

Long Term Government Bonds*

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January 2016

Abstract

We study how the issuance of long bonds affects optimal fiscal policy. Long bonds are usually modelled as having two features that are not found in the data: a) zero coupons and b) previously issued bonds are repurchased each period regardless of their time to maturity. The literature has found that under a) and b) issuing long bonds provides fiscal insurance. We show that these assumptions are not innocuous. Specifically we find that long bonds may not complete the markets even in the absence of uncertainty and under certain assumptions (namely those that are most empirically relevant) long bonds introduce additional tax volatility that offsets the attractiveness they provide through fiscal insurance. We find that introducing coupons helps alleviate the additional tax volatility but does so by reducing the ability of long bonds to provide insurance.

Under full commitment the government promises future tax changes in order to reduce current funding costs (interest rate twisting). This introduces additional tax volatility at different frequencies. If we remove assumptions a) and b) interest rate twisting takes a very different form, showing again that those assumptions matter. We also propose an alternative to full commitment that eliminates interest rate twisting, dramatically reduces the state space in a way that has relevance for a wide class of models and helps highlight the channels through which commitment affects the standard Ramsey equilibrium.

JEL Classification : E43, E62, H63

Keywords : Coupon Payments, Debt Management, Fiscal Policy, Government Debt, Long Bonds, Maturity Structure, Tax Smoothing

*Marcet acknowledges support from the European Research Council Advanced Grant (APMPAL) No. 324048, AGAUR (Generalitat de Catalunya), Plan Nacional project ECO2008-04785/ECON (Ministry of Science and Education, Spain), Axa Research Fund and Programa de Excelencia del Banco de España. Faraglia and Scott gratefully acknowledge funding from the ESRC's World Economy and Finance program. We are grateful to Alex Clymo for research assistance and Davide Debortoli, Gernot Müller and Mikhail Golosov for comments and seminar participants at EEA (Oslo), IAE, UPF, Cambridge, Monfispol, Queen Mary University, University of Southampton, St. Andrews University, UC Louvain Royal Economic Society Conference (Cambridge), Conference "Fiscal Policy under Fiscal Imbalance" (Becker Friedman Institute and Fed Chicago, Chicago), "Workshop on Policy Interaction: Fiscal Policy, Monetary Policy and Debt Management", (Bank of International Settlements, Basel), SED Annual Meeting.(Istanbul). All errors are our own. Corresponding Author: Rigas Oikonomou, IRES-CORE, Université Catholique de Louvain, Collège L. H. Dupriez, 3 Place Montesquieu 1348 Louvain la Neuve, Belgium.

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1 Introduction

We study optimal fiscal policy when the government issues real riskless long bonds. Dynamic equilibrium macro models under incomplete markets clearly require specific modelling assumptions about the type of government bonds available. Different assumptions obviously have the potential to lead to different results. Most of the vast literature on DSGE models considers the case of only one-period bonds. The same holds for the literature on optimal policy as in Barro (1979) and Aiyagari, Marcet, Sargent and Seppälä (2002). This focus on short bonds is surprising given that the share of U.S. government bonds issued with maturity longer than one year is, on average, 64%¹. However as we show, when long bonds are introduced a number of additional assumptions on bond policy have to be made compared to the case of one period bonds - especially issues around the repurchase of government bonds and whether to model bonds as paying coupons.

Most papers using long bonds assume zero-coupon payments and a full repurchase of previously issued bonds each period. This is the case for most of the work on optimal debt management as in Angeletos (2002), Barro (2003), Buera and Nicolini (2004) and Debortoli, Nunes and Yared (2015) in a complete market setting, or Nosbusch (2008), Lustig, Sleet and Yeltekin (2009) in an incomplete market setting. Other papers make simplifying assumptions aimed at making the model solution easier. For instance, a number of papers model long bonds as perpetuities that pay geometrically declining coupons - Woodford (2001), Broner, Lorenzoni and Schmulker (2013), Arellano and Ramanarayanan (2008), Chen, Curdia and Ferrero (2012) and Conesa and Kehoe (2015). Whilst tractable these assumptions are also not consistent with observed long run debt instruments.

In this paper we consider the impact on the optimal properties of long bonds by varying modelling assumptions around commitment, repurchasing bonds and coupon payments and find that doing so leads to non-trivial variations. The existing literature has tended to stress the advantages of issuing long bonds that arise from fiscal insurance (essentially the covariance of long bond prices with government expenditure shocks). We show that this advantage may be limited in some cases so that long bonds cannot complete the market, in contradiction to previous results. More generally, we find the advantages of the fiscal insurance that long bond provides may be offset by additional tax volatility induced by alternative assumptions over repurchase and coupons. These findings on the limitations of long bonds are strongest when the assumptions we make are closest to actual practice, namely when long bonds are not repurchased each and every period. Faraglia, Marcet, Oikonomou and Scott (2014b) document that most US government bonds are not repurchased or they are repurchased close to maturity date. Most government bonds are not perpetuities and they pay constant coupons.

As with many papers in the incomplete markets literature we take for granted the type of bonds that exist but introduce features that are present in actually issued bonds. Whether such practices can be justified² as optimal behaviour in response to market imperfections is in our view an important research agenda albeit one we do not pursue here. However our approach here is a normative one -

¹ This figure is for the period 1955-2011, taken from Faraglia, Marcet, Oikonomou and Scott (2014b).

² Intuitively a number of justifications spring to mind and are often cited by Debt Management Offices - transaction costs, rollover risks, market disturbances due to large scale government interventions, moral hazard and asymmetric information. What is essentially needed is a theory of market turnover in the secondary market - a substantive agenda not just for the debt management literature but the whole of finance.

examining the role of debt management and optimal fiscal policy under differing assumptions that arise when one starts to model long bonds. Both through simulations and analytic examples we show that long bonds potentially affect optimal policy in three ways - fiscal insurance, interest rate twisting and rollover risk. Under previously used assumptions only the first channel is noticeable and long bonds dominate short bonds in debt management. Under no repurchase the other effects become important and lessen the appeal of long bonds.

The plan of the paper is as follows. In Section 2 we start with the canonical model (zero coupon and with repurchase) but instead of considering the case where the government issues a single short term bond we consider a bond of maturity N . We show how the government has an incentive to twist interest rates by committing to vary tax rates at the redemption date for debt in order to minimise funding costs. Whilst the fiscal insurance properties of long bonds highlighted by Angeletos (2002), Barro (2003), Buera and Nicolini (2004) helps reduce tax volatility this interest rate twisting effect increases their volatility. In the usual case where only one period debt is considered this effect is conflated with the usual impact effect on taxes and is not observed. By focusing on a long bond we disentangle the impact effect on taxes from this intertemporal effect that occurs around redemption.

In Section 3 we consider the role of commitment and time consistency in this model with long term debt. A common numerical approach to solve for models of optimal taxation involving long bonds is to use a recursive solution that introduces as state variables past values of the Lagrange multipliers λ attached to the government's intertemporal budget constraint. This makes solving models with long bonds computationally challenging as the state space quickly becomes unwieldy with long maturities. From our reading of the debt management literature it is unclear why these Lagrange multipliers are needed or how they influence optimal policy and neither is there any explicit discussion of the role of commitment (with the notable exception of Lucas and Stokey (1983) and Derbortoli, Nunes and Yared (2015)). We show these two lacunae are related - the role of the co-state variables λ is to enforce in the appropriate continuation problem the promises for future taxes that drive optimal interest rate twisting. Having identified the channel through which the state space quickly becomes cumbersome in the presence of long bonds we can then modify the model set up to alleviate this problem. We do this by proposing a model of independent powers where the government sets taxes but takes interest rates as given. This removes the interest rate twisting channel, allows us to demonstrate the effect of commitment and dramatically reduces the state space.

In Section 4 we perform simulations and examine the magnitude of interest rate twisting and the differences between the full commitment model and our model of independent powers. Another way to reduce the state space of the model is to model long bonds as a perpetuity with decaying coupon payments (e.g Woodford (2001), Broner, Lorenzoni and Schmulker (2013), Arellano and Ramanarayanan (2008), Chen, Curdia and Ferrero (2012)). In an appendix we look at this approach in more detail and compare outcomes with the standard approach as well as our independent powers model. Whilst the decaying coupons approach is computationally straightforward it does achieve this by abstracting from features that long maturity bonds bring to debt management.

In Section 5 we move from the standard assumption that *every* period *every* outstanding bond is repurchased regardless of its outstanding maturity and instead make the opposite assumption that every bond once issued is only repurchased at its scheduled redemption date. Obviously in the case of one period bonds this issue does not arise as there is no option to buy back before redemption

so this is only a modelling issue when we consider long bonds. Because under incomplete markets the timing of cash flows matter it is however a non trivial assumption. Under no early buyback then long and short bonds differ both in their cash flow profiles and in the degree of fiscal insurance they offer. No early buyback induces additional rollover cycles in taxes with the same periodicity as the maturity of debt. We include a simple example where, even under certainty, a long bond does not complete the markets but it introduces tax volatility. Not only are taxes more volatile but debt displays more complicated dynamics than the martingale property documented by Aiyagari et al (2002). In this sense long bonds generate greater tax volatility introducing a trade off between fiscal insurance and roll over cycles.

In Section 6 we extend our analysis to whether bonds pay a coupon or not. Again this is not a relevant issue for short bonds because any coupon would be paid at the same date as redemption occurs. Introducing coupons enables us to introduce duration issues into our analysis of debt management. Duration reflects the average time over which the cash payments associated with a bond are paid. This is important both because the timing of cash flows matter under incomplete markets and because the responsiveness of bond prices to shocks is directly proportional to the duration of the bond. With short or long bonds with full repurchase or zero coupons there is no distinction between duration and maturity. With long bonds, no buy back and coupons duration can vary for a given maturity. Some analytic examples show how long bonds without repurchase do offer better tax smoothing opportunities if coupons are sufficiently large. Our numerical results also show that under both commitment and independent powers, the introduction of fixed coupon bonds helps reduce tax volatility. The longer the maturity of bonds the more introducing coupons helps to reduce tax volatility and lessens the impact of N cycles. However whilst coupons help reduce N period volatility they shorten the duration of a bond and so reduce the effectiveness of long bonds in achieving fiscal insurance.

A final section concludes. Modelling long bonds involves making a number of assumptions which are not required when modelling short bonds. Can the government commit to future promises? Does the government buy back debt before maturity? Should bonds pay coupons? Assuming no buy back and no coupons makes the cash flow implications of long bonds similar to short bonds and emphasises the importance of fiscal insurance. Removing these assumptions introduces additional channels and additional volatility which produces a trade off between fiscal insurance and N cycle volatility and lessens the attractiveness of long bonds.

2 Interest Rate Twisting

In this section we outline our base model, in essence an extension of Aiyagari et al (2002) to the case of a riskless real bond of maturity N where $N > 1$. We start with the standard modelling assumptions of zero-coupon bonds and assume the government buys back all previously issued bonds each period.

2.1 The Base Model

We assume the economy produces a single non-storable good with technology

$$c_t + g_t \leq A - x_t, \quad (1)$$

for all t , where x_t, c_t and g_t represent leisure, private consumption and government expenditure respectively. The exogenous stochastic process g_t is the only source of uncertainty. The consumer is endowed with A units of time that she allocates between leisure and labour. The representative consumer has utility function:

$$E_0 \sum_{t=0}^{\infty} \beta^t \{u(c_t) + v(x_t)\} \quad (2)$$

and faces a proportional tax rate τ_t on labor income. The representative firm maximizes profits and both consumers and firms act competitively by taking prices and taxes as given. Consumers, firms and government all have full information, i.e they observe all shocks up to the current period, and all variables dated t are chosen contingent on histories $g^t = (g_t, \dots, g_0)$. All agents, including the government, have rational expectations.

Agents can only borrow and lend in the form of a zero-coupon, risk-free, N -period bond so that the government budget constraint is:

$$g_t + p_{N-1,t} b_{N,t-1} = \tau_t (A - x_t) + p_{N,t} b_{N,t} \quad (3)$$

where $b_{N,t}$ denotes the number of bonds the government issues at time t . Each bond pays one unit of consumption good in N periods time with complete certainty. The price of an i -period bond at time t is $p_{i,t}$. As is standard in the literature on long bonds, we assume that at the end of each period the government buys back the existing stock of debt and then reissues new debt of maturity N , these repurchases are reflected in the left side of the budget constraint (3). In addition, government bonds have to remain within upper and lower limits $\underline{M}$ and $\overline{M}$ so as to rule out Ponzi schemes:³

$$\underline{M} \leq \beta^N b_{N,t} \leq \overline{M}. \quad (4)$$

The term β^N in this constraint reflects the value of the long bond at steady state so that the limits $\underline{M}$, $\overline{M}$ appropriately refer to the value of debt and are comparable across maturities.⁴

We assume that after purchasing a long bond the household entertains only two possibilities: one is to resell the government bond in the secondary market in the period immediately after having purchased it, the other possibility is to hold the bond until maturity.⁵ Letting $s_{N,t}$ be the sales in the secondary market the household's problem is to choose stochastic processes $\{c_t, x_t, s_{N,t}, b_{N,t}\}_{t=0}^{\infty}$ to maximize (2) subject to the sequence of budget constraints:

$$c_t + p_{N,t} b_{N,t} = (1 - \tau_t) (A - x_t) + p_{N-1,t} s_{N,t} + b_{N,t-N} - s_{N,t-N+1}$$

³ Similar debt constraints are assumed in Aiyagari et al (2002).

⁴ Obviously the actual value of debt is $p_{N,t} b_{N,t}$, we substitute $p_{N,t}$ by its steady state value β^N in this constraint for simplicity.

⁵ We need to introduce secondary market sales $s_{N,t}$ in order to price the repurchase price of the bond.

with prices and taxes $\{p_{N,t}, p_{N-1,t}, \tau_t\}$ taken as given. The household also faces debt limits analogous to (4). We assume for simplicity that these limits are less stringent than those faced by the government, so that in equilibrium the household's problem always has an interior solution.

The consumer's first order conditions of optimality are given by

$$\frac{v_{x,t}}{u_{c,t}} = 1 - \tau_t \quad (5)$$

$$p_{N,t} = \frac{\beta^N E_t(u_{c,t+N})}{u_{c,t}} \quad (6)$$

$$p_{N-1,t} = \frac{\beta^{N-1} E_t(u_{c,t+N-1})}{u_{c,t}}. \quad (7)$$

where $u_{c,t} \equiv u'(c_t)$.

2.1.1 The Ramsey problem

We follow a standard definition of Ramsey equilibrium, assuming the government has full commitment to implement the best sequence of (possibly time inconsistent) taxes and government debt knowing equilibrium relationships between prices, taxes and allocations. Using (5), (6) and (7) to substitute for taxes and consumption the Ramsey equilibrium can be found by solving

$$\max_{\{c_t, b_{N,t}\}} E_0 \sum_{t=0}^{\infty} \beta^t \{u(c_t) + v(x_t)\} \quad (8)$$

$$\text{s.t.} \quad \beta^{N-1} E_t(u_{c,t+N-1}) b_{N,t-1} = S_t + \beta^N E_t(u_{c,t+N}) b_{N,t} \quad (9)$$

and (4) with x_t implicitly defined by (1). $S_t = (u_{c,t} - v_{x,t})(c_t + g_t) - u_{c,t}g_t$ is the “discounted” surplus of the government.

We set up the Lagrangian

$$\begin{aligned} L = E_0 \sum_{t=0}^{\infty} \beta^t \{ & u(c_t) + v(x_t) + \lambda_t [S_t + \beta^N u_{c,t+N} b_{N,t} - \beta^{N-1} u_{c,t+N-1} b_{N,t-1}] \\ & + \nu_{1,t} (\bar{M} - \beta^N b_{N,t}) + \nu_{2,t} (\beta^N b_{N,t} - \underline{M}) \} \end{aligned}$$

where λ_t is the Lagrange multiplier associated with the government budget constraint, ie. the excess burden of taxation, and $\nu_{1,t}$ and $\nu_{2,t}$ are the multipliers associated with the debt limits.

The first-order conditions for the planner's problem with respect to c_t and $b_{N,t}$ are

$$\begin{aligned} u_{c,t} - v_{x,t} + \lambda_t (u_{cc,t} c_t + u_{c,t} + v_{xx,t} (c_t + g_t) - v_{x,t}) \\ + u_{cc,t} (\lambda_{t-N} - \lambda_{t-N+1}) b_{N,t-N} = 0 \end{aligned} \quad (10)$$

$$E_t(u_{c,t+N} \lambda_{t+1}) = \lambda_t E_t(u_{c,t+N}) + \nu_{2,t} - \nu_{1,t} \quad (11)$$

for all $t = 0, 1, \dots$, with $\lambda_{-1} = \dots = \lambda_{-N} = 0$.

Assuming g_t is a Markov process, Corollary 3.1 in Marcet and Marimon (2014) implies the solution

satisfies the recursive structure:

$$\begin{bmatrix} b_{N,t} \\ \lambda_t \\ c_t \end{bmatrix} = F(g_t, \lambda_{t-1}, \dots, \lambda_{t-N}, b_{N,t-1}, \dots, b_{N,t-N}) \quad (12)$$

$$\lambda_{-1} = \dots = \lambda_{-N} = 0, \text{ given } b_{N,-1} \quad (13)$$

for a time-invariant policy function F . Therefore the state vector in this recursive formulation has dimension $2N + 1$.⁶

These FOCs help characterize some features of optimal fiscal policy with long bonds. Following the discussion in Aiyagari et al. (2002) we see that, in the case where debt limits are non binding, i.e. for t such that $\nu_{1,t} = \nu_{2,t} = 0$, (11) implies λ_t is a risk-adjusted martingale, with risk-adjustment measure $\frac{u_{c,t+N}}{E_t(u_{c,t+N})}$, indicating that the presence of the state variable λ in the policy function imparts persistence in the variables of the model.

The term

$$\mathcal{D}_t = (\lambda_{t-N} - \lambda_{t-N+1}) b_{N,t-N} \quad (14)$$

in (10) is key for our analysis of long bonds and interest rate twisting as it captures the feature that what happened in period $t - N$ has a specific impact on today's taxes. In particular, as we shall see, this term captures the fact that governments when they issue debt at $t - N$ make (time inconsistent) commitments to influence future taxes in order to affect the interest rate payable on N period debt.

In order to build up intuition for the role of this term and to better understand this interest rate twisting we will in the following subsections show examples that can be solved analytically. However before doing so consider the following intuition. Since in the first best we have $u_{c,t} - v_{x,t} = 0$ and zero taxes this suggests that the higher is $\mathcal{D}_t$ the further the model is pulled away from the first best and taxes are higher, when the term $\mathcal{D}_t$ is positive it can be thought of as introducing a higher distortion in a given period. In periods when g_{t-N+1} is very high we have that the cost of the budget constraint is high so λ_{t-N+1} is high, and if the government is in debt $\mathcal{D}_t < 0$, implying that optimal policy is to lower taxes t . Of course this is not a tight argument, as λ_t also responds to the shocks that have happened between t and $t - N$ and λ_t also plays a role in (10), but this argument is at the core of the interest rate twisting policy we identify below.

2.2 Analytic Results

2.2.1 A model under certainty

Assume for now that government spending is constant, $g_t = \bar{g}$. In this case long bonds complete the market so that the standard result ensures that all equilibrium constraints are summarized in a single implementability constraint, namely

$$\sum_{t=0}^{\infty} \beta^t \frac{u_{c,t}}{u_{c,0}} \tilde{S}_t = b_{N,-1} p_0^{N-1}. \quad (15)$$

⁶ This allows for a simpler recursive formulation than the promised utility approach, as the co-state variables λ do not have to be restricted to belong to the set of feasible continuation variables so that the continuation problem is well defined. In section 3.1 we show this continuation problem explicitly.

for $\tilde{S}_t = \left(1 - \frac{v_{x,t}}{u_{c,t}}\right) (c_t + g_t)$. This is easily rewritten as

$$\sum_{t=0}^{\infty} \beta^t S_t = b_{N,-1} \beta^{N-1} u_{c,N-1}$$

Consider the case when the government is initially in debt such that $b_{N,-1} > 0$. It is clear that the funding costs of initial debt $b_{N,-1}^N > 0$ can be reduced by manipulating consumption so as to achieve $c_t < c_{N-1}$ for all $t \neq N$, as this lowers the total cost of initial debt on the right side of this equation. As long as the elasticity of consumption with respect to wages is positive, which would be the case for empirically reasonable calibrations, higher c_{N-1} will be achieved by promising a tax cut in period $N - 1$ relative to other periods. In other words, the planner sets

$$\begin{aligned} \tau_t &= \bar{\tau} \text{ for all } t \neq N - 1 \\ \bar{\tau} &> \tau_{N-1}. \end{aligned} \tag{16}$$

This promise achieves a reduction of $u_{c,N-1}$ and so reduces the cost of outstanding debt by twisting the long end of the yield curve downwards. This is the same interest rate manipulation channel noted by Lucas and Stokey (1983) except here it is shifted N periods forward due to the maturity of bonds. Note that even though there are no fluctuations in this economy, (16) shows optimal policy implies that the government desires to introduce variability in taxes. In particular, debt management concerns around cost of funding dominate the usual tax smoothing concerns.

2.2.2 A model with uncertainty at $t = 1$

We now introduce uncertainty into our model. In the interest of obtaining analytic results we assume uncertainty only in the first period, ie g is given by⁷:

$$\begin{cases} g_t = \bar{g} & \text{for } t = 0 \text{ and } t \geq 2 \\ g_1 \sim F_g \end{cases}$$

for some non-degenerate distribution F_g .

This is a special case of the model in Section 2.1 so the FOCs derived there apply. Since there is no more uncertainty for $t > 1$ we have $E_t(\lambda_{t+1}) = \lambda_{t+1}$ for all $t \geq 1$, so the martingale condition (11) implies $\lambda_{t+1} u_{c,t+N} = \lambda_t u_{c,t+N}$ and

$$\lambda_t = \lambda_1 \quad t > 1. \tag{17}$$

Therefore, in the case of short bonds ($N = 1$), (10) and feasibility imply c_t and τ_t constant for $t \geq 2$ reflecting the fact that even though markets are incomplete the government smoothes taxes after the shock is realized. However, clearly c_1 and τ_1 will be a function of the realization of g_1 .

For the case of long bonds when $N > 1$, letting $\mathcal{D}_t = (\lambda_{t-N} - \lambda_{t-N+1}) b_{N,t-N}$ the FOC with

⁷ The analytics of this economy are similar to those of Nosbusch (2008), except that this is an infinitely lived economy so debt is not cancelled in period $t = 2$, instead debt stays constant after period 2.

respect to consumption (10) is satisfied for

$$\mathcal{D}_t = 0 \quad \text{for } t \geq 0 \text{ and } t \neq N-1, N \quad (18)$$

$$\mathcal{D}_{N-1} = -\lambda_0 b_{N,-1}, \quad \mathcal{D}_N = (\lambda_0 - \lambda_1) b_{N,0}. \quad (19)$$

Combining this with feasibility, (17) and the fact that $g_2 = \bar{g}$ for all $t \geq 2$ means that equilibrium satisfies

$$c_t = c^*(g_1) \quad \text{for all } t \geq 2 \text{ and } t \neq N, N-1 \quad (20)$$

for a certain function c^* i.e consumption is the same in all periods $t \geq 2$ except $t = N, N-1$, although this level of constant consumption depends on the realization of the shock g_1 as clearly c_{N-1}, c_N do as well.

In this model, when the shock g_1 is realised the government optimally spreads out the taxation cost of this shock over current and future periods. Typically the government gets in debt in period 1 if g_1 is high, so all future taxes for $t \geq 2$ are higher and future consumption lower. This would also happen with short bonds $N = 1$. What is new with long bonds is that optimal policy introduces an additional source of tax volatility, since taxes vary in periods $N-1$ and N , even though by the time the economy arrives at these periods no more shocks have occurred for a long time.

To make this argument precise consider the utility function

$$\frac{c_t^{1-\gamma_c}}{1-\gamma_c} - B \frac{(1-x_t)^{1+\gamma_l}}{1+\gamma_l} \quad (21)$$

for $\gamma_c, \gamma_l, B > 0$, and $A = 1$.

Result 1 Assume utility (21) and $b_{N,-1} > 0$. Then

$$\tau_1 = \tau_t \text{ for all } t \geq 1, \quad t \neq N-1, N \quad (22)$$

Furthermore, for a high enough realization of g_1 we have

$$\tau_1 > \tau_{N-1}, \tau_N. \quad (23)$$

The inequalities are reversed if $b_{N,-1} < 0$ or if the realization of g_1 is sufficiently low.

Proof.

Towards (22) note first that from (20) we have $\tau_t = \tau_2$ for all $t \geq 2$ and $t \neq N-1, N$.

(10) and (17) give

$$\frac{u_{c,t}}{v_{x,t}} - \frac{B + (\gamma_l + 1)\lambda_1}{(1 + (-\gamma_c + 1)\lambda_1)B} + (\lambda_{t-N} - \lambda_{t-N+1})\mathcal{F}_t = 0 \quad \text{for } t \geq 1$$

where $\mathcal{F}_t \equiv \frac{u_{cc,t}b_{N,t-N}}{(1+(1-\gamma_c)\lambda_1)B}$. Consider $t = 1$. For any long maturity $N > 1$ we have that $\lambda_{t-N} = \lambda_{t-N+1} = 0$ when $t = 1$ so that

$$\frac{u_{c,1}}{v_{x,1}} = \frac{B + (\gamma_l + 1)\lambda_1}{(1 + (-\gamma_c + 1)\lambda_1)B}. \quad (24)$$

Therefore we can write

$$\frac{u_{c,t}}{v_{x,t}} - \frac{u_{c,1}}{v_{x,1}} = (\lambda_{t-N+1} - \lambda_{t-N}) \mathcal{F}_t = 0 \quad \text{for } t \geq 1. \quad (25)$$

For $N > 1$ and from (13) we have $\lambda_{t-N+1} = \lambda_{t-N} = 0$ when $t = 2$. This and (25) gives $\tau_1 = \tau_2$ so that we have (22).

Towards (23) we now show that $\mathcal{F}_t < 0$ for $t = N - 1, N$. Since $\lambda_1, B, \gamma_l > 0$ we have that $B + (\gamma_l + 1)\lambda_1 > 0$. Since $u_{c,1}, v_{x,1} > 0$ clearly (24) implies that $(1 + (-\gamma_c + 1)\lambda_1)B > 0$. Since we consider the case of initial government debt $b_{N,-1} > 0$ this leads to $b_{N,0} > 0$ and since $u_{cc,1} < 0$ we have $\mathcal{F}_t < 0$ for $t = N - 1, N$.

For $t = N - 1$ we have $\lambda_{t-N} - \lambda_{t-N+1} = -\lambda_0 < 0$ it follows

$$\frac{u_{c,N-1}}{v_{x,N-1}} < \frac{u_{c,1}}{v_{x,1}} \implies \tau_{N-1} < \tau_t \text{ for all } t > 1, \quad t \neq N - 1, N.$$

Also, it is clear from (24) that high g_1 implies a high λ_1 . Since the martingale condition implies $E_t(u_{c,N}\lambda_1) = \lambda_0 E_0(u_{c,N})$ for higher than average g_1 we have $\lambda_1 > \lambda_0$. Therefore, for $t = N$ and g_1 high enough we have $\lambda_{t-N} - \lambda_{t-N+1} = \lambda_0 - \lambda_1 < 0$ so that (25) implies

$$\frac{u_{c,N}}{v_{x,N}}, \frac{u_{c,N-1}}{v_{x,N-1}} < \frac{u_{c,1}}{v_{x,1}} \implies \tau_N, \tau_{N-1} < \tau_1.$$

■

Intuitively, in period $t = N - 1$ there is a tax cut for the same reasons as in Section 2.2.1. New in this section is the tax cut (for high g_1) at $t = N$. The intuition for this is clear: when an adverse shock to spending occurs at $t = 1$ the government uses debt as a buffer so $b_{N,1} > b_{N,0}$. This use of debt as a buffer is typical of incomplete market models as it allows tax smoothing by financing part of the adverse shock with higher future taxes. But since future surpluses are higher than expected as the higher interest payments have to be serviced, the government can lower the cost of existing debt by announcing a tax cut in period N , since this will reduce the price $p_{N-1,0}$ of period $t = 1$ outstanding bonds $b_{N,0}$. The tax cut at $t = N$ is a stochastic analog of the tax cut described in Section 2.2.1.

The above result shows that in this model tax policy is not independent of the maturity of government debt. In models of optimal policy the government usually desires to smooth taxes. Taxes would be constant in the above model if the government had access to complete markets. But we find that the government increases tax volatility in period N , long after the economy has received any shock. It is clear from this discussion that what will matter for the policy function is the term $\mathcal{D}_N = (\lambda_0 - \lambda_1)b_{N,0}$ which captures the government's commitment to alter future tax and interest rates. Therefore it is the interaction between past λ 's and past b 's that determines the size and the sign of today's tax cut.

To summarize, under incomplete markets and in the presence of an adverse shock to spending in period t the government has to take three actions: *i*) increase taxes permanently, *ii*) increase debt permanently, *iii*) announce a tax cut around the time when the outstanding debt matures, namely at $t + N$. Effects *i*) and *ii*) are well known in the literature of optimal taxation under incomplete markets, effect *iii*) is clearly seen in this model with long bonds since the promise is made N periods

ahead. Obviously in the case of short maturity $N = 1$ of Aiyagari et al. (2002) the effect of $\mathcal{D}_1$ would be felt in deciding optimally τ_1 but would be confounded with the fact g_1 is stochastic and influences demand for the consumption good. However when $N > 1$ the two effects are disentangled and we see how debt maturity introduces additional dynamics into taxes - *i)* reflects the usual increase in the excess burden of taxation given the adverse fiscal shocks, *ii)* is the usual incomplete market result that says the excess burden should follow a risk adjusted martingale leaving debt to fluctuate whereas *iii)* captures a distinct interest rate twisting channel due to debt maturity whereby governments induce additional tax volatility to reduce funding costs.

As mentioned earlier Lucas and Stokey (1983) also identify this interest rate twisting channel in their discussion of maturity. However because ours is an incomplete market model we identify this as a factor during *all* periods and not just the initial period, and because we have long bonds the interest rate twisting influences consumption N periods ahead.

It is also worth distinguishing this channel, which focuses on *real* interest rates and how future tax commitments influence current interest rates, from a number of related results in the literature that rely on nominal debt and the role of inflation surprises. Chari et al (1991) show how inflation surprises can bring about fluctuations in ex post real interest rates so as to achieve the complete market outcome and Schmitt-Grohe and Uribe (2004) and Siu (2004) extend this case to consider how this role is affected by introducing distortionary pricing. Lustig et al (2009) develop this approach yet further and like us consider the impact of introducing long term bonds. In their model long bonds have the attraction of postponing and concentrating the increase in nominal interest rates that adverse fiscal shocks produce. The Lustig et al model is one of incomplete markets, sticky prices, nominal bonds as well as long maturities. Their main focus is on extending the result of Chari et al (1991), about how inflation surprises influence nominal interest rates to achieve fiscal hedging in a model with long bonds.

3 Commitment and Independent Powers

We have so far followed the majority of the literature and assumed a Ramsey policy equilibrium with perfect commitment. This is important as the interest rate twisting mechanism that we identify is time inconsistent. Governments in Section 2 achieve lower current funding costs by promising lower future taxes but clearly this is a commitment governments would prefer to renege on. As well shall see it is this promise to cut future tax rates in order to influence current funding costs that is at the heart of why solving optimal tax models under incomplete markets and long bonds is so computationally demanding. Optimal time consistent policy requires keeping track of all these promises over the last N periods and as the maturity of the bond increases so too does the state space.

3.1 Time Inconsistency - a Continuation Problem

Previously we showed how the full commitment solution can be implemented recursively as in (12) by introducing N lags of b_N and λ as state variables. We now turn to discussing the role of these variables in characterizing the full commitment solution and their link to interest rate twisting and promises of future tax cuts. Doing so helps motivate our alternative institutional specification with its smaller state space.

Denote the Ramsey equilibrium in this subsection as $\{c_t^R, b_{N,t}^R\}_{t=0}^\infty$. Assume that the economy has been following the Ramsey equilibrium until some period $\bar{t} > 0$ and that, unexpectedly, in this period the government can follow alternative policies by maximizing

$$E_{\bar{t}} \sum_{t=0}^{\infty} \beta^t [u(c_{t+\bar{t}}) + v(x_{t+\bar{t}})] \quad (26)$$

subject to equilibrium constraints (9) and feasibility for $t = \bar{t}, \bar{t} + 1, \dots$. The only variable from the past that enters these constraints is $b_{N,\bar{t}-1}^R$. In general, this solution would be different from the continuation of the Ramsey policy $\{c_t^R, b_{N,t}^R\}_{t=\bar{t}}^\infty$ given $g_{\bar{t}}$.

The intuitive reason for this time inconsistency is that promises were made previously about future tax cuts (or tax increases) to promote interest rate twisting under the Ramsey Equilibrium. Formally, the optimal policy entails choices about c_t for $t \geq \bar{t}$ that enter in the expectations $E_t(u_{c,t+N-1})$ appearing in the government budget constraint (9) in previous periods $t = \bar{t} - N, \dots, \bar{t} - 1$. But these "promised" consumptions (related to promised taxes) will now be ignored if the government maximizes utility from the standpoint of period $\bar{t}$ as in (26).

It is possible to find a continuation problem that delivers the Ramsey allocation. Consider for now the case $N = 1$. If the government in period $\bar{t}$ maximizes

$$E_{\bar{t}} \sum_{t=0}^{\infty} \beta^t [u(c_{t+\bar{t}}) + v(x_{t+\bar{t}})] + \lambda_{\bar{t}-1}^R u_{c,\bar{t}} b_{N,\bar{t}-1}^R \quad (27)$$

subject to (9) for $t = \bar{t}, \bar{t} + 1, \dots$ the solution would be precisely $\{c_t^R, b_{N,t}^R\}_{t=\bar{t}}^\infty$. In other words, if we add the term $\lambda_{\bar{t}-1}^R u_{c,\bar{t}} b_{N,\bar{t}-1}^R$ to the utility function (26) the solution to this continuation problem is precisely the initially promised consumption and bond allocations in the Ramsey allocation from $\bar{t}$ onwards. The reader can convince herself of this statement by checking that the FOC derived from maximizing (27) coincide with the FOC from the Ramsey equilibrium. For a proof and a formal discussion see Marcet and Marimon (2014), Section 3.2 and Proposition 1.

The intuition for this result (for $N = 1$) is that interest rate twisting implies that under full commitment consumption at $\bar{t}$ reacts to shocks that occurred at $\bar{t} - 1$. If $g_{\bar{t}-1}$ is high and the government is in debt, then $\lambda_{\bar{t}-1}^R b_{N,\bar{t}-1}^R < 0$; since $u_{c,\bar{t}}$ is a decreasing function of consumption the additional term in (27) induces the government to choose a higher $c_{\bar{t}}$ than they would have chosen optimizing (26). This achieves a lower cost of bonds at $\bar{t} - 1$ which compensates in part for the higher $g_{\bar{t}-1}$, as required by optimal policy. But from the standpoint of period $\bar{t}$ a government maximizing (26) would rather forget about this promise unless the last term in (27) is introduced in the objective function. This is the reason that λ_{t-1} is a co-state variable that determines the choice at t .

For the general case $N > 1$ the equivalent continuation problem is to maximize

$$E_{\bar{t}} \left(\sum_{t=0}^{\infty} \beta^t [u(c_{t+\bar{t}}) + v(x_{t+\bar{t}})] + \sum_{t=0}^{N-2} \beta^t \mathcal{D}_{t+\bar{t}}^R u_{c,t+\bar{t}} + \beta^{N-1} \lambda_{\bar{t}-1}^R u_{c,\bar{t}+N-1} b_{N,\bar{t}-1}^R \right) \quad (28)$$

Here N lags of λ and b_N enter in the additional terms $\mathcal{D}_{t+\bar{t}}$, these terms have been defined in (14) and they modify the weight that consumptions c_t receive for $t = \bar{t}, \dots, \bar{t} + N - 1$. These consumptions are influenced by two terms, up to $t = \bar{t} + N - 2$ marginal utility is weighted by $\mathcal{D}$, while consumption at

$\bar{t} + N - 1$ is influenced only by the effect from the selling price $p_{N,\bar{t}-1}$. The result is that N lags of λ are part of the state vector since they influence the objective function of the continuation problem, along with N lags of b_N and, therefore, N lags of λ and b_N determine the solution at $\bar{t}$

Finally, the above discussion highlights why the lagrangean approach of Marcet and Marimon (2014) is convenient. The objective functions (27) and (28) are always well defined and they give a well defined maximization problem for any value of the λ 's, so that λ can be solved for without the computation of any complicated feasible set. On the other hand, the promised utility approach requires that the feasible set of promised utilities (or in this case promised marginal utilities $u_{c,t-i}$) is computed separately.

3.2 Independent Powers

The previous discussion shows that interest rate twisting arises because of the close connection between current interest rates and future tax policy in our model. Therefore, a policy authority that sets tax rates and internalizes interest rate costs will engage in interest rate twisting.

In this section we consider a different institutional set up, one of independent powers, such that governments cannot commit to influence future tax rates in order to affect current funding costs⁸. The result is a much simplified model that sheds insight on the relative importance of interest rate twisting, it simplifies computations and it can address issues on the current debate about the relative independence of fiscal, monetary and debt management authorities⁹.

More specifically, we relax the assumption of perfect coordination and assume the presence of a monetary policy authority¹⁰ that fixes interest rates in every period. The fiscal/debt management authority now takes interest rates as a given and implements optimal policy given these interest rates, knowing the relation between taxes and allocations given by (5) and feasibility. We examine an equilibrium where the two policy makers play a dynamic Markov Nash equilibrium with respect to the strategy of the other policy power and they both play Stackelberg leaders with respect to the consumer. More precisely, the fiscal authority chooses taxes and debt given a sequence for interest rates, the monetary authority simply chooses interest rates that clear the market and the fiscal authority maximizes the utility of agents. This assumption sidesteps the issues of commitment, now there is no room for interest rate twisting on the part of the fiscal authority since this agent takes interest rates as given.

It is easy to think of models where even if the monetary authority is independent it cannot deviate too much from equilibrium interest rates. Therefore we take a limit case and assume that

⁸ Debortoli, Nunes and Yared (2015) examine where governments cannot commit to future tax policies and focus on Markov Perfect Competitive Equilibrium rather than our institutional separation of powers. They also use the complete market solution method of Angeletos (2002) to solve for the optimal portfolio model rather than numerical state space based approaches.

⁹ See Greenwood et al (2014) for a discussion of these issues.

¹⁰ In practice there are three relevant agencies - a fiscal authority, a debt management office and a central bank. In our simple model we can think of interest rates as either being set independently by the central bank or the debt management office taking interest rates as given by the market and operating independently of the fiscal authority. What we propose in this section is not intended as an accurate description of how interest rates are set in the economy but merely to show the implications of independence between the two authorities.

the monetary authority simply sets interest rates in equilibrium as:

$$\begin{aligned} p_{N,t} &= \frac{\beta^N E_t(u_{c,t+N})}{u_{c,t}} \\ p_{N-1,t} &= \frac{\beta^{N-1} E_t(u_{c,t+N-1})}{u_{c,t}} \end{aligned} \quad (29)$$

given agent's consumption. Formally we use the following

Definition An equilibrium under independent powers (IP) is a sequence of bond prices $\{p_{N,t}, p_{N-1,t}\}$ each contingent on g^t , such that if the fiscal authority solves

$$\max_{\{c_t, b_{N,t}\}} E_0 \sum_{t=0}^{\infty} \beta^t \{u(c_t) + v(x_t)\} \quad (30)$$

$$\text{s.t.} \quad p_{N-1,t} b_{N,t-1} = \left(1 - \frac{v'(x_t)}{u'(c_t)}\right) (c_t + g_t) - g_t + p_{N,t} b_{N,t},$$

(1) and (4) taking bond prices as given then (29) holds.

We look for equilibria where bond prices are given by an interest rate policy function $\mathcal{R} : R^2 \rightarrow R^2$ that in equilibrium satisfies

$$(p_{N,t}, p_{N-1,t}) = \mathcal{R}(g_t, b_{N,t-1}), \quad (31)$$

although the relation $\mathcal{R}$ is ignored by the authority solving (30).

An advantage of this model is that within equilibria of the form (31) there is no longer any reason for longer lags to enter the state vector, as past Lagrange multipliers do not play a role. From the point of view of the fiscal authority the problem now is a standard dynamic programming problem with the vector of state variables $(b_{N,t-1}, g_t)$.

Multiplying both sides of the budget constraint by $u_{c,t}$ the Lagrangian of (30) becomes

$$\begin{aligned} L = E_0 \sum_{t=0}^{\infty} \beta^t \{ & u(c_t) + v(x_t) + \lambda_t [S_t + u_{c,t} (p_{N,t} b_{N,t} - p_{N-1,t} b_{N,t-1})] \\ & + \nu_{1,t} (\bar{M} - \beta^N b_{N,t}) + \nu_{2,t} (\beta^N b_{N,t} - \underline{M}) \}. \end{aligned} \quad (32)$$

The first order condition with respect to consumption combined with the budget constraint gives

$$u_{c,t} - v_{x,t} + \lambda_t (u_{cc,t} c_t + u_{c,t} + v_{xx,t} (c_t + g_t) - v_{x,t}) + u_{cc,t} \lambda_t S_t / u_{c,t} = 0 \quad (33)$$

In addition, the FOC with respect to bonds combined with (29) gives

$$\lambda_t E_t(u_{c,t+N}) + \nu_{2,t} - \nu_{1,t} = E_t(\lambda_{t+1} u_{c,t+N}). \quad (34)$$

An IP equilibrium can be computed using these two equations along with the government budget constraint, debt limits with their slackness conditions, feasibility and the fact that the only state variables are $(b_{N,t-1}, g_t)$.

Notice that (34) takes the same form as the FOC under full commitment (11). Therefore λ_t is

again a risk-adjusted martingale off corners. Obviously, the value of λ will be different than in the Ramsey equilibrium since consumption follows a different process under IP.

4 Stochastic Simulations

To better understand these channels and to appreciate their relative importance we now turn to a model where g_t is stochastic in all periods. We assume a utility function:

$$\frac{c_t^{1-\gamma_1}}{1-\gamma_1} + \eta \frac{x_t^{1-\gamma_2}}{1-\gamma_2}.$$

We choose $\beta = 0.98$, $\gamma_1 = 1$, $\gamma_2 = 2$ and $A = 100$. We set η such that if the government's deficit equals zero in the non stochastic steady state agents work a fraction of leisure equal to 30% of their time endowment. For the stochastic shock g we assume the following truncated AR(1) process:

$$g_t = \begin{cases} \bar{g} & \text{if } (1-\rho)g^* + \rho g_{t-1} + \varepsilon_t > \bar{g} \\ \underline{g} & \text{if } (1-\rho)g^* + \rho g_{t-1} + \varepsilon_t < \underline{g} \\ (1-\rho)g^* + \rho g_{t-1} + \varepsilon_t & \text{otherwise} \end{cases}.$$

We assume $\varepsilon_t \sim N(0, 1.44)^2$, $g^* = 25$, with an upper bound $\bar{g}$ equal to 35% and a lower bound $\underline{g} = 15\%$ of average GDP and $\rho = 0.95$. $\bar{M}$ is set equal to 80% of average GDP and $\underline{M} = -\bar{M}$.

4.1 Solving the Model with “Condensed PEA”

We solve the models applying the Parameterized Expectations Algorithm (hereafter PEA) of den Haan and Marcet (1990) to approximate numerically the terms that appear in the equilibrium conditions $E_t(u_{c,t+N})$, $E_t(u_{c,t+N-1})$ and $E_t(u_{c,t+N}\lambda_{t+1})$ as functions of the state variables. As highlighted before, in the model of Section 2 the dimension of the state vector is $2N + 1$ which even if we only consider bonds of 10 year maturities produces a state space of 21.

Faraglia, Marcet, Oikonomou and Scott (2014 a and b) suggest that in order to make the computation of models with large N manageable it is important to reduce the number of states which enter autonomously in the approximating polynomials. Using a refinement of the PEA called the “Condensed PEA”, their approach is to partition the state space into variables that are of primary importance for the solution and variables of secondary importance. The latter are introduced in the approximating functions as successive linear combinations. We apply this methodology to solve the commitment model and refer the reader to Faraglia et al. (2014 a and b) for an extensive discussion. The independent powers model with its state vector of only two variables (g_t, b_{t-1}^N) is solved applying the standard PEA.

To approximate the optimal policy accurately we make sure that we visit all possible realizations of the state vector with our simulations. This is more of an issue in our model since government debt is very persistent and therefore it may be expected that different realizations of spending, or different initial conditions of debt may make the debt and tax series follow considerably different paths. Our approximation to the parameterized expectation is based on 14000 samples each of 200 observations and with initial conditions for bonds uniformly distributed in the interval $[\underline{M}, \bar{M}]$. When

we later report our simulation results we change our sample and describe the model's performance over different horizons for given initial conditions.

4.2 Interest Rate Twisting

Figures 7 and 7 display the impulse response functions of key variables to an unexpected positive shock in g_t . The vertical axis is in units of each of the variables and expresses deviations from the value that would occur for the given initial condition if $g_t = g^*$. Each subplot shows two lines: the solid line represents the solution under full commitment of Section 2, the dashed line represents the case of the “independent powers” model of Section 3.2. Both figures are for a maturity $N = 10$.

[Figures 7 and 7 About Here]

Figure 7 presents the result when the government has zero inherited debt, $b_{N,-1} = 0$. The differences between the two models should highlight the effect of the government keeping past promises summarised by the variable $\mathcal{D}_t$. In this case there is no effect even under full commitment since $\mathcal{D}_N = 0$, the subsequent $\mathcal{D}$'s are negligible so they do not generate a significant difference between the two models. As the figure shows the rise in spending leads to an initially smaller but more persistent increase in taxes in the case of full commitment than under independent powers. However the effect is moderate leading to small differences in consumption, deficit and market value of government debt.¹¹ Clearly the two models are similar.

Figure 7 shows the same impulse response functions assuming the government inherits a positive debt equal to $b_{N,-1} = 0.5 y^* / \beta^N$ where y^* is steady state output. There is a blip in taxes at the time of maturity of the outstanding bonds $N = 10$. This is a reflection of the promise to cut taxes with the aim to twist interest rates as discussed in Section 2.2. Interest rate twisting, and the blip in period $t + N - 1$, occurs each period that g_t is high if the government is in debt. The size of the promised tax cut at $t + N - 1$ depends on how big is the relative past shocks, $(\lambda_{t-1} - \lambda_t)$, and the level of debt, $b_{N,t-1}$. Besides the stronger persistence, the tax rate in the model with commitment shows clearly the effect to reduce the tax rate and increase consumption $N - 1$ periods after the shock, in other words interest twisting. These anticipated changes affect also the deficit and the market value of government debt as illustrated by the bottom panels of the figure.

Obviously, the IP model does not show the blip in $N - 1$ periods, since interest rate twisting can not be found in a model where the fiscal authority does not understand the effect of taxes on interest rates. Other than that the responses are similar in the two models. The only notable difference is that in periods other than $N - 1$ the response of taxes is smoother under full commitment, reflecting the fact that interest rate twisting has the beneficial effect of smoothing taxes in periods other than $N - 1$.

4.3 The Impact of Maturity

To further illustrate the link between the maturity of debt and interest rate twisting, we plot in Figure 7, the response of taxes¹² to the shock under four different maturity structures, $N = \{5, 10, 15, 20\}$

¹¹ Differences in equilibrium interest rates between the two models are also minuscule.

¹² For the case $b_{N,-1} = 0.5 y^* / \beta^N$

The top left panel shows the case of commitment and zero initial debt, the top right high debt with commitment. The bottom panels illustrate the response of the tax schedule in the independent powers model.

[Figure 7 About Here]

Consistent with the previous results all tax responses in the top right panel show the interest rate twisting effect. Given our previous discussion it is clear why the blip in taxes keeps moving to the right as we increase the maturity. In the case of zero debt, as well as in the case of independent powers, the maturity structure shows little effect on optimal taxes.

4.4 Moments

We now evaluate the model properties reporting the first and second moments of some key model variables. In the first four rows of Table 1 and 2 we show the means of consumption, taxes, deficit and market value of debt for $N = 5, 10, 15, 20$. In the last four rows we report the standard deviations of these variables in our simulations. The means and standard deviations are evaluated over three different horizons: 40 periods (columns 1-4), 200 periods (columns 5-8) and 4500 periods¹³ (columns 9-12). These three cases enable us to clearly identify the influence of initial conditions on policy outcomes.

Table 1 reports the result for the model with commitment. With the exception of debt and deficit all the moments differ only to the second or third decimal place across maturities. Given that we know that tax policy changes when we change the maturity of government debt because of the existence of N -cycles in taxes this result seems surprising. However, with the government only issuing one type of bond in each case, smoothing taxes is mainly achieved by using debt as a buffer stock so that the fluctuations of the model variables are driven mostly by the strong low frequency fluctuations of debt leaving only a relatively minor impact of interest rate twisting on total variance.

The main exception are the levels of debt and deficit: the government in the long run holds assets, but average asset holdings are lower for higher maturities. As is well known, in models of optimal policy with incomplete markets, there is a force pushing the government to *accumulate* long bonds in the long run. More precisely, extending the results in Aiyagari et al (2002) Section III one can easily prove that in the case of linear utility ($u(c) = c$) the government would purchase a very large amount of private long bonds in the long run, enough to abolish taxes in the long run. This accounts for the negative means for debt shown in Table 1 and also it accounts for the significant differences in the means of the market value of debt which occur at longer horizons in the simulations.¹⁴ On the other hand, as argued in Angeletos (2002), Buera and Nicolini (2004) and Nosbusch (2008), if the term premium is negatively correlated with deficits (as it is in our model) it is optimal for the government to *issue* long bonds, as this provides fiscal insurance. Hence the government is aware that accumulating a very large amount of privately issued long bonds increases the volatility of taxes. This force accounts for the *lower* asset accumulation with *longer* maturities shown in Table 1.

To identify the effect of commitment we report the same moments for the “independent powers” models in Table 2. Comparing Table 1 and Table 2, it is evident that across all horizons and across

¹³ To get rid of the influence of initial conditions we dropped the first 500 observations from each sample in columns 8-12.

¹⁴ As the table shows the average market value becomes considerably more negative in the long run.

all maturity structures, the effect of the interest rate twisting channel is small. Both the means and the standard deviations of consumption and taxes show virtually no change relative to the model with commitment. Minor effects are found for the deficit and the market value of debt. This result makes us conjecture that interest rate twisting is of second order in terms of unconditional moments. The properties of the tax schedule are driven by fluctuations in government debt, but not by the effect of past promises (the first difference of the λ 's). The difference in tax volatility between the two models diminishes the longer the horizon of the simulations. The inherited debt effect can have some impact in tax volatility but it is short lived.

[Tables ?? and ?? About Here.]

To conclude, under the standard assumptions on long bonds, namely that they pay zero coupons and are purchased one period after issuance, the interest rate twisting policy channel is apparent but it does not substantially influence unconditional first and second moments. The model of “independent powers” may be a good model to have in the toolkit as it retains many of the interesting features of the Ramsey models, it has nearly the same moments, it avoids the technicalities arising from the very large state vector and it avoids discussion on the role to commitment at very long horizons. There are, however, issues of tax volatility showing up in small samples where the two models slightly differ.

5 Buy back and Rollover Cycles

In this section we extend our model further by assuming that the government never repurchases its previously issued bonds, so that N period government bonds are redeemed by the government N periods after issuance. This is an extreme assumption: the US government has sometimes repurchased its own bonds, but as shown in Faraglia et al (2014b) it only does so close to redemption date. Therefore, the no buy back assumption we use now is much closer to actual US bond issuance practice than the setup of Section 2.

As with interest rate twisting the implications of no buy back are absent with short bonds (if $N = 1$), since there is no room for the government to buy debt back before it matures. Only when debt lasts for more than one period ($N > 1$) is this distinction a relevant one. We show that no buyback provides additional intertemporal channels for debt management. In this specification interest twisting remains but takes on a more complex form. We show that taxes deviate from the usual martingale property and show in addition a cycle of N periods due to the need to rollover government debt. Therefore as with interest rate twisting we find that introducing long bonds adds additional volatility to taxes at a frequency connected to the debt maturity.

Under no buyback the budget constraint of the government simply becomes

$$b_{N,t-N} = \left(1 - \frac{v'(x_t)}{u'(c_t)}\right) (c_t + g_t) - g_t + p_{N,t} b_{N,t}. \quad (35)$$

Now a bond issued pays a given amount in N periods, while in Section 2 it paid an uncertain amount $p_{N-1,t+1}$ next period. Under incomplete markets this is not without loss of generality as the timing of cash flows matter and so assuming no early buy back will lead to different outcomes.

5.1 Impossibility of completing markets with a long bond

For a striking example of the impact of no early buyback, suppose as in the example of Section 2.2.1 that there is no uncertainty and $g_t = \tilde{g}$ for all t . To simplify even further, assume a long bond is of maturity $N = 2$ (although our results are valid for any N). Moreover assume the government inherits some non-zero debt $b_{2,-1}, b_{2,-2}$ and that the (finite) bond limits $\underline{M}, \overline{M}$ are large enough so as not to be binding in equilibrium.

Since we have no uncertainty it may seem at first sight that one bond completes the markets, so that all equilibrium constraints are summarized in a single implementability constraint

$$\sum_{t=0}^{\infty} \beta^t \frac{u_{c,t}}{u_{c,0}} \tilde{S}_t = b_{2,-1} \beta \frac{u_{c,1}}{u_{c,0}} + b_{2,-2}. \quad (36)$$

where $\tilde{S}_t$ has been defined after equation (15). But, perhaps surprisingly, it turns out that this constraint is not sufficient for an equilibrium and that a long bond can not complete the markets.

To see this consider the optimal allocation if (36) were be a sufficient implementability constraint and the bond issuance that would support such an allocation. It is clear that optimal taxes would be constant for $t \geq 2$ so that

$$\tilde{S}_t = \tilde{S}, \quad t = 2, 3, \dots \quad (37)$$

The analog of equation (36) at period t becomes $\frac{\tilde{S}}{1-\beta} = b_{2,t-1}\beta + b_{2,t-2}$ so that

$$b_{2,t} = \frac{\tilde{S}}{(1-\beta)\beta} - \frac{1}{\beta} b_{2,t-1} \quad t = 1, 2, \dots \quad (38)$$

Since $\frac{1}{\beta} > 1$ this is an explosive difference equation in $b_{2,t}$.

This shows that the optimal allocation if (36) were the only implementability constraint is not feasible under no buyback, because it implies that bond issuance oscillates between negative and positive amounts. As a result $b_{2,t}$ goes to infinity in absolute value, thereby violating the bond limits (4) for any finite $\underline{M}, \overline{M}$.¹⁵

What is going on? The problem is that the standard present value condition (36) is derived under the assumption that the total value of debt $b_{N,t-1}\beta + b_{N,t-2}$ remains bounded and, indeed, it does in this example. But this goes along with bond limits that explode in absolute value and this is ruled out by the bond limits (4). It is reasonable to rule out such allocations: if $b_{N,t-1}\beta + b_{N,t-2}$ remains bounded because $b_{N,t-1}$ and $b_{N,t-2}$ are eventually huge in absolute value and of opposite signs this means that the government is running a very large risk if the private sector defaults.

To summarize, under the bond limits (4) it is not true that (36) summarizes all equilibrium

¹⁵ Note that the initial condition $b_{2,-1}$ is independent on the constant $\frac{\tilde{S}}{1-\beta}$ so that no end condition holds to guarantee that the difference equation (38) is generically stable. There is one configuration of $b_{2,-1}, b_{2,-2}$ that does imply stability for a given wealth level $b_{2,-1}\beta \frac{u_{c,1}}{u_{c,0}} + b_{2,-2}$, but this would only happen by coincidence, most combinations of $b_{2,-1}, b_{2,-2}$ that give rise to the same wealth imply $|b_{2,t}| \rightarrow \infty$.

conditions. In fact, by forward substitution in (35) one can see that the following *two* conditions

$$\begin{aligned} b_{2,-2} &= \sum_{t=0}^{\infty} \beta^{2t} \frac{u_{c,2t}}{u_{c,0}} \tilde{S}_{2t} \\ b_{2,-1} &= \sum_{t=0}^{\infty} \beta^{2t} \frac{u_{c,2t+1}}{u_{c,1}} \tilde{S}_{2t+1} \end{aligned}$$

form a set of sufficient implementability conditions. These two conditions imply (36) but the converse is not true. If we only impose (36) these two constraints do not hold, that is why bond issuance goes to infinity and the bond limits are violated.

Intuitively: under no buyback and $N = 2$ there is no way to transfer income between odd periods and even periods. High income in, say $t = 1$, can be transferred to, say $t = 7$, but not to $t = 2$. Therefore, if $b_{2,-2} \neq b_{2,-1}$ we expect even and odd periods to have a different primary surplus so that (37) can not hold. This proves the following:

In the example of this section

$$\begin{aligned} \tau_t &= \tau^o \text{ for all } t \text{ odd} \\ \tau_t &= \tau^e \text{ for all } t \text{ even} \end{aligned}$$

and generically $\tau^o \neq \tau^e$.

Hence a zero-coupon long bond without repurchase cannot complete the markets even under certainty. In this setup, long bonds do not provide fiscal insurance. Taxes differ between odd and even periods, tax smoothing can only take place within odd periods or within even periods but not across all periods.

5.2 Some analytic examples

Maintaining certainty and for arbitrary N note that from the standard first order conditions it holds that¹⁶

$$\lambda_t = \lambda_{t+N} \text{ for all } t, \quad (39)$$

entailing that the multiplier λ_t follows an N cycle, since λ 's repeat every N periods but generally $\lambda_t \neq \lambda_{t+1} \neq \dots \neq \lambda_{t+N-1}$. Taxes and consumption inherit this N cycle property. Furthermore, interest rate twisting now causes taxes in the next N periods to respond to a shock differently depending on current debt. This is because now the government in period t holds bonds issued at $t - 1, \dots, t - N + 1$ and aims at twisting the interest rates of all these bonds to lower funding costs. We state these findings with the following result:

Result 2. *Assume no buyback, no uncertainty in g , and utility as in (21).*

Ramsey equilibrium is that there are cycles of order N in taxes for $t \geq N$. More precisely

$$\tau_t = \tau_{t+iN} \quad t = N, \dots, 2N - 1 \text{ for all } i = 1, 2, \dots \quad (40)$$

¹⁶ This follows from (49) and an argument parallel to the one leading to equation (17).

Furthermore, in the case $g_t = \tilde{g}$ for all t we have

$$b_{N,t} = b_{N,t+iN} \quad t = 0, \dots, N-1, \text{ for all } i = 1, 2, \dots \quad (41)$$

Assume further $b_{N,-i} > 0$ for $i = 1, \dots, N$ then

$$\tau_{i+N} > \tau_i \quad i = 0, \dots, N-1. \quad (42)$$

Proof.

We give details for $N = 2$, it is trivial to extend the proof to arbitrary N .

Equation (39) implies $\lambda_t = \lambda_{t-2}$ for all $t \geq 2$. Plug this in the first order condition (48) we find

$$\begin{aligned} u_{c,t} - v_{x,t} + \lambda_0 (u_{cc,t}c_t + u_{c,t} + v_{xx,t}(c_t + \tilde{g}) - v_{x,t}) &= 0 \text{ for all } t \geq 2, \quad t \text{ even} \\ u_{c,t} - v_{x,t} + \lambda_1 (u_{cc,t}c_t + u_{c,t} + v_{xx,t}(c_t + \tilde{g}) - v_{x,t}) &= 0 \text{ for all } t \geq 3, \quad t \text{ odd.} \end{aligned} \quad (43)$$

A standard derivation gives that for utility (21) we have

$$\begin{aligned} \tau_t &= \tau_2 \text{ for all } t > 2, \quad t \text{ even} \\ \tau_t &= \tau_3 \text{ for all } t > 3, \quad t \text{ odd.} \end{aligned} \quad (44)$$

This proves (40) for $N = 2$.

In the case $g_t = \tilde{g}$ (40) implies

$$\tilde{S}_2 = \tilde{S}_{2+2t} \text{ and } \tilde{S}_3 = \tilde{S}_{3+2t} \text{ for all } t = 1, 2, \dots$$

Substituting forward in the budget constraint (35) and using the debt limits we have

$$b_{2,t-2} = \sum_{j=0}^{\infty} \beta^{2j} \frac{u_{c,t+2j}}{u_{c,t}} S_{t+2j} \quad \text{for all } t = 0, 1, \dots$$

so that

$$\begin{aligned} b_{2,t} &= b_{2,0} = \frac{S_2}{1 - \beta^2} \text{ for all } t \geq 0, \quad t \text{ even} \\ b_{2,t} &= b_{2,1} = \frac{S_3}{1 - \beta^2} \text{ for all } t \geq 1, \quad t \text{ odd} \end{aligned}$$

the N period cycle emerges again.

Finally we show (42). For periods $t = 0, 1$ we have

$$\begin{aligned} u_{c,0} - v_{x,0} + \lambda_0 (u_{cc,0}c_0 + u_{c,0} + v_{xx,0}(c_0 + \tilde{g}) - v_{x,0}) - u_{cc,0} \lambda_0 b_{2,-2} &= 0 \\ u_{c,1} - v_{x,1} + \lambda_1 (u_{cc,1}c_1 + u_{c,1} + v_{xx,1}(c_1 + \tilde{g}) - v_{x,1}) - u_{cc,1} \lambda_1 b_{2,-1} &= 0 \end{aligned} \quad (45)$$

The only difference between equations (45) and (43) is the presence of two extra terms that are function of the initial condition of debt: $u_{cc,0} \lambda_0 b_{2,-2}$ and $u_{cc,1} \lambda_1 b_{2,-1}$. Since we have assumed that

$b_{2,-2}, b_{2,-1} > 0$ these terms are clearly positive, implying that

$$\begin{aligned}\tau_2 &> \tau_0 \\ \tau_3 &> \tau_1.\end{aligned}$$

■

The N -period cycle property highlighted in (40) and (41) is distinct and additional to the interest rate twisting mechanism outlined in previous sections. It is due to the fact that, as discussed before, a long bond without repurchase and bond limits does not complete the markets, all dates $t + iN$ are now isolated for different $t = 0, \dots, N - 1$. Therefore the N -period cycle arises because of the budget constraint, not because of the way we model policy, it is the same phenomenon we studied in Section 5.1. To show this more clearly, in an appendix we consider independent powers under no buyback and find N period cycles emerge in that model as well.

The inequality (42) shows that the government commits to twisting the interest rates for *all* N initial periods. This is in contrast with (23) in Result 1 without buyback, where interest rates were only twisted around the N -period. Under no buyback the government holds bonds issued in the last $N-1$ periods so it commits to lowering taxes for the next $N - 1$ periods in order to cut the cost of all bonds outstanding.

The following is a special case of Result 2 when g follows the certainty analog of the AR(1) process we use later in our simulations.

Result 3. *Consider the assumptions of Result 2 except $b_{N,-i} = 0$ for $i = 1, \dots, N - 1$. Assume in addition, that $g_t = (1 - \rho)g^* + \rho g_{t-1}$ for $g_0 > g^* > 0$ and $\rho \in (0, 1)$.*

Then

$$\tau_t > \tau_{t+1} \quad t = 0, \dots, N - 1. \quad (46)$$

Since (40) still applies, this means that taxes initially decrease for N periods, there is a jump at $t = N$ to set $\tau_N = \tau_0$, taxes decrease again until $t = 2N - 1$, there is a jump at $t = 2N$ and so on.

Proof

As argued in section 5.1 the budget constraints only link the periods of cycle N , so that the set of sufficient implementability conditions can be written as

$$0 = \sum_{i=0}^{\infty} \beta^{Ni} \frac{u_{c,t+Ni}}{u_{c,t}} [g_{t+Ni} - \tau_{t+Ni} (A - x_{t+Ni})] \quad \text{for } t = 0, \dots, N - 1. \quad (47)$$

For the g process assumed here it is clear that the discounted sum of expenditures decreases as t grows from 0 to $N - 1$, formally

$$\sum_{i=0}^{\infty} \beta^{Ni} \frac{u_{c,t+Ni}}{u_{c,t}} g_{t+Ni} > \sum_{i=0}^{\infty} \beta^{Ni} \frac{u_{c,t+1+Ni}}{u_{c,t+1}} g_{t+1+Ni}.$$

Therefore, if (47) must hold the discounted sum of tax revenues also has to go down as t grows from 0 to $N - 1$. Since τ has to be in the increasing part of the Laffer curve in order to be optimal tax, it means that taxes go down as i grows from 0 to $N - 1$. ■

This says that for the deterministic analog of the AR(1) process used in the simulations taxes

will go down within each N -period cycle.

5.3 Interest Rate Twisting under no buyback

We now argue that interest rate twisting also takes place under no buyback, but it takes a very different form. Now the presence of an adverse shock causes all taxes during the next N -period cycle to be slightly different than in previous cycles. This is because under no buyback the government owes bonds of maturities $1, \dots, N-1$, since long bonds issued $N-1$ periods ago have not yet been redeemed. Therefore the government promises cuts in taxes in order to affect consumption during this first cycle since each of them individually influences the value of currently held debt. We now show how this works using the optimality conditions of the Ramsey problem.

Building a lagrangean in an analogous way as we did in Section 2 gives that, off corners, the first-order conditions for the planner's problem with respect to c_t and $b_{N,t}$ are¹⁷

$$u_{c,t} - v_{x,t} + \lambda_t (u_{cc,t}c_t + u_{c,t} + v_{xx,t}(c_t + g_t) - v_{x,t}) + u_{cc,t}(\lambda_{t-N} - \lambda_t)b_{N,t-N} = 0 \quad (48)$$

$$E_t(u_{c,t+N}\lambda_{t+N}) = \lambda_t E_t(u_{c,t+N}) \quad (49)$$

There are only two differences with respect to the buyback case in (10) and (11), now we have

1. λ_{t+N} in (49) in place of λ_{t+1} in (11) and
2. λ_t in (48) in place of λ_{t-N+1} in (11).

The first difference implies that the martingale property of λ in Section 2 (see our discussion after (13)) now only holds every N -th period. This generates the N -period cycles that we have already discussed. The second difference produces a more subtle effect. Notice that the term that induces interest rate twisting is now $(\lambda_{t-N} - \lambda_t)b_{N,t-N}$ instead of $\mathcal{D}_t$ as defined in (14). The difference $(\lambda_{t-N} - \lambda_t)$ depends on *all* the shocks that have happened between $t-N$ and t , while in Section 2 we had $(\lambda_{t-N} - \lambda_{t-N+1})$ so that only the shock occurring at $t-N+1$ mattered. Due to (49) we should have $\lambda_{t-N} \simeq \lambda_t$ if all shocks are close to the mean between $t-N$ and t , but if negative (positive) shocks to g happen between $t-N$ and t the realized values will be $\lambda_{t-N} < \lambda_t$ ($>$) and the interest-rate twisting term will induce a lower (higher) consumption at t . This implies that a shock in period t induces interest rate twisting for all taxes in periods $t, \dots, t+N-1$. In this sense the effect of a shock to g on interest rates twisting is spread out over N periods. Furthermore, if g is a persistent process a shock to g accumulates over N periods. This is also reflected in the analytic result (42).

5.4 Simulations

Consider again our simulations of the previous section but now amend the model for the case of no buy back. Figure 7 shows how taxes respond to an adverse government expenditure shock when the government has initial debt equal to half of GDP. This Figure compares the case of buy back at the

¹⁷ See Faraglia, Marcet, Oikonomou and Scott (2014b) Section 6, for details on the Lagrangean and FOC.

end of each period (as in Section 2) and no buy back as in this Section. We see that a behavior analog to that in Figure XYZ arises: there are N -period cycles and taxes go down within each cycle. Under buy back we see clearly the interest rate twisting effect but under no buyback things are significantly different. The interest rate twisting effect is spread out across each of the periods and it can be barely seen, however the N -period cycles due to the cash flow requirements when debt is rolled over are obvious.

We see that long bonds add considerable additional intertemporal dynamics to taxes over and above the usual unit root properties. Table 5 shows stylised facts for the simulations from this model with no buy back. Comparing with Table 1 shows that under no buy back the deficit and market value of debt are larger on average and that taxes, deficit and consumption also become more volatile.

6 Coupon Bearing Bonds

A final issue to consider in modelling long bonds is the effect of introducing coupon payments. In practice long bonds invariably pay a coupon at fixed regular intervals (see FMOS (2014b) for documentary evidence). In the case of one period bonds coupons are unimportant - if coupons are paid at the end when the bond is redeemed all interest payments are paid at the maturity date and the duration of a bond is the same as its maturity. If we assume buyback then the impact of coupons is uninteresting as cash flows are unaffected.

Only if $N > 1$ and under no buyback will the existence of coupon payments make a difference. In this case coupons lead to a difference between maturity and duration. In terms of the rollover cycles of the previous section by spreading interest payments over the life of the bond and so reducing duration paying coupons should smooth taxes and reduce the N period cycles. However the volatility of the price of a bond is a direct function of its duration so whilst coupon payments will reduce the magnitude of N cycles they will also reduce the ability of long bonds to provide the fiscal insurance that the literature has to date emphasised. Given this it is worth investigating how the introduction of coupons affects the interest rate twisting and N period cycles we have identified above in a model with no buyback.

Let κ_t be the coupon payment of a bond issued at t , this payment is constant from t to $t + N$. The coupon may vary across issuance dates. In order to denote that non-zero coupon bonds have a different equilibrium price than zero coupon bonds considered previously, let q_t^N be the price of such a bond. It holds that:

$$q_t^N = \kappa_t \sum_{i=1}^{N-1} \beta^i E_t \left(\frac{u_{c,t+i}}{u_{c,t}} \right) + \beta^N E_t \left(\frac{u_{c,t+N}}{u_{c,t}} \right) \quad (50)$$

i.e. q_t^N is the sum of prices of zero coupon bonds of maturity $j < N$ ($p_t^j = \beta^j E_t \left(\frac{u_{c,t+j}}{u_{c,t}} \right)$) weighted by the coupon payments promised and a N period zero-coupon bond that pays one unit of consumption at maturity (a normalization). We call this a "fixed coupon bond" as the coupon κ_t is the same in all the periods that the bond is alive. Coupons, however, may differ across issuance dates and they may depend on the shocks g^t .

We normalize the payment at the end of the period to 1 unit of consumption. Therefore this 1-unit payment includes the principal ($1 - \kappa_t$) and the coupon paid in the period that the bond matures. This is a normalization, it simplifies formulas below, if the coupon would be paid separately, in

addition to a one unit promised, nothing would change.

To determine the size of the coupon we note that in US data long (non-zero-) coupon bonds trade at or close to par. In other words the debt management office designs coupons such that, under current market conditions the bond price equals the principal, ie. $q_t^N \approx 1 - \kappa_t$.

The government budget constraint is now

$$b_{N,t}q_t^N = b_{N,t-N} + \sum_{j=1}^{N-1} b_{N,t-j}\kappa_{t-j} + g_t - \left(1 - \frac{v_{x,t}}{u_{c,t}}\right)(A - x_t). \quad (51)$$

6.1 The Ramsey Program

The planner's objective is to maximize the agent's utility subject to (51), (50) and some ad hoc debt limits. The Lagrangian for the planner's program is now :

$$L = E_0 \sum \beta^t \left\{ u(c_t) + v(T - c_t - g_t) + \lambda_t \left[b_{N,t}(\beta^N u_{c,t+N} + \sum_{j=1}^{N-1} u_{c,t+j}\kappa_t) - b_{N,t-N}u_{c,t} - \sum_{j=1}^{N-1} b_{N,t-j}\kappa_{t-j}u_{c,t} + S_t \right] + v_{1,t}(\widetilde{M}_N - b_{N,t}) + v_{2,t}(b_{N,t} - \widetilde{M}) \right\}$$

where the appropriate debt limits are :

$$b_{N,t} \in \left[\frac{\underline{M}}{\sum_{j=1}^{N-1} \beta^j + \kappa \sum_{j=1}^{N-1} \sum_{i=1}^j \beta^i}, \frac{\overline{M}}{\sum_{j=1}^{N-1} \beta^j + \kappa \sum_{j=1}^{N-1} \sum_{k=1}^j \beta^k} \right] \equiv [\underline{\widetilde{M}}, \overline{\widetilde{M}}] \quad (52)$$

for $\kappa = E(\kappa_t)$. As in the zero-coupon model the limits are expressed in terms of the steady state market value of debt. In this case, the constraints $[\underline{\widetilde{M}}, \overline{\widetilde{M}}]$ are tighter in terms of new issuances. This is due to the fact that when debt is not bought back in every period a smaller number of bonds needs to be issued for the same market value of debt.¹⁸

From the above Lagrangian the first order condition for consumption is:

$$u_{c,t} - v_{x,t} + \lambda_t(u_{cc,t}c_t + u_{c,t} + v_{xx,t}(c_t + g_t) - v_{x,t}) + u_{cc,t}\kappa \sum_{j=1}^N (\lambda_{t-j} - \lambda_t)b_{N,t-j} + u_{cc,t}(\lambda_{t-N} - \lambda_t)b_{N,t-N} = 0$$

and the analogous condition for b_t^N :

$$\lambda_t E_t(\kappa \sum_{j=1}^N \beta^j u_{c,t+j} + \beta^N u_{c,t+N}) = E_t(\kappa \sum_{j=1}^N \beta^j u_{c,t+j} \lambda_{t+j} + \beta^N u_{c,t+N} \lambda_{t+N}) + v_{2,t} - v_{1,t}. \quad (53)$$

The optimal policies again satisfy

$$\begin{bmatrix} b_{N,t} \\ \lambda_t \\ c_t \end{bmatrix} = F(g_t, \lambda_{t-1}, \dots, \lambda_{t-N}, b_{N,t-1}, \dots, b_{N,t-N})$$

$$\lambda_{-1} = \dots = \lambda_{-N} = 0, \text{ given } b_{N,-1}$$

¹⁸ In FMOS (2014 a) we show that smaller issuances of long term bonds apply also under complete markets.

The state vector includes the lags of the multiplier λ and all the lags of the bond quantities so that the dimensionality of the state vector is again $2N + 1$.

Even when the government issues non-zero coupon bonds the incentive to twist interest rates is preserved. This may seem surprising since the per period budget constraint in (52) is a function only of one price, the issuance price. However, bonds which haven't matured in t affect the governments intertemporal constraint and its future income and financing needs.

To simplify, consider first an environment of no uncertainty, as in Section 2.1. We can derive the intertemporal constraint as follows:

$$\sum_{t=0}^{\infty} \beta^t \frac{u_{c,t}}{u_{c,0}} \tilde{S}_t = \sum_{j=1}^{N-1} b_{N,-j} \left(\sum_{k=1}^{N-j} \kappa p_0^k + p_0^{N-j} \right),$$

or using (50) we can rewrite this as:

$$\sum_{t=0}^{\infty} \beta^t S_t = \sum_{j=1}^{N-1} b_{N,-j} \left(\sum_{k=1}^{N-j} \kappa \beta^k u_{c,k} + \beta^{N-j} u_{c,N-j} \right). \quad (54)$$

From (54) it is clear that in the case of coupon bonds the government has the incentive to promise tax cuts in all periods from $t = 1$ to $N - 1$. Moreover, from (54) we can identify the term $\mathcal{D}_t$:

$$\mathcal{D}_t = \kappa \sum_{j=1}^N (\lambda_{t-j} - \lambda_t) b_{t-j}^N + (\lambda_{t-N} - \lambda_t) b_{t-N}^N$$

which highlights the fact that not only the level of debt issued from $t - 1$ to $t - N$ matters (as in the case of zero coupon long bonds and no buyback) but also the coupon payments matter to pin down the allocations in each period and in particular the level of taxation. For instance, in the case of a constant coupon bond and no buy back we have that (53) follows a complicated pattern which is a function of all the future terms $u_{c,t+j} \lambda_{t+j}$ for $j = 1, 2, \dots, N$ weighted by the promised payments.

6.2 Some Analytic Results

Before going to simulations we present a collection of analytic results. We already found in Section 5 that long bonds under no buyback may generate undesired tax volatility. We now show that this effect is alleviated if bonds pay sufficiently high coupons. However, in that case the bond positions are likely to be very volatile.

Sufficiency of Measurability Conditions

We have already pointed out in Section 5.1 that equilibrium constraints *cannot* be summarized in a standard implementability constraint even under certainty. In that case, a zero-coupon long bond without buyback cannot implement the complete market allocation. That example may have seemed contrived, as most long bonds in practice pay coupons, and coupons help transfer spending across periods. Now we explore in more generality the issue of how to write down a set of sufficient implementability conditions by considering coupon payments in a model with uncertainty.

Consider a feasible sequence of consumption $\{c_t\}$ and, associated with such a sequence, define

the discounted sum of surpluses z_t as

$$z_t = E_t \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} \left[\left(1 - \frac{v_{x,t+j}}{u_{c,t+j}} \right) (c_{t+j} + g_{t+j}) - g_{t+j} \right].$$

The literature has so far focused on three separate cases - complete markets, incomplete markets and effectively complete markets. We know that under complete markets where a full range of state contingent securities exists a necessary and sufficient condition for equilibrium is that

$$z_0 = b_{-1} \quad (55)$$

(see, for example, Lucas and Stokey (1983) and Chari, Christiano and Kehoe (1991)). If by contrast markets are incomplete and consist of only a real riskless one-period bond ($N=1$) Aiyagari et al. (2002) show that, in addition to (55) the following measurability conditions are needed for a set of sufficient conditions

$$z_t \text{ is a function of } g^{t-1} \text{ for all } t > 0. \quad (56)$$

A $\{c_t\}$ that satisfies these conditions is supported by a sequence of bonds $z_t = b_{1,t-1}$ for all $t > 0$. Finally Angeletos (2002) extends this result to the case of multiple riskless bonds when long bonds are bought back one period after issuance (as in our Section 3) and assuming that there are enough bonds to effectively complete the markets. For simplicity we only state this result for the case where g takes two values and there are two bonds, a one- and an N -period bond denoted $b_{1,t}, b_{N,t}$. Angeletos shows the sufficient equilibrium conditions are :

$$z_t = \mathbf{b}_{1,t-1} + E_t \left(\beta^N \frac{u_{c,t+N}}{u_{c,t}} \right) \mathbf{b}_{N,t-1} \quad (57)$$

for random variables $\mathbf{b}_{i,t}$ measurable with respect to g^t for all $t \geq 0$ $i = 1, N$. In this case the equilibrium is supported by bond positions $\mathbf{b}_{1,t} = b_{1,t}$ and $\mathbf{b}_{N,t} = b_{N,t}$ for all $t > 0$.

All three cases described above can be described as follows: sufficient equilibrium conditions require that private wealth in all periods must equal the discounted sum of primary surpluses z_t . We now show that when bonds are as in the current section an analog condition

$$\sum_{j=1}^{N-1} b_{N,t-j} \left(\kappa_{t-j} + q_t^{N-j} \right) + b_{N,t-N} = z_t \text{ for all } t \quad (58)$$

is *not* sufficient for an equilibrium, where q_t^{N-j} is as in (50) with maturity $N-j$ and coupon κ_{t-j} (instead of κ_t).

For this purpose we just need to find one case where (58) is not sufficient. We consider $N = 2$ so that (58) becomes

$$b_{2,t-1} \left(\kappa_{t-1} + \beta E_t \left(\frac{u_{c,t+1}}{u_{c,t}} \right) \right) + b_{2,t-2} = z_t. \quad (59)$$

The example in Section 5.1 showed that for $\kappa = 0$ this equation is not sufficient. One may think that removing the constraint $\kappa = 0$ could make (59) sufficient.

Indeed, if coupons are contingent on future information markets can be effectively completed.¹⁹ But contingent coupons are easily ruled out due to issues of moral hazard and because the fluctuations in coupons needed to complete the markets would be very large. Therefore we only consider fixed coupons in the remainder of the section, where κ_t is determined at the date of bond issuance as a function of g^t and the same coupon is paid during all periods $t + 1, \dots, t + N$.

We can offer the following set of sufficient conditions in our case

Result 4: *Consider risk-free real bonds of maturity $N = 2$ that pay fixed coupons $\{\kappa_t\}$ without buyback. Consider a consumption sequence $\{c_t\}$ such that the associated discounted sums of surpluses z_t satisfies*

1. $z_0 = b_{1,-1}(q_0^1 + \kappa_{-1}) + b_{2,-2}$ for given initial conditions $b_{1,-1}, b_{2,-2}, \kappa_{-1}$
2. (59) for all $t > 0$ a.s. for some random variables $b_{2,t}(g^t)$.

If, in addition,

3. *the random variables $b_{2,t}(g^t)$ mentioned in 2. satisfy bounds (52) for sufficiently large $\overline{M}, \underline{M}$ for all t a.s.*

then $\{c_t\}$ is a competitive equilibrium.

The proof follows a usual pattern and we do not offer details.²⁰ The main reason to write the above result is to highlight that in addition to the measurability constraints of Aiyagari et al. now condition 3 is needed under no buyback. It will turn out that condition 3 does not hold in many cases.²¹

A reference point will be coupons that are, roughly speaking, close to the net real rate of interest, so that bonds trade at (or close to) par, as most long bonds do in real markets (see Faraglia et al (2014b)). It follows from (50) that a coupon

$$\kappa_t^P = \frac{1 - \beta^2 E_t\left(\frac{u_{c,t+2}}{u_{c,t}}\right)}{1 + \beta E_t\left(\frac{u_{c,t+1}}{u_{c,t}}\right)} \quad (60)$$

causes the bond price to trade at par, namely $q_t^2 = 1 - \kappa_t^P$.

¹⁹ In particular, κ_{t-1} can always be chosen contingent on g_t so as to guarantee that (59) always holds for a correctly designed coupon. This can be done as follows. Take constant $b_{2,t-2} = b_{2,-1} = \bar{b}$. Then κ_{t-1} contingent on g_t can always be chosen to satisfy

$$\bar{b} = \frac{z_t}{1 + \kappa_{t-1} + \beta E_t\left(\frac{u_{c,t+1}}{u_{c,t}}\right)}$$

for all g^t so that (59) is certain to hold. In this case we are back to the standard case where (55) is the only implementability constraint.

²⁰ First prove necessity of (59). Then prove that if $b_{2,t}$ satisfies (59) the period-by-period budget constraints must hold.

²¹ In some cases fixed coupons can complete the markets and they can satisfy condition 3. For example, the portfolios of Angeletos, Buera and Nicolini can be implemented with a fixed κ_{t-1} , but this would require a very large and negative coupon. For the calibrated case of Buera and Nicolini a coupon of about minus 200% would implement the complete market allocation with a constant level of bonds. Again, we find this case of little interest as governments can not offer huge negative coupons.

Now, (59) can be rewritten as

$$\begin{aligned} b_{2,t-1} &= -z_t \delta_t + \delta_t b_{2,t-2} \\ \delta_t &= - \left(\kappa_{t-1} + \beta E_t \left(\frac{u_{c,t+1}}{u_{c,t}} \right) \right)^{-1} \end{aligned} \quad (61)$$

It should be clear that if $|\delta_t| < 1$ then it is "more likely" that a $b_{2,t}$ satisfying (61) does not explode and, therefore, it satisfies the boundedness condition 3 of Result 4.

This shows in a generic way that high coupons help to smooth taxes, since a high κ_{t-1} drives $|\delta_t|$ below 1. But the boundedness condition 3 is unlikely to hold if coupons are small.

Result 4 is "generic" because given a consumption sequence and a level of initial wealth $b_{1,-1} + (\beta + \kappa)b_{2,-2}$ there is always a value of $b_{1,-1}$ that guarantees that bond positions do not go to infinity if $|\delta|_1 < 1$, but for all other values of $b_{1,-1}$ and same wealth bonds go to infinity.

The following two results make this generic idea concrete in some special cases.

Result 5 *Consider, as in Section 5.1 the case of $g_t = \bar{g}$ and constant taxes. The boundedness condition 3 is satisfied (generically) if and only if the bond trades at a price higher than par, that is*

$$\kappa_t = \kappa \geq \kappa^P = 1 - \beta \quad (62)$$

It is clear that in this case $z_t = \bar{z}$ and $|\delta_t| = (\kappa + \beta)^{-1}$ so that if (62) holds then $b_{2,t}$ in (61) does not explode and condition 3 is valid. But for a low coupon $\kappa < \kappa^P = 1 - \beta$ then $b_{2,t-1}$ explodes and violates any bound limit. The example of Section 5.1 is a special case of this result for $\kappa = 0$.

If bonds trade at par $\kappa_t^P = 1 - \beta$ (59) gives

$$b_{2,t-1} = \bar{z} - b_{2,t-2}$$

and b oscillates in a two-period cycle: $b_{2,t} = b_{2,t-2}$.

The next result shows that high coupons also stabilize taxes in the case of uncertainty. It says that if coupons are sufficiently high long bonds and short bonds can implement the same allocations, but with low coupons long bonds implement fewer equilibria. Therefore one would expect less opportunities for tax smoothing in the presence of long bonds. For this result we make the following assumptions

A1- $u(c) = c$.

A2- g_t iid, stochastic and a.s. bounded: $\text{Prob}(|g_t| < K^g) = 1$ for $K^g < \mathcal{M}$ where $\mathcal{M} \equiv \max_x ((1 - v'(x))(A - x))$.

Notice that with this utility $1 - v'(x) = \tau$, therefore $\mathcal{M}$ represents the maximum tax revenue that can be generated in a given period in equilibrium (the maximum of the Laffer curve).

Define $\mathcal{CE}_\kappa^N$ as the set of all competitive equilibrium allocations $\{c_t\}_{t=0}^\infty$ for long bonds of maturity N , with a constant coupon κ . According to Result 4, allocations belong to $\mathcal{CE}_\kappa^N$ if and only if they satisfy conditions 1,2,3 in Result 4. For short bonds we write $\mathcal{CE}^1$ as the coupon is irrelevant.

Result 6

Assume A1 and A2 above. Consider two economies, one with a 1-period bond and another economy with a 2-period bond without buyback and coupon κ . The two economies have identical initial wealth, namely $b_{1,-1} = b_{2,-1}(\kappa + \beta) + b_{2,-2}$.

- a) If long bonds sell at higher than par, namely $\kappa > \kappa^P = 1 - \beta$, then $\mathcal{CE}^1 = \mathcal{CE}_\kappa^2$.
- b) For zero coupons $\mathcal{CE}_0^2 \subset \mathcal{CE}^1$ and $\mathcal{CE}_0^2 \neq \mathcal{CE}^1$

Proof

We first show that for any coupon κ we have $\mathcal{CE}_\kappa^2 \subset \mathcal{CE}^1$. Consider a given allocation $\{c_t\}_{t=0}^\infty \in \mathcal{CE}_\kappa^2$, with associated discounted sum of surpluses z_t and $\{b_{2,t}\}$ the bond sequence that implements this equilibrium with 2-period bonds. Since $z_t = b_{2,t-1}(\kappa + \beta) + b_{2,t-2}$ it is clear that z_t is measurable with respect to information at $t - 1$. It follows from proposition 1 in Aiyagari et al. (2002) that this allocation is also an equilibrium for an economy with one-period bonds given by $b_{1,t-1} = z_t$. Obviously, since $b_{2,t}$ is uniformly bounded so is $b_{1,t}$. Therefore all conditions in Result 4 for $N = 1$ (or equivalently, all conditions in Aiyagari et al. proposition 1) are satisfied.

All that remains for part a) is to show that $\mathcal{CE}^1 \subset \mathcal{CE}_\kappa^2$ for sufficiently high κ . Given $\{c_t\}_{t=0}^\infty \in \mathcal{CE}^1$ and the corresponding bond allocation $b_{1,t}$ we construct the following $b_{2,t}$

$$b_{2,t} = -b_{2,t-1} \frac{1}{\beta + \kappa} + b_{1,t} = \sum_{j=0}^{t+1} \left(-\frac{1}{\beta + \kappa} \right)^j b_{1,t-j} + \left(-\frac{1}{\beta + \kappa} \right)^{t+2} b_{2,-2}.$$

Clearly, since $b_{1,t-j}$ is uniformly bounded and $\frac{1}{\beta + \kappa} < 1$ this $b_{2,t}$ satisfies (59) and condition 3.

To show b) it is enough to find one allocation in $\mathcal{CE}^1$ that is not in $\mathcal{CE}_0^2$. Consider the case $b_{1,-1} = 0$. We construct such an allocation as follows: fix parameters $\alpha, \eta > 0$, given the state variables $(g_t, b_{1,t-1})$ tax revenue at t is given by

$$(1 - v'(x_t))(A - x_t) = E(g_t) + \alpha(g_t - E(g_t)) + \eta b_{1,t-1}. \quad (63)$$

This equation defines a level of hours worked for given values of the shock and past debt, hence a corresponding allocation c_t .

For this policy $\alpha \leq 1$ governs how much the deficit increases when g is higher than average, thus it governs how much of an adverse shock is absorbed by deficit and debt. If we set $\alpha = 1$ this leads to a balanced budget and no tax smoothing. However if $0 < \alpha < 1$ there is some tax smoothing, an adverse g causes a deficit and higher debt. Parameter η governs the effect of past debt on current primary deficit. We assume α, η are chosen so that the right side of (63) is lower than $\mathcal{M}$ so there is always an x_t that solves (63), more on this later.

The budget constraint implies that the corresponding bond sequence is

$$b_{1,t}\beta = (1 - \eta)b_{1,t-1} + (1 - \alpha)[g_t - E(g_t)]$$

so that

$$b_{1,t} = \sum_{j=0}^{t-1} \left(\frac{1 - \eta}{\beta} \right)^j \frac{1 - \alpha}{\beta} g_{t-j}$$

If

$$\eta > 1 - \beta$$

then $b_{1,t}$ is bounded above by $\frac{1-\alpha}{\beta+\eta-1} [K - E(g_t)]$ so that (63) belongs to $\mathcal{CE}^1$. Furthermore, from this equation it is clear that there are many values of α, η guaranteeing that the right side of (63) is lower than $\mathcal{M}$. It is also clear that this allocation belongs to $\mathcal{CE}^1$ for any $\alpha \in (0, 1)$ and $\eta > 1 - \beta$ such that revenue is feasible.

Now we check that this allocation does not belong to $\mathcal{CE}_0^2$. To implement the allocation (63) with $N = 2$ and $\kappa = 0$ we would need a $b_{2,t}$ such that $b_{1,t} = b_{2,t}\beta + b_{2,t-1}$ hence

$$\begin{aligned} b_{2,t} &= \sum_{j=0}^t \left(\frac{1}{\beta}\right)^j b_{1,t-j} = \sum_{j=0}^t \left(\frac{1}{\beta}\right)^j \sum_{i=0}^{t-j} \left(\frac{1}{\beta} - \eta\right)^i \frac{1-\alpha}{\beta} g_{t-j-i} \\ &= \left(\frac{1}{\beta}\right)^t [1 + (1 - \eta\beta) + (1 - \eta\beta)^2 + \dots + (1 - \eta\beta)^t] \frac{1-\alpha}{\beta} g_0 \\ &\quad + \left(\frac{1}{\beta}\right)^{t-1} [1 + (1 - \eta\beta) + (1 - \eta\beta)^2 + \dots + (1 - \eta\beta)^{t-1}] \frac{1-\alpha}{\beta} g_1 + \dots \\ &= \left(\frac{1}{\beta}\right)^t \frac{1-\alpha}{\beta} \sum_{j=0}^t \beta^j \frac{1 - (1 - \eta\beta)^{j+1}}{\eta\beta} g_{t-j} \end{aligned}$$

It is clear that

$$\begin{aligned} \text{var}(b_{2,t}) &= \left(\frac{1}{\beta}\right)^{2t} \left(\frac{1-\alpha}{\beta^2\eta}\right)^2 \left(\sum_{j=0}^t \beta^j [1 - (1 - \eta\beta)^{j+1}]\right)^2 \text{var}(g_t) \\ &> \left(\frac{1}{\beta}\right)^{2t} (1-\alpha)^2 \eta \text{var}(g_t) \end{aligned}$$

where the inequality comes from including only the term for $j = 0$ in the sum $\sum_{j=0}^t$.

This shows that $\text{var}(b_{2,t}) \rightarrow \infty$ as $t \rightarrow \infty$, therefore any bond limits will be violated eventually and we can not find a $b_{2,t}$ that implements the policy (63). ■

Volatility of positions

Although the previous results may suggest that selling long bonds at par may be enough for stability, the fact is that long bond positions that complete the markets can be very volatile. To see this, take the case used in Result 5. If bonds are sold at par this implies

$$b_{2,t} = \bar{z} - b_{2,t-1} \tag{64}$$

In cases where $b_{2,-1}$ is larger than $b_{2,-2}$ bonds would alternate forever between positive and negative numbers each period. Similarly, in a model with uncertainty as in Result 6 a shock in one period would cause higher debt in that period and changes in positions for all future periods if coupons are close to $1 - \beta$.

Such fluctuations of bond positions would cause large variations in gross issuance of debt from one period to the next, although strictly speaking these are not a problem in the current setup they are often seen as undesirable in actual debt management practice.

Summary

A summary of all these results is that long bonds without buyback do not provide much fiscal insurance. Coupons alleviate the problem but they may introduce large oscillations in gross bond issuance. If these oscillations are ruled out long bonds will not complete the markets even with coupons near par.

6.3 Simulations with coupons

To see more clearly the impact of paying coupons under uncertainty we need to resort once more to simulations. Consider again the case of Figure 7 based on persistent shocks, a ten year bond and positive levels of initial debt. In our simulations we set $\kappa_t = \kappa = 1 - \beta$ for all t , corresponding to coupons that trade approximately at par (exactly at par only in the risk neutral case).

Figure 7 shows the response of taxes to an adverse expenditure shock. We know from the previous example that no buy back induces greater volatility in taxes and produces a N period cycle. Under incomplete markets the timing of cash flow matters and so taxes spike with maturity and redemption. However as Figure 7 shows by allowing long term bonds to pay coupons we produce less volatile N cycles. The intuition is of course obvious - coupons spread the timing of cash payments from a bond and so reduce the magnitude of the N cycles.

An intuition for why coupons near par smooth taxes arises from Results 5 and 6, showing that in some cases coupons help sustain the same tax profile as with short bonds. Figure 7 does not show such an extreme case, we still find rollover cycles, unlike the case of short bonds $N = 1$, but the cycles are less pronounced than with zero coupons. What happens is that the bond limits we impose make it impossible for the government to smooth taxes as in short bonds because, as we explained in Section 6.2 we need a very large variation in bond positions in order for coupons to achieve tax smoothing as with short bonds. Therefore the case with coupons is somewhere between zero-coupon long bonds and short bonds.

To understand the behavior of optimal policy with coupons in more detail let us for simplicity assume shocks are i.i.d. and $\kappa = 0$. Moreover assume a temporary rise in g , $g_0 > g^*$ and $g_t = g^*$ for all periods different than zero. The government has to raise revenues in period 0 and subsequently set taxes equal to the steady state level for $N - 1$ periods until the debt issued in $t = 0$ matures and the principal repaid in $t = N$. On the other hand if $\kappa > 0$, the government promises to pay a coupon from $t = 1$ to $N - 1$ and the principal in N . These coupons are essentially short term debt and taxes can now be raised in all periods from 1 to N to finance the deficit. This suggests a fairly immediate explanation for why long term bonds pay coupons and short term bonds don't. Governments can use coupons as a way of reducing tax volatility. This is confirmed by comparing Tables 3 and 4 where paying coupons under no buy back reduces the volatility of taxes and consumption compared to the case of no coupons and no buy back.

6.4 Coupons, Commitment and Independent Powers

Having extended our model to allow for the empirically motivated features of no buy back and coupons we return again to considering the role of commitment in optimal debt management and fiscal policy. To that end, Figures 7 to 7 report the impulse responses of taxes and other model

variables to a one standard deviation fiscal shock under both full commitment and our model of independent powers. In both cases we assume that $N = 10$ and in Figure 7 we study the case of zero initial debt and in 7 positive initial debt. Figure 7 focuses on the impact of extending the maturity of debt for both commitment and independent powers.

Figure 7 shows that there are substantial oscillation in the tax rates in both models with evident N period cycles. When a shock hits, the planner issues debt to finance part of the extra deficit and every N years has to repay the principal. This leads to spikes in the repayment and the tax schedule for reasons explained in the previous section.

However, unlike the case where we assumed buyback at the end of each period we now see substantial differences in the magnitudes of the tax response between the commitment and independent power models (first panel in Figures 7 and 7). Under commitment taxes are much less volatile and indeed more persistent²². These differences persist also in the behaviour of the endogenous state variables of the models. In commitment models past government promises, summarized by the vector of lagged Lagrange multipliers $\lambda_{t-1}, \dots, \lambda_{t-N}$, proxy the policy objective to smooth taxes over time. In the independent power model, δ is not a state variable. Then the planner's tax policy is mainly driven by cash flow concerns making taxes a function of bonds and exogenous shocks solely. With no buyback available the government loses an adjustment channel and now the past sequence of debt issued in each past period covered by the maturity of debt exerts its impact.

Figure 7 shows the responses when initial government debt is positive and $b_{-1} = \dots = b_{-N}$. The government now wishes to reduce taxes between $t = 1$ and $t = N$ to reduce the value of the initial debt burden. This causes a mild drop in tax response in these initial periods relative to the case of zero initial debt. However the effect of interest rates twisting is overpowered by the considerably larger volatility originated by the N cycle property. Indeed even with positive government debt tax rates are way more volatile in the independent powers model.

[Figures 7, 7 and 7 About Here.]

Figure 7 extends these results to different maturities.²³

In Tables 3 and 4 we show sample moments generated by the simulations from the models of this section. The basic patterns of taxes, consumption and the market value of debt are similar to the ones generated by the model of the previous section. Debt is negative in the longer run because governments wish to accumulate precautionary savings for tax smoothing purposes. The additional volatility of taxes generated by independent powers however is larger now.

[Tables ?? and ?? About Here.]

To conclude, commitment matters more under no buyback. The key result is that in incomplete markets lumpy debt repayment profiles generate tax volatility causing spikes in the tax rates at redemption dates. These patterns are mitigated under commitment as the tax smoothing objective, summarized in the multipliers, influences the policy functions. In contrast, in an independent power model where the tax schedule is a function only of the exogenous shock and the level of debt the

²² High persistence is more clearly illustrated through extending the number of periods in the graph. We provide a longer time series of these responses in the Appendix. The results clearly demonstrate the differences in persistence and volatility.

²³ Clearly the N cycle property of the tax schedule coincides with the maturity date of debt in the figure. Across all values for N considered the variability of taxes in independent powers prevails.

lumpiness of government debt magnifies the volatility of taxes. Coupons help reduce the lumpiness and volatility of tax rates in both environments. Non-coordination between fiscal and monetary authorities induces greater tax volatility when governments do not buy back all outstanding debt at the end of each period.

7 Conclusions

In contrast to the case of short bonds, analysing long bonds requires numerous assumptions about the institutional setup. We have considered two such assumptions here - does the government buyback each period all outstanding government debt? do government bonds pay coupons? Depending on the assumptions made the behaviour of debt and the related optimal fiscal policy will be very different.

Our focus has been to understand how a government issuing long bonds impacts on optimal fiscal policy. It is well known in the literature that long bonds offer fiscal insurance in the sense that in response to adverse expenditure shocks their price declines. However we have shown that over and above fiscal insurance there are additional intertemporal channels - interest rate twisting and rollover cycles - that create additional tax volatility. The closer the assumptions we make regarding the structure of long bonds are to those we observe in practice (especially around no repurchase before maturity) the more important these additional channels are. Thus whilst long bonds may provide fiscal insurance they also may induce additional tax volatility lessening their attractiveness and suggesting a role for short term debt - an issue we examine in Farglia, Marcet, Oikonomou and Scott (2014b). Further not only is the attractiveness of long bonds offset by this additional tax volatility but we find cases where long bonds are incapable of completing the market.

We have also considered the nature of commitment that the government faces under optimal debt management. In the standard Ramsey institutional set up we have shown that the role of commitment for optimal policy under incomplete markets is a repeated attempt at lowering current interest rates by promising future tax cuts, depending on the current shock. It is this feature that makes modelling long bonds so computationally demanding and which motivates us to suggest an alternative institutional set up (independent powers) that is relatively easy to solve and has applicability in a wider class of models than public finance.

We show our insights both through analytical examples and simulations. Our analytic examples provide insight and intuition to the mechanisms at work with long bonds and these are born out in our simulations. With no buyback, taxes and outcomes are more volatile, coupon payments help reduce these problems. However many of our analytic examples have a form that emphasizes different factors from our business cycle focused simulations. In the face of one off large shocks (such as wars or financial crises) some of the complexities of long bonds that we highlight become more important. For instance, the impossibility result (that long bonds may not complete the market) requires an uneven debt structure before the current period, an outcome that is unlikely to occur in our simulations with highly persistent shocks.

The existing literature has taken a normative approach to debt management. It has assumed a certain structure for bonds and shown how under this structure long bonds excel in providing fiscal insurance and are key to optimal debt management. We examine different structures for long bonds and show that the ability to achieve fiscal insurance through long bonds is both reduced and

offset by additional tax volatility making long bonds less attractive. An obvious response is to argue that governments should not engage in no buyback. If governments simply repurchased all existing bonds regardless of maturity every period then the fiscal insurance benefits of long bonds would stand unalloyed. Whilst debt managers give many reasons why they do not repurchase every period and a few moments introspection can lead to many plausible theoretical candidates as to why they might not it is indeed an important lacunae in our understanding of debt management as to why governments repurchase only occasionally.

However until we have a better understanding of the reasons why debt managers don't repurchase and whether or not it is optimal to do so we are wary of restricting assumptions about long bonds to the case of zero coupon and repurchase. Inevitably incomplete market models of debt management have to assume some market imperfections exogenously. Most of the existing literature for instance has assumed that bonds are simply risk free or that the government can issue bonds of only one period and that period coincides with the frequency of government shocks. Similarly assuming that the government buys back each and every bond every period is not grounded in any optimality and neither is our opposite extreme assumption that governments never buyback until maturity. Our assumption does however have the merit of being the closest to what we observe in practice and the fact that we show it matters for the optimality of long bonds in debt management suggests the analysis of maturity in government bonds is more complex than just the standard fiscal insurance channel.

References

- [1] Aiyagari, R., A. Marcet, T.J. Sargent and J. Seppälä. (2002) “Optimal Taxation without State-Contingent Debt”, *Journal of Political Economy*, vol. 110, 1220-1254.
- [2] Angeletos, G-M (2002) “Fiscal policy with non-contingent debt and optimal maturity structure”, *Quarterly Journal of Economics*, vol. 117, 1105-1131
- [3] Arellano, C. and A. Ramanarayanan (2012) “Default and the Maturity Structure in Sovereign Bonds”, *Journal of Political Economy*, vol. 120(2), pages 187 - 232.
- [4] Barro, R.J. (1979) “On the determination of the public debt”, *Journal of Political Economy*, 87, 940-71.
- [5] Barro, R.J. (2003) “Optimal Management of Indexed and Nominal Debt” *Annals of Economics and Finance*, 4, 1-15
- [6] Bohn, H. (1990) “Tax smoothing with financial instruments” , *American Economic Review*, Vol. 80(5), 1217–30.
- [7] Broner, F., G. Lorenzoni, and S. Schmulker (2013) “Why do emerging markets borrow short term?” *Journal of the European Economic Association, European Economic Association*, vol. 11, 67-100.
- [8] Buera F. and J.P. Nicolini (2004) “Optimal Maturity of Government Debt with Incomplete Markets”, *Journal of Monetary Economics*, 51, 531-554 .
- [9] Chari, V.V , L.Christiano, and P.J. Kehoe (1991) ”Optimal Fiscal and Monetary Policy : Some Recent Results” *Journal of Money, Credit and Banking* 23, 519-539
- [10] Chen, H., V. Cúrdia and A. Ferrero (2012) ”The Macroeconomic Effects of Large-scale Asset Purchase Programmes”, *Economic Journal*, vol. 122(564), pages F289-F315.
- [11] Conesa, J.C. and T.J. Kehoe (2015) ”Gambling for Redemption and Self-Fulfilling Debt Crises”, mimeo Stony Brook University.
- [12] Debt Management Office (2003) “DMO Annual Review 2003/4” HM Treasury
- [13] Derbortoli, D., R. Nunes and P. Yared. (2015) ”Optimal Time-Consistent Debt Maturity”, Universitat Pompeu Fabra mimeo
- [14] den Haan, W. and A. Marcet. (1990) “Solving the stochastic growth model by parameterizing expectations”, *Journal of Business and Economic Statistics*, 8, 31-34.
- [15] Faraglia, E, A. Marcet and A. Scott (2008) “Fiscal Insurance and Debt Management in OECD Economies”, *The Economic Journal*, (118) 363-386
- [16] Faraglia, E., A. Marcet, and A. Scott (2010); “In Search of a Theory of Debt Management” *Journal of Monetary Economics*, 57, 821–836.

- [17] Faraglia, E., A. Marcet, R. Oikonomou, and A. Scott. (2014 a) “Optimal Fiscal Policy Problems Under Complete and Incomplete Financial Markets: A Numerical Toolkit ”, London Business School mimeo.
- [18] Faraglia, E., A. Marcet, R. Oikonomou and A. Scott. (2014 b) “Government Debt Management - The Long and the Short of it”, London Business School mimeo.
- [19] Greenwood, R., S. Hanson, J. Rudolph and L. Summers. (2014) ”Government Debt Management at the Zero Lower Bound”, Harvard Business School mimeo
- [20] Lucas, R.E. and N.L. Stokey. (1983) “Optimal Fiscal and Monetary Policy in an Economy without Capital”, *Journal of Monetary Economics*, 12, 55-93.
- [21] Lustig, H., C. Sleet and S. Yeltekin. (2009) “Fiscal hedging with nominal assets” *Journal of Monetary Economics*, 55, 710–727.
- [22] Marcet, A and Marimon, R (2014) “Recursive Contracts”, working paper, Institut d’Anàlisi Econòmica CSIC, Universitat Autònoma de Barcelona.
- [23] Marcet, A and A. Scott (2009) “Debt and Deficit Fluctuations and the Structure of Bond Markets” *Journal of Economic Theory* 144 (2009) 473–501
- [24] Nosbusch, Y. (2008) “Interest Costs and the Optimal Maturity Structure of Government Debt”, *Economic Journal*, 118, 477–498.
- [25] Schmitt-Grohe, S. and M. Uribe. (2004), ”Optimal Fiscal and Monetary Policy Under Sticky Prices”, *Journal of Economic Theory*, 114, 198-230.
- [26] Siu, H. (2004) ”Optimal Fiscal and Monetary policy with sticky prices”, *Journal of Monetary Economics*, 51, 575-607
- [27] Woodford, M. (2001) “Fiscal Requirements of Price Stability”, *Journal of Money Credit and Banking*, 33, 669-728.

Notes: The Figure plots the impulse response of taxes (top- left panel), consumption (top-right panel), deficit (bottom- left) and the market value of debt (bottom-right) to a one standard deviation shock in government spending. The maturity of long debt is $N = 10$ years. The quantities represented by the solid (blue) lines correspond to the optimal commitment allocation and the quantities plotted with dashed (red) lines correspond to the independent power model. The starting value of debt is zero.

Notes: The Figure plots the impulse response of taxes (top- left panel), consumption (top-right panel), deficit (bottom- left) and the market value of debt (bottom-right) to a one standard deviation shock in government spending. The maturity of long debt is $N = 10$ years. The quantities represented by the solid (blue) lines correspond to the optimal commitment allocation and the quantities plotted with dashed (red) lines correspond to the independent power model. The initial debt level is 50
par

Notes: The Figure plots the impulse response of taxes to a spending shock. The top panels show the commitment model under buyback with zero (left) and positive (right) debt levels. The solid line shows the response of the tax schedule when the maturity is $N = 5$. The dashed line corresponds to $N = 10$ and the crossed and dashed-dotted lines to $N = 15$ and $N = 20$ respectively. The bottom panels in the figure show the analogous responses in the case of independent powers.

		t=40				t=200				long run			
	<i>N</i>	<i>c</i>	τ	<i>deficit</i>	<i>MV</i>	<i>c</i>	τ	<i>deficit</i>	<i>MV</i>	<i>c</i>	τ	<i>deficit</i>	<i>MV</i>
mean	5	52.49	0.253	-0.233	-8.90	52.54	0.248	0.156	-15.98	52.58	0.245	0.317	-19.23
	10	52.50	0.252	-0.198	-7.18	52.54	0.248	0.120	-13.46	52.56	0.246	0.257	-16.28
	15	52.50	0.252	-0.184	-6.50	52.53	0.248	0.095	-11.76	52.55	0.247	0.209	-14.01
	20	52.50	0.252	-0.168	-5.83	52.52	0.249	0.076	-10.47	52.54	0.247	0.178	-12.48
std	5	2.27	0.0223	1.28	11.82	3.13	0.0357	1.58	23.82	3.48	0.0435	1.57	30.96
	10	2.27	0.0224	1.23	11.22	3.13	0.0357	1.59	24.23	3.48	0.0438	1.59	31.97
	15	2.27	0.0225	1.29	11.55	3.13	0.0357	1.60	24.86	3.48	0.0439	1.65	32.54
	20	2.27	0.0226	1.31	11.50	3.13	0.0357	1.61	25.03	3.48	0.0439	1.367	32.83

Notes: The table shows key moments from the optimal policy model with commitment under buyback. The first four rows plot the averages for consumption, taxes, deficit and the market value of debt in the case of four different maturities ($N = \{5, 010, 15, 20\}$). The last four rows show the standard deviations of these quantities. The table reports the moments over three different horizons: 1) 40 observations, 18000 samples. 2) 2000 observations, 18000 samples, and 3) 4500 observations 18000 samples (5000 observations were generated for each sample, 500 observations were dropped). The initial conditions for government debt are uniformly distributed over $[-M_N, M_N]$.

		t=40				t=200				long run			
	<i>N</i>	<i>c</i>	τ	<i>deficit</i>	<i>MV</i>	<i>c</i>	τ	<i>deficit</i>	<i>MV</i>	<i>c</i>	τ	<i>deficit</i>	<i>MV</i>
mean	5	52.49	0.253	-0.226	-8.23	52.54	0.248	0.156	-15.88	52.58	0.245	0.323	-19.49
	10	52.50	0.252	-0.198	-6.95	52.53	0.248	0.120	-13.33	52.56	0.246	0.260	-16.40
	15	52.50	0.252	-0.178	-6.09	52.53	0.249	0.095	-11.57	52.55	0.247	0.215	-14.23
	20	52.50	0.252	-0.161	-5.34	52.52	0.249	0.076	-10.01	52.54	0.247	0.175	-12.31
std	5	2.29	0.0224	1.14	11.05	3.14	0.0357	1.47	23.45	3.48	0.0436	1.51	31.12
	10	2.29	0.0224	1.13	11.00	3.14	0.0357	1.48	23.93	3.49	0.0438	1.54	32.20
	15	2.29	0.0223	1.12	10.97	3.14	0.0357	1.48	24.17	3.49	0.0440	1.55	32.78
	20	2.29	0.0223	1.12	10.94	3.14	0.0356	1.48	24.34	3.49	0.0440	1.56	33.20

Notes: The table shows key moments from the independent power model under buyback. The first four rows plot the averages for consumption, taxes, deficit and the market value of debt in the case of four different maturities ($N = \{5, 010, 15, 20\}$). The last four rows show the standard deviations of these quantities. The table reports the moments over three different horizons: 1) 40 observations, 18000 samples. 2) 2000 observations, 18000 samples, and 3) 4500 observations 18000 samples (5000 observations were generated for each sample, 500 observations were dropped). The initial conditions for government debt are uniformly distributed over $[-M_N, M_N]$.

		t=40				t=200				long run			
<i>N</i>	<i>c</i>	τ	<i>deficit</i>	<i>MV</i>		<i>c</i>	τ	<i>deficit</i>	<i>MV</i>	<i>c</i>	τ	<i>deficit</i>	<i>MV</i>
mean													
5	52.47	0.253	-0.252	-9.63		52.57	0.247	0.175	-17.39	52.61	0.244	0.336	-20.36
10	52.48	0.252	-0.225	-8.217		52.56	0.247	0.139	-14.98	52.59	0.245	0.278	-17.60
15	52.49	0.252	-0.200	-7.10		52.55	0.247	0.114	-13.15	52.59	0.245	0.239	-15.57
20	52.49	0.252	-0.186	-6.31		52.55	0.248	0.096	-11.78	52.58	0.246	0.211	-14.03
std													
5	2.28	0.0218	1.29	12.29		3.13	0.0359	1.60	23.84	3.49	0.0441	1.63	30.93
10	2.28	0.0216	1.30	12.51		3.13	0.0358	1.64	25.12	3.49	0.0445	1.70	33.22
15	2.29	0.0221	1.25	12.30		3.13	0.0361	1.62	25.19	3.50	0.0448	1.68	33.58
20	2.30	0.0228	1.19	11.99		3.14	0.0366	1.58	24.83	3.50	0.0452	1.64	33.29

Notes: The table shows key moments of consumption, taxes, deficit and the standard deviations of these quantities over 2000 observations, 18000 samples, were dropped). The initial conditions were dropped).

		t=40				t=200				long run			
<i>N</i>	<i>c</i>	τ	<i>deficit</i>	<i>MV</i>		<i>c</i>	τ	<i>deficit</i>	<i>MV</i>	<i>c</i>	τ	<i>deficit</i>	<i>MV</i>
mean													
5	52.47	0.253	-0.241	-15.69		52.58	0.246	0.232	-19.26	52.61	0.244	0.350	-20.16
10	52.47	0.253	-0.244	-14.70		52.57	0.246	0.208	-17.98	52.60	0.244	0.316	-18.74
15	52.47	0.253	-0.252	-13.30		52.57	0.246	0.195	-17.82	52.60	0.244	0.313	-18.98
20	52.47	0.253	-0.253	-11.84		52.57	0.246	0.180	-16.94	52.60	0.245	0.301	-18.43
std													
5	2.30	0.0296	1.84	12.40		3.19	0.0419	1.71	15.91	3.53	0.0471	1.59	17.98
10	2.27	0.0284	1.94	13.72		3.17	0.0429	2.04	19.46	3.52	0.0491	2.00	22.64
15	2.26	0.0245	1.69	13.91		3.15	0.0393	1.85	22.18	3.50	0.0465	1.82	26.74
20	2.27	0.0237	1.53	13.68		3.14	0.0385	1.77	23.20	3.50	0.0460	1.77	28.48

Notes: The table shows key moments from the independent power model with coupons and no buyback. The first four rows plot the averages for consumption, taxes, deficit and the market value of debt in the case of four different maturities ($N = \{5, 010, 15, 20\}$). The last four rows show the standard deviations of these quantities. The table reports the moments over three different horizons: 1) 40 observations, 18000 samples. 2) 2000 observations, 18000 samples, and 3) 4500 observations 18000 samples (5000 observations were generated for each sample, 500 observations were dropped). The initial conditions for government debt are uniformly distributed over $[-M_N, M_N]$.

Notes: The Figure plots the impulse response of taxes to a spending shock. The solid line is the response under the assumption that debt is bought back in every period, the dashed line shows the case of no buyback and zero coupons. Finally, the crossed line shows the case of non zero coupon bonds.

Notes: The Figure plots the impulse response of taxes (top- left panel), consumption (top-right panel), deficit (bottom- left) and the market value of debt (bottom-right) to a one standard deviation shock in government spending. The maturity of long debt is $N = 10$ years. The quantities represented by the solid (blue) lines correspond to the optimal commitment allocation and the quantities plotted with dashed (red) lines correspond to the independent power model. The starting value of debt is zero.

A Derivations- No Commitment Models

A.1 Independent Powers - Constant Coupons and No Buyback

Consider the first model of Section 3. In the no-commitment case (independent powers) we have the following Lagrangian:

$$\begin{aligned} \mathcal{L} = E_0 \sum \beta^t \left\{ u(c_t) + v(T - c_t - g_t) + \lambda_t [b_t^N q_t^N u_{c,t} - b_{t-N}^N u_{c,t} - \kappa \sum_{j=1}^N b_{t-j}^N u_{c,t} + S_t] \right. \\ \left. + v_{1,t}(\widetilde{M}_N - b_t^N) + v_{2,t}(b_t^N - \widetilde{M}_N) \right\} \end{aligned}$$

and taking first order conditions we have:

$$u_{c,t} - v_{x,t} + \lambda_t(u_{cc,t}c_t + u_{c,t} + v_{xx,t}(c_t + g_t) - v_{x,t} - u_{cc,t}\lambda_t[b_t^N q_t^N - \kappa \sum_{j=1}^N b_{t-j}^N - b_{t-N}^N]) = 0$$

Notes: The Figure plots the impulse response of taxes (top- left panel), consumption (top-right panel), deficit (bottom- left) and the market value of debt (bottom-right) to a one standard deviation shock in government spending. The maturity of long debt is $N = 10$ years. The quantities represented by the solid (blue) lines correspond to the optimal commitment allocation and the quantities plotted with dashed (red) lines correspond to the independent power model. The initial debt level is 50

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Notes: The Figure plots the impulse response of taxes to a spending shock. The top panels show the commitment model under decaying coupons with zero (left) and positive (right) debt levels. The solid line shows the response of the tax schedule when the maturity is $N = 5$. The dashed line corresponds to $N = 10$ and the crossed and dashed-dotted lines to $N = 15$ and $N = 20$ respectively. The bottom panels in the figure show the analogous responses in the case of independent powers.

which through substitution of the budget constraint gives:

$$u_{c,t} - v_{x,t} + \lambda_t(u_{cc,t}c_t + u_{c,t} + v_{xx,t}(c_t + g_t) - v_{x,t} - u_{cc,t}\lambda_t[g_t - (1 - \frac{v_{x,t}}{u_{c,t}})(c_t + g_t)]) = 0 \quad (65)$$

Moreover, off corners the analogous condition b_t^N is:

$$\lambda_t E_t(\kappa \sum_{j=1}^N \beta^j u_{c,t+j} + \beta^N u_{c,t+N}) = E_t(\kappa \sum_{j=1}^N \beta^j u_{c,t+j} \lambda_{t+j} + \beta^N u_{c,t+N} \lambda_{t+N})$$

We utilize the above first order conditions to solve the optimal no-commitment model. Notice that in order to solve this model we have to apply the 'Condensed PEA' methodology of FMOS (2014 b). The state vector for this model includes g_t $b_{N,t-j}$, $j = 1, 2, \dots, N$. This gives $N + 1$ state variables.

B Appendix: Decaying Coupon Perpetuities

In order to overcome the problem of dimensionality some authors model long bonds as perpetuities with decaying coupon payments where the rates of decay mimic differences in maturity (e.g Woodford (2001), Broner, Lorenzoni and Schmulker (2013), Arellano and Ramanarayanan (2008), Chen, Curdia and Ferrero (2012)). Our independent powers model can be considered as an alternative means of achieving the same computational parsimony.

In this case governments issues perpetuities, b , with coupon payments that decay geometrically i.e a bond with decay factor δ_L pays a coupon equal to $\delta_L^j b_j$ in period j . The decay rate determines effective bond maturity as duration is defined by $1/(1 - \delta_L)$ so that a bond of effective maturity 10 years has $\delta_L = 0.1$. In this case total payments from all previously and currently issued perpetuities are then given by $B_t = b_t + \delta_L b_{t-1} + \delta_L^2 b_{t-2} + \dots + \delta_L^t b_0$ which follows the recursive structure $B_t = \delta_L B_{t-1} + b_t$. Treating this as the outstanding stock of the perpetuity we have a convenient way of dealing with long maturity bonds which dramatically reduces the state space as it is only necessary to keep track of the total number of bonds issued and not the number of bonds issued in each period. This reduction in the state space means that the “Condensed PEA” is no longer required and the model can be solved using more conventional methods.

Whilst assuming decaying coupon payments has great computational merit it is not without modelling consequences. One justification for assuming decaying payoffs is that it mimics a bond portfolio with fixed shares that decay with maturity. This however does not seem to comply with the empirical evidence of US debt management whereby shares of long and short bonds are indeed time varying, though highly persistent and don't decline with maturity (see Section 2 in FMOS (2014 b)). Further, modelling bond payoffs in this way is contrary to the structure of most government portfolios where the majority of the payoff occurs at the time of maturity, as we have modelled in this paper, whereas with decaying coupons the majority of cash flow is paid out in the early years. Moreover if our goal is to build a model of debt management where the object is precisely to study the appropriate portfolio weights, assuming fixed portfolio weights would seem inappropriate. However this approach has been used in the literature and does offer substantial computational efficiency so it is worth comparing it with our own approaches.

Under perpetual bonds the government budget constraint becomes:

$$B_{t-1} = \tilde{S}_t + p_t (B_t - \delta_L B_{t-1})$$

where $B_t - \delta_L B_{t-1} = b_t$ is the amount of bonds that the government issues in period t and B_{t-1} is the amount of coupons and maturing bonds that the government has to repay in the same period.

The Ramsey problem then becomes:

$$\begin{aligned} \max_{\{c_t, B_t, p_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \{u(c_t) + v(x_t)\} \\ B_{t-1} = \tilde{S}_t + p_t (B_t - \delta_L B_{t-1}) \\ p_t = \frac{\beta E_t (u_{c,t+1} (1 + \delta_L p_{t+1}))}{u_{c,t}}. \end{aligned}$$

The price of the bond can be rewritten as $p_t = \frac{\beta E_t(\sum_{j=0}^{\infty} (\beta \delta_L)^{j-1} u_{c,t+j})}{u_{c,t}}$, that shows that it is a function of all the future marginal utilities since the bond will pay an income for the rest of the time.

B.1 The Ramsey Program

We can rewrite the Lagrangian of the problem as:

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \{ u(c_t) + v(x_t) + \lambda_t [S_t + u_{c,t} p_t (B_t - \delta_L B_{t-1}) - u_{c,t} B_{t-1}] \\ + u_{c,t} (\mu_t p_t - \beta \mu_{t-1} (1 + \delta_L p_t)) \}$$

dropping the debt limits for brevity.

The first order conditions of the decaying coupon model are as follows:

$$u_{c,t} - v_{x,t} + \lambda_t (u_{cc,t} c_t + u_{c,t} + v_{xx,t} (c_t + g_t) - v_{x,t}) \\ + u_{cc,t} [\lambda_t p_t (B_t - \delta_L B_{t-1}) - \lambda_t B_{t-1} + \mu_t p_t - \mu_{t-1} (1 + \delta_L p_t)] = 0 \quad (66)$$

$$\lambda_t u_{c,t} p_t = \beta E_t (\lambda_{t+1} u_{c,t+1} (1 + \delta_L p_{t+1})) \quad (67)$$

$$\mu_t = \delta_L \mu_{t-1} - \lambda_t (B_t - \delta_L B_{t-1}). \quad (68)$$

A new state variable, μ_t , emerges and the state space becomes $\{g_t, \mu_{t-1}, B_{t-1}\}$. Using (68) μ_t may be expressed as:

$$\mu_t = - \sum_{j=0}^{\infty} \delta_L^j \lambda_{t-j} (B_{t-j} - \delta_L B_{t-1-j}) \quad (69)$$

where μ_t is a function of all the past government promises, λ 's. Equation (69) can be substituted in (70)

$$u_{c,t} - v_{x,t} + \lambda_t (u_{cc,t} c_t + u_{c,t} + v_{xx,t} (c_t + g_t) - v_{x,t}) \\ - u_{cc,t} \left[B_{t-1} \lambda_t - \sum_{j=0}^{\infty} \delta_L^j \lambda_{t-j-1} (B_{t-j-1} - \delta_L B_{t-2-j}) \right] = 0$$

which can be further rearranged into:

$$u_{c,t} - v_{x,t} + \lambda_t (u_{cc,t} c_t + u_{c,t} + v_{xx,t} (c_t + g_t) - v_{x,t}) \\ - u_{cc,t} \left[\sum_{j=0}^{\infty} \delta_L^j (\lambda_{t-j} - \lambda_{t-j-1}) B_{t-j-1} \right] = 0 \quad (70)$$

According to (70) we have: $\mathcal{D}_t = \sum_{j=0}^{\infty} \delta_L^j (\lambda_{t-j} - \lambda_{t-j-1}) B_{t-j-1}$. This implies that interest rate twisting now concerns the entire history $j = 0, 1, 2, \dots$ of issuances weighted by δ_L^j , since bonds payout coupons forever.

This result can be further simplified if we follow the same steps taken in Section 2. We can write

the implementability constraint as:

$$E_t \sum_{j=0}^{\infty} \beta^j S_{t+j} = B_{t-1} E_t \sum_{j=0}^{\infty} (\delta_L \beta)^j u_{c,t+j} = \left(\sum_{i=1}^t \delta_L^{i-1} b_{t-i} \right) E_t \sum_{j=0}^{\infty} (\delta_L \beta)^j u_{c,t+j}.$$

If we assume no uncertainty this becomes:

$$\sum_{j=0}^{\infty} \beta^j S_{t+j} = \left(\sum_{i=1}^t \delta_L^{i-1} b_{t-i} \right) \sum_{j=0}^{\infty} (\delta_L \beta)^j u_{c,t+j}.$$

It becomes clear that the government has an incentive to affect the interest rates and consequently the taxes on an infinite horizon with decaying weights.²⁴ On the other hand under independent powers these terms have no direct influence on fiscal policy.

B.2 Independent Powers - Decaying Coupons and No Buyback

Consider now the second model of Section 3, the decaying coupon model. Under no-commitment we have:

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \{ u(c_t) + v(x_t) + \lambda_t [S_t + u_{c,t} p_t (B_t - \delta_L B_{t-1}) - u_{c,t} B_{t-1}] \}$$

therefore, the planner is assumed to not control bond prices p_t and we have dropped for brevity the debt limits.

The first order conditions are as follows:

$$\begin{aligned} u_{c,t} - v_{x,t} + \lambda_t (u_{cc,t} c_t + u_{c,t} + v_{xx,t} (c_t + g_t) - v_{x,t}) \\ + u_{cc,t} [\lambda_t p_t (B_t - \delta_L B_{t-1}) - \lambda_t B_{t-1}] = 0 \end{aligned}$$

and

$$\lambda_t u_{c,t} p_t = \beta E_t (\lambda_{t+1} u_{c,t+1} (1 + \delta_L p_{t+1}))$$

The appropriate state variables for this model are obviously B_{t-1}, g_t .

B.3 Simulation Results

We now consider the properties of the numerical solution to the decaying coupon model. In Figures B.4 , B.4 and B.4 we plot the responses of taxes and other key model variables in the case of decaying coupons under both commitment and independent powers.

[Figures B.4 , B.4 and B.4 About Here]

²⁴ This is exactly the same as the case of a model with no buyback. The budget constraint there becomes:

$$\sum_{j=0}^{\infty} \beta^j S_{t+j} = \sum_{i=0}^N p_{N-i,t} b_{N,t-i}$$

Following a spending shock tax rates are now much smoother than the ones generated by the constant coupons and no buyback model analysed in the text, even though the assumption of no buyback is maintained. This smoothness (relative to the responses shown in figures 7 and 7) obviously derives from the timing of payments which now is considerably different. Given this is an incomplete markets setting the timing of cash flows is crucial and the decaying coupon bonds smooth the cash flow. When the government issues b_0 in $t = 0$ they have to pay b_0 in $t = 1$ using taxes and new debt, $b_0\delta_L + b_1$ in $t = 2$ and $b_0\delta_L^t + b_1\delta_L^{t-1} + \dots + b_{t-1}$ in any generic t . This explains the mild hump shape response we see in the figures.

Note that there are significant differences between the allocations generated by the full commitment and the independent powers model, even when initial debt is zero (Figure B.4). In the independent powers model the government has no incentive by construction to engage in interest rate twisting. The differences are due to the different set of state variables of each model. As we have seen earlier, in the commitment model the state vector includes the entire history of the λ s, in the independent powers model taxes are a function of the inherited debt stock and the level of spending²⁵. These results persist also when initial debt is positive (Figure B.4). The effects of interest rate twisting are vaguely noticeable, the dominant force behind the tax response is the timing of payments. Figure B.4 shows that these results survive when other maturities are considered.

[Tables ?? and ?? About Here.]

Now considering the short and long horizon simulations, Table 7 shows that taxes are more volatile under independent powers and the standard deviation increases with the horizon of the simulations. In short samples when the two models generate a similar (negative) average market value of debt, tax volatility of the independent power model is 50% higher than the one of the commitment model. At longer horizons and under independent powers the government accumulates a larger stock of precautionary savings. This brings the standard deviations of the tax rates closer to the commitment model. Nonetheless only part of the gap is closed: taxes are 20% more volatile in the independent power model.

B.4 Model Comparisons

We now compare the behaviour of the three models. We simulate all our models (buyback, decaying coupon, no buy back with coupons) under full commitment and independent power models, with the same 200 period shock sample and assuming b_0 to be 60% of GDP (the average of government debt in the US economy in the period 1955-2011).

In Figure B.4 we show tax rates under commitment in the top panel and independent powers in the bottom one. The solid line represents the buyback model of Section 2, the dashed line shows taxes under constant coupons and no buyback and the dashed dotted line shows the case of the decaying coupons model. In the case of commitment the tax schedules exhibit very similar behavior mirroring our previous results that under full commitment tax smoothing is a key policy objective. The variables which mainly influence taxes are the level of government spending and the level of debt. However, in the independent power model tax rates exhibit dramatically different behaviors

²⁵ Notice that this reasoning helps explain the differences in taxes in the first period across the two models, since these initial conditions influence one allocation but not the other.

due to the buyback assumption. The volatility of tax rates increases and the difference from the commitment models is much greater under no buyback.

In Figure B.4 we show the behavior of the market value of debt expressed as a percentage of GDP for the same shock sample that generated the tax rates. The evolution of the debt aggregate confirms the previous results. In the case of full commitment the three debt levels in the top panel track closely one another. But under no commitment the deviations are considerable. Under long bonds, changes in the institutional set up can generate substantially different debt behaviour.

[Figures B.4 and B.4 About Here.]

Notes: The Figure plots the impulse response of taxes (top- left panel), consumption (top-right panel), deficit (bottom- left) and the market value of debt (bottom-right) to a one standard deviation shock in government spending. The maturity of long debt is $N = 10$ years. The quantities represented by the solid (blue) lines correspond to the optimal commitment allocation and the quantities plotted with dashed (red) lines correspond to the independent power model. The starting value of debt is zero.

Notes: The Figure plots the impulse response of taxes (top- left panel), consumption (top-right panel), deficit (bottom- left) and the market value of debt (bottom-right) to a one standard deviation shock in government spending. The maturity of long debt is $N = 10$ years. The quantities represented by the solid (blue) lines correspond to the optimal commitment allocation and the quantities plotted with dashed (red) lines correspond to the independent power model. The initial debt level is 50
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Notes: The Figure plots the impulse response of taxes to a spending shock. The top panels show the commitment model under decaying coupons with zero (left) and positive (right) debt levels. The solid line shows the response of the tax schedule when the maturity is $N = 5$. The dashed line corresponds to $N = 10$ and the crossed and dashed-dotted lines to $N = 15$ and $N = 20$ respectively. The bottom panels in the figure show the analogous responses in the case of independent powers.

Notes: The Figure plots a simulated path for tax rates. The top panel show the case of the commitment model and the bottom panels the case of no commitment. We assume a starting value of government debt equal to 60 per cent of steady state GDP.

Notes: The Figure plots a simulated path for the market value of debt. The top panel show the case of the commitment model and the bottom panels the case of no commitment. We assume a starting value of government debt equal to 60 per cent of steady state GDP.

t=40				t=200				long run				
<i>N</i>	<i>c</i>	τ	<i>deficit</i>	<i>MV</i>	<i>c</i>	τ	<i>deficit</i>	<i>MV</i>	<i>c</i>	τ	<i>deficit</i>	<i>MV</i>
mean												
5	52.49	0.253	-0.214	-7.76	52.56	0.247	0.148	-14.26	52.57	0.246	0.274	-17.30
10	52.49	0.253	-0.201	-7.03	52.56	0.248	0.119	-13.01	52.56	0.246	0.258	-16.50
15	52.49	0.253	-0.194	-6.71	52.56	0.248	0.119	-11.77	52.56	0.246	0.253	-16.21
20	52.50	0.253	-0.188	-6.54	52.56	0.248	0.115	-12.73	52.56	0.246	0.246	-15.98
std												
5	2.27	0.0220	1.18	10.47	3.13	0.0356	1.54	21.12	3.48	0.0435	1.59	29.70
10	2.28	0.0220	1.16	10.24	3.13	0.0356	1.52	21.33	3.48	0.0436	1.58	30.16
15	2.28	0.0220	1.14	10.05	3.14	0.0357	1.51	21.36	3.48	0.0437	1.58	30.41
20	2.28	0.0220	1.14	9.94	3.14	0.0357	1.51	21.33	3.48	0.0438	1.58	30.63

Notes: The table shows key moments from the optimal policy model with decaying coupons. The first four rows plot the averages for consumption, taxes, deficit and the market value of debt in the case of four different maturities ($N = \{5, 010, 15, 20\}$). The last four rows show the standard deviations of these quantities. The table reports the moments over three different horizons: 1) 40 observations, 18000 samples. 2) 2000 observations, 18000 samples, and 3) 4500 observations 18000 samples (5000 observations were generated for each sample, 500 observations were dropped). The initial conditions for government debt are uniformly distributed over $[-M_N, M_N]$.

t=40				t=200				long run				
<i>N</i>	<i>c</i>	τ	<i>deficit</i>	<i>MV</i>	<i>c</i>	τ	<i>deficit</i>	<i>MV</i>	<i>c</i>	τ	<i>deficit</i>	<i>MV</i>
mean												
5	52.53	0.249	0.049	-12.12	52.78	0.233	1.13	-26.36	52.88	0.225	1.696	-34.97
10	52.53	0.249	0.039	-11.15	52.76	0.234	1.05	-24.53	52.86	0.226	1.591	-32.92
15	52.53	0.249	0.035	-10.61	52.75	0.235	1.00	-23.51	52.84	0.227	1.530	-31.69
20	52.53	0.249	0.032	-10.27	52.74	0.235	0.98	-22.88	52.83	0.228	1.494	-30.98
std												
5	2.32	0.0266	1.19	10.54	3.23	0.0445	1.69	20.43	3.59	0.0533	1.73	25.91
10	2.32	0.0265	1.16	10.30	3.23	0.0445	1.70	20.80	3.60	0.0542	1.80	27.44
15	2.32	0.0265	1.15	10.18	3.23	0.0445	1.70	20.97	3.61	0.0546	1.84	28.22
20	2.32	0.0265	1.14	10.10	3.23	0.0445	1.70	21.07	3.61	0.0549	1.86	28.68

Notes: The table shows key moments from the independent power model with decaying coupons. The first four rows plot the averages for consumption, taxes, deficit and the market value of debt in the case of four different maturities ($N = \{5, 010, 15, 20\}$). The last four rows show the standard deviations of these quantities. The table reports the moments over three different horizons: 1) 40 observations, 18000 samples. 2) 2000 observations, 18000 samples, and 3) 4500 observations 18000 samples (5000 observations were generated for each sample, 500 observations were dropped). The initial conditions for government debt are uniformly distributed over $[-M_N, M_N]$.