A Stock-Flow Theory of Unemployment with Endogenous Match Formation*

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Abstract

We develop a tractable equilibrium model of stock-flow matching in the labor market. The economy consists of many labor markets, and workers and jobs continually flow into each labor market. Within a labor market, potential worker-job matches differ in quality. Accordingly, a worker newly arrived in a labor market may not find any acceptable matches; if so, she becomes part of the stock of unemployed workers and must wait for the arrival of a new vacancy in the flow which offers her a sufficiently high quality match. When labor market conditions change, the set of acceptable matches changes in response, a feature not allowed for in previous work on stock-flow matching. Our model is consistent with several stylized facts about the labor market, such as the importance of flows, as well as stocks, for matching rates, as well as with duration dependence in unemployment.

Keywords: Job search, unemployment, stock-flow matching, matching function, duration dependence.

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1 Introduction

Why does a worker remain unemployed when jobs remain vacant in the same labor market? The explanation offered by the stock-flow matching paradigm (Taylor, 1995; Coles and Muthoo, 1998; Coles and Smith, 1998; Ebrahimy and Shimer, 2010) is that the quality of the worker's potential match with any available job is not acceptable. The worker chooses to remain unmatched in order to wait for the arrival of better opportunities in the inflow of new vacancies into the market. This approach is attractive since it accounts naturally for two striking facts: first, unemployment exit rates depend on the flows of newly unemployed workers and jobs as well as on the stocks of existing unemployed workers and vacancies (Coles and Smith, 1998; Burdett and Cunningham, 1999; Coles and Petrongolo, 2008), and second, the hazard rate at which an unemployed worker finds a job falls with the duration of her unemployment spell (Clark and Summers, 1979; Machin and Manning, 1999; Imbens and Lynch, 2006; Kroft, Lange, and Notowidigdo, 2013). Stock-flow matching can account for these even when workers and jobs are ex ante homogeneous: a newly unemployed worker may quickly find a suitable match with a pre-existing vacancy, but if this does not happen, the hazard rate at which the worker becomes employed is lower since she must wait for a suitable vacancy to become available.

Despite the intuitive and empirical appeal of the stock-flow paradigm, its use in studying applied macroeconomic questions has been limited. One reason for this is that in existing models of stock-flow matching the set of acceptable job matches is an exogenous object. This stylized notion of match quality heterogeneity represents an important limitation for two reasons. First, it limits the ability of the model to provide a quantitatively reasonable account of how the durations of workers' unemployment spells vary with labor market conditions. Second, and more fundamentally, it rules out by assumption that changes in labor market conditions or policies might alter the set of acceptable matches. This means that, for example, the model cannot readily explain differences in matching efficiency over the business cycle or across countries.

In this paper, therefore, we enrich the stock-flow paradigm to allow for rich heterogeneity in match quality. Our paper makes three main contributions. First, we propose an equilibrium stock-flow matching model which allows for endogenous match formation and labor market participation. The tractability of our model makes it suitable for investigating applied macroeconomic questions. Second, we show that allowing for the set of acceptable matches to vary with labor market conditions is important in accounting for the changing shape of the unemployment duration distribution in good and bad labor market conditions. Our model can explain a significant fraction of the rise of long-term unemployment between the low-unemployment boom years preceding the Great Recession and the high-unemployment years following. Third, we highlight a novel source of inefficiency in the labor market. While an appropriate wage determination protocol can ensure that the set of matches created is efficient conditional on the sets of firms and workers already present in the market, in particular if a newly unmatched vacancy conducts a second-price auction with an appropriate reserve price to choose the worker who will match with it, this protocol does not give firms the correct incentives to create new jobs in the first place.

In more detail, the economy we study consists of many distinct labor markets, which we refer to as islands. Unemployed workers and vacant jobs located on the same island match according to a stock-flow matching process. When a new job is created, it may match with unemployed workers waiting for a job; similarly, when a new unemployed worker arrives on the island, she may match with the stock of vacancies. Different worker-job matches are of different qualities, and not all potential matches are sufficiently productive to be worth forming. This heterogeneity arises both because different workers can perform different sets of jobs, and also because workers' productivities vary across the jobs they can perform. If one interprets islands as industries, for example, the first type of heterogeneity arises due to skill or geographical location mismatch. The second type of heterogeneity arises due to differences in match-specific productivity. While in our baseline model, all the islands in the economy are homogeneous, we can easily allow for islands to differ, in that they offer workers different sets of jobs and different match productivity distributions.[1] Worker mobility across islands is due to idiosyncratic reasons and is modelled as an exogenous reallocation process that occurs from time to time.

A worker who arrives in an island and does not find a suitable match (either because she cannot perform any of the available jobs or because she rejects available offers in order to wait for a better employment opportunity) is no longer part of the flow of new unemployed workers, and instead becomes part of the stock. Once in the stock, such a worker exits unemployment only when a new job arrives on the island, and then only if she can perform the job and if she is able to generate output in that match that both exceeds what can be produced by other applicants, and also exceeds a cutoff value. Similarly, a newly arrived vacancy becomes part of the stock if it is not immediately filled; if so, it must wait for new arrival unemployed workers to match with.

Our explanation for why unemployment and vacant jobs coexist is quite different from the one proposed by the Diamond-Mortensen-Pissarides (DMP) framework (Diamond, 1982; Mortensen, 1982; Pissarides, 1985). In the DMP framework, even though productive matches for an unemployed worker may exist, an exogenously imposed black-box matching friction (perhaps intended to model difficulty in finding out about the identity or location of potential trading partners) prevents immediate pairing. The worker is unemployed not because no suitable matches are available to her, but because she has been unable to contact any. That this is the key friction in the labor market seems somewhat implausible. If it is, it is hard to account not only for the small amount of time spent by the unemployed on job search (Krueger and Mueller, 2012), but also for the fact that the drastic fall in the cost of gathering information about potential matches induced by the rise of internet-based job search in the last 15 years has not been associated with a corresponding reduction in unemployment and vacancy rates.

Moreover, the reduced-form matching function approach is a shaky foundation for our theories of employment and unemployment for two further reasons. First, as Petrongolo and Pissarides (2001)

[1] This is an important difference from the model of Shimer (2007), in which no island has both unemployed workers and vacant jobs, so that if all islands were identical, either the aggregate unemployment or vacancy rate would necessarily equal zero. In our model unemployed workers and vacant jobs coexist on each island, and the unemployment and vacancy rates are both strictly positive even when all islands are identical.

note in their survey of the matching function: '. . . its usefulness depends on its empirical viability and on how successful it is in capturing the key implications of the heterogeneities and frictions in macro models' (p. 392). But in reality, this approach has numerous empirical shortcomings. It cannot explain why the flows of unemployed workers and vacancies should matter for matching rates, in addition to the stocks. A version of the DMP model without heterogeneity cannot account for duration dependence in unemployment, for the rise of long-term unemployment in recessions (Elsby, Hobijn, and Şahin, 2010; Pissarides, 2013; Wiczer, 2013), or for shifts in the Beveridge curve or for changes in the measured efficiency of the matching function (Elsby, Hobijn, and Şahin, 2010). While aggregation across heterogeneous workers and labor markets does allow the DMP model some ability to account for these facts, the explanations are at best partial (Wiczer, 2013; Şahin et al., 2014), so that researchers who use the DMP model to investigate questions for which the unemployment duration distribution are important often need to impose duration dependence in an ad hoc way (for example, Kroft et al., 2014).

Further, even if standard matching functions did a better job of describing the process of matching between unemployed workers and vacant jobs in at least some dimensions, the black-box nature of the matching function in standing in for a more microfounded account of 'heterogeneities and frictions' would still be troubling. The Lucas critique should give us caution about deriving policy conclusions when the matching function is treated as an invariant structural object. There is no way of telling whether the matching function will remain invariant if the economic environment changes, either due to cyclical or policy changes. Indeed, the fact that matching efficiency seems to have declined in the most recent recession suggests that assuming it is invariant may more generally have misleading policy and welfare implications. The current paper is a first step in building a microfounded model of matching which may be more immune to such critiques.

Finally, it is instructive to compare our approach with that of models based on mismatch. In models such as those of Lagos (2000) and Shimer (2007), unlike in the current model, unemployed workers and vacant jobs do not coexist within the same narrowly defined labor market: for example, in Shimer's model, only agents in the long side of the local labor market are unmatched.[2] In our model, as well as in earlier work in the stock-flow paradigm, on the other hand, unemployed workers coexist with vacancies since the quality of the potential match that can be formed is inadequate. That is, unemployment in our model is more reminiscent of Jovanovic's (1987) concept of "wait" (or "rest") unemployment: a worker who does not match while in the flow prefers to wait in the stock for better job opportunities to arrive in the future, rather than accept any match available now. While the difference between these approaches can be subtle (it hinges, in essence, on how large a labor market is), we believe that our approach is more naturally suited to studying standard labor market datasets in which unemployed workers and vacant jobs coexist in even the most disaggregated labor markets which can be observed.

The structure of the remainder of the paper is as follows. Section 2 introduces our model frame-

[2] Also related is Sattinger's (2010) model of queueing and searching, in which unemployed workers queue at firms, but no firm has both vacant jobs and queueing workers.

work. Section 3 investigates its implications for duration dependence in unemployment. Section 4 studies the normative implications of different bargaining protocols.

2 General Framework

Time is continuous and the economy is in steady state. There is a measure $N_w > 0$ of workers and a measure N_f of firms that populate the economy. All agents are risk neutral and discount the future at rate $r > 0$. Workers can be either employed or unemployed. When unemployed each worker receives a flow payoff of $b > 0$. The wages of employed workers will be determined below. The objective of any worker is to maximize his expected lifetime utility. Any firm consists of only one job, which can be either vacant or filled, and uses labor as the only input. Let $\gamma > 0$ denote the flow cost of posting a job opening. The objective of any firm is to maximize expected lifetime profits.

The economy is divided into a measure one of distinct labor markets. As in Shimer (2007), one can think of each of these labor markets as representing a particular combination of occupation, industry, and location. We refer to these labor markets as islands and index them by $i \in [0, 1]$. We assume that these islands are also in their respective steady states. Let U_i denote the measure of unemployed workers, V_i the measure of vacancies and M_i the measure of worker-firm matches in island i. These measures refer to the stock of workers and vacant jobs.

Employed and unemployed workers are subject to exogenous reallocation shocks. Let s denote the Poisson rate at which these reallocation shocks arrive. When a worker receives a reallocation shock, this worker leaves his current island and arrives to a new, randomly chosen island. These shocks intent to capture the reallocation frictions preventing workers from moving across islands even if local labor market conditions are unfavourable.[3] We assume that if the worker that receives the reallocation shock is employed, the job remains on its current island. Let λ denote the inflow rate of workers into a given island i that is due to reallocation shocks.

Jobs are subject to exogenous destruction shocks. Let δ denote the Poisson rate at which each job, vacant or filled, is destroyed. If the job is filled, then the worker remains on his current island. At any point in time inactive firms can become active in the economy by paying a cost c. Once a new firm becomes active, it gets randomly assigned to an island i. Let μ denote the inflow rate of vacancies into a given island due to firm entry.

2.1 Stock-Flow Matching

Matching on any island follows a stock-flow process. Newly arrived workers constitute part of the flow of agents into the island and can only match with the stock of existing vacancies. In particular, a worker that arrives into an island i on which there is a stock of measure V_i vacancies,

[3] These frictions can arise, for example, due to the large investment in occupational (re-)training that workers have to undertake if they want to change labor markets. These investments inhibit rapid mobility across islands relative to job availability in a given island (see Mortensen (2009) for such an interpretation).

can potentially match with $k \geq 0$ jobs, where k is a finite integer. This assumption captures the notion that some workers might not have the required set of skills for all vacancies in their current labor market. Given that we have a continuum of agents, we assume k is distributed according to a Poisson distribution with parameter V_i, such that

$$e^{-V_i}\frac{V_i^k}{k!},$$

gives the probability that the worker can potentially match with precisely k jobs in island i.

This modeling assumption requires some comment, since it differs from stock-flow models, such as those of Taylor (1995), Coles and Muthoo (1998), and Ebrahimy and Shimer (2010), in which the sets of traders are finite. In that framework, a worker newly arrived in the flow observes *all* the finite number of vacant jobs in the stock, and can choose to accept any one of these potential matches or to reject all of them. Here, instead, the measures of the sets of unmatched agents are real numbers rather than integers; but, consistent with the spirit of the stock-flow approach, rather than observing a continuum of vacancies, a newly arrived unemployed worker in the flow observes an integer number of jobs in the stock. This modeling approach has two important benefits for the tractability of our model, analogously to how Walrasian equilibrium is more tractable than game-theoretic models with large finite numbers of players. First, it ensures that conditional on the stocks U_i, V_i, and M_i, the evolution of the labor market on each island is deterministic. Second, it avoids complicated strategic considerations in which the set of acceptable jobs to a worker depend in a complicated way on the particular draws of match quality for every worker-firm pair in the market, so that, for example, a particular worker's view of the acceptability of a particular job match changes when another unemployed worker flows into the market.[4]

Similarly, newly arrived vacancies constitute part of the flow of agents into the island and can only match with the stock of existing unemployed workers. A newly arrived firm with a job vacancy on an island i, which has a stock of U_i unemployed workers, can potentially match with k workers from the stock of unemployment, such that

$$e^{-U_i}\frac{U_i^k}{k!},$$

gives the probability that a vacant job can potentially match with precisely k workers. From the perspective of workers and vacancies in the stock, the probability of observing k vacancies or workers, respectively, is iid across arrivals. We rule out stock-stock matching by assumption.[5]

[4] If we had assumed a discrete number of workers the natural assumption would have been to use a binomial distribution to describe the probability that a worker can perform a set of jobs. The above Poisson distribution would be the limit of this binomial as the sample size goes to infinity. Coles (1999) makes a similar assumption. An important difference with that paper, however, is that meetings in our model are not bilateral but rather many-to-one: a single worker in the flow may contact arbitrarily many jobs in the stock, with the expected number of contacts being given by V_i.

[5] If we assume that a worker in the flow has unacceptably low match quality with all vacancies in the stock except the k she samples, then this restriction does not affect matching patterns in steady state, since a potential match between the worker and a job in the stock which was rejected when one of the two was in the flow will not become

We assume that at rate χ^u unemployed workers in the stock return to the flow in the same island on which they are already located. This shock captures the notion that workers might get additional skills (retrain within their occupation) while in the stock and can potentially perform a new set of jobs in their current island. Similarly, at rate χ^v vacant jobs in the stock return to the flow of their respective islands. The latter shock can be interpreted as firms changing the skill requirement of their vacancies. We also assume that when a worker experiences a reallocation shock the worker becomes part of the flow in a random island. In this case we will assume that, if the worker was employed, his employer becomes part of the flow in the current island. If the worker's employer experiences a job destruction shock δ, we assume that worker becomes part of the flow in his current island.

2.2 Match-Specific Heterogeneity

A key assumption in our model is that workers have different match-specific productivities, x, across all those jobs they are able to match with within the island. The match-specific productivity $x \geq 0$ measures the (constant) amount of output per unit of time that can be produced by a particular firm-worker match, in units of the numeraire. For any specific worker-job pair on island i, it is a random variable, distributed according to an exogenous cdf F_i. The distribution of match-specific productivities can differ across islands. Match-specific productivities are jointly independent across worker-job pairs.

We assume that each F_i has full support on a closed interval $X \subseteq [0, \infty)$ which is common to all i. Write $\underline{x} = \min\{X\}$ and $\overline{x} = \sup\{X\}$ for the left and right limits of this interval. We also assume that F_i has no atoms. These assumptions are not critical to our analysis but they simplify the exposition. For $k \geq 0$, denote by X^k the set of k-tuples $(x_1, \ldots, x_k)$ of elements of X; X^0 is the set containing only the empty tuple $\emptyset$. Also denote by X^* the set of finite tuples of elements of X, that is, $X^* = \bigcup_{k=0}^{\infty} X^k$. We will denote a typical element of X^* by $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ for some $k \geq 0$. Denote by $F_i^k(\cdot)$ the product measure on X^k. Finally, we need a boundedness condition on F_i to ensure that workers and jobs have finite values in our model; more precisely, we assume that, for any $z \geq 0$, if $k \geq 0$ is distributed according to a Poisson distribution with parameter z the expectation of the maximum of a sample of k independent random draws each distributed F^i is finite:

$$\sum_{k=1}^{\infty} \frac{e^{-z} z^k}{k!} \int_{X^k} \max\{\mathbf{x}\}\, dF_i^k(\mathbf{x}) = \sum_{k=1}^{\infty} \frac{e^{-z} z^k}{k!} \int_X x k F_i^{k-1}(x)\, dF_i(x) < +\infty \qquad \text{for any } k \geq 0. \tag{1}$$

(1) is clearly satisfied if $\overline{x} < +\infty$, but it is a strictly weaker condition.

Once a worker arrives on an island and observes a subset of vacant jobs, she also observes her match-specific productivity x for each of these jobs. All the realizations of these match-specific productivities are common knowledge to the worker and to all the jobs she observes. Given the

acceptable thereafter. The restriction is more substantive out of steady state, if the set of acceptable matches changes over time.

tuple of realized productivities $\mathbf{x}$, one of two possible events will occur: either the worker and one of the jobs will form a match, or else the worker will not match and will then become part of the stock of unemployed workers on the island. We will discuss the match formation decision and the determination of wages and profits in more detail in Section 2.4 below.

Similarly, when a new job arrives and observes a subset of workers, each worker draws a match-specific productivity for that job. Again, the realization of the tuple of match-specific productivity draws, $\mathbf{x}$, is common knowledge. Either the job matches with one of the workers, or else it does not, in which case it enters the stock.

For simplicity we assume no recall of previously rejected jobs or workers. Once a worker enters the stock, she is no longer able to contact the jobs she sampled when in the flow. In steady state, this assumption is without loss of generality.

In general each island i can be characterised by a vector of characteristics

$$\omega_i = (U_i, V_i, M_i, F_i, \chi_i^u, \chi_i^v, \delta_i, s_i, \mu_i, \lambda_i, b_i, \gamma_i).$$

To save notation, we will summarise these island-specific objects by the subscript i.

2.3 Value Functions

Consider an island i with U_i unemployed workers and V_i vacancies in the stock. Let W_i^s denote the expected value of an unemployed worker in the stock; W_i^f denote the expected value for a worker of being in the flow; and $W_i^m(w)$ denote the expected value if matched with a job and paid wage w. Similarly, let J_i^s denote the expected value of a firm with a vacant job in the stock; J_i^f denote the expected value for the firm of being in the flow; and $J_i^m(\pi)$ denote the expected value if matched with a worker and earning flow profit π. We now characterize these values.

Matched Agents. First, consider a worker on island i who is currently employed in a job paying w. The expected present discounted value of income for this agent satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$rW_i^m(w) = w + s_i \left[\mathbb{E}_{i'} W_{i'}^f - W_i^m(w)\right] + \delta_i \left[W_i^f - W_i^m(w)\right]. \tag{2}$$

The second term on the right side of (2) describes the expected capital gain (or loss, if negative) from reallocation, which returns the worker to the flow on another, randomly drawn, island and occurs at rate s_i. The third term is the expected gain from job destruction (which returns the worker to the flow on island i and occurs at rate δ_i). It is immediate from (2) that $W_i^m(w)$ is an affine function of w: a unit increase in w raises $W^m(w)$ by $(r + s_i + \delta_i)^{-1}$ units.

Similarly, the HJB equation for a filled job with profit π takes the form

$$rJ_i^m(\pi) = \pi + s_i \left[J_i^f - J_i^m(\pi)\right] - \delta_i J_i^m(\pi). \tag{3}$$

Recall that job destruction leaves no scrap value.

Agents in the Flow. Next, consider the expected value of an unemployed worker in the flow on island i. His value exceeds that of a worker in the stock on the same island because of the possibility that he may match with a vacant job while in the flow. Denote by G_i^f this increase in value associated with being in the flow, rather than the stock, on island i; then

$$W_i^f = W_i^s + G_i^f. \tag{4}$$

We will discuss how the gain G_i^f is determined in Section 2.4 below. Similarly, the value of a job in the flow exceeds that of a job in the stock; where H_i^f is the increase in value, we can write

$$J_i^f = J_i^s + H_i^f. \tag{5}$$

Agents in the Stock. Last, consider an unemployed worker in the stock. His value is characterized by the HJB equation

$$rW_i^s = b_i + (\mu_i + \chi_i^v V_i + s_i M_i) G_i^s + \chi_i^u G_i^f + s_i \left[\mathbb{E}_{i'} W_{i'}^f - W_i^s \right]. \tag{6}$$

The last two terms on the right side of (6) represent the capital gains for the worker from returning to the flow either on his own island, at rate χ_i^u, or on another island, at rate s_i. (Notice that the gain in the former case is G_i^f, according to (4).) The rate at which the agent is contacted by a vacancy in the flow is $\mu_i + \chi_i^v V_i + s_i M_i$, and G_i^s denotes the expected capital gain for the worker from such a contact, relative to his current value of being unemployed in the stock. Again, we will be more concrete about these gain terms in Section 2.4 below.

Following similar logic, the HJB equation for a vacant job in the stock takes the form

$$rJ_i^s = -\gamma_i + (\lambda_i + \chi_i^u U_i + \delta_i M_i) H_i^s - \delta_i J_i^s + \chi_i^v \left[J_i^f - J_i^s \right]. \tag{7}$$

The last two terms on the right side of (7) arise from job destruction (which is associated with a zero continuation payoff) and the return of the job to the flow on the current island. H_i^s is the expected capital gain associated with the job being contacted by a worker in the flow, which occurs at rate $\lambda_i + \chi_i^u U_i + \delta_i M_i$.

Match Surplus. We can now define the *surplus* associated with forming a match with productivity x on island i as

$$S_i(x) = [W_i^m(w) - W_i^s] + [J_i^m(x-w) - J_i^s]; \tag{8}$$

here the wage w is arbitrary. Substituting from (2), (3), (6), and (7), we see that

$$\begin{aligned}(r + s_i + \delta_i) S_i(x) = x - b_i + \gamma_i - (\mu_i + \chi_i^v V_i + s_i M_i) G_i^s - (\lambda_i + \chi_i^u U_i + \delta_i M_i) H_i^s \\ - (\chi_i^u - \delta_i) G_i^f - (\chi_i^v - s_i) H_i^f.\end{aligned} \tag{9}$$

Notice that (9) establishes that the surplus is well defined, since the wage w (which is a transfer from the firm to the worker) cancels out.[6] Notice also that higher-x matches deliver higher surplus; more precisely, a unit increase in x increases $S_i(x)$ by $(r + s_i + \delta_i)^{-1}$ units, so that $S_i(x)$ is an increasing affine function of x.

2.4 Match Formation and Bargaining

We can now be more concrete about the types of match formation and wage and profit determination mechanisms we consider.

Match Formation. We assume that match formation is *privately efficient*. By this we mean that when an agent in the flow encounters a set of agents in the stock and draws a tuple of match qualities $\mathbf{x}$, only the match which generates the maximal surplus $\max_{x \in \mathbf{x}} S_i(x)$ forms, and this only if that surplus is non-negative. That is, when there is a multilateral meeting between, say, a worker in the flow and a tuple of jobs in the stock, a match forms when there are gains from trade, and if so, those (private) gains from trade are fully harvested.

Because $S_i(x)$ is increasing in x, the conditions that the match surplus $S_i(x)$ be positive is equivalent to the existence of a *reservation match quality* $\tilde{x}_i$, so that an agent in the flow will match if and only if at least one of her match quality draws $x \in \mathbf{x}$ exceeds $\tilde{x}_i$. (If she encounters more than one, she will match with the agent with whom she has the highest match quality draw). The reservation match quality on island i solves $S_i(\tilde{x}_i) = 0$; more concretely, we have

$$\tilde{x}_i = b_i - \gamma_i + (\mu_i + \chi_i^v V_i + s_i M_i)\, G_i^s + (\lambda_i + \chi_i^u U_i + \delta_i M_i)\, H_i^s + (\chi_i^u - \delta_i)\, G_i^f + (\chi_i^v - s_i)\, H_i^f. \quad (10)$$

(10) is a key equation of our model. It relates how selective firms and workers will be about which matches to form to the opportunity costs of forming a match. These latter are of two types: first, the flow unemployment income the worker forgoes and the saving on vacancy-posting costs the job enjoys, and second, the capital gains agents forgo by being matched relative to unmatched.

There are several interesting points to note about this equation. First, even though when a match is formed, one of the agents is in the flow, the outside option of both agents should no match be formed is to be unmatched in the stock. This symmetry ensures that the surplus and the reservation match quality do not depend on whether the worker or the job is in the flow. Second, the gains accruing to agents contacted while in the stock, G_i^s and H_i^s, enter (10) symmetrically except insofar as the rate of arrivals of new jobs in the flow, $\mu_i + \chi_i^v V_i + s_i M_i$, differs from that of new workers in the flow, $\lambda_i + \chi_i^u U_i + \delta_i M_i$. Third, the capital gains G_i^f enjoyed by workers in the flow are only relevant to the match quality threshold $\tilde{x}_i$ if $\chi_i^u - \delta_i \neq 0$. To understand why, recall that χ_i^u is the rate at which unemployed workers in the stock are reallocated to the flow, while δ_i is the rate at which matched workers lose their jobs and return to the flow. If $\chi_i^u > \delta_i$, then forming a match requires the worker to accept a reduction in the rate at which she expects to receive the

[6] This is because the wage does not play an allocative role within the match: in particular, since there is no on-the-job search, the wage does not affect the length of the match.

gain G_i^f associated with being in the stock; all else equal, this makes agents less willing to form a marginal match. If $\chi_i^u < \delta_i$, the reverse applies. A similar consideration applies for H_i^f depending on the value of $\chi_i^v - s_i$.

Closely related to the reservation match quality are the *reservation wage* of a worker, $\tilde{w}_i$, and the *reservation profit* of a job, $\tilde{\pi}_i$. A job on island i delivers a worker a higher value than that of being unmatched in the stock if it pays a wage w satisfying $W_i^m(w) - W_i^s \geq 0$. Since $W_i^m(w)$ is an increasing function, this is equivalent to the condition that $w \geq \tilde{w}_i$, where $\tilde{w}_i$ is the unique solution to $W_i^m(\tilde{w}_i) = W_i^s$. Similarly, for a job, being matched is better than being unmatched in the stock if the flow profit π exceeds $\tilde{\pi}_i$, defined as the unique solution to $J_i^m(\tilde{\pi}_i) = J_i^s$. It is immediate from (8) that the reservation match quality is the sum of the reservation wage and the reservation profit:

$$\tilde{x}_i = \tilde{w}_i + \tilde{\pi}_i. \tag{11}$$

Gains from Match Formation. Under the assumption that match formation is privately efficient, we can be more concrete about the capital gains enjoyed by workers and jobs at the time of match formation.

First, suppose that a worker in the flow contacts a set of jobs in the stock with which she draws the tuple $\mathbf{x}$ of match qualities. Private efficiency implies that a match forms when $\max\{\mathbf{x}\} \geq \tilde{x}_i$. In this case, denote by $w_i^f(\mathbf{x})$ and $\pi_i^s(\mathbf{x})$ the wage and profit (per unit time) for the agents who match (that is, those with the maximal x draw). We assume that these are the only transfers between agents associated with such a multilateral meeting: in particular, there are no transfers between agents who do not match and no other transfers between those who do. Denote by $G_i^f(\mathbf{x})$ and $H_i^s(\mathbf{x})$ the capital gains associated with this multilateral meeting respectively for the worker in the flow and for the job in the stock with the maximal match quality $x \in \mathbf{x}$. (A job in the stock whose match quality x is not maximal in $\mathbf{x}$ does not receive any benefit from being contacted.) These gains are related to the flow wage and profit as follows:

$$G_i^f(\mathbf{x}) = \begin{cases} W_i^m(w_i^f(\mathbf{x})) - W_i^s & \text{if } \max\{\mathbf{x}\} \geq \tilde{x}_i \\ 0 & \text{otherwise,} \end{cases} \quad \text{and } H_i^s(\mathbf{x}) = \begin{cases} J_i^m(\pi_i^s(\mathbf{x})) - J_i^s & \text{if } \max\{\mathbf{x}\} \geq \tilde{x}_i \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

The gain that a worker enjoys from being in the flow rather than the stock, G_i^f, and the expected gain from being contacted by a worker in the flow for a job in the stock, can now be written by taking expectations of $G_i^f(\mathbf{x})$ and $H_i^s(\mathbf{x})$ over tuples $\mathbf{x} \in X^*$:

$$G_i^f = \sum_{k=1}^{\infty} \frac{e^{-V_i} V_i^k}{k!} \int_{X^k} G_i^f(\mathbf{x})\, dF_i^k(\mathbf{x}) \quad \text{and} \quad H_i^s = \sum_{k=1}^{\infty} \frac{e^{-V_i} V_i^{k-1}}{(k-1)!} \frac{1}{k} \int_{X^k} H_i^s(\mathbf{x})\, dF_i^k(\mathbf{x}). \tag{13}$$

Both these expressions require some comment. First, in the expression for G_i^f, we started the summation at $k = 1$; while it is true that with probability e^{-V_i} a worker in the flow fails to contact any vacancies, in that case a zero capital gain results.[7] Second, in writing the expression for H_i^s given by (13), we have used the fact that, from the point of view of a job in the stock which is contacted by a worker in the flow, the number of *other* jobs in the stock simultaneously contacted by the firm is distributed Poisson with parameter V_i.[8] The factor $1/k$ arises from the fact that each of the k workers contacted is equally likely to have the highest match quality x.

The situation in which the job, rather than the worker, is in the flow, is similar. Denote by $w_i^s(\mathbf{x})$ and $\pi_i^f(\mathbf{x})$ the wages and profits earned by the job and by the maximal-x worker if a match forms (that is, if $\max\{\mathbf{x}\} \geq \tilde{x}_i$); then the associated gains to this pair of agents satisfy

$$G_i^s(\mathbf{x}) = \begin{cases} W_i^m(w_i^s(\mathbf{x})) - W_i^s & \text{if } \max\{\mathbf{x}\} \geq \tilde{x}_i \\ 0 & \text{otherwise,} \end{cases} \quad \text{and } H_i^f(\mathbf{x}) = \begin{cases} J_i^m(\pi_i^f(\mathbf{x})) - J_i^s & \text{if } \max\{\mathbf{x}\} \geq \tilde{x}_i \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

The expected gain from being contacted for a worker in the stock, G_i^s, and the gain associated with a job being in the flow rather than the stock, H_i^f, are then given by

$$G_i^s = \sum_{k=1}^{\infty} \frac{e^{-U_i} U_i^{k-1}}{(k-1)!} \frac{1}{k} \int_{X^k} G_i^s(\mathbf{x})\, dF_i^k(\mathbf{x}) \quad \text{and} \quad H_i^f = \sum_{k=1}^{\infty} \frac{e^{-V_i} V_i^k}{k!} \int_{X^k} H_i^f(\mathbf{x})\, dF_i^k(\mathbf{x}). \tag{15}$$

Wage and Profit Determination. It remains to discuss how wages and profits are determined. Our model is general enough to accommodate many different protocols for splitting the match surplus; however, in order to make progress studying equilibrium, we find it useful to restrict attention to wage and profit functions which satisfy certain regularity properties. Our strategy in this section is as follows. Since the key aggregate objects of our model—the surplus $S_i(\cdot)$ and the reservation match quality $\tilde{x}_i$—depend on the primitives of the model—the wage and profit functions $w_i^s(\cdot)$, $\pi_i^s(\cdot)$, $w_i^f(\cdot)$, and $\pi_i^f(\cdot)$—only through the expected capital gain terms G_i^s, H_i^s, G_i^f, and H_i^f, we first specify the properties we require these last objects to satisfy. We then give sufficient conditions on the model primitives to ensure that these properties are satisfied.

Definition 1. Wages and profits are *regular* if there exist continuous functions G^f, H^f, G^s, H^s : $\mathbb{R}^2 \times \Delta(X) \to [0, \infty)$ such that for all i and for all parameter constellations $\{\omega_i\}_{i=0}^1$,

$$G_i^f = \frac{1}{r + \delta_i + s_i} G^f(\tilde{x}_i, V_i, F_i), \qquad G_i^s = \frac{1}{r + \delta_i + s_i} G^f(\tilde{x}_i, U_i, F_i), \tag{16a}$$

[7] We could extend the sum to include the term $k = 0$ if we use the convention that $F^0(\cdot)$ is the measure putting probability 1 on the empty tuple $\mathbf{x} = \emptyset$, with $G_i^f(\emptyset) = 0$.

[8] To see why, recall that we assumed that the number of jobs k contacted by a worker in the flow is distributed Poisson with parameter V_i. However, a given contact is more likely to arise from a job with a high value of k. The probability that a particular contact arises from a job which contacts exactly k workers is zero for $k = 0$, while for $k \geq 1$ it is

$$\frac{k \frac{e^{-V_i} V_i^k}{k!}}{\sum_{j=0}^{\infty} j \frac{e^{-V_i} V_i^j}{j!}} = \frac{\frac{e^{-V_i} V_i^k}{(k-1)!}}{\sum_{j=1}^{\infty} \frac{e^{-V_i} V_i^j}{(j-1)!}} = \frac{\frac{e^{-V_i} V_i^k}{(k-1)!}}{V_i} = \frac{e^{-V_i} V_i^{k-1}}{(k-1)!},$$

that is, $k - 1$ is distributed Poisson with parameter V_i.

$$H_i^f = \frac{1}{r+\delta_i+s_i} G^f(\tilde{x}_i, U_i, F_i), \quad \text{and} \quad H_i^s = \frac{1}{r+\delta_i+s_i} G^f(\tilde{x}_i, V_i, F_i), \tag{16b}$$

and such that

1. $G^s(\cdot)$, $H^s(\cdot)$, $G^f(\cdot)$, and $H^f(\cdot)$ are all strictly decreasing in the first argument;

2. $G^s(\cdot)$ and $H^s(\cdot)$ are strictly decreasing and $G^f(\cdot)$ and $H^f(\cdot)$ are strictly increasing in the second argument;

3. for any $U, V \geq 0$, $\lim_{\tilde{x}\to\overline{x}} G^s(\tilde{x}, U) = \lim_{\tilde{x}\to\overline{x}} H^s(\tilde{x}, V) = \lim_{\tilde{x}\to\overline{x}} G^f(\tilde{x}, V) = \lim_{\tilde{x}\to\overline{x}} H^f(\tilde{x}, U) = 0$;

This definition requires some comment. First consider the expected capital gain accruing to a worker (job) in the flow. If wages and profits are regular, then this gain can be written as a fixed function of the reservation match quality, $\tilde{x}_i$, and of the measure of jobs (workers) in the stock, V_i (U_i); in essence, this amounts to an assumption that wages and profits be determined the same way on all islands. Moreover, only the two key endogenous variables $\tilde{x}_i$ and V_i ($\tilde{x}_i$ and U_i) matter for wage (profit) determination; all other characteristics of the island and of the aggregate economy matter only through their effect on these two variables. The higher the reservation match quality, the less the potential gain from becoming matched, because the higher the value of being unmatched which is forgone. The larger the stock of jobs (workers) available to match with, the more match quality draws the worker (job) in the flow will have available (at least in expectation), and hence the higher the quality of the best potential match.

For a worker (job) in the stock, on the other hand, the expected gain from matching depends not on the measure of jobs (workers) in the stock on the *other* side of the market, but rather on the measure of workers (jobs) in the stock on the *same* side. The more agents in the stock on the same side, the more simultaneous contacts in expectation have been made by a job (worker) in the flow who contacts the agent under consideration. Therefore, the less likely that this agent will succeed in matching, and the lower the gain associated with a contact. The gain associated with a contact also falls the higher the reservation match quality, for a similar reason to before.

Since whether wage and profit determination is regular depends on properties of the expected gain terms G^f, H^f, G^s, and H^s, it is useful to give conditions on the flow wages and profits that are sufficient to ensure regularity obtains. To do this, it is useful to introduce some notation. First, for $k \geq 1$, $i \in \{1, 2, \ldots, k\}$, and $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in X^k$, denote by $x_{[i]}$ the ith-greatest element of $\mathbf{x}$.[9] Second, define a partial order on X^* by

$$(x_1, x_2, \ldots, x_k) \succeq (x'_1, x'_2, \ldots, x'_l) \quad \Leftrightarrow \quad k \geq l \text{ and } x_{[i]} \geq x'_{[i]} \text{ for } 1 \leq i \leq l\};$$

that is, a tuple $\mathbf{x}$ is greater than another tuple $\mathbf{x}'$ if for every element of $\mathbf{x}'$ we can find a corresponding element of $\mathbf{x}$ which is greater (with a different element of $\mathbf{x}$ for each element of $\mathbf{x}'$). Next,

[9]This differs from the usual notion of order statistics, in that $x_{[i]}$ is the ith-greatest rather than the ith-smallest element of $\mathbf{x}$. In terms of standard order statistics, $x_{[i]} = x_{(k-i+1)}$.

define operators $M : X^* \setminus \{\emptyset\} \to X$ and $m : X \setminus \{\emptyset\} \to X^*$ by

$$M((x_1, x_2, \ldots, x_k) = \max\{x_1, x_2, \ldots, x_k\} = x_{[1]} \quad \text{and} \quad m(x_1, x_2, \ldots, x_k) = (x_{[2]}, x_{[3]}, \ldots, x_{[k]});$$

that is, $M(\mathbf{x})$ is the maximal element of $\mathbf{x}$ and $m(\mathbf{x})$ is the (possibly empty) tuple obtained by discarding the maximal element and ordering the remaining elements from largest to smallest. Finally, define a binary operator $\ominus : X^* \times \mathbb{R} \to X^*$ such that

$$(x_1, x_2, \ldots, x_k) \ominus y = (x_i - y)_{i|x_i \geq y};$$

that is, $\mathbf{x} - y$ is the tuple obtained by subtracting y from each element of $\mathbf{x}$, then discarding any strictly negative elements which result. For example, $(2, 6, 4) - 1 = (1, 5, 3)$; $(2, 6, 4) - 4 = (2, 0)$, $(2, 6, 4) - 5 = (1)$, and $(2, 6, 4) - 7 = \emptyset$.

With this notation, we can state the following sufficient conditions for regularity.

Proposition 1. *Suppose that there are continuous functions $\hat{w}^f, \hat{w}^s, \hat{\pi}^f, \hat{\pi}^s : X^* \to [0, \infty)$ such that for all i and for all parameter constellations $\{\omega_i\}_{i=0}^1$,*

$$w_i^f(\mathbf{x}) = \hat{w}^f(\mathbf{x} \ominus \tilde{x}_i) + \tilde{w}_i, \qquad w_i^s(\mathbf{x}) = \hat{w}^s(\mathbf{x} \ominus \tilde{x}_i) + \tilde{w}_i, \tag{17a}$$

$$\pi_i^f(\mathbf{x}) = \hat{\pi}^f(\mathbf{x} \ominus \tilde{x}_i) + \tilde{\pi}_i, \quad \text{and} \quad \pi_i^s(\mathbf{x}) = \hat{\pi}^s(\mathbf{x} \ominus \tilde{x}_i) + \tilde{\pi}_i, \tag{17b}$$

with the properties that

(a) for any $\mathbf{x} \in X$, $\hat{w}^f(\mathbf{x}) + \hat{\pi}^s(\mathbf{x}) = M(\mathbf{x})$ and $\hat{w}^s(\mathbf{x}) + \hat{\pi}^f(\mathbf{x}) = M(\mathbf{x})$;

(b) for any $\mathbf{x} \in X^*$, $\hat{w}^f(\mathbf{x} \ominus z)$, $\hat{\pi}^f(\mathbf{x} \ominus z)$, $\hat{w}^s(\mathbf{x} \ominus z)$, and $\hat{\pi}^s(\mathbf{x} \ominus z)$ are decreasing in $z \geq 0$;

(c) for any $\mathbf{x}, \mathbf{y} \in X$ with $M(\mathbf{x}) \geq M(\mathbf{y})$ and $m(\mathbf{x}) = m(\mathbf{y})$, then $\hat{w}^f(\mathbf{x}) \geq \hat{w}^s(\mathbf{y})$, $\hat{\pi}^f(\mathbf{x}) \geq \hat{\pi}^f(\mathbf{y})$, $\hat{w}^s(\mathbf{x}) \geq \hat{w}^s(\mathbf{y})$, and $\hat{\pi}^s(\mathbf{x}) \geq \hat{\pi}^s(\mathbf{y})$;

(d) for any $\mathbf{x}, \mathbf{y} \in X$ with $M(\mathbf{x}) = M(\mathbf{y})$ and $m(\mathbf{x}) \succeq m(\mathbf{y})$, then $\hat{w}^f(\mathbf{x}) \geq \hat{w}^f(\mathbf{y})$, $\hat{\pi}^f(\mathbf{x}) \geq \hat{\pi}^f(\mathbf{y})$, $\hat{w}^s(\mathbf{x}) \leq \hat{w}^s(\mathbf{y})$, and $\hat{\pi}^s(\mathbf{x}) \leq \hat{\pi}^s(\mathbf{y})$.

Then wages and profits are regular.

The proof of Proposition 1, as well as all other omitted proofs, can be found in Appendix A.

The hypotheses of Proposition 1 are of several types. First, the fixed functional form for wages and profits ensures that bargaining proceeds in the same way on all islands. Second, since $\hat{w}^f(\cdot)$, $\hat{\pi}^f(\cdot)$, $\hat{w}^s(\cdot)$, and $\hat{\pi}^s(\cdot)$ are non-negative, (17) requires that whenever the match surplus is positive, the wage and flow profit must exceed their reservation values. Property (a) requires that the entirety of the match surplus be split between the worker and the job (that is, there is budget balance and no money burning). Property (b) guarantees that a rise in the reservation match quality $\tilde{x}_i$, which reduces the surplus associated with a match of a given quality x, reduces the excess of both wages and profits over their reservation values.[10] Property (c) ensures that an increase in the match quality of the match which is actually formed (that is, an increase in $M(\mathbf{x})$) leads to both a higher wage and a higher profit for the agents that match. Finally, Property (d) ensures that an improvement in the set of *other* match quality draws encountered by the agent in the flow, not including the best, raises the payoff of the agent in the flow and lowers that of the agent who matches from the stock. The intuition for this last property is that, for example, when a job contacts multiple workers in the stock, competition with workers whose match quality is closer to his own may reduce the wage of the worker with the highest match quality.[11].

Examples. The notion of regular wage and profit determination is general enough to incorporate many familiar bargaining models used in the literature. Two obvious examples are as follows.

1. *Nash bargaining.* A worker and the job who form the match split the flow surplus $x - \tilde{x}_i$ in ratio $\beta : (1-\beta)$, regardless of the other possible match qualities $m(\mathbf{x})$ rejected at the time of match formation, and regardless of which agent was in the flow. In this case,

$$\hat{w}^f(\mathbf{x}) = \hat{w}^s(\mathbf{x})) = \beta M(\mathbf{x}) \qquad \text{and} \qquad \hat{\pi}^f(\mathbf{x}) = \hat{\pi}^s(\mathbf{x}) = (1-\beta)M(\mathbf{x}).$$

It is immediate that the hypotheses of Proposition 1 are satisfied for Nash bargaining.

2. *Bertrand competition among agents in the stock.* In the case where a job in the flow contacts a tuple of workers in the stock, and multiple x draws exceed the reservation match quality $\tilde{x}_i$, then the highest-x worker matches at a wage such that the job is indifferent between hiring him at that wage and hiring the worker with the second-highest x draw at her reservation wage, that is, $x_{[1]} - w_i^s(\mathbf{x}) = x_{[2]} - \tilde{w}_i$. If only one draw exceeds the reservation quality, then the worker earns all the surplus, that is, $x_{[1]} - w_i^s(\mathbf{x}) = \tilde{\pi}_i$. Since $\tilde{x}_i = \tilde{w}_i + \tilde{\pi}_i$, we can unify these two expressions by writing $x_{[1]} - w_i^s(\mathbf{x}) = \max\{x_{[2]}, \tilde{x}_i\} - \tilde{w}_i$, that is,

$$\begin{aligned} & w_i^s(\mathbf{x}) = x_{[1]} - \max\{x_{[2]}, \tilde{x}_i\} + \tilde{w}_i \\ \text{and} \qquad & \pi_i^f(\mathbf{x}) = x_{[1]} - w_i^s(\mathbf{x}) = \max\{x_{[2]}, \tilde{x}_i\} - \tilde{w}_i = \max\{x_[2] - \tilde{x}_i, 0\} + \tilde{\pi}_i, \end{aligned}$$

that is

$$\hat{w}^s(\mathbf{x}) = M(\mathbf{x}) - \max\{m(\mathbf{x})\} \qquad \text{and} \qquad \hat{\pi}^f(\mathbf{x}) = \max\{m(\mathbf{x})\}$$

When the worker is in the flow, a similar form of competition ensues between the jobs in the stock, so that $\pi_i^s(\mathbf{x}) = w_i^s(\mathbf{x})$ and $w_i^f(\mathbf{x}) = \pi_i^f(\mathbf{x})$, or equivalently

$$\hat{w}^f(\mathbf{x}) = \max\{m(\mathbf{x})\} \qquad \text{and} \qquad \hat{\pi}^s(\mathbf{x}) = M(\mathbf{x}) - \max\{m(\mathbf{x})\}.$$

[10]Depending on whether the increase in $\tilde{x}_i$ is associated with an increase in $\tilde{w}_i$, $\tilde{\pi}_i$, or both, its implications for wages and profits themselves are more ambiguous.

[11]Property (d) can be stated more concisely, since given Property (a), the assumption that $\hat{w}^f(\mathbf{x})$ and $\hat{\pi}^f(\mathbf{x})$ are increasing in $m(\mathbf{x})$ conditional on $M(\mathbf{x})$ implies that $\hat{w}^s(\mathbf{x})$ and $\hat{\pi}^s(\mathbf{x})$ are decreasing in $m(\mathbf{x})$

Again, it is straightforward to verify that the hypotheses of Proposition 1 are satisfied.

We will discuss alternative wage determination models further in Section 5 below.

2.5 Flow Equations

Given that agents' job acceptance strategies are characterised by a common reservation match-specific productivity, $\tilde{x}$, and that agents form a match with the $\max(\mathbf{x})$ such that $\max(\mathbf{x}) \geq \tilde{x}$, we now proceed to derive the number of agents in the stock. In particular, the number of unemployed workers, vacant jobs and the number of matched pairs attached to an island are given by:

$$\dot{U} = -(\mu + \chi^v V + sM)\left(1 - e^{-U(1-F(\tilde{x})))}\right) + (\lambda + \chi^u U + \delta M)e^{-V(1-F(\tilde{x}))} - (s + \chi^u)U \tag{18}$$

$$\dot{V} = (\mu + \chi^v V + sM)e^{-U(1-F(\tilde{x}))} - (\lambda + \chi^u U + \delta M)\left(1 - e^{-V(1-F(\tilde{x}))}\right) - (\delta + \chi^v)V \tag{19}$$

$$\dot{M} = (\mu + \chi^v V + sM)\left(1 - e^{-U(1-F(\tilde{x})))}\right) + (\lambda + \chi^u U + \delta M)\left(1 - e^{-V(1-F(\tilde{x}))}\right) - (s + \delta)M, \tag{20}$$

where once again we have left the subscript i implicit.

To understand these flow equations, note that from the inflow of $(\mu + \chi^v v + sM)$ jobs, a fraction

$$\sum_{k=0}^{\infty} e^{-U}\frac{U^k}{k!}\left[1 - F(\tilde{x})^k\right] = 1 - e^{-U(1-F(\tilde{x}))}$$

match immediately, so adding an additional match and subtracting from the stock of unemployed workers. The remaining jobs add to the stock of vacancies.[12] Similarly, of the inflow of $(\lambda + \chi^u U + \delta M)$ workers, a fraction $1 - e^{-V(1-F(\tilde{x}))}$ match immediately, adding to M and subtracting from the stock of vacant jobs. The remaining inflow of workers adds to the stock of unemployed workers. A matched pair is affected by two kinds of shocks, all of which reduce M but have different effects on U and V: (i) worker reallocation (at rate s), which has no effect on U since the worker is moved to another island but increases V as although jobs return to the flow, some of them might not be able to form a match; (ii) job destruction (at rate δ), which has no effect on V, but increases U as although workers return to the flow, some of them might not be able to form a match. Finally, note that an unemployed worker in the stock is reallocated to a new island at rate s and becomes part of the flow in his current island at rate χ^u, while a vacancy in the stock is destroyed at rate δ and becomes part of the flow at rate χ^v.

The Matching Function Our stock-flow structure implies that on each island, the (reduced form) matching function is given by the first two terms in equation (20),

$$F(U, V, M, \tilde{x}) = [\mu + \chi^v V + sM]\left(1 - e^{-U(1-F(\tilde{x}))}\right) + [\lambda + \chi^u U + \delta M]\left(1 - e^{-V(1-F(\tilde{x}))}\right), \tag{21}$$

[12] Given that $F(x)^k$ is the probability that x is the maximum of k iid draws from F, the density of the maximum is given by $kF(x)^{k-1}f(x)$.

where the first term describes the number of new matches between firms in the flow and unemployed workers in the stock; while the second term describes the number of new matches between workers in the flow and vacant jobs in the stock. Relative to Coles and Smith (1998), the key insight here is that the matching function on any given island depends on the acceptance strategies of workers and firms, summarised by $\tilde{x}$, as well as on the flow and stock of vacancies and unemployed workers and the number of current matches. This implies that any changes in local labor market conditions, will influence the shape of the matching functions at the level of individual labor markets and at the aggregate.

2.5.1 Free Entry

Firms decide whether to create a vacancy in this economy or stay inactive. As in Shimer (2007), creating a vacancy in the economy implies paying a sunk cost c, after which the vacancy is randomly allocated on an island i. We assume that there is free-entry, such that at any point in time, the number of firms in the economy, N_f, is determined by

$$c = \mathbb{E}_{i'} J_{i'}^{f} = \mathbb{E}_{i'}[J_{i'}^{s} + H_{i'}^{f}], \tag{22}$$

where the expectation is taken with respect to the distribution of islands' characteristics.

Note that (90) is a special case of the participation decision described in Diamond (1982) and used in Coles (1999) in a stock-flow context. In those papers firms encounter project opportunities from time to time. Once one project is encountered the sunk cost to undertake it, c, is a random draw from some known continuous distribution. If the realised cost is sufficiently small relative to the expected benefit of participating, then the firm enters. Otherwise it waits until a new project is found some time in the future. A reservation cost then describes firms' entry decisions. The condition (90) simplifies these entry decisions by assuming projects are encountered with probability one during each time interval and that the cost distribution degenerates at a single value of $c > 0$. In future versions of this paper we intend to replace the free entry condition with

2.6 Equilibrium

Definition: A Stock-Flow Matching (SFM) equilibrium is a measure of firms N_f, a set of measures of unemployed workers in the stock, vacant jobs in the stock, and matches $\{U_i, V_i, M_i\}_{i \in I}$, and a set of value functions $\left\{W_i^m(\cdot), W_i^f(\cdot), W_i^s(\cdot), J_i^m(\cdot), J_i^f(\cdot), J_i^s(\cdot)\right\}_{i \in I}$, such that:

1. The value functions satisfy the HJB equations (2) - (7).
2. Optimal match formation implies $x \geq \tilde{x}_i$, where $\tilde{x}_i$ solves $S_i(\tilde{x}) = 0$ and $S_i(\cdot)$ satisfies (9).
3. Wages and profits are regular;
4. Optimal entry (90) determines N_f.

5. The measures U_i, V_i and M_i are the steady state solution to the transition equations described in equations (18) -(20) .

2.6.1 Characterisation and Existence

Equilibrium in our model reduces to solving for the measure of matches M_i and the reservation productivity $\tilde{x}_i$ for each island i and then solving the free-entry condition for N_f. Therefore, we first consider a steady state equilibrium on a island i, taking as given N_f. We will assume that $\{\chi_i^u, \chi_i^v, \delta_i, s_i, \mu_i, \lambda_i, b_i, \gamma_i\}$ are common across islands and once again we drop the subscript i to save notation.

Single Island Denote by N^w the measure of workers, both matched and unemployed, and by N^f the measure of jobs, both filled and vacant. In steady state $\dot{N}^w = \dot{N}^f = \dot{U} = \dot{V} = \dot{M} = 0$; adding (18) and (20) makes clear that $N^w = U + M = \lambda/s$. Similarly, combining (19) and (20) establishes that $N^f = V + M = \mu/\delta$.

Substituting these expressions into (20) allows us to determine the value of M that is consistent with flow balance given that the reservation match quality used by agents is $\tilde{x}$:

$$(s+\delta)M = F\left(M, N^w - M, N^f - M, \tilde{x}\right) \tag{23}$$

$$\begin{aligned} &\equiv \left[\mu + \chi^v N^f - (\chi^v - s)M\right]\left[1 - e^{-(N^w - M)(1-F(\tilde{x}))}\right] \\ &\quad + \left[\lambda + \chi^u N^w - (\chi^u - \delta)M\right]\left[1 - e^{-(N_f - M)(1-F(\tilde{x}))}\right], \end{aligned} \tag{24}$$

where $F(\cdot)$ is the matching function defined by (21). (23) equates outflows from employment (on the left side) to inflows to employment (on the right side). (23) can also be written

$$\begin{aligned} 0 = &\left[\mu + \chi^v(N^f - M)\right]\left[1 - e^{-(N^w - M)(1-F(\tilde{x}))}\right] - sMe^{-(N^w - M)(1-F(\tilde{x}))} \\ &+ \left[\lambda + \chi^u(N^w - M)\right]\left[1 - e^{-(N_f - M)(1-F(\tilde{x}))}\right] - \delta M e^{-(N_f - M)(1-F(\tilde{x}))}; \end{aligned}$$

The right side of this equation is decreasing in both $\tilde{x}$ and M, so that the equation describes a downward-sloping locus in $(\tilde{x}, M)$-space. This is intuitive: as agents in the stock become more picky, fewer matches are formed, and so M is lower.

The other key equation characterizing the equilibrium when entry is exogenous is (9), which determines the threshold match quality conditional on U, V, and M. After substituting the steady-state conditions discussed above, (9) can be written

$$\tilde{x} - b + \gamma = \left[\lambda + \chi^u N^w - (\chi^u - \delta)M\right] H^s + \left[\mu + \chi^v N^f - (\chi^v - s)M\right] G^s + (\chi^u - \delta)G^f + (\chi^v - s)H^f. \tag{25}$$

To understand the implications of this equation, first consider the special case in which $\chi^u = \delta$ and $\chi^v = s$. This case delivers two important simplifications: first, the arrival rate of new jobs and new workers into the labor market does not depend on the matching pattern, and second, the terms G^f

and H^f do not appear in the determination of the surplus, since an agent's probability of returning to the flow (on some island) is now independent of her match status. In this case, (48) takes the simpler form

$$\tilde{x} - b + \gamma = [\lambda + \chi^u N^w] H^s + [\mu + \chi^v N_f] G^s. \tag{26}$$

If wages and profits are regular, $G^s = G^s(\tilde{x}, N^w - M)$ and $H^s = H^s(\tilde{x}, N^f - M)$ are both strictly decreasing in $\tilde{x}$ and strictly increasing in M. In this case, (48) defines an upward-sloping locus in $(\tilde{x}, M)$-space. Because (23) and (48) define curves of opposite slope in $(\tilde{x}, M)$ space, we then immediately have the following result

Proposition 2. *Suppose that wages and profits are regular and that $\chi^u = \delta$ and $\chi^v = s$. Then there is a unique steady-state equilibrium on any island.*

Notice that Proposition 2 does not guarantee that the equilibrium on the island will have positive activity; this requires that the surplus from forming a match be strictly positive for the highest match quality $\bar{x}$. It also relies on a parametric restriction to ensure that (48) defines an upward-sloping locus in the $(\tilde{x}, M)$ plane. However, we can show using the implicit function theorem that there continues to be a unique steady-state equilibrium for parameters in an open neighborhood of the restriction defined in the statement of the Proposition. Also, given $(\tilde{x}, M)$, the values of U and V consistent with equilibrium can of course be constructed immediately from flow balance, that is, $U = \lambda/s - M$ and $V = \mu/\delta - M$.

We now turn to analyse the existence of equilibrium taking into account the participation margin of firms.

Free Entry When the number of firms in the aggregate economy is endogenized, the equilibrium is characterized not only by (23) and (48) but in addition by the free entry condition (90). It is straightforward to establish the following result.

Proposition 3. *Suppose that wages and profits are regular and that $\chi^u = \delta$ and $\chi^v = s$. Then there is a unique equilibrium with free entry of jobs.*

A sketch of the proof is as follows. [We will formalize this argument in a later version of the paper.] Suppose for simplicity that all islands are homogeneous; it is straightforward to extend the argument to allow for heterogeneity across islands. Write the free-entry condition (90) as

$$c = \frac{1}{r+\delta}\left[-\gamma + (\lambda + \chi^u U + \delta M) H^s + (r + \delta + \chi^v) H^f\right]. \tag{27}$$

If $\tilde{x}$ did not vary, it would be straightforward to establish the uniqueness of equilibrium as follows: an increase in N^f can be shown to imply an increase in M, an increase in V, and a decrease in U using the flow-balance condition (23). (This is intuitive: holding constant the probability that a given match is acceptable, when more jobs enter, workers are more likely to be matched and jobs less likely.). Because wages and profits are regular, this lowers $H^s = H^s(\tilde{x}, V)$ and also lowers $H^f = H^f(\tilde{x}, V)$; thus, the right side of (27) would be decreasing in N^f. When $\tilde{x}$ changes in response to an increase in N^f, this effect is augmented by two others. First, the regularity of wages and profits implies that the change in $\tilde{x}$ directly affects H^s and H^f. Second, the change in $\tilde{x}$ affects U and V in the same direction. However, we can show that the intuition from the simple case in which $\tilde{x}$ does not change carries over, so that equilibrium continues to be unique.

3 On the Duration of Unemployment and Vacancy Spells

This section investigates the implication of our stock-flow matching model for the duration distribution of unemployment and vacancy spells. We first show that, in the baseline version of our model in which islands are homogeneous, the survivor functions for both jobs and workers are fully determined by observations on three quantities that are easily observed by the econometrician: the unemployment and vacancy rates and the rate of job destruction. In Section 3.2 below we discuss the implications of heterogeneity across islands for the duration distribution.

In this section we assume that $\chi^u = \delta$ and $\chi^v = s$; this simplifies the analysis by imposing that the rate of arrival of new agents in the flow does not depend on the matching pattern.

3.1 Homogeneous Islands

We begin by assuming that islands are homogeneous. We therefore omit the i subscripts in this section.

Denote by $u = U/N^w = U/(U+M)$ and $v = V/N^f = V/(V+M)$ the unemployment and vacancy rates. Notice that because matching is one-to-one, we have that $N^w - U = M = N^f - V$, which implies that

$$(1-u)N^w = (1-v)N^f,$$

so that, since in steady state $N^w = \lambda/s$ and $N^f = \mu/\delta$, we have that

$$(1-u)\frac{\lambda}{s} = (1-v)\frac{\mu}{\delta}$$

and

$$\frac{N^f}{N^w} = \frac{1-u}{1-v}.$$

It is also useful to introduce notation for the vacancy-unemployment ratio

$$\theta = \frac{V}{U} = \frac{vN^f}{uN^w} = \frac{v(1-u)}{u(1-v)}. \tag{28}$$

Of course, because the matching function does not exhibit constant returns to scale, unlike in the DMP model, θ is not a sufficient statistic for the probability that an unmatched agents finds a job; it is still a useful quantity to define. Notice that θ can be written only in terms of u and v and not on other characteristics of the economy.

Next, denote by p^u and p^v respectively the probabilities that an unemployed worker and a vacant job successfully match while in the flow, and by ϕ^u and ϕ^v the Poisson rates at which unemployed workers and vacant jobs in the stock match conditional on not receiving a reallocation shock and returning to the flow. Notice that the true Poisson rate at which a stock-unemployed worker or a stock vacancy find a new match are not ϕ^u and ϕ^v, but incorporate also the rate at which the agent returns to the flow on this or another island

$$\psi^u = \phi^u + (s + \chi^u)p^u = \phi^u + (s + \delta)p^u;$$
$$\psi^v = \phi^v + (\delta + \chi^v)p^v = \phi^v + (s + \delta)p^v.$$

From the matching function, we know that

$$p^u = 1 - e^{-V(1-F)} \tag{29}$$
$$p^v = 1 - e^{-U(1-F)} \tag{30}$$
$$\phi^u = (s + \delta)\frac{\mu}{\delta}\frac{1}{U}\left[1 - e^{-U(1-F)}\right] = (s + \delta)\frac{N^f}{U}\left[1 - e^{-U(1-F)}\right] \tag{31}$$
$$\phi^v = (s + \delta)\frac{\lambda}{s}\frac{1}{V}\left[1 - e^{-V(1-F)}\right] = (s + \delta)\frac{N^w}{V}\left[1 - e^{-V(1-F)}\right] \tag{32}$$

where for now we write F for $F(\tilde{x})$ to save on notation. Using this notation, we have the following characterization of p^u and p^v.

Lemma 1. *Suppose that $\chi^u = \delta$ and $\chi^v = s$. Then p^u and p^v, the probabilities that a newly unemployed worker (respectively, a newly vacant job) match while in the flow are given by the unique solution to*

$$(1 - p^u)^{\frac{u}{1-u}} = (1 - p^v)^{\frac{v}{1-v}}; \tag{33}$$
$$\frac{p^u}{1-u} + \frac{p^v}{1-v} = 1. \tag{34}$$

Lemma 1 allows us to illustrate graphically the determination of p^u and p^v. In the unit square in (p^u, p^v)-space very nicely, the locus of points satisfying (34) is a downward-sloping line segment which intersects the p^u-axis at $p^u = 1 - u$ and the p^v-axis at $p^v = 1 - v$. (33) characterizes $1 - p^v$ as a power of $1 - p^u$, so the locus defined by this equation takes the form of a power function with origin at $(p^u, p^v) = (1, 1)$. If $\theta = 1$, that is, $v = u$, then the locus defined by (33) is the 45 degree line. If $\theta > 1$, that is, $v > u$, this locus is convex, while if $v < u$, the empirically reasonable case for the US economy, it is concave. Figure 1 shows this. Since the position of the downward-sloping line will not move much for u and v in the empirically relevant range (say, less than 0.15), the changes in p^u and p^v are essentially all driven by movements of the curve defined by (33), that is, by movements in θ.

The graphical representation given by Figure 1 is also useful to determine comparative statics of p^u and p^v with respect to labor market conditions, summarized by u and v. In particular, holding

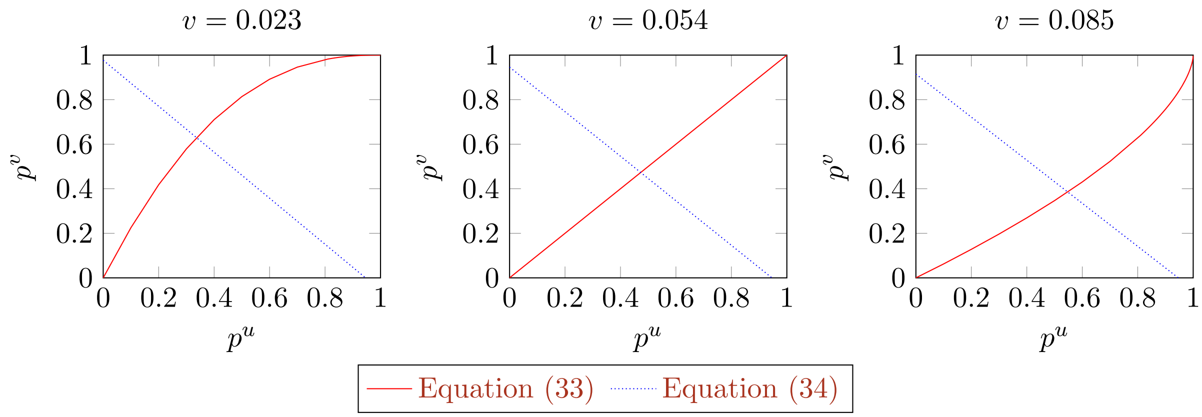

Figure 1. Determining p^u and p^v. Plots are all calculated for $u = 0.054$ as in Shimer (2007).

constant θ, a decrease in both u and v jointly leaves (33) unchanged but shifts the locus defined by (34) away from the origin. This raises both p^u and p^v, consistent with the reduction in matching efficiency required to increase the measures of unmatched agents on both sides of the market. The comparative statics with respect to a change in u or v alone have effects that are less easy to sign analytically—there are two effects of opposite signs in of increasing u on the left side of (108), the direct effect and the effect via a change in θ. However, graphical analysis establishes that in the empirically relevant range at least, an increase in v raises p^u and lowers p^v; an increase in u has the opposite effect.

A final point of interest about the determination of p^u and p^v is that the values of these probabilities depend *solely* on u and v, and not on any other parameters at all, including $s + \delta$.

Understanding the unemployment duration distribution, or, equivalently, the survivor function for unemployment spells, requires understanding not only the job-finding probability of workers in the flow, p^u, but also the Poisson rate ψ^u at which stock-unemployed workers find jobs. However, given the relationship between p^u, p^v, ϕ^u, and ϕ^v implied by (29), (30), (31), and (32), it is straightforward to see that these too are uniquely determined. More precisely, substituting from (30) into (31) shows immediately that

$$\phi^u = (s + \delta)\frac{\theta}{v}p^v,$$

so that the Poisson job-finding rate for a worker in the flow is given by

$$\psi^u = (s + \delta)\left(p^u + \frac{\theta}{v}p^v\right).$$

We have a similar relation for the Poisson job-finding rate for a vacancy in the flow. Since p^u and p^v are determined uniquely in terms of u and v according to Lemma 1, and since θ depends only on u and v according to (28), we have therefore established the following result.

Proposition 4. *Suppose that $\chi^u = \delta$ and $\chi^v = s$. Then p^u and p^v, the matching probabilities for a worker and a job in the flow respectively, and ϕ^u and ϕ^v, the Poisson matching rates for a worker and a job in the stock respectively, can be uniquely determined in terms of the unemployment and vacancy rates u and v and the job destruction rate $s + \delta$.*

Proposition 4 is a striking result. It says that the distribution of durations of unemployment spells in a homogeneous-islands version of our stock-flow matching economy is entirely determined by the unemployment and vacancy rates, together with the rate of job destruction. The dependence of the survivor function on the job destruction rate is in fact fairly weak: notice that p^u, the job-finding probability for a newly-unemployed worker while in the flow, does not depend on $s+\delta$, while ψ^u is proportional to it. Since flow balance requires that the inflow to unemployment equal the outflow, an increase in the job destruction rate $s+\delta$, holding constant u and v, must be matched by a corresponding increase in the matching rate for unemployed workers. In our model, this occurs simply through a proportional increase in ψ^u. A similar set of statements are true for vacancies.

It is not straightforward to see graphically how ϕ^u will vary with u and v. However, it is straightforward to see this sign of some such effects from the flow balance equation (106): an increase in p^u, holding constant u, must be associated with a decrease in ϕ^u, so that an increase in v, which raises p^u, must lower ϕ^u. To characterize the relationship more precisely, we solve the model numerically.

Figure 2 illustrates the dependence of p^u, ϕ^u, and ψ^u on changes in u or v, holding constant the job destruction rate. The left panel of the Figure shows the comparative static effects on these quantities of changing u alone, holding v constant at the mean value seen in data from the Job Openings and Labor Turnover Survey (JOLTS) for the US economy for the period from March 2002 to February 2004, that is, $v = 0.027$. As can be seen, for this vacancy rate, an increase in u, holding v constant, is consistent with a reduction both in the probability a newly unemployed worker matches while in the flow, as well as in the Poisson rate at which she matches while in the stock. The right panel of the Figure shows the comparative static effect of changing v alone, holding the unemployment rate constant at $u = 0.054$, the mean value of the unemployment rate observed in the Current Population Survey (CPS) for the same time period. An increase in the vacancy rate is consistent with a rise in the job-finding probability for newly unemployed workers, intuitively since such workers now sample from a larger stock of vacancies. Holding the unemployment rate constant requires that if p^u rises, then the Poisson job-finding rate for stock-unemployed workers, ψ^u, must fall.

The interpretation of the comparative statics shown in Figure 2 is complicated, however, by the fact that the unemployment and vacancy rates are endogenous variables. In fact, changes in u and v in our model can be generated in three ways: by changes in the measures of workers and firms, N^w and N^f, and by changes in the probability that a match is acceptable, $F(\tilde{x})$. Assuming that the size of the labor force is relatively slow-moving, the most interesting effects to consider are those induced by changes in firm entry and in the job acceptance probability. Although N^w, N^f, and F are not uniquely determined by u and v, it is easy to verify that the effect on the unemployment and

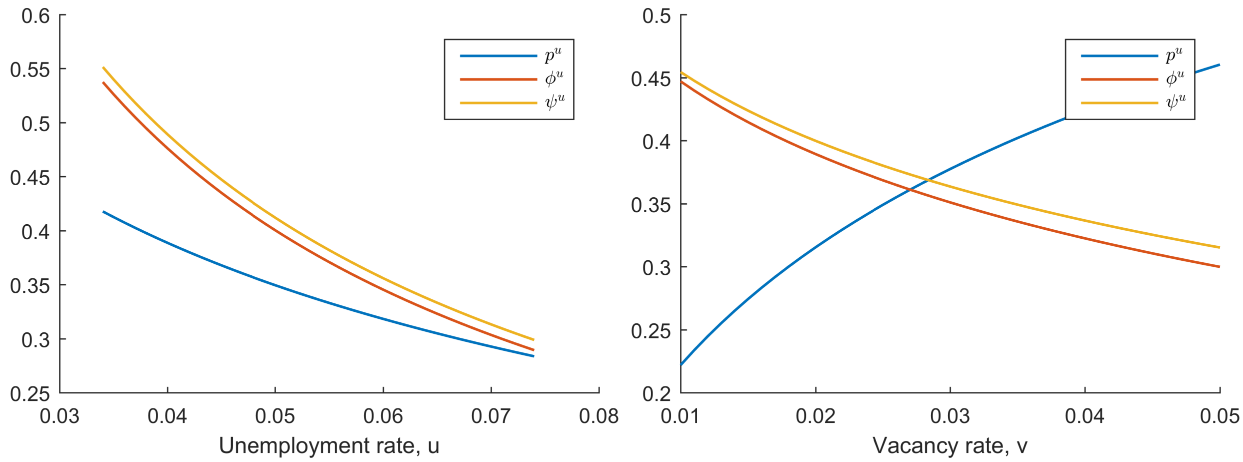

Figure 2. Left panel: effect of changing u alone on p^u, ϕ^u, and ψ^u. Right panel: effect of changing v alone.

vacancy rates u and v and on the shape of the unemployment and vacancy duration distributions of proportional changes in N^f or in $1-F$ do not depend on N^w, N^f, and F independently of u and v. We can therefore determine uniquely the effect of proportional changes in N^f, that is, the effect of a change in the number of firms active in the market, holding constant the probability that a match is acceptable, $F(\tilde{x})$. Similarly, we can determine the effect of proportional changes in $1-F$, the probability that a given match is acceptable, holding constant N^f.

The results of these two exercises are shown in Figure 3 and Figure 4. While either an increase in N^f or in $1-F$ is consistent with a fall in the unemployment rate, the two driving forces have strikingly different effect on the unemployment duration distribution, summarized by the survivor function. The left panel of Figure 3 shows that an increase in N^f leads to a rise in both p^u and ψ^u; the right panel shows that this is associated with a relatively uniform fall in the survivor function. Figure 4 shows that the effect of a change in $1-F$, on the other hand, leads to a fall in p^u (associated with a fall in the stock of vacant jobs available to be sampled by a newly unemployed worker in the flow) but a rise in ϕ^u. This leads to a disproportionate fall in the fraction of long-duration unemployment spells, as shown in the right panel of the figure: for $1-F$ large, that is, for u and v small, the survivor function falls much more steeply and asymptotes to zero more rapidly.

3.2 Heterogeneous Islands

We now investigate what happens when labor markets are no longer homogeneous. Intuitively, this provides a force for duration dependence in unemployment due to duration-dependent sampling: when some labor markets have lower exit rates from stock-unemployment, long-duration unemployed workers observed in the data are particularly likely to be associated with such labor markets, and so have a lower hazard rate of exiting unemployment. That is, allowing for heterogeneity across labor markets adds to the true duration dependence *within* a homogeneous labor market generated by the stock-flow structure of our model additional duration dependence resulting

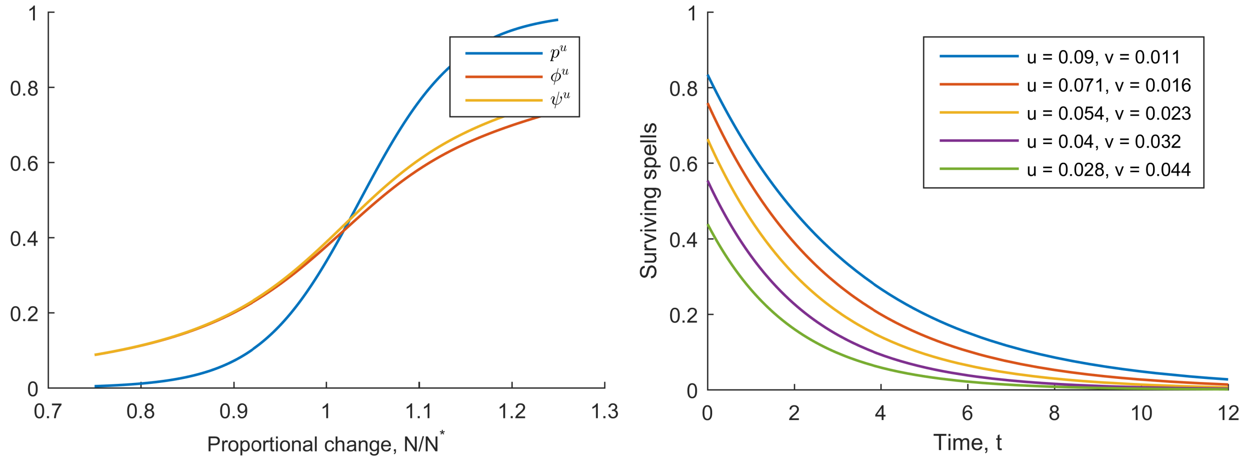

Figure 3. Left panel: effect of changing N^f on p^u, ϕ^u, and ψ^u. Right panel: effect of changing N^f on survivor function for unemployment spells.

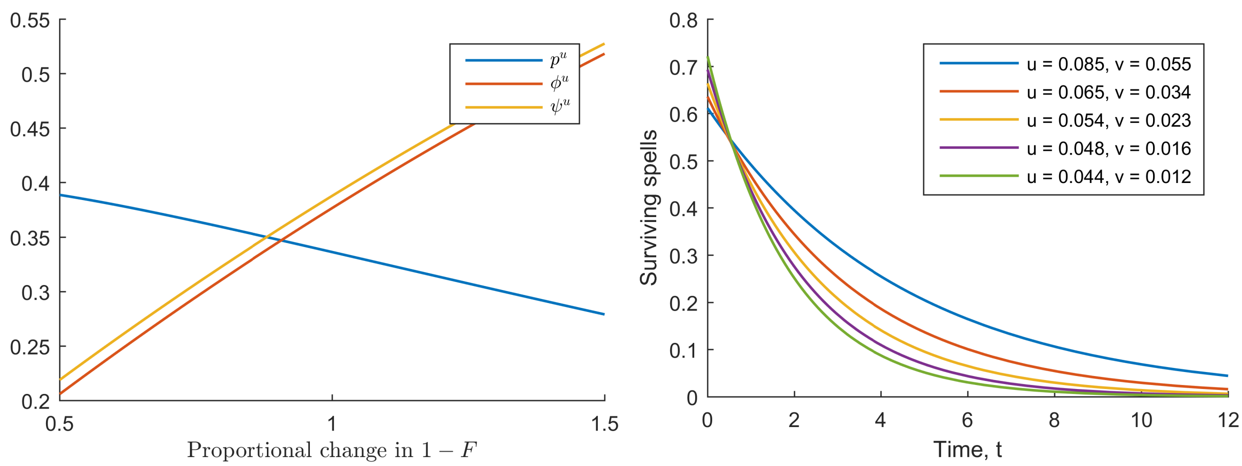

Figure 4. Left panel: effect of changing $1 - F(\tilde{x})$ on p^u, ϕ^u, and ψ^u. Right panel: effect of changing $1 - F(\tilde{x})$ on survivor function for unemployment spells.

from aggregation across ex ante heterogeneous workers (Heckman and Singer, 1984).

To allow for heterogeneity, we augment the basic structure of our model by positing that there are K archipelagos, each homogeneous among themselves but different from each other. Suppose that reallocation shocks only reallocate workers within an archipelago.[13] Suppose that there is $(\alpha_1, \ldots, \alpha_K) \in \mathbb{R}^K$ with $\alpha_k \geq 0$ for all k and $\sum_{k=1}^K \alpha_k = 1$, such that fraction α_k of the total population of the economy live in archipelago k. Also, write the job-finding probability in the flow and the Poisson job-finding rate in the stock in archipelago k as p_k and ψ_k.

Denote the unemployment and vacancy rates in archipelago k by u_k and v_k. Then the aggregate unemployment and vacancy rates are given by the appropriate weighted averages

$$u = \frac{\sum_{k=1}^K \alpha_k u_k}{\sum_{k=1}^K \alpha_k} = \sum_{k=1}^K \alpha_k u_k$$

and

$$v = \sum_{k=1}^K \alpha_k v_k.$$

Notice that by appropriate choice of the populations α_k, the aggregate unemployment and vacancy rates can lie anywhere in the convex hull of the set $\{(u_k, v_k)\}_{k=1}^K$, and the aggregate labor market tightness can lie anywhere between $\min_k v_k/u_k$ and $\max_k v_k/u_k$.

To study the survivor functions and hazards, we need to know the fraction of a cohort of newly unemployed workers who are found in archipelago k. The EU transition rate in archipelago k is $(s_k + \delta_k)(1 - p_k^u)$, while the measure of employed workers is $\alpha_k(1 - u_k)$. That is, the flow of workers into stock unemployment in archipelago k is the product of these, $(s_k+\delta_k)(1-p_k^u)(1-u_k)\alpha_k$. Notice that flow balance requires that $(s_k + \delta_k)(1 - p_k^u)(1 - u_k) = \psi_k u_k$, we can write this as $\psi_k \alpha_k u_k$. Thus, the share of a cohort of newly unemployed workers in archipelago k is

$$\beta_k = \frac{\psi_k \alpha_k u_k}{\sum_{l=1}^K \psi_l \alpha_l u_l}.$$

Of a cohort of workers who lose a job at a particular time, denoted 0, the share of the cohort continuing in unemployment at some time $t > 0$ (that is, the value of the survivor function at time t) is

$$y_t = \sum_{k=1}^K \beta_k (1 - p_k) e^{-\psi_k t},$$

so that the hazard is given by

$$-\frac{\dot{y}_t}{y_t} = \frac{\sum_{k=1}^K \beta_k (1 - p_k) e^{-\psi_k t} \cdot \psi_k}{\sum_{k=1}^K \beta_k (1 - p_k) e^{-\psi_k t}},$$

[13] Equivalently, we could imagine that there are K unrelated economies, each of which is a different instance of a homogeneous economy of the type we have in mind; the econometrician only observes data for the aggregate of all these K economies.

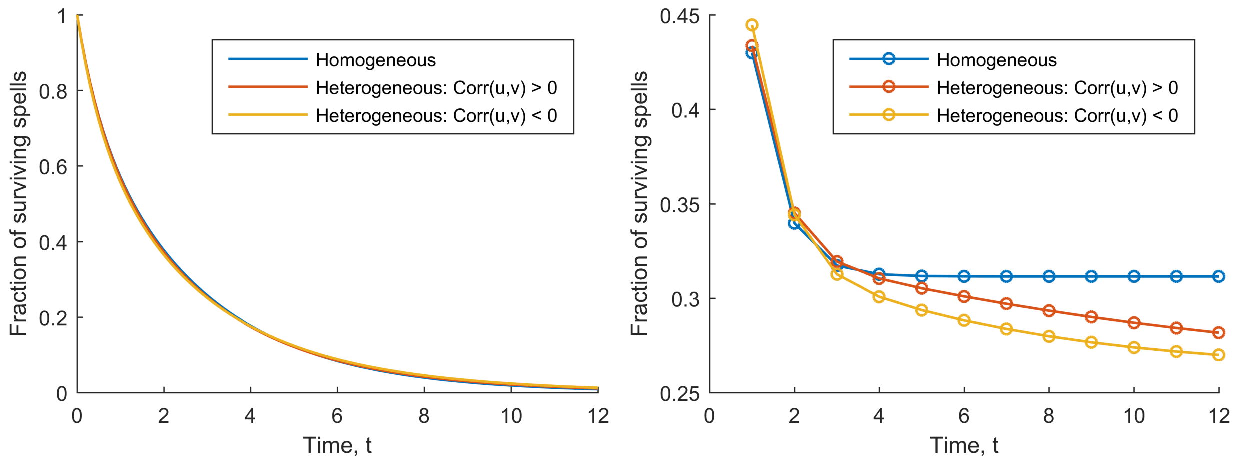

Figure 5. Survivor functions and hazards under stylized heterogeneity.

a weighted average of the ψ_k with weights for t small proportional to $\beta_k(1-p_k)$, and with weights for t large putting all the mass on the k for which ψ_k takes its lowest value. That is, the hazard is a decreasing function of time: it declines smoothly over time from an initial value

$$y_{0^+} = \frac{\sum_{k=1}^{K} \beta_k(1-p_k) \cdot \psi_k}{\sum_{k=1}^{K} \beta_k(1-p_k)}$$

and converges to $\min_k \psi_k$ as $t \to +\infty$. This is well known: because agents with lower job-finding rates have on average longer spells, they are represent an ever larger share of the pool of unemployed workers with longer duration spells.

We can investigate the effect of heterogeneity numerically in several ways. First, we compare the survivor functions and hazard rates for economies with the same baseline unemployment and vacancy rates ($u = 0.054$, $v = 0.027$), but in which islands are heterogeneous in different ways.

1. a case in which all islands are identical;

2. a case in which there are two types of islands, with $\mathrm{Corr}(u_k, v_k) > 0$—specifically, $\alpha_1 = \alpha_2 = 0.5$, $u_1/u = v_1/v = 0.8$, and $u_2/u = v_2/v = 1.2$;

3. a case in which there are two types of islands, with $\mathrm{Corr}(u_k, v_k) < 0$—specifically, $\alpha_1 = \alpha_2 = 0.5$, $u_1/u = v_2/v = 1.5$, and $u_2/u = v_1/v = 0.5$;

The resulting survivor and hazard functions are shown in Figure 5. As can be seen both the cases with heterogeneity exhibit a declining hazard, and are qualitatively similar. It seems relatively challenging to identify the distribution of island heterogeneity from this kind of exercise.

A second approach is to attempt to see whether the empirical hazard rate can be matched by our model allowing for appropriate heterogeneity. We consider two separate approaches to this. First, we see whether, given arbitrary heterogeneity, our stock-flow matching model is able to match the empirical survivor function, which we take from CPS data for the baseline period under

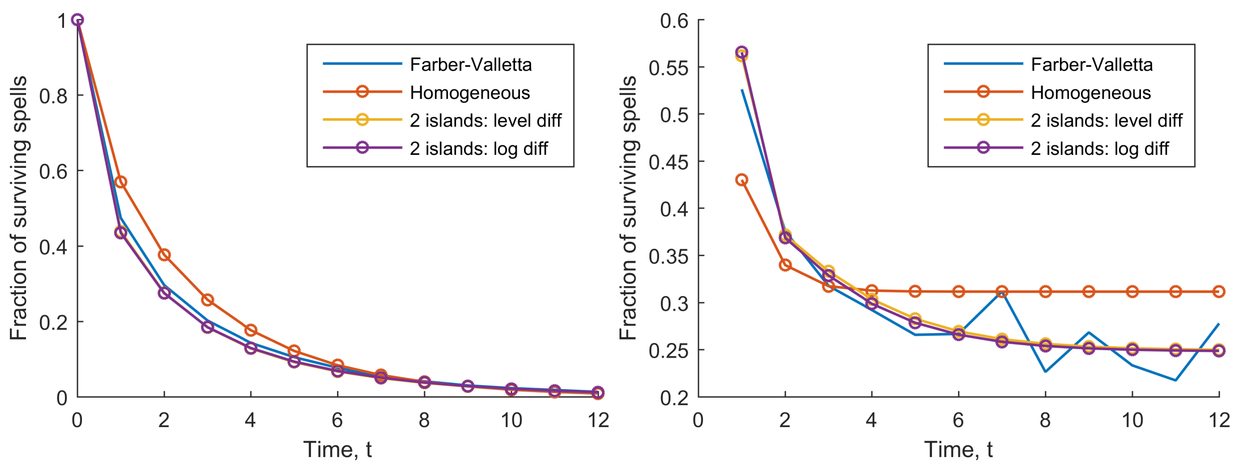

Figure 6. Survivor functions and hazards under heterogeneity estimated to match empirical hazard functions.

consideration, that is, March 2002 through February 2004 Farber and Valletta (2013). It turns out that allowing for only two types of islands allows us to match the empirical survivor function and hazard function almost exactly. More formally, to do this, we estimate the parameters of the archipelagos, that is, $\{(\alpha_k, u_k, v_k)\}_{k=1}^{K}$, subject to the constraints that $\sum_{k=1}^{K} \alpha_k = 1$ and that the average unemployment and vacancy rates match the baseline targets $u = 0.054$ and $v = 0.027$. We estimate these parameters by minimizing the the square norm of the distance between the empirical hazard function for the baseline period and its model-generated counterpart. If we allow only for $K = 2$, the model-generated hazard function already matches very well its empirical counterpart: the best-fitting hazard function is generated by the parameter vector

$$(\alpha_1, u_1, v_1) = (0.611, 0.063, 0.044); \qquad (\alpha_2, u_2, v_2) = (0.389, 0.039, 1.5 \times 10^{-9}).$$

It is interesting that this calibration relies on an island with almost no vacancies. (Mechanically, the reason that ψ^u can be reasonably high even with $v \approx 0$ is that reallocation shocks mean that unemployed workers in the stock encounter a reasonable number of job openings; vacancies, on the other hand, are filled with probability essentially 1 when in the flow.) The resulting survivor and hazard functions are shown in Figure 6.

A second approach to investigating the ability of our stock-flow model to match the empirical hazard rate is to use data on the cross-sectional dispersion of unemployment and vacancies. For now, to investigate this, we disaggregate the aggregate unemployment and vacancy rates using the CPS to assign unemployed workers to their most recent industry and JOLTS to assign vacancies to industries; in this we follow the only exercise in Şahin et al. (2014) that can be replicated using publicly accessible data. The results for our baseline period of 2002-2004 are shown in Figure 7. It is apparent that allowing even for this limited degree of disaggregation already helps our model substantially in matching empirical hazard rates.

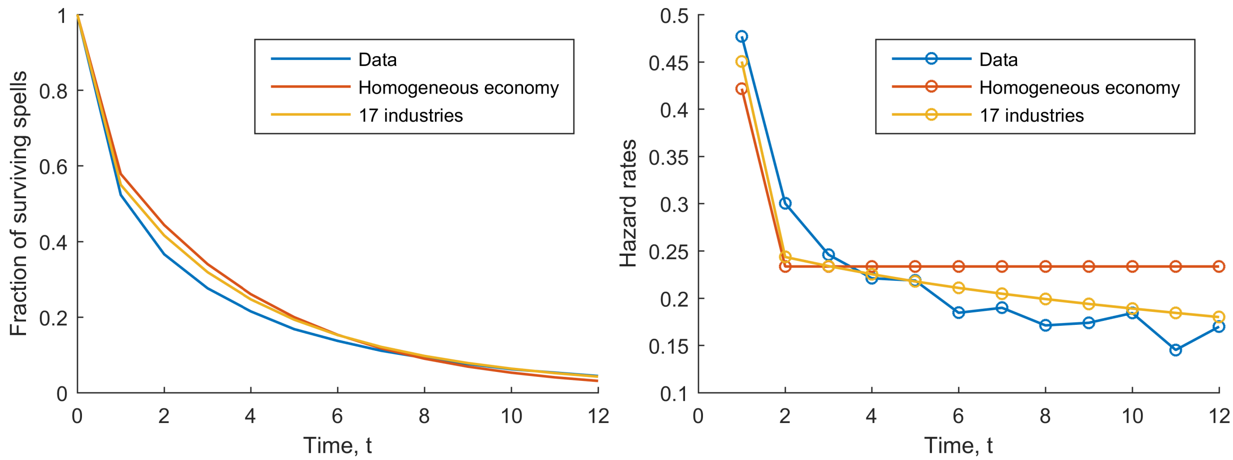

Figure 7. Effect of allowing for heterogeneity across 17 major industries. Left panel: survivor functions. Right panel: hazard functions.

4 Planner's Problem

This section characterizes efficiency of the equilibrium in our model. It shows that the equilibrium is generically inefficient. There is only one very special and limited case where the equilibrium is efficient for any distribution $F(x)$ of match heterogeneity: only if entry is exogenous, only if $\chi^u = \delta$ and $\chi^v = s$, and only if agents in the stock compete Bertrand, in the sense that the agent with the highest x draw receives flow payoff equal to the difference between her x draw and either the second-best, or $\tilde{x}$, whichever is higher.

Notice that in this section we use the notation u, v, and m, rather than U, V, and M, for the measures of unemployed workers, vacant jobs, and matched worker-job pairs.

4.1 Planner's problem with fixed entry

For concreteness, first consider the easiest case in which entry is exogenous.

The planner's problem can be represented as

$$\max_{m(\cdot),u(\cdot),v(\cdot),\tilde{x}(\cdot)} \int_0^\infty \Bigg[bu(t) - \gamma v(t) - c\mu(t) + \frac{\mu(t)+\chi^v v + sm}{r+s+\delta} \int_{\tilde{x}(t)}^\infty x u(t) e^{-u[1-F(x)]} f(x)\,dx \\ + \frac{\lambda + \chi^u u + \delta m}{r+s+\delta} \int_{\tilde{x}(t)}^\infty x v(t) e^{-v[1-F(x)]} f(x)\,dx \Bigg] e^{-rt}\,dt \tag{35}$$

subject to the flow constraints

$$\dot{u}(t) = -(\mu(t) + \chi^v v(t) + sm(t)) \left(1 - e^{-u(t)(1-F(\tilde{x}(t)))}\right) \\ + (\lambda + \chi^u u(t) + \delta m(t)) e^{-v(t)(1-F(\tilde{x}(t)))} - (s+\chi^u) u(t) \tag{36}$$

$$\begin{aligned}\dot{v}(t) &= (\mu(t) + \chi^v v(t) + sm(t))e^{-u(t)(1-F(\tilde{x}(t)))} \\ &\quad - (\lambda + \chi^u u(t) + \delta m(t))\left(1 - e^{-v(t)(1-F(\tilde{x}(t)))}\right) - (\delta + \chi^v)v(t)\end{aligned} \tag{37}$$

$$\begin{aligned}\dot{m}(t) &= (\mu(t) + \chi^v v(t) + sm(t))\left(1 - e^{-u(t)(1-F(\tilde{x}(t)))}\right) \\ &\quad + (\lambda + \chi^u u(t) + \delta m(t))\left(1 - e^{-v(t)(1-F(\tilde{x}(t)))}\right) - (s + \delta)\, m(t).\end{aligned} \tag{38}$$

The form of the objective function of the planner requires some comment. $\tilde{x}(t)$ is the cutoff match quality the planner uses to decide whether matches formed at t are acceptable. Note that this is a jump variable: it affects only new matches. The planner accounts for the value of the output formed by such matches at the time of their formation; since the rate of pure time discounting is r and the matches end at rate $s + \delta$, this means that the present value of the output from a newly formed match of quality x to the planner is $x/(r + s + \delta)$. The output from the stock of existing matches does not enter the objective because the present value of the output of each match was already accounted for at the time that match was formed. (Writing the objective this way assumes that the planner will never terminate existing matches; this is without loss of generality in an optimal steady state since the optimal cutoff $\tilde{x}(t)$ will in fact be constant over time.)

Dropping the time arguments, necessary conditions to characterize an optimal steady state can be found by writing the Hamiltonian

$$\begin{aligned}\mathcal{H} &= bu - \gamma v - c\mu \\ &\quad + \frac{\mu + \chi^v v + sm}{r + s + \delta} \int_{\tilde{x}}^{\infty} xue^{-u[1-F(x)]} f(x)\, dx \\ &\quad + \frac{\lambda + \chi^u u + \delta m}{r + s + \delta} \int_{\tilde{x}}^{\infty} xve^{-v[1-F(x)]} f(x)\, dx \\ &\quad + \eta^u \left[-(\mu + \chi^v v + sm)\left[1 - e^{-u(1-F(\tilde{x}))}\right] + (\lambda + \chi^u u + \delta m)e^{-v(1-F(\tilde{x}))} - (s + \chi^u)u\right] \\ &\quad + \eta^v \left[(\mu + \chi^v v + sm)e^{-u(1-F(\tilde{x}))} - (\lambda + \chi^u u + \delta m)\left[1 - e^{-v(1-F(\tilde{x}))}\right] - (\delta + \chi^v)v\right] \\ &\quad + \eta^m \left[(\mu + \chi^v v + sm)\left[1 - e^{-u(1-F(\tilde{x}))}\right] + (\lambda + \chi^u u + \delta m)\left[1 - e^{-v(1-F(\tilde{x}))}\right] - (s + \delta)\, m\right]\end{aligned} \tag{39}$$

We can now take first order conditions. The FOC with respect to $\tilde{x}$ takes the form

$$\begin{aligned}0 &= -\frac{\mu + \chi^v v + sm}{r + s + \delta} \tilde{x}ue^{-u[1-F(\tilde{x})]} f(\tilde{x}) + \frac{\lambda + \chi^u u + \delta m}{r + s + \delta} \tilde{x}ve^{-v[1-F(\tilde{x})]} f(\tilde{x}) \\ &\quad + (\eta^u + \eta^v - \eta^m)\left[(\mu + \chi^v v + sm)uf(\tilde{x})e^{-u(1-F(\tilde{x}))} + (\lambda + \chi^u u + \delta m)vf(\tilde{x})e^{-v(1-F(\tilde{x}))}\right]\end{aligned} \tag{40}$$

or equivalently

$$\frac{\tilde{x}}{r + s + \delta} = \eta^u + \eta^v - \eta^m. \tag{41}$$

This equation is intuitive. The left side is the present value of the output from a marginally acceptable match, while the right side is the opportunity cost from forming that match, which is the value of an unemployed worker and a vacancy in the stock less the value (net of output in the current match) of a matched firm-worker pair.

The FOC with respect to u takes the form

$$\begin{aligned}r\eta^u = b &+ \frac{\mu+\chi^v v+sm}{r+s+\delta}\int_{\tilde{x}}^{\infty} x\left[1-u(1-F(x))\right]e^{-u[1-F(x)]}f(x)\,dx \\ &+ \frac{\chi^u}{r+s+\delta}\int_{\tilde{x}}^{\infty} xve^{-v[1-F(x)]}f(x)\,dx \\ &- (\eta^u+\eta^v-\eta^m)\left[(\mu+\chi^v v+sm)[1-F(\tilde{x})]e^{-u(1-F(\tilde{x}))}+\chi^u\left[1-e^{-v(1-F(\tilde{x}))}\right]\right] \\ &- s\eta^u.\end{aligned} \tag{42}$$

Substitute for $\eta^u+\eta^v-\eta^m$ using (41) and use that

$$\int_{\tilde{x}}^{\infty}\left[1-u(1-F(x))\right]e^{-u[1-F(x)]}f(x)\,dx = \left[-(1-F(x))e^{-u(1-F(x))}\right]\Big|_{x=\tilde{x}}^{\infty} = (1-F(\tilde{x}))e^{-u(1-F(\tilde{x}))}$$

and

$$\int_{\tilde{x}}^{\infty} ve^{-v[1-F(x)]}f(x)\,dx = e^{-v(1-F(x))}\Big|_{x=\tilde{x}}^{\infty} = 1-e^{-v(1-F(\tilde{x}))} \tag{43}$$

to write (42) as

$$\begin{aligned}(r+s)\eta^u = b &+ \frac{\mu+\chi^v v+sm}{r+s+\delta}\int_{\tilde{x}}^{\infty}(x-\tilde{x})\left[1-u(1-F(x))\right]e^{-u[1-F(x)]}f(x)\,dx \\ &+ \frac{\chi^u}{r+s+\delta}\int_{\tilde{x}}^{\infty}(x-\tilde{x})\,ve^{-v[1-F(x)]}f(x)\,dx.\end{aligned} \tag{44}$$

A similar set of manipulations establishes that the FOC with respect to v takes the form

$$\begin{aligned}(r+\delta)\eta^v = -\gamma &+ \frac{\chi^v}{r+s+\delta}\int_{\tilde{x}}^{\infty}(x-\tilde{x})\,ue^{-u[1-F(x)]}f(x)\,dx \\ &+ \frac{\lambda+\chi^u u+\delta m}{r+s+\delta}\int_{\tilde{x}}^{\infty}(x-\tilde{x})\left[1-v(1-F(x))\right]e^{-v[1-F(x)]}f(x)\,dx.\end{aligned} \tag{45}$$

and that the FOC with respect to m takes the form

$$\begin{aligned}(r+s+\delta)\eta^m = s\eta^v+\delta\eta^u &+ \frac{s}{r+s+\delta}\int_{\tilde{x}}^{\infty}(x-\tilde{x})\,ue^{-u[1-F(x)]}f(x)\,dx \\ &+ \frac{\delta}{r+s+\delta}\int_{\tilde{x}}^{\infty}(x-\tilde{x})\,ve^{-v[1-F(x)]}f(x)\,dx.\end{aligned} \tag{46}$$

Add (44) and (45), subtract (46), and substitute into (41) to obtain that

$$\begin{aligned}\tilde{x} - b + \gamma = & \frac{\lambda + \chi^u u + \delta m}{r+s+\delta} \int_{\tilde{x}}^{\infty} (x - \tilde{x}) \left[1 - v(1-F(x))\right] e^{-v[1-F(x)]} f(x)\, dx \\ & + \frac{\mu + \chi^v v + sm}{r+s+\delta} \int_{\tilde{x}}^{\infty} (x - \tilde{x}) \left[1 - u(1-F(x))\right] e^{-u[1-F(x)]} f(x)\, dx \\ & + \frac{\chi^u - \delta}{r+s+\delta} \int_{\tilde{x}}^{\infty} (x - \tilde{x})\, v e^{-v[1-F(x)]} f(x)\, dx \\ & + \frac{\chi^v - s}{r+s+\delta} \int_{\tilde{x}}^{\infty} (x - \tilde{x})\, u e^{-u[1-F(x)]} f(x)\, dx. \end{aligned} \tag{47}$$

For comparison, the corresponding equation characterizing $\tilde{x}$ in the decentralized equilibrium takes the form

$$\tilde{x} - b + \gamma = (\lambda + \chi^u u + \delta m) H^s + (\mu + \chi^v v + sm) G^s + (\chi^u - \delta) G^f + (\chi^v - s) H^f. \tag{48}$$

Case 1: $\chi^u = \delta$ and $\chi^v = s$. If $\chi^u = \delta$ and if $\chi^v = s$, then efficiency requires that

$$\tilde{x} - b + \gamma = \frac{\lambda(s+\delta)}{s} \hat{H}^s(\tilde{x}, v) + \frac{\mu(s+\delta)}{\delta} \hat{G}^s(\tilde{x}, u). \tag{49}$$

where

$$\hat{G}^s(\tilde{x}, u) \equiv \frac{1}{r+s+\delta} \int_{\tilde{x}}^{\infty} (x - \tilde{x}) \left[1 - u(1-F(x))\right] e^{-u[1-F(x)]} f(x)\, dx \tag{50}$$

and

$$\hat{H}^s(\tilde{x}, v) \equiv \frac{1}{r+s+\delta} \int_{\tilde{x}}^{\infty} (x - \tilde{x}) \left[1 - v(1-F(x))\right] e^{-v[1-F(x)]} f(x)\, dx. \tag{51}$$

The equation characterizing $\tilde{x}$ in the decentralized equilibrium, on the other hand, takes the form

$$\tilde{x} - b + \gamma = \frac{\lambda(s+\delta)}{s} H^s(\tilde{x}, v) + \frac{\mu(s+\delta)}{\delta} G^s(\tilde{x}, u). \tag{52}$$

Thus, it is clear that a sufficient condition for the equilibrium $\tilde{x}$ to be efficient is that

$$G^s(\tilde{x}, u) = \hat{G}^s(\tilde{x}, u) \tag{53}$$

for all $\tilde{x}$ and u, and

$$H^s(\tilde{x}, v) = \hat{H}^s(\tilde{x}, v) \tag{54}$$

for all $\tilde{x}$ and v.

On the other hand, suppose that the equilibrium $\tilde{x}$ is constrained efficient for all values of the exogenous parameters λ, μ, $\chi^u = \delta$, $\chi^v = s$, b, γ, and r. We would like to show that in this case $G^s(\cdot) \equiv \hat{G}^s(\cdot)$ and $H^s(\cdot) \equiv \hat{H}^s(\cdot)$. I don't know whether this is true in general.

The obvious proof strategy, which doesn't quite work, is the following. First, observe that the steady-state flow balance equations take the same form for the optimum and the equilibrium, conditional on $\tilde{x}$:

$$0 = -\frac{\mu}{\delta}\left(1 - e^{-u(1-F(\tilde{x})))}\right) + \frac{\lambda}{s}e^{-v(1-F(\tilde{x}))} - u \tag{55}$$

$$0 = \frac{\mu}{\delta}e^{-u(1-F(\tilde{x}))} - \frac{\lambda}{s}\left(1 - e^{-v(1-F(\tilde{x}))}\right) - v \tag{56}$$

$$0 = \frac{\mu}{\delta}\left(1 - e^{-u(1-F(\tilde{x})))}\right) + \frac{\lambda}{s}\left(1 - e^{-v(1-F(\tilde{x}))}\right) - m. \tag{57}$$

Note that these equations depend only on λ/s and μ/δ (and not on λ and s separately or on μ and δ separately.

Next, consider any $(u, v, \tilde{x})$ tuple that is consistent with the flow balance equation (55), (56), and (57) for some $(\lambda/s, \mu/\delta)$. When is this possible? (55) and (56) are a pair of linear equations in λ/s and μ/δ, and have solution

$$\frac{\lambda}{s} = \frac{Au + (1-A)v}{A+B-1} \qquad \text{and} \qquad \frac{\mu}{\delta} = \frac{(1-B)u + Bv}{A+B-1} \tag{58}$$

where

$$A = e^{-u(1-F(\tilde{x}))} \qquad \text{and} \qquad B = e^{-v(1-F(\tilde{x}))}. \tag{59}$$

Thus, the solution for λ/s and μ/δ exists provided $A + B \neq 1$, and, since $0 < A, B < 1$, both λ/s and μ/δ are positive whenever $A + B > 1$, that is, whenever

$$e^{-u(1-F(\tilde{x}))} + e^{-v(1-F(\tilde{x}))} > 1. \tag{60}$$

That is, if u and v are large, then the flow equations can only account for this if $1 - F(\tilde{x})$ is small enough, that is, if $F(\tilde{x})$ is large enough, that is, if a small enough fraction of matches is acceptable.[14] Given a $(u, v, \tilde{x})$ satisfying (60), write $L = L(u, v, \tilde{x})$ and $M = M(u, v, \tilde{x})$ for the corresponding values of λ/s and μ/δ for which $(u, v, \tilde{x})$ satisfy (55) and (56). Choose $b - \gamma$ and $s + \delta$ such that the planner's FOC for $\tilde{x}$, (49), is satisfied. (The choice of $s + \delta$ can be made independently of L and M by scaling $\lambda = Ls$ and $\mu = M\delta$.) For this set of parameter values, the equilibrium $\tilde{x}$ is determined by (52). Comparing this equation with (49), it follows that

$$LH^s(\tilde{x}, v) + MG^s(\tilde{x}, u) = L\hat{H}^s(\tilde{x}, v) + M\hat{G}^s(\tilde{x}, u). \tag{61}$$

If L and M could be chosen arbitrarily, this equation would imply that $H^s(\tilde{x}, v) = \hat{H}^s(\tilde{x}, v)$ and $G^s(\tilde{x}, u) = \hat{G}^s(\tilde{x}, u)$. But unfortunately L and M are not arbitrary: they are fully determined by $(u, v, \tilde{x})$. Thus, this argument is not valid.

[14] One can show also that given the above values for λ/s and μ/δ, then (57) defines

$$m = \frac{(1-B)u + (1-A)v}{A+B-1},$$

which is also positive provided that $A + B > 1$.

To make progress, restrict attention to the case in which bargaining is *symmetric*, in the sense that $G^s(\cdot) \equiv H^s(\cdot)$. (We know that this is true of the planner's counterparts $\hat{G}^s(\cdot) \equiv \hat{H}^s(\cdot)$, but we haven't assumed this about the decentralized equilibrium up to now.) Next, choose $\tilde{x}$ and u. First suppose that we can choose $L > 0$ such that the symmetric allocation $(u, u, \tilde{x})$ is consistent with $\lambda/s = \mu/\delta = L$ in the flow balance equations (55) and (56). Simplifying the conditions above, this is possible when

$$2e^{-u(1-F(\tilde{x}))} > 1 \tag{62}$$

and requires setting

$$L = \frac{u}{2e^{-u(1-F(\tilde{x}))} - 1}.$$

Next, choose $b - \gamma$ and $s + \delta$ such that $\tilde{x}$ and u are consistent with the planner's FOC for $\tilde{x}$, (49), that is,

$$\frac{\tilde{x} - b + \gamma}{s + \delta} = 2L\hat{G}^s(\tilde{x}, u).$$

The right side uses the fact that $\hat{H}^s(\tilde{x}, u) = \hat{G}^s(\tilde{x}, u)$. In order for the equilibrium $\tilde{x}$ to be constrained efficient, the same value of $\tilde{x}$ must solve the corresponding equilibrium condition

$$\frac{\tilde{x} - b + \gamma}{s + \delta} = 2LG^s(\tilde{x}, u).$$

It follows immediately that

$$G^s(\tilde{x}, u) = \hat{G}^s(\tilde{x}, u). \tag{63}$$

It remains to deal with $(\tilde{x}, u)$ which do not satisfy (62). Given such a pair, choose a v small enough that

$$e^{-u(1-F(\tilde{x}))} + e^{-v(1-F(\tilde{x}))} > 1. \tag{64}$$

Since $e^{-u(1-F(\tilde{x}))} \leq 1/2$, this implies in particular that $e^{-v(1-F(\tilde{x}))} > 1/2$, so that

$$2e^{-v(1-F(\tilde{x}))} > 1.$$

It follows from the preceding argument and specifically from (63) that

$$G^s(\tilde{x}, v) = \hat{G}^s(\tilde{x}, v). \tag{65}$$

Now choose $L = \lambda/s$ and $M = \mu/\delta$ consistent with the tuple $(u, v, \tilde{x})$. Choose $b - \gamma$ and $r + s$ such that $\tilde{x}$ satisfies the planner's FOC (49) for these values of u and v. If the equilibrium is to be efficient, we have that the symmetric analog to (61) must hold, that is,

$$LG^s(\tilde{x}, v) + MG^s(\tilde{x}, u) = L\hat{G}^s(\tilde{x}, v) + M\hat{G}^s(\tilde{x}, u).$$

But substituting from (65), it follows immediately that also

$$G^s(\tilde{x}, u) = \hat{G}^s(\tilde{x}, u). \tag{66}$$

This completes the argument.

Case 2: χ^u and χ^v arbitrary

In case we do not restrict attention only to the case where $\chi^u = \delta$ and $\chi^v = s$, then the planner's FOC for $\tilde{x}$, (47), can be written

$$\tilde{x}-b+\gamma = (\lambda+\chi^u u+\delta m)\hat{H}^s(\tilde{x}, v)+(\mu+\chi^v v+sm)\hat{G}^s(\tilde{x}, u)+(\chi^u-\delta)G^f(\tilde{x}, v)+(\chi^v-s)H^f(\tilde{x}, u) \tag{67}$$

where $\hat{G}^s(\cdot)$ and $\hat{H}^s(\cdot)$ are as in (50) and (51) respectively, and

$$\hat{G}^f(\tilde{x}, u) = \frac{1}{r+s+\delta}\int_{\tilde{x}}^{\infty} (x-\tilde{x})\, ve^{-v[1-F(x)]} f(x)\, dx \tag{68}$$

and

$$\hat{H}^f(\tilde{x}, v) = \frac{1}{r+s+\delta}\int_{\tilde{x}}^{\infty} (x-\tilde{x})\, ue^{-u[1-F(x)]} f(x)\, dx. \tag{69}$$

It is clear that a sufficient condition for efficiency is that $G^s(\cdot) = \hat{G}^s(\cdot)$, $H^s(\cdot) = \hat{H}^s(\cdot)$, $G^f(\cdot) = \hat{G}^f(\cdot)$, and $H^f(\cdot) = \hat{H}^f(\cdot)$. I claim that if the equilibrium and the planner's choice of $\tilde{x}$ correspond for all parameter tuples $(b, \gamma, \lambda, \mu, s, \delta, \chi^u, \chi^v, r)$, and if the equilibrium payoffs for agents in the stock are symmetric in the sense that $G^s(\cdot) = H^s(\cdot)$ then in fact $G^s(\cdot) = \hat{G}^s(\cdot)$, $G^f(\cdot) = \hat{G}^f(\cdot)$, and $H^f(\cdot) = \hat{H}^f(\cdot)$. (Note that there is no assumption of symmetry for the flow payoffs; that is, I do not need to assume that $G^f(\cdot) = H^f(\cdot)$.)

To prove the first of these claims, it is sufficient to restrict attention to the case where $\chi^u = \delta$ and $\chi^v = s$; then the argument given above for Case 1 applies, so that we can deduce that $G^s(\cdot) = \hat{G}^s(\cdot)$.

To complete the proof, let $\tilde{x}$ and v be given, and choose some $u > 0$ and some parameter tuple $(\lambda, \mu, s, \delta, \chi^u, \chi^v)$ with $\chi^v = s$ and $\chi^u \neq \delta$ such that $(u, v, \tilde{x})$ solve the flow balance equations

$$0 = -(\mu+\chi^v v+sm)\left[1-e^{-u(1-F(\tilde{x}))}\right]+(\lambda+\chi^u u+\delta m)e^{-v(1-F(\tilde{x}))}-(s+\chi^u)u \tag{70}$$

$$0 = (\mu+\chi^v v+sm)e^{-u(1-F(\tilde{x}))}-(\lambda+\chi^u u+\delta m)\left[1-e^{-v(1-F(\tilde{x}))}\right]-(\delta+\chi^v)v \tag{71}$$

$$0 = (\mu+\chi^v v+sm)\left[1-e^{-u(1-F(\tilde{x}))}\right]+(\lambda+\chi^u u+\delta m)\left[1-e^{-v(1-F(\tilde{x}))}\right]-(s+\delta)\, m. \tag{72}$$

To see that this can be done, write $\Delta = (\chi^u - \delta)/(s+\delta)$. Simplify the flow balance equations (70) and (71) by substituting for $\chi^v = s$ and solve (this is straightforward using the similar structure of the equations to (55) and (56)), to see that a generalization of (58) holds:

$$\frac{\lambda}{s} = \frac{Au+(1-A)v+(1-B)\Delta u}{A+B-1} \qquad \text{and} \qquad \frac{\mu}{\delta} = \frac{(1-B)u+Bv+(1-B)\Delta u}{A+B-1} \tag{73}$$

where A and B are as in (59). If $\Delta > 0$, then both these expressions are again positive provided that $A + B > 1$. Thus, the desired parameter choice can be made provided that u is small enough that

$$e^{-u(1-F(\tilde{x}))} + e^{-v(1-F(\tilde{x}))} > 1$$

and provided that $\Delta > 0$. Notice that (72) can then be solved for

$$m = \frac{\mu}{\delta}(1 - A) + \left(\frac{\lambda}{s} + \Delta u\right)(1 - B),$$

which is also positive since $A, B < 1$.

Now, for this parameter tuple, choose $b - \gamma$ and $s + \delta$ so that (67) holds, and then choose $\chi^u > \delta$ accordingly so that $\Delta = (\chi^u - \delta)/(s + \delta)$. Then for the equilibrium to be efficient for this parameter tuple, we must have that

$$\begin{aligned} 0 = &\left[(\lambda + \chi^u u + \delta m)\hat{H}^s(\tilde{x}, v) + (\mu + \chi^v v + sm)\hat{G}^s(\tilde{x}, u) + (\chi^u - \delta)\hat{G}^f(\tilde{x}, v)\right] \\ &- \left[(\lambda + \chi^u u + \delta m)H^s(\tilde{x}, v) + (\mu + \chi^v v + sm)G^s(\tilde{x}, u) + (\chi^u - \delta)G^f(\tilde{x}, v)\right], \end{aligned}$$

that is,

$$\begin{aligned} 0 = &\left[\left(\frac{\lambda}{s} + \Delta u\right)\hat{H}^s(\tilde{x}, v) + \frac{\mu}{\delta}\hat{G}^s(\tilde{x}, u) + \Delta\hat{G}^f(\tilde{x}, v)\right] \\ &- \left[\left(\frac{\lambda}{s} + \Delta u\right)H^s(\tilde{x}, v) + \frac{\mu}{\delta}G^s(\tilde{x}, u) + \Delta G^f(\tilde{x}, v)\right]. \end{aligned}$$

Since we know from Case 1 that efficiency under symmetry requires that $G^s(\cdot) = H^s(\cdot) = \hat{G}^s(\cdot) = \hat{H}^s(\cdot)$, and since $\Delta > 0$, this immediately implies that

$$G^f(\tilde{x}, v) = \hat{G}^f(\tilde{x}, v).$$

Since $(\tilde{x}, v)$ was arbitrary, it follows that $G^f(\cdot) \equiv \hat{G}^f(\cdot)$.

A very similar argument setting $\chi^u = \delta$ and $\chi^v > s$ can be used to prove that $H^f(\cdot) \equiv \hat{H}^f(\cdot)$. This completes the proof.

Summarizing, we can state the result that we have proved formally as follows.

Proposition 5. *1. If $G^s(\cdot) = H^s(\cdot) = \hat{G}^s(\cdot)$ and $G^f(\cdot) = H^f(\cdot) = \hat{G}^f(\cdot)$, then the equilibrium $\tilde{x}$ is constrained efficient for any $(b, \gamma, \lambda, \mu, s, \delta, \chi^u, \chi^v)$.*

2. Suppose that $G^s(\cdot) \equiv H^s(\cdot)$. If the equilibrium $\tilde{x}$ is constrained efficient for all $(b, \gamma, \lambda, \mu, s, \delta, \chi^u, \chi^v)$ with $\chi^u = \delta$ and $\chi^v = s$, then $G^s(\cdot) \equiv \hat{G}^s(\cdot)$.

3. Suppose that $G^s(\cdot) \equiv H^s(\cdot)$. If the equilibrium $\tilde{x}$ is constrained efficient for all $(b, \gamma, \lambda, \mu, s, \delta, \chi^u, \chi^v)$, then in addition $G^f(\cdot) = H^f(\cdot) = \hat{G}^f(\cdot)$.

4.2 Planner's problem with endogenous entry

When entry is endogenous, and when all islands are symmetric, the planner's problem can again be written as (35) subject to (36), (37), and (38), with the sole modification that the planner also chooses the jump variable $\mu(t)$ at all times t. The necessary conditions for an interior optimum can again be found by solving the Hamiltonian (39), with an additional FOC for the choice of μ, which takes the form

$$0 = -c + \frac{1}{r+s+\delta} \int_{\tilde{x}}^{\infty} xue^{-u[1-F(x)]} f(x)\, dx - (\eta^u + \eta^v - \eta^m) \left[1 - e^{-u(1-F(\tilde{x}))}\right] + \eta^v \tag{74}$$

Substitute for $\eta^u + \eta^v - \eta^m$ from (41) and use (43) to write this in the form

$$\begin{aligned} c &= \frac{1}{r+s+\delta} \int_{\tilde{x}}^{\infty} (x - \tilde{x}) ue^{-u[1-F(x)]} f(x)\, dx + \eta^v & (75) \\ &= \hat{H}^f(\tilde{x}, u) + \eta^v. & (76) \end{aligned}$$

The free-entry condition in the equilibrium, on the other hand, takes the form

$$c = J^f(\tilde{x}, u, v) = H^f(\tilde{x}, u) + J^s(\tilde{x}, u, v).$$

Since we showed in Proposition 5 that efficiency of $\tilde{x}$ requires that $H^f(\cdot) = \hat{H}^f(\cdot)$, it follows that entry will be efficient iff $\eta^v = J^s$. (45) shows that

$$(r+\delta)\eta^v = -\gamma + \chi^v \hat{H}^f(\tilde{x}, u) + (\lambda + \chi^u u + \delta m)\hat{H}^s(\tilde{x}, v); \tag{77}$$

the corresponding HJB equation in the equilibrium is

$$(r+\delta)J^s = -\gamma + \chi^v H^f(\tilde{x}, u) + (\lambda + \chi^u u + \delta m) H^s(\tilde{x}, v). \tag{78}$$

Comparing (77) and (78), and using the fact that $H^f(\cdot) \equiv \hat{H}^f(\cdot)$ and $H^s(\cdot) \equiv \hat{H}^s(\cdot)$ if $\tilde{x}$ is efficient, implies the following result, which generalizes Proposition 5.

Proposition 6. *1. If $G^s(\cdot) = H^s(\cdot) = \hat{G}^s(\cdot)$ and $G^f(\cdot) = H^f(\cdot) = \hat{G}^f(\cdot)$, then the equilibrium $\tilde{x}$ and the equilibrium entry rate μ are constrained efficient for any $(b, \gamma, \lambda, \mu, s, \delta, \chi^u, \chi^v)$.*

2. Suppose that $G^s(\cdot) \equiv H^s(\cdot)$. If the equilibrium $\tilde{x}$ is constrained efficient for all $(b, \gamma, \lambda, \mu, s, \delta, \chi^u, \chi^v)$, then in addition $G^f(\cdot) = H^f(\cdot) = \hat{G}^f(\cdot)$ and the entry rate determined by free entry is also constrained efficient.

Note that a slight weakness of this statement of the result is that the hypothesis of part 2 of the Proposition is that $\tilde{x}$ is constrained efficient for all parameter tuples, treating μ as exogenous. However, if entry is endogenous, then μ should be a monontonic function of the entry cost c. While we could work harder to tighten this up, I'm not sure what the payoff is.

4.3 Bertrand competition among agents in the stock

Can the efficiency requirement that $G^s(\tilde{x}, u) \equiv \hat{G}^s(\tilde{x}, u)$ be satisfied? In this section we show that it can, if bargaining takes the following form: the agents in the stock compete Bertrand. More precisely, what this means is that when a job in the flow meets k workers in the stock, then the payoffs are determined as follows. Without loss of generality, suppose that the realizations are ordered, $x_1 \geq x_2 \geq \ldots \geq x_k$.

- If $x_1 < \tilde{x}$, then no match is formed, and all agents receive zero gain.
- If $x_1 \geq \tilde{x} > x_2$, then the agent in the flow matches with stock agent 1; the agent in the flow receives flow payoff $\tilde{x}$ and the agent in the stock receives $x_1 - \tilde{x}$ as long as the match continues.
- If $x_1 \geq x_2 \geq \tilde{x}$, then the agent in the flow matches with stock agent 1; the agent in the flow receives flow payoff x_2 and the agent in the stock receives $x_1 - x_2$ as long as the match continues.

I claim that this protocol delivers $G^s(\tilde{x}, u) = \hat{G}^s(\tilde{x}, u)$. To see this, the simplest way is to notice that the expected total flow surplus $S(\tilde{x}, u)$ produced when a job arrives in the flow is given by the expectation of $(x_1 - \tilde{x})$, that is,

$$\begin{aligned} S(\tilde{x}, u) &= \sum_{k=0}^{\infty} \frac{e^{-u} u^k}{k!} \int_{\tilde{x}}^{\infty} (x - \tilde{x}) k F(x)^{k-1} f(x)\, dx \\ &= \int_{\tilde{x}}^{\infty} (x - \tilde{x}) \sum_{k=1}^{\infty} \frac{e^{-u} u^{k-1} F(x)^{k-1}}{(k-1)!} u f(x)\, dx \\ &= \int_{\tilde{x}}^{\infty} (x - \tilde{x}) u e^{-u(1-F(x))} f(x)\, dx. \end{aligned} \tag{79}$$

The expected gain for the agent in the flow, $H^f(\tilde{x}, u)$ can be calculated as the expected value of $x_{[2]} - \tilde{x}$, where $x_{[2]}$ is the second-highest realization.

$$\begin{aligned} H^f(\tilde{x}, u) &= \sum_{k=0}^{\infty} \frac{e^{-u} u^k}{k!} \int_{\tilde{x}}^{\infty} (x - \tilde{x}) k(k-1) F(x)^{k-2} (1 - F(x)) f(x)\, dx \\ &= \int_{\tilde{x}}^{\infty} (x - \tilde{x}) \sum_{k=2}^{\infty} \frac{e^{-u} u^{k-2} F(x)^{k-2}}{(k-2)!} u^2 (1 - F(x)) f(x)\, dx \\ &= \int_{\tilde{x}}^{\infty} (x - \tilde{x}) u^2 e^{-u(1-F(x))} (1 - F(x)) f(x)\, dx. \end{aligned} \tag{80}$$

Note the summation starts from $k = 2$ since there is no gain for the job in the flow unless the second-highest x realization exceeds $\tilde{x}$. Also note that the density of the second-highest realization is $k(k-1)F(x)^{k-2}(1-F(x))f(x)$ since for x to be the second-highest realization, $k-2$ realizations must be lower than x (with probability $F(x)^{k-2}$ for any particular ordering of the x_i), one must be higher (with probability $1 - F(x)$), and one must be equal to x (with density $f(x)$); there are k choices of which x_i will be highest and then $k-1$ choices of which will be second-highest.

Finally, $G^s(\tilde{x}, u)$ is defined as the expected gain for an agent in the stock, conditional on being met by an agent in the flow. Since in expectation each agent in the flow meets u agents in the stock, and since each of these has ex ante the same gain, the expected gain per meeting per agent in the stock satisfies

$$S(\tilde{x}, u) = H^f(\tilde{x}, u) + uG^s(\tilde{x}, u).$$

It follows immediately that

$$G^s(\tilde{x}, u) = \int_{\tilde{x}}^{\infty} (x - \tilde{x}) e^{-u(1-F(x))} \left[1 - u(1 - F(x))\right] f(x)\, dx = \hat{G}^s(\tilde{x}, u). \tag{81}$$

As an aside, notice that we can calculate the gain for an agent in the stock directly also, in two different ways. First, notice that the distribution of that agent's x is $F(\cdot)$, independently of how many other agents in the stock are contacted. The expected gain conditional on $x > \tilde{x}$ is a decreasing function of the number of other agents contacted. To see this, note that conditional on $x > \tilde{x}$ and k, that expected gain is equal to the expected value of the difference between x and the larger of $\tilde{x}$ and the second-largest realization of match quality y, conditional on that realization being smaller than x:

$$G^s(\tilde{x} \,|\, x, k) \int_{\underline{x}}^{\infty} \max\{x - \max[y, \tilde{x}], 0\} k F(y)^{k-1} f(y)\, dy = \int_{\underline{x}}^{\infty} \max\{x - \max[y, \tilde{x}], 0\}\, dF^k(y)$$

The function $y \mapsto \max\{x - \max[y, \tilde{x}], 0\}$ is constant at $x - \tilde{x}$ for $y < \tilde{x}$, linearly decreasing for $\tilde{x} \leq y \leq x$, and constant at 0 for $y > x$. The distributions $F^k(\cdot)$ increase in the sense of FOSD as k increases (and always have full support on $\underline{x}, \bar{x}$. Thus, $G^s(\tilde{x} \,|\, x, k)$ is strictly decreasing in k. Since the distribution of k is increasing in the sense of FOSD as u increases, the expected value of the gain earned by an agent in the stock conditional only on his own realization x is also decreasing in u:

$$G^s(\tilde{x}, u \,|\, x) = \mathbf{E}_u \left[G^s(\tilde{x} \,|\, x, k) \,|\, x \sim \text{Poisson}(u) \right].$$

This can also be proved directly:

$$\begin{aligned} G^s(\tilde{x}, u \,|\, x) &= \sum_{k=0}^{\infty} \frac{e^{-u} u^k}{k!} \int_{\underline{x}}^{\infty} \max\{x - \max[y, \tilde{x}], 0\} k F(y)^{k-1} f(y)\, dy \\ &= \int_{\underline{x}}^{\infty} \max\{x - \max[y, \tilde{x}], 0\} e^{-u(1-F(y))} u f(y)\, dy \\ &= \int_{\underline{x}}^{\infty} \max\{x - \max[y, \tilde{x}], 0\}\, d\left[e^{-u(1-F(y))} \right]. \end{aligned}$$

Again, the distributions $y \mapsto e^{-u(1-F(y))}$ are increasing in u in a FOSD sense. Finally, we have that the expected gain of an agent in the stock is the expectation over x of the gain conditional on x, so it too is decreasing in u.

$$G^s(\tilde{x}, u) = \mathbf{E}_x \left[G^s(\tilde{x}, u \,|\, x) \,|\, x \sim F(\cdot) \right]$$

That is $G^s(\tilde{x}, u)$ is strictly decreasing in u.

Also notice that the function $y \mapsto \max\{x - \max[y, \tilde{x}], 0\}$ is pointwise weakly decreasing in $\tilde{x}$, and strictly decreasing on $y \leq \tilde{x}$; it follows that $G^s(\tilde{x}, u)$ is strictly decreasing in $\tilde{x}$ also.

Finally, notice that the results in this section, together with Proposition 5, ensure that if wage determination is according to Bertrand competition among agents in the stock, then the equilibrium decentralizes the constrained efficient choice of $\tilde{x}$ in the case where $\chi^u = \delta$ and $\chi^v = s$. However, the choice of $\tilde{x}$ will not in general be efficient if this parameter condition does not hold, and nor will the choice of entry. This is apparent since, according to (80), $H^f(\cdot)$ is not in general equal to $\hat{H}^f(\cdot)$, as characterized by (69). In fact, it is clear that this result is much more general. The entire surplus to be split between the workers in the stock and the job in the flow is given by (79), and it is apparent that $S(\tilde{x}, u)$ defined by (79) is equal to $\hat{H}^f(\tilde{x}, u)$ defined by (69). That is, efficient entry in particular requires that agents in the flow earn the entire surplus. But since efficiency also requires that agents in the flow earn their marginal contribution to that surplus (that is, the gain in the surplus associated with the presence of the highest draw of x in the distribution), these two requirements are inconsistent given budget balance. This is an intuitive result. When there is random search, efficiency requires that agents not earn the entire surplus, so they internalize the congestion externality they cause. Here there is no congestion, so efficient entry requires that the agents in the flow earn the full surplus. If there is a non-trivial intensive margin choice of x, however, then agents in the stock must also earn something.

This suggests that when entry is endogenous, the inefficiency of the equilibrium in our model arises from a tradeoff between two competing sources of inequality. The equilibrium cannot simultaneously deliver efficient choices of both $\tilde{x}$ and N^f.

As a final remark, if there were no intensive margin choice of $\tilde{x}$, our model *would* deliver efficient entry if all the surplus went to the agent in the flow. This would not be true if we endogenized worker entry as well: in that case, the increasing returns to scale in the matching function would lead to an inefficient equilibrium, for the same reason as in Coles (1999).

5 Wage Determination by Sequential Auctions

In our framework different wage determination protocols may have different implications on agents' match formation and entry decisions, and hence on the number of matches being formed at each point in time. However, the appropriate wage determination protocol to study in the case of many-to-one meetings, as occur in our model, is not a settled question. In this section we show how to adapt the approach to trilateral bargaining used by (Cahuc, Postel-Vinay, and Robin, 2006) to our model. We think that this wage determination protocol is interesting in its own right, but we also consider the effect of allowing for Nash bargaining.

In what follows we will focus on the case in which x is exponentially distributed with parameter a, such that $F(x) = 1 - e^{-ax}$. This assumption will be very useful as it allows us to obtain close form solutions for the expression H^f, H^s, G^f and G^s in HJB equations described above.

5.1 Sequential Auctions

First, denote by β the exogenous bargaining power of an agent in the flow.

Suppose that a worker in the flow samples $k \geq 1$ vacancies (workers) in the stock in island i. Although not essential to the argument, for simplicity assume that all these draws are above $\tilde{x}$ and order the draws $\tilde{x} < x_1 < x_2 < \ldots < x_k$. If $k = 1$, then the worker earns $\beta(x_1 - \tilde{x})/(r + s + \delta)$ and the firm $(1 - \beta)(x_1 - \tilde{x})/(r + s + \delta)$ above their respective outside options, of being in the stock. If $k = 2$, then the worker bargains with the x_2 firm, taking as outside option that she will receive $\beta(x_1 - \tilde{x})/(r + s + \delta)$. The surplus in this case is

$$\frac{x_2 - \tilde{x}}{r + s + \delta} - \frac{\beta(x_1 - \tilde{x})}{r + s + \delta},$$

and the worker receives share β of it. She also receives her outside option, for a total return relative to being in the stock of

$$\beta \left[\frac{x_2 - \tilde{x}}{r + s + \delta} - \frac{\beta(x_1 - \tilde{x})}{r + s + \delta} \right] + \frac{\beta(x_1 - \tilde{x})}{r + s + \delta} = \beta \frac{x_2 - \tilde{x}}{r + s + \delta} + \beta(1 - \beta) \frac{x_1 - \tilde{x}}{r + s + \delta}.$$

Continuing inductively, the worker in the flow gets

$$\beta \sum_{j=1}^{k} (1 - \beta)^{k-j} \left(\frac{x_j - \tilde{x}}{r + s + \delta} \right)$$

and the firm in the stock gets

$$\left(\frac{x_k - \tilde{x}}{r + s + \delta} \right) - \beta \sum_{j=1}^{k} (1 - \beta) \left(\frac{x_j - \tilde{x}}{r + s + \delta} \right).$$

If we drop the assumption that the x_j are ordered and that they all exceed $\tilde{x}$, the worker's gain is

$$\beta \sum_{j=1}^{k} (1 - \beta)^{k-j} \max \left(\frac{x_{(j)} - \tilde{x}}{r + s + \delta}, 0 \right), \tag{82}$$

where $x_{(j)}$ denotes the jth order statistic and for the sake of clarity, $x_{(1)}$ is the minimum and $x_{(k)}$ is the maximum.

Using equation (82) then $W_i^f - W_i^s$ is given by

$$W_i^f - W_i^s = \sum_{k=0}^{\infty} e^{-v} \frac{v^k}{k!} \int \max \left[W_i^m(w_i^f(\mathbf{x})) - W_i^s, 0 \right] dF_i^{(k)}(\mathbf{x})$$

$$= \sum_{k=0}^{\infty} e^{-v} \frac{v^k}{k!} \int \cdots \int_{x_1 < \cdots < x_k} \beta \sum_{j=1}^{k} (1-\beta)^{k-j} \frac{x - \tilde{x}}{r+s+\delta} \, dF(x_1) \ldots dF(x_k)$$
$$= \frac{1}{r+s+\delta} \sum_{k=0}^{\infty} e^{-v(1-F(\tilde{x}))} \frac{[v(1-F(\tilde{x}))]^k}{k!} \beta \frac{1}{a} \sum_{j=1}^{k} \frac{1-(1-\beta)^j}{j}$$
$$= \frac{1}{a(r+s+\delta)} \operatorname{Ein}(\beta(1-F(\tilde{x}))v),$$

where the function

$$\operatorname{Ein}(\beta(1-F(\tilde{x}))v) \equiv \int_0^{\beta(1-F(\tilde{x}))v} \frac{1-e^{-t}}{t} \, dt,$$

is strictly increasing in v and β, and strictly decreasing in $\tilde{x}$.

The expected gain for a job in the stock can then be computed by subtraction. When there are k jobs with x values exceeding $\tilde{x}$, the expected value of the maximum x is

$$\tilde{x} + \frac{1}{ka} + \frac{1}{(k-1)a} + \cdots + \frac{1}{a} = \tilde{x} + \frac{1}{a} \sum_{j=1}^{k} \frac{1}{j}$$

so the expected value of the total surplus created by a worker in the flow is

$$\sum_{k=1}^{\infty} e^{-v(1-F(\tilde{x}))} \frac{(v(1-F(\tilde{x})))^k}{(r+s+\delta)k!} \frac{1}{a(r+s+\delta)} \sum_{j=1}^{k} \frac{1}{j} = \frac{1}{a(r+s+\delta)} \operatorname{Ein}((1-F(\tilde{x}))v).$$

Since there are in expectation v firms contacted by the worker in the flow, and each is equally likely to be the maximum, the expected gain associated with being contacted for a firm in the stock is

$$\frac{1}{a(r+s+\delta)v} \left[\operatorname{Ein}((1-F(\tilde{x}))v) - \operatorname{Ein}(\beta(1-F(\tilde{x}))v) \right].$$

Using these insights and using $F(x) = 1 - e^{-ax}$, we can express G^s, H^s, G^f and H^f as

$$G^s = \frac{1}{(r+s+\delta)au} \left[\operatorname{Ein}\left(ue^{-a\tilde{x}}\right) - \operatorname{Ein}\left(\beta ue^{-a\tilde{x}}\right) \right] \tag{83}$$
$$H^s = \frac{1}{(r+s+\delta)av} \left[\operatorname{Ein}\left(ve^{-a\tilde{x}}\right) - \operatorname{Ein}\left(\beta ve^{-a\tilde{x}}\right) \right] \tag{84}$$
$$G^f = \frac{1}{(r+s+\delta)a} \operatorname{Ein}\left(\beta ve^{-a\tilde{x}}\right) \tag{85}$$
$$H^f = \frac{1}{(r+s+\delta)a} \operatorname{Ein}\left(\beta ue^{-a\tilde{x}}\right) \tag{86}$$

Reservation productivity Substituting out the equations that described the corresponding value functions into (9) and noting that $S(\tilde{x}) = 0$ we have that $\tilde{x}$ solves:

$$\begin{aligned}\tilde{x} = b - \gamma + \frac{\chi^u - \delta}{a(r+s+\delta)} \operatorname{Ein}\left(\beta v e^{-a\tilde{x}}\right) + \frac{\chi^v - s}{a(r+s+\delta)} \operatorname{Ein}\left(\beta u e^{-a\tilde{x}}\right) \\ + \frac{(\mu + \chi^v v + sm)}{a(r+s+\delta)u}\left[\operatorname{Ein}\left(u e^{-a\tilde{x}}\right) - \operatorname{Ein}\left(\beta u e^{-a\tilde{x}}\right)\right] \\ + \frac{(\lambda + \chi^u u + \delta m)}{a(r+s+\delta)v}\left[\operatorname{Ein}\left(v e^{-a\tilde{x}}\right) - \operatorname{Ein}\left(\beta v e^{-a\tilde{x}}\right)\right].\end{aligned}$$

Letting $\chi^u = \delta$ and $\chi^v = s$, noting that $u = \lambda/s - m$, $v = \mu/\delta - m$, $N_w = \lambda/s$ and $N_f = \mu/\delta$, then the above equation simplifies to

$$\begin{aligned}\tilde{x} - b + \gamma = \frac{N_f(s+\delta)}{a(r+s+\delta)(N_w - m)}\left[\operatorname{Ein}\left((N_w - m)e^{-a\tilde{x}}\right) - \operatorname{Ein}\left(\beta(N_w - m)e^{-a\tilde{x}}\right)\right] \\ + \frac{N_w(s+\delta)}{a(r+s+\delta)(N_f - m)}\Big[\operatorname{Ein}[(N_f - m)e^{-a\tilde{x}}] - \operatorname{Ein}[\beta(N_f - m)e^{-a\tilde{x}}]\Big].\end{aligned} \tag{87}$$

Implicit differentiation shows that (87) describes an increasing function in $(\tilde{x}, m)$ space.

Turnover Steady state turnover and $\chi^u = \delta$ and $\chi^v = s$ imply that

$$m = N_f\left(1 - e^{-(N_w - m)e^{-a\tilde{x}}}\right) + N_w\left(1 - e^{-(N_f - m)e^{-a\tilde{x}}}\right), \tag{88}$$

which describes a decreasing function in $(\tilde{x}, m)$ space.

Free-entry In the case in which all islands are identical, the free entry condition is given by

$$c = J^f = J^s + \frac{1}{a(r+s+\delta)} \operatorname{Ein}(\beta u e^{-a\tilde{x}}),$$

where J^s is given by

$$\begin{aligned}(r+\delta)J^s = -\gamma + \frac{(\lambda + \chi^u u + \delta m)}{a(r+s+\delta)v}\Big[\operatorname{Ein}[v e^{-a\tilde{x}}] - \operatorname{Ein}[\beta v e^{-a\tilde{x}}]\Big] \\ + \left[\frac{\chi^v}{a(r+s+\delta)}\Big[\operatorname{Ein}[\beta u e^{-a\tilde{x}}]\Big]\right].\end{aligned} \tag{89}$$

Substituting this expression in the free-entry condition and letting $\chi^u = \delta$ and $\chi^v = s$, we obtain that

$$\begin{aligned}\gamma + (r+\delta)c = \frac{N_w(s+\delta)}{a(r+s+\delta)(N_f - m)}\Big[\operatorname{Ein}[(N_f - m)e^{-a\tilde{x}}] - \operatorname{Ein}[\beta(N_f - m)e^{-a\tilde{x}}]\Big] \\ + \frac{1}{a}\Big[\operatorname{Ein}[\beta(N_w - m)e^{-a\tilde{x}}]\Big].\end{aligned} \tag{90}$$

A free-entry equilibrium is then characterised by the tuple $(\tilde{x}^*, m^*, N_f^*)$ that solves for (87), (88) and (90).

5.2 Nash Bargaining

We now consider the case in which a worker-firm pair determines a wage using Nash Bargaining at the moment the match is form. The pair commits to this wage for the duration of the match on the employment island. In the case of a worker in the flow, the agreed wage, w^f, maximizes the Nash product

$$\left[W_i^m(w^f) - W_i^s\right]^\beta [J_i^m(\pi^s) - J_i^s]^{1-\beta}.$$

In the case in which the firm was in the flow w^s solves

$$[W_i^m(w^s) - W_i^s]^{1-\beta}\left[J_i^m(\pi^f) - J_i^s\right]^\beta.$$

The crucial assumption we are making by adopting Nash Bargaining in this way is that when a worker and a vacancy decide to form a match, they do not consider other workers or vacancies in the stock.

Dropping the i subscript, the precise functional forms for G^s, H^s, $W^f - W^s$, and $J^f - J^s$ under Nash bargaining are

$$\begin{aligned}
G^s &= \sum_{k=0}^{\infty} e^{-u}\frac{u^k}{k!}\frac{1}{k+1}\int \max\left[W_i^m(w_i^s(\mathbf{x})) - W_i^s, 0\right] dF^{(k+1)}(\mathbf{x}) \\
&= \sum_{k=0}^{\infty} e^{-u}\frac{u^k}{k!}\frac{1}{k+1}\int_{\tilde{x}}^{\infty}(1-\beta)\frac{x-\tilde{x}}{r+s+\delta}(k+1)F^k(x)f(x)\,dx \\
&= \frac{1-\beta}{r+s+\delta}\sum_{k=0}^{\infty} e^{-u}\frac{u^k}{k!}\int_{\tilde{x}}^{\infty}(x-\tilde{x})F^k(x)f(x)\,dx \\
&= \frac{1-\beta}{r+s+\delta}\int_{\tilde{x}}^{\infty}(x-\tilde{x})e^{-u(1-F(x))}f(x)\,dx
\end{aligned}$$

When $F(x) = 1 - \exp(-ax)$, that is, x is distributed exponential with parameter a, then this integral can be written

$$G(s) = \frac{1-\beta}{r+s+\delta}\int_{\tilde{x}}^{\infty}(x-\tilde{x})e^{-ue^{-ax}}ae^{-ax}\,dx.$$

This expression can be written in terms of exponential integrals also. Specifically, if we use the special function

$$\mathrm{E}_1(x) \equiv \int_x^{\infty}\frac{e^{-t}}{t}\,dt,$$

then it is easy to verify that

$$\int (x-\tilde{x})e^{-ue^{-ax}} ae^{-ax}\,dx = \frac{1}{u}(x-\tilde{x})\,e^{-ue^{-ax}} - \frac{1}{au}\,\mathrm{E}_1\left(ue^{-ax}\right) + c,$$

where c is a constant of integration. We have that

$$\mathrm{E}_1(x) = -\gamma - \log x + \mathrm{Ein}(x),$$

so that we can also write the integral above (using that $\log(ue^{-ax}) = \log u - ax$) as

$$\int (x-\tilde{x})e^{-ue^{-ax}} ae^{-ax}\,dx = \frac{1}{u}\left[(x-\tilde{x})\,e^{-ue^{-ax}} - x\right] + \frac{1}{au}\,\mathrm{Ein}\left(ue^{-ax}\right) + c,$$

It follows that

$$\int_{\tilde{x}}^{\infty} (x-\tilde{x})e^{-ue^{-ax}} ae^{-ax}\,dx = \frac{1}{au}\,\mathrm{Ein}\left(ue^{-a\tilde{x}}\right).$$

To get this last equation, it is useful to note that as $x \to +\infty$, we have that $\mathrm{Ein}\left(ue^{-ax}\right) \to \mathrm{Ein}(0) = 0$ and that $x\left(e^{-ue^{-ax}} - 1\right) \to 0$. (To verify the second of these is a tricky use of L'Hôpital's rule: write $t = e{-}ax$; then we want to calculate

$$\begin{aligned}
\lim_{x\to+\infty} x\left(e^{-ue^{-ax}} - 1\right) &= \frac{1}{a}\lim_{t\to 0^+} \log t\left(1-e^{-ut}\right) \\
&= \frac{1}{a}\lim_{t\to 0^+} \frac{\log t}{\frac{1}{1-e^{-ut}}} \\
&= \frac{1}{a}\lim_{t\to 0^+} \frac{\frac{1}{t}}{\frac{ue^{-ut}}{(1-e^{-ut})^2}} \\
&= \frac{1}{au}\lim_{t\to 0^+} \frac{\left(1-e^{-ut}\right)^2}{te^{-ut}} \\
&= \frac{1}{au}\lim_{t\to 0^+} \frac{2\left(1-e^{-ut}\right)ue^{-ut}}{e^{-ut}\left(1-ut\right)} \\
&= \frac{2}{a}\lim_{t\to 0^+} \frac{1-e^{-ut}}{1-ut} \\
&= 0.
\end{aligned}$$

The third and fifth equalities both use L'Hôpital's rule.

In summary, when x is distributed $\mathrm{Exp}(a)$, then

$$G^s = \frac{1-\beta}{(r+s+\delta)au}\,\mathrm{Ein}\left(ue^{-a\tilde{x}}\right) \tag{91}$$

We can verify directly that this is decreasing in $\tilde{x}$ and in u. It suffices to show that

$$f(z,u) = \frac{1}{u}\,\mathrm{Ein}(zu) = \frac{1}{u}\int_0^{zu} \frac{1-e^{-t}}{t}\,dt$$

is increasing in z and decreasing in u. Since the integrand is positive, the first of these is immediate. The second follows by calculating

$$
\begin{aligned}
f_u(z,u) &= \frac{1}{u^2}\left[u \cdot z \frac{1-e^{-zu}}{zu} - \int_0^{zu} \frac{1-e^{-t}}{t}\, dt\right] \\
&= \frac{1}{u^2}\left[\int_0^{zu} e^{-t}\, dt - \int_0^{zu} \frac{1-e^{-t}}{t}\, dt\right] \\
&= -\frac{1}{u^2}\int_0^{zu} \frac{e^{-t}}{t}\left[e^t - (1+t)\right] dt
\end{aligned}
$$

which is negative since $e^t > 1+t$ for $t > 0$.

We similarly have

$$H^s = \frac{1-\beta}{r+s+\delta}\int_{\tilde{x}}^{\infty} (x-\tilde{x})e^{-v(1-F(x))} f(x)\, dx.$$

Notice that the bargaining power is also $1-\beta$ in this calculation, since the job is in the stock. For $x \sim \mathrm{Exp}(a)$, we have

$$H^s = \frac{1-\beta}{(r+s+\delta)av}\,\mathrm{Ein}\left(ve^{-a\tilde{x}}\right). \tag{92}$$

Now turning to the gains from being in the flow, we have

$$
\begin{aligned}
G^f &= \sum_{k=1}^{\infty} e^{-v}\frac{v^k}{k!}\int \max\left[W^m(w^f(\mathbf{x})) - W^s, 0\right] dF^{(k)}(\mathbf{x}) \\
&= \sum_{k=1}^{\infty} e^{-v}\frac{v^k}{k!}\int_{\tilde{x}}^{\infty} \beta\frac{x-\tilde{x}}{r+s+\delta} kF^{k-1}(x)f(x)\, dx \\
&= \frac{\beta v}{r+s+\delta}\int_{\tilde{x}}^{\infty} (x-\tilde{x})e^{-v(1-F(x))} f(x)\, dx.
\end{aligned}
$$

When $x \sim \mathrm{Exp}(a)$, we have

$$G^f = \frac{\beta}{(r+s+\delta)a}\,\mathrm{Ein}\left(ve^{-a\tilde{x}}\right). \tag{93}$$

This is decreasing in $\tilde{x}$ similarly to the argument for G^s and H^s. It has the same comparative statics with respect to v as with respect to $e^{-a\tilde{x}}$, so it is increasing in v.

Similarly, the gains from being in the flow for a job are

$$H^f = \frac{\beta u}{r+s+\delta}\int_{\tilde{x}}^{\infty} (x-\tilde{x})e^{-u(1-F(x))} f(x)\, dx.$$

For $x \sim \mathrm{Exp}(a)$, we have

$$H^f = \frac{\beta}{(r+s+\delta)a}\,\mathrm{Ein}\left(ue^{-a\tilde{x}}\right). \tag{94}$$

Reservation productivity Putting all of this together gives the reservation match quality equation under Nash bargaining in the form

$$\begin{aligned}(r+s+\delta)(\tilde{x}-b+\gamma) = (\mu+sm+\chi^v v)(1-\beta)\int_{\tilde{x}}^{\infty}(x-\tilde{x})e^{-u(1-F(x))}f(x)\,dx \\ +(\lambda+\delta m+\chi^u u)(1-\beta)\int_{\tilde{x}}^{\infty}(x-\tilde{x})e^{-v(1-F(x))}f(x)\,dx \\ +(\chi^v-s)\,\beta u\int_{\tilde{x}}^{\infty}(x-\tilde{x})e^{-u(1-F(x))}f(x)\,dx \\ +(\chi^u-\delta)\,\beta v\int_{\tilde{x}}^{\infty}(x-\tilde{x})e^{-v(1-F(x))}f(x)\,dx.\end{aligned} \tag{95}$$

When $\chi^u=\delta$ and $\chi^v=s$, we have

$$\begin{aligned}(r+s+\delta)(\tilde{x}-b+\gamma) = \frac{\mu(s+\delta)(1-\beta)}{\delta}\int_{\tilde{x}}^{\infty}(x-\tilde{x})e^{-u(1-F(x))}f(x)\,dx \\ +\frac{\lambda(s+\delta)(1-\beta)}{s}\int_{\tilde{x}}^{\infty}(x-\tilde{x})e^{-v(1-F(x))}f(x)\,dx.\end{aligned} \tag{96}$$

When $x\sim\text{Exp}(a)$, we have that

$$(r+s+\delta)(\tilde{x}-b+\gamma)=\frac{(s+\delta)(1-\beta)}{a}\left[\frac{\mu}{\delta u}\,\text{Ein}\left(ue^{-a\tilde{x}}\right)+\frac{\lambda}{sv}\,\text{Ein}\left(ve^{-a\tilde{x}}\right)\right]. \tag{97}$$

Finally using the fact that $N_f=\mu/\delta$ and that $N_w=\lambda/s$ we have that

$$\tilde{x}-b+\gamma=\frac{(s+\delta)(1-\beta)}{a(r+s+\delta)}\left[\frac{N_f}{(N_w-m)}\,\text{Ein}\left((N_w-m)e^{-a\tilde{x}}\right)+\frac{N_w}{(N_f-m)}\,\text{Ein}\left((N_f-m)e^{-a\tilde{x}}\right)\right]. \tag{98}$$

Implicit differentiation shows that (98) describes an increasing function in $(\tilde{x},m)$ space.

Turnover Steady state turnover and $\chi^u=\delta$ and $\chi^v=s$ imply that

$$m=N_f\left(1-e^{-(N_w-m)e^{-a\tilde{x}}}\right)+N_w\left(1-e^{-(N_f-m)e^{-a\tilde{x}}}\right), \tag{99}$$

which is the same as before and describes a decreasing function in $(\tilde{x},m)$ space.

Free-entry In the case in which all islands are identical, recall that free entry implies

$$\gamma+(r+\delta)c=(\lambda+\delta m+\chi_u^u u)\,H^s+(r+\delta+\chi^v)\,H^f.$$

If we assume that $\chi^u=\delta$ and $\chi^u=s$, we obtains that

$$\gamma+(r+\delta)c=\frac{\lambda(s+\delta)}{s}H^s+(r+s+\delta)H^f. \tag{100}$$

Under Nash bargaining, this can be written[15]

$$\gamma + (r+\delta)c = \frac{\lambda(s+\delta)(1-\beta)}{s(r+s+\delta)} \int_{\tilde{x}}^{\infty} (x-\tilde{x})e^{-v(1-F(x))} f(x)\,dx + \beta u \int_{\tilde{x}}^{\infty} (x-\tilde{x})e^{-u(1-F(x))} f(x)\,dx.$$

If $x \sim \text{Exp}(a)$, we have that

$$\gamma + (r+\delta)c = \frac{N_w(s+\delta)(1-\beta)}{(r+s+\delta)a(N_f-m)} \operatorname{Ein}\left((N_f-m)e^{-a\tilde{x}}\right) + \frac{\beta}{a} \operatorname{Ein}\left((N_w-m)e^{-a\tilde{x}}\right). \tag{101}$$

A free-entry equilibrium with Nash bargaining is then characterised by the tuple $(\tilde{x}^*, m^*, N_f^*)$ that solves for (98), (99) and (101).

6 Numerical Investigation

Now consider illustrative comparative statics using numerical examples (not a "proper" calibration). We present the results for the case of wage bargaining and auctions. The results for Nash Bargaining are qualitative similar. We explore the case in which all islands are identical. We let the time period be a quarter and let $x \sim \text{Exp}(a)$, $F(x) = 1-\exp(-ax)$, $a = 3$. Table 1 shows the rest of the baseline parameter values.

Table 1. Baseline Parametrization

b	γ	r	s	δ	χ_u	χ_v	λ	μ	β
0	4.35	0	0.05	0.05	0.05	0.05	3	2.90	0.5

These parameters imply that in steady state the number of workers and firms are given by $N_w = 60$ and $N_f = 58.1$, respectively. Further, the number of unemployed workers and vacant jobs in the stock are given by $u = 3.2$ and $v = 1.3$. The number of matches $m = 56.8$. The unemployment and vacancy rates are $\frac{u}{N_w} = 5.4\%$ and $\frac{v}{N_w} = 2.3\%$, respectively. The proportion of acceptable match-specific productivities above $\tilde{x}$ is 0.307. This implies that the matching probability in flow is 0.34 for workers and 0.63 for jobs; while the matching Poisson rate in stock is 1.15 for workers and 1.54 for jobs.

Figure 8 shows the survivor function for unemployed workers that is implied by our stock-flow

[15] Notice that there are interesting comparative statics here when $\beta = 1$, that is, the agent in the flow has all the bargaining power, in which case the equation becomes

$$\gamma + (r+\delta)c = u \int_{\tilde{x}}^{\infty} (x-\tilde{x})e^{-u(1-F(x))} f(x)\,dx.$$

The right side here is increasing in u. There are also interesting comparative statics when $\beta = 0$, that is, the agent in the stock has all the bargaining power, in which case the equation becomes

$$\gamma + (r+\delta)c = \frac{\lambda(s+\delta)}{s(r+s+\delta)} \int_{\tilde{x}}^{\infty} (x-\tilde{x})e^{-v(1-F(x))} f(x)\,dx.$$

Now the right side is decreasing in v. We conjecture that, all else equal, raising the entry rate tends to raise v and lower u, so for fixed $\tilde{x}$, this equation delivers an intuitive free entry condition.

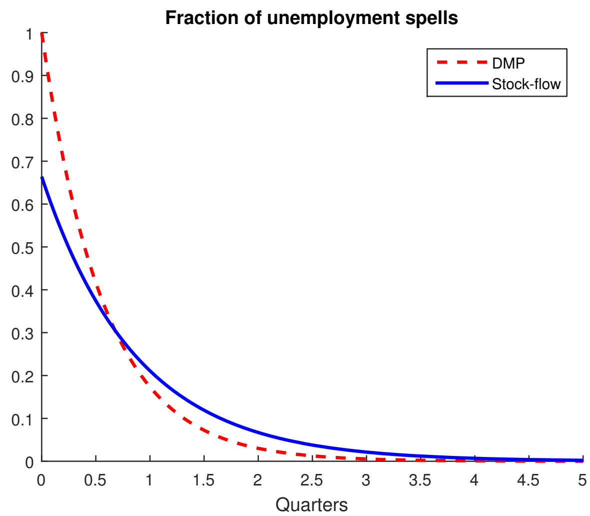

Figure 8. Survivor Functions: Stock-Flow vs. DMP

model, relative to the DMP model, which exhibits a constant hazard rate.[16] It is clear from this figure that the stock-flow model generates more short-term and more long-term unemployment. Table 2 shows this comparison by dividing the spell lengths according to the definition of short-term, medium-term and long-term unemployment used in the US.

Table 2. Spell Durations: Stock-Flow vs DMP

Spell duration	Stock-flow	Constant hazard rate
Short-term: < 4 weeks	0.533	0.417
Medium-term: 5 to 26 weeks	0.400	0.553
Long-term: > 26 weeks	0.067	0.030

6.1 Cross-section Heterogeneity with No Free-entry

Now consider comparative statics in the cross-section. We assume that islands differ in their flow match surplus, indexed by worker's opportunity cost b_i or a_i. Other parameters are constant and common to all islands. In particular, the inflow rates λ and μ are held constant, which implies that all islands have the same measures of workers and jobs. Figures 2 and 3 show the results of these exercises. The third panel (from the left) in the top row of these figures shows that islands with higher match surplus have higher $\tilde{x}$ and hence agents on those islands become less picky when forming a match. The first panel in the top row shows that in islands with higher flow match surplus the number (and rates) of unemployed workers and firms with vacant jobs in the stock is lower than in islands with lower flow match surplus. Hence, as shown in the second panel of the top row, the number of long-term unemployed (defined as those workers in the stock with spells of more

[16] Constant hazard rate model is parameterized to match overall unemployment rate given same Poisson rate of match destruction.

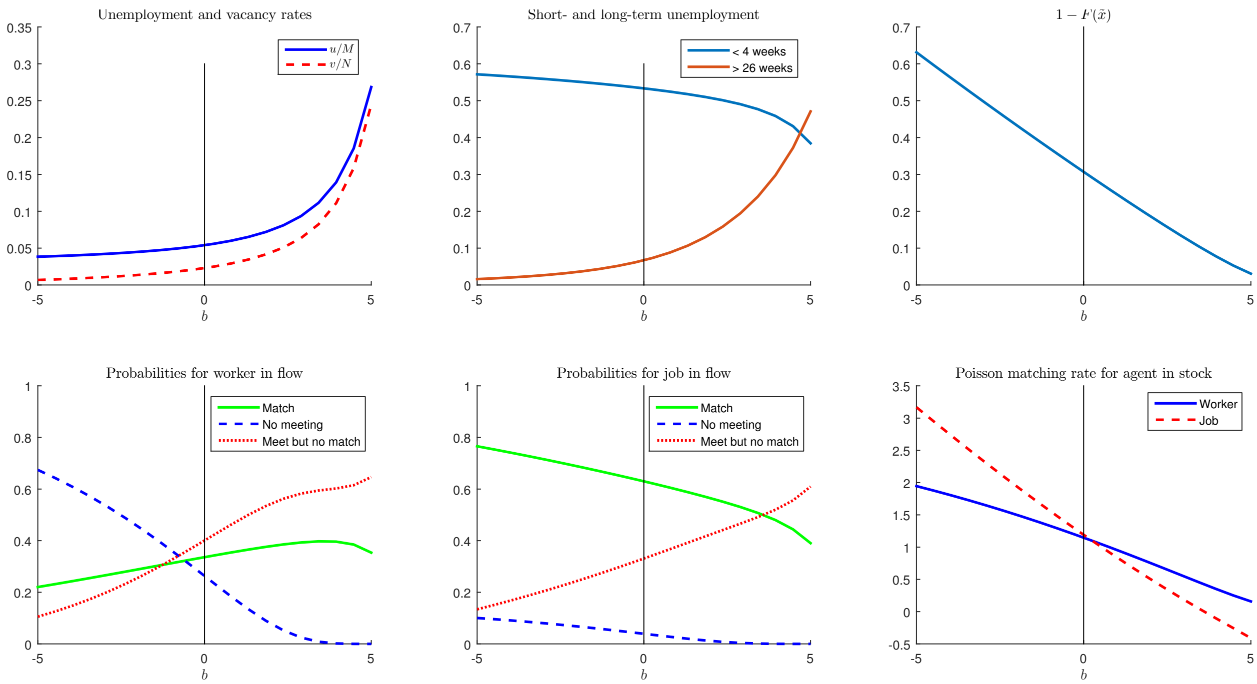

Figure 9. Cross-Sectional Heterogeneity in Flow Surplus: Heterogeneity in b

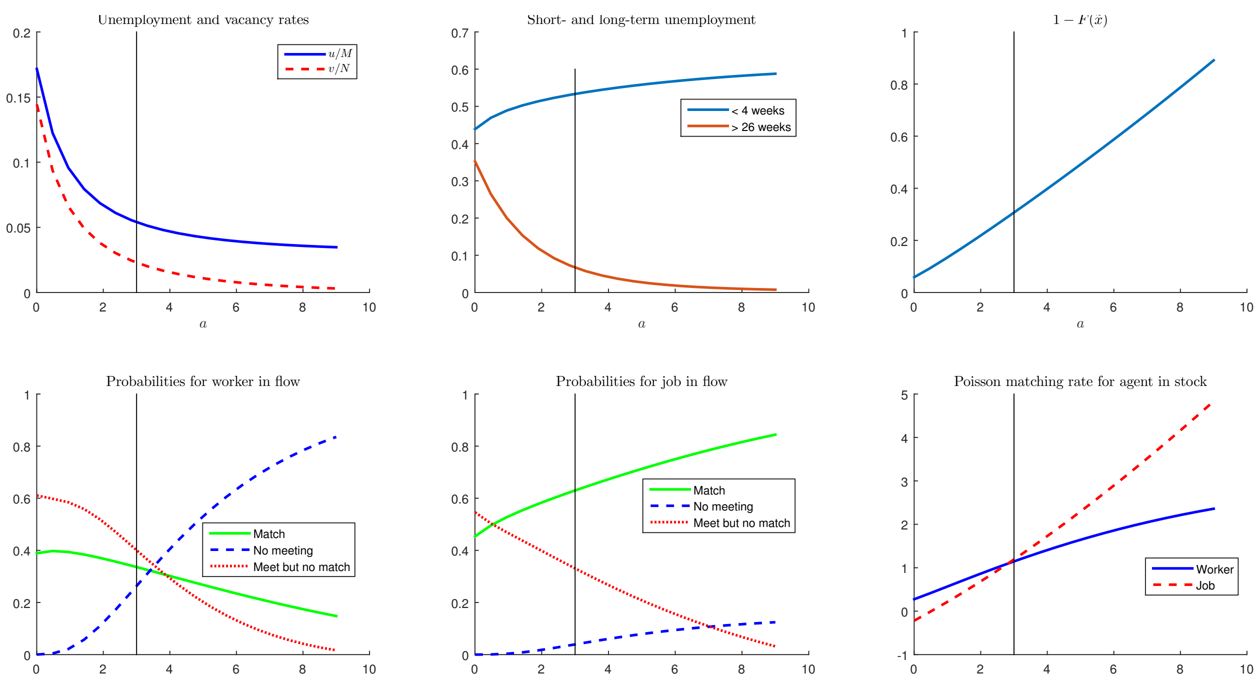

Figure 10. Cross-Sectional Heterogeneity in Flow Surplus: Heterogeneity in a

than 26 weeks) decrease with flow surplus, while the number of short-term unemployed (defined as those workers in the stock with spells of less than 5 weeks) is increasing in match surplus.

The bottom three panels of Figures 2 and 3 show the matching probabilities. For example, in the first and second panels (from the left) of the bottom row, we find that workers and firms in the flow on islands with higher flow surplus have a higher probability of being mismatch. This occurs because these islands exhibit relatively small numbers of vacancies and unemployed workers in the stock and hence the probability of being mismatch for workers and firms, e^{-v_i} and e^{-u_i}, respectively, is low on these islands. These panels also show that the probability of meeting agents in the flow and drawing an unacceptable match also increases with flow surplus. This seems to occur because, although $\tilde{x}$, decreases, the number of draws, k, from where to choose from also decreases. Together this implies that the probability of forming a match decreases for workers in the flow. The probability of forming a match for vacancies in the flow increases seems to occur because there is relative more unemployed in the stock.

Note that when changing the flow match surplus and holding firm entry fixed, the number of unemployed workers and firms with vacancies in the stock are perfectly positively correlated. This is by construction: $U_i = \frac{\lambda}{s} - M_i$, $V_i = \frac{\mu}{\delta} - M_i$. Allowing for endogenous match formation affects the cross-sectional *dispersion* in (U_i, V_i) but not the correlation. To relax the correlation, we need heterogeneity in λ, μ, s, or δ.

Figure 4 shows comparative statics when we vary the inflow rate of firms to the economy, μ. One can interpret this comparative statics as describing the case in which for some islands receive a higher number of vacancies in the flow than other islands. The first panel (from the left) on the top row shows the negative correlation between the unemployment and vacancies rates as we increase the inflow rate of firms. This is intuitive. Islands that receive a higher number of firms in the flow exhibit lower numbers of workers unemployed in the stock as there is a higher matching rate between of workers in the stock. At the same time the lower values of u, imply that the probability of mismatch for vacancies in the flow goes up and this seems to be strong enough to increase the number of vacancies in the stock. Note that because the number of vacancies in the stock increases with μ, workers in the flow on islands with high μ face lower mismatch probabilities and hence match probabilities, helping to further reduce the number of unemployed workers in the stock.

6.2 Comparative Statics: Free-entry

We now turn to analyse changes in aggregate parameters. In particular, for this exercise we will assume that all islands are identical and we will consider changes in the parameter b on all islands and allow the entry rate of jobs μ to adjust consistent with free entry. Figure 5 shows the outcomes of this exercise. In this case, we can see that much of the same results obtained from changing μ shown in Figure 4 hold here as well. That is, when flow match surplus increases and firms decide to enter in greater quantities to take advantage of the higher surplus, the unemployment rate of workers in the stock decreases, while the vacancy rate of firms in the stock goes up for the same reasons described above. If we interpret the comparison of steady states when b changes as

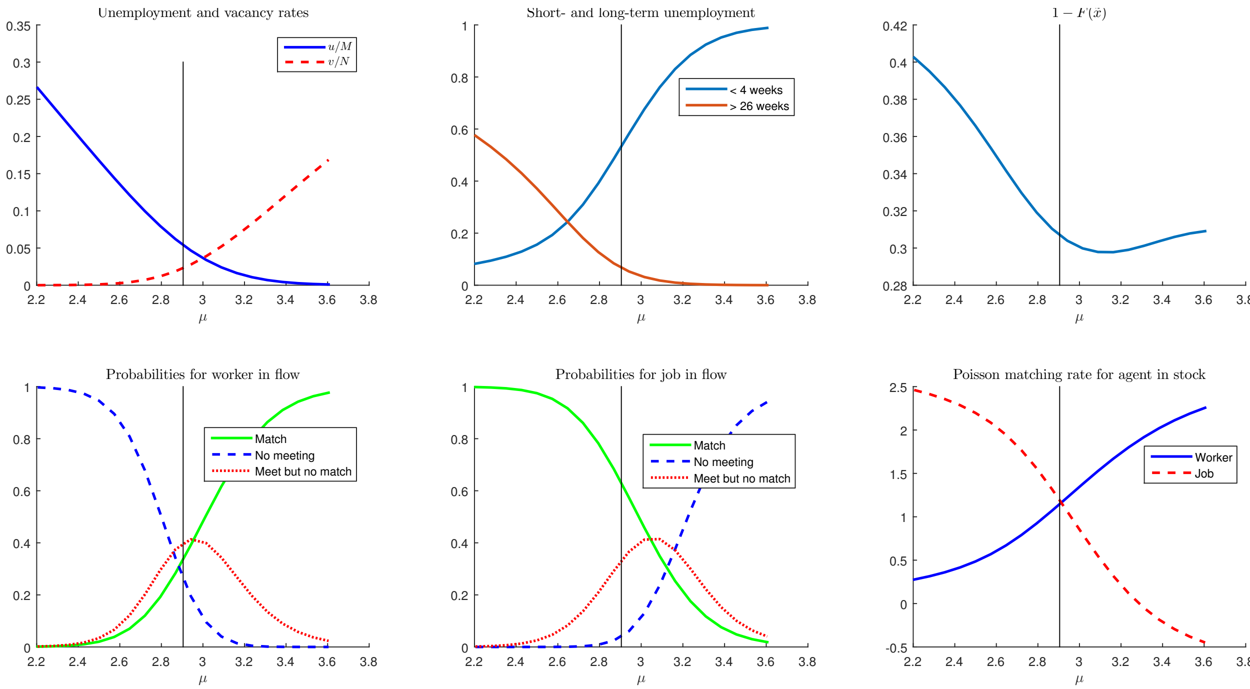

Figure 11. Cross-Sectional Heterogeneity in Job Inflow Rate

approximations describing economic recessions and expansions, we find that in recessions (higher b) unemployment goes up mainly because of the increase in the probability of mismatch. Even though agents become more picky in recessions, the probability of workers in the flow meeting a set of vacancies in the stock and deciding not to match and wait in the stock goes down. This implies that in our stock-flow model when aggregate match surplus decreases, 'wait' unemployed decreases and mismatch unemployment increases leading to a higher numbers of long-term unemployed.

7 Conclusion

To be written.

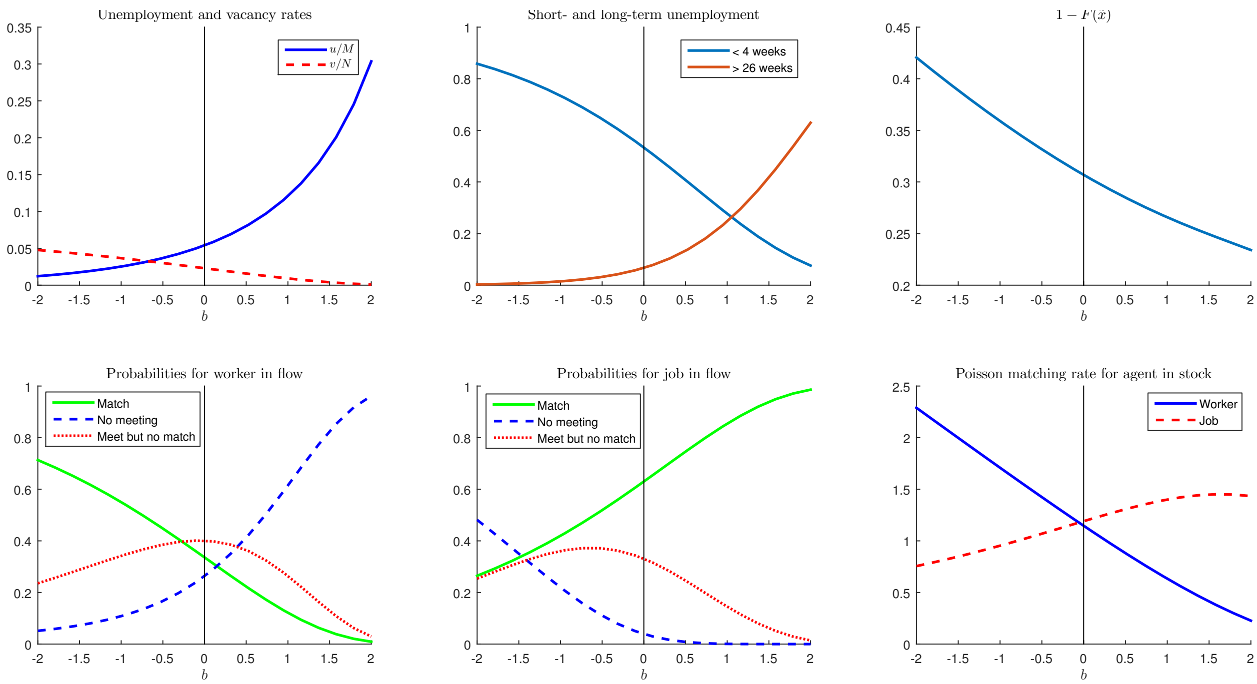

Figure 12. Changes in Aggregate Flow Surplus

A Omitted Proofs

Proof of Proposition 1. First consider the gain associated with a worker being in the flow rather than the stock. Because of the definition of the reservation wage $\tilde{w}_i$ and because (2) shows that $W_i^m(w)$ increases by $(r+\delta_i+s_i)^{-1}$ units per unit increase in w, we can write

$$W_i^m(w) - W_i^s = \frac{w - \tilde{w}_i}{r+\delta_i+s_i}.$$

Substituting in (12) and using (17a), we can write $G_i^f(\mathbf{x}) = \hat{w}^f(\mathbf{x} \ominus \tilde{x}_i)/(r+\delta_i+s_i)$ if $M(\mathbf{x}) \geq \tilde{x}_i$; this expression also applies when $M(\mathbf{x}) < \tilde{x}_i$ if we use the convention that $\hat{w}^f(\emptyset) = 0$. (13) then shows that G_i^f can be written as an expectation over $\mathbf{x} \in X^*$:

$$G_i^f = \frac{1}{r+\delta_i+s_i} \sum_{k=1}^{\infty} \frac{e^{-V_i} V_i^k}{k!} \int_{X^k} \hat{w}^f(\mathbf{x} \ominus \tilde{x}_i)\, dF_i^k(\mathbf{x}).$$

This satisfies (16a) when

$$G^f(\tilde{x}, v, F) = \sum_{k=1}^{\infty} \frac{e^{-v} v^k}{k!} \int_{X^k} \hat{w}^f(\mathbf{x} \ominus \tilde{x})\, dF^k(\mathbf{x}).$$

We need to establish that this function is continuous and that $G^f(\tilde{x}, v, F)$ is decreasing in $\tilde{x}$, has limit 0 as $\tilde{x} \to \overline{x}$, and is increasing in v. The continuity of $G^f(\cdot)$ is immediate from the continuity of $\hat{w}^f(\cdot)$. Similarly, Property (b) in the statement of Proposition 1 shows that $\hat{w}^f(\mathbf{x} \ominus \tilde{x})$ is decreasing in $\tilde{x}$ for all $\mathbf{x} \in X^*$, so that $G^f(\cdot)$ is decreasing in $\tilde{x}$. It remains to verify the other two properties.

To show that $\lim_{\tilde{x}\to\overline{x}} G^f(\tilde{x}, v, F) = 0$, first observe from Property (a) that for any $\mathbf{x} \in X^*$, we have $0 \leq \hat{w}^f(\mathbf{x} \ominus \tilde{x}) \leq M(\mathbf{x} \ominus \tilde{x}) = \max\{\max\{\mathbf{x}\} - \tilde{x}, 0\}$. It is clear that as $\tilde{x} \to \overline{x}$, we have

$$0 \leq (r + \delta_i + s_i)G_i^f \leq \sum_{k=1}^{\infty} \frac{e^{-V_i} V_i^k}{k!} \int_{X^k} \max\{\max\{\mathbf{x}\} - \tilde{x}, 0\}\, dF_i^k(\mathbf{x}). \tag{102}$$

Thus, to establish that $G_i^f \to 0$, it suffices to show that the right side of (102) converges to zero as $\tilde{x} \to \overline{x}$. This is immediate if $\overline{x} < +\infty$, since the integrand is everywhere zero when $\tilde{x} = \overline{x}$. When $\overline{x} = +\infty$, we need to use the boundedness condition (1). Let $\varepsilon > 0$, and use (1) to choose $\xi > 0$ large enough that[17]

$$\sum_{k=1}^{\infty} \frac{e^{-V_i} V_i^k}{k!} \int_{X^k} M(\mathbf{x}) \mathbf{1}(M(\mathbf{x}) \geq \xi)\, dF_i^k(\mathbf{x}) < \varepsilon.$$

Then we have $\max\{M(\mathbf{x} \ominus \xi), 0\} = 0$ if $M(\mathbf{x}) \leq \xi$, so that

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{e^{-V_i} V_i^k}{k!} \int_{X^k} \max\{M(\mathbf{x} \ominus \tilde{x}), 0\}\, dF_i^k(\mathbf{x}) &= \sum_{k=1}^{\infty} \frac{e^{-V_i} V_i^k}{k!} \int_{X^k} \max\{M(\mathbf{x} \ominus \xi), 0\} \mathbf{1}(M(\mathbf{x}) \geq \xi)\, dF_i^k(\mathbf{x}) \\
&\leq \sum_{k=1}^{\infty} \frac{e^{-V_i} V_i^k}{k!} \int_{X^k} M(\mathbf{x}) \mathbf{1}(M(\mathbf{x}) \geq \xi)\, dF_i^k(\mathbf{x}) \leq \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the claim follows.

To see that $G^f(\cdot)$ is increasing in its second argument, first notice that as v increases, the Poisson distribution of the number of contacts, k, increases in the sense of first-order stochastic dominance. Thus, it suffices to show that $\int_{X^k} \hat{w}^f(\mathbf{x} \ominus \tilde{x})\, dF^k(\mathbf{x})$ is increasing in k. For simplicity, set $\tilde{x} = 0$; the argument is unchanged for other values of $\tilde{x}$. Since all the match quality draws are independent, for this it suffices to show that adding elements to a tuple $\mathbf{x}$ must increase $\hat{w}^f(\mathbf{x})$, since, if so,

$$\begin{aligned}
&\int_{X^{k+1}} \hat{w}^f((x_1, \ldots, x_{k+1}))\, dF^{k+1}((x_1, \ldots, x_{k+1})) \\
&\quad = \int_{X^k} \int_X \hat{w}^f(x_1, \ldots, x_{k+1}))\, dF(x_{k+1})\, dF^k((x_1, \ldots, x_k)) \\
&\quad \geq \int_{X^k} \int_X \hat{w}^f(x_1, \ldots, x_k))\, dF(x_{k+1})\, dF^k((x_1, \ldots, x_k)) \\
&\quad = \int_{X^k} \hat{w}^f(x_1, \ldots, x_k))\, dF^k((x_1, \ldots, x_k)).
\end{aligned}$$

Thus, let $\mathbf{x} = (x_1, \ldots, x_k) \in X^*$ and $x_{k+1} \in X$ be arbitrary, and without loss of generality, assume $\tilde{x} = 0$ to simplify the notation. Denote by $\mathbf{x}' = (x_1, \ldots, x_k, x_{k+1})$ the tuple obtained by appending x_{k+1} to $\mathbf{x}$. We want to show that $\hat{w}^f(\mathbf{x}') \geq \hat{w}^f(\mathbf{x})$. Suppose that x_{k+1} is the lth-largest element of $\mathbf{x}'$. If $l \geq 2$, then the maximal element of the tuple is unchanged by the inclusion of x_{k+1}, that

[17] $\mathbf{1}(\cdot)$ denotes an indicator function which takes the value 1 if the condition in parentheses is true and 0 otherwise.

is, $M(\mathbf{x}) = M(\mathbf{x}')$, while we have that $m(\mathbf{x}') \succeq m(\mathbf{x})$, since the second through $(l-1)$st highest elements of $\mathbf{x}'$ are the same as those of $\mathbf{x}$, the lth through kth are all strictly increased, and $\mathbf{x}'$ is a longer tuple. The claim is then immediate from Property (d). If $l = 1$, on the other hand, then define $\mathbf{x}'' = (x_1, x_2, \ldots, x_k, x_{[1]})$ to be the tuple obtained from $\mathbf{x}$ by appending $x_{[1]} = M(\mathbf{x})$, the largest element of $\mathbf{x}$. Comparing $\mathbf{x}$ with $\mathbf{x}''$, we have that $M(\mathbf{x}'') = M(\mathbf{x})$ and $m(\mathbf{x}'') \succeq m(\mathbf{x})$ using the same argument just given, so that $\hat{w}^f(\mathbf{x}'') \geq \hat{w}^f(\mathbf{x})$ according to Property (d). Comparing $\mathbf{x}''$ with $\mathbf{x}'$, we have that $M(\mathbf{x}') > M(\mathbf{x}'')$ and $m(\mathbf{x}') = m(\mathbf{x}'')$, so that according to Property (c), we have $\hat{w}^f(\mathbf{x}') \geq \hat{w}^f(\mathbf{x}'')$. The claim then follows by transitivity.

This completes the proof of the parts of the Proposition relating to the case when a worker is in the flow. The case when a job is in the flow can be handled very similarly.

Now consider the case where the worker is in the stock. Arguing as for the previous case, it is straightforward to see using (14), (15), and (17a) that the function $G^s(\cdot)$ defined by

$$G^s(\tilde{x}, u, F) = \sum_{k=1}^{\infty} \frac{e^{-u} u^{k-1}}{(k-1)!} \frac{1}{k} \int_{X^k} \hat{w}^s(\mathbf{x} \ominus \tilde{x})\, dF^k(\mathbf{x}). \tag{103}$$

satisfies the relevant part of (16a). We can argue very similarly to before to establish that this function is continuous, decreasing in its first argument, and satisfies $\lim_{\tilde{x} \to \overline{x}} G^s(\tilde{x}, u, F) = 0$.

It remains to prove that $G^s(\tilde{x}, u, F)$ is decreasing in u. Analogously to before, notice that as u increases, the Poisson distribution of k increases in the sense of first-order stochastic dominance; also, for simplicity, set $\tilde{x} = 0$. It then suffices to prove that the expected flow gain associated with being contacted by a job in the flow which contacts a total of k workers in the stock, that is,

$$\omega_k \equiv \int_{X^k} \frac{1}{k} \hat{w}^s(\mathbf{x})\, dF^k(\mathbf{x}),$$

is decreasing in k. In the expression just given, ω_k is expressed as the expectation over k-tuples $\mathbf{x}$ of the flow gain, $\hat{w}^s(\mathbf{x})$ for the worker who draws the highest match quality $M(\mathbf{x})$, multiplied by the probability, $1/k$, that the worker under consideration is that worker. An alternative expression is given by first conditioning on x, the match quality draw of the worker under consideration, then taking an expectation over $\mathbf{y}$, the tuple of the remaining $k-1$ draws, and noting that the worker under consideration only enjoys a positive gain if x is maximal, that is, if $x \geq M(\mathbf{y})$. That is,

$$\omega_k = \int_X \int_{X^{k-1}} \hat{w}^s((x)^\frown \mathbf{y}) \mathbf{1}(x \geq M(\mathbf{y}))\, dF^{k-1}(\mathbf{y})\, dF(x),$$

where $^\frown$ denotes concatenation, that is, $(x)^\frown(y_1, \ldots, y_{k-1}) = (x, y_1, \ldots, y_{k-1})$ and, as above, $\mathbf{1}(\cdot)$ is an indicator function.

We need to show that the expected flow gain ω_k is decreasing in the number of contacts k. It suffices to show the stronger result that, given the x draw of the worker under consideration and the tuple $\mathbf{y} \in X^{k+1}$ of the match quality draws the job enjoys with the $k-1$ other workers also contacted, then adding an additional, $(k+1)$st, worker contact must weakly decrease the gain of the worker under consideration: that is, it suffices to show that

$$\hat{w}^s((x)^\frown \mathbf{y})\mathbf{1}(x \geq M(\mathbf{y})) \geq \hat{w}^s((x)^\frown \mathbf{y}^\frown (z))\mathbf{1}(x \geq M(\mathbf{y}^\frown (z))). \tag{104}$$

To see why (104) is sufficient, use the independence of the match quality draws to write

$$\begin{aligned}
\omega_{k+1} &= \int_X \int_{X^k} \hat{w}^s((x)^\frown \mathbf{y}')\mathbf{1}(x \geq M(\mathbf{y}'))\, dF^k(\mathbf{y}')\, dF(x) \\
&= \int_X \int_{X^{k-1}} \int_X \hat{w}^s((x)^\frown \mathbf{y}^\frown (z))\mathbf{1}(x \geq M(\mathbf{y}^\frown (z)))\, dF(z)\, dF^{k-1}(\mathbf{y})\, dF(x) \\
&\leq \int_X \int_{X^{k-1}} \int_X \hat{w}^s((x)^\frown \mathbf{y})\mathbf{1}(x \geq M(\mathbf{y}))\, dF(z)\, dF^{k-1}(\mathbf{y})\, dF(x) \\
&= \int_X \int_{X^{k-1}} \hat{w}^s((x)^\frown \mathbf{y}^\frown (z))\mathbf{1}(x \geq M(\mathbf{y}^\frown (z)))\, dF^{k-1}(\mathbf{y})\, dF(x) = \omega_k;
\end{aligned}$$

here the second line decomposes $\mathbf{y}' \in X^k$ into the $(k-1)$-tuple $\mathbf{y} \in X^{k-1}$ and the singleton $z \in X$, the third line uses (104), and the fourth line uses that $\int_Z dF(z) = 1$. To see why (104) is true, consider several cases. First, if $x < M(\mathbf{y})$, then the indicator functions on both sides are zero and the inequality holds weakly. If $x \geq M(\mathbf{y})$ but $x < z$, then the left side is a product of two non-negative terms, so is non-negative while the indicator function on the right side is zero. Finally, if $x \geq \max\{M(\mathbf{y}), z\}$, then both indicator functions take the value 1, and the claim follows using Property (d) and the observation that $M((x)^\frown \mathbf{y}^\frown (z)) = M((x)^\frown \mathbf{y}) = x$ and $m((x)^\frown \mathbf{y}^\frown (z)) \succeq m((x)^\frown \mathbf{y})$.

Proof of Proposition 2. Given that (23) decreases while (48) increases in $(\tilde{x}, m)$-space when $\chi^u = \delta$ and $\chi^v = s$, continuity implies that there is a unique pair $(\tilde{x}_i^*, m_i^*)$ that solves both equations if (23) intersects (48) from below. To verify the latter we analyse the limits of these functions using Properties 1 and 2.

(i) First let $\tilde{x} \to 0$ and consider the flow equation (23). In this case the number of matches m achieves its maximum, $\overline{m}$, such that the latter solves the equation

$$(s+\delta)\overline{m} = (\mu + \chi^v N_f)(1 - e^{-(N_w - \overline{m})}) + (\lambda + \chi^u N_w)(1 - e^{-(N_f - \overline{m})}).$$

Note that $\overline{m} \leq min\{N_w, N_f\}$ as when $\tilde{x} = 0$ there could be agents in the stock due to mismatch. Now consider the reservation equation (48). As $\tilde{x} \to 0$ the left side of this equation decreases and goes to $-b + \gamma$. Since H^s and G^s are increasing in $\tilde{x}$, the right side of (48) increases and converges to its maximum for a given m, u and v. Note that Properties 1 and 2 imply that this maximum is decreasing in u and v and hence is increasing in m. Since the left side of (48) is strictly smaller than its right side in this limit, both u and v must increase and hence m must decrease to reduce the right side and satisfy (48), such that the resulting $m = \hat{m}$ satisfies $\hat{m} \leq \overline{m}$. The latter then establishes that as $\tilde{x} \to 0$, equation (48) lies below (23).

(ii) Now let $\tilde{x} \to \infty$. In this limit $F(\tilde{x})$ converges to one and hence the right side of (23) converges to zero, such that the solution to (23) in this limit implies $m = 0$, $u = N_w$ and $v = N_f$. Now consider equation (48). As $\tilde{x} \to \infty$, Properties 1 and 2 imply its right side converges to zero because H^s and G^s converge to zero for a given m, u and v, while it left side becomes unboundedly large. Since H^s and G^s are decreasing in u and v, respectively, to satisfy (48), u and v must decrease and hence m must increase such that the resulting m is strictly positive. The latter then establishes that as $\tilde{x} \to \infty$, equation (48) lies above (23).

(iii) Continuity and monotonicity of (48) and (23) then establishes that there exists a unique pair $(\tilde{x}, m)$ that solves these equations. This completes the proof of Lemma 1.

Proof of Lemma 1. (29) implies that

$$1 - p^u = e^{-V(1-F)}$$

and (30) implies that

$$1 - p^v = e^{-U(1-F)},$$

so that

$$(1 - p^v)^V = (1 - p^u)^U.$$

Given (28), this immediately implies (33). We can also rearrange the equation to write p^v in terms of p^u by noting that

$$1 - p^v = (1 - p^u)^{\frac{1}{\theta}},$$

so that

$$p^v = 1 - (1 - p^u)^{\frac{V}{U}} = 1 - (1 - p^u)^{\frac{1}{\theta}}. \tag{105}$$

To establish (34), the first step is to use the flow balance equations to relate u and v to p^u, p^v, ϕ^u, and ϕ^v. We can write the law of motion of the stock of unemployed workers as follows:

$$\dot{U} = -(s + \delta)N^f \left[1 - e^{-U(1-F)}\right] + (s + \delta)N^w e^{-V(1-F)} - (s + \delta)U.$$

Restricting to steady state and using the characterizations of p^u and ϕ^u given by (29) and (31), this can be rewritten as

$$0 = -\phi^u U + (s + \delta)N^w(1 - p^u) - (s + \delta)U = 0,$$

or

$$(s + \delta + \phi^u)u = (s + \delta)(1 - p^u),$$

or

$$u = \frac{s + \delta}{s + \delta + \phi^u}(1 - p^u). \tag{106}$$

An alternative derivation notes that the fraction of the population on the island which flows into unemployment per unit time is $(1 - p^u)(s + \delta)(1 - u)$, while the fraction flowing out is $\psi^u u =$

$(\phi^u + (s+\delta)p^u)u$. Equating these gives (106).

To proceed further, use (29), (30), (31), and (32) to write

$$\phi^u = (s+\delta)\frac{N^f}{U}p^v = (s+\delta)\frac{N^f}{N^w}\frac{N^w}{U}p^v = (s+\delta)\frac{1-u}{1-v}\frac{1}{u}p^v = (s+\delta)\frac{\theta}{v}p^v.$$

and similarly

$$\phi^v = (s+\delta)\frac{N^w}{V}p^u = (s+\delta)\frac{N^w}{N^f}\frac{N^f}{V}p^u = (s+\delta)\frac{1-v}{1-u}\frac{1}{v}p^u = (s+\delta)\frac{\frac{1}{\theta}}{u}p^u.$$

Notice that these expressions allow all the four terms p^u, p^v, ϕ^u, and ϕ^v to be written in terms of any of the others. For example, in terms of p^u in particular, we have that

$$\phi^u = (s+\delta)\frac{\theta}{v}p^v = (s+\delta)\frac{\theta}{v}\left[1-(1-p^u)^{\frac{1}{\theta}}\right]. \tag{107}$$

Now substitute from this equation into (106) to see that

$$u = \frac{1-p^u}{1+\frac{\theta}{v}p^v}.$$

After substituting for θ from (28) and doing a couple of lines of algebra, this implies (34).

To see that the solution to the system given by (33) and (34) is unique, substitute from (105) to see that

$$\frac{p^u}{1-u} + \frac{1-(1-p^u)^{\frac{1}{\theta}}}{1-v} = 1. \tag{108}$$

This equation uniquely determines p^u, since the left side is a strictly increasing function of p^u which takes the value 0 when $p^u = 0$ and $(1-u)^{-1} + (1-v)^{-1} > 1$ when $p^u = 1$.

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