicontinuous. Let me take for example the space of Lipschitz continuous functions that have the same Lipschitz constant K . In other words, $\tilde{\mathbb{C}} = \{\psi \in \mathbb{C} : |\psi(x_1) - \psi(x_2)| < K|x_1 - x_2|, \forall (x_1, x_2) \in [0, 1]^2\}$.

For $\Psi \in \mathbb{C}$, I define $v^*(\cdot, \cdot; \Psi)$ as the optimal creditor value function, solution of the fixed point problem:

$$\begin{aligned} v^*(\Lambda, \Sigma; \Psi) = \mathbb{E}^{\Lambda, \Sigma} \left[\int_0^\tau e^{-\rho t} r_d dt + e^{-\rho \tau} 1_{\{\tau=\tau_\lambda\}} \max(1, v^*(\Lambda(\tau), \Sigma(\tau); \Psi)) \right. \\ \left. + e^{-\rho \tau} 1_{\{\tau=\tau_\phi\}} \min(1, \Sigma(\tau)) + e^{-\rho \tau} 1_{\{\tau=\tau_b\}} \min(1, \alpha \Sigma(\tau)) \right] \quad (33) \end{aligned}$$

v^* denote the value function for a given creditor, evaluated using the creditor's best response, when all other creditors use strategy Ψ . **Proposition 4** shows that v^* is appropriately defined, and is continuous and bounded. Let $\kappa > 0$ be "small enough" (precise statement to follow). Let $\mathbb{T}$ be the following functional map:

$$\begin{aligned} \mathbb{T} : \tilde{\mathbb{C}} &\rightarrow \tilde{\mathbb{C}} \\ \Psi &\rightarrow (\mathbb{T}\Psi)(\Lambda) = \Psi(\Lambda) + \kappa(1 - v^*(\Lambda, \Psi(\Lambda); \Psi)) \end{aligned}$$

I need to show that this functional map is properly defined. In other words, I need to show that for any $\Psi \in \tilde{\mathbb{C}}$, (a) $\mathbb{T}\Psi$ is Lipschitz with constant K , and (b) the image of $\mathbb{T}\Psi$ is in $[0, A]$.

Part (a) is difficult to prove. Since I have established (via the contraction mapping theorem) that $v^*(\cdot, \cdot; \Psi)$ is continuous in both argument, it means that if $\Psi \in \mathbb{C}$, it must also be the case that $\Lambda \rightarrow v^*(\Psi(\Lambda), \Lambda; \Psi)$ is continuous on $[0, 1]$. In other words, the mapping $\mathbb{T}$ maps continuous functions into continuous functions. Does it map Lipschitz functions with Lipschitz constant K into Lipschitz functions with Lipschitz constant K ? More work needs to be done on this.

Part (b) requires to show that the image of the mapped function $\mathbb{T}\Psi$ is in $[0, A]$. For an arbitrary $\Psi \in \mathbb{C}$ and an arbitrary $\Lambda \in [0, 1]$, if $0 \leq \Psi(\Lambda) < \underline{\Sigma}$, I know that $\kappa(1 - v^*(\Lambda, \Psi(\Lambda); \Psi)) > 0$ due to the existence of the lower dominance region; in other words it must be the case that $(\mathbb{T}\Psi)(\Lambda) > 0$. For an arbitrary $\Psi \in \mathbb{C}$ and an arbitrary $\Lambda \in [0, 1]$, if $A \geq \Psi(\Lambda) > \bar{\Sigma}$, I know that $\kappa(1 - v^*(\Lambda, \Psi(\Lambda); \Psi)) < 0$ due to the existence of the higher dominance region;

in other words it must be the case that $(\mathbb{T}\Psi)(\Lambda) < A$. Finally, note that for any Λ , and any Ψ , $\kappa(1 - v^*(\Lambda, \Psi(\Lambda); \Psi)) \in [-\kappa \frac{r_d - \rho}{\rho + \phi}, \kappa]$. I can thus pick κ small enough such that for any $\Psi \in \mathbb{C}$, the image of $\mathbb{T}\Psi$ is in $[0, A]$, which is what I need to establish that $\mathbb{T}$ is properly defined.

The next step is to show that $\mathbb{T}$ is a continuous mapping, in other words, for any sequence of functions $\{\Psi_n\}_{n \geq 1}$ such that $\Psi_n \in \tilde{\mathbb{C}}$ for any n , if $\Psi_n \rightarrow \Psi$, then $\mathbb{T}\Psi_n \rightarrow \mathbb{T}\Psi$. Pick a sequence of Lipschitz continuous functions Ψ_n that satisfy $\Psi_n \rightarrow \Psi$. Pick $\Lambda \in [0, 1]$, and notice that:

$$\begin{aligned} |\mathbb{T}\Psi_n(\Lambda) - \mathbb{T}\Psi(\Lambda)| &\leq |\Psi_n(\Lambda) - \Psi(\Lambda)| + \kappa | (v(\Lambda, \Psi_n(\Lambda); \Psi_n) - v(\Lambda, \Psi(\Lambda); \Psi_n)) | \\ &\quad + \kappa | (v(\Lambda, \Psi(\Lambda); \Psi_n) - v(\Lambda, \Psi(\Lambda); \Psi)) | \end{aligned}$$

Since by assumption $\Psi_n \rightarrow \Psi$, it is immediate to see that $|\Psi_n(\Lambda) - \Psi(\Lambda)| \rightarrow 0$ when n is large. Similarly, by the continuity of v in its second argument, it must be the case that $|(v(\Lambda, \Psi_n(\Lambda); \Psi_n) - v(\Lambda, \Psi(\Lambda); \Psi_n))| \rightarrow 0$.

I can then use Schauder's fixed point theorem to conclude: since $\mathbb{T}$ is a continuous map from $\tilde{\mathbb{C}}$ into itself, $\mathbb{T}$ must have a fixed point, i.e. there must exist a function Ψ such that $\Psi = \mathbb{T}\Psi$. For such function, for any Λ , I have:

$$1 = v^*(\Psi(\Lambda), \Lambda; \Psi) \tag{34}$$

□

Proof – Part B[Incomplete]: In this section I prove that for any point (Σ, Λ) such that $\Sigma \geq \Psi(\Lambda)$, I must have $v^*(\Sigma, \Lambda; \Psi) \geq 1$. First, note that **Proposition 7** provides the two boundary points of any function Ψ satisfying **equations (34)**:

$$\begin{aligned} \Psi(0) &= 1/\alpha \\ \Psi(1) &= c^* \end{aligned}$$

Now, take any point (Σ, Λ) such that $\Sigma > \Psi(\Lambda)$. Let $\tilde{\tau} = \inf\{t \geq 0 : \Sigma(t) = \Psi(\Lambda(t))\}$ – in other words, the first time at which the state reaches the boundary $\Sigma = \Psi(\Lambda)$. $\tilde{\tau} = \infty$ if the stopping time occurs after a debt maturity τ_λ , an asset maturity τ_ϕ , or a default τ_b . Note $\tau = \tilde{\tau} \wedge \tau_\lambda \wedge \tau_\phi \wedge \tau_b$. The value function v^* can be written:

$$\begin{aligned} v^*(\Sigma, \Lambda; \Psi) &= \mathbb{E}^{\Sigma, \Lambda} \left[\int_0^\tau e^{-\rho t} r_d dt + e^{-\rho \tau} 1_{\{\tau = \tau_\lambda\}} \max(1, v^*(\Sigma(\tau), \Lambda(\tau); \Psi)) + e^{-\rho \tau} 1_{\{\tau = \tilde{\tau}\}} \right. \\ &\quad \left. + e^{-\rho \tau} 1_{\{\tau = \tau_\phi\}} \min(1, \Sigma(\tau)) + e^{-\rho \tau} 1_{\{\tau = \tau_b\}} \min(1, \alpha \Sigma(\tau)) \right] \end{aligned}$$

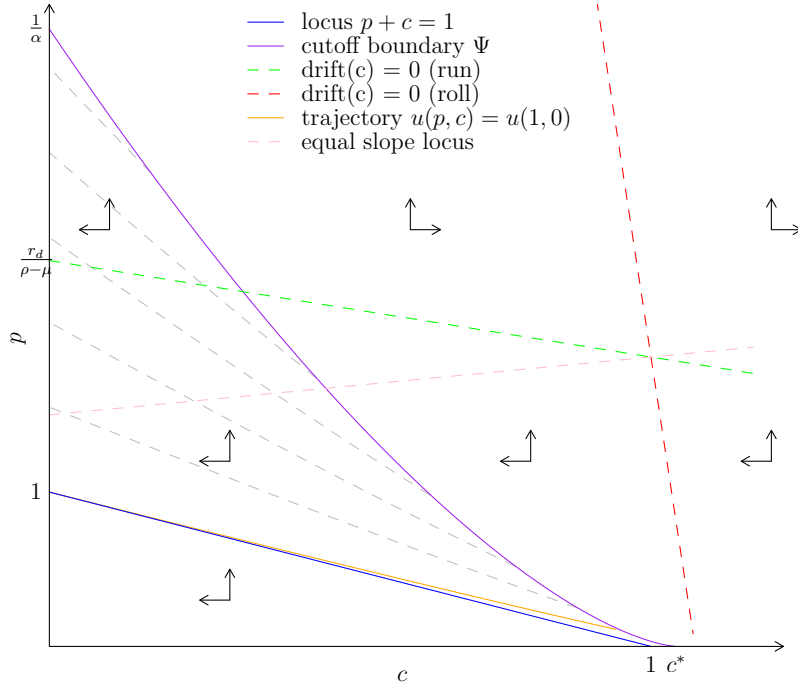
The equality above is obtained after noticing that $v^*(\Sigma(\tilde{\tau}), \Lambda(\tilde{\tau}); \Psi) = 1$.

Proof of Proposition 9: In the case where $\sigma = 0$, the dynamics of the state variables

are as follows:

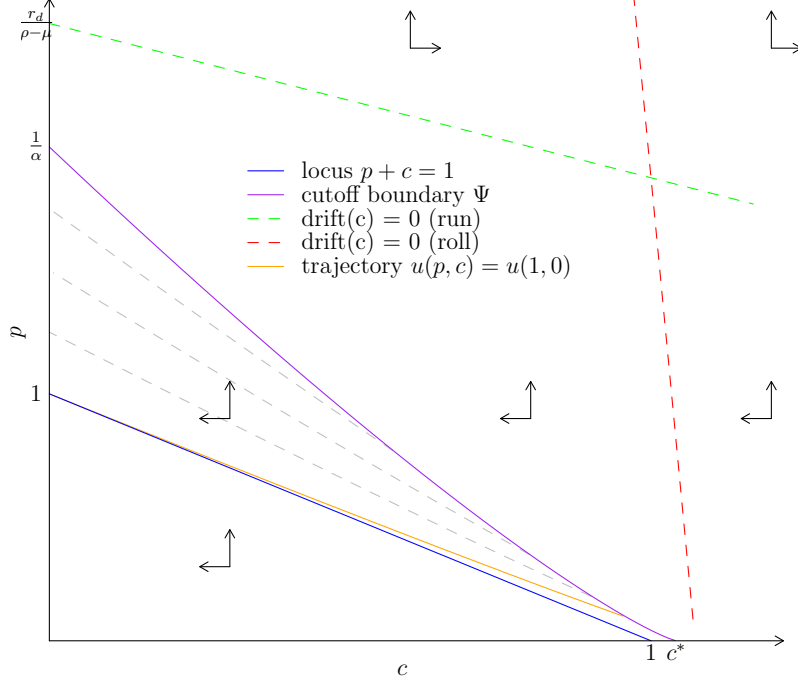
$$\begin{aligned} dp(t) &= (\mu + \lambda 1_{\{(c(t), p(t)) \in \mathcal{R}_S\}}) p(t) dt \\ dc(t) &= ((\rho - \mu) + (r_c + \lambda 1_{\{(c(t), p(t)) \in \mathcal{R}_S\}}) c(t) - (r_d + \lambda 1_{\{(c(t), p(t)) \in \mathcal{R}_S\}})) dt \end{aligned}$$

Figure 28: State Space – $\sigma = 0$ – Case 1



The proof of the proposition is broken down as follows. I first prove that irrespective of the strategy S followed by all creditors, when the state is in the domain $\{(c, p) \in \mathbb{R}_+^2, p + c \leq 1\}$ (the region below the blue line in [Figure 28](#) and [Figure 29](#)), it stays there. Using the fact that the region near the boundary $\{c = 0, p \leq 1\}$ belongs to the lower dominance region, I then solve the HJB equation satisfied by the optimized value function v^* in this region (assuming creditors are running), and show that v^* is strictly less than 1 on the domain $\{(c, p) \in \mathbb{R}_+^2, p + c \leq 1\}$. Using the continuity of the optimized value function v^* at the boundary $\{(c, p) \in \mathbb{R}_+^2 : p + c = 1\}$, I then solve the HJB equation satisfied by v^* in the run domain where $p + c \geq 1$, but where the trajectory $(c(t), p(t))$ conditioned on the initial state (c, p) intersects the locus of points $p + c = 1$ in finite time (the region above the blue line but below the orange line in [Figure 28](#) and [Figure 29](#)). On such domain, I will show that v^* is increasing in (c, p) , leading to the existence of a continuous and decreasing cutoff boundary

Figure 29: State Space – $\sigma = 0$ – Case 2



encoded via the decreasing function Ψ_1 (the portion of the purple line that is below the orange line in Figure 28 and Figure 29). Any equilibrium of the game will have to feature creditors who are running when $p \leq \Psi_1(c)$ and rolling otherwise. I will then focus on the region of the state space $\{(c, p) \in \mathbb{R}_+^2 : p + c \geq 1\}$ where the trajectory $(c(t), p(t))$ conditioned on the initial state (c, p) does not intersect the locus of points $\{(c, p) \in \mathbb{R}_+^2 : p + c = 1\}$ in finite time (the region above the orange line in Figure 28 and Figure 29). For any equilibrium S , I will show that when the initial state is inside this region and when creditors are running, it must be the case that the state $(c(t), p(t))$ stays inside the run region until the deterministic stopping time τ at which $c(\tau) = 0$. With this insight, I solve the HJB equation satisfied by v^* on this domain, assuming that creditors run. On such domain, I will show that v^* is increasing in (c, p) , leading to the existence of a continuous and decreasing cutoff boundary encoded via the decreasing function Ψ_2 (the portion of the purple line that is above the orange line in Figure 28 and Figure 29). Constructing the continuous (and differentiable) decreasing function Ψ from Ψ_1 and Ψ_2 , I then establish that one equilibrium of the game consists of creditors running in the region $\{(c, p) \in \mathbb{R}_+^2 : p \leq \psi(c), c \leq c^*\}$. I then show a procedure to construct other equilibria indexed by some value $\tilde{c} < c^*$, in which agents' cutoff boundary will be (a) for $c < \tilde{c}$, the state trajectory conditioned on a run and which goes through the

point $(\Psi(\tilde{c}), \tilde{c})$, and (b) for $c \geq \tilde{c}$, the function Ψ .

LEMMA 6. Given $\sigma = 0$, for any strategy S followed by all creditors, if $p_0 + c_0 \leq 1$, then $p(t) + c(t) \leq 1$ for all t .

This holds for any strategy S , whether S is an equilibrium strategy or not. In the region $\{(c, p) \in \mathbb{R}_+^2 : p + c \leq 1\}$, the drift of cash is negative, irrespective of whether creditors run or roll. The slope of state trajectories $\frac{dp}{dc}$ when creditors roll is:

$$\frac{dp}{dc} = \frac{\mu p}{(\rho - \mu)p + r_c c - r_d}$$

This slope is negative, but strictly greater than -1 when $p, c \geq 0$ and $p + c \leq 1$. Similarly, the slope of state trajectories $\frac{dp}{dc}$ when creditors run is equal to:

$$\frac{dp}{dc} = \frac{(\mu + \lambda)p}{(\rho - \mu)p + (r_c + \lambda)c - (r_d + \lambda)}$$

This slope is also negative and strictly greater than -1 when $p, c \geq 0$ and $p + c \leq 1$. In other words, once the state is inside the region $p + c \leq 1$, it never leaves such region. $\square$

LEMMA 7. Given initial conditions $(p_0, c_0) \in \mathcal{R}_S$, trajectories of the state vector (c, p) in the run region are described by the equation $u(c, p) = u(c_0, p_0)$, where the function u is defined as follows:

$$u(c, p) := \left(\frac{\rho - \mu}{\mu - r_c} p + \frac{r_d + \lambda}{r_c + \lambda} - c \right) p^{-\frac{r_c + \lambda}{\mu + \lambda}} \quad (35)$$

Integrate the ordinary differential equations satisfied by $p(t), c(t)$ to obtain:

$$p(t) = e^{(\mu + \lambda)t} p_0 \quad (36)$$

$$c(t) = e^{(r_c + \lambda)t} c_0 + \frac{\rho - \mu}{\mu - r_c} (e^{(\mu + \lambda)t} - e^{(r_c + \lambda)t}) p_0 - \frac{r_d + \lambda}{r_c + \lambda} (e^{(r_c + \lambda)t} - 1) \quad (37)$$

Thus, in the (c, p) space, for a given starting level (p_0, c_0) , when a run occurs, the trajectory of the state variables can be described by the locus of points satisfying:

$$c = \left(\frac{p}{p_0} \right)^{\frac{r_c + \lambda}{\mu + \lambda}} c_0 + \frac{\rho - \mu}{\mu - r_c} \left(\frac{p}{p_0} - \left(\frac{p}{p_0} \right)^{\frac{r_c + \lambda}{\mu + \lambda}} \right) p_0 - \frac{r_d + \lambda}{r_c + \lambda} \left(\left(\frac{p}{p_0} \right)^{\frac{r_c + \lambda}{\mu + \lambda}} - 1 \right)$$

This can be re-written: $u(c, p) = u(c_0, p_0)$. $\square$

LEMMA 8. Given $\sigma = 0$, for any strategy S followed by all creditors, the optimized payoff function $v^*(\cdot, \cdot; S)$ is continuous on the domain $\{(c, p) \in \mathbb{R}_+^2 : p + c \leq 1\}$.

The continuity of the optimized value function on the interior of the domain $\{(c, p) \in \mathbb{R}_+^2 : p + c \leq 1\}$ follows from standard contraction mapping arguments. Indeed, v^* is the fixed point of a contraction operator that maps continuous bounded functions into itself. This is the case since the flow payoff of the expectation defining v^* is constant and thus continuous. State trajectories are continuous functions, irrespective of the strategy S employed by creditors. In the neighborhood of the locus $\{(c, p) : c = 0, p \leq 1\}$, the slope of any state trajectory is negative and strictly greater than -1 , and at $c = 0$, the terminal value αp is continuous in (c, p) . Finally, in the neighborhood of the locus $\{(c, p) : p + c = 1\}$, the slope of any state trajectory is also negative and strictly greater than -1 , meaning that if the initial state is in a neighborhood of $\{(c, p) : p + c = 1\}$ but *outside* (i.e. if there exists $\epsilon > 0$ small such that $p_0 + c_0 = 1 + \epsilon$), it enters the domain $\{(c, p) \in \mathbb{R}_+^2 : p + c \leq 1\}$ “shortly” thereafter irrespective of the strategy S followed by creditors. Thus, v^* must be continuous on the domain $\{(c, p) \in \mathbb{R}_+^2 : p + c \leq 1\}$, and specifically at the boundaries of such domain. $\square$

Since $v(0, p; s, S) = \alpha p < 1$ for any s, S , there must exist a neighborhood $\mathcal{O}$ of points near the locus $\{(c, p) : c = 0, p \leq 1\}$ such that $v(c, p; s, S) < 1$ for any $(c, p) \in \mathcal{O}$. Thus $\mathcal{O}$ belongs to the dominance region $\mathcal{D}_l$. I can thus solve the HJB equation satisfied by v^* in such neighborhood, assuming creditors run. I do so in the following lemma.

LEMMA 9. Given $\sigma = 0$, in any equilibrium characterized by strategy S , the region $\{(c, p) : p + c \leq 1\}$ belongs to the run region $\mathcal{R}_S$. The *equilibrium* value function v^* in such region is equal to:

$$v^*(c, p) = \frac{\phi}{\rho + \phi - \mu} \left(1 + \frac{\rho - \mu}{\rho + \phi - r_c} \right) p + \frac{\phi}{\rho + \phi - r_c} c + \frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \phi - r_c)(\rho + \phi + \lambda)} + p^{\frac{\rho + \phi + \lambda}{\mu + \lambda}} G_0(u(c, p)) \quad (38)$$

where the function G_0 is defined as follows:

$$G_0(x) = \left[\left(\alpha - \frac{\phi}{\rho + \phi - \mu} \left(1 + \frac{\rho - \mu}{\rho + \phi - r_c} \right) \right) \zeta(x) - \frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \phi - r_c)(\rho + \phi + \lambda)} \right] \zeta(x)^{-\frac{\rho + \phi + \lambda}{\mu + \lambda}},$$

with the function ζ defined implicitly on the interval $\left[\frac{(r_d + \lambda)(\mu + \lambda)}{(\mu - r_c)(r_c + \lambda)} \left(\frac{r_d + \lambda}{\rho - \mu} \right)^{-\frac{r_c + \lambda}{\mu + \lambda}}, +\infty \right)$ via:

$$x = \left(\frac{\rho - \mu}{\mu - r_c} \zeta(x) + \frac{r_d + \lambda}{r_c + \lambda} \right) \zeta(x)^{-\frac{r_c + \lambda}{\mu + \lambda}} \quad (39)$$

In the domain $\{(c, p) \in \mathbb{R}_+^2 : p + c \leq 1\}$, the function v^* is strictly increasing in (c, p) , strictly less than 1, and reaches its maximum at the point $p = 1, c = 0$.

In the neighborhood $\mathcal{O}$ previously defined, the optimized payoff function $v^*(\cdot, \cdot; S)$ must

satisfy the following HJB (omitting the dependence on the strategy S for simplicity):

$$\begin{aligned} \rho v^*(c, p) = r_d + (\mu + \lambda)p \frac{\partial v^*}{\partial p}(c, p) + ((\rho - \mu)p + (r_c + \lambda)c - (r_d + \lambda)) \frac{\partial v^*}{\partial c}(c, p) \\ + \lambda(1 - v^*(c, p)) + \phi(p + c - v^*(c, p)) \end{aligned} \quad (40)$$

This is a linear first order partial differential equation that can be solved analytically with the method of characteristics. The characteristic system can be written:

$$\frac{dp}{(\mu + \lambda)p} = \frac{dc}{(\rho - \mu)p + (r_c + \lambda)c - (r_d + \lambda)} = \frac{dv^*}{(\rho + \lambda + \phi)v^* - (r_d + \lambda + \phi(p + c))}$$

I first focus on the equality:

$$\frac{dp}{(\mu + \lambda)p} = \frac{dc}{(\rho - \mu)p + (r_c + \lambda)c - (r_d + \lambda)}$$

The general solution to this ODE is:

$$c(p) = K_1 p^{\frac{r_c + \lambda}{\mu + \lambda}} + \frac{\rho - \mu}{\mu - r_c} p + \frac{r_d + \lambda}{r_c + \lambda}$$

Thus the constant of integration K_1 can be expressed as:

$$K_1 = \left(c - \frac{\rho - \mu}{\mu - r_c} p - \frac{r_d + \lambda}{r_c + \lambda} \right) p^{-\frac{r_c + \lambda}{\mu + \lambda}}$$

Consider now the transformed variable $z = v^* - \frac{\phi}{\rho + \phi - \mu} \left(1 + \frac{\rho - \mu}{\rho + \phi - r_c} \right) p - \frac{\phi}{\rho + \phi - r_c} c$. For this variable, I have:

$$\frac{dz}{(\rho + \phi + \lambda)z - \frac{(r_d + \lambda)(\rho - r_c)}{\rho + \phi - r_c}} = \frac{dp}{(\mu + \lambda)p}$$

The general solution to this ODE is:

$$z(p) = \frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \phi - r_c)(\rho + \lambda + \phi)} + K_2 p^{\frac{\rho + \phi + \lambda}{\mu + \lambda}}$$

Thus the constant of integration K_2 can be expressed as:

$$\begin{aligned} K_2 &= \left(z - \frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \phi - r_c)(\rho + \lambda + \phi)} \right) p^{-\frac{\rho + \phi + \lambda}{\mu + \lambda}} \\ &= \left(v^* - \frac{\phi}{\rho + \phi - \mu} \left(1 + \frac{\rho - \mu}{\rho + \phi - r_c} \right) p - \frac{\phi}{\rho + \phi - r_c} c - \frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \phi - r_c)(\rho + \phi + \lambda)} \right) p^{-\frac{\rho + \phi + \lambda}{\mu + \lambda}} \end{aligned}$$

The general solution of [equation \(40\)](#) can then be written as follows:

$$v^*(c, p) = \frac{\phi}{\rho + \phi - \mu} \left(1 + \frac{\rho - \mu}{\rho + \phi - r_c} \right) p + \frac{\phi}{\rho + \phi - r_c} c + \frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \phi - r_c)(\rho + \phi + \lambda)} + p^{\frac{\rho + \phi + \lambda}{\mu + \lambda}} G_0 \left(\left(\frac{\rho - \mu}{\mu - r_c} p + \frac{r_d + \lambda}{r_c + \lambda} - c \right) p^{-\frac{r_c + \lambda}{\mu + \lambda}} \right)$$

For a function $G_0 : \mathbb{R} \rightarrow \mathbb{R}$ to be determined using boundary conditions. For $c = 0$ and $p < 1$, I know that $v^*(0, p) = \alpha p$. Define the function $u_0(\cdot)$ as follows:

$$u_0 : \left(0, \frac{r_d + \lambda}{\rho - \mu} \right) \longrightarrow \left(\frac{(r_d + \lambda)(\mu + \lambda)}{(\mu - r_c)(r_c + \lambda)} \left(\frac{r_d + \lambda}{\rho - \mu} \right)^{-\frac{r_c + \lambda}{\mu + \lambda}}, +\infty \right) \\ p \longrightarrow u_0(p) := \left(\frac{\rho - \mu}{\mu - r_c} p + \frac{r_d + \lambda}{r_c + \lambda} \right) p^{-\frac{r_c + \lambda}{\mu + \lambda}} \quad (41)$$

Using this new function u_0 , the boundary condition $v^*(0, p) = \alpha p$ can be expressed as follows:

$$G_0(u_0(p)) = \left[\left(\alpha - \frac{\phi}{\rho + \phi - \mu} \left(1 + \frac{\rho - \mu}{\rho + \phi - r_c} \right) \right) p - \frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \phi - r_c)(\rho + \phi + \lambda)} \right] p^{-\frac{\rho + \phi + \lambda}{\mu + \lambda}}$$

Note that the function $u_0(\cdot)$ is strictly decreasing on $(0, \frac{r_d + \lambda}{\rho - \mu}]$, which means that on that interval, its inverse $\zeta := u_0^{-1}$ is appropriately defined. In other words, ζ is implicitly defined on $\left[\frac{(r_d + \lambda)(\mu + \lambda)}{(\mu - r_c)(r_c + \lambda)} \left(\frac{r_d + \lambda}{\rho - \mu} \right)^{-\frac{r_c + \lambda}{\mu + \lambda}}, +\infty \right)$ via:

$$x = \left(\frac{\rho - \mu}{\mu - r_c} \zeta(x) + \frac{r_d + \lambda}{r_c + \lambda} \right) \zeta(x)^{-\frac{r_c + \lambda}{\mu + \lambda}}$$

I conclude that G_0 can be written:

$$G_0(x) = \left[\left(\alpha - \frac{\phi}{\rho + \phi - \mu} \left(1 + \frac{\rho - \mu}{\rho + \phi - r_c} \right) \right) \zeta(x) - \frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \phi - r_c)(\rho + \phi + \lambda)} \right] \zeta(x)^{-\frac{\rho + \phi + \lambda}{\mu + \lambda}}$$

The function $\zeta(\cdot)$ is strictly decreasing over its interval of definition, with values between $\frac{r_d + \lambda}{\rho - \mu}$ and 0. I also have $\zeta'(x) = \frac{(\mu + \lambda)\zeta(x)^{1 + \frac{r_c + \lambda}{\mu + \lambda}}}{(\rho - \mu)\zeta(x) - (r_d + \lambda)}$ on the relevant interval. Finally, for $p \rightarrow 0$, I have $u_0(p) \underset{p \rightarrow 0}{=} \frac{r_d + \lambda}{r_c + \lambda} p^{-\frac{r_c + \lambda}{\mu + \lambda}} + o(p^{-\frac{r_c + \lambda}{\mu + \lambda}})$, which means that for $x \rightarrow +\infty$:

$$\zeta(x) \underset{x \rightarrow +\infty}{=} \left(\frac{r_c + \lambda}{r_d + \lambda} x \right)^{-\frac{\mu + \lambda}{r_c + \lambda}} + o\left(x^{-\frac{\mu + \lambda}{r_c + \lambda}}\right) \\ G_0(x) \underset{x \rightarrow +\infty}{=} -\frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \phi - r_c)(\rho + \phi + \lambda)} \left(\frac{r_c + \lambda}{r_d + \lambda} x \right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} + o\left(x^{\frac{\rho + \lambda + \phi}{r_c + \lambda}}\right)$$

But this means that:

$$\lim_{p \rightarrow 0} p^{\frac{\rho+\lambda+\phi}{\mu+\lambda}} G_0 \left(\left(\frac{\rho-\mu}{\mu-r_c} p + \frac{r_d+\lambda}{r_c+\lambda} - c \right) p^{-\frac{r_c+\lambda}{\mu+\lambda}} \right) = -\frac{(r_d+\lambda)(\rho-r_c)}{(\rho+\phi-r_c)(\rho+\phi+\lambda)} \left(1 - \frac{r_c+\lambda}{r_d+\lambda} c \right)^{\frac{\rho+\lambda+\phi}{r_c+\lambda}}$$

In other words, for $p = 0$, I retrieve [equation \(14\)](#) on the interval $c \in [0, 1]$:

$$v^*(c, 0) = \left(\frac{(r_d+\lambda)(\rho-r_c)}{(\rho+\lambda+\phi)(\phi+\rho-r_c)} \right) \left[1 - \left(1 - \frac{r_c+\lambda}{r_d+\lambda} c \right)^{\frac{\rho+\lambda+\phi}{r_c+\lambda}} \right] + \frac{\phi}{\phi+\rho-r_c} c$$

Note also that G_0 admits the following derivative:

$$G'_0(x) = \frac{1}{\mu+\lambda} \left[\frac{(r_d+\lambda)(\rho-r_c)}{\rho+\phi-r_c} - \zeta(x) \left(\alpha(\rho+\phi-\mu) - \phi \left(1 + \frac{\rho-\mu}{\rho+\phi-r_c} \right) \right) \right] \zeta'(x) \zeta(x)^{-\frac{\rho+\phi+\lambda}{\mu+\lambda}-1}$$

Since I know that $\zeta(x) \in \left(0, \frac{r_d+\lambda}{\rho-\mu} \right]$, it means that the term in brackets above is between $\frac{(r_d+\lambda)(\rho-r_c)}{\rho+\phi-r_c}$ (when $\zeta(x) \rightarrow 0$) and $(r_d+\lambda)(1-\alpha) \left(1 + \frac{\phi}{\rho-\mu} \right)$ (when $\zeta(x) = \frac{r_d+\lambda}{\rho-\mu}$). Since ζ is strictly decreasing, it means that G_0 is strictly decreasing over the interval of interest. The function u admits the following partial derivatives:

$$\begin{aligned} \frac{\partial u}{\partial p}(c, p) &= \frac{p^{-\frac{r_c+\lambda}{\mu+\lambda}-1}}{\mu+\lambda} ((\rho-\mu)p + (r_c+\lambda)c - (r_d+\lambda)) < 0 \\ \frac{\partial u}{\partial c}(c, p) &= -p^{-\frac{r_c+\lambda}{\mu+\lambda}} < 0 \end{aligned}$$

The function v^* obtained in [equation \(38\)](#) is thus strictly increasing in c and strictly increasing in p . In particular, on the compact set $\{(c, p) \in \mathbb{R}_+^2 : p+c \leq 1\}$, v^* must reach its maximum at a point $(1-p, p)$, for some $p \in [0, 1]$. I show that this maximum is reached at $p = 1, c = 0$. Indeed, note that $v^*(1-p, p)$ is equal to:

$$\begin{aligned} v^*(1-p, p) &= \frac{\phi(\rho-r_c)}{(\rho+\phi-\mu)(\rho+\phi-r_c)} p + \frac{\phi}{\rho+\phi-r_c} \\ &\quad + \frac{(r_d+\lambda)(\rho-r_c)}{(\rho+\phi-r_c)(\rho+\phi+\lambda)} + p^{\frac{\rho+\phi+\lambda}{\mu+\lambda}} G_0(u(p, 1-p)) \end{aligned}$$

Since $u(1-p, p) = \left(\frac{\rho-r_c}{\mu-r_c} p + \frac{r_d-r_c}{r_c+\lambda} \right) p^{-\frac{r_c+\lambda}{\mu+\lambda}}$, the function $p \rightarrow u(1-p, p)$ is strictly decreasing for $p \in (0, 1)$, and since G_0 is decreasing, it must be the case that $p \rightarrow G_0(u(1-p, p))$ is increasing for $p \in [0, 1]$. I thus conclude that $p \rightarrow v^*(1-p, p)$ is increasing in p , which means that v^* attains its maximum at the point $c = 0, p = 1$. Such maximum is equal to $\alpha < 1$, meaning that $v^*(c, p) < 1$ in $\{(c, p) : p+c \leq 1\}$. $\square$

I have thus characterized completely the *equilibrium* value function v^* on the domain $\{(c, p) \in \mathbb{R}_+^2 : p+c \leq 1\}$ for *any* equilibrium of the game. Using the continuity of the optimized value function v^* at the boundary $p+c = 1$, I now solve the HJB equation

satisfied by v^* in the run domain where $p + c \geq 1$, but where the trajectory $(c(t), p(t))$ conditioned on the initial state (c, p) intersects the locus of points $p + c = 1$ in finite time (the region above the blue line $p + c = 1$ but below the orange line $u(c, p) = u(1, 0)$ in [Figure 28](#) and [Figure 29](#)). Using [Lemma 7](#), this region can be characterized as follows: $\{(c, p) \in \mathbb{R}_+^2 : p + c \geq 1, u(c, p) \geq u(0, 1)\}$.

LEMMA 10. Given $\sigma = 0$, in any equilibrium characterized by strategy S , there exists a continuous, once differentiable and decreasing function $\Psi_1 : [\underline{c}, c^*] \rightarrow \mathbb{R}_+$ (for some $\underline{c} \in (0, c^*)$) such that the region $\{(c, p) : p + c \geq 1, u(c, p) \geq u(0, 1), p < \Psi_1(c)\}$ belongs to the run region $\mathcal{R}_S$. The *equilibrium* value function v^* in such region is equal to:

$$v^*(c, p) = \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} + p^{\frac{\rho + \lambda + \phi}{\mu + \lambda}} G_1(u(c, p)) \quad (42)$$

where the function G_1 is defined as follows:

$$G_1(x) = G_0(x) + \frac{\phi}{\rho + \phi - r_c} \left(\frac{\rho - r_c}{\rho + \phi - \mu} \chi(x) - \frac{r_d - r_c}{\rho + \phi + \lambda} \right) \chi(x)^{-\frac{\rho + \lambda + \phi}{\mu + \lambda}},$$

with the function χ defined implicitly on the interval $\left[\frac{(r_d - r_c)(\mu + \lambda)}{(\mu - r_c)(r_c + \lambda)} \left(\frac{r_d - r_c}{\rho - r_c} \right)^{-\frac{r_c + \lambda}{\mu + \lambda}}, +\infty \right)$ via:

$$x = \left(\frac{\rho - r_c}{\mu - r_c} \chi(x) + \frac{r_d - r_c}{r_c + \lambda} \right) \chi(x)^{-\frac{r_c + \lambda}{\mu + \lambda}} \quad (43)$$

In the domain $\{(c, p) : p + c \geq 1, u(c, p) \geq u(0, 1), p < \Psi_1(c)\}$, the function v^* is strictly increasing in (c, p) , less than 1, and the cutoff function Ψ_1 satisfies $v^*(c, \Psi_1(c)) = 1$ for any $c \in [\underline{c}, c^*]$.

Thanks to [Lemma 8](#), I know that the value function v^* is continuous at the boundary $p + c = 1$. In the neighborhood of such boundary but where $p + c > 1$, since the optimized payoff function v^* is strictly less than 1, it must satisfy the following HJB (omitting the dependence on the strategy S for notational simplicity):

$$\begin{aligned} \rho v^*(c, p) = r_d + (\mu + \lambda)p \frac{\partial v^*}{\partial p}(c, p) + ((\rho - \mu)p + (r_c + \lambda)c - (r_d + \lambda)) \frac{\partial v^*}{\partial c}(c, p) \\ + \lambda(1 - v^*(c, p)) + \phi(1 - v^*(c, p)) \end{aligned} \quad (44)$$

This is a linear first order partial differential equation that can be solved analytically. The characteristic system of this equation is:

$$\frac{dp}{(\mu + \lambda)p} = \frac{dc}{(\rho - \mu)p + (r_c + \lambda)c - (r_d + \lambda)} = \frac{dv^*}{(\rho + \lambda + \phi)v^* - (r_d + \lambda + \phi)}$$

I first focus on the equality:

$$\frac{dp}{(\mu + \lambda)p} = \frac{dc}{(\rho - \mu)p + (r_c + \lambda)c - (r_d + \lambda)}$$

The general solution to this ODE is:

$$c(p) = K_1 p^{\frac{r_c + \lambda}{\mu + \lambda}} + \frac{\rho - \mu}{\mu - r_c} p + \frac{r_d + \lambda}{r_c + \lambda}$$

Thus the constant of integration K_1 can be expressed as:

$$K_1 = \left(c - \frac{\rho - \mu}{\mu - r_c} p - \frac{r_d + \lambda}{r_c + \lambda} \right) p^{-\frac{r_c + \lambda}{\mu + \lambda}}$$

Consider then the equality:

$$\frac{dp}{(\mu + \lambda)p} = \frac{dv^*}{(\rho + \lambda + \phi)v^* - (r_d + \lambda + \phi)}$$

The general solution to this ODE is:

$$v^*(p) = K_2 p^{\frac{\rho + \lambda + \phi}{\mu + \lambda}} + \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi}$$

Thus the constant of integration K_2 can be expressed as:

$$K_2 = \left(v^* - \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} \right) p^{-\frac{\rho + \lambda + \phi}{\mu + \lambda}}$$

Thus, the general solution of [equation \(44\)](#) can be written as follows:

$$v^*(c, p) = \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} + p^{\frac{\rho + \lambda + \phi}{\mu + \lambda}} G_1 \left(\left(\frac{\rho - \mu}{\mu - r_c} p + \frac{r_d + \lambda}{r_c + \lambda} - c \right) p^{-\frac{r_c + \lambda}{\mu + \lambda}} \right)$$

For a function $G_1 : \mathbb{R} \rightarrow \mathbb{R}$ to be determined using boundary conditions. Using the continuity of v^* at the locus of points $p + c = 1$ and [equation \(38\)](#), I obtain the following boundary condition:

$$\begin{aligned} \frac{\phi}{\rho + \phi - \mu} \left(1 + \frac{\rho - \mu}{\rho + \phi - r_c} \right) p + \frac{\phi}{\rho + \phi - r_c} (1 - p) \\ + \frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \phi - r_c)(\rho + \phi + \lambda)} + p^{\frac{\rho + \lambda + \phi}{\mu + \lambda}} G_0(u(1 - p, p)) \\ = \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} + p^{\frac{\rho + \lambda + \phi}{\mu + \lambda}} G_1(u(1 - p, p)) \end{aligned}$$

This means that the function G_1 verifies, for $0 \leq p \leq 1$:

$$G_1(u(1-p, p)) = G_0(u(1-p, p)) + \frac{\phi}{\rho + \phi - r_c} \left(\frac{\rho - r_c}{\rho + \phi - \mu} p - \frac{r_d - r_c}{\rho + \phi + \lambda} \right) p^{-\frac{\rho + \phi + \lambda}{\mu + \lambda}}$$

Note that $u(1-p, p) = \left(\frac{\rho - r_c}{\mu - r_c} p + \frac{r_d - r_c}{r_c + \lambda} \right) p^{-\frac{r_c + \lambda}{\mu + \lambda}}$. The function $p \rightarrow u(1-p, p)$ is thus strictly decreasing for $p \in \left(0, \frac{r_d - r_c}{\rho - r_c} \right]$, which means that its reciprocal function χ is appropriately defined on $\left[\frac{(r_d - r_c)(\mu + \lambda)}{(\mu - r_c)(r_c + \lambda)} \left(\frac{r_d - r_c}{\rho - r_c} \right)^{-\frac{r_c + \lambda}{\mu + \lambda}}, +\infty \right)$, strictly decreasing, and it verifies:

$$x = \left(\frac{\rho - r_c}{\mu - r_c} \chi(x) + \frac{r_d - r_c}{r_c + \lambda} \right) \chi(x)^{-\frac{r_c + \lambda}{\mu + \lambda}}$$

Note also that the derivative of χ verifies $\frac{\chi'(x)}{\chi(x)} = \frac{(\mu + \lambda)\chi(x)^{\frac{r_c + \lambda}{\mu + \lambda}}}{(\rho - r_c)\chi(x) - (r_d - r_c)}$. The image of the function χ is $\left(0, \frac{r_d - r_c}{\rho - r_c} \right]$. I can then conclude that G_1 takes the following form:

$$G_1(x) = G_0(x) + \frac{\phi}{\rho + \phi - r_c} \left(\frac{\rho - r_c}{\rho + \phi - \mu} \chi(x) - \frac{r_d - r_c}{\rho + \phi + \lambda} \right) \chi(x)^{-\frac{\rho + \phi + \lambda}{\mu + \lambda}}$$

Note that G_1 is a strictly decreasing function, since:

$$G'_1(x) = G'_0(x) + \frac{\phi}{\rho + \phi - r_c} \frac{\chi'(x)}{\chi(x)} \frac{\chi(x)^{-\frac{\rho + \phi + \lambda}{\mu + \lambda}}}{\mu + \lambda} ((r_d - r_c) - (\rho - r_c)\chi(x))$$

This is negative since G_0 is decreasing, χ is decreasing, and since the image of χ is $\left(0, \frac{r_d - r_c}{\rho - r_c} \right]$. Thus v^* is strictly increasing in c . Note then that u is decreasing in c . Since $u(c, p) \geq u(0, 1)$, I must have $p \leq 1$ and the following inequality:

$$c \leq \frac{\rho - \mu}{\mu - r_c} p + \frac{r_d + \lambda}{r_c + \lambda} - \left(\frac{\rho - \mu}{\mu - r_c} + \frac{r_d + \lambda}{r_c + \lambda} \right) p^{\frac{r_c + \lambda}{\mu + \lambda}}$$

Using this inequality, I can sign the partial derivative of u w.r.t. p :

$$\begin{aligned} \frac{\partial u}{\partial p}(c, p) &= \frac{p^{-\frac{r_c + \lambda}{\mu + \lambda} - 1}}{\mu + \lambda} ((\rho - \mu)p + (r_c + \lambda)c - (r_d + \lambda)) \\ &\leq \frac{p^{-\frac{r_c + \lambda}{\mu + \lambda} - 1}}{\mu + \lambda} \left(\left(\rho - \mu + \frac{(\rho - \mu)(r_c + \lambda)}{\mu - r_c} \right) p - \left(r_d + \lambda + \frac{(\rho - \mu)(r_c + \lambda)}{\mu - r_c} \right) p^{\frac{r_c + \lambda}{\mu + \lambda}} \right) \\ &< 0 \end{aligned}$$

The latter inequality is derived using [Assumption 1](#) and $p \leq 1$. Thus u is decreasing in (c, p) , and it must be the case that the function v^* is increasing in both p and c . This directly gives the existence of a decreasing function $\Psi_1(c)$ such that $v^*(c, p) = 1$ for $p = \Psi_1(c)$. Ψ_1 is

defined implicitly via:

$$1 = \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} + \Psi_1(c)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} G_1(u(c, \Psi_1(c))) \quad (45)$$

Note that I also need to impose/verify that $u(c, \Psi_1(c)) \geq u(0, 1)$, since the form of value function v^* determined above is only valid for $u(c, p) \geq u(0, 1)$. In other words, noting $\underline{c} = \min\{c : 1 = \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} + p^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} G_1(u(c, p)), u(c, p) \geq u(0, 1)\}$, the function Ψ_1 is implicitly defined on the interval $[\underline{c}, c^*]$. I can invoke the implicit function theorem using [equation \(45\)](#) to establish that Ψ_1 is continuous and differentiable on $(\underline{c}, c^*)$. Finally and as an aside, for $p \rightarrow 0$, I have $u(1 - p, p) \underset{p \rightarrow 0}{=} \frac{r_d - r_c}{r_c + \lambda} p^{-\frac{r_c + \lambda}{\mu + \lambda}} + o(p^{-\frac{r_c + \lambda}{\mu + \lambda}})$, which means that for $x \rightarrow \infty$:

$$\chi(x) \underset{x \rightarrow +\infty}{=} \left(\frac{r_c + \lambda}{r_d - r_c} x \right)^{-\frac{\mu + \lambda}{r_c + \lambda}} + o\left(x^{-\frac{\mu + \lambda}{r_c + \lambda}}\right)$$

But this means that:

$$\begin{aligned} \lim_{p \rightarrow 0} p^{\frac{\rho + \lambda + \phi}{\mu + \lambda}} G_1 \left(\left(\frac{\rho - \mu}{\mu - r_c} p + \frac{r_d + \lambda}{r_c + \lambda} - c \right) p^{-\frac{r_c + \lambda}{\mu + \lambda}} \right) = \\ - \left[\frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \phi - r_c)(\rho + \phi + \lambda)} + \frac{\phi(r_d - r_c)}{(\rho + \phi - r_c)(\rho + \phi + \lambda)} \left(\frac{r_d + \lambda}{r_d - r_c} \right)^{\frac{\rho + \phi + \lambda}{r_c + \lambda}} \right] \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c \right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} \end{aligned}$$

In other words, for $p = 0$, I retrieve [equation \(14\)](#) on the interval $c \in [1, c^*]$:

$$v^*(0, c) = H_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c \right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} + \frac{r_d + \phi + \lambda}{\rho + \phi + \lambda}$$

This confirms that $\Psi(c^*) = 0$, where c^* is the cutoff determined in [Proposition 5](#). $\square$

The previous lemma provides a complete characterization of the *equilibrium* value function v^* in the region $\{(c, p) \in \mathbb{R}_+^2 : p + c \geq 1, u(c, p) \geq u(0, 1)\}$ illustrated in [Figure 28](#) and [Figure 29](#) as the region above the blue line ($p + c = 1$) but below the orange line ($u(c, p) = u(1, 0)$). The next step is to show that for any equilibrium, if the initial state is such that $u(c, p) \leq u(0, 1)$ (i.e. above the orange line in [Figure 28](#) and [Figure 29](#)) and creditors are running, the state never leaves the run region.

LEMMA 11. In any equilibrium of the game with $\sigma = 0$ and characterized by the equilibrium strategy S , if $(c_0, p_0) \in \mathcal{R}_S$ (in other if the initial state is in the run region) and if $u(c_0, p_0) \leq u(0, 1)$, then $(c(t), p(t)) \in \mathcal{R}_S$ for all t (in other words the deterministic trajectory of the state stays inside the run region).

Take any equilibrium of the game, characterized by its equilibrium strategy S . Take $(c_0, p_0) \in \mathcal{R}_S$. Agents have perfect foresight, and can thus predict perfectly the trajectory of the state $(c(t), p(t))$ given initial conditions (c_0, p_0) . Imagine that there exists a time $t > 0$

such that $(c(t), p(t)) \in \mathcal{NR}_S$, in other words at time t , the state is in the roll region. Since the trajectory is a continuous function of time, there must exist a finite deterministic stopping time $\tilde{\tau} = \min\{t : (c(t), p(t)) \in \mathcal{NR}_S\}$. Since $(c_0, p_0) \in \mathcal{R}_S$, I must have $v^*(c_0, p_0; S) < 1$. Since both $\tilde{\tau}$ and τ_b (the bankruptcy time, potentially infinite) are deterministic stopping times, it must be the case that $\tilde{\tau} < \tau_b$. Noting $\tau = \tilde{\tau} \wedge \tau_\lambda \wedge \tau_\phi$, I can then express $v^*(c_0, p_0; S)$ as follows:

$$\begin{aligned} v^*(c_0, p_0; S) = \mathbb{E}^{c_0, p_0} & \left[\int_0^\tau e^{-\rho t} r_d dt + e^{-\rho \tau} 1_{\{\tau = \tau_\lambda\}} \max(1, v^*(c(\tau_\lambda), p(\tau_\lambda); S)) \right. \\ & \left. + e^{-\rho \tau} 1_{\{\tau = \tau_\phi\}} \min(1, p(\tau_\phi) + c(\tau_\phi)) + e^{-\rho \tau} 1_{\{\tau = \tilde{\tau}\}} v^*(c(\tilde{\tau}), p(\tilde{\tau}); S) \right] \end{aligned}$$

Given the definition of $\tilde{\tau}$, I must have $v^*(c(\tilde{\tau}), p(\tilde{\tau}); S) \geq 1$. For $t < \tilde{\tau}$, creditors are running, meaning that the trajectory of the state space can be described for $t < \tilde{\tau}$ by $u(c(t), p(t)) = u(c_0, p_0)$. Since by assumption $u(p_0, c_0) \leq u(0, 1)$, it must be the case that for $t < \tilde{\tau}$, $p(t) + c(t) \geq 1$, meaning that conditional on $\tau_\phi \leq \tilde{\tau} \wedge \tau_\lambda$, $p(\tau_\phi) + c(\tau_\phi) \geq 1$. Using the law of iterated expectations, the value function for creditor i must satisfy the following inequality:

$$\begin{aligned} v^*(c_0, p_0; S) \geq & \Pr(\tilde{\tau} \leq \tau_\lambda \wedge \tau_\phi) \times \mathbb{E} \left[\frac{r_d}{\rho} (1 - e^{-\rho \tilde{\tau}}) + e^{-\rho \tilde{\tau}} | \tilde{\tau} \leq \tau_\lambda \wedge \tau_\phi \right] \\ & + \Pr(\tau_\lambda \leq \tilde{\tau} \wedge \tau_\phi) \times \mathbb{E} \left[\frac{r_d}{\rho} (1 - e^{-\rho \tau_\lambda}) + e^{-\rho \tau_\lambda} | \tau_\lambda \leq \tilde{\tau} \wedge \tau_\phi \right] \\ & + \Pr(\tau_\phi \leq \tilde{\tau} \wedge \tau_\lambda) \times \mathbb{E} \left[\frac{r_d}{\rho} (1 - e^{-\rho \tau_\phi}) + e^{-\rho \tau_\phi} | \tau_\phi \leq \tilde{\tau} \wedge \tau_\lambda \right] \end{aligned}$$

In other words, I have expressed $v^*(c_0, p_0; S)$ as a weighted average of 1 and $r_d/\rho > 1$, meaning that I must have $v^*(c_0, p_0; S) > 1$. This is the contradiction I was seeking. $\square$

The previous lemma will prove useful in characterizing the value function v^* in the domain $\{(c, p) \in \mathbb{R}_+^2 : p + c \geq 1, u(c, p) \leq u(0, 1)\}$ without worrying about what HJB equation to solve (i.e. whether assuming creditors run or roll). Indeed, once the state is in the run regime, it must stay there, meaning that v^* can be computed by solving the HJB equation corresponding to the run regime. I can now tackle the value function v^* when $p + c \geq 1$ and when $u(c, p) \leq u(0, 1)$. Before doing so, I prove an interim useful lemma.

LEMMA 12. Given $\sigma = 0$, the firm default stopping time τ_b is a deterministic strictly increasing function of the initial state (c, p) .

The proof is straightforward, using [Equation \(37\)](#), which describes the evolution of c as a function of time. Indeed, τ_b is defined implicitly as the smallest positive root of:

$$c + \frac{\rho - \mu}{r_c - \mu} (1 - e^{-(r_c - \mu)\tau_b}) p = \frac{r_d + \lambda}{r_c + \lambda} (1 - e^{-(r_c + \lambda)\tau_b}) \quad (46)$$

Viewed as a function of τ_b , the left handside of [Equation \(46\)](#) is an increasing and convex function, intersecting the vertical axis at c . The left handside is also an increasing function of

both c and p . Viewed as a function of τ_b , the right handside of Equation (46) is an increasing and concave function, intersecting the vertical axis at 0. The right handside is independent of both c and p . Thus, if τ_b is finite (in other words if there exists a solution to Equation (46)), such solution must be increasing in c and p . $\square$

LEMMA 13. Given $\sigma = 0$, there exists an equilibrium characterized by the function Ψ , constructed from a continuous, once differentiable and strictly decreasing function $\Psi_2 : [0, \underline{c}] \rightarrow \mathbb{R}_+$ and from the function $\Psi_1 : [\underline{c}, c^*] \rightarrow \mathbb{R}_+$ defined via equation (45). In such equilibrium, the domain $\{(c, p) : p + c \geq 1, u(c, p) \leq u(1, 0), p < \Psi_2(c)\}$ belongs to the run region $\mathcal{R}_S$. The *equilibrium* value function v^* in such region is equal to:

$$v^*(c, p) = \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} + p^{\frac{\rho + \lambda + \phi}{\mu + \lambda}} G_2(u(c, p)) \quad (47)$$

where the function G_2 is defined as follows:

$$G_2(x) = \left(\alpha \zeta(x) - \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} \right) \zeta(x)^{-\frac{\rho + \lambda + \phi}{\mu + \lambda}},$$

In the domain $\{(c, p) : p + c \geq 1, u(c, p) \leq u(0, 1), p < \Psi_2(c)\}$, the function v^* is strictly increasing in (c, p) , less than 1, and the function Ψ_2 verifies $v^*(\Psi_2(c), c) = 1$. The domains $\{(c, p) : p + c \leq 1\}$ and $\{(c, p) : p + c \geq 1, u(c, p) \geq u(0, 1), p < \Psi_1(c)\}$ complete the run region $\mathcal{R}_S$, and the optimized value function in those regions is the one established in Lemma 9 and Lemma 10. The roll region $\mathcal{NR}_S$ is the complement of the run region, and within $\mathcal{NR}_S$, the creditor's *equilibrium* value function v^* is greater than or equal to 1.

Since the drift of cash is negative for $c = 0$ and $p \in \left(1, \min\left(1/\alpha, \frac{r_d}{\rho - \mu}\right)\right)$ (irrespective of whether creditors run or roll), the value function v and its optimized counterpart v^* must be continuous at such boundary. Since $\alpha p < 1$, the optimized value function v^* must be strictly less than 1 on such boundary, and it therefore solves equation (44) on a neighborhood of such boundary. The general solution of this equation was showed to be:

$$v^*(c, p) = \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} + p^{\frac{\rho + \phi + \lambda}{\mu + \lambda}} G_2\left(\left(\frac{\rho - \mu}{\mu - r_c} p + \frac{r_d + \lambda}{r_c + \lambda} - c\right) p^{-\frac{r_c + \lambda}{\mu + \lambda}}\right) \quad (48)$$

For a function $G_2 : \mathbb{R} \rightarrow \mathbb{R}$ to be determined using boundary conditions. Consider states (c, p) in the run region and such that $u(0, 1/\alpha) < u(c, p) \leq u(0, 1)$. Starting from such states, the trajectory $(c(t), p(t))$ must stay within the run region and must hit the boundary $c = 0$ at a time τ such that $1 < p(\tau) < 1/\alpha$. I can then use the boundary condition $v^*(0, p) = \alpha p$, which gives:

$$G_2(x) = \left(\alpha \zeta(x) - \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} \right) \zeta(x)^{-\frac{\rho + \phi + \lambda}{\mu + \lambda}}$$

Establishing the strict monotonicity of v^* in c and p is then straightforward, using Lemma 11: once the state is in the run region, it never leaves, meaning that the value function v^* that

solves [equation \(44\)](#) can also be expressed as the following expectation:

$$v^*(c, p) = \mathbb{E}^{c, p} \left[\int_0^\tau e^{-\rho t} r_d dt + e^{-\rho \tau} 1_{\{\tau \neq \tau_b(c, p)\}} + e^{-\rho \tau} 1_{\{\tau = \tau_b(c, p)\}} \alpha p e^{(\mu + \lambda) \tau_b(c, p)} \right],$$

where $\tau := \tau_\phi \wedge \tau_\lambda \wedge \tau_b(c, p)$, where I have introduced $\tau_b(c, p) := \inf\{t : c(t) = 0 \mid (c(0), p(0)) = (c, p)\}$, and where $\alpha p e^{(\mu + \lambda) \tau_b(c, p)} < 1$. [Lemma 12](#) shows that the stopping time $\tau_b(p, c)$ is increasing in c and in p , which means that the terminal payoff in the expectation above is also increasing in (c, p) , and therefore that the entire expectation above is increasing in both c and p . The strict monotonicity of v^* in c and p means that for a given $c \in [0, \underline{c}]$, there is exists a unique $p = \Psi_2(c)$ such that $v^*(c, \Psi_2(c)) = 1$. The function Ψ_2 satisfies:

$$1 = \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} + \Psi_2(c)^{\frac{\rho + \phi + \lambda}{\mu + \lambda}} G_2(u(c, \Psi_2(c))) \quad (49)$$

Note that since $c \rightarrow u(c, p)$ is decreasing in c , since $p \rightarrow u(c, p)$ is decreasing in p , and since G_2 is decreasing, the function Ψ_2 must be strictly decreasing in c . The implicit function theorem finally provides for the differentiability of the function Ψ_2 on $(0, \underline{c})$.

Consider then the strategy consisting of running when $c < c^*$ and $p < \Psi(c)$, with Ψ the function constructed from Ψ_1 and Ψ_2 . Clearly such strategy is an equilibrium: the proof above shows that $v^*(c, p; S) < 1$ when $c < c^*$ and $p < \Psi(c)$. Now take any state (c, p) such that $c \geq c^*$ or $p \geq \Psi(c)$. If the cash drift $(\rho - \mu)p + r_c c - r_d$ is positive, it will stay positive, and the cash reserve of the firm diverges to $+\infty$, leading to an optimized value function $v^*(c, p; S) = \frac{r_d + \phi}{\rho + \phi} \geq 1$. If the cash drift is negative at such point (c, p) , let $\tau_1 := \inf\{t : (\rho - \mu)p(t) + r_c c(t) - r_d = 0\}$ be the first time at which the cash drift is zero. Let $\tau_2 := \inf\{t : p(t) = \Psi(c(t))\}$ be the first time at which the state hits the boundary encoded by the function Ψ . Both τ_1 and τ_2 are deterministic stopping times. Note that if $\tau_2 > \tau_1$, then for any $t > \tau_1$, the cash drift is strictly positive, meaning that the value function $v^*(p(\tau_1), c(\tau_1)) = \frac{r_d + \phi}{\rho + \phi}$. Note also that by definition of τ_2 , I must have $v^*(p(\tau_2), c(\tau_2)) = 1$. Finally, note that either (i) the bankruptcy time τ_b is greater than τ_2 (if $1/\alpha > \frac{r_d}{\rho - \mu}$), or (ii) at the bankruptcy date τ_b , $\alpha p(\tau_b) \geq 1$. In other words, noting $\tau = \tau_1 \wedge \tau_2 \wedge \tau_\lambda \wedge \tau_\phi \wedge \tau_b$:

$$\begin{aligned} v^*(c, p; S) = \mathbb{E}^{p, c} & \left[\int_0^\tau e^{-\rho t} r_d dt + e^{-\rho \tau} 1_{\{\tau = \tau_\lambda\}} \max(1, v^*(p(\tau_\lambda), c(\tau_\lambda); S)) \right. \\ & + e^{-\rho \tau} 1_{\{\tau = \tau_\phi\}} \min(1, p(\tau_\phi) + c(\tau_\phi)) + e^{-\rho \tau} 1_{\{\tau = \tau_1\}} v^*(p(\tau_1), c(\tau_1); S) \\ & \left. + e^{-\rho \tau} 1_{\{\tau = \tau_2\}} v^*(p(\tau_2), c(\tau_2); S) + e^{-\rho \tau} 1_{\{\tau = \tau_b\}} \min(1, \alpha p(\tau_b)) \right] \end{aligned}$$

This can be bounded below by:

$$\begin{aligned}
v^*(p_0, c_0; S) &\geq \Pr(\tau = \tau_\lambda) \times \mathbb{E} \left[\frac{r_d}{\rho} (1 - e^{-\rho\tau_\lambda}) + e^{-\rho\tau_\lambda} | \tau = \tau_\lambda \right] \\
&+ \Pr(\tau = \tau_\phi) \times \mathbb{E} \left[\frac{r_d}{\rho} (1 - e^{-\rho\tau_\phi}) + e^{-\rho\tau_\phi} | \tau = \tau_\phi \right] \\
&+ \Pr(\tau = \tau_1) \times \mathbb{E} \left[\frac{r_d}{\rho} (1 - e^{-\rho\tau_\phi}) + e^{-\rho\tau_\phi} \frac{r_d + \phi}{\rho + \phi} | \tau = \tau_1 \right] \\
&+ \Pr(\tau = \tau_2) \times \mathbb{E} \left[\frac{r_d}{\rho} (1 - e^{-\rho\tau_\phi}) + e^{-\rho\tau_\phi} | \tau = \tau_2 \right] \\
&+ \Pr(\tau = \tau_b) \times \mathbb{E} \left[\frac{r_d}{\rho} (1 - e^{-\rho\tau_\phi}) + e^{-\rho\tau_\phi} | \tau = \tau_b \right]
\end{aligned}$$

The above bound is greater than or equal to 1, which was to be proven. $\square$

The previous lemma describes one symmetric Markov perfect equilibrium in cutoff strategies. It turns out that under certain parameter configurations, this equilibrium is not unique, as the next lemma shows.

LEMMA 14. Given $\sigma = 0$, if $\frac{r_d - \mu}{\rho - \mu} > \frac{1}{\alpha}$, there exists a unique symmetric Markov perfect equilibrium characterized by the strictly decreasing function Ψ . Otherwise, there is a continuum of symmetric Markov perfect equilibria, indexed by a value $\tilde{c} < c^*$, in which creditors' cutoff boundary consists of (a) for $c < \tilde{c}$, the state trajectory $u(c, p) = u(\Psi(\tilde{c}), \tilde{c})$ (i.e. the state trajectory conditioned on a run and which goes through the point $(\Psi(\tilde{c}), \tilde{c})$), and (b) for $c \geq \tilde{c}$, the function Ψ .

Notice first that the locus of points (c, p) such that the slope of state trajectories is the same, whether creditors run or roll, satisfies $p = \frac{1}{\rho - \mu} [(\mu - r_c)c - (\mu - r_d)]$. Assume for now that $\frac{r_d - \mu}{\rho - \mu} < \frac{1}{\alpha}$. Let us define two key threshold values $\hat{c}_1$ and $\hat{c}_2$ as follows. $\hat{c}_1$ is defined via:

$$\frac{1}{\rho - \mu} ((\mu - r_c)\hat{c}_1 - (\mu - r_d)) = \Psi(\hat{c}_1)$$

Graphically, $\hat{c}_1$ is the c -coordinate of the intersection of (i) the locus of points of “equal slopes” (in other words the locus of points such that the slope of state trajectories is the same, whether creditors run or roll – the dashed pink line in [Figure 28](#)), and (ii) the cutoff boundary Ψ . Then define $\hat{c}_2$ via:

$$\hat{c}_2 = \frac{\rho - \mu}{\mu - r_c} \Psi(\hat{c}_2) + \frac{r_d}{r_c} - \frac{r_d \mu}{r_c(\mu - r_c)} \left(\frac{\rho - \mu}{r_d} \Psi(\hat{c}_2) \right)^{r_d/\mu}$$

Graphically, $\hat{c}_2$ is the c -coordinate of the intersection of (i) the state trajectory conditioned on creditors rolling and intersecting the axis $c = 0$ at the point $p = \frac{r_d}{\rho - \mu}$, and (ii) the cutoff boundary Ψ . Then take an arbitrary $0 < \tilde{c} < \hat{c}_1 \wedge \hat{c}_2$. Construct the function $\Psi_{\tilde{c}}$, consisting of (a) for $c < \tilde{c}$, the state trajectory $u(c, p) = u(\tilde{c}, \Psi(\tilde{c}))$ (i.e. the downward sloping state

trajectory conditioned on a run and which goes through the point $(\tilde{c}, \Psi(\tilde{c}))$, and (b) for $c \geq \tilde{c}$, the function $p = \Psi(c)$. Note that $\Psi_{\tilde{c}} \leq \Psi$. Consider then the strategy $S_{\tilde{c}}$ consisting of running when $c < c^*$ and $p < \Psi_{\tilde{c}}(c)$, and rolling otherwise. Such strategy must be an equilibrium. The various lemmas above show that $v^*(c, p; S_{\tilde{c}}) < 1$ when $c < c^*$ and $p < \Psi_{\tilde{c}}(c) \leq \Psi(c)$. Now take any state (c, p) such that $c \geq c^*$ or $p \geq \Psi_{\tilde{c}}(c)$. If either $c \geq c^*$ or $p \geq \Psi(c)$, I can mimic the proof in [Lemma 13](#) to show that $v^*(c, p; S_{\tilde{c}}) \geq 1$. The only case that needs to be studied is when $c < c^*$ and $\Psi(c) \geq p \geq \Psi_{\tilde{c}}(c)$. In such case however, since creditors are rolling, the state trajectory satisfies, for some given u :

$$c = \frac{\rho - \mu}{\mu - r_c} p + \frac{r_d}{r_c} - u p^{r_d/\mu}$$

By construction, such state trajectory never “re-enters” the run region $\mathcal{R}_{S_{\tilde{c}}}$ since I picked $\tilde{c} < \hat{c}_1$, meaning that the slope of the state trajectory $\frac{dp}{dc}$ is “steeper” (i.e. more negative in the (c, p) plane) than the slope of the state trajectory conditioned on a run. By construction, such state trajectory will intersect the “roll zero cash drift” locus $\{(c, p) : (\rho - \mu)p + r_c c - r_d = 0\}$ before hitting $c = 0$ since I picked $\tilde{c} < \hat{c}_2$. At the time the state trajectory hits the “roll zero cash drift” locus, the value function must be equal to $\frac{r_d + \phi}{\rho + \phi}$ since after that time, the drift of cash will stay positive forever, with a cash reserve increasing monotonically. Thus the value function at (c, p) must also satisfy $v^*(c, p; S_{\tilde{c}}) = \frac{r_d + \phi}{\rho + \phi}$. $\square$

Proof of Lemma 3: When p or c are very large, the probability that a run occurs before the illiquid asset matures converges to zero. It is thus clear that when p or c tends to $+\infty$, the value function e verifies:

$$e(c, p) = \mathbb{E}^{p, c} [e^{-\rho\tau_\phi} (p(\tau_\phi) + c(\tau_\phi) - 1)] + o(1)$$

Where the expectation is taken under the following dynamics for p and c :

$$\begin{aligned} dp(t) &= \mu p(t) dt + \sigma p(t) dB(t) \\ dc(t) &= ((\rho - \mu)p(t) + r_c c(t) - r_d) dt \end{aligned}$$

Given $p(0) = p$, since $e^{-\rho t} p(t) = p e^{(\mu - \rho - \frac{1}{2}\sigma^2)t + \sigma B(t)}$, I have:

$$\mathbb{E}^{p, c} [e^{-\rho\tau_\phi} p(\tau_\phi)] = p \int_0^{+\infty} \phi e^{-\phi x} e^{(\mu - \rho)x} dx = \frac{\phi}{\rho + \phi - \mu} p$$

Note also that for any fixed time t , I have:

$$\begin{aligned} \mathbb{E}^{p, c} [e^{-\rho t} c(t)] &= c + \mathbb{E}^{p, c} \left[\int_0^t ((\rho - \mu)p(s) + (r_c - \rho)c(s) - r_d) e^{-\rho s} ds \right] \\ &= c + (1 - e^{-(\rho - \mu)t}) p + (r_c - \rho) \int_0^t \mathbb{E}^{p, c} [e^{-\rho s} c(s)] ds - \frac{r_d}{\rho} (1 - e^{-\rho t}) \end{aligned}$$

Thus, I have:

$$\frac{d}{dt} (\mathbb{E}^{p,c} [e^{-\rho t} c(t)]) = (\rho - \mu)p e^{-(\rho - \mu)t} + (r_c - \rho) \mathbb{E}^{p,c} [e^{-\rho t} c(t)] - r_d e^{-\rho t}$$

This enables me to conclude that:

$$\mathbb{E}^{p,c} [e^{-\rho t} c(t)] = \left(c + \frac{\rho - \mu}{\mu - r_c} (e^{(\mu - r_c)t} - 1) p - \frac{r_d}{r_c} (1 - e^{-r_c t}) \right) e^{-(\rho - r_c)t}$$

And finally:

$$\begin{aligned} \mathbb{E}^{p,c} [e^{-\rho \tau_\phi} c(\tau_\phi)] &= \int_0^{+\infty} \phi \left[\left(c + \frac{\rho - \mu}{\mu - r_c} (e^{(\mu - r_c)t} - 1) p - \frac{r_d}{r_c} (1 - e^{-r_c t}) \right) e^{-(\rho - r_c)t} \right] e^{-\phi t} dt \\ &= \frac{\phi}{\rho + \phi - r_c} \left(c + \frac{\rho - \mu}{r_c - \mu} p - \frac{r_d}{r_c} \right) + \frac{\phi}{\rho + \phi - \mu} \frac{\rho - \mu}{\mu - r_c} p + \frac{\phi}{\rho + \phi} \frac{r_d}{r_c} \end{aligned}$$

I can then conclude that when p, c are large, I have:

$$e(c, p) = \frac{\phi}{\rho + \phi - r_c} c + \frac{\phi}{\rho + \phi - \mu} \left(1 + \frac{\rho - \mu}{\rho + \phi - r_c} \right) p - \frac{\phi}{\rho + \phi} \left(\frac{r_d}{\rho + \phi - r_c} + 1 \right) + o(1)$$

Note finally that when $c > \frac{r_d}{r_c}$, since c is monotone and increasing irrespective of the strategy followed by creditors, I know that creditors never run thereafter, which means that the approximation above is actually an equality. $\square$

Proof of Lemma 4: First, when the firm is subject to a run, the dynamic equation for cash holdings leads to:

$$\begin{aligned} d(e^{-r_c t} C(t)) &= e^{-r_c t} ((\rho - \mu)P(t) - (r_d + \lambda)D(t)) dt \\ C(t) &= C + e^{r_c t} \int_0^t e^{-r_c s} \left((\rho - \mu)P e^{(\mu - \frac{1}{2}\sigma^2)s + \sigma B(s)} - (r_d + \lambda)D e^{-\lambda s} \right) ds \end{aligned}$$

Thus, I can compute $\mathbb{E}^{P,D,C} [C(t)]$ as follows:

$$\begin{aligned} \mathbb{E}^{P,D,C} [C(t)] &= C + e^{r_c t} \int_0^t e^{-r_c s} ((\rho - \mu)P e^{\mu s} - (r_d + \lambda)D e^{-\lambda s}) ds \\ &= C + \frac{\rho - \mu}{r_c - \mu} P (e^{r_c t} - e^{\mu t}) - \frac{r_d + \lambda}{r_c + \lambda} D (e^{r_c t} - e^{-\lambda t}) \end{aligned}$$

Giving the above preliminary calculations, some algebra leads me to conclude that the net

expected cash-outflow over a time period Δ has the following form:

$$\begin{aligned}\mathbb{E}^{P,D,C} \left[\int_0^\Delta ((r_d + \lambda)D(s) - (\rho - \mu)P(s) - r_c C(s)) ds \right] \\ = \frac{r_d + \lambda}{r_c + \lambda} D(e^{r_c \Delta} - e^{-\lambda \Delta}) - \frac{\rho - \mu}{r_c - \mu} P(e^{r_c \Delta} - e^{\mu \Delta}) - r_c \Delta C\end{aligned}$$

Thus, the requirement that the liquidity reserve C be greater than the expected net cash outflow over the time period $[t, t + \Delta]$ is equivalent to:

$$C \geq \frac{1}{1 + r_c \Delta} \left[\frac{r_d + \lambda}{r_c + \lambda} D(e^{r_c \Delta} - e^{-\lambda \Delta}) - \frac{\rho - \mu}{r_c - \mu} P(e^{r_c \Delta} - e^{\mu \Delta}) \right]$$

When $\Delta \rightarrow 0$, I obtain:

$$C \geq (r_d + \lambda)D\Delta - (\rho - \mu)P\Delta$$

□

Proof of Proposition 11: I will assume that there is a threshold $\hat{c}$ such that for $c \leq \hat{c}$, it is optimal for the large creditor to run, while for $c \geq \hat{c}$, it is optimal for the large creditor to continue rolling. The threshold $\hat{c}$ will need to verify $w_0(c) - 1 < (c - 1)w'_0(c)$ for $c < \hat{c}$, and $w_0(c) - 1 > (c - 1)w'_0(c)$ for $c > \hat{c}$. Note that I do not necessarily impose the condition $w_0(\hat{c}) - 1 = (\hat{c} - 1)w'_0(\hat{c})$ since w_0 is a-priori not continuously differentiable at $c = \hat{c}$. I will establish that $\hat{c} \in (1, \frac{r_d + \lambda}{r_c + \lambda})$, but for the time being, no specific assumption is made on the value of $\hat{c}$.

1. $c \in (0, 1 \wedge \hat{c})$

On this interval, the large creditor is not rolling over its debt claim. The value function w_0 must satisfy:

$$(\rho + \lambda)w_0(c) = r_d + \lambda + ((r_c + \lambda)c - (r_d + \lambda))w'_0(c) + \phi(\min(1, c) - w_0(c))$$

Since on this interval, $c < 1$, I have:

$$w'_0(c) = \frac{\rho + \lambda + \phi}{(r_c + \lambda)c - (r_d + \lambda)} w_0(c) - \frac{r_d + \lambda + \phi c}{(r_c + \lambda)c - (r_d + \lambda)}$$

Given the boundary $w_0(0) = 0$, this ordinary differential equation admits the following solution:

$$w_0(c) = \left(\frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \lambda + \phi)(\phi + \rho - r_c)} \right) \left[1 - \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c \right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} \right] + \frac{\phi}{\phi + \rho - r_c} c$$

Note that the expression for $w_0(c)$ on this interval is identical to the expression of v_0 . Given the expression for w_0 , I can immediately conclude that w_0 is strictly increasing

on $(0, 1 \wedge \hat{c})$. I then compute $w_0(c) - 1 - (c - 1)w'_0(c)$:

$$\begin{aligned} w_0(c) - 1 - (c - 1)w'_0(c) &= \left(\frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \lambda + \phi)(\phi + \rho - r_c)} \right) \left[1 - \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c \right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} \right] \\ &\quad + \frac{\phi}{\phi + \rho - r_c} c - 1 - (c - 1) \left[\frac{\rho - r_c}{\rho + \phi - r_c} \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c \right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda} - 1} + \frac{\phi}{\phi + \rho - r_c} \right] \end{aligned}$$

This simplifies to:

$$\begin{aligned} \left(\frac{\rho - r_c}{r_c + \lambda} \right) \left[\frac{r_d + \lambda}{\rho + \phi + \lambda} \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c \right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} - \frac{r_d - r_c}{\rho + \phi - r_c} \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c \right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda} - 1} \right. \\ \left. - \frac{(\rho + \phi - r_d)(r_c + \lambda)}{(\rho + \phi + \lambda)(\rho + \phi - r_c)} \right] \end{aligned}$$

It can then be showed that $w_0(c) - 1 - (c - 1)w'_0(c)$ is strictly decreasing on $[0, 1]$, with value zero when $c = 0$. In other words, on $(0, 1]$, $w_0(c) - 1 < (c - 1)w'_0(c)$, justifying a posteriori our assumption that the large creditor runs on this interval.

2. $c \in (1, \hat{c} \wedge \frac{r_d + \lambda}{r_c + \lambda})$

On this interval, the value function w_0 satisfies:

$$(\rho + \lambda)w_0(c) = r_d + \lambda + ((r_c + \lambda)c - (r_d + \lambda)) w'_0(c) + \phi (\min(1, c) - w_0(c))$$

Since I am now focused on $\hat{c} > c > 1$, I have:

$$w'_0(c) = \frac{\rho + \lambda + \phi}{(r_c + \lambda)c - (r_d + \lambda)} w_0(c) - \frac{r_d + \lambda + \phi}{(r_c + \lambda)c - (r_d + \lambda)}$$

Using value matching at $c = 1$, this ordinary differential equation admits the following solution:

$$w_0(c) = H_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c \right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} + \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi}$$

Where the constant H_1 was determined previously to be equal to:

$$H_1 = - \frac{\phi(r_d - r_c)}{(\rho + \lambda + \phi)(\phi + \rho - r_c)} \left(1 - \frac{r_c + \lambda}{r_d + \lambda} \right)^{-\frac{\rho + \lambda + \phi}{r_c + \lambda}} - \frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \lambda + \phi)(\phi + \rho - r_c)}$$

I thus obtain the following functional form for w_0 on $(1, \hat{c})$:

$$w_0(c) = H_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c \right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} + \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi}$$

Note that once again the expression for $w_0(c)$ on this interval is identical to the ex-

pression for v_0 . Since $H_1 < 0$, it must also be the case that w_0 is strictly increasing on this interval. I then find a convenient form for the function $w_0(c) - 1 - (c - 1)w'_0(c)$ on $(1, \hat{c} \wedge \frac{r_d + \lambda}{r_c + \lambda})$:

$$\begin{aligned} w_0(c) - 1 - (c - 1)w'_0(c) &= -H_1 \frac{\rho + \phi - r_c}{r_c + \lambda} \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c\right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} \\ &\quad + H_1 \frac{(r_d - r_c)(\rho + \phi + \lambda)}{(r_c + \lambda)(r_d + \lambda)} \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c\right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda} - 1} + \frac{r_d - \rho}{\rho + \lambda + \phi} \end{aligned}$$

This function of c can be showed to be strictly increasing on $(1, \frac{r_d + \lambda}{r_c + \lambda})$. For $c = 1$, it takes value $w_0(1) - 1 < 0$, inequality already established in my study of the funtion v_0 . For $c = \frac{r_d + \lambda}{r_c + \lambda}$, the function $w_0(c) - 1 - (c - 1)w'_0(c)$ is equal to $\frac{r_d - \rho}{\rho + \lambda + \phi} > 0$. By the intermediate value theorem, there must exists a threshold $\hat{c} \in (1, \frac{r_d + \lambda}{r_c + \lambda})$ such that $w_0(\hat{c}) - 1 = (\hat{c} - 1)w'_0(\hat{c})$. Note finally that that $\hat{c} > c^*$. This inequality is due to the fact that:

$$\begin{aligned} v_0(c^*) - 1 &= 0 \\ w_0(\hat{c}) - 1 &= (\hat{c} - 1)w'_0(\hat{c}) > 0 \end{aligned}$$

3. $c \in (\hat{c}, \frac{r_d}{r_c})$

On this interval, the value function w_0 satisfies:

$$\rho w_0(c) = r_d + (r_c c - r_d)w'_0(c) + \phi(\min(1, c) - w_0(c))$$

Since $\hat{c} > 1$, I have for $c \in (\hat{c}, \frac{r_d}{r_c})$:

$$w'_0(c) = \frac{\rho + \phi}{r_c c - r_d} w_0(c) - \frac{r_d + \phi}{r_c c - r_d}$$

This ordinary differential equation admits the following solution:

$$w_0(c) = H'_2 \left(1 - \frac{r_c}{r_d} c\right)^{\frac{\rho + \phi}{r_c}} + \frac{r_d + \phi}{\rho + \phi}$$

H'_2 is a constant to be determined. Value matching at $c = \hat{c}$ gives me the following equation for H'_2 :

$$\begin{aligned} H_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda} \hat{c}\right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} + \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} &= H'_2 \left(1 - \frac{r_c}{r_d} \hat{c}\right)^{\frac{\rho + \phi}{r_c}} + \frac{r_d + \phi}{\rho + \phi} \\ \Rightarrow H'_2 &= \left(1 - \frac{r_c}{r_d} \hat{c}\right)^{-\frac{\rho + \phi}{r_c}} \left[H_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda} \hat{c}\right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} - \frac{\lambda(r_d - \rho)}{(\rho + \phi)(\rho + \lambda + \phi)} \right] \end{aligned}$$

$H'_2 < 0$, meaning that $w_0(\cdot)$ is increasing on $(\hat{c}, \frac{r_d}{r_c})$. It now remains to verify that on

this interval $(\hat{c}, \frac{r_d}{r_c})$, $w_0(c) - 1 > (c - 1)w'_0(c)$. I compute:

$$\begin{aligned} w_0(c) - 1 - (c - 1)w'_0(c) &= -H'_2 \frac{\rho + \phi - r_c}{r_c} \left(1 - \frac{r_c}{r_d}c\right)^{\frac{\rho + \phi}{r_c}} \\ &\quad + H_2 \frac{(r_d - r_c)(\rho + \phi)}{r_c r_d} \left(1 - \frac{r_c}{r_d}c\right)^{\frac{\rho + \phi}{r_c} - 1} + \frac{r_d - \rho}{\rho + \phi} \end{aligned}$$

This function of c can be showed to be strictly increasing on $(\hat{c}, \frac{r_d}{r_c})$, with value zero for $c = \hat{c}$. In other words, I verify a-posteriori that the large creditor's choice of run boundary $\hat{c}$ is optimal. I also obtain a result that was a-priori not imposed: the function w_0 is continuously differentiable at $c = \hat{c}$.

4. $c \in (\frac{r_d}{r_c}, +\infty)$

$\bar{c}(t)$ is a strictly increasing function of time since the initial cash reserve is above $\frac{r_d}{r_c}$. The large creditor constantly rolls over its debt claim, and the value function w_0 is constant, equal to:

$$w_0(c) = \frac{r_d + \phi}{\rho + \phi}$$

Of course on that interval, $w'_0(c) = 0$ and $w_0(c) > 1$, meaning that the optimality condition $w_0(c) - 1 > (c - 1)w'_0(c)$ is satisfied.

To summarize, the value function of the large creditor w_0 is equal to:

$$w_0(c) = \begin{cases} \left(\frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \lambda + \phi)(\phi + \rho - r_c)} \right) \left[1 - \left(1 - \frac{r_c + \lambda}{r_d + \lambda}c\right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} \right] + \frac{\phi}{\phi + \rho - r_c}c & \text{for } 0 < c < 1 \\ H_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda}c\right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} + \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} & \text{for } 1 < c < \hat{c} \\ H'_2 \left(1 - \frac{r_c}{r_d}c\right)^{\frac{\rho + \phi}{r_c}} + \frac{r_d + \phi}{\rho + \phi} & \text{for } \hat{c} < c < \frac{r_d}{r_c} \\ \frac{r_d + \phi}{\rho + \phi} & \text{for } c > \frac{r_d}{r_c} \end{cases}$$

□

Proof of Proposition 10: Assume that it is optimal for shareholders to default when $p \leq p^*$, for some value p^* to be determined endogeneously. The debt value $d(p; p^*)$ (where the dependence on p^* is made explicit) thus satisfies the following Hamilton-Jacobi-Bellman equation, for $p > p^*$:

$$(\rho + \lambda + \phi)d = r_d + \lambda + \phi \min(1, p) + \mu p \frac{\partial d}{\partial p} + \frac{1}{2} \sigma^2 p^2 \frac{\partial^2 d}{\partial p^2}$$

The boundary conditions for d are $d(p^*; p^*) = \alpha p^*$ and $\lim_{p \rightarrow +\infty} b(p; p^*) = \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi}$. I introduce

the roots $x_1 < 0 < x_2$ of the second order polynomial:

$$x^2 + \frac{2}{\sigma^2} \left(\mu - \frac{1}{2} \sigma^2 \right) x - \frac{2}{\sigma^2} (\rho + \lambda + \phi)$$

If $p^* < 1$, the debt value d thus satisfies:

$$d(p; p^*) = \begin{cases} \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} + k_1 p^{x_1} & p \geq 1 \\ \frac{\phi p}{\rho + \lambda + \phi - \mu} + \frac{r_d + \lambda}{\rho + \lambda + \phi} + l_1 p^{x_1} + l_2 p^{x_2} & 1 > p \geq p^* \end{cases}$$

The constants k_1, l_1, l_2 are found using value matching and smooth pasting at $p = 1$, as well as value matching at $p = p^*$:

$$\begin{aligned} \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} + k_1 &= \frac{\phi}{\rho + \lambda + \phi - \mu} + \frac{r_d + \lambda}{\rho + \lambda + \phi} + l_1 + l_2 \\ x_1 k_1 &= \frac{\phi}{\rho + \lambda + \phi - \mu} + x_1 l_1 + x_2 l_2 \\ \frac{\phi p^*}{\rho + \lambda + \phi - \mu} + \frac{r_d + \lambda}{\rho + \lambda + \phi} + l_1 (p^*)^{x_1} + l_2 (p^*)^{x_2} &= \alpha p^* \end{aligned}$$

This system can be written in matrix format as follows:

$$\begin{pmatrix} 1 & 1 & -1 \\ x_1 & x_2 & -x_1 \\ (p^*)^{x_1} & (p^*)^{x_2} & 0 \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \\ k_1 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ A_3(p^*) \end{pmatrix}$$

In the above, I have noted:

$$\begin{aligned} A_1 &:= \phi \left(\frac{1}{\rho + \lambda + \phi} - \frac{1}{\rho + \lambda + \phi - \mu} \right) \\ A_2 &:= -\frac{\phi}{\rho + \lambda + \phi - \mu} \\ A_3(p^*) &:= \left(\alpha - \frac{\phi}{\rho + \lambda + \phi - \mu} \right) p^* - \frac{r_d + \lambda}{\rho + \lambda + \phi} \end{aligned}$$

This linear system yields the following constants:

$$\begin{aligned} l_1(p^*) &= A_3(p^*) (p^*)^{-x_1} + \frac{x_1 A_1 - A_2}{x_2 - x_1} (p^*)^{x_2 - x_1} \\ l_2(p^*) &= \frac{A_2 - x_1 A_1}{x_2 - x_1} \\ k_1(p^*) &= A_3(p^*) (p^*)^{-x_1} + \frac{x_1 A_1 - A_2}{x_2 - x_1} (p^*)^{x_2 - x_1} + \frac{A_2 - x_2 A_1}{x_2 - x_1} \end{aligned}$$

If $p^* > 1$, the debt value d simply satisfies:

$$d(p; p^*) = \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} \left(1 - \left(\frac{p}{p^*} \right)^{x_1} \right) + \alpha p^* \left(\frac{p}{p^*} \right)^{x_1}$$

Consider then the present value of bankruptcy costs $b(p; p^*) := \mathbb{E}^p [e^{-\rho\tau_b}(1 - \alpha)p(\tau_b)]$. When $p > p^*$, $b(\cdot; p^*)$ solves the following Hamilton-Jacobi-Bellman equation:

$$(\rho + \phi) b = \mu p \frac{\partial b}{\partial p} + \frac{1}{2} \sigma^2 p^2 \frac{\partial^2 b}{\partial p^2}$$

The boundary conditions for b are $b(p^*; p^*) = (1 - \alpha)p^*$ and $\lim_{p \rightarrow +\infty} b(p; p^*) = 0$. I introduce the roots $y_1 < 0 < y_2$ of the second order polynomial:

$$y^2 + \frac{2}{\sigma^2} \left(\mu - \frac{1}{2} \sigma^2 \right) y - \frac{2}{\sigma^2} (\rho + \phi)$$

The present value of bankruptcy costs b thus satisfies:

$$b(p; p^*) = (1 - \alpha) p^* \left(\frac{p}{p^*} \right)^{y_1}$$

Finally, the equity value $e(p; p^*)$ is simply equal to the value of the illiquid asset p minus (a) the value of the debt d and (b) the value of bankruptcy costs b :

$$e(p; p^*) = p - d(p; p^*) - b(p; p^*)$$

The shareholders set the default boundary p^* optimally, by setting:

$$\frac{\partial e}{\partial p} \Big|_{p=p^*} = 0$$

This equation has an explicit solution under the assumption that $p^* > 1$:

$$p^* = \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} \frac{-x_1}{1 - \alpha x_1 - (1 - \alpha) y_1}$$

A sufficient condition for the quantity above to be greater than 1 is $(1 - \alpha)(y_1 - x_1) > 1$, which is equivalent to:

$$\frac{1 - \alpha}{\sigma^2} \left(\mu - \frac{1}{2} \sigma^2 \right) \left[\sqrt{1 + \frac{2\sigma^2(\rho + \lambda + \phi)}{(\mu - \frac{1}{2}\sigma^2)^2}} - \sqrt{1 + \frac{2\sigma^2(\rho + \phi)}{(\mu - \frac{1}{2}\sigma^2)^2}} \right] > 1$$

For the parameter configurations I will be considering, λ is large compared to ρ and ϕ , such that the sufficient condition above is approximately equivalent to:

$$\frac{(1 - \alpha)\sqrt{2\lambda}}{\sigma} > 1$$

This inequality will be satisfied in practice. Note however that in the other case (i.e. when $p^* < 1$), I obtain a non-linear equation that p^* is solution of:

$$1 - (\alpha x_1 + (1 - \alpha)y_1) - \frac{\phi(1 - x_1)}{\rho + \lambda + \phi - \mu} ((p^*)^{x_2-1} - 1) + \frac{x_1/p^*}{\rho + \lambda + \phi} (r_d + \lambda + \phi(p^*)^{x_2}) = 0$$

I then compute par CDS spread $CDS(c, p; T)$ as follows:

$$\begin{aligned} \text{Num}(p; T) &:= \mathbb{E}^p \left[1_{\{\tau_b < T \wedge \tau_\phi\}} e^{-\rho \tau_b} \max(0, 1 - \alpha p(\tau_b)) \right] \\ \text{Denom}(p; T) &:= \mathbb{E}^p \left[\int_0^{\tau_b \wedge \tau_\phi \wedge T} e^{-\rho t} dt \right] \\ CDS(p; T) &:= \frac{\text{Num}(p; T)}{\text{Denom}(p; T)} \end{aligned}$$

It is straightforward to establish the partial differential equations that $\text{Num}(p; t)$ and $\text{Denom}(p; t)$ solve:

$$\begin{aligned} (\rho + \phi) \text{Num} &= -\frac{\partial \text{Num}}{\partial t} + \mu p \frac{\partial \text{Num}}{\partial p} + \frac{1}{2} \sigma^2 p^2 \frac{\partial^2 \text{Num}}{\partial p^2} \\ (\rho + \phi) \text{Denom} &= 1 - \frac{\partial \text{Num}}{\partial t} + \mu p \frac{\partial \text{Num}}{\partial p} + \frac{1}{2} \sigma^2 p^2 \frac{\partial^2 \text{Num}}{\partial p^2} \end{aligned}$$

The boundary conditions are as follows:

$$\begin{aligned} \text{Num}(p; 0) &= 0 \forall p > p^* \\ \lim_{p \rightarrow +\infty} \text{Num}(p; t) &= 0 \\ \text{Num}(p^*; t) &= 1 - \alpha p^* \\ \text{Denom}(p; 0) &= 0 \forall p > p^* \\ \lim_{p \rightarrow +\infty} \text{Num}(p; t) &= \frac{1}{\rho + \phi} (1 - e^{-(\rho + \phi)T}) \\ \text{Num}(p^*; t) &= 0 \end{aligned}$$

B Key Tables and Figures

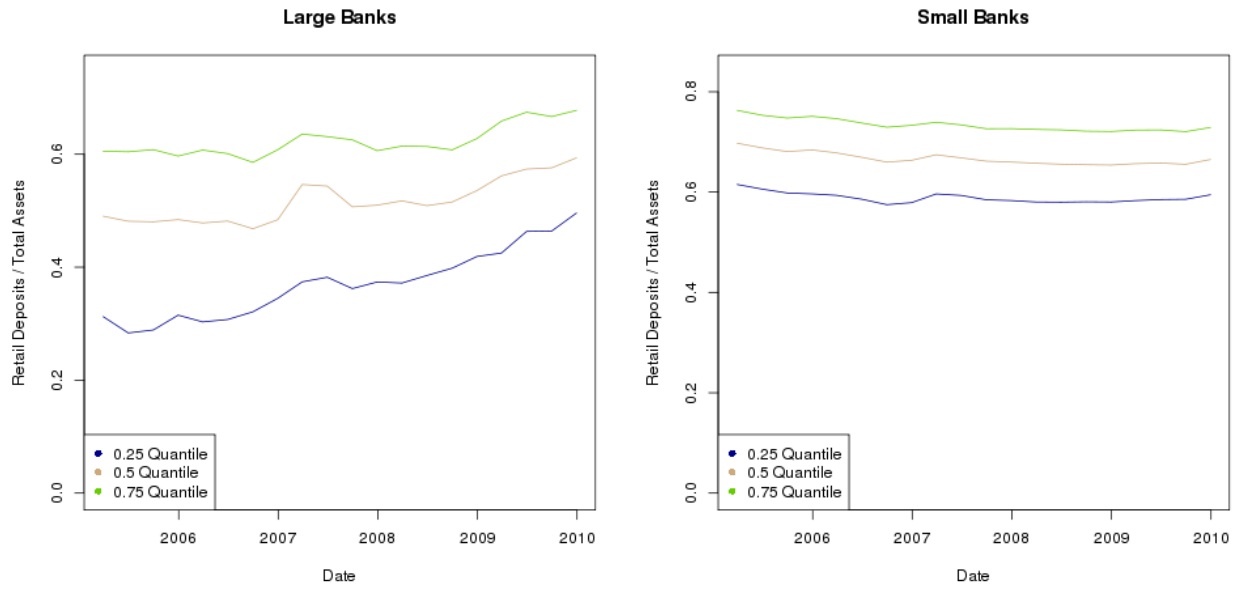
Asset Category	Asset Classification	Regulatory Haircut
Cash and Bank Notes	Level 1	0%
Sovereign Bonds with 0% Basel II risk-weights	Level 1	0%
Domestic Currency Sovereign Bonds	Level 1	0%
Sovereign Bonds with 20% Basel II risk-weights	Level 2A	15%
AA (or higher) Non-Financial Corporate Bonds	Level 2A	15%
AA (or higher) Non-Financial Covered Bonds	Level 2A	15%
AA (or higher) RMBS	Level 2B	25%
A or BBB Non-Financial Corporate Bonds	Level 2B	50%
Common Stocks	Level 2B	50%

Table 2: Basel III HQLA: Regulatory Haircuts

Liability Type	Runoff Probability
Insured/“Stable” Retail Deposits	3%
“Less Stable” Retail Deposits	10%
Non-Finance Wholesale Funding	20 - 40%
Financial Unsecured Wholesale Funding	100%
Secured Wholesale Funding vs. Level 1 Assets	0%
Secured Wholesale Funding vs. Level 2A Assets	15%
Secured Wholesale Funding vs. Level 2B Assets	50%
Secured Wholesale Funding vs. other Assets	100%
Derivatives Liabilities Funding Needs	100%
Collateral Posting Upon Downgrade	100%
ABCP Liquidity Facilities	100%

Table 3: Basel III NCO: Liability Run-off Rates

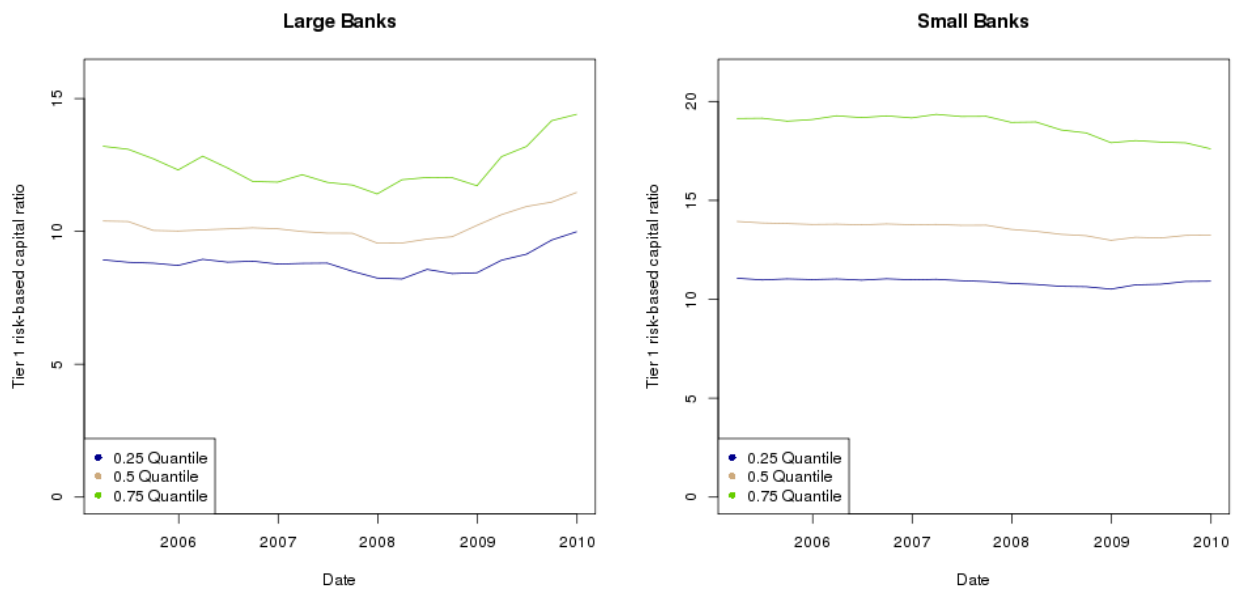
Figure 30: Retail Deposits - Total Assets Ratio



(a) Banks with Total Assets > \$10bn

(b) Banks with Total Assets < \$10bn

Figure 31: Tier 1 Risk Based Capital Ratio



(a) Banks with Total Assets > \$10bn

(b) Banks with Total Assets < \$10bn