A Theory of Repurchase Agreements, Collateral Re-use, and Repo Intermediation*

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Abstract

This paper characterizes repurchase agreements as equilibrium contracts starting from first principles. We show that repos trade-off the borrower's desire to augment its consumption today with the lender's desire to hedge against future market risk. As a result, safer assets will command a lower haircut and a higher liquidity premium relative to riskier assets. Haircuts may also be negative. When lenders can re-use the asset they receive in a repo, we show that collateral constraints are relaxed and borrowing increases. Re-usable assets should command low haircuts. Finally, endogenous credit intermediation arises whereby trusty counterparties re-use collateral and borrow on behalf of riskier counterparties. These findings are helpful to rationalize trading patterns observed between the bilateral and the tri-party repo markets.

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1 Introduction

According to Gorton and Metrick (2012), the financial panic of 2007-08 started with a run on the market for repurchase agreements (repos). Their paper was very influential in shaping our understanding of the crisis. It was quickly followed by many attempts to understand repo markets more deeply, both empirically and theoretically as well as calls to regulate these markets.[1]

A repo is the sale of an asset combined with a forward contract that requires the original seller to repurchase the asset at a given price. Repos are different from simple collateralized loans in (at least) one important way. A repo lender obtains the legal title to the pledged collateral and can thus use the collateral during the length of the forward contract. This practice is known as re-use or re-hypothecation.[2] With standard collateralized loans, borrowers must agree to grant the lender similar rights. This special feature of repos has attracted a lot of attention from economists and regulators alike.[3]

Repos are extensively used by market makers and dealer banks as well as other financial institutions as a source of fundings, to acquire securities that are on specials, or simply to obtain a safe return on idle cash. As such, they are closely linked to market liquidity and so they are important to understand from the viewpoint of Finance. The Federal Reserve Bank, the central bank in the United States, and other central banks use repos to steer the short term nominal

[1] See Acharya (2010) "A Case for Reforming the Repo Market" and (FRBNY 2010)

[2] There is a subtle difference between US and EU law. Under EU law, a repo is a transfer of the security's title to the lender. However, as Comotto (2014) explains, a repo in the US falls under New York law, and "Under the law of New York, which is the predominant jurisdiction in the US, the transfer of title to collateral is not legally robust. In the event of a repo seller becoming insolvent, there is a material risk that the rights of the buyer to liquidate collateral could be successfully challenged in court. Consequently, the transfer of collateral in the US takes the form of the seller giving the buyer (1) a pledge, in which the collateral is transferred into the control of the buyer or his agent, and (2) the right to re-use the collateral at any time during the term of the repo, in other words, a right of re-hypothecation. The right of re-use of the pledged collateral (...) gives US repo the same legal effect as a transfer of title of collateral."

[3] Aghion and Bolton (1992) argue that securities are characterize by cash-flow rights but also control rights. Collateralized loans grant neither cash-flow rights nor control rights over the collateral to the lender. To the contrary, a repo that allows re-use, gives the lender full control rights over the security as well as over its cash-flows (the interest rate usually controls for cash-flows that accrue during the length of the contract). There is a legal difference between re-use and rehypothecation, but economically they are equivalent, see e.g. Singh (2011).

interest rate. The Fed's newly introduced reverse repos are considered an effective tool to increase the money market rate when there is large excess reserves. As such, repos are essential to the conduct of monetary policy. Finally, firms also rent capital and use collateralized borrowing and some forms of repos to finance their activities or hedge exposures (notably interest rate risk, see BIS, 1999). This affects real activities, and so repos are also an important funding instrument for the macroeconomy.

Most existing research papers study specific aspects of the repo markets, e.g. exemption from automatic stay, fire sales, etc., taking the repo contract and most of its idiosyncrasies as given. These theories leave many fundamental questions unanswered, such as why are repos different from collateralized loans? What is the nature of the economic problem solved by the repo contract? If one is to answer these questions, to understand the repo market and the effect of regulations, one cannot presume the existence or the design of repo contracts. In this paper we thus study repos from first principles.

More precisely, starting from simple preferences and technology, we study why repos is the funding instrument of choice, the impact of repos on asset prices and their liquidity premium, as well as collateral re-use. We do not take the structure of financial instruments as given, but we borrow techniques from security design to show how repos arise endogenously in equilibrium. In our model the equilibrium contract shares many features with repos: The borrower sells an asset spot combined with a forward contract promising a re-purchase at an agreed price.

The model has three periods and has two types of agents, a natural borrower and a natural lender. The borrower is risk neutral is more or less trustworthy, while the lender is risk averse. The borrower is endowed with an asset that yields an uncertain payoff in the last period. The payoff realization becomes known in the second period. Therefore, the second period price of the asset will fully reflect its payoff. In the first period the borrower wants to consume more than his resources. He could sell the asset to the lender, but the lender will not pay much for it, as it exposes him to price risk. Instead, the borrower can obtain resources from the lender by selling the asset combined with a forward contract promising to repurchase the asset in period 2. The lender prefers this contract to an outright sale, whenever the forward contract offers him some insurance. The equilibrium contract will trade-off the borrowing motive of the borrower, which tends to increase the repayment profile, with the insurance motive of the borrower, which tends to flatten the repayment profile. We show that the equilibrium forward contract is a flat schedule as soon as the payoff of the asset is high enough, thus perfectly insuring the lender against the price risk. However, a constant repurchase price may trigger default in the lowest states and the optimal design of the forward contract controls for this.[4]

When the asset is abundant and the borrower is trustworthy, the equilibrium contract is a standard repo with a fixed repayment. The constant repurchase price perfectly hedges the lender against price risk. When the asset is scarce and/or the borrower is not so trustworthy, the optimal repo contract trades off the willingness to increase the size of the loan with the hedging motive.

Using this equilibrium contract we derive equilibrium properties of haircuts, and liquidity premium. We show that haircuts are increasing with asset risk. Also, we show that safer assets command higher liquidity premium than riskier assets. This is intuitive: safer assets are better collateral as the forward contract will better insure the lender. This property increases the equilibrium value of safe assets relative to risky ones. This theory of haircuts allows us to analyze the effect of allowing collateral re-use. In particular, we show that re-use relaxes borrowing constraints whenever the asset is particularly scarce. To understand this, suppose the collateral is so special that it commands a negative haircut. This means that the borrower can borrow \$110 say, for an asset which can be sold for \$100. So let us suppose the borrower conducts this trade and so gets \$110 from the lender. With this \$110, the borrower can then purchase the asset for \$100 leaving him with \$10 and the asset. But then he can re-pledge the asset with the lender (or another one), to get another \$110, etc. The borrower can then generate more resources this way and get closer to his goals. Here this is possible as we assumed that the haircut is negative. In the model, haircuts are endogenous and positively depend on the borrower's trustworthiness, among other things. When haircuts are low but positive, the lender pockets these gains from re-use. Interestingly, when

[4]In practice, even in the absence of outright default, traders opportunistically delay the settlment of transactions, as documented by Fleming and Garbade (2005). In our model, these failures or delay to settle ultimately introduce state contingency in the contract payoff.

the borrower cannot pledge more income than the asset used in a repo, haircuts are large and allowing re-use does not affect the equilibrium allocation. Therefore, one implication of our model is that collateral re-use should be more prevalent for assets that command low haircuts.

Finally, we analyze repo intermediation, an issue that we think is important to understand the connection between bilateral repo and tri-party repos. A simplified tri-party repo transaction involves a dealer bank, the tri-party agent, and a money market mutual fund (MMF). Typically, the tri-party agent manages the collateral pledged by the dealer bank. MMFs agree to repo only with specific dealer banks that they deem safe enough. So not every financial market participants, typically hedge funds, can participate in this arrangement. This is where repo intermediation comes into the picture: If a hedge fund needs cash, he will first (bilaterally) repo with a dealer bank with access to the tri-party agent. This dealer bank will then repo the cash from an MMF via the tri-party agent. Our paper shows that this trade is not necessarily driven by regulatory requirements, but can also be driven by differences in counterparty quality. In particular, we show how a repo intermediary, and by extension a chain of intermediation, can arise endogenously.

Relation to the literature

Gorton and Metrick (2012) argue that the recent crisis started with a run on repo whereby funding dropped dramatically for many financial institutions. Subsequent studies by Krishnamurty et al. (2014) and Copeland et al. (2014) have qualified this finding by showing that the run was specific to the - large - bilateral segment of the repo market. Recent theoretical works have indeed highlightedt some features of repo contracts as sources of funding fragility. As a short-term debt instrument to finance long-term assets, Zhang (2014) and Martin et al. (2014) show that repos are subject to roll-over risk. Antinolfi et al. (2015) emphasize the trade-off from the exemption from automatic stay for repo collateral. Lenders easy access and to collateral from defaulted loans enhancse market liquidity but creates the potential for fire sales, a point also made by Infante (2013). These papers take repurchase agreements as given while we want to understand why they emerge as a funding instrument. In this respect, we share the objective of Narajabad and Monnet (2012), Tomura (2013) and Parlatore (2015) to explain why repos prevail over outright sale of the collateral. In these papers, agents choose collateralized loans or repos because lenders might not be able to extract the full value of the asset due to bargaining frictions on spot markets[5]. In our competitive model, repos are essential because the repurchase price provides insurance to the initial buyer. Our theory thus rationalizes haircuts since the hedging components explains why borrowers choose repos when they could obtain more cash in the spot market[6]. To derive the repo contract, we follow Geanakoplos (1996) , Araújo et al. (2000) and Geanakoplos and Zame (2014) where collateralized promises traded by agents are selected in equilibrium. Our model differs from theirs as we allow for an extra non-pecuniary penalty for default in the spirit of Dubey et al. (2005).

In the second part of the paper, we explore a crucial feature of repurchase agrements whereby the borrower tranfers the legal title to the collateral to the lender. Singh and Aitken (2010) and Singh (2011) argue that collateral re-use or rehypothecation lubricates transactions in the financial system[7] but rehypothecation entails risks for collateral pledgors (see Monnet, 2011). While Bottazzi et al. (2012) or Andolfatto et al. (2014) abstract from the limited commitment problem of the collateral receiver, Maurin (2015) shows in a general environment that re-use risk seriously mitigates the benefits from circulation. In our model indeed, re-use relaxes collateral constraints only thanks to the extra penalty for default for borrowers besides the collateral loss. Asset re-use then plays a role similar to pyramiding (see Gottardi and Kubler, 2015) whereby lenders pledge the face value of a debt owed rather than re-pledging the asset backing the debt. Muley (2015) rationalizes the choice of rehypothecation over pyramiding by the difference in income pledgeability across types of agents. We highlight the connecton between collateral re-use and repo market intermediation as in Infante (2015) and Muley (2015). Unlike these papers, intermediation arises endogenously as trustworthy agents re-use the collateral from risky counterparties to borrow on their behalf. Endogenous credit intermediation is efficient since the intermediation rent-seeking behavior in Farboodi (2015) is absent from our competitive model.

The structure of the paper is as follows. We present the model and the complete

[5]In Parlatore (2015), cash in the market pricing delivers the same result.

[6]In particular, we do not need transactions costs as suggested by Duffie (1996).

[7]Fuhrer et al. (2015) estimate an average 5% re-use rate in the Swiss repo market over 2006-2013.

market benchmark in Section 2. We analyze the optimal repo contracts, including properties for haircuts, liquidity premiums, and repo rates in Section 3. In Section 4, we study the equilibrium when agents are allowed to re-use the collateral and we study intermediation. Finally, Section 5 concludes.

2 The Model

2.1 Setting

The economy lasts three dates, $t = 1, 2, 3$. There are two agents $i = 1, 2$ and only one good each period. Agent 1 is endowed with ω units of the good in each of the three periods, while agent 2 is endowed with ω in all but the last period. Agent 1 is also endowed with a units of an asset while agent 2 has none. This asset pays dividend s in date 3. The dividend is distributed according to a cumulative distribution function $F(s)$ with support $\mathcal{S} = [\underline{s}, \bar{s}]$ and with mean $E[s] = 1$. In date 2, all agents receive a perfect signal on the realization of s in date 3. This is an easy way to model price risk at date 2.

Let c_t^i denote agent i consumption in period t. Preferences from consumption profile (c_1, c_2, c_3) for agent 1 and 2 respectively are

$$\begin{aligned} v^1(c_1, c_2, c_3) &= c_1 + \delta(c_2 + c_3) \\ v^2(c_1, c_2, c_3) &= c_1 + u(c_2) \end{aligned}$$

where $\delta < 1$ and $u(.)$ is a strictly concave function that satisfies Inada conditions. We assume $u'(\omega) > \delta$ and $u'(2\omega) < \delta$, so that there are gains from trade in date 2 and the optimal allocation is interior. These preferences contain two important elements: First, as $\delta < 1$, agent 1 prefers to consume in date 1 and would like to borrow from agent 2. Second, agent 2 does not care for consumption in date 3 and dislikes consumption variability in date 2.

Agents might not be able to fully commit to future promised payments. When an agent defaults, he incurs a non-pecuniary penalty proportional to the value of the default, that is he suffers a utility cost θr when he defaults on a repayment r, where $\theta \in [0, 1]$. There are no other stigma attached to default, and while he incurs a penalty, a defaulting agent still has market access. As will be clear below, $\theta = 1$ corresponds to the case with full-commitment for agent 1, and $\theta = 0$ is the case with no commitment, when loans are non-recourse (See Geanakoplos, 1996).

The environment is a simple set-up where a repo contract arises naturally: Limited commitment implies that the value of the borrowers' debt must be in line with the (uncertain) market value of their collateral. But the lender dislikes this market risk and will try to hedge it using a forward contract. A repo is different from a standard collateralized loans, as the latter does not (necessarily) grant the lender the right to re-use the collateral. When agents can re-use the asset, we will show that they strictly prefer to use a repo to a collateralized loan.

2.2 Full commitment

In this section we study the benchmark allocation when agents can fully commit to future promises. An Arrow-Debreu equilibrium of this economy is a system of consumption prices $(q_2, \{q_3(s)\}_{s\in\mathcal{S}})$ and allocations $(c_1^i, c_2^i, c_3^1(s)\}_{s\in\mathcal{S}})^{i=1,2}$ such that agents solve their maximization problem,

$$\max_{c_1^i, c_2^i, \{c_3^i(s)\}_{s\in\mathcal{S}} \in \mathbb{R}^+} \int_s v^i(c_1^i, c_2^i, c_3^i(s))dF(s)$$

subject to

$$c_1^i + q_2 c_2^i + \int_s q_3(s)c_3^i(s)dF(s) \leq (1+q_2)\omega + \int_s q_3(s)(\mathbb{I}_{\{i=1\}}\omega + a_0^i s)dF(s),$$

and markets clear

$$\begin{aligned} c_t^1 + c_t^2 &= 2\omega \qquad \text{for } t = 1, 2, \\ c_3^1(s) + c_3^2(s) &= \omega + as. \end{aligned}$$

Clearly agent 2 will not consume at date 3. Considering the case with strictly positive consumptions, the necessary and sufficient first order conditions for both agents give $q_2 = q_3(s) = \delta$, for all s as well as $c_2^2 = c_{2,*}^2$ and $c_2^1 = c_{2,*}^1$ where

$(c^1_{2,*}, c^2_{2,*})$ is the unique solution to

$$\begin{cases} u'(c^2_{2,*}) = \delta \\ c^1_{2,*} + c^2_{2,*} = 2\omega \end{cases} \tag{1}$$

Since $u'(\omega) > \delta$ and $u'(2\omega) < \delta$ both agents consume positive amounts in date 2. The date 1 consumption of agent 1 is given by his budget constraint, $c^1_1 = \omega + \delta(c^2_{2,*} - \omega)$, and $c^1_1 \geq 0$ follows from $c^2_{2,*} > \omega$. Turning to c^2_1, and using the BC of agent 2 we obtain $c^2_1 = \omega - \delta(c^2_{2,*} - \omega)$, so that $c^2_1 > 0$ requires

$$\frac{(1+\delta)}{\delta}\omega \geq c^2_{2,*} \tag{2}$$

We maintain this assumption in the rest of the paper, which implicitly defines a restriction on δ. In the following, we introduce limited commitment and we show that a repo dominates simple spot trades to implement this allocation.

2.3 Limited commitment, spot trades, and repos

In this section, we introduce limited commitment so that agents cannot write unsecured contracts. Agents cannot achieve the full commitment benchmark allocation by using only spot trades, as it exposes agent 2 to market risk. Then we allow agents to trade repo contracts and we show conditions under which agents prefer to trade repo only.

2.3.1 Spot Transactions and Efficiency

Here we show that agents cannot reach the full commitment benchmark allocation (or competitive market equilibrium allocation) when they are constrained to trade spot. In any equilibrium with only spot market trades, agent 2 who purchased a^2_1 units of the asset in period 1 will consume $c^2_2(s) = \omega + p_2(s)a^2_1$, as he is endowed with ω and holds a^2_1 units of the asset with unit-price $p_2(s)$. So the dividend risk at date 3 generates price risk at date 2 when agent 2 sells back his assets, and he cannot achieve the efficient allocation which is independent of s. We further characterize the equilibrium where only spot markets are available in the Appendix. If we let

p_1 be the equilibrium price of the asset in period 1 then we can define the asset liquidity premium in the standard way,

$$\mathcal{L} = p_1 - \delta\mathbb{E}[p_2(s)] \tag{3}$$

where $p_2(s) = s$ is the equilibrium spot price of the asset at date 2, and so $\delta E[s]$ is the fundamental value of the asset at date 1. Proposition 1 states the equilibrium properties of the liquidity premium and the importance of asset availability for the agents' ability to borrow and save. Recall that agent 1 is endowed with a units of the asset.

Proposition 1. *When agents can only trade spot, the equilibrium is as follows:*

1. Low asset quantity: Suppose (a, ω) is such that $\mathbb{E}\{s[u'(\omega + sa) - \delta]\} > 0$ then agent 1 sells his entire asset holdings at date 1. The liquidity premium is $\mathcal{L} = \mathbb{E}\{s[u'(\omega + sa) - \delta]\} > 0$.

2. High asset quantity: Suppose (a, ω) is such that $\mathbb{E}\{s[u'(\omega + sa) - \delta]\} \leq 0$ then agent 1 sells less than a at date 1. The liquidity premium is $\mathcal{L} = 0$.

As we observed, agent 1 wants to consume at date 1. When agent 1's asset endowment is low, he sells his entire asset holding and then buys it all back at date 2. Agent 1 would like to sell more, but he cannot short the asset. In this case the asset commands a liquidity premium which is related to the severity of agent 1's short-selling constraint. When a is high enough, agent 1 does not sell everything in date 1 and the asset price is determined by agent 1 according to its fundamental value. So in this case there is no liquidity premium.[8]

As we will show below the sale in a repo will allow agent 1 to borrow at date 1 while the commitment to a repurchase price insures agent 2 against price risk.

2.3.2 Trading in Spot and Repo Markets

In this section, we specify the agents' problem when they can trade spot and repo contracts. Since trading repo requires writing a forward contract, we first derive

[8] If agent 2 was alive in period 3, he may want to save whenever $\omega + sa > c^2_{2*}$. However, we should emphasize that allowing for this possibility would not modify the result, even if agent 2 had risk neutral preferences for period 3 consumption.

the conditions under which agents will make good on their promises. Then we derive the optimal solution to agents' problem when they take the repo contract as given. In the following sections, we derive the equilibrium repo contract, defined as the contract such that in equilibrium no two agents benefit from trading another contract.

We first define a repo contract between two agents i and j where i is the borrower and j is the lender. A repo contract at date 1 between i and j is the pledge (or sale) of one unit of asset by i to j at price $p_{F,ij}$ combined with a forward contract $F_{ij} = (\{\bar{p}_{ij}(s)\}_{s\in\mathcal{S}}, \nu_{ij})$ where $\bar{p}_{ij}(s)$ is the price i agrees to pay j at date 2 to repurchase the asset in state s and $\nu_{ij} \in [0, 1]$ the share of collateral that agent j can re-use.[9] To be clear, if agent j obtained a from *entering a repo* with agent i, agent j can re-use at most an amount $\nu_{ij}a$ from that collateral in later trades. However, if agent j *purchased* a from agent i, agent j can naturally use all of a later. We assume that the regulator imposes $\nu_{ij} \leq \nu$ for some $\nu \in [0, 1]$ and we will do comparative statics on ν. If $\nu_{ij} = \nu = 1$ the asset can be used in the same way, whether it was acquired through a loan or an outright purchase. In the sequel, it will be clear that agents will want to re-use collateral as much as possible and so we set $\nu_{ij} = \nu$ for all pairs (i, j). Our repo contract will be different from a simple collateralized loan when $\nu > 0$ and the two are indistinguishable otherwise. We endogenously characterize the equilibrium schedule $\bar{p}_{ij} = \{\bar{p}_{ij}(s)\}_{s\in\mathcal{S}}$. Typically, $\bar{p}_{ij}$ will differ from $\{p_2(s)\}_{s\in\mathcal{S}}$ and, as a result, $p_{F,ij}$ will differ from p_1: a repo will typically modify agent 1's borrowing capacity.[10]

For any state s, the price of the asset at date 2 is $p_2(s) = s$. Indeed, at date 2, agent 2 is ready to sell all of his asset holdings at any price $p_2(s) \geq 0$, as he derives no utility of consumption in period 3. Agent 1 is ready to purchase any amount of the asset as long as $p_2(s) \leq s$. But $p_2(s) < s$ would yield excess demand for the asset and excess supply otherwise, hence the only possible equilibrium is $p_2(s) = s$.

[9]Observe that in particular a repo can replicate two spot transactions if $\bar{p}(s)$ is set to $p_2(s)$. This is actually the only feasible trades with two-sided limited commitment and without punishment for default besides the loss of collateral.

[10]In this model, a spot transaction achieve the same allocation as the repo contract F with $\bar{p}(s) = s$ for all s and $\nu = 1$. The difference is that a spot transaction does not require the buyer of the asset to resell it later.

For each forward contract F_{ij}, we denote by $b^{ij} \geq 0$ the quantity of assets that agent i pledges as collateral if he borrows from j, and $\ell^{ij} \geq 0$ the collateral agent i receives if he lends to j. We will write the agents problem when they only trade one contract F^{ij} and F^{ji} and we will verify that in equilibrium agents do trade only one contract. Of course, market clearing will imply $b^{ij} = \ell^{ji}$. Having differentiated borrowing from lending positions, we can write the agents' borrowing constraint as

$$a_1^i + \nu_{ji}\ell^{ij} \geq b^{ij}. \tag{4}$$

The collateral constraint (4) takes a simple form. Collateral can come from two sources (LHS of (4)): asset purchase or re-usable collateral. The sum of the two must be sufficient to cover the needs of agent i when he borrows (RHS). Here, it is assumed that re-usable collateral can also be sold in the spot market, and (4) implies that a_1^i can now be negative.[11]

We assume that no agent can fully commit to future promises. If an agent defaults, he loses his entitlement (the collateral or the cash) and incurs a penalty.[12] We assume this additional penalty is non-pecuniary, meaning that there is no hidden collateral that agents can use besides their utility. This penalty is proportional to the value of the default: when borrower i defaults on a repayment $\bar{p}(s)\tilde{a}$ he incurs a non-pecuniary loss $\theta_i\bar{p}(s)\tilde{a}$, where $\theta_i \in [0, 1]$. Similarly, when lender i does not return collateral $\nu\tilde{a}$ in state s, he incurs a penalty $\theta_i\nu\tilde{a}s$, equal to a fraction θ_i of the value of the collateral he should return.

For tractability we set $\theta_2 = 0$ for agent 2, while θ_1 can take any value in $[0, 1]$. We can naturally extend the analysis to the case where $\theta_2 > 0$, but we do not gain much additional insights for a substantial increase in complexity. Furthermore, setting $\theta_2 = 0$ is harmless for the analysis, as agent 2 is the natural lender and

[11] When only re-pledging is possible, the collateral constraints breaks into two parts

$$a_1^i \geq 0,$$
$$a_1^i + \nu_{ji}\ell^{ij} \geq b^{ij}.$$

[12] The penalty is also lost to the lender. Without this assumption, the penalty would be enough to transfer consumption across time and collateral would play no role. We might also want to see the penalty as a non-pecuniary cost to avoid hitting the constraint of positivity of consumption. In that case the actual penalty would be $\min\{\theta\bar{p}(s)\tilde{a}, \omega\}$.

setting $\theta_2 > 0$ would only facilitate re-use. Finally, although he incurs a penalty, a defaulter can access the spot market in period 2 to trade the asset.

Therefore, given $b^{ij} > 0$ and $\theta_2 = 0$, borrower i does not default on his promise whenever $p_2(s)b^{ij} + \theta_i \bar{p}_{ij}(s)b^{ij} \geq \bar{p}_{ij}(s)b^{ij}$, where the LHS is the cost of defaulting comprising the loss of the asset market value plus the non-pecuniary cost where again $\theta_2 = 0$, while the RHS is the cost of repaying the loan. Therefore, the borrower makes good on his promise to pay $\bar{p}_{ij}(s)$ in state s whenever,

$$\bar{p}_{ij}(s) \leq \frac{s}{1-\theta_i} \tag{5}$$

Conveniently, $\theta_1 = 1$ corresponds to the case with full commitment for agent 1 and any repayment schedule $\bar{p}(s)$ is incentive compatible. Setting $\theta_1 = 0$ corresponds to the case with no commitment since the borrower does not incur any loss besides that of the collateral.[13] Hence, as long as $\theta_1 > 0$ selling the asset in a repo potentially allows agent 1 to increase his borrowing capacity with respect to a standard spot sale. Since the penalty is lost to the lender, there is no loss in generality in focusing on forward contracts without default. As $\theta_2 = 0$, we sometimes use $\theta \equiv \theta_1$ when there is no risk of confusion.

The lender should also make good on his promise to return the asset. When the lender defaults, the segregated (or encumbered) fraction $1-\nu_{ij}$ of the collateral goes back to the borrower. In addition, to maintain the analogy with borrowers, we assume that a lender with creditworthiness θ_j not returning collateral $\nu_{ij}a$ in state s incurs a penalty $\theta_j \nu_{ij} as$, equal to a fraction θ_j of the value of the collateral he should return. So, with re-use, the lender makes an implicit promise to return $\nu_{ij}a$ units of the asset, as the return of the fraction $1-\nu_{ij}$ is automatic. The no-default constraint of the lender thus writes:[14]

$$\bar{p}_{ij}(s) \geq \nu_{ij}(1-\theta_j)s \tag{6}$$

[13] We refer to cases with $\theta_1 \in (0,1)$ as cases with limited commitment.

[14] Assuming $\theta_2 = 0$ makes it more difficult for agent 2 to return the collateral, and so goes against re-use in equilibrium. When $\theta_2 > 0$ the no-default constraint of agent 2 becomes

$$u(\bar{p}_{12}(s)a + X) \geq u(\nu_{12}as + X) + \theta_2 a \nu_{12} s$$

where X is agent 2's net wealth. It turns out that this constraint does not bind when evaluated at the solution with $\theta_2 = 0$.

This constraint implies that a borrower who is defaulted upon (off equilibrium) will have the incentive to purchase the collateral in the spot market to satisfy his obligations as a lender. Indeed, the payoff from this is $\bar{p}_{ij}(s,\nu)-\nu_{ij}s$ and the payoff of reneging is $-\theta_j\nu_{ij}s$, which give us again the no default constraint (6). Finally, as is customary, the default of one party to a specific repo voids the obligation of the other party of, e.g. returning the collateral, for that particular repo.

A feasible repo contract $\tilde{p} = \{\tilde{p}(s)\}_{s\in\mathcal{S}}$ is a continuous function $\tilde{p}(s) : \mathcal{S} \to \mathbb{R}^+$ verifying constraint (5) and (6). In the sequel, we show that we can limit our analysis to just two contracts per pair of agents: one contract $F^{12} = \{\bar{p}_{12}(s)\}_{s\in\mathcal{S}}$ where agent 2 is the lender and agent 1 is the borrower, and another contract $F^{21} = \{\bar{p}_{21}(s)\}_{s\in\mathcal{S}}$ where agent 2 is the borrower and agent 1 is the lender. With this in mind, we can now write the problem of agent i in the following concise way:

$$\max_{a_t^i,\{b^{ij}\},\{\ell^{ij}\}} \omega + p_1(a_0^i - a_1^i) - p_{F,ji}\ell^{ij} + p_{F,ij}b^{ij} \tag{7}$$

$$+\mathbb{E}\left[u_i\left(\omega + s(a_1^i - a_2^i) + \bar{p}_{ji}(s)\ell^{ij} - \bar{p}_{ij}(s)b^{ij}\right)\right]$$

$$+\mathbb{E}\left[\mathbb{I}_1(i)u_i\left(\omega + sa_2^i\right)\right] \tag{8}$$

subject to:

$$a_1^i + \nu\ell^{ij} \geq b^{ij} \qquad (\gamma_1^i) \tag{9}$$

$$b^{ij} \geq 0 \qquad (\xi_b^{ij}) \tag{10}$$

$$\ell^{ij} \geq 0 \qquad (\xi_\ell^{ij}) \tag{11}$$

where $u_1(x) = \delta x$, $u_2(x) = u(x)$ and $\mathbb{I}_1(i)$ equals 1 when $i = 1$ and zero otherwise. At date 1, agents have resources $\omega + p_1 a_0^i$, where $a_0^1 = a$ and $a_0^2 = 0$, and choose their asset holding a_1^i, lending ℓ^{ij} using contract F^{ji} and borrowing b^{ij} using contract F^{ij} . Given these decisions, their resources at date 2 is their endowment ω and the value of their asset holdings $p_2(s)a_1^i$ as well the net value of their forward contracts positions $\bar{p}_{ij}(s)\ell^{ij} - \bar{p}_{ji}(s)b^{ij}$. Agent 2 has no lust for consumption at date 3 and so will not save, i.e. $a_2^2 = 0$. Agent 1 however will save a_2^1 to consume at date 3 and market clearing implies $a_2^1 = a$. Finally, notice that (9) and (10) imply a no-short sale constraint on the asset when $\nu = 0$.

An equilibrium is a list of forward contracts $\{F^{ij}\}$, prices $\{p_1, p_{F,ij}, p_2(s)\}$ and allocations $\{c_t^i(s), a_t^i, \ell^{ij}, b^{ij}\}$ such that $\{c_t^i(s), a_t^i, \ell^{ij}, b^{ij}\}$ solves the agents' problem given prices and contracts, markets clear, and there are equilibrium prices for all other feasible forward contracts so that no two agents want to trade them.

In the following section, we characterize the equilibrium contracts when re-use is not allowed, $\nu = 0$. Then we move on to characterize the equilibrium when $\nu > 0$.

3 Equilibrium contracts with no re-use ($\nu = 0$)

In this section, we characterize the equilibrium when agents cannot re-use collateral. We conjecture that agents do not trade spot and agent 1 only sells a repo $\bar{p} = \{\bar{p}(s)\}_{s \in \mathcal{S}}$ to agent 2. In this section, repo contracts are indistinguishable from collateralized loans as $\nu = 0$. When agents are collateral constrained, they strictly prefer to trade the repo contract than trading spot. Otherwise they are indifferent and could use both contracts in equilibrium.

The construction of the repo contract relies on a trade-off between a borrowing motive and a risk-sharing motive. A forward contract smoothes consumption variability at date 2 when the repurchase price is constant. But limited commitment imposes the no default-constraint (5) which puts an upper bound on the repurchase price. This hinders the borrowing capacity of the repo seller. Then agent 1's desire to borrow might overcome that of smoothing agent 2's consumption across states at date 2. The optimal contract solves this trade-off. First we define s^* as the state where agent 2 can reach $c^2_{2,*}$ when he repos the entire stock of assets, that is s^* satisfies

$$u'\left(\omega + \frac{s^*}{1-\theta}a\right) = \delta. \tag{12}$$

Clearly, s^* is decreasing with a, θ, or δ. Therefore, it is easier to achieve the first best level of consumption the larger the stock of asset, the more agent 1 is able to commit, or the more agent 1 values the asset in period 2. We have the following result (all proofs are in the Appendix).

Proposition 2. *There is a unique equilibrium allocation supported by the repo contract $F = \{\bar{p}(s)\}_{s \in \mathcal{S}}$ where:*

If $s^* \geq \bar{s}$ (a is low), $\bar{p}(s) = s/(1-\theta)$ for all $s \in \mathcal{S}$

If $s^* \in [\underline{s}, \bar{s}]$ (a is intermediate),

$$\bar{p}(s) = \begin{cases} \frac{s}{1-\theta} & \text{for } s \leq s^* \\ \frac{s^*}{1-\theta} & \text{for } s \geq s^* \end{cases} \tag{13}$$

If $s^* \leq \underline{s}$ (a is high), $\bar{p}(s) = p^*$ for all $s \in \mathcal{S}$ and any $p^* \in [s^*/(1-\theta), \bar{s}/(1-\theta)]$.

In equilibrium, agents strictly prefer to trade repo over any combination of repo and spot trades when a is low or intermediate. They weakly prefer to trade repo when a is high.

In the first two cases, agents trade all the available asset in a repo and they do not achieve the first best allocation. However, when the asset is plentiful, agents can achieve the first best allocation by using some of the asset in a repo. In this case, the contract is essentially unique since agents can choose several p^* in the specified interval to attain the same allocation. If p^* is higher than the lower bound, agents just pledge less asset. The feasible contracts are all the continuous locus below the line $s/(1-\theta)$. The blue line in Figure 1 shows the equilibrium repo contract when the asset level is in the intermediate region. The increasing part of the repo contract corresponding to the asset payoffs where the borrowing motive of agent 1 dominates the hedging motive of agent 2. The opposite is trye for the flat part of the blue line, starting from s^*. Intuitively, agent 2 is willing to pay up to $\Delta u'(c_2^2)$ at date 1 to increase his consumption by Δ at date 2. As $\delta < 1$, agent 1 is always willing to borrow and promise Δ as long as what he gets at date 1 $\Delta u'(c_2^2)$ is higher than what he has to pay, $\delta\Delta$. Hence, the borrowing motive stops dominating the hedging motive at s^*.

Now, it may be that agents prefer to trade other contracts. Below, we give an intuitive account for why this contract is the unique equilibrium contract and we refer the interested reader to the Appendix for the details of the proof.

We need to show that agents do not want to trade any other contract $\tilde{p}$. To do so, we can use the equilibrium conditions to derive the marginal willingness to buy and sell contract $\tilde{p}$ for agent i at date 1, that is the forward prices $\tilde{p}^i_{F,b}$ and $\tilde{p}^i_{F,s}$ respectively. From the first order conditions of agent 1 we obtain

$$p_F - \delta \mathbb{E}\left[\bar{p}(s)\right] - \gamma_1^1 = 0, \tag{14}$$

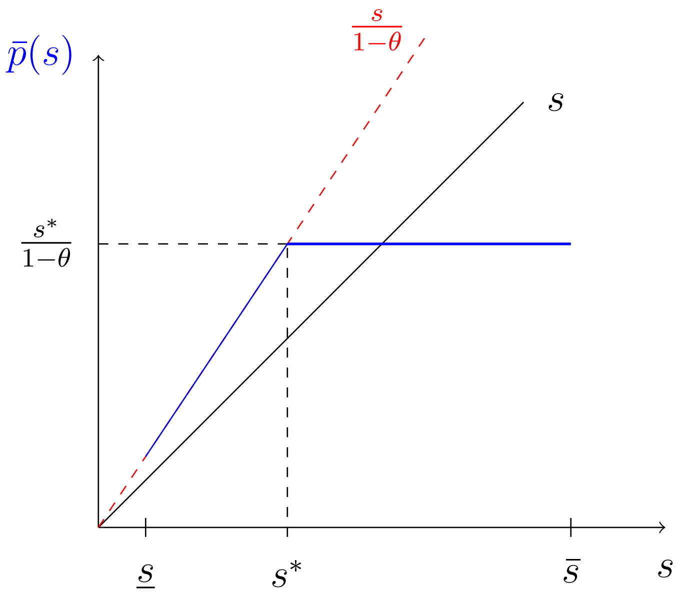

Figure 1: Equilibrium repo contract

where γ_1^1 is the Lagrange multiplier on the collateral constraint of agent 1, and from the ones of agent 2,

$$-p_F + \mathbb{E}\left[\bar{p}(s)u'(\omega + \bar{p}(s)a)\right] = 0. \tag{15}$$

where we have used the equilibrium $\ell^{21} = a$. In equilibrium, the willingness of agent 2 to buy another contract $\tilde{p}$ is $\tilde{p}_F^2 = E\left[\tilde{p}(s)u'(\omega + \bar{p}(s)a)\right]$, while the willingness of agent 1 to sell this contract is $\tilde{p}_F^1 = \delta E\left[\tilde{p}(s)\right] + \gamma_1^1$. Therefore agents will not trade this contract whenever[15] $\tilde{p}_F^2 \leq \tilde{p}_F^1$, or:

$$\mathbb{E}\left[\big(\bar{p}(s) - \tilde{p}(s)\big)\big(u'(\omega + \bar{p}(s)a) - \delta\big)\right] \geq 0 \tag{16}$$

where we used (14)-(15) to obtain the expression for γ_1^1. Condition (16) is a necessary and sufficient no-trade condition for contract $\tilde{p}$ when contract $\bar{p}$ is available. It should be clear that the equilibrium contract $\bar{p}(s)$ represented by the blue line in 1 satisfies (16). The reason is that it is $s/(1-\theta)$ whenever $u'(\omega + \bar{p}(s)a) > \delta$ so

[15] In the Appendix, we also check that agent 2 does not want to sell this contract and agent 1 does not want to buy it.

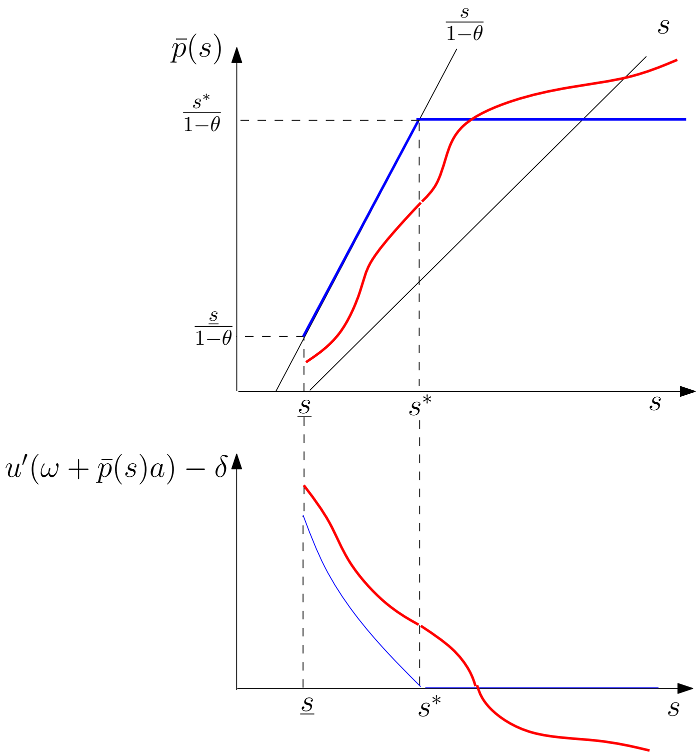

Figure 2: Non-equilibrium repo contract

that there is no other contract such that $\bar{p}(s) < \tilde{p}(s)$ whenever $u'(\omega + \bar{p}(s)a) > \delta$. Also, whenever $p(s) < s/(1-\theta)$, we have $u'(\omega+\bar{p}(s)a) = \delta$, so that (16) is satisfied for any other feasible repo contract $\tilde{p}(s)$.

Now, suppose that a candidate price schedule is $\hat{p}$ depicted by the red curve on Figure 2. This $\hat{p}$ is feasible as it lies below the default line. However, it does not satisfy (16). Indeed, take the alternative contract $\tilde{p}$ that equals $\bar{p}(s)$ for all $s < s'$ such that $\bar{p}(s') = \hat{p}(s')$ and $\tilde{p} = \hat{p}(s)$ for all $s > s'$. The contract $\tilde{p}$ so constructed clearly violates (16), as $\big(\hat{p}(s) - \tilde{p}(s)\big)\big(u'(\omega + \hat{p}(s)a) - \delta\big) < 0$ for all $s < s'$ and zero otherwise.[16] Here the borrowing motive dominates: By increasing the repayment schedule, agent 1 can borrow more at date 1, while also making sure agent 2 increases its revenue at date 2. Agent 1 surely prefers this, and agent 2 consumes more at date 2, thus making him better off as well. This argument implies that the equilibrium contract must have $\bar{p}(s) = s/(1-\theta)$ for all $s \leq s^*$.

[16] In the Appendix we take care of the fact that $\tilde{p}$ should be continuous and monotonically increasing schedule.

Now suppose there is s' such that $\hat{p}(s) > s^*/(1-\theta)$ for some $s > s'$ as for the red curve shown on Figure 2. In this case, construct another contract $\tilde{p}$ for which $\tilde{p}(s) = \hat{p}(s)$ for all $s < s'$ and $\tilde{p}(s) = s^*/(1-\theta)$ otherwise. Then (16) once again does not hold. The reason is that when $\hat{p}(s) > s^*/(1-\theta)$ agent 1 pays more in all states $s > s'$ than what he receives at date 1 for it, that is $u'(\omega + \hat{p}(s)a) < \delta$, and he would rather decrease the repayment schedule. Here the hedging motive dominates and agent 1 will extract more from a contract that insures agent 2 than from a contract that gives him too much consumption. Finally, the proof makes clear that when agents are not attaining the first best allocation, they strictly prefer to repo the entire quantity of asset to any combination of spot and repo trades, which is the case when a is relatively small.[17]

3.1 Haircuts, liquidity premium, and repo rates

In this section, we derive the equilibrium properties of the liquidity premium and repo haircuts. We compare the haircuts and liquidity premiums of two assets with different risk profile in a given economy, and we do comparative statics with respect to counterparty risk, as measured by θ.

The liquidity premium is given by (3). In our economy, the liquidity premium is positive because the short sales constraint is binding: When there is only spot trades, agent 1 would like to sell more of the asset than what he has, while with repo, his collateral constraint is binding. The repo haircut is

$$\mathcal{H} \equiv p_1 - p_F = \delta \left(\mathbb{E}[s] - \mathbb{E}[\bar{p}(s)] \right), \tag{17}$$

as it costs p_1 to obtain 1 unit of the asset, which can be pledged as collateral to borrow p_F. So to purchase 1 unit of the asset, an agent needs $p_1 - p_F$ which is the downpayment or haircut.[18] Finally, the repo rate is

$$1 + r = \frac{\mathbb{E}[\bar{p}(s)]}{p_F} = \frac{1}{\delta} \frac{\delta \mathbb{E}[s] - \mathcal{H}}{\mathcal{L} + \delta \mathbb{E}[s] - \mathcal{H}}. \tag{18}$$

[17] When the amount of asset is large however, agents can attain the first best allocation using either only repos or both repo and spot trades.

[18] An alternative definition is $(p_1 - p_F)/p_1$.

Two forces determine the repo rate. Lower haircuts increase the repo rate. However, fixing the haircut, a higher liquidity premium reduces the repo rate below the rate prevalent under the first best. Indeed, the liquidity premium is symptomatic of agents being (collateral) constrained. But the collateral constraint is binding when agent 1 has a strong desire to borrow, that is when borrowing is relatively cheap or $r < 1/\delta$. In other words, when the asset is expensive relative to its fundamental value, agent 1 would only use it as collateral rather than sell it outright if repo is cheap. Interestingly, repo rates r could even be negative for assets with large haircuts and large liquidity premium. This is consistent with market data as reported in ICMA (2013).[19]

We can derive the haircut and liquidity premium for the optimal price schedule $\bar{p}(s)$.

Corollary 3. *The equilibrium haircut and liquidity premium are given by:*

If $s^* \geq \bar{s}$ (a is low),

$$\mathcal{H} = -\frac{\delta\theta}{1-\theta}\mathbb{E}[s],$$
$$\mathcal{L} = \delta\mathbb{E}\left[\frac{s}{1-\theta}\left(u'(\omega + \frac{s}{1-\theta}a) - \delta\right)\right].$$

If $s^* \in [\underline{s}, \bar{s}]$ (a is intermediate),

$$\mathcal{H} = \delta\left[-\int_{\underline{s}}^{s^*}\frac{\theta}{1-\theta}s\, dF(s) + \int_{s^*}^{\bar{s}}\left(s - \frac{s^*}{1-\theta}\right)\mathrm{d}F(s)\right],$$
$$\mathcal{L} = \delta\int_{\underline{s}}^{s^*}\frac{s}{1-\theta}\left(u'(\omega + \frac{s}{1-\theta}a) - \delta\right) dF(s).$$

If $s^* \leq \underline{s}$ (a is high),

[19]The ICMA (2013) reports that "The demand for some assets can become so strong that the repo rate on that particular asset falls to zero or even goes negative. The repo market is the only financial market in which a negative rate of return is not an anomaly." (p.12) and in footnote 6 "negative repo rates have been a frequent occurrence and can be deeply negative." Also, see Duffie (1996), or Vayanos and Weill (2008).

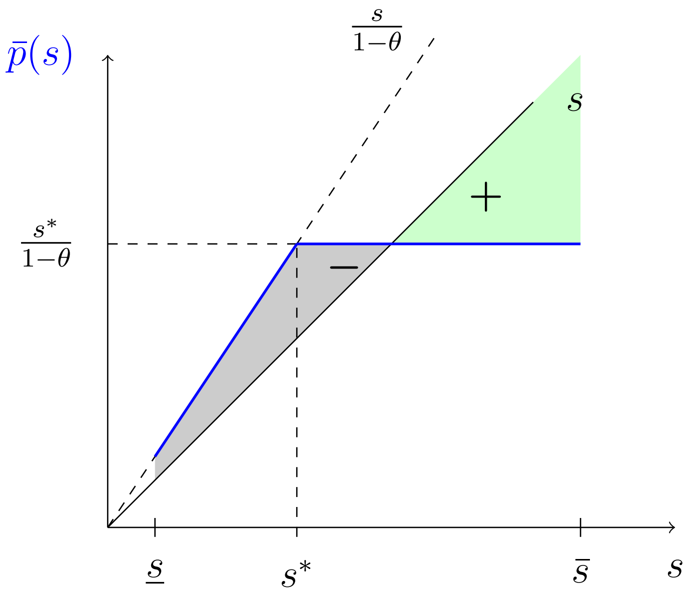

Figure 3: Repo haircuts

$$
\mathcal{H} \in \left[\delta\mathbb{E}[s] - \delta\frac{\underline{s}}{1-\theta}, \delta\mathbb{E}[s] - \delta\frac{s^*}{1-\theta}\right],
$$
$$
\mathcal{L} = 0.
$$

The corollary implies that haircuts must be negative when the asset is sufficiently scarce, and can be either negative or positive otherwise. As Figure 3 shows, the borrowing and hedging motives contribute in opposite ways to the size of the haircut. First, the borrowing motive contributes negatively to the haircut. The reason is that in low states, the repo contract pays more to the asset holder than a spot sale. So if the repo lender anticipated a low state, he would be willing to pay more for the repo (p_F) than the period 1 spot price, implying a negative haircut. In the opposite, hedging contributes positively to the haircut. Again, hedging limits the payment to agent 2 in high states, and if the repo lender expects s to fall in this region, he is willing to pay less than the period 1 spot price to acquire the asset in a repo, hence implying a positive haircut. To summarize, if $\theta = 0$ the borrower cannot borrow more than the present value of the asset and the haircut

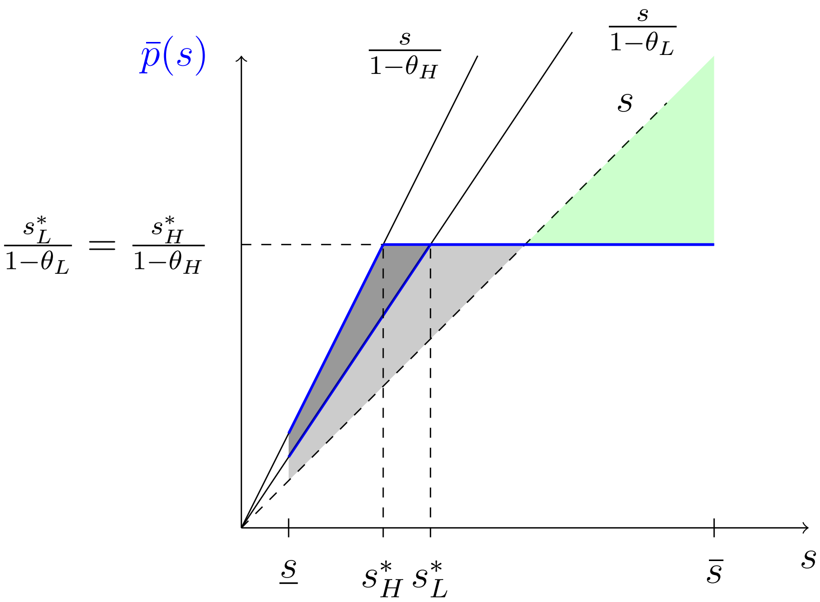

Figure 4: Influence of θ

is positive. As θ increases, the borrower increases its borrowing capacity and the haircut declines. Finally, observe that the haircut is not pinned down when $s^* \leq \underline{s}$ since several (constant) repurchase prices $\bar{p}$ are possible in equilibrium.

3.1.1 Counterparty risk

We now perform a comparative static exercise varying θ a proxy for counterparty risk. The risk that the borrower defaults is higher as θ is low since the non-pecuniary default loss is proportional to θ. Of course, there is never default in equilibrium, bu the equilibrium contract reflects default risk. θ. In particular, s^* depends negatively on θ.

Building on our previous expression for the haircut $\mathcal{H}$, we obtain that haircuts increase with counterparty risk, as

$$\frac{\partial \mathcal{H}}{\partial \theta} = -\frac{\delta}{(1-\theta)^2} \int_{\underline{s}}^{s^*} s dF(s) \leq 0$$

Indeed, as Figure 4 shows, a higher θ increases the amount a borrower can raise per unit of the asset pledged. This naturally leads to a decrease in the haircut, by increasing the size of the region where $\bar{p}(s) > s$ while leaving the other region unchanged.

Turning now to liquidity, it would natural to expect that an increase in θ has the same effect as an increase in a, as they both raise the borrowing capacity of agent 1. Hence, we would expect the liquidity premium to decrease since agents become less constrained. This argument however does not fully take into account the optimal contract design. Indeed, the slope of the optimal repurchase schedule, $\bar{p}(s)$, increases with θ for $s \in [\underline{s}, s^*]$ where the agent is constrained. This has the adverse effect of raising agent 2 consumption variability in the lowest states. Therefore, the overall effect of a higher θ on the liquidity premium is ambiguous and depends on the risk aversion of agent 2, as the expression for $\mathcal{L}$ shows. We have, with $c_2^2(s) = \omega + \frac{s}{1-\theta}a$,

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{1}{(1-\theta)^2} \int_{\underline{s}}^{s^*} s \left[u'\left(c_2^2(s)\right) - \delta + \frac{u''\left(c_2^2(s)\right)}{1-\theta} \right] dF(s)$$

The first term captures the positive effect of θ on the liquidity premium: a higher borrowing capacity for agent 1 brings agent 2 consumption closer to the first best level of consumption in states $s \in [\underline{s}, s^*]$. The second term shows the effect of the steeper slope of the repurchase profile $\bar{p}(s)$.

3.1.2 Asset risk

In this section, we compare repo terms given a richer asset structure.[20] To do this, we introduce two assets with different risk profiles but perfectly correlated payoffs. We compute the liquidity premium of the safer asset relative to the riskier, the haircuts that both assets carry, and the repo rate for both assets.

As before, $s \sim F[\underline{s}, \bar{s}]$ but there are now two assets $i = A, B$ with payoffs $\rho_i(s)$:

$$\rho_i(s) = s + \alpha_i(s - \mathbb{E}[s]),$$

[20] In the Appendix we consider the effect of an increase in risk, in the sense that the support $[\underline{s}, \bar{s}]$ changes when s is uniformly distributed.

where $\alpha_i \geq -1$, and $\alpha_B > \alpha_A$. When $\alpha_i = -1$, the asset is perfectly safe and pays off $\mathbb{E}[s]$. An asset with a higher α has the same mean but a higher variance than an asset with a lower α. Indeed $Var[\rho_\alpha] = (1+\alpha)Var[s]$. Therefore, asset B is relatively riskier than A. We choose to consider two assets with perfectly correlated payoff, as in this way the structure of the repo contract does not change. Otherwise holding both assets may offer some risk sharing which will modify the structure of the optimal contract we derived above. Agent 1 is endowed with a units of asset A and b units of asset B, while agent 2 does not hold any of the assets.

It is relatively straightforward to extend the equilibrium analysis of the previous section to this new economy with two assets. The set of available contracts consists of feasible repos using assets A and B. The structure of the repo contracts is essentially the same as before, defining the threshold s^{**} above which agent 2 consumes $c^2_{2,*}$, as

$$u'\left(\omega + \frac{a\rho_A(s^{**}) + b\rho_B(s^{**})}{1-\theta}\right) = \delta.$$

Then the repayment schedule for asset i is

$$\bar{p}_i(s) = \begin{cases} \frac{\rho_i(s)}{1-\theta} & \text{for } s \leq s^{**}, \\ \frac{\rho_i(s^{**})}{1-\theta} & \text{for } s \geq s^{**}. \end{cases}$$

Then we have the following result.

Proposition 4. *The safer asset A always has a higher liquidity premium and a lower haircut than the riskier asset B. The repo rate spread is indeterminate.*

The riskier asset carries a larger haircut because its repo contract contains a larger hedging component for high states, and this increases the borrowing cost, i.e. agent 2 is willing to pay less to accept this asset in a repo. Also, because the safer asset has a lower haircut, agent 1 is willing to pay a higher price to acquire this asset to later pledge it as collateral. This implies that the (gross) repo rate r_i defined in (18) may be lower or higher for the different risk profile. As we showed, the liquidity term is lower for more risky assets, so that tends to increase the rate on risky asset but this increase is partially (or totally) offset by the increase in haircut, so that the overall effect on rates is indeterminate. So our model suggests that repo rates are not a good statistics to determine the risk profile of an asset.

4 Collateral re-use and collateral intermediation

4.1 Collateral re-use

In this section, we analyze the equilibrium implications of allowing agents to re-use a fraction of the repo collateral. Re-use has been very much under scrutiny following the crisis (see Singh and Aitken, 2010) since a default on re-used collateral may affect several agents along a credit chain. While we do not model the consequence of such default cascades, we provide the foundations for this analysis by highlighting the benefits of allowing re-use.

To solve this problem we will guess and verify the structure of the equilibrium contracts by building on the intuition from the case without re-use. It also helps to think of the implicit sequence of trades that yields the result. Without re-use, agent 2 (the lender) is constrained: he would like to sell some of the asset to agent 1 but he only holds collateral pledged by agent 1. When we allow re-use, agent 2 has to segregate at least a fraction $1-\nu$ of agent 1's collateral a, but he can sell a fraction ν back to agent 1 (or some other agent 1). Agent 1 can then use νa as collateral to borrow from agent 2 again. It is important to notice that agent 1 can use his entire asset holding at this stage, νa, as he *bought* it from agent 2. Also, after each round of such trades, the asset available to agent 1 decreases. So the trading process goes on until he extinguishes the available asset. It should now be intuitive that allowing re-use relaxes both agents' constraints.

We now guess that agents use only one repo contract $F^{12} = (\bar{p}, \nu)$ and agent 2 resells the fraction ν of the collateral he obtains on the spot market. Let $\bar{p}(s,\nu)$ be defined by

$$\bar{p}(s,\nu) = \begin{cases} \frac{s}{1-\theta} & \text{if } s < s^*(\nu) \\ \frac{s^*(\nu)}{1-\theta} + \nu(s - s^*(\nu)) & \text{if } s \geq s^*(\nu) \end{cases} \tag{19}$$

where we define $s^*(\nu)$ implicitly as follows

$$u'\left(\omega + \frac{as^*(\nu)}{1-\nu}\left[\frac{1}{1-\theta} - \nu\right]\right) = \delta.$$

Then we show in the Appendix

Proposition 5. *Suppose agent 2 can re-use at most a fraction $\nu \in [0,1]$ of the collateral. The (essentially) unique equilibrium forward contract underlying the repo is $\bar{p}_\nu = \{\bar{p}(s,\nu)\}_{s\in\mathcal{S}}$ defined in (19). Collateral re-use is essential whenever $\theta > 0$ and the complete market benchmark allocation is achieved for any $\nu \geq \nu^*$ defined as*

$$\nu^* = \frac{s^* - \underline{s}}{s^* - (1-\theta)\underline{s}}.$$

The proof makes clear that the benefit of re-use is to allow agent 2 to short-sell the asset. Looking back at (9) suppose that agents are constrained, so that their collateral constraint binds. We thus have

$$a_1^1 = b^{12} \tag{20}$$

$$a_1^2 = -\nu \ell^{21} \tag{21}$$

and agent 2 goes short in the asset whenever $\ell^{21} > 0$, that is, he sells the collateral that he obtained from lending. Naturally, even with re-use, there is a constraint on the amount of physical asset available. Precisely we have $a_1^1 + a_1^2 = a$. Also market clearing requires $b^{12} = \ell^{21}$. So summing (20) and (21) we obtain

$$a_1^1 = \frac{a}{1-\nu} = b^{12}$$

$$a_1^2 = -\frac{\nu}{1-\nu} a = -\nu b^{12}$$

To get a grasp for these expressions, let us decompose the pledging and re-using process as we explain above. In the first step, agent 1 pledges a units of collateral. Agent 2 must segregate a fraction $1-\nu$ but resells a fraction ν to agent 1 who repeats the first operation. Through this mechanism, agent 1 can leverage his asset holdings a by a factor $1/(1-\nu)$. Agent 2 effectively has a short position in the asset because he sold collateral received on his loan to agent 1. But it enables agent 2 to consume more at date 2.

Finally, notice that re-use is only essential when agents are to some extent trustworthy, i.e. $\theta > 0$. To see this, let us assume that $\theta = 0$, and allow re-use $\nu = 1$. Since $\theta = 0$ the repo repayment schedule is $\bar{p}(s)$ lies below s for all s. This implies that agent 2 will only get less than s from the repo loan, while he has to pay s for each unit of assets he sold short at date 1. Thus agent 2 is unable to increase his date 2 consumption beyond what is feasible with no re-use. Hence re-use is redundant when $\theta = 0$.[21] However, with $\theta > 0$ agent 2 can increase his consumption in the low states as agent 1 can promise $s/(1-\theta)$ per unit of the asset, while it only costs s to agent 2 to cover a unit short position.

Haircuts can again be positive or negative with re-use, but increasing re-use always lowers haircut. Indeed, using (17) and replacing the repayment schedule (19), we find

$$\mathcal{H}/\delta = \mathbb{E}[s] - \int_{\underline{s}}^{s^*(\nu)} \frac{s}{1-\theta} dF(s) - \int_{s^*(\nu)}^{\bar{s}} \left[\frac{s^*(\nu)}{1-\theta} + \nu \left(s - s^*(\nu)\right)\right] dF(s)$$

which is always negative when $\nu = 1$. Here the chain of re-use is infinite which explains why the haircut is negative with $\nu = 1$. In the chain of re-use is rarely infinite and so we should not expect that re-use is associated with negative haircuts. Still in practice re-use is either prohibited or allowed with $\nu = 1$. Also, our model predicts that re-use is most helpful when collateral is most scarce and there is evidence that this is indeed the case (see Fuhrer et al. (2015)). Since re-use relaxes the collateral constraint, the liquidity premium is naturally decreasing in ν and is null whenever $\nu \geq \nu^*$.[22]

4.2 Intermediating repo

A prominent feature of the tri-party repo market, although one that has not been studied, is the intermediation of repo loans. As we already explained in the introduction, a simplified tri-party repo trade involves a dealer bank (the cash borrower), the tri-party agent, and a money market mutual fund (MMF, the cash lender). Typically, the tri-party agent manages the collateral pledged by the dealer bank. MMFs agree to repo only with specific dealer banks that they deem safe enough. So not every financial market participants, typically hedge funds, can participate in this arrangement. This is where repo intermediation comes into the fore: If a hedge fund needs cash and wants to tap in the MMF cash pool, he will do

[21]See Maurin (2015) for a proof in a more general environment in this case.

[22]With re-use the liquidity premium is simply $\mathcal{L} = \delta \int_{\underline{s}}^{s^*(\nu)} \frac{s}{1-\theta} \left(u'(\omega + \frac{s}{1-\theta}a) - \delta\right) dF(s)$.

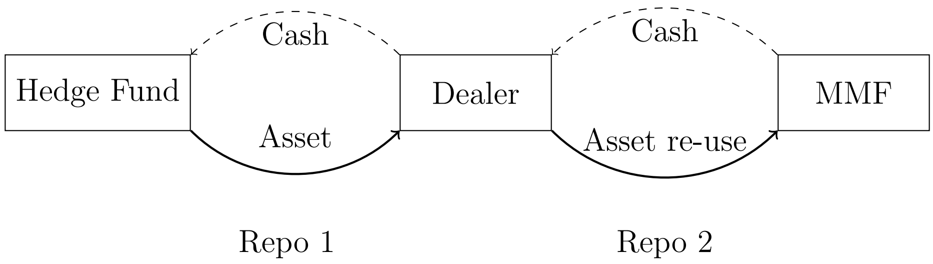

Figure 5: Intermediation with Repo

so indirectly, through a dealer bank that has access to the tri-party agent. Figure 5 illustrates this pattern of repo intermediation.

In this section we show that equilibrium with intermediation exist, thus illustrating another benefit of collateral re-use. To do so, we need to modify the economy slightly. We now assume there are two agents of type 1, L and H, who differ with respect to their trustworthiness $\theta_L \leq \theta_H$ and their marginal utility for consumption in period 2 and 3, δ_H and $\delta_L \leq \delta_H$ respectively.[23] Agent L initially owns all the asset. We say that there is intermediation when L trades with H rather than with agent 2 directly. Intuitively, H could play the role of intermediary since $\theta_H \geq \theta_L$ implies he is a more reliable counterparty for 2 than agent L, even though agent L is as bad a counterparty for agent H than for agent 2. This section derives conditions under which intermediation arises in equilibrium when agents are free to choose contracts.

4.2.1 Feasible Contracts

We build on our analysis of re-use to describe feasible contracts. As in the previous section a feasible repo contract $F_{ij} = (\bar{p}_{ij}, \nu_{ij})$ is a promise by borrower i to deliver $\bar{p}_{ij}(s)$ in state s together with a right to re-use a fraction ν_{ij} of the collateral for lender j, that verifies the no default constraints (5)-(6). We have shown in the previous section that unlimited re-use, that is $\nu_{ij} = 1$, can deliver the first best allocation. In this section, we assume re-use is limited, that is agent i can re-

[23]Agent 1_H may also have an additional balance sheet cost τ of owning assets, but we ignore this possibility here as this gives us essentially the same results.

use at most a fraction ν_i of the collateral he receives. In order to focus on the intermediation role of repo, we set the following exogenous limits: $\nu_L = 0$, $\nu_H < 1$ and $\nu_2 < 1$. Also, we assume agent L is endowed with all the asset $a_0^L = a$, while agents H and 2 have none, $a_0^H = a_0^2 = 0$. Agent L is a natural borrower and, as he will not lend, there is no loss in generality in preventing him to re-use collateral.

4.2.2 Agents' problem

We conjecture an equilibrium with intermediation with the following trade pattern: Agent L borrows from agent H using a repo. Agent H can then use the collateral pledged by agent L in two ways. He can resell it to agent L to increase the asset supply as in the previous section, or he can re-pledge it in a repo with agent 2. In the latter case, agent H stands between agents L and 2.

Using our notation, we guess that two contracts are traded in this equilibrium: a repo $F_{LH} = (\bar{p}_{LH}, \nu_H)$ between agents L and H and a repo $F_{H2} = (\bar{p}_{H2}, \nu_2)$ between agents H and 2. In the equilibrium we conjecture, we can use (7) to rewrite the problem of all agents (see the Appendix for details).[24] As in the previous sections, we then derive the price $\tilde{p}_F$ for non-traded repo contracts $\tilde{F}$ to support the equilibrium.

Notice that b^{H2}, the amount agent H borrows from agent 2, naturally affects the amount of asset that agent 2 can re-sell. Prices will therefore adjust endogenously to b^{H2}, and the repo contract will reflect this adjustment. Therefore b^{H2} can only be implicitly defined. More precisely, for a given amount of asset b, let us define $s^*(b, \theta_i, \nu_2)$ for $i = H, L$, as the solution in s to:

$$u'\left(\omega + \frac{bs}{1-\nu_2}\left[\frac{1}{1-\theta_i} - \nu_2\right]\right) = \delta_i,$$

and for $i = H, L$, define the contract $\bar{p}_{i2}(s, \nu_2)$ as

$$\bar{p}_{i2}(s, \nu_2) = \begin{cases} \frac{s}{1-\theta_i} & \text{if } s < s^*(b, \theta_i, \nu_2) \\ \frac{s^*(b,\theta_i,\nu_2)}{1-\theta_i} + \nu_2(s - s^*(b, \theta_i, \nu_2)) & \text{if } s \geq s^*(b, \theta_i, \nu_2) \end{cases} \tag{22}$$

[24] Since agents L and H have linear utility and different discount factors, corner solutions for consumption are possible, but we guess and verify throughout that this is not the case. They would hit the constraint if, e.g., ν_H was high.

and b_*^{H2} as the solution to

$$\int_{\underline{s}}^{s^*(b^{H2},\theta_H,\nu_2)} \left[u'\left(\omega + \frac{b^{H2}s}{1-\nu_2}\left[\frac{1}{1-\theta_H} - \nu_2\right]\right) - \delta_H\right] \mathrm{s}dF(s) \quad = \quad \frac{(\delta_H - \delta_L)\theta_L}{(1-\nu_H)(1-\theta_L)}\left[\frac{1-(1-\theta_H)\nu_2}{(1-\theta_H)(1-\nu_2)}\right]^{-1} \tag{23}$$

In the Appendix we show the main result of this section,

Proposition 6. *If $\theta_H - \theta_L \geq \Delta\bar{\theta}$ and $\delta_H - \delta_L \leq \Delta\bar{\delta}$ the only equilibrium contracts are, the repo contract $\{\bar{p}_{LH}(s)\}_{s\in\mathcal{S}}$ where agent L borrows from agent H, with*

$$\bar{p}_{LH}(s) = \frac{s}{1-\theta_L}$$

and the repo contract $\{\bar{p}_{H2}(s,\nu_2)\}_{s\in\mathcal{S}}$ defined in (22) where agent H borrows b_*^{H2} from agent 2.

Notice that the repo contract between agents L and H does not reflect any hedging motive as agent H is risk neutral. Also, in this equilibrium agent L does not trade with 2 in spite of being endowed with the asset. The reason is that his ability to commit to repay is too low. So agent L cannot obtain a large loan from agent 2 because of this inability to commit but also because the equilibrium repo contract with agent 2 contains a hedging component (the flat part passed s^*). This hedging component is absent from the repo contract between agent L and H as agent H is risk neutral. So, although agent L cannot commit more toward agent H than toward agent 2, everything else equal, he prefers to borrow from agent H. Now this equilibrium exists only if agent H can borrow from agent 2 substantially more than agent L which requires that θ_H be sufficiently larger than θ_L. The final condition for intermediation is that the gains from trade between agents L and H are low enough. Indeed, if the gains from trade between agents L and H as measured by $\delta_H - \delta_L$ are too large, then agent H would just re-sell the re-usable fraction of the collateral back to L until the collateral is entirely segregated[25] and

[25] Agent H could also run out of resources, but we consider the case where all consumption levels are positive.

agent 2 would not enter the picture. An alternative but equivalent requirement for the existence of the intermediation equilibrium is that $a \leq \bar{a}$ where the upper bound $\bar{a}$ is a decreasing function of $\delta_H - \delta_L$.

Corollary 7. *If $\theta_L = \theta_H$ and $\delta_L = \delta_H$ then the unique forward contract underlying the repo is $\{\bar{p}_{L2}(s, \nu_2)\}_{s\in\mathcal{S}}$ defined in (19) where agent L borrows directly from agent 2 by pledging collateral a.*

In this equilibrium there is no intermediation because agents L and H are alike. Agent L has the benefit of being endowed with the asset that he can pledge as collateral in the repo contract with agent 2. Agent H would also like to borrow from agent 2 but he does not have the necessary collateral and so he cannot trade.

Corollary 8. *Let $\nu_2 = 0$. If $\theta_L = \theta_H$ and $\delta_L < \delta_H$ then the only repo contracts in equilibrium are: the repo contract $\{\bar{p}_{LH}(s)\}_{s\in\mathcal{S}}$ where agent L borrows b^{LH} from agent H, with*

$$\bar{p}_{LH}(s) = \frac{s}{1-\theta_L}$$

and the repo contract $\{\bar{p}_{L2}(s, 0)\}_{s\in\mathcal{S}}$ defined in (22) where agent L borrows b^{L2} directly from agent 2.

In this equilibrium, agent L's willingness to borrow is higher than agent H's, while both agents have the same ability to commit. Agent L can then borrow from agent H and agent 2 alike and so he has to be indifferent between both options in equilibrium. We simplified the analysis by imposing $\nu_2 = 0$ so that agent 2 cannot sell the collateral back to agent L, otherwise he would. Here there is no intermediation as agent L deals directly with both agents.

5 Conclusion

We analyzed a simple model of repurchase agreement. We derived the equilibrium lending contracts with limited commitment and showed that this contract shares many features of a repo contract: It is a spot sale combined with a forward contract. The forward contract contains a fixed repayment schedule that insures the lender against the risk that the price of the asset varies. We found that the repo haircut is an increasing function of counterparty risk as measured by the borrowers' ability to commit and a decreasing function of the asset inherent risk. Also, safe assets naturally command a higher liquidity premium than risky ones. We use the model to analyze the consequences of allowing re-use. We find that re-use are most useful when the asset is special, in the sense that its haircut is negative. Then re-use helps agents to achieve the first best allocation. Finally, we found conditions under which intermediation can arise endogenously in repo markets, thus giving us first elements to understand the endogenous formation of repo chains.

Obviously, we made simplifying assumptions. We considered a finite economy in which the limited commitment problem is especially binding. Extending the model to an infinite horizon is however not trivial, as the optimal contract will become dynamic, unless we resort to the usual subtleties. Also, we considered a real economy and abstracted from money on purpose. While traders active in the repo market trade cash, we felt that it would be too difficult in a first pass to model repo and money seriously. However, given the results and the recent progress of the monetary literature we do not doubt that embedding money is feasible. In the end, the model is rather tractable and could be used to understand current policy issues such as minimum haircuts, collateral transformation, or the effect of a large asset purchase by the monetary authority on the repo market. We leave these important topics for future research.

Appendices

A Proofs

A.1 Equilibrium analysis of spot trade only

We now discuss the equilibrium allocation when only spot market trades are feasible. Since there is no market for forward contracts, the equilibrium conditions become

$$\begin{aligned} p_2(s) &= s \\ -p_1(1+\lambda_1^1) + \delta E[s] + \xi_1^1 &= 0, \\ -p_1(1+\lambda_1^2) + E\left[su'(\omega + sa_1^2)\right] + \xi_1^2 &= 0, \\ \lambda_1^1\lambda_1^2 &= 0 \\ \xi_1^1\xi_1^2 &= 0 \\ c_2^1(s) &= \omega + s(a_1^1 - a) \\ c_3^1(s) &= 2\omega + as \end{aligned}$$

As we have seen, the complete market trade amounts to a loan from agent 2 to agent 1. When only spot trades are available, agent 1 should sell his asset in period 1 and buy it back in period 1 to borrow implicitly from agent 2. A simple argument by contradiction shows that $\xi_1^2 = 0$ (agent 2 saves asset from date 1 to date 2). Also we guess and verify later that $\lambda_1^i = 0$. Then the equilibrium conditions are

$$\begin{aligned} -p_1 + \delta E[s] + \xi_1^1 &= 0, \\ -p_1 + E\left[su'(\omega + sa_1^2)\right] &= 0, \\ c_2^1(s) &= \omega - sa_1^2 \\ c_3^1(s) &= 2\omega + as \end{aligned}$$

so

$$\xi_1^1 = E\left\{s\left[u'(\omega + sa_1^2) - \delta\right]\right\} \tag{24}$$

Hence, either $\xi_1^1 = 0$ and a_1^2 solves (24), or $\xi_1^1 > 0$ and $a_1^2 = a$ where $E\left\{s\left[u'(\omega + sa) - \delta\right]\right\} > 0$. We now verify that consumption is non-negative for both agents at date 1. Agent 1 sells assets at date 1 and so trivially his consumption is positive. Agent 2 purchases a_1^2 at price p_1 and consumes $c_1^2 = \omega - p_1 a_1^2$. When $\xi_1^1 = 0$, $p_1 = \delta E[s]$ and $c_1^2 = \omega - \delta E[s] a_1^2(\omega)$. Notice that the solution $a_1^2(\omega)$ to (24) is decreasing in ω. When $\xi_1^1 > 0$, $p_1 = E\left[su'(\omega + sa)\right]$ and so $c_1^2 = \omega - aE\left[su'(\omega + sa)\right]$. So in both case $c_1^2 > 0$ whenever ω is large enough.

Finally, notice that the liquidity premium is given by (24).

B Proof of Proposition 2

As we observed before, preferences are such that $p_2(s) = s$ and agent 1 holds all the asset at date 3, that is $a_2^1(s) = a$ for all s. To characterize the optimal contract $\bar{p}$, we derive conditions on $\bar{p}$ so that agents are not willing to trade a different contract $\tilde{p}$ with $\tilde{p} \neq \bar{p}$ in the sense that there is a positive measure of states s such that $\tilde{p}(s) \neq \bar{p}(s)$. We can use the equilibrium conditions to derive the marginal willingness to buy and sell contract $\tilde{p}$ for agent i at date 1, that is the forward prices $\tilde{p}_{F,b}^i$ and $\tilde{p}_{F,s}^i$ respectively. We obtain

$$
\begin{aligned}
\tilde{p}_{F,b}^1 &= \delta E\left[\tilde{p}(s)\right] \\
\tilde{p}_{F,s}^1 &= \delta E\left[\tilde{p}(s)\right] + \eta_1^1 \\
\tilde{p}_{F,b}^2 &= E\left[\tilde{p}(s) u'(\omega + sa_1^2 + \bar{p}(s)\ell^2)\right] \\
\tilde{p}_{F,s}^2 &= E\left[\tilde{p}(s) u'(\omega + sa_1^2 + \bar{p}(s)\ell^2)\right] + \eta_1^2
\end{aligned}
$$

Note that the marginal willingness to buy and sell $\tilde{p}$ can differ for agent i when he hits the collateral constraint, that is $\eta_1^i > 0$. For agents not to trade contract $\tilde{p}$ the following inequalities must hold:

$$
\tilde{p}_{F,b}^1 \leq \tilde{p}_{F,s}^2 \tag{25}
$$

$$
\tilde{p}_{F,b}^2 \leq \tilde{p}_{F,s}^1 \tag{26}
$$

We first assume that $\ell^2 \geq 0$ in equilibrium.[26]. This means that $\eta_1^2 = 0$. So the equilibrium conditions are

[26]This is the natural case since agent 2 wants to lend. However, if $\bar{p}$ is low, agent 2 can combine a short position $a_F^2 < 0$ in repo with a spot purchase $a_1^2 > 0$ to increase his consumption in period 2. We discuss the case $a_F^2 < 0$ later to show that it cannot be an equilibrum if agents may also trade the contract described in Proposition 2.

$$
\begin{aligned}
-p_1 + \delta E[s] + \eta_1^1 + \xi_1^1 &= 0, \\
-p_F + \delta E\left[\bar{p}(s)\right] + \eta_1^1 &= 0. \\
-p_1 + E\left[su'(c_2^2(s))\right] + \xi_1^2 &= 0, \\
-p_F + E\left[\bar{p}(s)u'(c_2^2(s))\right] &= 0. \\
\xi_1^1\xi_1^2 &= 0 \\
c_2^1(s) &= \omega + s(a_1^1 - a) - \bar{p}(s)b^1 \\
c_2^2(s) &= \omega + sa_1^2 + \bar{p}(s)\ell^2
\end{aligned}
$$

To start with we show that, without loss of generality, we can focus on the case where $a_1^2 = 0$ and $\ell^2 = b^1 = a$. That is any combination of spot trades repo trade with $\bar{p}$ can be replicated by an appropriate $\hat{p}$ without additional spot trade. Consider indeed $\hat{p}$ defined for all $s \in \mathcal{S}$ by:

$$
\hat{p}(s) = \frac{a_1^2 s + \ell^2 \bar{p}(s)}{a}
$$

Clearly, $\hat{p}(s) \geq 0$ for all s and $\hat{p}$ is weakly increasing since $\ell^2 \geq 0$. To show that $\hat{p}(s)$ is feasible, we need to show $\hat{p}(s) < s/(1-\theta)$ for all s. But market clearing requires $\ell^2 = b^1$. Then $\eta_1^1 \geq 0$ gives us $a_1^1 \geq b^1$. Since $a = a_1^2 + a_1^1$ we get $a \geq a_1^2 + b^1 = a_1^2 + \ell^2$. And as $\bar{p}(s) \leq s/(1-\theta)$ we obtain $\hat{p}(s) < s/(1-\theta)$. Therefore our candidate $\hat{p}(s)$ is also feasible. We now verify that the payoff for agents are the same in dates 1 and 2 under both price and allocations. Clearly, the allocation of agent 2 is the same in date 2 for all states s. Therefore, it is also identical for agent 1 in date 2 in all states. Turning to date 1, agent 2 obtains under the new trading pattern:

$$
\hat{c}_1^2 = \omega - \hat{p}_F a
$$

where $\hat{p}_F = E\left[\hat{p}(s)u'(c_2^2(s))\right]$. Hence,

$$
\begin{aligned}
\hat{c}_1^2 &= \omega - E\left[\left[\frac{a_1^2 s + \ell^2 \bar{p}(s)}{a}\right] u'(c_2^2(s))\right] a \\
&= \omega - a_1^2 E\left[su'(c_2^2(s))\right] - E\left[\bar{p}(s)u'(c_2^2(s))\right]\ell^2 \\
&= \omega - a_1^2 p_1 - \ell^2 p_F
\end{aligned}
$$

as $\xi_1^2 = 0$ whenever $a_1^2 > 0$. Therefore, $\hat{c}_1^2 = c_1^2$ which concludes the argument. Hence, from now on, we set $a_1^2 = 0$ and $\ell^2 = a$.

Case 1: $\eta_1^1 = 0$

Suppose first that no agent is constrained, that is $\eta_1^1 = 0$. We must then have $\tilde{p}_{F,s}^1 = \tilde{p}_{F,s}^2$ for every contract $\tilde{p}$ by (25)-(26). Therefore $\tilde{p}_{F,s}^1 - \tilde{p}_{F,s}^2 = 0$ which implies that $E\left[\tilde{p}(s)\left(u'(\omega + \bar{p}(s)a) - \delta\right)\right] =$

0 holds for any $\tilde{p}$. Since $u'(\omega + \bar{p}(s)a)$ is monotone it must be that $u'(\omega + \bar{p}(s)a) = \delta$ almost surely. In other words, $\omega + \bar{p}(s)a = c^2_{2,*}$, that is the optimal price schedule implements the first best allocation $(c^1_{2,*}, c^2_{2,*})$. Then $\bar{p}(s)$ should not depend on s, that is $\bar{p}(s) = p^*$ such that $p^*a = c^2_{2,*} - \omega$. Observe that any repurchase price $p \in [p^*, \frac{s}{1-\theta}]$ and quantity pledged $\ell^2 \leq a$ such that $pa = c^2_{2,*} - \omega$ implements the first best allocation. Thus, our guess that $\eta^1_1 = 0$ may hold if and only if $s^* \leq \underline{s}$.

Case 2: $\eta^1_1 > 0$

Suppose next that agent 1 is constrained, i.e. $\eta^1_1 > 0$. Then (26) implies that the following inequality must hold for any $\tilde{p}$.

$$E\left[\big(\bar{p}(s) - \tilde{p}(s)\big)\big(u'(\omega + \bar{p}(s)a) - \delta\big)\right] \geq 0 \tag{27}$$

Suppose first that $\forall s \in \mathcal{S}$, $u'(\omega + \bar{p}(s)a) - \delta > 0$. Then the inequality above can only hold if $\bar{p}(s) = s/(1-\theta)$ for all $s \in \mathcal{S}$. If that is not the case, let us denote:

$$s_{min} := \min\left\{s \in \mathcal{S} \mid u'(\omega + \bar{p}(s)a) - \delta = 0\right\},$$

where $s_{min} > \underline{s}$, in the case considered. We have $u'(\omega + \bar{p}(s)a) > \delta$ for any $s \in [\underline{s}, s_{min}]$, as $\bar{p}(s)$ is weakly increasing. Define $s_1 \leq s_{min}$ such that $s_1/(1-\theta) = \bar{p}(s_{min})$. If $s_1 < \underline{s}$, then we must have $\bar{p}$ constant over $[\underline{s}, s_{min}]$. Otherwise, inequality (27) would be violated with

$$\tilde{p}(s) = \begin{cases} \bar{p}(s_{min}) & \text{for } s \leq s_{min} \\ \bar{p}(s) & \text{for } s \geq s_{min} \end{cases}$$

If $s_1 \geq \underline{s}$, then a similar argument shows that $\bar{p}$ must be constant over $[s_1, s_{min}]$. This is compatible with the definition of s_{min} if and only if $s_1 = s_{min}$. which implies $\bar{p}(s_{min}) = s_{min}/(1-\theta)$. Then we must also have $\bar{p}(s) = s/(1-\theta)$ for all $s \in [\underline{s}, s_{min}]$. Otherwise, inequality (27) would be violated with

$$\tilde{p}(s) = \begin{cases} s/(1-\theta) & \text{for } s \leq s_{min} \\ \bar{p}(s) & \text{for } s \geq s_{min} \end{cases}$$

This shows that $\bar{p}(s) = s/(1-\theta)$ for all $s \leq s_{min}$. We now take this property as given. Define now

$$s_{max} := \max\left\{s \in \mathcal{S} \mid u'(\omega + \bar{p}(s)a) - \delta = 0\right\}, \tag{28}$$

We show that $s_{max} = \bar{s}$. Suppose otherwise. Then, we have $\bar{p}(s) > \bar{p}(s_{max})$ and $u'(\omega + \bar{p}(s)a) < \delta$ for $s \in (s_{max}, \bar{s})$ and condition (27) would be violated with

$$\tilde{p}(s) = \begin{cases} \bar{p}(s) & s \leq s_{max} \\ \bar{p}(s_{max}) & s \geq s_{max} \end{cases}$$

Finally, on the interval $[s_{min}, s_{max}]$, by definition $\bar{p}$ is constant. Therefore, putting pieces together, we find that the optimal forward contract is

$$\bar{p}(s) = \begin{cases} \frac{s}{1-\theta} & \text{for } s \leq s^* \\ \frac{s^*}{1-\theta} & \text{for } s \geq s^* \end{cases}$$

where s^* solves

$$u'\left(\omega + \frac{s^*}{1-\theta}a\right) = \delta.$$

B.1 Optimal Repurchase Contract

Analysis of $b^2 > 0$.

We now consider a price schedule $\bar{p}$ such that the equilibrium has $b^2 > 0$, that is agent 2 is borrowing in a repo. We can focus on the case where $s^* < \bar{s}$ as otherwise agents could reach the equilibrium allocation (where $b^2 > 0$) with the contract described in Proposition 2. In any such equilibrium:

$$\begin{aligned} -p_1 + \delta E[s] + \xi_1^1 &= 0, \\ -p_F + \delta E\left[\bar{p}(s)\right] &= 0. \\ -p_1 + E\left[su'(c_2^2(s))\right] + \eta_1^2 &= 0, \\ -p_F + E\left[\bar{p}(s)u'(c_2^2(s))\right] + \eta_1^2 &= 0. \\ c_2^1(s) &= \omega + a_1^2(\bar{p}(s) - s) \\ c_2^2(s) &= \omega + a_1^2(s - \bar{p}(s)) \end{aligned}$$

where we have that $\eta_1^2 > 0$. Otherwise agents would not be constrained and would attain the first best allocation. Using the equilibrium condition that $\tilde{p}_{F,s}^2 \geq \tilde{p}_{F,b}^1$, for any continuous and increasing price schedule $\tilde{p}$, the following condition must hold

$$E[(\tilde{p}(s) - \bar{p}(s))(u'(c_2^2(s)) - \delta)] \geq 0 \tag{29}$$

We cannot reproduce the argument of the previous case since $c_2^2(s)$ needs not be monotonic in s. Instead, we use local properties to make our claim. Define indeed the following subsets of $\mathcal{S}$:

$$S_0 = \{s \in \mathcal{S} | u'(c_2^2(s)) - \delta = 0\}$$
$$S_+ = \{s \in \mathcal{S} | u'(c_2^2(s)) - \delta > 0\}$$
$$S_- = \{s \in \mathcal{S} | u'(c_2^2(s)) - \delta < 0\}$$

Let us define a sequence of sets $\mathcal{A} = \{A_n\}_{n\in\mathbb{N}}$ such that $\forall n \in \mathbb{N}$, $A_n \in \mathcal{S}_0 \cup \mathcal{S}_- \cup \mathcal{S}_+$ and $\mathcal{S} = \cup_n A_n$. By continuity of $c_2^2(s)$, we know that elements of $\mathcal{S}_-$ and $\mathcal{S}_+$ are open intervals while $\mathcal{S}_0$ may contain isolated points. Practically, we divide the support into intervals where $u'(c_2^2(s)) - \delta$ has a constant sign.

Let us first consider $A = (s_1, s_2) \in \mathcal{S}_+$. Then $\bar{p}$ must be constant over A. If not, let us construct $\tilde{p}$ such that $\tilde{p}(s) = \bar{p}(s)$ for $s \leq s_1$ and $s \geq s_2$ and $\tilde{p}(s) < \bar{p}(s)$ for $s \in A$ with $\tilde{p}$ increasing. With such $\tilde{p}$, inequality (29) is violated. Then let us prove that $\underline{s} \in \mathcal{S}_+$. Indeed $c_2^2(\underline{s}) = \omega + a_1^2(\underline{s} - \bar{p}(\underline{s})) < \omega + a\underline{s} < c_{2,*}^2$ because $s^* > \underline{s}$. This implies that $u'(c_2^2(\underline{s})) - \delta > 0$. Thus applying our previous argument for intervals in $\mathcal{S}_+$, we know that $\exists s_1 > \underline{s}$ such that $\bar{p}(s) = 0$, for $s \in [\underline{s}, s_1)$. Indeed $\bar{p}$ must be constant on this interval. If the constant is not zero we can construct a profile $\tilde{p}$ which would violate (29).

Consider next $A = (s_1, s_2) \in \mathcal{S}_0$ such that $s_1 < s_2$. By definition of S_0 , we must have $c_2^2(s)$ constant over A which implies that $\bar{p}(s) - s$ is constant over A.

Let us finally consider $A = (s_1, s_2) \in \mathcal{S}_-$. Then we can have either to two cases: (i) there exists $s_0 \in (s_1, s_2]$ such that $\bar{p}(s) = s/(1-\theta)$ over $[s_1, s_0]$ and $\bar{p}$ is constant over $[s_0, s_2]$; or (ii) $\bar{p}$ is constant over $[s_1, s_2]$. Indeed, if neither of these is true, we can construct $\tilde{p} > \bar{p}$ on A to find a contradiction with (29). Next we argue that that the first case is not possible. Indeed, there would exist $\epsilon > 0$ such that $(s_1 - \epsilon, s_1) \in \mathcal{S}_0 \cup \mathcal{S}_-$. Then, given our previous conclusions, $\bar{p}$ cannot be continuous and satisfy the no-default constraint (5). This rules out case (i).

With these elements in hand, we can characterize the optimal $\bar{p}$.

First, we show that there cannot be $(s_1, s_3) \in \mathcal{S}$ such that $\exists s_2 \in (s_1, s_3)$ with $(s_1, s_2) \in \mathcal{S}_0$ and $(s_2, s_3) \in \mathcal{S}_+$. Indeed, we know that $\bar{p}(s) - s$ is constant over (s_1, s_2) and $\bar{p}$ is constant over (s_2, s_3). Then, we can construct $\tilde{p}$ such that $\tilde{p}(s) = \bar{p}(s)$ for $s < s_1$ and $s > s_3$ and $\tilde{p}(s) = \bar{p}(s_1) + (s - s_1)/(s_3 - s_1)$ for $s \in (s_1, s_3)$.

Second we show that there cannot be $A = (s_1, s_2) \in \mathcal{S}_-$. Let otherwise A be such an interval with $s_1 = \min\{s \in \mathcal{S}_-\} > \underline{s}$. Then $\bar{p}$ is constant over A equal to $\bar{p}(s_1)$ and $s \leq s_1$ implies that $s \in \mathcal{S}_0 \cup \mathcal{S}_+$. If $\bar{p}(s_1) > 0$, it is easy to find a deviation with $\tilde{p} < \bar{p}$. But if $\bar{p}(s_1) = 0$, we have a contradiction, since then $A \in \mathcal{S}_0 \cup \mathcal{S}_+$.

Then, we can argue that the optimal profile for $\bar{p}$ in such an equilibrium must have the following form:

$$\bar{p}(s) = \begin{cases} 0 & s \leq s_* \\ s - s_* & s > s_* \end{cases}$$

Now we can use condition (26) to show that agents would like to deviate. Indeed, it writes

$$E[\tilde{p}(s)(u'(c_2^2(s)) - \delta)] \leq 0$$

Obviously, this inequality would not hold for any $\tilde{p}$ with strictly positive values. In particular, it does not hold for the candidate optimal $\bar{p}$. QED.

The optimal price schedule balances the two desirable features of a repo over a combination of spot sales: the higher borrowing capacity and the possibility of hedging. Consider indeed the second case when $s^* \in [\underline{s}, \bar{s}]$. When $s \leq s^*$, the asset payoff is low. Thus, at the optimal price-schedule, the no-default constraint (5) binds for agents to maximize the consumption transfer across time. When $s \geq s^*$, the asset payoff is high enough to achieve the first best levels of consumption $(c_{2,*}^1, c_{2,*}^2)$. The price schedule becomes constant as agent 2 dislikes variability in consumption.

The optimal price schedule is essentially unique as all optimal $\bar{p}$ finance the same allocation. This multiplicity materializes when the asset is abundant, that is $s^* \leq \underline{s}$. First, the constant price schedule can take any value $p^* \in [s^*/(1-\theta), \underline{s}/(1-\theta)]$. Indeed, the optimality condition only pins down the level of borrowing, that is the product $p^* a_F^2 = c_{2,*}^2 - \omega$ but not the repurchase price p^*. An agent may thus use less of the asset in the repo if he promises a higher repurchase price. Second and more subtly, agent 2 may also be the borrower in the repo while he is ultimately "lending" overall. Suppose indeed that $s^* \leq \underline{s}(1-\theta)$ and let $p^* = s^*/(1-\theta)$. Then any promise by agent 1 to repay p^* could be replicated by (i) a spot sale by agent 1 to agent 2 (ii) a repo sale from agent 2 to agent 1 with schedule $p(s) = s - p^*$. Combining these two operations, agent 2 ultimately transfers p^* units of consumption good per unit of asset to period 2.

C Proof of Proposition 4

Proof. To show that the liquidity premium that the safer asset is higher than the one of the riskier asset, we take the difference in the liquidity premium. The liquidity premium for asset $i = A, B$ is

$$\mathcal{L}_i(\alpha) = \delta \int_{\underline{s}}^{s^{**}} \frac{\rho_i(s)}{1-\theta} \left[u' \left(\omega + \frac{a\rho_{\alpha_A}(s) + b\rho_{\alpha_B}(s)}{1-\theta} \right) - \delta \right] dF(s)$$

and so

$$
\begin{aligned}
\mathcal{L}_{a,b}(\alpha) &= \mathcal{L}_a(\alpha) - \mathcal{L}_b(\alpha) \\
&= \delta \int_{\underline{s}}^{s^{**}} \frac{\rho_{\alpha_a}(s) - \rho_{\alpha_b}(s)}{1-\theta} \left[u'\left(\omega + \frac{a\rho_{\alpha_A}(s) + b\rho_{\alpha_B}(s)}{1-\theta}\right) - \delta \right] dF(s) \\
&= \frac{\delta(\alpha_a - \alpha_b)}{1-\theta} \int_{\underline{s}}^{s^{*}} (s - \mathbb{E}[s]) \left[u'\left(\omega + \frac{a\rho_{\alpha_A}(s) + b\rho_{\alpha_B}(s)}{1-\theta}\right) - \delta \right] dF(s) \\
&> 0
\end{aligned}
$$

where the inequality follows from the fact that $\alpha_a < \alpha_b$, and the integral is negative over the integration range.[27]

The haircut as a function of α then is (using linearity of ρ_i in terms of s):

$$
\begin{aligned}
\mathcal{H}_i(\alpha) &= p_{1,i} - p_{F,i} \\
&= \delta \mathbb{E}\left[\rho_i(s) - \bar{p}_i(s)\right] \\
\frac{\mathcal{H}_i(\alpha)}{\delta} &= \mathbb{E}[s] - \int_{\underline{s}}^{s^{**}} \frac{\rho_i(s)}{1-\theta} dF(s) - \int_{s^{**}}^{\bar{s}} \frac{\rho_i(s^{**})}{1-\theta} dF(s) \\
&= \mathbb{E}[s] - \int_{\underline{s}}^{s^{**}} \frac{(1+\alpha_i)s - \alpha_i \mu}{1-\theta} dF(s) - \int_{s^{**}}^{\bar{s}} \frac{(1+\alpha_i)s^{**} - \alpha_i \mu}{1-\theta} dF(s) \\
&= \mathbb{E}[s] + \frac{\alpha_i \mu}{1-\theta} - \frac{(1+\alpha_i)}{1-\theta} \left[\int_{\underline{s}}^{s^{**}} s dF(s) + \int_{s^{**}}^{\bar{s}} s^{**} dF(s) \right]
\end{aligned}
$$

The term in brackets is less than $\mathbb{E}[s]$ therefore, for all assets A and B such that $\alpha_A < \alpha_B$ we obtain

$$
\mathcal{H}_A(\alpha_A, \alpha_B) < \mathcal{H}_B(\alpha_A, \alpha_B)
$$

i.e. the safe asset always commands a lower haircut than the risky asset.

□

[27] Notice that we can expand this analysis for many more assets where s^{**} would then be given by

$$
u'\left(\omega + \frac{\sum_{i=1}^{n} \rho_i(s^{**}) a_i}{1-\theta}\right) = \delta
$$

where a_i would be the number of asset of type i pledged as collateral. The liquidity premium could be defined in the same way.

D Mean Preserving Spread

Obviously, a higher $\bar{s}$ means that the asset is riskier. Given the distribution for s we obtain

$$
\begin{aligned}
\mathbb{E}[\bar{p}(s)] &= \frac{1}{\bar{s}-\underline{s}}\left[\int_{\underline{s}}^{s^*} \frac{s}{1-\theta} ds + \int_{s^*}^{\bar{s}} \frac{s^*}{1-\theta} ds\right] \\
&= \frac{1}{2(\bar{s}-1)(1-\theta)}\left[2(\bar{s}-1) - \frac{1}{2}\left(\bar{s}-s^*\right)^2\right] \\
&= \frac{1}{1-\theta} - \frac{\left(\bar{s}-s^*\right)^2}{4(\bar{s}-1)(1-\theta)}
\end{aligned}
$$

Then the haircut $\mathcal{H}$ is always increasing in $\bar{s}$. First, notice that when $s^* \geq \bar{s}$, $\mathcal{H}$ depends only on $\mathbb{E}[s]$ but not on the distribution of s. Hence the haircut is not affected marginally by a mean-preserving spread. Secondly, when $s^* \leq \underline{s}$, it is clear from the expression above that the the lower bound for admissible haircut $\delta\mathbb{E}[s] - \delta\frac{2-\bar{s}}{1-\theta}$ increases with $\bar{s}$. Finally, when $s^* \in [\underline{s}, \bar{s}]$, we obtain the following expression for the haircut

$$
\begin{aligned}
\mathcal{H} &= \delta(1 - \mathbb{E}[\bar{p}(s)]) \\
&= \frac{\delta\theta}{1-\theta} + \frac{\delta\left(\bar{s}-s^*\right)^2}{4(\bar{s}-1)(1-\theta)}
\end{aligned}
$$

The derivative with respect to $\bar{s}$ is

$$
\frac{\partial \mathcal{H}}{\partial \bar{s}} = \frac{\bar{s}-s^*}{4(\bar{s}-1)^2(1-\theta)}\left[2(\bar{s}-1) - (\bar{s}-s^*)\right] = \frac{(\bar{s}-s^*)(s^*-\underline{s})}{4(\bar{s}-1)^2(1-\theta)} \geq 0
$$

Intuitively, with a mean-preserving spread, the collateral quality deteriorates and hence commands higher haircut to compensate the lender. Haircut increases with asset riskiness only because of the hedging motive of the lender.

The impact of an increase in risk on the liquidity premium $\mathcal{L}$ is however not clear. Indeed, in the case where $s^* \in [\underline{s}, \bar{s}]$, we have

$$
\mathcal{L} = \frac{1}{2(\bar{s}-1)(1-\theta)} \int_{\underline{s}}^{s^*} s\left(u'(\omega + \frac{s}{1-\theta}a) - \delta\right) ds
$$

Hence, we obtain

$$
\frac{\partial \mathcal{L}}{\partial \bar{s}} = \frac{1}{2(\bar{s}-1)(1-\theta)}\left[-\frac{1}{\bar{s}-1}\int_{\underline{s}}^{s^*} s\left(u'(\omega + \frac{s}{1-\theta}a) - \delta\right) ds + \underline{s}\left(u'(\omega + \frac{\underline{s}}{1-\theta}a) - \delta\right)\right]
$$

and the sign of this expression depends in particular on the coefficient of relative risk aversion of u.

Another variable of interest is the (gross) repo rate that can be defined as

$$r := \frac{\mathbb{E}[\bar{p}(s)]}{p_F} = \frac{1}{\delta} \frac{-\mathcal{H} + \delta \mathbb{E}[s]}{-\mathcal{H} + \delta \mathbb{E}[s] + \mathcal{L}}$$

Again, the effect of the mean preserving spread on the repo rate are not clear. However, numerical computations seem to indicate that the rate increase with riskiness.

Actually, we do not need to make distributional assumptions on s to get the result. As before, $s \sim F[\underline{s}, \bar{s}]$ but the asset payoff is now:

$$\rho_\alpha(s) = s + \alpha(s - \mathbb{E}[s]), \qquad \alpha \geq -1$$

When α=-1, the asset is perfectly safe and pays off $\mathbb{E}[s]$. An asset with a higher α has the same mean but a higher variance than an asset with a lower α. Indeed $Var[\rho_\alpha] = (1+\alpha)Var[s]$.

Let us now call $s^*(\alpha)$ the threshold such that $\rho_\alpha(s^*(\alpha)) = s^*$. Exactly as before, the optimal repo contract for asset α is

$$\bar{p}_\alpha(s) = \begin{cases} \frac{\rho_\alpha(s)}{1-\theta} & \text{for } s \leq s^*(\alpha) \\ \frac{\rho_\alpha(s^*)}{1-\theta} & \text{for } s \geq s^*(\alpha) \end{cases}$$

Remember that $\mathcal{H}_\alpha = \delta(\mathbb{E}[\rho_\alpha(s)] - \mathbb{E}[\bar{p}_\alpha(s)]) = \delta E[s] - \delta \mathbb{E}[\bar{p}_\alpha(s)]$. We have (assuming that $s^*(\alpha) \in [\underline{s}, \bar{s}]$)

$$\mathbb{E}[\bar{p}_\alpha(s)] = \frac{1}{1-\theta}\left[\int_{\underline{s}}^{s^*(\alpha)} \rho_\alpha(s) dF(s) + \rho_\alpha(s^*(\alpha))\left(1 - F\left(s^*(\alpha)\right)\right)\right]$$

Hence, we obtain

$$\begin{aligned} \frac{\partial \mathcal{H}_\alpha}{\partial \alpha} &= -\frac{\delta}{1-\theta}\left[\int_{\underline{s}}^{s^*(\alpha)} (s - \mathbb{E}[s]) dF(s) + \frac{\partial s^*(\alpha)}{\partial \alpha} \rho_\alpha\left(s^*(\alpha)\right) f\left(s^*(\alpha)\right) - \rho_\alpha\left(s^*(\alpha)\right) \frac{\partial s^*(\alpha)}{\partial \alpha} f\left(s^*(\alpha)\right)\right] \\ &= -\frac{\delta}{1-\theta}\int_{\underline{s}}^{s^*(\alpha)} (s - \mathbb{E}[s]) dF(s) \geq 0 \end{aligned}$$

Here the asset risk is also the fundamental risk in the economy. With two assets (see below), we can compare haircuts for assets with different risk while keeping the economy unchanged. Actually this result together with that for two assets suggests that when fundamental risk increases, the difference in haircuts between risky and safe assets increase as well (a bit mechanical maybe).

E Proof of Proposition 5

Proof. If agents are constrained, the collateral constraint binds. We thus have

$$a_1^1 = b^{12} \tag{30}$$
$$a_1^2 = -\nu \ell^{21} \tag{31}$$

Even with re-use, there is a constraint on the amount of physical asset available. Precisely we have $a_1^1 + a_1^2 = a$. Also market clearing requires $b^{12} = \ell^{21}$. Summing (30) and (31) we obtain

$$a_1^1 = \frac{a}{1-\nu} = b^{12}$$
$$a_1^2 = -\frac{\nu}{1-\nu}a = -\nu b^{12}$$

Using agent 2's budget constraint in period 2, we can write his consumption as

$$c_2^2(s) = \omega + \frac{a}{1-\nu}(\bar{p}(s,\nu) - \nu s)$$

Notice that while $\bar{p}(s,\nu)$ is of the general form (13), insurance requires that $\bar{p}(s,\nu)$ corrects for the cost agent 2 incurs from purchasing the asset to cover his short position in period 2. So following our earlier argument $\bar{p}(s,\nu)$ is

$$\bar{p}(s,\nu) = \begin{cases} \frac{s}{1-\theta} & \text{if } s < s^*(\nu) \\ \frac{s^*(\nu)}{1-\theta} + \nu(s - s^*(\nu)) & \text{if } s \geq s^*(\nu) \end{cases}$$

where we can define $s^*(\nu)$ implicitly as follows

$$u'\left(\omega + \frac{as^*(\nu)}{1-\nu}\left[\frac{1}{1-\theta} - \nu\right]\right) = \delta.$$

It should be clear that $s^*(\nu)$ is decreasing in ν, thus a higher ν makes it easier to reach the first best. Notice that

$$s^*(\nu) = \frac{(1-\nu)}{1-\nu(1-\theta)}s^*$$

where $s^* = s^*(0)$ defined in (12). In addition, contract $\bar{p}(s,\nu)$ satisfies the incentive constraint of the lender for any ν. Therefore, the first best is achieved for any $\nu > \nu^*$ where ν^* is given by

$$\nu^* = \frac{s^* - \underline{s}}{s^* - (1-\theta)\underline{s}}$$

Finally, notice that re-use is essential only when $\theta > 0$.[28] Re-use has a collateral multiplier effect. Indeed, allowing agent 2 to re-use has the same effect as increasing the asset quantity a by a factor $s^*/s^*(\nu)$.

□

F Proof of Proposition 6

Proof. The problem of agent L writes :

$$\max_{a_1^L, b^{HL}} \omega + p_1(a - a_1^L) + p_{F,LH} b^{LH} + \delta_L E\left[2\omega + s a_1^L - \bar{p}_{LH}(s) b^{LH}\right]$$

$$\begin{aligned} \text{s.to} \quad a_1^L \geq & \; b^{LH} \quad (\gamma_1^L) \\ b^{LH} \geq & \; 0 \quad (\xi_{LH}) \end{aligned}$$

While the problem of agent H is:

$$\begin{aligned} & \max_{a_H^1, l^{HL}, b^{H2}} \omega - p_1 a_H^1 - p_{F,LH} \ell^{HL} + p_{F,H2} b^{H2} \\ & + \delta_H E\left[2\omega + s a_H^1 + \bar{p}_{LH}(s) \ell^{HL} - \bar{p}_{H2}(s) b^{H2}\right] \\ & \text{s.to} \quad a_1^H + \nu_H \ell^{HL} \geq b^{H2} \quad (\gamma_1^H) \\ & \qquad \ell^{HL} \geq 0 \quad (\xi_{HL}) \\ & \qquad b^{H2} \geq 0 \quad (\xi_{H2}) \end{aligned}$$

Recall that a_1^H is the spot market trade of agent H. The variable ℓ^{HL} is the amount agent H lends to L and every unit of loans yields agent H a fraction ν_H of re-usable asset. These units can be re-sold spot, which decreases a_1^H, or re-pledged to agent 2, which increases b^{H2}. Finally, agent 2 solves

$$\max_{a_1^L, l^{2H}} \omega - p_1 a_1^2 - p_{F,H2} \ell^{2H} + E\left[u\left(\omega + s a_1^2 + \bar{p}_{H2}(s) \ell^{2H}\right)\right]$$

$$\begin{aligned} \text{s.t.} \quad a_1^2 + \nu_2 \ell^{2H} & \geq 0 \quad (\gamma_1^2) \\ \ell^{2H} & \geq 0 \quad (\xi_{2H}) \end{aligned}$$

where ℓ^{2H} is the loan of agent 2 to agent H.

[28] Maurin (2015) shows that re-use is redundant in the no commitment case $\theta = 0$.

Let us now write down the first order conditions for our 3 agents:

$$
\begin{aligned}
-p_1 + \delta_H E[s] + \gamma_1^H &= 0 && (32)\\
-p_{F,LH} + \delta_H E[\bar{p}_{LH}(s)] + \nu_H \gamma_1^H &= 0 && (33)\\
+p_{F,H2} - \delta_H E\left[\bar{p}_{H2}(s)\right] - \gamma_1^H &= 0 && (34)\\
-p_1 + \delta_L E[s] + \gamma_1^L &= 0 && (35)\\
p_{F,LH} - \delta_L E[\bar{p}_{LH}(s)] - \gamma_1^L &= 0 && (36)\\
-p_1 + E\left[su'(c_2^2(s))\right] + \gamma_1^2 &= 0 && (37)\\
-p_{F,H2} + E\left[\bar{p}_{H2}(s)u'(c_2^2(s))\right] + \nu_2\gamma_1^2 &= 0 && (38)
\end{aligned}
$$

Market clearing implies that $b^{ij} = \ell^{ji}$ for each pair of agents (i,j). Hence, we only use the notation b in the following. A quick examination shows that all three collateral constraints must bind, that is $\gamma_1^L > 0$, $\gamma_1^H > 0$ and $\gamma_1^2 > 0$. This implies that:

$$
\begin{gathered}
a_1^L = b^{LH}\\
a_1^H + \nu_H b^{HL} = b^{H2}\\
a_1^2 + \nu_2 b^{H2} = 0
\end{gathered}
$$

while market clearing for the asset implies:

$$a_1^L + a_1^H + a_1^2 = a$$

Agent 2 consumption in period 2 is:

$$
\begin{aligned}
c_2^2(s) &= \omega + a_1^2 s + b^{H2}\bar{p}_{H2}(s)\\
&= \omega + b^{H2}(\bar{p}_{H2}(s) - v_2 s)
\end{aligned}
$$

Hence for given repurchase contracts $F_{LH}$and F_{H2}, these equations pin down the allocation up to the amount b^{H2} that agent H borrows from agent 2.

We solve for period 1 prices and Lagrange multipliers. Equations (32) and (35) give

$$\gamma_1^L = (\delta_H - \delta_L)\, E[s] + \gamma_1^H$$

Also from (33) and (36) we obtain

$$(1-\nu_H)\gamma_1^H = (\delta_H - \delta_L) E[\bar{p}_{LH}(s) - s]$$

Hence these two equations pin down γ_1^L and γ_1^H. From equations (32), (34), (37), and 38 we obtain

$$\gamma_1^2 = \frac{1}{(1-\nu_2)} E\left[(\bar{p}_{H2}(s) - s)\left(u'(c_2^2(s)) - \delta_H\right)\right]$$
$$\gamma_1^H = \frac{1}{1-\nu_2} E\left[\bar{p}_{H2}(s)\left(u'(c_2^2(s)) - \delta_H\right)\right] - \frac{\nu_2}{1-\nu_2} E\left[s\left(u'(c_2^2(s)) - \delta_H\right)\right]$$

This last equation pins down b^{H2} the amount agent H borrows from 2. The prices naturally obtain from the expression of the Lagrange multipliers.

We now derive the equilibrium contracts. In particular, since agents are free to trade any feasible contract, we derive conditions for F_{LH} and F_{H2} to be the contracts the agents select in equilibrium. First we consider the **equilibrium contracts between agents L and H.**

Agents L and H are not willing to trade contract $\tilde{F}_{LH}$ if and only if

$$\delta_L E[\tilde{p}_{LH}(s)] + \gamma_1^L \geq \delta_H E[\tilde{p}_{LH}(s)] + \nu_H \gamma_1^H$$

Agent L marginal willingness to sell this contract exceeds agent H marginal willingness to buy it. Using the expressions we derived for the Lagrange multipliers, we obtain:

$$(\delta_H - \delta_L) E\left[\bar{p}_{LH}(s) - \tilde{p}_{LH}(s)\right] \geq 0$$

which may only hold if

$$\bar{p}_{LH}(s) = \frac{s}{1-\theta_L}.$$

We can use this expression to derive the values of the Lagrange multipliers:

$$\gamma_1^L = \frac{(\delta_H - \delta_L)(1 - \nu_H(1-\theta_L))}{(1-\nu_H)(1-\theta_L)} \tag{39}$$
$$\gamma_1^H = \frac{(\delta_H - \delta_L)\theta_L}{(1-\nu_H)(1-\theta_L)} \tag{40}$$

We now turn to the **equilibrium contracts between agents H and 2.** Agents H and 2 trade as in the previous section. However, now b^{H2} is the (endogenous) amount agent H uses to borrow from agent 2 (instead of a). Notice that agent 2 also re-sells the collateral agent H pledges. For a given amount of asset b^{H2}, let us define $s^*(b^{H2}, \theta_H, \nu_2)$ as the solution in s to:

$$u'\left(\omega + \frac{b^{H2}s}{1-\nu_2}\left[\frac{1}{1-\theta_H} - \nu_2\right]\right) = \delta_H.$$

Then the contract between agents H and 2 takes the following form:

$$\bar{p}_{H2}(s, \nu_2) = \begin{cases} \frac{s}{1-\theta} & \text{if } s < s^*(b^{H2}, \theta_H, \nu_2) \\ \frac{s^*(b^{H2}, \theta_H, \nu_2)}{1-\theta} + \nu_2(s - s^*(b^{H2}, \theta_H, \nu_2)) & \text{if } s \geq s^*(b^{H2}, \theta_H, \nu_2) \end{cases}$$

To sustain intermediation, we need conditions for b^{H2} to be interior. Given the pattern of trades we consider, the bounds on b^{H2} are

$$0 \leq b^{H2} \leq \frac{\nu_H a}{1-\nu_2}$$

The lower bound is the case where agent agent H re-sells all the collateral to agent L and pledges nothing to agent 2. The upper bound is the case where agent H re-pledges all the re-usable collateral to agent 2, which equals to $\nu_H a$ times the re-use multiplier $1/(1-\nu_2)$. Hence, the intermediation equilibrium requires:

$$\frac{1-(1-\theta_H)\nu_2}{(1-\theta_H)(1-\nu_2)} E\left[\left\{u'\left(\omega + \frac{\nu_H a s}{1-\nu_2}\left[\frac{1}{1-\theta_H} - \nu_2\right]\right) - \delta_H\right\} s\right] \leq \gamma_1^H \leq \frac{1-(1-\theta_H)\nu_2}{(1-\theta_H)(1-\nu_2)}(u'(\omega) - \delta_H) \tag{41}$$

The RHS inequality says that the gains from trade between agents H and 2 measured by the difference in marginal utility $u'(\omega) - \delta_H$ must be large enough. Otherwise, agent H only trades with agent L: he receives and sells back the collateral until it is completely segregated. The left inequality says that these gains must not be too large either. Otherwise, it would be too costly to segregate collateral in the trade between agents L and H since the asset would be put to a better use by increasing borrowing from agent 2. When these inequalities hold, the equation implicitly defining b^{H2} is the following:

$$\frac{1-(1-\theta_H)\nu_2}{(1-\theta_H)(1-\nu_2)} \int_{\underline{s}}^{s^*(b^{H2},\theta_H,\nu_2)} \left[u'\left(\omega + \frac{b^{H2}s}{1-\nu_2}\left[\frac{1}{1-\theta_H} - \nu_2\right]\right) - \delta_H\right] \mathrm{s}\mathrm{d}F(s) = \frac{(\delta_H - \delta_L)\theta_L}{(1-\nu_H)(1-\theta_L)} \tag{42}$$

The RHS is agent H's marginal willingness to sell the collateral to agent L and depends on the difference in discount factors, θ_L and the re-usable fraction by agent H. The LHS is the marginal willingness to borrow from agent 2. In equilibrium, these must be equal.

Finally, we consider the **equilibrium contract between agents** L **and** 2. We need to check that intermediation is optimal, that is agents L and 2 do not want to trade directly a contract $\tilde{F}_{L2}$. This requires

$$\delta_L E[\tilde{p}_{L2}(s)] + \gamma_1^L \geq E[\tilde{p}_{L2}(s) u'(c_2^2(s))] + \nu_2 \gamma_1^2$$

where agent 2 marginal willingness to pay also reflects his ability to re-use collateral through the term $\nu_2\gamma_1^2$. We can rewrite the condition as

$$(1-\nu_2)\gamma_1^2 \geq E\left[(\tilde{p}_{L2}(s) - s)\left(u'(c_2^2(s)) - \delta_L\right)\right]$$

This inequality is most binding for $\tilde{p}_{L2}(s) = s/(1-\theta_L)$, and replacing this expression we obtain:

$$E\left[(\bar{p}_{H2}(s) - s)\left(u'(c_2^2(s)) - \delta_H\right)\right] \geq \frac{\theta_L}{1-\theta_L} E\left[s\left(u'(c_2^2(s)) - \delta_L\right)\right]$$

Using the variable $s^*(b^{H2}, \theta_H, \nu_2)$ defined above, we can rewrite the following inequality as :

$$\left(\frac{\theta_H}{1-\theta_H} - \frac{\theta_L}{1-\theta_L}\right) \int_{\underline{s}}^{s^*(b^{H2},\theta_H,\nu_2)} s\left(u'\left(c_2^2(s)\right) - \delta_H\right) \geq \frac{\theta_L(\delta_H - \delta_L)}{1-\theta_L} \tag{43}$$

It should be clear from the inequality above that intermediation is only possible when $\theta_H \geq \theta_L$, that is when agent H offers better guarantees as a borrower than agent L. To interpret it, let us rewrite this constraint as :

$$(1-\nu_H)\gamma_1^H \leq E\left[(\bar{p}_{H2}(s) - \bar{p}_{LH}(s))\left(u'(c_2^2(s)) - \delta_H\right)\right]$$

The LHS measures the cost of intermediation which is proportional to the shadow price of collateral for agent H times the fraction he has to segregate. The RHS measures the benefit from intermediation which is proportional to $\bar{p}_{H2}(s) - \bar{p}_{LH}(s)$, that is, how much more (relative to agent L) agent H can borrow from agent 2 per unit of the asset. In equilibrium the shadow price γ_1^H is proportional to $(\delta_H - \delta_L)$, a measure of the gains from trade between agents L and H.

Finally, we check that **consumption remains positive.** For agent L, we need to check that $c_2^L(\bar{s}) \geq 0$, this writes:

$$\omega - \frac{\theta_L}{1-\theta_L} b^{LH} \bar{s} \geq 0$$

A sufficient condition obtains when replacing b^{LH} by $a/(1-\nu_H)$ in the expression above. Agent 2's consumption is positive in period 1 whenever

$$c_1^2 = \omega - (p_{F,H2} - \nu_2 p_1) b^{H2} \geq 0$$

or

$$\omega \geq b^{H2} E\left[(\bar{p}_{H2}(s) - \nu_2 s)\, u'(c_2^2(s))\right]$$
$$\omega \geq \frac{1-\nu_2(1-\theta_H)}{1-\theta_H} b^{H2} \left[\int_{\underline{s}}^{s^*(b^{H2},\theta_H,\nu_2)} s u'(c_2^2(s)) \mathrm{d}F(s) + \delta_H s^*(b^{H2},\theta_H,\nu_2)\left\{1 - F(s^*(b^{H2},\theta_H,\nu_2))\right\}\right]$$

Turning to agent H, we must check that $c_1^H \geq 0$, that is:

$$\omega - p_1 a_1^H - p_{F,LH} b^{LH} + p_{F,H2} b^{H2} \geq 0$$
$$\omega + (p_{F,H2} - p_1) b^{H2} - (p_{F,LH} - \nu_H p_1) b^{LH} \geq 0$$
$$\omega + \delta_H \frac{\theta_H b^{H2}}{1-\theta_H} \int_{\underline{s}}^{s^*(b^{H2},\theta_H,\nu_2)} s \mathrm{d}F(s) - \delta_H \frac{1-\nu_H(1-\theta_L)}{1-\theta_L} b^{LH} \geq 0$$

A sufficient condition for this to hold is then,

$$\omega \geq \delta_H \frac{[1-\nu_H(1-\theta_L)]\, a}{(1-\theta_L)(1-\nu_H)}$$

Which can never hold when $\nu_H \to 1$. So, in general, we need to have ω big enough.

To recapitulate, let $s^*(a,\theta,\nu)$ be defined implicitly by :

$$u'\left(\omega + \frac{as^*(a,\theta,\nu)}{1-\nu}\left[\frac{1}{1-\theta} - \nu\right]\right) = \delta_H.$$

and b^{H2} defined by :

$$\frac{1-(1-\theta_H)\nu_2}{(1-\theta_H)(1-\nu_2)} \int_{\underline{s}}^{s^*(b^{H2},\theta_H,\nu_2)} \left[u'\left(\omega + \frac{b^{H2}s}{1-\nu_2}\left[\frac{1}{1-\theta_H} - \nu_2\right]\right) - \delta_H\right] s\mathrm{d}F(s) = \frac{(\delta_H - \delta_L)\theta_L}{(1-\nu_H)(1-\theta_L)}$$

We obtain an equilibrium with intermediation iff

$$b^{H2} \in \left[0, \frac{\nu_H a}{1-\nu_2}\right]$$
$$\left(\frac{\theta_H}{1-\theta_H} - \frac{\theta_L}{1-\theta_L}\right) \int_{\underline{s}}^{s^*(b^{H2},\theta_H,\nu_2)} s\left(u'\left(c_2^2(s)\right) - \delta_H\right) \geq \frac{\theta_L(\delta_H - \delta_L)}{1-\theta_L}$$

$\square$

References

Aghion, P. and P. Bolton (1992). An incomplete contracts approach to financial contracting. *The Review of Economic Studies 59*(3), 473–494. 3

Andolfatto, D., F. Martin, and S. Zhang (2014). Rehypothecation. Technical report. 1

Antinolfi, G., F. Carapella, C. Kahn, A. Martin, D. Mills, and E. Nosal (2015). Repos, fire sales, and bankruptcy policy. *Review of Economic Dynamics 18*(1), 21 – 31. Money, Credit, and Financial Frictions. 1

Araújo, A., J. Orrillo, and M. R. Páscoa (2000). Equilibrium with default and endogenous collateral. *Mathematical Finance 10*(1), 1–21. 1

BIS (1999). Implications of repo markets for central banks. Cgfs publication. 1

Bottazzi, J.-M., J. Luque, and M. R. Páscoa (2012). Securities market theory: Possession, repo and rehypothecation. *Journal of Economic Theory 147*(2), 477–500. 1

Comotto, A. (2014). Repo, re-use and re-hypothecation. Technical report, ICMA. 2

Copeland, A., A. Martin, and M. Walker (2014). Repo runs: Evidence from the tri-party repo market. *The Journal of Finance 69*(6), 2343–2380. 1

Dubey, P., J. Geanakoplos, and M. Shubik (2005). Default and punishment in general equilibrium1. *Econometrica 73*(1), 1–37. 1

Duffie, D. (1996). Special repo rates. *The Journal of Finance 51*(2), 493–526. 6, 19

Farboodi, M. (2015). Intermediation and voluntary exposure to counterparty risk. Technical report. 1

Federal Reserve Bank of New York (2010). Tri-party repo infrastructure reform. White paper. 1

Fleming, M. J. and K. D. Garbade (2005). Explaining settlement fails. *Current Issues in Economics and Finance 11*(Sep). 4

Fuhrer, L. M., B. Guggenheim, and S. Schumacher (2015). Re-use of collateral in the repo market. Working Papers 2015-02, Swiss National Bank. 7, 4.1

Geanakoplos, J. (1996). Promises Promises. Cowles Foundation Discussion Papers 1143, Cowles Foundation for Research in Economics, Yale University. 1, 2.1

Geanakoplos, J. and W. Zame (2014). Collateral equilibrium, I: a basic framework. *Economic Theory 56*(3), 443–492. 1

Gorton, G. and A. Metrick (2012). Securitized banking and the run on repo. *Journal of Financial Economics 104*(3), 425 – 451. Market Institutions, Financial Market Risks and Financial Crisis. 1

Gottardi, P. and F. Kubler (2015). Dynamic competitive economies with complete markets and collateral constraints. *The Review of Economic Studies 82*(3), 1119–1153. 1

ICMA (2013). Frequently Asked Questions on repo. Technical report, International Capital Market Association. 19

Infante, S. (2013). Repo collateral fire sales: the effects of exemption from automatic stay. Technical report. 1

Infante, S. (2015). Liquidity Windfalls: The Consequences of Repo Rehypothecation. Finance and Economics Discussion Series 2015-22, Board of Governors of the Federal Reserve System (U.S.). 1

Krishnamurty, A., S. Nagel, and D. Orlov (2014). Sizing up repo. *The Journal of Finance 69*(6), 2381–2417. 1

Martin, A., D. Skeie, and E.-L. v. Thadden (2014). Repo runs. *Review of Financial Studies 27*(4), 957–989. 1

Maurin, V. (2015). Re-using collateral, a general equilibrium model of rehypothecation. Technical report, European University Institute. 1

Monnet, C. (2011). Rehypothecation. *Business Review* (Q4), 18–25. 1

Muley, A. (2015). Collateral reuse in shadow banking and monetary policy. Technical report, MIT. 1

Narajabad, B. N. and C. Monnet (2012). Why Rent When You Can Buy? A Theory of Repurchase Agreements. Technical report. 1

Parlatore, C. (2015). Collateralizing liquidity. Working paper, New York University, Stern School of Business. 1, 5

Singh, M. (2011). Velocity of pledged collateral: Analysis and implications. IMF Working Papers 11/256, International Monetary Fund. 3, 1

Singh, M. and J. Aitken (2010). The (sizable) role of rehypothecation in the shadow banking system. IMF Working Papers 10/172, International Monetary Fund. 1, 4.1

Tomura, H. (2013). Investment Horizon and Repo in the Over-the-Counter Market. UTokyo Price Project Working Paper Series 013, University of Tokyo, Graduate School of Economics. 1

Vayanos, D. and P.-O. Weill (2008). A search-based theory of the on-the-run phenomenon. *Journal of Finance 63*(3), 1361–1398. 19

Zhang, S. (2014). Collateral risk, repo rollover and shadow banking. Working paper series, London School of Economics. 1