Advance Information and Distorted Beliefs in Macroeconomic and Financial Fluctuations*

Kyle Jurado
Duke University

February 3, 2016

Abstract

Fluctuations in the beliefs of economic agents can be driven by current fundamentals, advance information about future fundamentals, or distortions resulting from informational or psychological limitations. This paper presents a dynamic stochastic general equilibrium (DSGE) model that jointly considers all three possibilities and estimates their relative importance for explaining macroeconomic and financial data. To discipline the estimation, direct data on subjective forecasts is included in the set of observable variables. Results indicate that both advance information and distorted beliefs are important. On average about two-thirds of the fluctuations in endogenous variables can be attributed to these two sources. While they are equally important for output and employment, advance information is most important for explaining inflation and investment, and distorted beliefs are most important for explaining stock returns and consumption.

JEL classification: D84, E10, E44, G12, C63
Keywords: News shocks, near-rationality, change of measure, recursive utility preferences, Bayesian estimation

*I would like to thank Ricardo Reis and Michael Woodford for their advice and support over the course of this project. I would also like to thank Ryan Chahrour, Serena Ng, José Scheinkman, Stephanie Schmitt-Grohé, Jón Steinsson, and Martín Uribe for helpful discussions. For part of time I was working on this project, I was funded by the National Science Foundation under grant number DGE-07-07425.

1 Introduction

An old idea in macroeconomics is that the beliefs of economic agents can be an important cause, and not merely the consequence, of developments in current economic conditions.[1] This causation can be proximate, as when agents receive advance information about future fundamentals. Or it can be ultimate, as when informational or psychological factors lead agents to hold beliefs that are distorted relative to the beliefs that would have been justified by fundamentals alone. Each of these possibilities has received renewed interest in recent years as a potentially important source of fluctuations in macroeconomic or financial data. However, most existing studies have not considered both possibilities at the same time. It therefore remains unclear whether one is more important than the other from a quantitative perspective. Or, given how similar the two hypotheses are, whether it is even possible to tell them apart from aggregate data alone.

This paper argues that it is possible to separately identify the effects of advance information and distorted beliefs, and presents estimates of their relative importance in explaining fluctuations in macroeconomic and financial data. The estimates come from a medium-scale dynamic stochastic general equilibrium (DSGE) model populated by forward-looking agents whose beliefs may reflect advance information about future fundamentals or may be subject to exogenous distortions. Taken together, these two sources of exogenous variation in beliefs are responsible for two-thirds of the variation in the data. They are equally important for explaining output growth and the growth rate of hours worked. However, distorted beliefs are most important for explaining stock returns and consumption growth, and advance information is most important for explaining inflation and investment growth.

It is possible to separately identify advance information from distorted beliefs given data on endogenous forward-looking variables and the fundamentals relevant for determining those variables. At first, this may seem surprising, because agents can behave in exactly the same way whether they have received accurate good news about the future or whether they happen to be unjustifiably optimistic. What matters for their decisions today is *what* they believe about the future, not *why* they believe it. While that is true, the key for identification is that these two hypotheses generate different types of co-movement between current actions and the subsequent

[1] This idea is often traced back to Pigou (1927).

realizations of future fundamentals. The variation in current actions that results from advance information should be systematically related to future fundamentals, while the variation that results from distorted beliefs should not. With the benefit of hindsight, an econometrician can therefore examine the relationship between current actions and future fundamentals and determine the extent to which current actions are affected by these two different underlying causes.

In principle, any endogenous forward-looking variable should reveal information about the importance of advance information and distorted beliefs. Nevertheless, it is desirable to discipline the estimation procedure by incorporating data that reflect these types of belief-driven fluctuations as directly as possible. The two types of data selected in this study to satisfy that desideratum are subjective forecasts and stock returns. For the former, agents' explicit reported forecasts of future activity are valuable because they represent a direct measure of agents' beliefs. For the latter, there is a large literature in financial economics suggesting that fluctuations in the stock market reflect agents' expectations about future activity. Indeed, the recent resurgence of advance information as a live hypothesis in macroeconomics has been based on that very idea.

A key methodological contribution of the paper is in the way that belief distortions are specified. I propose to directly parameterize the Radon-Nikodým derivative process that governs the change of measure from agents' beliefs to the beliefs implied by their model environment. This approach has the theoretical advantage that it does not require a complete specification of the information that agents have at their disposal, or of the particular inferential procedure they use to update their beliefs. At a practical level, it circumvents the need to solve agents' (often complicated) filtering problems in order to obtain their equilibrium policy rules. It also makes it computationally straightforward to include belief distortions in a wide class of nonlinear equilibrium models. The idea of parameterizing the Radon-Nikodým process has been fruitfully employed in empirical asset pricing for handling stochastic risk adjustments, but has not yet been introduced into macroeconomic equilibrium models as a way of quantifying belief distortions.

The theoretical model developed in this paper integrates several features that have been separately identified as important for explaining macroeconomic and financial data. These include six real frictions: recursive utility preferences, internal habit formation in consumption, investment adjustment costs, variable capacity utilization, and imperfect competition in labor and product markets. They also include two nominal frictions: time-dependent staggered price and wage setting. The real frictions are important for jointly explaining the behavior of stock returns and macroeconomic aggregates. The nominal frictions are important for explaining nominal wages, interest rates, and inflation. Together, these features help the model generate empirical predictions that are as consistent with the data as possible.

Solving and estimating models with recursive preferences introduces some technical challenges. To address those challenges this paper presents a new framework for obtaining a linear approximation to the model that preserves the theoretically appealing features of this preference structure while maintaining the tractability that comes from working with a linear solution. The main insight underlying the method is that under the assumption of normal disturbances, the conditional expectation of a log-linear expression automatically incorporates information on conditional variances. By exploiting this is well-known property of log-normal processes, it is possible to incorporate what is typically regarded as "second-moment" information into a linear approximation. The method is internally consistent in that it does not require approximating some equilibrium conditions differently than others. This is because the model itself naturally prioritizes the second-moment information that is particularly important for describing the dynamics in stochastic equilibrium. Furthermore, the method can be applied to many modern equilibrium models, including those with time-varying uncertainty.

This paper makes a contribution to the literature on the sources of fluctuations in key macroeconomic and financial aggregates. If beliefs play an important role in the determination of those aggregates, it is of interest to understand what causes beliefs to change. In addition, there are normative reasons why the distinction between advance information and distorted beliefs can be important. For example, a monetary authority would want to respond differently to exogenous changes in private sector beliefs depending on which of these two sources were ultimately responsible. If it could tell the difference, it would want to accommodate advance information more than distorted beliefs, because the later do not reflect any actual change in future economic fundamentals.[2] Of course telling the difference in real time can be difficult. During the stock market boom of the 1990s, former U.S. Federal Reserve Chairman Alan

[2]Appendix D provides a New-Keynesian example where this is the case. It is recommended to read Section 3 before consulting this appendix.

Greenspan highlighted this point in his now famous remark concerning the difficulty of determining whether "irrational exhuberance" had unduly escalated asset values. While the positive analysis of this paper does not attempt to solve this difficulty, it does bring us one step closer by providing a systematic way to tell the difference between these underlying causes after the fact.

As an overview of the paper, Section 2 discusses some studies that are related to this one. Section 3 describes the precise way that advance information and distorted beliefs are modeled, together with a simple permanent-income example to illustrate the main intuition behind identification. Section 4 presents the theoretical environment that serves as the laboratory for this paper. Section 5 presents the solution and estimation methods that are applied to the model. Section 6 presents the data, estimates, and discusses the main results. Section 7 performs several robustness checks of these results. Section 8 explores how the model accounts for the joint behavior of the real economy and the stock market during two historical episodes: the Dot-Com boom and bust of the 1990s and the recent Great Recession. Section 9 concludes and discusses useful extensions of the analysis in this paper.

2 Related Literature

The idea that advance information about future fundamentals can be important for driving fluctuations in macroeconomic and financial data has received considerable attention in recent years. In the macroeconomic literature, Beaudry and Portier (2006) and Beaudry and Lucke (2010) use evidence from vector autoregressions (VARs) to argue that innovations to productivity growth are mostly anticipated, and have been a significant driving force of U.S. business cycles.[3] Schmitt-Grohé and Uribe (2012) estimate a fully-specified DSGE model and find that advance information can account for half of the variation in output, consumption, investment, and employment. In the financial economic literature, Bansal and Yaron (2004) have pioneered a line of work that emphasizes the importance of "long-run risks" — advance information about future consumption growth — for explaining the notoriously difficult risk-free rate,

[3] I refrain from using the word "news" in this paper to avoid any equivocation. In some contexts, "news" also refers to the realization of the *current* innovation to fundamentals, which would have been unforecastable one period before. I also refrain from using "long-run risks" because that label is less useful for emphasizing the informational content of this type of disturbance.

equity premium, and excess volatility puzzles.[4]

At the same time, exogenous distortions in agents' beliefs have also played a prominent role in explaining business cycles and asset prices. Both Angeletos and La'O (2010) and Lorenzoni (2009) demonstrate how informational frictions can open the door for "noise shocks" — expectational errors arising from imperfect information about underlying fundamentals — to be an important cause of typical business cycle co-movement. In current ongoing work, Angeletos, Collard, and Dellas (2014) present a model with dispersed information and heterogeneous priors where agents' higher-order beliefs are subject to exogenous distortions. They find that this type of higher-order uncertainty, which they interpret as "confidence," has the potential to explain a large fraction of business cycle fluctuations.[5]

Belief distortions have also played a prominent role in the asset-pricing literature. Cechetti, Lam, and Mark (2000) argue that an exogenously distorted pessimism during expansions and optimism during contractions enables their consumption-based asset-pricing model to explain the first and second moments of the equity premium and risk-free rate puzzles. Connecting distorted beliefs about asset valuations to psychological factors has been the cornerstone of a large body of behavioral literature (e.g. Shiller, 2000 and Akerlof and Shiller, 2009). More recently, Hassan and Mertens (2014) emphasize the potential importance of belief distortions for the interaction between financial markets and the real economy. Their focus is on how small, correlated belief distortions about fundamentals can reduce the ability of stock prices to convey accurate information about the state of the economy.

A few studies explicitly consider advance information and distorted beliefs together as this paper does. Blanchard, L'Huillier, and Lorenzoni (2013) employ a similar information structure to Lorenzoni (2009) to identify the role of news and noise in macroeconomic data. While they find that noise appears to be a more important driver of business cycles, technically what is called "news" in those papers does not represent what is called advance information here. This is because their news

[4]For a more comprehensive discussion of the business cycle literature, see Beaudry and Portier (2013); for the long-run risks literature see Bansal, Kiku, and Yaron (2012).

[5]Appendix B discusses the type of disturbance proposed by Angeletos, Collard, and Dellas (2014) along with two others that are closely related to the type of belief distortions described in the present paper. Through simple examples, it clarifies the main conceptual differences and explains how those differences might be identified in the data. Incorporating direct data on subjective beliefs turns out to be important.

shock is not independent of current productivity. Instead, the key distinction in those papers is between permanent and transitory components of productivity. This paper will feature advance information and distorted beliefs about both the permanent and transitory components of productivity, in addition to a number of other exogenous fundamental processes.

Barsky and Sims (2012) also make a distinction between advance information and distorted beliefs; they call the former "news" and the latter "animal spirits." They use forward-looking measures of consumer confidence from the Michigan Survey of Consumers to help separately identify the two, and find that advance information is more important. By comparison, this paper uses direct data on expectations from the Federal Reserve Bank of Philadelphia's Survey of Professional Forecasters (SPF). Measures of consumer confidence may have the advantage of being representative of consumers' beliefs more generally, with the disadvantage of being a less precise measure of expectations. Data from the SPF has also been more widely used for the purposes of estimating DSGE models.[6]

Another important difference with existing studies is that this paper features an alternative formulation of distorted beliefs which does not require any explicit behavioral assumptions about how agents update their beliefs in the presence of new information.[7] Specifically, I follow Hansen and Sargent (2005, 2007) and Woodford (2010) and characterize belief distortions according to the similarity of the probabilities that agents assign to future outcomes relative to the probabilities implied by their model environment. However, instead of using measures of statistical discrepancy to bound the degree of distortion, I propose to directly parameterize the Radon-Nikodým derivative process which summarizes the change from the distorted to historical probability measures. Given that the focus of this paper is primarily positive rather than normative, this approach proves to be particularly useful both theoretically and computationally. The insight behind this added flexibility is that the goal of many learning models is essentially to solve for a particular change of measure, given an exogenously specified information structure. By specifying the change of measure directly, it is possible to avoid solving an often complicated intertemporal

[6]For example, see Del Negro and Eusepi (2011), Milani (2011), and Miyamoto and Nguyen (2014).

[7]The models in the previous paragraph all assume that agents update their beliefs in a Bayesian manner. For models where agents use non-Bayesian procedures, see Eusepi and Preston (2011) and Milani (2011).

inference problem.

The idea of directly parameterizing belief distortions draws upon the practice in empirical asset pricing of parameterizing the change between the historical and *risk-neutral* measures. For example, this is the approach used in the term-structure models of Dai and Singleton (2000) and Le, Singleton, and Dai (2010). This literature has shown that direct parameterization can be a useful way of uncovering many important properties of the equilibrium pricing kernel. In their study of bond risk premia, Piazzesi, Salomao, and Schneider (2013) also extend this approach to belief distortions as in this paper. However, they do not focus on the distinction between advance information and distorted beliefs, and do not allow for independent exogenous variation in beliefs to arise through the Radon-Nikodým derivative process. They also do not apply this approach in a dynamic equilibrium setting, where optimizing agents choose their policy rules using a potentially distorted probability measure.

A related situation involving distorted beliefs is when agents fear that their model of the economy is misspecified. Hansen, Sargent, and Tallarini (1999), Cagetti, Hansen, Sargent, and Williams (2002), and Bidder and Smith (2011) study equilibrium models where agents have a concern for robustness. Similarly, Illut and Schneider (2012) endow agents with multiple priors utility to determine whether ambiguity about future fundamentals is responsible for business cycles. While these studies all involve belief distortions, those distortions reflect the degree to which agents are confident in *their own model* of the economy. According to the interpretation in this paper, however, belief distortions do not necessarily arise due to any fear of model misspecification. Moreover, because distortions arising out of a concern for robustness reflect agents' beliefs about "worst case" outcomes, they may not be present in reported survey forecasts. Using forecast data therefore provides an important way of isolating distortions that are not the result of these types of risk adjustments.[8]

In terms of the quantitative model, this paper is related to a rapidly growing literature on the integration between asset markets and the broader macroeconomy in a general equilibrium setting. Jermann (1998) and Boldrin, Christiano, and Fisher (2001) both emphasize the importance of habit formation and real frictions in investment for jointly explaining asset returns and macroeconomic quantities in a production economy. More recently, a number of papers have focused on the role that recursive preferences can play in production-based economies, as opposed to habit for-

[8]See Appendix B for further discussion of this point.

mation; for example, Kaltenbrunner and Lochstoer (2010), Croce (2014), Rudebusch and Swanson (2012), Gourio (2012), and Li and Palomino (2014). One contribution of this paper is to combine several different features from these studies. For example, it allows for both habits and recursive preferences, incorporates real frictions in production such as investment adjustment costs, and incorporates nominal rigidities and therefore a role for monetary policy. A second contribution is that the model in this paper is estimated using likelihood-based methods. Most of the existing literature in this area has focused exclusively on calibration.

The solution method proposed in this paper is closely related to the recent contribution of Malkhozov (2014). He also exploits the properties of log-normal random vectors to adjust policy functions for risk. However, when applied to homoskedastic models, his procedure only delivers a constant risk adjustment. That is, risk premia do not affect the impulse responses of endogenous variables to any of the exogenous disturbances. By contrast, the impulse response functions in this paper depend on risk (and therefore risk preferences) because of a steady-state consistency criterion. Specifically, the steady state around which the approximation is constructed is required to coincide with the steady state implied by that approximation. Another difference is that the equilibrium conditions in this paper are not separated into two groups before the solution procedure is applied — all equations are treated in the same way. Lastly, the presentation in this paper explicitly derives the coefficients of the approximate solution from an arbitrary nonlinear model using matrix calculus. This facilitates comparison with other work such as Schmitt-Grohé and Uribe (2004).

3 Sources of Fluctuations in Beliefs

This section presents the mathematical details for how advance information and distorted beliefs are modeled. To highlight the intuition, the exposition is carried out through a familiar permanent income model of consumption and savings, which has only one fundamental process: income. In addition to being familiar, it has the advantage of admitting a closed-form solution, so that issues related to approximation can be postponed until Section 5. Nevertheless, it is important to keep in mind that this discussion applies to any forward-looking relation of the type described here; nothing is special about this consumption-based example in that respect. This section is meant to show three things: first, how both advance information and distorted beliefs can be included in an optimizing equilibrium model. Second, how both sources of fluctuations in beliefs can elicit similar responses from endogenous variables. And third, how it is nevertheless possible to separately identify the importance of these sources.

3.1 A Permanent-Income Example

The main distinctive aspect of this example (and the model of Section 4) is that it features two probability measures instead of one. To be precise about this distinction, let $(\Omega, \mathfrak{F})$ denote the measurable space upon which the model economy is built. The set Ω contains all possible outcomes and $\mathfrak{F}$ is a sigma-algebra of its subsets. On this space introduce two probability measures, both of which assign probabilities to events $\mathcal{A} \in \mathfrak{F}$. The first is the objective measure, denoted by $\mathbb{P}$, which defines the actual probability that any event will occur. The other is the distorted measure, denoted by $\tilde{\mathbb{P}}$, which represents the subjective beliefs of economic agents within the model and may differ from the objective measure. In the case that these two measures coincide, agents are said to have model-consistent (or "rational") expectations.

The history of events in the economy is recorded by a sequence of non-decreasing sigma-algebras $\{\mathfrak{F}_t\}$ for $t \geq 0$. At each date, a representative agent chooses consumption c_t and debt holdings d_t so as to maximize his lifetime utility, which can be expressed recursively as:

$$V_t = u(c_t) + \beta \tilde{E}_t[V_{t+1}].$$

The expectation in this expression is the one induced by $\tilde{\mathbb{P}}$, conditional on $\mathfrak{F}_t$. Note that the fact that $\tilde{\mathbb{P}}$ is a probability measure means that while the agent's beliefs may be distorted, he nevertheless always maintains an internally-consistent set of beliefs. In particular, his subjective beliefs satisfy the law of iterated expectations and standard results in dynamic programming can be used to characterize his optimal actions.[9]

Each period, the agent receives a stochastic endowment of income, z_t. Exogenous innovations to z_t represent the "fundamental" disturbances driving the dynamics of the model. The agent is permitted to borrow or lend using a risk-free bond that pays a constant gross real interest rate $R > 1$. Under the simplifying assumptions of Hall

[9]This differs, for example, from situations in which agents update their beliefs using simple econometric models. See Preston (2005) for a careful discussion of this point.

(1978), namely that $u(c) = -\frac{1}{2}(c - \bar{c})^2$ for a bliss level of consumption $\bar{c} \geq 0$ and $\beta R = 1$, the optimal consumption choice satisfies[10]

$$c_t = (1 - \beta) \left[\sum_{\tau=0}^{\infty} \beta^\tau \tilde{E}_t[z_{t+\tau}] - R d_{t-1} \right]. \tag{1}$$

At each date, consumption equals the annuity value of expected total wealth: the present discounted value of expected lifetime income minus interest payments on outstanding debt. In order to complete the model, it is necessary to specify the evolution of z_t under $\mathbb{P}$ and to clarify the relationship between $\mathbb{P}$ and $\tilde{\mathbb{P}}$.

3.2 Current Fundamentals

Without advance information or distorted beliefs, the only thing that can cause a change in beliefs is a change in current fundamentals. To see why this is the case, suppose that income is a first-order Markov process with independent, identically distributed normal innovations under the objective measure:

$$z_t = \rho_z z_{t-1} + e_t^z, \quad e_t^z \overset{\mathbb{P}}{\sim} \mathcal{N}(0, \sigma_z^2). \tag{2}$$

Under the hypothesis that the agent's beliefs are model-consistent (so that $\mathbb{P}$ and $\tilde{\mathbb{P}}$ coincide), he forecasts future outcomes using the conditional distribution:

$$z_{t+1} | \mathfrak{F}_t \overset{\mathbb{P}}{\sim} \mathcal{N}(\rho_z z_t, \sigma_z^2).$$

The only source of variation in the right-hand side of this expression at date t relative to date $t-1$ comes from the contemporaneous innovation e_t^z, which enters through z_t. Because the income process is persistent, high current fundamentals signal high future fundamentals.

Under these assumptions, it is possible to solve (1) and show that consumption follows:

$$c_t = \left(\frac{1 - \beta}{1 - \beta \rho_z} \right) z_t - \left(\frac{1 - \beta}{\beta} \right) d_{t-1}.$$

In response to a positive innovation in income, part of the increase is consumed, part is used to pay off existing debt, and the rest is saved. Importantly, it follows that

[10] As usual, this condition was derived under the assumption that when the agent decides on his debt holdings, he cannot expect to run a Ponzi scheme: $\lim_{j \to \infty} \beta^j \tilde{E}_t[d_{t+j}] \leq 0$.

innovations in consumption only come as a result of innovations in current income. By introducing advance information and distorted beliefs, it is possible to break this tight link and allow changes in consumption to occur without any corresponding change in current income.

3.3 Advance Information

To allow for the possibility that the agent may be able to partially predict changes in future fundamentals, I introduce the new stochastic process a_t, which is related to z_t in the following way:[11]

$$\begin{aligned} z_t &= \rho_z z_{t-1} + a_{t-1} + e_t^z \\ a_t &= \rho_a a_{t-1} + e_t^a, \quad e_t^a \overset{\mathbb{P}}{\sim} \mathcal{N}(0, \sigma_a^2). \end{aligned}$$

The disturbance e_t^a, which is independent and identically distributed over time and independent of e_t^z, represents advance information. It is unrelated to current fundamentals: $E[e_t^a z_t] = 0$, but it is related to future fundamentals: $E[e_t^a z_{t+\tau}] \neq 0$ for all $\tau > t$. The new conditional distribution relevant for forecasting future outcomes is:

$$z_{t+1} | \mathfrak{F}_t \overset{\mathbb{P}}{\sim} \mathcal{N}(\rho_z z_t + a_t, \sigma_z^2).$$

Now beliefs about z_{t+1} may fluctuate due to e_t^a as well as e_t^z, where the former disturbance enters through a_t. If $e_t^a = 0$ at all times, then no advance information about z_t is ever available and beliefs only fluctuate as a result of current developments.

With advance information, the equilibrium consumption rule from the permanent income model in (1) becomes:

$$c_t = \left(\frac{1-\beta}{1-\beta\rho_z} \right) \left[z_t + \left(\frac{\beta}{1-\beta\rho_a} \right) a_t \right] - \left(\frac{1-\beta}{\beta} \right) d_{t-1}.$$

Now, consumption may increase at date t either because the agent is wealthier due to an innovation in z_t, or because he has received advance information that he will be wealthier in the future due to an innovation in a_t. The interesting aspect of modeling

[11] This dynamic structure is also present in Barksy and Sims (2009, 2011), Jinnai (2013), Barsky, Basu, and Lee (2014), Bansal and Yaron (2004), and Bansal, Kiku and Yaron (2007, 2012). Alternative specifications include those by Christiano, et al. (2010) and Schmitt-Grohé and Uribe (2012), who fix $\rho_a = 0$ and allow the process a_t to enter with a lag greater than one period: $z_t = \rho_z z_{t-1} + a_{t-\ell} + e_t^z$, with $\ell \geq 1$.

advance information in this way is that it allows the link between current actions and current fundamentals to be broken while still maintaining the model-consistency of beliefs. However, it still implies that beliefs are entirely determined by developments in fundamentals, once future developments are taken into account. Beliefs exert no genuinely ultimate causal influence on economic outcomes. The introduction of exogenous distortions in beliefs relaxes this restriction.

3.4 Distorted Beliefs

To introduce distortions in the agent's beliefs, I relax the restriction that subjective and objective probabilities must coincide. Specifically, I build $\tilde{\mathbb{P}}$ using a strictly positive $\mathfrak{F}_t$-measurable stochastic process M_t. At any date $t \geq 0$, $\tilde{\mathbb{P}}$ assigns probabilities to events $\mathcal{A} \in \mathfrak{F}_t$ according to the definition:

$$\tilde{\mathbb{P}}(\mathcal{A}) \equiv E[1_{\mathcal{A}} M_t],$$

where $1_{\mathcal{A}}$ is the indicator function for the set $\mathcal{A}$. In order for $\tilde{\mathbb{P}}$ to be a well-defined probability measure on the space $(\Omega, \mathfrak{F})$ according to this definition, the process M_t must satisfy the following two criteria:

$$E_t[M_{t+1}] = M_t \quad \text{and} \quad E[M_t] = 1. \tag{3}$$

The first says that M_t is a martingale with respect to $\mathbb{P}$ and the filtration $\{\mathfrak{F}_t\}_{t\geq 0}$. This is necessary so that $\tilde{\mathbb{P}}$ satisfies the Law of Iterated Expectations. The second guarantees that $\tilde{\mathbb{P}}(\Omega) = 1$. The economic content of this specification of distortions is that the agent agrees with the model about which events are *impossible* (occur with probability zero), but he can arbitrarily disagree about how likely it is that any *possible* event may occur. The process M_t is known as a Radon-Nikodým derivative process, and represents the degree of disagreement between the agent and the model concerning these probabilities. In the case that $M_t = 1$ at all times, there are no distortions, and the agent's beliefs are model-consistent.[12]

Even given these restrictions, the class of admissible distortions M_t is still too broad to be useful from the perspective of constructing economic models. One ap-

[12] This approach to modeling belief distortions is used, for example, by Hansen and Sargent (2005) in their work on robust control, and by Woodford (2010) in his formulation of "near-rational" expectations.

proach to restrict them further has been non-parameteric: the extent to which subjective beliefs may be distorted relative to objective beliefs can be bounded on the basis of statistical discrepancy measures.[13] An alternative approach, which is the one I pursue here, is to impose additional parametric structure on M_t. The parametric approach has been fruitfully applied in asset-pricing contexts to model risk-adjustments generated by stochastic discounting.[14] For the example in this section, I assume that $M_0 = 1$ and that

$$M_{t+1} = M_t \exp\left(-\frac{1}{2}\frac{b_t^2}{\sigma_z^2} + \frac{b_t}{\sigma_z^2}e_{t+1}^z\right) \tag{4}$$

for an $\mathfrak{F}_t$-measurable stochastic process b_t. Notice that the first term inside the exponential guarantees that M_t is a martingale. Together with the initial condition, this implies that the required conditions in (3) are satisfied.

This parametric form is common in asset-pricing contexts and has a clear economic interpretation. Under the distorted measure induced by this process, it can be shown that the fundamental in (2) has the following dynamics under the distorted probability measure:

$$z_{t+1}|\mathfrak{F}_t \overset{\tilde{\mathbb{P}}}{\sim} \mathcal{N}(\rho_z z_t + b_t, \sigma_z^2).$$

That is, b_t represents a distortion in the conditional mean of the fundamental process. Higher values of b_t indicate that the agent is overly optimistic about future economic prospects relative to the model. Notice that under the restriction in (4), the agent still agrees with the model about the conditional variance of future random disturbances. A more general parametric class that also allows for distortions in the conditional variance is discussed in Section 5. However, there are two advantages to focusing on the conditional mean specification. First, because advance information is assumed only to affect conditional means and not conditional variances, this approach has the advantage of treating both sources of fluctuations in beliefs symmetrically; belief

[13]See Borovička, Hansen, and Scheinkman (2014) for a discussion of the class of discrepancy measures that take the form:

$$\frac{1}{\theta(1+\theta)}E_t\left[\left(\frac{M_{t+1}}{M_t}\right)^{1+\theta} - 1\right],$$

and their usefulness for measuring the martingale component of stochastic discount factors. Woodford (2010) focuses on the limiting case of $\theta \to 0$, where this measure reduces to conditional relative entropy: $E_t[M_{t+1}/M_t \ln(M_{t+1}/M_t)]$.

[14]See Dai and Singleton (2000) and Le, Singleton, and Dai (2010) for discussions of affine term-structure models which rely on parameterizations of the stochastic discount factor.

distortions resemble "mistaken" advance information. Second, with this restriction the model remains conditionally homoskedastic under both $\mathbb{P}$ and $\tilde{\mathbb{P}}$, which allows me to avoid some technical issues involved in solving models with time-varying volatility that are not the focus of this paper.

Finally, it remains to specify a law of motion for the process b_t. Because it is the analogue of a_t, I endow it with the same dynamic structure:

$$b_t = \rho_b b_{t-1} + e_t^b, \quad e_t^b \overset{\mathbb{P}}{\sim} \mathcal{N}(0, \sigma_b^2).$$

The disturbance e_t^b is independent and identically distributed over time, independent of e_t^z, and captures genuinely exogenous variation in beliefs. Like advance information, it is unrelated to current fundamentals: $E[e_t^b z_t] = 0$. But in contrast, it is unrelated to future fundamentals: $E[e_t^b z_{t+\tau}] = 0$ for all $\tau > t$. With belief distortions of this type, the equilibrium consumption rule in (1) becomes

$$c_t = \left(\frac{1-\beta}{1-\beta\rho_z}\right)\left[z_t + \left(\frac{\beta}{1-\beta\rho_b}\right) b_t\right] - \left(\frac{1-\beta}{\beta}\right) d_{t-1}.$$

Consumption increases at date t either because the agent is wealthier today due to an innovation in z_t, or because he incorrectly believes that he will be wealthier in the future due to an innovation in b_t.

3.5 Combined System

Bringing together the three different potential sources of fluctuations in beliefs, I arrive at the following system of equations:

$$\begin{aligned}
z_t &= \rho_z z_{t-1} + a_{t-1} + e_t^z, \quad e_t^z \overset{\mathbb{P}}{\sim} \mathcal{N}(0, \sigma_z^2) \overset{\tilde{\mathbb{P}}}{\sim} \mathcal{N}(b_{t-1}, \sigma_z^2) \\
a_t &= \rho_a a_{t-1} + e_t^a, \quad e_t^a \overset{\mathbb{P},\tilde{\mathbb{P}}}{\sim} \mathcal{N}(0, \sigma_a^2) \\
b_t &= \rho_b b_{t-1} + e_t^b, \quad e_t^b \overset{\mathbb{P},\tilde{\mathbb{P}}}{\sim} \mathcal{N}(0, \sigma_b^2).
\end{aligned} \tag{5}$$

Under these exogenous dynamics, an observed increase in consumption can come from one of three sources: an increase in current income, a correctly anticipated change in future fundamentals, or an incorrectly anticipated change in future fundamentals. For completeness, the policy rule combining (1) and (5) is:

$$c_t = \left(\frac{1-\beta}{1-\beta\rho_z}\right)\left[z_t + \left(\frac{\beta}{1-\beta\rho_a}\right) a_t + \left(\frac{\beta}{1-\beta\rho_b}\right) b_t\right] - \left(\frac{1-\beta}{\beta}\right) d_{t-1}. \tag{6}$$

Before moving on, it is worth discussing some implicit restrictions imposed by this system, as well as the interpretation of the processes a_t and b_t. First, advance information and distorted beliefs are defined only relative to the fundamental process z_t. This means that there is no advance information about future belief distortions, and that there are no belief distortions about advance information. Moreover, there is also no advance information about future advance information, and there are no belief distortions about future belief distortions. While such multiple layers of fluctuations in beliefs would be theoretically straightforward to include, the more parsimonious structure employed here is sufficient for the purposes of this paper. Nevertheless, that is technically a restriction that will be imposed throughout.

A second and related restriction is that the three innovations e_t^z, e_t^a, and e_t^b are independent. This rules out, for example, situations in which an innovation to current fundamentals *systematically* leads to a more than proportional increase in agents' optimism concerning future fundamentals (whether that optimism is justified or not). In the context of a structural model such as the example in this section, it is sometimes possible to separately identify some forms of correlation between these disturbances. I will show how that can be done in the following subsection. The quantitative model in Section 4, however, will feature independent disturbances. This is in keeping with the business cycle literature, which typically assumes that all exogenous processes are independent.[15]

Third, it is important to understand the interpretation that is being given to the processes a_t and b_t. First consider b_t. This process is not meant to represent any variable that is "out there" in the model, which agents observe and incorrectly believe to be informative concerning future fundamentals. Rather, it is a useful mathematical way of representing the degree of correspondence between agents' beliefs and the ones implied by the model, no matter how agents arrived at those beliefs.[16] Similarly, on this interpretation the process a_t should not be interpreted as a variable that is literally observed by agents and known to be perfectly informative concerning future fundamentals. Instead, it is only meant to *reflect* that component of agents' beliefs

[15]However, see Cúrdia and Reis (2010) for a discussion of why this might not be such a good idea, along with a methodological proposal for how to generalize that assumption in empirical work.

[16]See Woodford (2010) for a similar point. As in his analysis, the interpretation taken here implies that the usual assumption of model-consistent beliefs does not require that agents "know the model" and correctly solve its equations, but only that their beliefs coincide with the ones predicted by their model environment.

regarding the future which are systematically correlated with subsequent realizations of fundamental activity. Indeed, as I mentioned in the introduction, one advantage to modeling beliefs in this way is precisely to avoid making statements about what agents observe and what behavioral rules they use to update their beliefs based on that information.[17]

3.6 Identification: Key Moments

Given the similarity between advance information and distorted beliefs affect actions, how is it possible to separately identify their relative importance? To make this question precise, consider an econometrician who understands the structure of the physical economy and is charged with estimating the relative importance of current fundamentals, advance information, and distorted beliefs. In terms of the the example of this section, he knows that in equilibrium consumption and debt holdings are related to income according to (6) and the budget constraint $c_t + d_{t-1}/\beta = z_t + d_t$, and that income and beliefs about income can be represented by the system (5). He also knows the value of β and the persistence parameters ρ_z, ρ_a, and ρ_b. Because he knows the persistence parameters, fix them at zero; this makes the key moment conditions especially stark. He is endowed with a sequence of observations of consumption and income generated by the model, and is charged with estimating the parameters σ_z, σ_a, and σ_b.

With a sample of observations of c_t and z_t, it is possible to separately identify these parameters. The intuition is that the three different disturbances induce different dynamic correlations between consumption and income. To isolate the key moments, note that in the case of no persistence, consumption growth satisfies:[18]

$$\Delta c_t = (1-\beta)\Big[z_t + \beta(a_t + b_t) - (a_{t-1} + b_{t-1})\Big].$$

First, the covariance between consumption growth and future income pins down the importance of advance information. Second, given σ_a, the variance of income pins down σ_z.[19] Third, given σ_a and σ_z, the variance of consumption growth reveals the

[17]Appendix C discusses the type of belief distortions that can arise from making informational assumptions of these types.

[18]Recall that consumption is not stationary in this model; it is a martingale under $\tilde{\mathbb{P}}$.

[19]Note that it is also possible to use the covariance between consumption growth and income to identify σ_z (or to help identify β if it is also unknown): $E[\Delta c_t z_t] = (1-\beta)\sigma_z^2$.

importance of distorted beliefs. Explicity, these three moments are:

$$\begin{aligned} E[\Delta c_t z_{t+1}] &= \beta(1-\beta)\sigma_a^2 \\ E[z_t^2] &= \sigma_a^2 + \sigma_z^2 \\ E[\Delta c_t^2] &= (1-\beta)^2 \left[\sigma_z^2 + \beta^2\sigma_a^2 + (1+\beta^2)\sigma_b^2\right]. \end{aligned} \tag{7}$$

In fact, it turns out that if the econometrician can directly observe data on subjective beliefs, it may be possible for him to identify these parameters even without any knowledge of the economic model. Suppose that his sample also includes observations of income forecasts, $\tilde{E}_t[z_{t+1}] = a_t + b_t$. Then he can infer the values of σ_z, σ_a, and σ_b from the three moments:

$$\begin{aligned} E[\tilde{E}_t[z_{t+1}]z_{t+1}] &= \sigma_a^2 \\ E[z_t^2] &= \sigma_a^2 + \sigma_z^2 \\ E[\tilde{E}_t[z_{t+1}]^2] &= \sigma_a^2 + \sigma_b^2. \end{aligned} \tag{8}$$

Notice that this identification scheme does not require any knowledge of the structural relation between consumption and income (or any parameters in that relation, such as β). This highlights one of the main advantages of using direct evidence on subjective beliefs: in general, subjective forecasts of model variables provide additional moment restrictions without requiring any additional structural assumptions. This is because the dynamics of the structural model already contain predictions about the behavior of agents' forecasts. Later in this section, I will show in a Monte Carlo experiment that adding data on subjective forecasts can help to increase the precision of one's estimates of the relative importance of advance information and distorted beliefs.

In summary, the key moment for determining the relative importance of advance information is the covariance between a current endogenous variable (in this case either consumption or expected income) and future fundamentals. The key moment for determining the relative importance of current fundamentals is the variance of current fundamentals. And lastly, the key moment for determining the relative importance of distorted beliefs is the variance of the endogenous variable.

3.7 Correlations and Measurement Errors

This subsection briefly addresses two possible complications related to identifying the different sources of fluctuations in beliefs. First is the situation when either a_t or b_t is affected by current fundamental disturbances e_t^z; in other words, the innovation to a_t or b_t is correlated with the current innovation to fundamentals. For example suppose that

$$b_t = \rho_b b_{t-1} + \tau e_t^z + e_t^b,$$

where $\tau \neq 0$. If $\tau > 0$ this implies that in response to a positive innovation in current fundamentals, agents' beliefs systematically become overly optimistic. In this case, is it possible to identify the parameter τ along with σ_z, σ_a, and σ_b?

The answer is yes. The key additional moment is the correlation between the current endogenous variable and current fundamentals. In the consumption-income example of this section, still assuming for simplicity that $\rho_z = \rho_a = \rho_b = 0$ but now allowing for nonzero values of τ, it can be shown that

$$E[\Delta c_t z_t] = (1-\beta)(1+\beta\tau)\sigma_z^2.$$

The mechanics behind the identification here is that σ_a and σ_z are already determined by the first two moments in system (7). If consumption growth is more correlated with income than these two parameters suggest based on the model, then it must be that the agent systematically overreacts to current fundamental developments. That is, that $\tau > 0$.

A second situation of interest is when variables are observed with error. To entertain this possibility, suppose that the econometrician only observes the following noisy measures of consumption, income, and expected income:

$$\begin{aligned}
\Delta\tilde{c}_t &= \Delta c_t + u_{\Delta c,t}, \quad u_{\Delta c,t} \overset{\mathbb{P}}{\sim} \mathcal{N}(0, \omega_{\Delta c}^2) \\
\tilde{z}_t &= z_t + u_{z,t}, \quad u_{z,t} \overset{\mathbb{P}}{\sim} \mathcal{N}(0, \omega_z^2) \\
\tilde{\tilde{E}}_t[z_{t+1}] &= \tilde{E}_t[z_{t+1}] + u_{E,t}, \quad u_{E,t} \overset{\mathbb{P}}{\sim} \mathcal{N}(0, \omega_E^2).
\end{aligned}$$

In this case, is it possible to identify the three parameters $\omega_{\Delta c}$, ω_z, and ω_E in addition to σ_z, σ_a, and σ_b?

Again the answer is yes.[20] The reason is that classical measurement errors of the type assumed here only affect one observed series, and are uncorrelated with every other disturbance in the model. By contrast, each of the other three disturbances

[20] The answer is no in this example if the econometrician observes income and only one of either consumption or expected income. In general it is important that he observe more than one variable that is affected by belief distortions.

affect more than one variable. They generate particular patterns of *comovement* that are not affected by the measurement errors. Specifically, observe that the following six moments can be solved sequentially for all six unknown parameters:[21]

$$\begin{aligned}
E[\tilde{E}_t[z_{t+1}]z_{t+1}] &= \sigma_a^2 \\
E[\Delta c_t z_t] &= (1-\beta)\sigma_z^2 \\
E[\tilde{E}_t[z_{t+1}]\Delta c_t] &= (1-\beta)\beta(\sigma_a^2+\sigma_b^2) \\
E[z_t^2] &= \sigma_a^2+\sigma_z^2+\omega_z^2 \\
E[\tilde{E}_t[z_{t+1}]^2] &= \sigma_a^2+\sigma_b^2+\omega_E^2 \\
E[\Delta c_t^2] &= (1-\beta)^2[\sigma_z^2+\beta^2\sigma_a^2+(1+\beta^2)\sigma_b^2]+\omega_{\Delta c}^2
\end{aligned}$$

This subsection has therefore demonstrated that it is in principle possible to separately identify the relative importance of distorted beliefs and advance information even in the presence of some forms of correlation between the sources of fluctuations in beliefs, or classical measurement error. The next subsection presents an example of how these parameters can be estimated in practice.

3.8 Two Monte Carlo Experiments

The discussion of identification so far has focused on the case with no persistence in order to make the key moment conditions as stark as possible. In the more general case that the persistence parameters themselves must also be estimated from the data, each parameter can no longer be associated with just one moment condition. Instead, all the parameters must be estimated jointly from a set of moment conditions. The approach of this paper is to prioritize different moments according to the way they are encoded in the likelihood function. This subsection presents two Monte Carlo experiments that illustrate how the different parameters of the consumption-income example can be identified based on the likelihood function. It also illustrates the additional precision afforded by exploiting subjective forecasts.

The true parameters are fixed at $(\rho_z,\rho_a,\rho_b,\sigma_z,\sigma_a,\sigma_b) = (0.9, 0.7, 0.5, 0.4, 0.6, 0.8)$. The subjective discount factor is assumed to be known and is set to $\beta = 0.995$. Given these true parameters, I simulate 10,000 artificial samples of consumption growth Δc_t,

[21] In fact, one additional moment is still available either to identify β if it is unkown, or to allow for one additional correlation parameter. That moment is $E[\Delta c_t z_{t+1}]$.

income growth Δz_t, and expected income growth $\tilde{E}_t[\Delta z_{t+1}]$ from the consumption-income example in this section. Each sample consists of 219 observations of each variable, which is the same size as the actual data sample that will be used in Section 6. All parameter estimates are obtained by numerically maximizing the likelihood function implied by the model.

In the first experiment, suppose that the econometrician only observes consumption growth and income growth but no subjective forecasts. The dashed lines in Figure 1 represent the distribution of parameter estimates based on these two variables. In each case the mean is close to the true value (represented by the vertical dotted line), indicating that the likelihood function does a relatively good job at revealing the values of these parameters for a sample of this size. Note however that the parameters related to belief distortions (the third column) are the most imprecisely estimated.

In the second experiment, suppose that the econometrician is also given data on subjective forecasts. In particular, he observes subjective expectations of future income growth in addition to consumption growth and income growth. The solid lines in Figure 1 represent the distribution of parameter estimates based on these three variables. The main difference between this and the first experiment is clear: by including data on subjective forecasts, the precision of estimates notably increases. The distributions are still centered around their true values, but now the dispersion of the distribution is reduced, especially for the persistence parameters and for the conditional variance of belief distortions.

4 Quantitative Equilibrium Model

This section presents the central model of the paper. It has three distinctive features. First, agents have beliefs that may reflect advance information or exogenous distortions about about future fundamentals. Second, "fundamentals" are defined by a rich set of nine exogenous processes. This is to ensure that advance information and distorted beliefs are not merely standing in for other omitted processes commonly considered in the literature. Third, households have recursive utility preferences, which are included to help the model jointly explain macroeconomic quantities and asset returns.

4.1 Household

The model features a representative household with preferences that satisfy the recursion:[22]

$$V_t = \left\{ (1-\beta)\zeta_t^C U_t^{1-1/\xi} + \beta \tilde{E}_t \left[V_{t+1}^{1-\gamma} \right]^{\frac{1-1/\xi}{1-\gamma}} \right\}^{\frac{1}{1-1/\xi}}. \tag{9}$$

The variable $U_t \geq 0$ denotes the period utility kernel, and ζ_t^C is an exogenous stochastic disturbance to preferences. The parameter $\beta \in (0,1)$ determines the marginal rate of time preference, $\gamma > 0$ controls the degree of risk aversion, and $\xi > 0$ controls the degree to which the household is willing to substitute period utility flows over time.[23] In the special case that $\gamma = 1/\xi$, these preferences collapse to the standard power utility form. If $\gamma > 1/\xi$, the household has a preference for early resolution of uncertainty, and if $\gamma < 1/\xi$ it has a preference for late resolution. As in the model of the previous section, expectations are computed according to the distorted probability measure $\tilde{\mathbb{P}}$, which is common to all agents in the model.

The period utility kernel U_t is a function of current and past consumption, C_t, and hours worked H_t:

$$U_t = C_t - \phi C_{t-1} - \vartheta H_t^{\psi} D_t. \tag{10}$$

The variable D_t is a geometric average of current and past habit-adjusted consumption.[24] Its law of motion is assumed to be:

$$D_t = (C_t - \phi C_{t-1})^{\nu} D_{t-1}^{1-\nu}. \tag{11}$$

The parameter $\phi \in [0,1)$ controls the degree of internal habit formation in preferences, $\psi > 1$ determines the Frisch elasticity of labor supply when $\nu = \phi = 0$, and $\vartheta > 0$ scales the disutility of labor.

[22] Preferences of this type are also often called "Epstein-Zin" or "Epstein-Zin-Weil" preferences in reference to Epstein and Zin (1989) and Weil (1989). The original framework for recursive utility preferences was developed by Kreps and Porteus (1978).

[23] When $\xi = 1$ preferences are defined by the pointwise limit:

$$V_t = \left\{ U_t^{(1-\beta)\zeta_t^C} \tilde{E}_t \left[V_{t+1}^{1-\gamma} \right]^{\frac{\beta}{1-\gamma}} \right\}^{\frac{1}{(1-\beta)\zeta_t^C + \beta}}.$$

[24] The introduction of this process is due to Jaimovich and Rebelo (2009), and Schmitt-Grohé and Uribe (2012) modify it to incorporate habit persistence.

These preferences have two appealing features. First, they nest as special cases two widely used utility functions in the business cycle literature. If $\nu = 1$, $\phi = 1$, and $\gamma = 1/\xi$, preferences are of the type considered by King, Plosser, and Rebelo (1988); if $\nu = 0$, $\phi = 1$, and $\gamma = 1/\xi$, there are no wealth effects on labor supply as in Greenwood, Hercowitz, and Huffman (1988). Second, they nest as special cases two widely used utility functions in the asset pricing literature. If $\phi > 0$, $\gamma = 1/\xi$, and $\vartheta = 0$, utility is derived from current consumption relative to a (one period) habit stock of consumption and the household is indifferent concerning the intertemporal resolution of uncertainty as in Campbell and Cochrane (1999). If $\phi = 0$, $\gamma \neq 1/\xi$, and $\vartheta = 0$, there are no habits but the household is not indifferent about how uncertainty is resolved as in Bansal and Yaron (2004). A byproduct of the estimation will therefore be to determine the quantitative impact of each of these four prominent features of preferences.

The household has access to a complete set of Arrow securities, each of which represents a claim to a real state-contingent return in the subsequent period. The aggregate resources devoted to these securities at date t is denoted by A_t, and they deliver the real state-contingent return R^A_{t+1} at date $t+1$. The household also receives wage income W^s_t from supplying its raw labor services to a monopolistic union. Letting T_t denote net lump-sum transfers, the household's flow budget constraint is:

$$C_t + A_t \leq W^s_t H_t + R^A_t A_{t-1} + T_t.$$

The absence of arbitrage opportunities in this economy requires that there exist a unique stochastic discount factor S_{t+1} with the property that the state-contingent return on any asset must satisfy the relation $\tilde{E}_t[S_{t+1} R^A_{t+1}] = 1$. Letting $\Lambda_t U_t^{-1/\xi}$ denote the household's marginal utility of income, the real stochastic discount factor in this economy is:

$$S_{t+1} = \beta \frac{\Lambda_{t+1}}{\Lambda_t} \left(\frac{U_{t+1}}{U_t} \right)^{-1/\xi} \left(\frac{V_{t+1}}{\tilde{E}_t[V_{t+1}^{1-\gamma}]^{1/(1-\gamma)}} \right)^{1/\xi - \gamma}. \tag{12}$$

Between t and $t+\tau$, the stochastic discount factor is $S_{t,t+\tau} \equiv \prod_{s=1}^{\tau} S_{t+\tau}$.

When there are no habits in consumption ($\phi = 0$), the utility kernel is separable in consumption and leisure ($\nu = 0$), and there is no exogenous variation in preferences ($\zeta^C_t \equiv 1$), the marginal utility of income is equal to $U_t^{-1/\xi}$. On the other hand when those features are present, the marginal utility of income depends on how income will affect the habit stock and disutility of labor in subsequent periods. This forward-looking aspect can be seen in the optimality condition with respect to consumption:

$$\Lambda_t = (1-\beta)\zeta_t^C - \nu\Gamma_t \frac{D_t}{C_t - \phi C_{t-1}} - \phi \tilde{E}_t \left[S_{t+1}^* \left((1-\beta)\zeta_{t+1}^C - \nu\Gamma_{t+1} \frac{D_{t+1}}{C_{t+1} - \phi C_t} \right) \right], \tag{13}$$

where $S_{t+1}^* \equiv S_{t+1}\Lambda_t/\Lambda_{t+1}$. The household internalizes the effects of its consumption choices on the stock variable D_t. The marginal value of increasing this variable by one unit is denoted $\Gamma_t U_t^{-1/\xi}$, where Γ_t follows:

$$\Gamma_t = \vartheta\psi H_t^{\psi}(1-\beta)\zeta_t^C + (1-\nu)\tilde{E}_t \left[S_{t+1}^* \Gamma_{t+1} \frac{D_{t+1}}{C_{t+1} - \phi C_t} \right]. \tag{14}$$

Lastly, the household takes wages as given, and supplies labor up to the point where the ratio of its marginal disutility of work to its marginal utility of income equals the real wage:

$$\frac{\vartheta\psi H_t^{\psi-1} D_t (1-\beta)\zeta_t^C}{\Lambda_t} = W_t^s. \tag{15}$$

4.2 Labor Union

A centralized labor union receives raw labor input H_t at a real marginal cost of $\zeta_t^W W_t^s$, and specializes it for use in a continuum of labor markets indexed by $j \in [0,1]$. The union supplies this specialized labor monopolistically, and ζ_t^W captures exogenous variation in marginal costs that creates a wedge between the the wage it charges in labor market j and the wage it pays to the household. In each labor market, the union supplies enough specialized labor to satisfy demand:

$$H_t(j) = \left(\frac{W_t(j)}{W_t} \right)^{-\eta_W} H_t^d.$$

The variable $W_t(j)$ denotes the real wage charged by the union in labor market j, W_t is an index of wages in the economy, and H_t^d represents the aggregate level of specialized labor demanded by productive firms.

In each period, the union is only capable of resetting wages optimally in a randomly chosen fraction $1-\theta_W \in [0,1)$ of labor markets.[25] In the remaining θ_W markets,

[25] For a discussion of the differences between this approach to modeling nominal wage rigidity and other common approaches such as the one in Erceg, et al. (2000), see Schmitt-Grohé and Uribe (2007). The main advantage here is that the assumption of a centralized market for the raw household labor input ensures that equilibrium heterogeneity in labor supply does not also lead to heterogeneity in consumption.

nominal wage growth is indexed to past gross inflation according to rule:

$$\frac{W_t(j)}{W_{t-1}(j)}\Pi_t = (\mu_W \Pi_{t-1})^{\chi_W}.$$

The constant term μ_W is the steady-state growth rate of real wages, and is determined in equilibrium. The variable Π_t denotes the gross inflation rate from period $t-1$ to t. The parameter $\chi_W \in [0,1]$ controls the degree of wage indexation, with $\chi_W = 0$ indicating no indexation.

In each labor market where the union is able to optimally set its posted wage, it maximizes the present discounted value of future real profits,

$$\tilde{E}_t \sum_{\tau=0}^{\infty} \theta_W^{\tau} S_{t,t+\tau} \left\{ (W_{t+\tau}(j) - \zeta_{t+\tau}^W W_{t+\tau}^s) H_{t+\tau}(j) \right\},$$

subject to the demand curve and indexation rule above. All profits are re-distributed lump sum to households, so state-contingent profit streams are valued using the stochastic discount factor $S_{t,t+\tau}$. Marginal costs are common across all labor markets, so the union will post the same wage W_t^* and supply the same amount of labor H_t^* in all markets where it is able to re-optimize. Optimality requires that W_t^* equates the expected marginal gain of having that wage persist forever in terms of additional revenue per unit of labor demand to a constant markup over the marginal loss in terms of reduced demand:

$$\frac{W_t^* E_t^{\mathbb{Q}} \sum_{\tau=0}^{\infty} \theta_W^{\tau} S_{t,t+\tau} H_{t+\tau}^d W_{t+\tau}^{\eta_W} \left(\prod_{s=1}^{\tau} \frac{(\mu_W \Pi_{t+s-1})^{\chi_W}}{\Pi_{t+s}} \right)^{1-\eta_W}}{\tilde{E}_t \sum_{\tau=0}^{\infty} \theta_W^{\tau} S_{t,t+\tau} H_{t+\tau}^d W_{t+\tau}^{\eta_W} \zeta_{t+\tau}^W W_{t+\tau}^s \left(\prod_{s=1}^{\tau} \frac{(\mu_W \Pi_{t+s-1})^{\chi_W}}{\Pi_{t+s}} \right)^{-\eta_W}} = \left(\frac{\eta_W}{\eta_W - 1} \right).$$

In the case that $\theta_W \to 0$, this condition says that the optimal reset wage is equal to a time-varying markup over the wage paid to the household:

$$W_t^* = \left(\frac{\eta_W}{\eta_W - 1} \right) \zeta_t^W W_t^s.$$

It is possible to express this optimal pricing condition recursively by introducing an auxiliary forward-looking variable Υ_t^W. This variable satisfies the two relations:

$$\Upsilon_t^W = \left(\frac{\eta_W}{\eta_W - 1} \right) \zeta_t^W W_t^s H_t^d W_t^{\eta_W} + \theta_W \tilde{E}_t \left[S_{t+1} \left(\frac{(\mu_W \Pi_t)^{\chi_W}}{\Pi_{t+1}} \right)^{-\eta_W} \Upsilon_{t+1}^W \right] \tag{16}$$

$$\Upsilon_t^W = W_t^* H_t^d W_t^{\eta_W} + \theta_W \tilde{E}_t \left[S_{t+1} \left(\frac{(\mu_W \Pi_t)^{\chi_W}}{\Pi_{t+1}} \right)^{1-\eta_W} \frac{W_t^*}{W_{t+1}^*} \Upsilon_{t+1}^W \right] \tag{17}$$

4.3 Productive Firm

To produce wholesale goods, a representative firm hires labor, accumulates capital, and controls how intensely it uses its currently installed capital. Its production function is:

$$Y_t = \zeta_t^Y (u_t K_{t-1})^{\alpha_K} (Z_t H_t^d)^{\alpha_H} (Z_t L)^{1-\alpha_K-\alpha_H}. \tag{18}$$

The variable ζ_t^Y is an exogenous, transitory productivity process, and Z_t is a permanent one. A fixed amount of land L is used in production, which generates decreasing returns to scale in the other factors of production. In the special case that $\alpha_K + \alpha_H = 1$, this production function exhibits constant returns to scale in capital and labor. Variable capacity utilization is captured by u_t.

The labor input used by the firm is a composite of labor from all different labor markets:

$$H_t^d = \left[\int_0^1 H_t(j)^{1-1/\eta_W} dj\right]^{1/(1-1/\eta_W)},$$

where $\eta_W > 1$ controls the degree of substitutability across different labor types. The labor demand function faced by the centralized union is the solution to the firm's problem of minimizing total labor cost subject to this aggregation technology.

Capital accumulation is subject to convex investment adjustment costs:

$$K_t = (1 - \delta(u_t))K_{t-1} + \left[1 - \Phi\left(\frac{I_t}{I_{t-1}}\right)\right] \zeta_t^I I_t, \tag{19}$$

where ζ_t^I represents an exogenous disturbance in the rate of transformation from investment goods I_t to capital goods, and $\delta(u_t) \equiv \delta_0 + \delta_1(u_t - 1) + \delta_2/2(u_t - 1)^2$ is the stochastic rate of depreciation, which increases if the firm employs its existing capital stock more intensely. Adjustment costs are quadratic in deviations of investment growth from its steady-state level: $\Phi(I_t/I_{t-1}) \equiv \frac{\kappa}{2}(I_t/I_{t-1} - \mu_I)^2$ with $\kappa > 0$. The total cost of investment goods in terms of the consumption good is $X_t I_t$, where X_t is a non-stationary exogenous process that represents the "relative price of investment."[26]

Output is sold in a competitive, centralized market at price P_t^s. Profits earned by the firm are rebated lump sum to the household. Its objective is to maximize the present discounted value of its profits from production, where future payouts are

[26] Alternatively, because X_t captures changes in the ability of firms to transform investment goods into consumption goods, it is sometimes also referred to as "investment-specific technological progress" or "capital-embodied technological progress."

valued according to the household's stochastic discount factor. The (cum-dividend) value of the firm obeys the recursion:

$$J_t = P_t^s Y_t - W_t H_t^d - X_t I_t + \tilde{E}_t[S_{t+1} J_{t+1}]. \tag{20}$$

Equity in this model is defined as a levered claim on the productive firm. Letting $\lambda \geq 1$ capture the degree of leverage, the gross real return on equity is therefore a function of the market return on the firm:[27]

$$R_{t+1}^e = \left(\frac{J_{t+1}}{J_t}\right)^\lambda. \tag{21}$$

Turning to the optimal decisions of the firm, labor is demanded up to the point that its marginal product is equals the aggregate wage:

$$\alpha_H \frac{P_t^s Y_t}{H_t^d} = W_t, \tag{22}$$

and the desired level of capital must satisfy the no-arbitrage condition:

$$1 = \tilde{E}_t\left[S_{t+1}\left(\frac{\alpha_K Y_{t+1}/K_t + Q_{t+1}(1-\delta(u_{t+1}))}{Q_t}\right)\right], \tag{23}$$

where the term in parentheses is the gross rate of return on investment. The variable Q_t represents (marginal) Tobin's Q — the value of installed capital in terms of its replacement cost. In the absence of any adjustment costs or any transitory investment-specific technological change ($\zeta_t^I \equiv 1$) Tobin's Q is equal to the replacement cost of capital in terms of the consumption good: $Q_t = X_t$. In the presence of adjustment costs however, the replacement cost of capital also depends on the growth rate of investment:

$$\begin{aligned} X_t = \zeta_t^I Q_t &\left[1 - \Phi\left(\frac{I_t}{I_{t-1}}\right) - \Phi'\left(\frac{I_t}{I_{t-1}}\right)\frac{I_t}{I_{t-1}}\right] \\ &+ \tilde{E}_t\left[S_{t+1}\zeta_{t+1}^I Q_{t+1}\Phi'\left(\frac{I_{t+1}}{I_t}\right)\left(\frac{I_{t+1}}{I_t}\right)^2\right]. \end{aligned} \tag{24}$$

Lastly, the level of intensity at which the firm utilizes its installed capital equates the marginal benefit of higher output and the marginal cost of a higher rate of depreciation:

$$\alpha_K \frac{Y_t}{K_{t-1}} = Q_t u_t \delta'(u_t). \tag{25}$$

[27]Related studies such as Kaltenbrunner and Lochstoer (2010) and Croce (2013) instead associate equity returns with levered returns to a capital claim. With constant returns to scale, these two approaches are the same (for example, see Restoy and Rockinger, 1994). However, the model in this paper allows for non-constant returns to scale; therefore this definition is the appropriate one.

4.4 Retail Firm

A monopolistic, centralized retailer purchases wholesale goods from productive firms at a per-unit cost $\zeta_t^P P_t^s$, and specializes them for sale in a continuum of retail markets indexed by $i \in [0,1]$. The variable ζ_t^P captures exogenous variation in marginal costs and creates a wedge between the price that the retailer charges in market i and the price it pays the productive firm. The demand curve in each market is given by

$$Y_t(i) = P_t(i)^{-\eta_P} Y_t^d,$$

where $P_t(i)$ denotes the price posted in market i relative to an aggregate price index, and Y_t^d is the aggregate amount of final goods demanded in the economy.

As with the labor union, the retailer is only capable of optimally resetting wages in a randomly chosen fraction $1-\theta_P \in [0,1)$ of product markets. In the remaining θ_P markets, nominal product price growth is indexed to past inflation according to the rule:

$$\frac{P_t(i)}{P_{t-1}(i)}\Pi_t = \Pi_{t-1}^{\chi_P},$$

where $\chi_P \in [0,1]$ controls the degree of price indexation.

In each product market the retailer is able to optimally adjust its posted price, it maximizes the present discounted value of all future real profits,

$$\tilde{E}_t \sum_{\tau=0}^{\infty} \theta_P^{\tau} S_{t,t+\tau} \left\{ (P_{t+\tau}(i) - \zeta_{t+\tau}^P P_{t+\tau}^s) Y_{t+\tau}(i) \right\},$$

subject to the demand curve and indexation rule above. All profits are redistributed lump sum to households. Because marginal costs are common across all product markets, the price will be the same in all markets that are re-optimized in period t. This price, P_t^*, satisfies the optimality condition:

$$\frac{P_t^* \tilde{E}_t \sum_{\tau=0}^{\infty} \theta_P^{\tau} S_{t,t+\tau} Y_{t+\tau}^d \left(\prod_{s=1}^{\tau} \frac{(\Pi_{t+s-1})^{\chi_P}}{\Pi_{t+s}} \right)^{1-\eta_P}}{\tilde{E}_t \sum_{\tau=0}^{\infty} \theta_P^{\tau} S_{t,t+\tau} Y_{t+\tau}^d \zeta_{t+\tau}^P P_{t+\tau}^s \left(\prod_{s=1}^{\tau} \frac{(\mu_P \Pi_{t+s-1})^{\chi_P}}{\Pi_{t+s}} \right)^{-\eta_P}} = \left(\frac{\eta_P}{\eta_P - 1} \right).$$

This optimality condition can also be expressed recursively by introducing the auxiliary forward-looking variable Υ_t^P, which satisfies the two relations:

$$\Upsilon_t^P = \left(\frac{\eta_P}{\eta_P - 1}\right) \zeta_t^P P_t^s Y_t^d + \theta_P \tilde{E}_t \left[S_{t+1} \left(\frac{\Pi_t^{\chi_P}}{\Pi_{t+1}}\right)^{-\eta_P} \Upsilon_{t+1}^P \right] \tag{26}$$

$$\Upsilon_t^P = P_t^* Y_t^d + \theta_P \tilde{E}_t \left[S_{t+1} \left(\frac{\Pi_t^{\chi_P}}{\Pi_{t+1}}\right)^{1-\eta_P} \frac{P_t^*}{P_{t+1}^*} \Upsilon_{t+1}^P \right]. \tag{27}$$

4.5 Government

The government consumes an exogenous amount of goods, G_t, each period. The fiscal authority finances that consumption through lump-sum taxation of the household, and maintains a balanced budget each period. The monetary authority targets the return on nominally risk-free bonds according to a feedback rule of the form:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\varphi_R} \left[\left(\frac{\Pi_t}{\Pi}\right)^{\varphi_\Pi} \left(\frac{Y_t^d / Y_{t-1}^d}{\mu_Y}\right)^{\varphi_{Y_d}} \right]^{1-\varphi_R} \zeta_t^R. \tag{28}$$

The nominal rate is a function of deviations of gross inflation from its long-run target level Π, deviations of aggregate demand growth from its steady state level μ_Y, and potentially also past rates. The variable ζ_t^R captures exogenous variation in the nominal rate.

By no arbitrage, the equilibrium ex-post real interest rate between period t and $t+1$, denoted R_t/Π_{t+1}, must satisfy the pricing condition:

$$1 = \tilde{E}_t \left[S_{t+1} \frac{R_t}{\Pi_{t+1}} \right]. \tag{29}$$

4.6 Market Clearing and Equilibrium

There are four non-financial markets in this economy: the markets for unspecialized labor, specialized labor, wholesale goods, and retail goods. Market clearing requires that aggregate supply equal aggregate demand in each of these markets.

The amount of unspecialized labor supplied by households must equal the amount demanded by the union, $H_t = \int_0^1 H_t(j)dj$, and in turn the amount of specialized labor supplied by the union must equal the amount demanded by productive firms, $\int_0^1 H_t(j)dj = H_t^d \int_0^1 (W_t(j)/W_t)^{-\eta_W} dj$. Defining $\Delta_t^W \equiv \int_0^1 (W_t(j)/W_t)^{-\eta_W} dj$, these relations imply that a fraction of the labor supplied by households is lost due to the inefficient wage dispersion:

$$H_t = \Delta_t^W H_t^d. \tag{30}$$

Similarly, a fraction of the total output supplied by productive firms is lost as a result of inefficient price distortions. Defining terms analogously to the labor market:

$$Y_t = \Delta_t^P Y_t^d, \tag{31}$$

where the aggregate demand for final goods is given by

$$Y_t^d = C_t + X_t I_t + G_t. \tag{32}$$

The nature of the time-dependent rigidites in these markets allows the laws of motion for the aggregate wage index, price index, wage dispersion, and price dispersion to be written in recursive form. In that order, the laws of motion are:

$$W_t^{1-\eta_W} = \theta_W \left(\frac{(\mu_W \Pi_t)^{\chi_W}}{\Pi_{t+1}} \right)^{1-\eta_W} W_{t-1}^{1-\eta_W} + (1-\theta_W)(W_t^*)^{1-\eta_W} \tag{33}$$

$$1 = \theta_P \left(\frac{\Pi_{t-1}^{\chi_P}}{\Pi_t} \right)^{1-\eta_P} + (1-\theta_P)(P_t^*)^{1-\eta_P} \tag{34}$$

$$\Delta_t^W = \theta_W \left(\frac{W_{t-1}}{W_t} \frac{(\mu_W \Pi_{t-1})^{\chi_W}}{\Pi_t} \right)^{-\eta_W} \Delta_{t-1}^W + (1-\theta_W) \left(\frac{W_t^*}{W_t} \right)^{-\eta_W} \tag{35}$$

$$\Delta_t^P = \theta_P \left(\frac{\Pi_{t-1}^{\chi_P}}{\Pi_t} \right)^{-\eta_P} \Delta_{t-1}^P + (1-\theta_P)(P_t^*)^{-\eta_P}. \tag{36}$$

Having laid out all the necessary conditions, I can now define the competitive equilibrium:

Definition 1. *A competitive equilibrium in the model from Section 4 is a set of stochastic processes* $\{C_t, I_t, Y_t, Y_t^d, H_t, H_t^d, K_t, U_t, D_t, Q_t, V_t, J_t, \Lambda_t, \Gamma_t, u_t, S_t, R_t, R_t^e, W_t^s, W_t^*, W_t, P_t^s, P_t^*, \Pi_t, \Upsilon_t^W, \Upsilon_t^P, \Delta_t^P, \Delta_t^W\}_{t=0}^{\infty}$ *satisfying the system of equations* (9)-(36)*, given the set of exogenous stochastic processes* $\{\zeta_t^C, \zeta_t^I, \zeta_t^Y, \zeta_t^W, \zeta_t^P, \zeta_t^R, G_t, Z_t, X_t\}_{t=0}^{\infty}$ *and initial conditions.*

4.7 Fundamental Processes

The model features nine fundamental processes, representing exogenous variation in permanent neutral productivity (Z_t), transitory neutral productivity (ζ_t^Y), permanent investment-specific productivity (X_t), transitory investment-specific productivity (ζ_t^I), labor market marginal costs (ζ_t^W), product market marginal costs (ζ_t^P), preferences (ζ_t^C), government spending (G_t), and monetary policy (ζ_t^R). The two permanent shocks are assumed to have stationary growth rates:

$$\mu_{Z,t} \equiv \frac{Z_t}{Z_{t-1}} \quad \text{and} \quad \mu_{X,t} \equiv \frac{X_t}{X_{t-1}}.$$

These two permanent shocks generate trend growth in several endogenous variables. The trends in output and investment, for example are given by:

$$Z_t^Y \equiv Z_t X_t^{\frac{\alpha_K}{\alpha_K - 1}} \quad \text{and} \quad Z_t^I \equiv \frac{Z_t^Y}{X_t}.$$

Government spending is assumed to be cointegrated with aggregate output, but with a potentially smoother trend. Letting Z_t^G denote the trend in government spending, its law of motion is:

$$Z_t^G = (Z_{t-1}^G)^{\rho_{zg}} (Z_{t-1}^Y)^{1-\rho_{zg}},$$

where $\rho_{zg} \in [0, 1)$ controls the degree of smoothness in the trend level of government spending.[28] To induce stationarity, all model variables are divided by their associated long-run trends. As will be explained in the next section, the equilibrium of the model will be expressed in terms of deviations from these trends.

I assume that the law of motion for each of the nine exogenous processes can be expressed in the form of system (5) from Section 3 in natural logarithms. Specifically, let $z_{\omega,t} \equiv \ln(\omega_t/\omega)$ for $\omega \in \{\zeta_C, \zeta_I, \zeta_Y, \zeta_W, \zeta_P, \zeta_R, g, \mu_Z, \mu_X\}$. The law of motion for each $z_{\omega,t}$ can be written as:

$$\begin{aligned}
z_{\omega,t} &= \rho_{z,\omega} z_{\omega,t-1} + a_{\omega,t-1} + e^z_{\omega,t}, \quad e^z_{\omega,t} \overset{\tilde{\mathbb{P}}}{\sim} \mathcal{N}(b_{\omega,t-1}, \sigma^2_{z,\omega}) \\
a_{\omega,t} &= \rho_{a,\omega} a_{\omega,t-1} + e^a_{\omega,t}, \quad e^a_{\omega,t} \overset{\tilde{\mathbb{P}}}{\sim} \mathcal{N}(0, \sigma^2_{a,\omega}) \\
b_{\omega,t} &= \rho_{b,\omega} b_{\omega,t-1} + e^b_{\omega,t}, \quad e^b_{\omega,t} \overset{\tilde{\mathbb{P}}}{\sim} \mathcal{N}(0, \sigma^2_{b,\omega}).
\end{aligned} \tag{37}$$

5 Solution Methods

This section describes the solution method applied to the model from Section 4. Two features of that model that are out of the ordinary from a computational standpoint are the inclusion of distorted beliefs, and the presence of recursive preferences. The

[28] Schmitt-Grohé and Uribe (2012) employ this same specification of trend growth in government spending.

main insight for handling distorted beliefs is to realize that equilibrium policy rules for endogenous variables do not depend on the objective measure — that measure is only relevant for estimation. Therefore the model can be solved under the distorted measure ($\tilde{\mathbb{P}}$) and the policy rules can be converted back to the objective measure ($\mathbb{P}$) for estimation. The main insight related to recursive preferences is that most dynamic equilibrium models like the one in this paper can be written in a form that makes it possible to use the properties of lognormal random vectors in a self-consistent way to obtain linear-approximate policy rules that do not satisfy certainty equivalence. The presentation is carried out in somewhat general terms, to illustrate that these methods can be applied to a wide class of equilibrium models, including those with neither belief distortions nor recursive preferences.

5.1 Solving Models with Distorted Beliefs

Beginning with the filtered probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P})$, consider an economic model of the form:

$$\begin{aligned} y_t &= g(w_t, z_t) \\ w_t &= h(w_{t-1}, z_{t-1}), \end{aligned} \tag{38}$$

where z_t is an $n_z \times 1$ vector of all exogenous processes in the model. At the risk of some abuse in notation, this vector includes all exogenous processes, including those that represent advance information or distorted beliefs. The $n_y \times 1$ vector y_t contains all endogenous, non-predetermined variables and the $n_w \times 1$ vector w_t contains all endogenous predetermined variables. I assume that the law of motion for z_t is known and of the form

$$z_t = \lambda^{\mathbb{P}}(z_{t-1}) + e_t, \quad e_t \overset{\mathbb{P}}{\sim} \mathcal{N}(0, \Sigma_e). \tag{39}$$

The vector-valued function $\lambda^{\mathbb{P}} : \mathsf{R}^{n_z} \to \mathsf{R}^{n_z}$ is known, but the functions $g : \mathsf{R}^{n_w+n_z} \to \mathsf{R}^{n_y}$ and $h : \mathsf{R}^{n_w+n_z} \to \mathsf{R}^{n_w}$ are unknown and will be implicitly determined by the set of equilibrium conditions that characterize the equilibrium of the model economy. It is assumed that these conditions can be summarized by a system of non-linear expectational difference equations of the form:

$$\tilde{E}_t \left[F(y_{t+1}, y_t, w_{t+1}, w_t, z_{t+1}, z_t) \right] = 1, \tag{40}$$

where $F : \mathsf{R}^{2n_y+2n_w+2n_z} \to \mathsf{R}_+^{n_y+n_w}$ is a known vector-valued function, and the expectation is computed according to the distorted probability measure $\tilde{\mathbb{P}}$ which may differ from $\mathbb{P}$. This reflects the fact that the beliefs agents hold at the time they make their decisions may not coincide with the beliefs implied by their model environment.

While agents do not need to assign probabilities to future states in a way that exactly coincides with $\mathbb{P}$, they are required to hold beliefs that are not to dissimilar from those implied by that measure. Mathematically, these two measures are assumed to be locally equivalent. That is, at every date $t \geq 0$,

$$\tilde{\mathbb{P}}(\mathcal{A}) = 0 \Leftrightarrow \mathbb{P}(\mathcal{A}) = 0, \quad \text{for all } \mathcal{A} \in \mathfrak{F}_t.$$

This means that at each point in time, both probability measures agree on which future events will occur with zero probability. Letting $\mathbb{P}_t$ and $\tilde{\mathbb{P}}_t$ denote the restriction of these probability measures to $\mathfrak{F}_t$, define the Radon-Nikodým derivative process as

$$M_t \equiv \frac{d\tilde{\mathbb{P}}_t}{d\mathbb{P}_t} \quad t \geq 0,$$

which is a unique, strictly positive (a.s.) martingale under $\mathbb{P}$.

This process defines a sequence of conditional distributions for e_{t+1} under $\tilde{\mathbb{P}}$. By imposing a particular parametric structure on M_t, it is possible to obtain a closed-form solution for that conditional distribution. Specifically, let $M_0 = 1$ and

$$\frac{M_{t+1}}{M_t} = \left(\frac{|\Psi(z_t)|}{|\Sigma_e|}\right)^{-\frac{1}{2}} \exp\Bigg(-\frac{1}{2}\left[\psi(z_t)'\Psi(z_t)^{-1}\psi(z_t) + e_{t+1}'(\Psi(z_t)^{-1} - \Sigma_e^{-1})e_{t+1}\right] \\ + \psi(z_t)'\Psi(z_t)^{-1}e_{t+1}\Bigg) \tag{41}$$

where $\psi : \mathsf{R}^{n_z} \to \mathsf{R}^{n_z}$ is a known vector function and $\Psi : \mathsf{R}^{n_z} \to \mathsf{R}^{n_z \times n_z}$ is a known, symmetric positive definite matrix function. It turns out that z_t remains a conditionally normal, exogenous first-order Markov process if and only if M_t is defined in this way. Stating this formally:

Theorem 1. *Under the assumption that* $e_t \overset{\mathbb{P}}{\sim} \mathcal{N}(0, \Sigma_e)$*, if* $M_t \equiv d\tilde{\mathbb{P}}_t/d\mathbb{P}_t$ *is defined according to the recursion in* (41)*, then* $e_t|\mathfrak{F}_{t-1} \overset{\tilde{\mathbb{P}}}{\sim} \mathcal{N}(\psi(z_{t-1}), \Psi(z_{t-1}))$ *and the law of motion for* z_t *under* $\tilde{\mathbb{P}}$ *is:*

$$z_t = \lambda^{\tilde{\mathbb{P}}}(z_{t-1}) + \epsilon_t, \quad \epsilon_t \overset{\tilde{\mathbb{P}}}{\sim} \mathcal{N}(0, \Psi(z_{t-1})),$$

where $\lambda^{\tilde{\mathbb{P}}}(z_{t-1}) \equiv \lambda^{\mathbb{P}}(z_{t-1}) + \psi(z_{t-1})$ *and* $\epsilon_t \equiv e_t - \psi(z_{t-1})$*. The converse is also true.*

In the special case that $\psi(z_t) = 0$ and $\Psi(z_t) = \Sigma_e$, the distorting martingale $M_{t+1} = 1$ at all times, so the conditional distribution of e_t under $\tilde{\mathbb{P}}$ coincides with its distribution under $\mathbb{P}$ (i.e. there are no belief distortions). In the model from Section 4, I implicitly assumed $\Psi(z_t) = \Sigma_e$ so that

$$\frac{M_{t+1}}{M_t} = \exp\bigg(- \frac{1}{2}\psi(z_t)'\Sigma_e^{-1}\psi(z_t) + \psi(z_t)'\Sigma_e^{-1}e_{t+1}\bigg).$$

As pointed out in Section 3, this implies that distortions in agents' beliefs affect conditional means but not conditional variances, and is desirable for two reasons. First, it ensures that distorted beliefs and advance information are symmetric, because the latter are assumed only to affect conditional means. Second, it ensures that the model remains conditionally homoskedastic under both probability measures.[29] Under this parameterization, the law of motion for z_t under $\tilde{\mathbb{P}}$ is

$$z_t = \lambda^{\tilde{\mathbb{P}}}(z_{t-1}) + \epsilon_t, \quad \epsilon_t \overset{\tilde{\mathbb{P}}}{\sim} \mathcal{N}(0, \Sigma_e). \tag{42}$$

Combining the laws of motion in (38) and (42) with the equilibrium conditions in (40), it follows that the unknown transition rules g and h solve the functional equation:

$$\tilde{E}_t\bigg[F(g(h(w_t, z_t), \lambda^{\tilde{\mathbb{P}}}(z_t) + \epsilon_{t+1}), g(w_t, z_t), h(w_t, z_t), w_t, \lambda^{\tilde{\mathbb{P}}}(z_t) + \epsilon_{t+1}, z_t)\bigg] = 1. \tag{43}$$

While exact solutions for g and h are typically not available, it is possible to approximate these functions using standard numerical methods. Therefore, modeling distorted beliefs in this way does not introduce any computational complexity above what is normally present when solving non-linear dynamic models of this type. The only difference is that once g and h have been obtained (or approximated), estimation is carried out with the dynamics of z_t expressed under the objective measure, as in (39).

[29] To the extent that homoskedasticity is desirable from an estimation point of view, it is much less troublesome to have conditional heteroskedasticity under the distorted measure than under the objective measure. This is because the model is estimated under the objective measure. Nevertheless, there would be something of a theoretical asymmetry in supposing that subjective volatility is time-varying, while objective volatility is not. And while it would be interesting to investigate situations in which either type of time-variation in volatility is present, that investigation is beyond the scope of this paper.

5.2 Linear Approximations and Recursive Preferences

The goal of this subsection is to obtain a linear approximation to the model's equilibrium dynamics in which elements such as recursive preferences affect the determination of endogenous variables.[30] The starting point is to compute a log-linear approximation of the function F from (43) in terms of the variables w_t, z_t, and ϵ_{t+1}. Letting $f(\tilde{y}, y, \tilde{w}, w, \tilde{z}, z) \equiv \ln(F(\tilde{y}, y, \tilde{w}, w, \tilde{z}, z))$, and using the chain rule together with (38) and (39),

$$f(g(h(w_t, z_t), \lambda^{\tilde{\mathbb{P}}}(z_t) + \epsilon_{t+1}), g(w_t, z_t), h(w_t, z_t), w_t, \lambda^{\tilde{\mathbb{P}}}(z_t) + \epsilon_{t+1}, z_t)$$

$$\begin{aligned}
&\approx f + [f_{\tilde{y}} g_w h_w + f_y g_w + f_{\tilde{w}} h_w + f_w](w_t - \bar{w}) \\
&+ [f_{\tilde{y}}(g_w + g_z \lambda_z^{\tilde{\mathbb{P}}}) + f_y g_z + f_{\tilde{w}} h_z + f_{\tilde{z}} \lambda_z^{\tilde{\mathbb{P}}} + f_z](z_t - \bar{z}) \\
&+ [f_{\tilde{y}} g_z + f_{\tilde{z}}]\epsilon_{t+1},
\end{aligned}$$

where $f_x \equiv \frac{\partial f}{\partial x'}$, and it is understood that all constant terms are evaluated at the steady state levels $w_t = \bar{w}$, $z_t = \bar{z}$, and $\epsilon_{t+1} = 0$.[31] Substituting this approximation into (43), and taking natural logarithms of both sides implies that:

$$\begin{aligned}
0 = f &+ [f_{\tilde{y}} g_w h_w + f_y g_w + f_{\tilde{w}} h_w + f_w](w_t - \bar{w}) \\
&+ [f_{\tilde{y}}(g_w + g_z \lambda_z^{\tilde{\mathbb{P}}}) + f_y g_z + f_{\tilde{w}} h_z + f_{\tilde{z}} \lambda_z^{\tilde{\mathbb{P}}} + f_z](z_t - \bar{z}) \\
&+ \frac{1}{2}\text{diag}\left([f_{\tilde{y}} g_z + f_{\tilde{z}}]\Sigma_e [f_{\tilde{y}} g_z + f_{\tilde{z}}]'\right).
\end{aligned} \tag{44}$$

The last term in this expression makes use of the fact that ϵ_{t+1} is conditionally normal under $\tilde{\mathbb{P}}$, together with the properties of lognormally distributed random vectors. For any square matrix A, The operator diag(A) collects the diagonal elements of A into a column vector.

Because expression (44) must hold for any realization of w_t and z_t, it must be the case that each of the constant terms and two coefficient terms are equal to zero. Consider first the term multiplying w_t:

$$f_{\tilde{y}} g_w h_w + f_y g_w + f_{\tilde{w}} h_w + f_w = 0. \tag{45}$$

[30] To further clarify, the main aim is really to obtain an approximation where *risk premia* have non-zero effects on the approximate dynamics. Recursive preferences simply provide an additional degree of freedom to allow those premia to better fit the data.

[31] This subsection retains the distinction between the distorted and historical probability measures. For models in which those two measures coincide, replace $\tilde{\mathbb{P}}$ by $\mathbb{P}$ throughout.

As long as the number of stable generalized eigenvalues of the matrices $-[f_{\tilde{w}}, f_{\tilde{y}}]$ and $[f_w, f_y]$ equals the number of endogenous predetermined variables n_w, it is possible to solve this polynomial matrix equation for g_w and h_w.[32]

Next, the coefficient on z_t is used to determine g_z and h_z:

$$f_{\tilde{y}}(g_w + g_z\lambda_z^{\tilde{\mathbb{P}}}) + f_y g_z + f_{\tilde{w}} h_z + f_{\tilde{z}}\lambda_z^{\tilde{\mathbb{P}}} + f_z = 0, \tag{46}$$

where in this expression, g_w and h_w are known (in addition to $\lambda_z^{\tilde{\mathbb{P}}}$). Vectorizing this system, and using the fact that $\text{vec}(ABC) = (C' \otimes A)\text{vec}(B)$,

$$[(I_{n_z} \otimes f_y) + ((\lambda_z^{\tilde{\mathbb{P}}})' \otimes f_{\tilde{y}}), \quad (I_{n_z} \otimes [f_{\tilde{w}} + f_{\tilde{y}} g_w])] \begin{bmatrix} \text{vec}(g_z) \\ \text{vec}(h_z) \end{bmatrix} = -\text{vec}(f_z + f_{\tilde{z}}\lambda_z^{\tilde{\mathbb{P}}}).$$

This linear system determines g_z and h_z.[33]

Lastly, the constant terms $\bar{w}$ and $\bar{y} \equiv g(\bar{w}, \bar{z})$ in (46), which are arguments of the known derivative matrices, satisfy the system of $n_y + n_w$ nonlinear equations:

$$f + \frac{1}{2}\text{diag}\left([f_{\tilde{y}} g_z + f_{\tilde{z}}]\Sigma_e[f_{\tilde{y}} g_z + f_{\tilde{z}}]'\right) = 0. \tag{47}$$

The usual result that $f(\bar{y}, \bar{y}, \bar{w}, \bar{w}, \bar{z}, \bar{z}) = 0$ no longer holds, and it is generally not possible to obtain closed form solutions for all these constant terms. Instead, these equations can be solved numerically. One consideration that greatly simplifies the task of solving this system is that it is possible to show that the rows of $f_{\tilde{y}}$ and $f_{\tilde{z}}$ corresponding to some of the equilibrium conditions is zero. Using the fact that for these conditions, the corresponding element of f must still equal zero, it is often possible to derive closed-form expressions for a number of steady state relations that hold almost surely under $\tilde{\mathbb{P}}$ (and therefore also under $\mathbb{P}$). This reduces the number of unknown values that must be determined numerically.

How is this method helpful in cases with recursive preferences? Primarily because it delivers a linear solution where the matrices g_w, g_z, h_w, h_z, and the steady-state values $\bar{y}$, $\bar{w}$ are functions of the parameter γ, which adjusts the household's continuation value for risk. The risk adjustment affects the steady state of the model through the second term on the left-hand side of (47), which then affects the transition dynamics through (45) and (46). If the second term on the left-hand side of (47) is omitted,

[32]For details on the Schur decomposition method employed in this paper, see Klein (2000).

[33]An alternative approach to solving (46) writes the system in the form of a generalized Sylvester equation. For details on this approach, see Gomme and Klein (2011).

then this solution method is equivalent to a first-order perturbation approximation of the model around its non-stochastic steady state, as in Schmitt-Grohé and Uribe (2004) for example. Formally,

Theorem 2. *The linear-approximate dynamics derived in this subsection are equivalent to a first-order perturbation approximation of the model's dynamics around its non-stochastic steady state if and only if the steady-state values $\bar{y}$ and $\bar{w}$ satisfy the $n_y + n_w$ system of equations*

$$f(\bar{y}, \bar{y}, \bar{w}, \bar{w}, \bar{z}, \bar{z}) = 0,$$

taking $\bar{z}$ as given.

As an example of how this solution method works, consider the hypothetical asset-pricing exercise of obtaining a linear solution to the real risk-free rate based on the following equilibrium condition:

$$1 = \tilde{E}_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-1/\xi} \left(\frac{V_{t+1}}{\tilde{E}_t[V_{t+1}^{1-\gamma}]^{1/(1-\gamma)}} \right)^{1/\xi - \gamma} R_t^f \right],$$

where $\Delta c_{t+1} \equiv \ln(C_{t+1}/C_t)$ and $vc_{t+1} \equiv \ln(V_{t+1}/C_{t+1})$ are conditionally linear *exogenous* processes. A standard first order perturbation approximation of this expression around the non-stochastic steady state implies that

$$r_t^f = -\ln(\beta) + \frac{1}{\xi} \tilde{E}_t[\Delta c_{t+1}].$$

Note that γ does not appear in this expression, so no amount of data on r_t^f, Δc_t, or vc_t would identify that parameter when this approximation is used.

In order to apply the approximation method of this subsection, it is first necessary to rewrite the nonlinear condition in the form of (40). Introducing the auxiliary variable $X_t \equiv \tilde{E}_t[V_{t+1}^{1-\gamma}]^{1/(1-\gamma)}$, the system is

$$1 = \tilde{E}_t \begin{bmatrix} \exp\left(\ln(\beta) - \frac{1}{\xi}\Delta c_{t+1} + \left(\frac{1}{\xi} - \gamma\right) [vc_{t+1} + \Delta c_{t+1} - x_t] \right) \\ \exp\left((1-\gamma)[vc_{t+1} + \Delta c_{t+1} - x_t] \right) \end{bmatrix},$$

where $x_t \equiv \ln(X_t)$. In this case the function F is log-linear, so no approximations are necessary. Using the properties of lognormal random vectors and substituting x_t out of the result implies that

$$\begin{aligned} r_t^f = & -\ln(\beta) - \frac{1}{\xi}\tilde{E}_t[\Delta c_{t+1}] - \frac{1}{2}\gamma^2\sigma_{\Delta c}^2 \\ & - \frac{1}{2}\left(\frac{1}{\xi} - \gamma\right)\left[(1-\gamma)\sigma_{\Delta c}^2 + \left(1 - \frac{1}{\xi}\right)\sigma_{vc}^2 + \sigma_{\Delta c, vc}\right], \end{aligned}$$

where $\sigma_{\Delta c}^2$ is the conditional variance of Δc_t, σ_{vc}^2 is the conditional variance of vc_t, and $\sigma_{\Delta c,vc}$ is their conditional covariance. From this expression, it is clear that γ influences the equilibrium level of $\tilde{E}[r_t^f]$. This expression for r_t^f is exact because Δc_t and vc_t are assumed to be linear processes for the purpose of this example. If that were not the case, the solution procedure entails replacing their exact laws of motion with linear approximations.

6 Empirical Analysis

In this section, the model from Section 4 is confronted with data. A subset of its parameters are calibrated to conventional values from related studies, and the remaining parameters are estimated using likelihood-based methods. The primary focus is on the importance of belief-driven fluctuations, and the relative importance of advance information and distorted beliefs for different series. As a brief summary of the main findings, the estimates point to a large role for belief-driven fluctuations. In particular, belief distortions are most important for explaining stock returns and consumption growth, and advance information is most important for explaining inflation and investment growth.

6.1 Estimation Method

The first step in the estimation procedure is to describe the mapping from a collection of observable time series to their model counterparts. The approximation method in Section 5 produces a linear system characterizing the model's dynamics. The state vector in the economy is given by $x_t \equiv (w_t', z_t')'$. An $n_o \times 1$ observation vector y_t^o is related to this state vector by the equation

$$y_t^o = \bar{y}^o + Ax_t^* + \eta_u u_t, \quad u_t \overset{\mathbb{P}}{\sim} \mathcal{N}(0, \Sigma_u), \tag{48}$$

where $x_t^* \equiv x_t - \bar{x}$. The $n_u \times 1$ vector u_t contains independent and identically distributed measurement errors, which affect y_t^o according to the $n_o \times n_u$ selection matrix η_u. The matrix A mapping the observation vector to the latent states can be written as

$$A \equiv S_o \begin{bmatrix} g_x \\ h_x \\ g_x h_x \\ h_x h_x \end{bmatrix}, \quad \text{where} \quad g_x \equiv [g_w, g_z], \quad h_x \equiv \begin{bmatrix} h_w & h_z \\ 0 & \lambda_z^{\tilde{\mathbb{P}}} \end{bmatrix},$$

and S_o is an $n_o \times 2(n_y + n_x)$ selection matrix. The constant term $\bar{y}^o$ is defined as $\bar{y}^o \equiv A\bar{x}$. The presence of the last two blocks in A permit expectations of future model variables to be included in the observation vector. To complete the state-space representation of the model in terms of y_t^o and x_t^*, the dynamics of x_t^* must be expressed under the objective measure:

$$x_t^* = \begin{bmatrix} h_w & h_z \\ 0 & \lambda_z^{\mathbb{P}} \end{bmatrix} x_{t-1}^* + \begin{bmatrix} 0 \\ I_{n_e} \end{bmatrix} e_t, \quad e_t \overset{\mathbb{P}}{\sim} \mathcal{N}(0, \Sigma_e). \tag{49}$$

The transition and covariance matrices in (48) and (49) are functions of the underlying parameter vector Θ. The goal of estimation is to approximate the posterior distribution of Θ conditional on the complete sample of data $\{y_t^o\}_{t=1}^T$. By Bayes' rule, the posterior combines information from the likelihood function and the prior distribution:

$$p(\Theta | \{y_t^o\}_{t=1}^T) \propto p(\{y_t^o\}_{t=1}^T | \Theta) p(\Theta).$$

Because the system is linear, the likelihood function can be evaluated using the Kalman filter. This is a main advantage of the solution method employed in this paper. An estimate of the mode of the posterior distribution is first obtained by numerical optimization, and then the entire distribution is simulated using Markov Chain Monte Carlo (MCMC) methods.[34]

6.2 Data

The model is estimated using quarterly U.S. data from 1954:Q3 – 2009:Q1. This sample excludes the period after 2009:Q1 when the nominal interest rate was at its lower bound. The fourteen data series used in estimation, along with the model concepts associated with these series, are listed in Table 1. Listing them here for

[34] See An and Schorfheide (2007) for a detailed description of this approach.

convenience as well, they are: growth in real per capita GDP, consumption, investment, and government spending; growth in per capita hours worked, real wages, total factor productivity, and the relative price of investment; inflation, real stock returns, the nominal risk-free rate; and finally expected growth in per capita real GDP, inflation, and nominal risk-free returns. Three series do not extend all the way back to 1954:Q3. Expected output and inflation only extend to 1968:Q4 and the expected nominal risk-free rate only extends back to 1981:Q3. In these cases, observations before the available start date are treated as missing observations. All remaining variables contain observations over the entire sample.

Three different groups of these variables have been included to help discipline the estimation. First, the real stock market return and three expectational series contain the most direct evidence on agents' beliefs about the future. They are important for plausibly identifying the relative importance of advance information and distorted beliefs, which is the main focus of the empirical analysis. Second, the growth rates of real government consumption, total factor productivity, and the relative price of investment are fundamentals in the model. Including observations on these variables is helpful to ensure that the postulated fundamental processes (and implicitly the beliefs about them) behave consistently with the available observations. Third, wage growth, inflation, and the nominal interest rate discipline the nominal rigidities in the model: the imperfect price-setting structure in labor and retail markets, and the monetary authority's feedback rule.

6.3 Calibrated Parameters and Priors

Some parameters contained in Θ are calibrated. These parameters, their fixed values, and a brief description are listed in Table 2. The marginal rate of time preference β is fixed at 0.995, which corresponds to a real interest rate of about 2% in the non-stochastic steady state. A value of one for the elasticity of intertemporal substitution is standard within the business-cycle literature. The share of non-labor inputs (capital and land) in production is assumed to be 0.325, and the labor share is 0.675. As in Jaimovich and Rebelo (2009) and Schmitt-Grohé and Uribe (2012), the degree of decreasing returns to scale in production is fixed at 10 percent, which implies that the capital share is 0.225. In the depreciation function, $\delta(u) = \delta_0 + \delta_1(u-1) + \delta_2/2(u-1)^2$, the parameter δ_1 is chosen to ensure that capacity utilization, u, equals one in the steady state. In the steady state, capital depreciates at a rate of 10 percent per year, which corresponds to $\delta_0 = 0.025$.

The steady-state growth rates μ^Y and μ^X, the share of government purchases G/Y^d, and the steady-state inflation rate Π are chosen to match their average levels over the sample period. The parameter ϑ, which scales the disutility of labor in the period utility kernel, is chosen to ensure that the steady-state fraction of time spent working (h) is equal to 20 percent. The steady-state wage and price markups are fixed at 15 percent, which is a value in line with Justiniano, Primiceri, and Tambalotti (2011). The interest rate feedback rule of the monetary authority does not feature any endogenous persistence, $\varphi_R = 0$. A common alternative is to assume an "inertial" policy rule ($\varphi_R > 0$), with independent and identically distributed errors. The specification chosen here allows all exogenous disturbances in the model to be treated symmetrically. Moreover, Rudebusch (2002, 2006) and Carrillo, Fève, and Matheron (2007) provide empirical evidence that a policy rule with serially correlated disturbances and little internal persistence may be more consistent with the dynamic behavior of interest rates.

The remaining parameters in Θ that are not calibrated are assigned prior distributions. These all fall closely in line with the existing literature, and are reported in Table 3. The prior for risk aversion, γ is assumed to be a (generalized) gamma distribution with a mean of five and standard deviation of two. Without habit persistence or labor supply, γ represents the risk aversion parameter of the household over atemporal wealth gambles. A value of five for this parameter would in that case lie in the center of the conventional range for this parameter in the asset pricing literature since Mehra and Prescott (1985). The leverage parameter λ is assigned a gamma distribution with a mean of 2 and standard deviation of 0.75. According to this prior, aggregate stock returns are twice as volatile as the growth rate in the value of productive firms. This is consistent with the amount of financial leverage measured by Rauh and Sufi (2012), and is consistent with the calibration of Croce (2013).

For the exogenous processes, the persistence parameters of contemporaneous fundamental disturbances are assigned (generalized) beta priors with a mean of 0.6 and standard deviation of 0.2. The persistence of permanent technology and the relative price of investment are assigned a lower mean of 0.2, to reflect the view that these series (absent any advance information) are close to being independent and identically distributed across time. To allow for the possibility that these parameters may be exactly zero, or even somewhat negative, the priors are bounded below by negative one rather than zero. The remaining persistence parameters associated with advance information and distorted beliefs are assigned beta priors centered at zero.

The prior distribution for the standard deviation of each contemporaneous disturbance is inverse gamma with a mean of 0.5 and a standard deviation of 1. The standard deviations associated with advance information and distorted beliefs are instead centered at 0.1. In the event that all persistence parameters were fixed at zero, these priors imply that contemporaneous disturbances are responsible for about 93 percent of the variation in fundamentals under the $\tilde{\mathbb{P}}$-measure, and about 96 percent under the $\mathbb{P}$-measure.

6.4 Estimation Results

Table 4 displays estimates of the structural parameters in the model, and Tables 5 and 6 display estimates of the parameters that govern the exogenous processes in the model. For comparison, the prior means are included in the table as well. The two asset-pricing parameters in the model are γ, which controls the risk preferences of the household, and λ, which controls the degree of leverage in equity returns relative to the growth in firm value. The former is estimated to be close to ten: somewhat higher than the prior mean and near the upper limit of the conventional range $(0, 10)$ typically imposed on this parameter in the case without labor supply or habit persistence. The leverage parameter is slightly higher than its prior mean at 2.7, but still consistent with existing studies.

The remaining estimates of preference and technology parameters are consistent with existing literature. For example, the parameter ν, which controls the degree of wealth effects on labor supply, is small at 0.06. This is not quite as small as the value estimated by Schmitt-Grohé and Uribe (2012), who find a posterior median estimate of $\nu = 0.003$. Nevertheless, this estimate still confirms their finding that wealth effects are small, and that preferences are closer to the type of Greenwood, Hercowitz, and Huffman (1988).

The degree of nominal rigidity in the model is estimated to be small. The probabilities that wages and prices are reset each period are 0.97 and 0.92, respectively. These imply that the average time between optimal readjustments in labor and product markets is close to three months. By contrast, Justiniano, Primiceri, and Tambalotti (2011) find an average duration around one year for optimal wages and prices. The degree of price indexation is also small, at 26 percent. Some studies have estimated smaller values: Justiniano, Primiceri, and Tambalotti (2011) find only ten percent indexation in both markets. But, others have worked with higher values: in their empirical model, Altig, Christiano, Eichenbaum, and Lindé (2011) calibrate both indexation parameters to one. The estimated feedback rule contains a strong response to increases in inflation, and a smaller response to output growth.

To gauge the ability of the model to fit the data, Table 7 presents some selected moments from the model and the data. While the estimation procedure attempts to match the entire autocovariance function of the vector of observable variables, this comparison can be helpful for understanding the model's limitations. Overall, the fit is comparable to other related studies. According to the model, consumption growth is less volatile than output growth, investment growth is much more volatile than output growth, equity returns are much more volatile than the risk-free rate, and there is an ex-post equity premium of about four percent, which almost exactly matches the 4.05% equity premium observed in the data.

Nevertheless, the model does fall short along several dimensions. For one thing, it overstates the standard deviation of almost every variable. It also overstates the first-order autocorrelation of output and consumption by about an order of two. At the same time, the model understates the autocorrelation of hours, as well as its correlation with output. In terms of nominal variables, the model overstates the degree of negative correlation between the nominal risk-free rate and output growth.

6.5 Decomposing Fluctuations

Table 8 displays the share of the unconditional variance of each variable used in estimation that can be attributed to current fundamental disturbances, advance information, and distorted beliefs. The first main finding is that for most endogenous variables (the first eleven rows), beliefs play an important role. For example, over 40 percent of the fluctuations in output growth and 70 percent of the fluctuations in consumption growth are due either to advanced information or distorted beliefs. On average across all endogenous variables, two thirds of the variation can be attributed to one of these two sources. The second finding is that both advance information and distorted beliefs play an economically significant role. Neither type of disturbance completely crowds out the other across the board. In particular, both are equally important for explaining output growth and hours worked.

Advance information and distorted beliefs are not equally important for all series however. Advance information is most important for inflation (and expected inflation), at over 50%. The fact that inflation reflects disturbances that were correctly anticipated is consistent with the idea that the actions of the Federal Reserve are largely known in advance. Indeed, anticipated monetary policy disturbances are responsible for over half of the variation in inflation (53%). This can be seen in Table 9, which displays the contributions from the nine exogenous processes that are most important for explaining the observed endogenous variables. Advance information is also particularly important for real investment (44%). Nearly all of this anticipated component comes from advance information related to transitory investment-specific technological change. Advance information about changes in investment-specific technology also explain almost all of the anticipated component of GDP growth.

On the other hand, distorted beliefs are much more important for explaining stock returns and consumption growth. This result is noteworthy because those two series are the main focus of many modern consumption-based asset-pricing models. It is consistent with the idea that the prices of financial assets are less tied to fundamental developments than are real investment, hiring, and production. What are these distorted beliefs about? In the case of stock returns, distorted beliefs about future marginal costs in product markets represent the largest factor (51%), while distorted beliefs about the relative price of investment are the second largest (19%). Distorted beliefs about the relative price of investment are also important for consumption growth (36%), but future marginal costs play less of a role there (2%). Instead, distortions related to future government purchases are more important (17%).

Among the contemporaneous innovations to current fundamentals, those to investment-specific technology and temporary productivity are the most important for explaining output (36%), investment (42%), and hours (33%). Innovations to product-market marginal costs play an important role in explaining wage growth (30%). These are presented in the first three columns of Table 9. These findings are consistent with those in the existing literature. In particular, the importance of investment-specific technology for explaining these variables is also documented by Justiniano, Primiceri, and Tambalotti (2011) and Schmitt-Grohé and Uribe (2012). Also consistent with the second of these studies is the finding that the anticipated component of investment-specific technology plays a large role. Indeed, for output, investment, and hours, the anticipated innovation in this fundamental process is even slightly more important than the unanticipated innovation. Finally, a new contribution to the discussion of investment-specific disturbances is the finding in Table 9 that distorted beliefs concerning the relative price of investment explain a substantial fraction of the variation in output (10%), consumption (36%), hours (20%), wages (30%), stock returns (19%) and nominal interest rates (12%). Indeed, distorted beliefs about this process are the single most important source of distortions for the real economy.

In contrast to existing studies (e.g. Smets and Wouters, 2007), exogenous variation in labor market marginal costs does not represent a particularly large source of fluctuations in the data. Current innovations in this process make up less than one percent of the variation in all endogenous variables. The advance information component is larger, but still explains less than ten percent of the variation in each of the endogenous variables. Schmitt-Grohé and Uribe (2012), for example, find that the anticipated component of this process explains 67 percent of the variation in hours, and between 15 and 20 percent of the variation in output and consumption. This is a particularly appealing result given the amount of criticism leveled against labor supply shocks of this type (e.g. by Chari, Kehoe, and McGrattan, 2009, and Shimer, 2009).

Last but not least, what about the survey forecasts? For the most part, Table 6 indicates that the importance of current fundamentals, advance information, and distorted beliefs for these variables largely mirror their realized counterparts. Expected inflation largely reflects advance information as does realized inflation. Belief distortions are important for expected nominal risk-free interest rates, as they are for realized interest rates. However, distorted beliefs are somewhat more important for expected output than they are for realized output (34% vs 22%). On average, 62 percent of the variation in the three forecast series is attributable to either current or (correctly) anticipated changes in fundamentals.

As a point of comparison, Table 10 displays the same variance decomposition as Table 8, but with the parameters set to their prior means. In every case disturbances to current fundamentals play the largest role. Stock returns and consumption are not as strongly affected by belief distortions, and inflation does not have such a large anticipated component. While these observations are informative, it is also important to keep in mind that the prior distributions were chosen to reflect uncertainty surrounding the parameters themselves, not about the model-implied variance decompositions.

7 Robustness

This section explores the robustness of the findings from Section 6 to four independent perturbations of interest. First, I remove stock returns from the set of observable variables. Second, I add classical measurement error to the three survey forecasts. Third I remove belief distortions from the analysis altogether to see what the data have to say when the only alternative to current innovations is advance information. Fourth, I increase the degree of wage and price rigidity in the model by fixing the parameters θ_W and θ_P to their prior means. In each case, results are only reported for the posterior mode. In the baseline case for example, Tables 4-6 shows that the posterior mode does a good job of capturing the central tendency of the underlying parameter draws.

7.1 No Stock Returns

Stock returns have been included in the analysis so far for three main reasons. First, because a large literature in financial economics suggests that the stock market should contain useful information about the market's expectations of future developments in the economy. Second, because conclusions about the relative importance of advance information and distorted beliefs may be (and in fact are) different for macroeconomic and financial variables. And third, because it is of independent interest whether a medium-scale model of the type developed in this paper is capable of explaining the joint dynamics of financial and macroeconomic variables — especially those aspects that have been difficult for existing models, such as the equity premium and excess volatility of stock returns. However, because standard practice in the business cycle literature is not to force the model to be consistent with data on stock returns, this subsection considers how things change if stock returns are omitted from the set of observable variables.

Table 11 displays the variance decomposition based on the estimates without stock returns. The decomposition of stock returns are still included in this table even though they are no longer an observable series, for comparison with Table 8. For the most part the results are similar to the baseline case. The most notable deviations are that without data on stock returns, the importance of belief distortions for stock returns and the nominal risk-free rate both fall by about ten percent. Instead, these variations are now attributed to innovations in current fundamentals. In particular, the fraction that was previously attributed to distorted beliefs about marginal costs in product markets is now attributed to current innovations in transitory productivity. Nevertheless it still remains true that belief distortions are most important for stock returns and consumption, and that advance information is most important for inflation and investment.

7.2 Measurement Errors in Survey Forecasts

A main emphasis of this paper has been on the importance of using direct data on subjective beliefs to determine the relative importance of advance information and distorted beliefs about fundamentals. However, it is possible to entertain the view that while such direct data is valuable in principle, the forecasts used in practice for the empirical analysis of this paper may be subject to errors in measurement. To allow for that possibility, I introduce classical measurement errors on each of the three survey forecasts and re-estimate the variances of those errors along with the other parameters in the model. The standard deviations of these errors are assigned inverse gamma priors with a standard deviation of one; the means are chosen so that at the prior mean, the measurement error variance is one percent of its observable counterpart.

Table 12 displays the variance decomposition of all observable variables after the measurement errors have been included. The results are essentially unchanged from the baseline case. Evidently the data favors the model's structural driving processes over measurement errors for explaining the observed variation in survey forecasts. For the forecasts of output growth and the nominal risk-free rate, the measurement error explains less than one percent of the series' total variation. This number is slightly higher for expected inflation, at about six percent, but still small compared to the other disturbances in the model. The results are therefore robust to having measurement errors of this type in the survey forecasts.

7.3 No Distorted Beliefs

The focus of this paper is on the relative importance of advance information *and* distorted beliefs for explaining macroeconomic and financial data. Nevertheless, it would be interesting to know what different conclusions might have been reached if distorted beliefs were excluded from the start as an alternative hypothesis. To investigate this possibility, I eliminate the distorted belief component of each fundamental processes so that the subjective probabilities coincide with those implied by the model. I then re-estimate the remaining parameters using the same priors listed in Table 3.

The variance decomposition based on this exercise are reported in Table 13. The main result is that without distorted beliefs, advance information is responsible for about half of the variation in almost every endogenous variable, and more than half for expected output growth (64%) and expected inflation (83%). Moreover, the predictable component is now larger for all three of the exogenous processes. Only 11% in the case of total factor productivity, but nearly forty percent for government spending. These findings are roughly consistent with those of Schmitt-Grohé and Uribe (2012).

However, it is advance information about transitory neutral technology and investment-specific technology rather than marginal costs in the labor market that are responsible for the bulk of the anticipated components of output, consumption, hours, and wages.[35] Inflation is still largely driven by advance information about the exogenous component of monetary policy, and stock returns by advance information about marginal costs in product markets. Current innovations to marginal costs in the labor market do play a larger role in explaining hours at 17 percent, compared to less than one percent in the baseline case with distorted beliefs.

7.4 More Nominal Rigidity

One unusual feature of the results in Table 4 relative to related business cycle models with nominal rigidities is that the degree of wage and price rigidity is estimated to be low: wages and prices may be reset optimally almost every quarter. The purpose of this subsection is to examine the extent to which the main findings in Section 6 are robust to imposing a greater degree of nominal price and wage rigidity. To do so,

[35]The decomposition by exogenous process are not shown; the salient points are reported here in the text.

I fix both θ_P and θ_W at their prior mean value of 0.66 and re-estimate the model. These values imply that the average time between optimal readjustments in labor and product markets is around nine months.

Table 14 displays the variance decomposition results. In this case, the importance of advance information and distorted beliefs generally declines for all variables. For example, half of consumption growth is explained by distorted beliefs (down from 62% in the baseline case), and only two percent is explained by advance information (down from 14% in the baseline case). On average over all eleven endogenous variables, these two belief-related sources explain around 40 percent of the observed variation. It is still true that belief distortions are most important for explaining stock returns and consumption growth, although they are now relatively more important for consumption growth.

Which current fundamental has become a more plausible explanation in the presence of larger nominal rigidity? Labor market marginal costs. Current innovations to this exogenous process are now responsible for 20 percent of the observed variation in output, 15 percent in consumption, 15 percent in hours worked, 13 percent in real wages, and 10 percent of inflation. By comparison, this innovation explained less than one percent of the variation in each endogenous variable according to the baseline results.

8 Two Historical Episodes

So far, the relative importance of advance information and distorted beliefs has been discussed in terms of the forecast error variance decomposition implied by the estimated model. Another way to discuss their role in driving fluctuations is to consider the roles they have played in specific historical episodes. This section presents estimates of the sources of fluctuations in the real economy and the stock market during two historical episodes: the Dot-Com boom and bust of the mid-late 1990s and the recent Great Recession. These episodes are given special attention because the first is one in which developments in the stock market took center stage, and because the second is the largest downturn in the real economy since the Great Depression. Therefore, they provide interesting case studies of both macroeconomic and financial fluctuations. Before describing the results, I briefly describe the procedure used to construct the historical decompositions.

8.1 Construction of Historical Decompositions

The state-space model in (48) and (49) implies a specific conditional density for the exogenous disturbance vector $\varepsilon_t \equiv (u_t', e_t')'$. Given the set of observable variables and a value for the parameter vector, it is possible to compute estimates of these disturbances using the Kalman smoother. By performing counterfactual experiments where some disturbances are "shut off" (set to zero) and others are not, it is possible to isolate the contribution of each shock to the different observable variables. The details of this procedure are presented in Algorithm 1.[36]

Algorithm 1 Historical decomposition of fluctuations

(1) Use the Kalman smoother to estimate the sequence of exogenous disturbances $\{\hat{\varepsilon}_t\}_{t=1}^T$ and the initial state $\hat{x}_0^*$ based on the posterior mean parameter estimate $\hat{\Theta}$, where

$$\hat{x}_0^* \equiv E[x_0^* | \{y_t^o\}_{t=1}^T, \hat{\Theta}] \quad \text{and} \quad \hat{\varepsilon}_t \equiv E[\varepsilon_t | \{y_t^o\}_{t=1}^T, \hat{\Theta}].$$

(2) For each disturbance $i = 1, \ldots, n_\varepsilon$ (the i-th element of ε_t),

(a) Construct a new sequence $\{\hat{\varepsilon}_t(i)\}_{t=1}^T$ by replacing all elements but those of the i-th series in $\hat{\varepsilon}_t$ by zeros.

(b) Beginning with $\hat{x}_0^*$, use the state dynamics in equation (49) and the observation equation (48) to generate a sequence of counterfactual observations $\{\hat{y}_t^o(i)\}_{t=1}^T$ based on $\{\hat{\varepsilon}_t(i)\}_{t=1}^T$.

(3) The value $\hat{y}_t^o(i)$ represents the estimated contribution of disturbance i at date t to the observed variation in y_t^o.

8.2 The Dot-Com Boom and Bust

In the 1990's U.S. stock prices increased by almost a factor of five — the largest increase during any decade in the nation's history. The "Dot-Com Boom," as this

[36] For a discussion of how to compute smoothed estimates of the exogenous disturbances, which is the first step of the algorithm, see Durbin and Koopman (2012).

period has come to be known, was marked by unprecedented growth in information technology, computers, and internet-based companies. Following this period of growth was a sharp and rapid bust. From its peak at the turn of the century, the stock market lost nearly half its value over the next year and a half. What caused these dramatic fluctuations? The estimated model of this paper provides a precise answer to that question.

Figure 2 shows the sources of fluctuations in per capita output growth and aggregate stock returns over the ten-year period from 1992 to 2002. The relative importance of current fundamentals, advance information, and distorted beliefs during this episode are generally consistent with the "unconditional" importance of these three sources reported in Table 6. Most of the fluctuations in output growth are the result of developments in current fundamentals, followed by advance information and distorted beliefs. By contrast, most of the fluctuations in stock returns are the result of belief distortions; however, current fundamentals also play a non-negligible role.

What type of picture does the model paint for the stock market boom and bust during these years? For most of the period from 1995 to 1999, the stock market was buffeted by a series of positive innovations in current fundamentals — specifically, expansionary innovations to current transitory productivity and product market marginal costs ($e^z_{\zeta_Y,t}$ and $e^z_{\zeta_P,t}$ in the notation of Section 4). These capture the positive effects of technological development during the period. After this sequence of innovations, the stock market was affected by three quarters of particularly excessive optimism: in the last quarter of 1998, the second quarter of 1999, and the last quarter of 1999. The peak of the boom was in the first quarter of 2000. The bust was fueled by a sequence of large negative belief distortions about marginal costs in product markets ($e^b_{\zeta_P,t}$). These distortions appear to have had something like a grain of truth to them. In each quarter that beliefs about future marginal costs were excessively pessimistic, *actual* (current) marginal costs were also negative.

In the real economy, the positive innovations in current productivity that contributed to the boom in the stock market also led to a boom in GDP growth. In fact, the persistent effects of these innovations on output lasted until even after the peak of the stock market in 2000. At that time, a sequence of adverse information about future fundamentals led to negative output growth, ultimately culminating in the 2001 recession. The belief distortions about future marginal costs that drove the stock market bust were less important for the real economy. That is because those innovations generate offsetting effects on output growth. The expectation of persistently higher marginal costs in product markets lead households to expect higher current and future prices, which reduces consumption. The fact that households are averse to late resolution of uncertainty implies that they discount risky projects more heavily as a result. Equilibrium in asset markets requires the expected return on investment to increase, leading to an increase in investment by productive firms. This increase in investment offsets the initial decrease in consumption leading to only a small response in aggregate output.

Taken together, the picture is one of technological innovations leading to a boom in both the real economy and the stock market. A wave of optimism at the end of the boom helped to propel the stock market into the stratosphere. A sudden wave of pessimism at the turn of the century, accompanied by some actual negative contemporaneous developments in firms' marginal costs, drove the crash. Because underlying productivity in the economy was still fairly strong, however, aggregate output did not suffer that much. The subsequent recession was only a mild one by historical standards.

8.3 The Great Recession

The real economy only experienced a mild downturn after the Dot-Com boom of the 1990s, but it would experience a severe collapse just a few years later. In the fourth quarter of 2008 alone, per capita real GDP fell by almost ten percent on an annualized basis — the largest single quarter decline in over half a century. At the same time, the stock market collapse around this time was just as severe as the one that occurred in 2000. Figure 3 displays the joint behavior of per capita GDP growth and aggregate stock returns from 2002 to 2009, together with the model-implied account of the sources of fluctuations in these series.

According to the model, output began falling in 2006 and 2007 due to a series of negative realizations in current transitory neutral productivity ($e^{z}_{\zeta_Y,t}$) from 2006 to the beginning of 2008. The largest of these occurred in the first quarter of 2008. The cataclysmic drop in output growth that followed in the fourth quarter of 2008 was primarily caused by a large negative innovation to current transitory investment-specific productivity ($e^{z}_{\zeta_I,t}$). Investment-specific productivity captures the economy's ability to transform current savings into future capital inputs. Justiniano, Primiceri, and Tambalotti (2011) argue that this type of disturbance is likely to proxy for more fundamental disturbances to the functioning of the financial sector. That interpretation is consistent with the idea that the bankruptcy of the large U.S. investment bank Lehman Brothers in the fourth quarter of 2008 severely impeded the functioning of the financial sector, leading to a sharp drop in output growth.

Belief distortions also played an important role. During the downturn, the real economy experienced a wave of pessimism concerning future permanent neutral productivity ($\mu_{Z,t}$) and product market marginal costs (ζ_t^P). This pessimism had two effects. First, it amplified the drop in output growth beyond what would have occurred with current fundamental disturbances alone. Second, it served as the primary cause for the collapse of the stock market. This is seen from the second panel of Figure 3. Even though current disturbances in investment specific productivity had large effects in the real economy (output, consumption, investment, and hours worked), they had almost no impact on the stock market.

Turning to the period before the recession, both output growth and stock returns remained fairly close to their respective trends. In 2003, a few positive innovations in transitory neutral productivity contributed to a small upturn in both series. A conspicuous feature of the decomposition during this period is that there is no strong upward pressure coming from monetary policy shocks. Mechanically, this is because the estimated degree of nominal rigidity in the model is low. Although nominal interest rates were low during most of this period, that was a result of the monetary authority responding to low inflation. In fact, policymakers in the U.S. at that time were quite explicit about their concern over low inflation. For example, in his 2003 testimony before the U.S. Congress, Federal Reserve Chairman Alan Greenspan justified low policy rates as an appropriate response to low inflation, and even expressed concern over the possibility of a deflationary episode.[37]

[37] His exact words were:

> "...[W]e face new challenges in maintaining price stability, specifically to prevent inflation from falling too low. This is one reason the FOMC has adopted a quite accommodative stance of policy. A very low inflation rate increases the risk that an adverse shock to the economy would be more difficult to counter effectively. Indeed, there is an especially pernicious, albeit remote, scenario in which inflation turns negative against a backdrop of weak aggregate demand, engendering a corrosive deflationary spiral."

While this description of the sources of fluctuations behind the Great Recession may be plausible, it is necessarily incomplete. In particular, there are two important things to keep in mind. First, the model in this paper does not feature a housing sector, so it has nothing to say about the boom in house prices that occurred during this period. Second, while the model is capable of capturing disruptions in the financial sector in a reduced-form way — through innovations to transitory investment-specific productivity — it would certainly be desirable to incorporate a more microfounded model of financial frictions. With those considerations in mind, the conclusion that comes through from this decomposition analysis is that the Great Recession was primarily driven by negative contemporaneous developments in transitory neutral and investment-specific productivity, together with a wave of pessimism about future permanent neutral productivity and marginal costs in product markets.

9 Conclusion

This paper has investigated the role of advance information and distorted beliefs in explaining the observed fluctuations in a range of macroeconomic and financial time series, and has presented three main findings. First, advance information and distorted beliefs are important. About two-thirds of the variation in endogenous variables can be attributed to one of these two sources. Second, both sources are separately important; neither one completely swamps the other. Third, these two sources important in different ways for different variables. Advance information is most important for explaining inflation and investment, while distorted beliefs are most important for explaining stock returns and consumption. They are roughly equally important for explaining output growth and hours worked.

In arriving at these results, this paper has built and estimated a new medium-scale DSGE model that incorporates recursive utility preferences, internal habit formation, capital accumulation, and nominal rigidities, and which is capable of delivering an empirically adequate description of a set of key U.S. macroeconomic and financial data. It has also introduced a flexible way of incorporating belief distortions into a class of optimizing equilibrium models that is useful for empirical work. Specifically, it has proposed to directly parameterize the Radon-Nikodým process that links agents' subjective beliefs to the objective ones implied by their model environment. Finally, it has developed a new internally consistent framework for obtaining a linear approximation to models that naturally incorporates important “higher-order” information. This framework is shown to be particularly useful for models in which agents are not indifferent about the intertemporal resolution of uncertainty.

The findings in this paper also suggest several fruitful areas of future research. For one, given the importance of advance information and distorted beliefs, what are the precise mechanisms responsible for generating these underlying sources of fluctuations? One of the main reasons for developing structural models like the one in this paper is to perform counterfactual experiments that can be helpful for characterizing the effects of different policy actions. But to perform experiments of that type, it is important to know how the economy's technology for producing advance information or distorted beliefs would change under different policy regimes. The exogenous specifications employed in this paper represent a first approximation that is helpful for arriving at positive results, but not for performing counterfactual policy analysis.

A second interesting question involves the role of uncertainty. Several recent papers have emphasized the importance of time-varying uncertainty for explaining macroeconomic phenomena.[38] This paper has abstracted from those types of disturbances in order to focus more sharply on different sources of fluctuations in beliefs about conditional means. However, essentially all of the arguments and methods presented here would directly apply to cases when agents have advance information and distorted beliefs about conditional variances. In fact, it turns out that the solution procedures developed in Section 6 can be directly applied to obtain conditionally linear approximations of models with time-varying uncertainty. This may seem either surprising or unhelpful, given that conventional wisdom says that linear approximations are not capable of capturing the effects of time-varying volatility. But the main intuition for why this is not necessarily the case is as follows: if time-varying uncertainty is of “first-order” importance, it should be possible to construct a linear (i.e. first-order) approximation which captures those effects. The solution method in this paper can be extended to provide one internally consistent way of doing just that.

[38] For example, Bloom (2009), Fernández-Villaverde, Guerrón-Quintana, Rubio-Ramírez, and Uribe (2011), and Bloom, et al. (2012).

References

[1] Adam, Klaus, and Michael Woodford (2012) "Robustly Optimal Monetary Policy in a Microfounded New Keynesian Model," *Journal of Monetary Economics*, 59(5):468-487.

[2] Altig, David, Lawrence J. Christiano, Martin Eichenbaum, and Jesper Lindé (2011) "Firsm-Specific Capital, Nominal Rigidities, and the Business Cycle," *Review of Economic Dynamics*, 14(2):225-247.

[3] An, Sungbae and Frank Schorfheide (2007) "Bayesian Analysis of DSGE Models," *Econometric Reviews*, 26(2-4):113-172.

[4] Anderson, T.W. (1984) *An Introduction to Multivariate Statistical Analysis*, (Wiley: New York)

[5] Angeletos, George-Marios, Fabrice Collard, and Harris Dellas (2014) "Quantifying Confidence," MIT Working Paper.

[6] Angeletos, George-Marios and Jennifer La'O (2010) "Noisy Business Cycles," NBER Macroeconomics Annual 2009, 24:319-378.

[7] Bansal, Ravi, Dana Kiku, and Amir Yaron (2007) "Risks for the Long Run: Estimation and Inference," Working Paper, University of Pennsylvania.

[8] Bansal, Ravi, Dana Kiku, and Amir Yaron (2012) "An Empirical Evaluation of the Long-Run Risks Model for Asset Prices," *Critical Finance Review*, 1(1):183-221.

[9] Bansal, Ravi, and Amir Yaron (2004) "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles," *Journal of Finance*, 59(4):1481-1509.

[10] Barro, Robert J., and Robert B. King (1984) "Time-Separable Preferences and Intertemporal-Substitution Models of Business Cycles," *Quarterly Journal of Economics*, 99(4):817-839.

[11] Barsky, Robert B., Susanto Basu, and Keyoung Lee (2014) "Whither News Shocks," Forthcoming in *NBER Macroeconomics Annual 2014*, 29.

[12] Barsky, Robert B., and Eric R. Sims (2011) "News shocks and business cycles," *Journal of Monetary Economics*, 58(3):273-289.

[13] Barsky, Robert B., and Eric R. Sims (2012) "Information, Animal Spirits, and the Meaning of Innovations in Consumer Confidence," *American Economic Review*, 102(4):1343-1377.

[14] Basu, Susanto, John G. Fernald, and Miles S. Kimball (2006), "Are Technology Improvements Contractionary?" *American Economic Review*, 96(5):1418-1448.

[15] Beaudry, Paul and Bernd Lucke (2010) "Letting Different Views about Business Cycles Compete," in *NBER Macroeconomics Annual 2009*, 24:413-455.

[16] Beaudry, Paul, and Franck Portier (2004) "An Exploration into Pigou's theory of cycles," *Journal of Monetary Economics*, 51(6):1183-1216.

[17] Beaudry, Paul, and Franck Portier (2006) "Stock Prices, News, and Economic Fluctuations," *American Economic Review*, 94(6):1293-1307.

[18] Beaudry, Paul, and Franck Portier (2013) "News Driven Business Cycles: Insights and Challenges," NBER Working Paper No. 19411.

[19] Beeler, Jason, and John Y. Campbell (2012) "The Long-Run Risks Model and Aggregate Prices: An Empirical Assessment," *Critical Finance Review*, 1(1):141-182.

[20] Benhabib, Jess, and Roger E.A. Farmer (1999) "Chapter 6: Indeterminacy and sunspots in macroeconomics," in John B. Taylor and Michael Woodford, eds, *Handbook of Macroeconomics*, 1(A):387-448.

[21] Bidder, Rhys and Matthew E. Smith (2012) "Robust Animal Spirits," *Journal of Monetary Economics*, 59(8):738-750.

[22] Blanchard, Olivier J., Jean-Paul L'Huillier, and Guido Lorenzoni (2013) "News, Noise, and Fluctuations: An Empirical Exploration," *American Economic Review*, 103(7):3045-3070.

[23] Bloom, Nicholas (2009) "The Impact of Uncertainty Shocks," *Econometrica*, 77(3):623-685.

[24] Bloom, Nicholas, Max Floetotto, Nir Jaimovich, Itay Saporta-Eksten, and Stephen J. Terry (2012) "Really Uncertain Business Cycles," NBER Working Paper No. 18245.

[25] Boldrin, Michele, Lawrence J. Christiano, and Jonas D. M. Fisher (2001) "Habit Persistence, Asset Returns, and the Business Cycle," *American Economic Review*, 91(1):149-166.

[26] Borovička, Jaroslav, Lars Peter Hansen, and José Scheinkman (2014) "Misspecified Recovery," University of Chicago Working Paper.

[27] Cagetti, Marco, Lars Peter Hansen, Thomas Sargent, and Noah Williams (2002) "Robustness and Pricing with Uncertain Growth," *Review of Financial Studies*, 15(2):363-404.

[28] Campbell, John Y., and John H. Cochrane (1999) "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior," *Journal of Political Economy*, 107(2):205-251.

[29] Carillo, Julio, Patrick Fève, and Julien Matheron (2007) "Monetary POlicy Inertia or Persistent Shocks: A DSGE Analysis," *International Journal of Central Banking*, 3(2):1-38.

[30] Cass, David, and Karl Shell (1983) "Do Sunspots Matter?" *Journal of Political Economy*, 91(2):193-227.

[31] Chari, V. V., Patrick J. Kehoe, and Ellen R. McGrattan (2009) "New Keynesian Models: Not Yet Useful for Policy Analysis," *American Economic Journal: Macroeconomics*, 1(1):242-266.

[32] Christiano, Lawrence, Cosmin Ilut, Robert Motto, and Massimo Rostagno (2010) "Monetary policy and stock market booms," Proceedings: Economic Policy Symposium, Jackson Hole, Federal Reserve bank of Kansas City, pp. 85-145.

[33] Croce, Mariano Massimiliano (2014) "Long-run productivity risk: A new hope for production-based asset pricing?" *Journal of Monetary Economics*, 66:13-31.

[34] Cúrida, Vasco, and Ricardo Reis (2010) "Correlated Disturbances and U.S. Business Cycles," NBER Working Paper No. 15774.

[35] Dai, Qiang, and Kenneth J. Singleton (2000) "Specification Analysis of Affine Term Structure Models," *Journal of Finance*, 55(5):1943-1978.

[36] Del Negro, Marco, and Stefano Eusepi (2011) "Fitting observed inflation expectations," *Journal of Economic Dynamics and Control*, 35(12):2105-2131.

[37] Epstein, Larry G., and Stanley E. Zin (1989) "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57(4):937-969.

[38] Erceg, Christopher J., Dale W. Henderson, and Andrew T. Levin (2000) "Optimal Monetary Policy with Staggered Wage and Price Contracts," *Journal of Monetary Economics*, 46(2):281-313.

[39] Eusepi, Stefano, and Bruce Preston (2011) "Expectations, Learning, and Business Cycle Fluctuations," *American Economic Review*, 101(6):2844-2872.

[40] Fernald, John (2012) "A quarterly, utilization-adjusted series on total factor productivity," Federal Reserve Bank of San Francisco Working Paper No. 2012-19.

[41] Fernández-Villaverde, Jesús, Pablo Guerrón-Quintana, Juan F. Rubio-Ramírez, and Martín Uribe (2011) "Risk Matters: The Real Effects of Volatility Shocks," *American Economic Review*, 101(6):2530-2561.

[42] Gilboa, Itzhak, and David Schmeidler (1989) "Maxmin expected utility with non-unique prior," *Journal of Mathematical Economics*, 18(2):141-153.

[43] Gomme, Paul and Paul Klein (2011) "Second-order approximation of dynamic models without the use of tensors," *Journal of Economic Dynamics and Control*, 35(4):604-615.

[44] Gourio, François (2012) "Disaster Risk and Business Cycles," *American Economic Review*, 102(6):2734-2766.

[45] Greenwood, Jeremy, Zvi Hercowitz, and Gregory W. Huffman (1988) "Investment, Capacity Utilization, and the Real Business Cycle," *American Economic Review*, 78(3):402-417.

[46] Greenspan, Alan (2003) "Testimony of Chairman Alan Greenspan," remarks made before the Committee on Financial Services, U.S. House of Representatives,

as part of the presentation of the Federal Reserve Board's semiannual "Monetary Policy Report to the Congress," July 15, 2003.

[47] Hall, Robert E. (1978) "Stochastic Implications of the Life Cycle-Permanent Income Hypothesis: Theory and Evidence," *Journal of Political Economy*, 86(6):971-987.

[48] Hansen, Lars Peter, and Thomas J. Sargent (2005) "Robust estimation and control under commitment," *Journal of Economic Theory*, 124(2):258-301.

[49] Hansen, Lars Peter, and Thomas J. Sargent (2007) "Recursive robust estimation and control without commitment," *Journal of Economic Theory*, 136(1):1-27.

[50] Hansen, Lars Peter, Thomas J. Sargent, and Thomas D. Tallarini, Jr. (1999) "Robust Permanent Income and Pricing," *Review of Economic Studies*, 66(4):873-907.

[51] Hassan, Tarek A. and Thomas M. Mertens (2011) "The Social Cost of Near-Rational Investment," NBER Working Paper No. 17027.

[52] Hassan, Tarek A. and Thomas M. Mertens (2014) "Information Aggregation in a DSGE Model," Forthcoming in *NBER Macroeconomics Annual 2014*, 29.

[53] Ilut, Cosmin, and Martin Schneider (2014) "Ambiguous Business Cycles," *American Economic Review*, 104(8):2368-99.

[54] Jaimovich, Nir, and Sergio Rebelo (2009) "Can News about the Future Drive the Business Cycle?" *American Economic Review*, 99(4):1097-1118.

[55] Jermann, Urban J. (1998) "Asset Pricing in Production Economies," *Journal of Monetary Economics*, 41(2):257-275.

[56] Jinnai, Ryo (2013) "News shocks and inflation," *Economics Letters*, 119(2):176-179.

[57] Justiniano, Alejandro, Giorgio E. Primiceri, and Andrea Tambalotti (2011) " Investment Shocks and the Relative Price of Investment," *Review of Economic Dynamics*, 14(1):102-121.

[58] Kaltenbrunner, Georg and Lars A. Lochstoer (2010) "Long-Run Risk Through Consumption Smoothing," *Review of Financial Studies*, 23(8):3190-3224.

[59] Klein, Paul (2005) "Using the generalized Schur form to solve a multivariate linear rational expectations model," *Jounral of Economic Dynamics and Control*, 24(10):1405-1423.

[60] Komunjer, Ivana, and Serena Ng (2011) "Dynamic Identification of Dynamic Stochastic General Equilibrium Models," *Econometrica*, 79(6):1995-2032.

[61] Kreps, David M., and Evan L. Porteus (1978) "Temporal Resolution of Uncertainty and Dynamic Choice Theory," *Econometrica*, 46(1):185-200.

[62] Le, Anh, Kenneth J. Singleton, and Qiang Dai (2010) "Discrete-Time Affine$^{\mathbb{Q}}$ Term Structure Models with Generalized Market Prices of Risk," *Review of Financial Studies*, 23(5):2184-2227.

[63] Lorenzoni, Guido (2009) "A Theory of Demand Shocks," *American Economic Review*, 99(5):2050-2084.

[64] Malkhozov, Aytek (2014) "Asset prices in affine real business cycle models," *Journal of Economic Dynamics and Control*, 45:180-193.

[65] Mehra, Rajnish and Edward Prescott (1985) "The Equity Premium: a Puzzle," *Journal of Monetary Economics*, 15(2):145-161.

[66] Milani, Fabio (2011) "Expectation Shocks and Learning as Drivers of the Business Cycle," *The Economic Journal*, 121(552):379-401.

[67] Miyamoto, Wataru, and Thuy Lan Nguyen (2014) "News Shocks and Business Cycles: Evidence from Forecast Data," Working Paper, Columbia University.

[68] Nakamura, Emi, Jón Steinsson, and Dmitriy Sergeyev (2012) "Growth-Rate Uncertainty Shocks in Consumption: Cross-Country Evidence," Working Paper, Columbia University.

[69] Piazzesi, Monika, Juliana Salomao, and Martin Schneider (2013) "Trend and Cycle in Bond Premia," Working Paper, Stanford University.

[70] Pigou, Arthur C. (1927) *Industrial Fluctuations*, MacMillan and Co.: London.

[71] Preston, Bruce (2005) "Learning about Monetary Policy Rules when Long-Horizon Expectations Matter," *International Journal of Central Banking*, 1(2):81-126.

[72] Rauh, Joshua D. and Amir Sufi (2012) "Explaining Corporate Capital Structure: Product Markets, Leases, and Asset Similarity," *Review of Finance*, 16(1):115-155.

[73] Restoy, Fernando and G. Michael Rockinger (1994) "On Stock Market Returns and Returns on Investment," *Journal of Finance*, 49(2):543-556.

[74] Rudebusch, Glenn D. (2002) "Term Structure Evidence on Interest Rate Smoothing and Monetary Policy Inertia," *Journal of Monetary Economics*, 55(4):677-691.

[75] Rudebusch, Glenn D. (2006) "Monetary Policy Inertia: Fact or Fiction?" *International Journal of Central Banking*, 2(4):85-135.

[76] Rudebusch, Glenn D. and Eric T. Swanson (2012) "The Bond Premium in a DSGE Model with Long-Run Real and Nominal Risks," *American Economic Journal: Macroeconomics*, 4(1):105-143.

[77] Schmitt-Grohé, Stephanie, and Martín Uribe (2004) "Solving dynamic general equilibrium models using a second-order approximation to the policy function," *Journal of Economic Dynamics and Control*, 28(4):755-775.

[78] Schmitt-Grohé, Stephanie, and Martín Uribe (2007) "Comparing Two Variants of Calvo-Type Wage Stickiness," NBER Working Paper No. 12740.

[79] Schmitt-Grohé, Stephanie, and Martín Uribe (2012) "What's News in Business Cycles," *Econometrica*, 80(6):2733-2764.

[80] Shimer, Robert (2009) "Convergence in Macroeconomics: The Labor Wedge," *American Economic Journal: Macroeconomics*, 1(1):280-297.

[81] Smets, Frank and Rafael Wouters (2007) "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach," *American Economic Review*, 97(3):586-606.

[82] Woodford, Michael (2010) "Robustly Optimal Monetary Policy with Near-Rational Expectations," *American Economic Review*, 100(1):274-303.

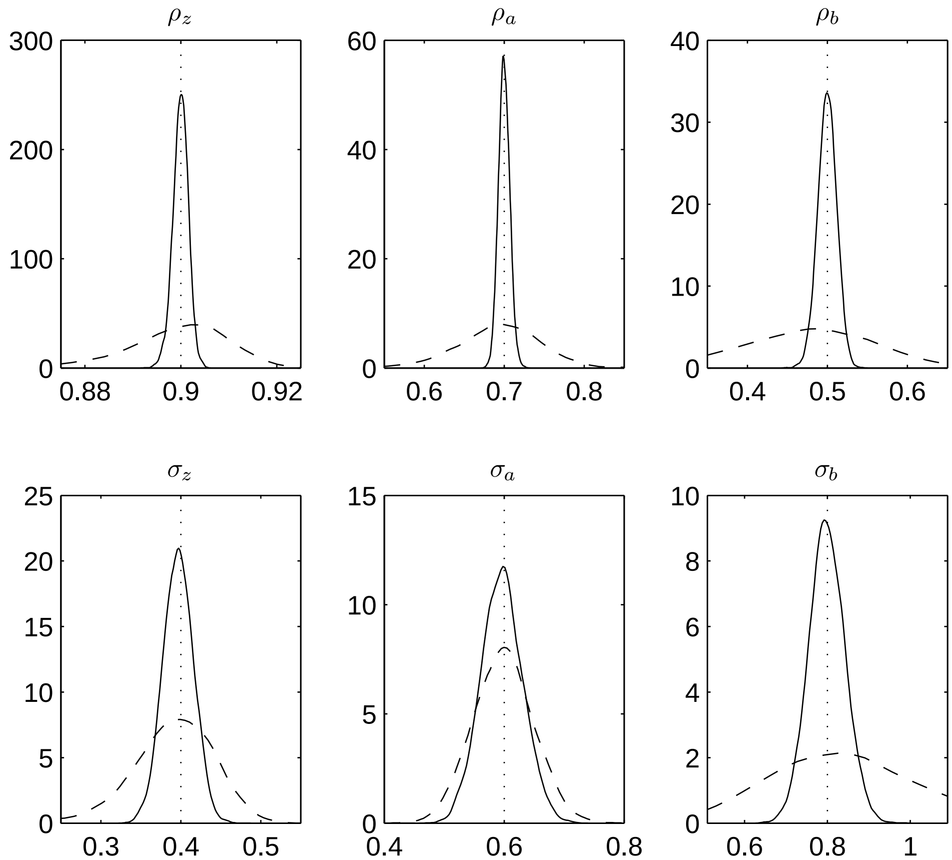

Figure 1: Monte Carlo experiments. This figure shows the distribution of parameter estimates from the consumption-income example discussed in Section 3. In each subplot, the vertical dotted line indicates the true value of the parameter. Based on these values, 10,000 samples of 219 observations are simulated for estimation. Estimates are obtained by numerically maximizing the likelihood function implied by the model. The dashed line is the distribution of estimates based only on observations of consumption growth and income growth. The solid line is the distribution when observations of subjective forecasts of income growth are included in addition to consumption growth and income growth. The first column of panels corresponds to the parameters related to contemporaneous innovations in income (persistence ρ_z and conditional volatility σ_z), the second column to the parameters related to advance information about future income (ρ_a,σ_a), and the third to the parameters related to distorted beliefs about future income (ρ_b,σ_b).

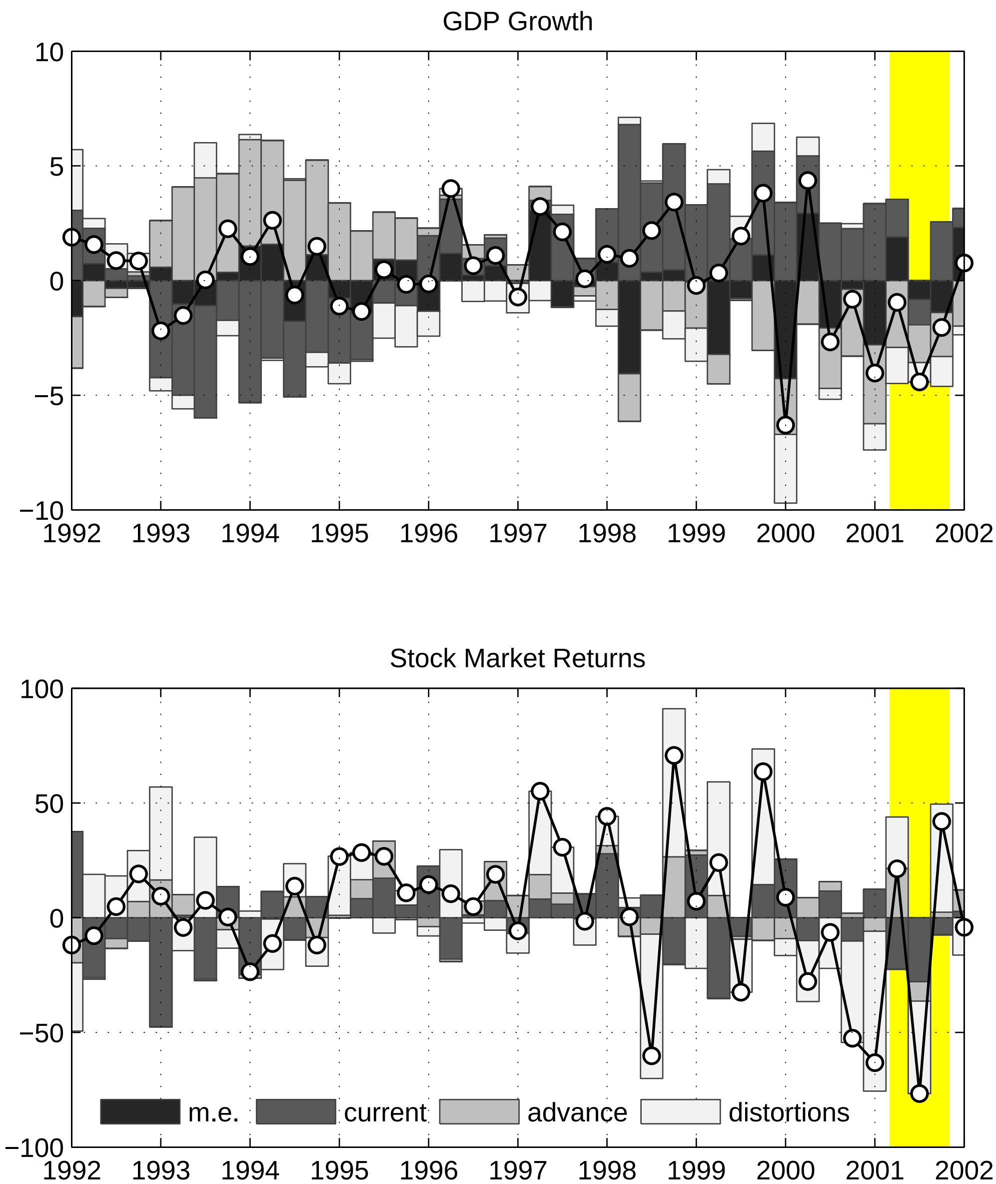

Figure 2: Dot-Com Boom and Bust (1992-2002). The top panel is the annualized growth rate of real per capita GDP and the bottom panel is the annualized ex-post return on the aggregate stock market. Each quarterly observation (white circle) is decomposed into one of four sources: measurement error (m.e.), current fundamentals (current) advance information about future fundamentals (advance), or belief distortions about future fundamentals (distortions). These add the contributions from all exogenous processes of that respective category. Contributions from individual disturbances are discussed in the main text. The yellow shaded region is a recessionary period according to the NBER Business Cycle Dating Committee.

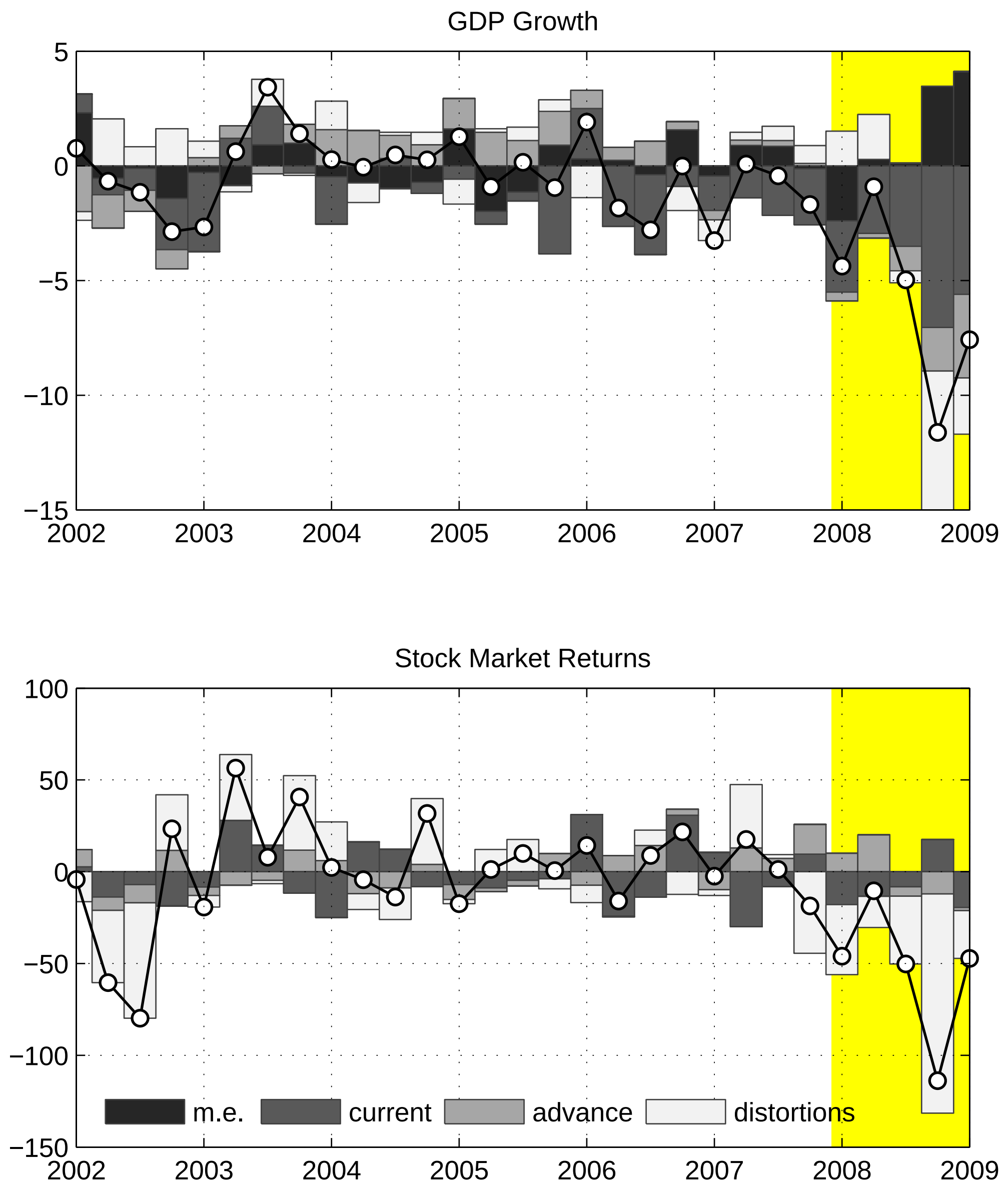

Figure 3: Run-up to the Great Recession (2002-2009). The top panel is the annualized growth rate of real per capita GDP and the bottom panel is the annualized ex-post return on the aggregate stock market. Each quarterly observation (white circle) is decomposed into one of four sources: measurement error (m.e.), current fundamentals (current) advance information about future fundamentals (advance), or belief distortions about future fundamentals (distortions). These add the contributions from all exogenous processes of that respective category. Contributions from individual disturbances are discussed in the main text. The yellow shaded region is a recessionary period according to the NBER Business Cycle Dating Committee.

Model Concept	Description
$\Delta \ln(Y_t^d)$	Real per capita GDP growth
$\Delta \ln(C_t)$	Real per capita consumption growth
$\Delta \ln(X_t I_t)$	Real per capita investment growth
$\Delta \ln(H_t)$	Growth in per capita hours worked
$\Delta \ln(W_t)$	Real wage growth
$\ln(\Pi_t)$	Inflation
$\ln(R_t^e)$	Real stock market return
$\ln(R_t)$	Nominal risk-free return
$\tilde{E}_t[\Delta \ln(Y_{t+1}^d)]$	Expected per capita real GDP growth
$\tilde{E}_t[\ln(\Pi_{t+1})]$	Expected inflation
$\tilde{E}_t[\ln(R_{t+1})]$	Expected nominal risk-free return
$\Delta \ln(G_t)$	Real per capita government spending growth
$\Delta \ln(\mathrm{TFP}_t)$	Total factor productivity growth
$\Delta \ln(X_t)$	Growth in relative price of investment

Table 1: Data. The variable $\mathrm{TFP}_t \equiv \zeta_t^Y Z_t^{1-\alpha_K}$ denotes (capacity-adjusted) total factor productivity, and Δ is the linear first-difference operator. The sample period is 1954:Q3 – 2009:Q1. A more detailed description of the data and sources is included in Appendix A.

Parameter	Value	Description
β	0.995	Marginal rate of time preference
ξ	1	Elasticity of intertemporal substitution
α_K	0.225	Capital share in production
α_H	0.675	Labor share in production
δ_0	0.025	Steady-state depreciation rate
u	1	Steady-state utilization rate
μ^Y	1.0044	Steady-state gross growth rate of per capita GDP
μ^X	0.9968	Steady state gross growth rate of investment price
G/Y^d	0.2103	Steady state share of government purchases in GDP
h	0.2	Steady-state level of hours
$\eta_W/(\eta_W - 1)$	1.15	Steady-state wage markup
$\eta_P/(\eta_P - 1)$	1.15	Steady-state price markup
φ_R	0	Persistence in nominal interest rate rule
Π	1.0086	Steady-state inflation rate

Table 2: Calibrated parameters. The time unit is one quarter.

Parameter	Description	Dist.	Mean	S.D.	L.B.	U.B.
γ	risk aversion	$\mathcal{G}$	5	2	1	$+\infty$
λ	equity leverage	$\mathcal{G}$	2	0.75	1	$+\infty$
ν	wealth elasticity of labor	$\mathcal{U}$	0.5	$1/(2\sqrt{3})$	0	1
ϕ	habit persistence	$\mathcal{B}$	0.5	0.1	0	1
ψ	labor supply elasticity	$\mathcal{G}$	2	0.75	1	$+\infty$
κ	investment adj costs	$\mathcal{G}$	4	1	0	$+\infty$
δ_2/δ_1	utilization cost elasticity	$\mathcal{IG}$	0.5	0.5	0	$+\infty$
θ_W	wage no reset probability	$\mathcal{B}$	0.66	0.1	0	1
θ_P	price no reset probability	$\mathcal{B}$	0.66	0.1	0	1
χ_W	wage indexation	$\mathcal{B}$	0.5	0.15	0	1
χ_P	price indexation	$\mathcal{B}$	0.5	0.15	0	1
φ_Π	feedback rule inflation	$\mathcal{G}$	1.7	0.3	1	$+\infty$
φ_{Y_d}	feedback rule output growth	$\mathcal{G}$	0.125	0.05	0	$+\infty$
ρ_{ZG}	smooth trend govt spending	$\mathcal{B}$	0.6	0.2	-1	1
$\rho_{z,i}$	persistence current fundamentals	$\mathcal{B}$	0.6	0.2	-1	1
ρ_{z,μ_Z}	persistence perm tech growth	$\mathcal{B}$	0.2	0.2	-1	1
ρ_{z,μ_X}	persistence rel price of inv	$\mathcal{B}$	0.2	0.2	-1	1
$\rho_{a,i}$	persistence advance information	$\mathcal{B}$	0	0.2	-1	1
$\rho_{b,i}$	persistence belief distortions	$\mathcal{B}$	0	0.2	-1	1
$\sigma_{z,i}$	std dev current fundamentals	$\mathcal{IG}$	0.5	1	0	$+\infty$
$\sigma_{z,i}$	std dev advance information	$\mathcal{IG}$	0.1	1	0	$+\infty$
$\sigma_{z,i}$	std dev belief distortions	$\mathcal{IG}$	0.1	1	0	$+\infty$
σ_{u,Y^d}	measurement error GDP growth	$\mathcal{B}$	0.1	0.05	0	0.2923

Table 3: Prior distributions. The symbol $\mathcal{U}$ denotes a uniform distribution, $\mathcal{B}$ is a generalized beta distribution, $\mathcal{G}$ is a generalized gamma distribution, and $\mathcal{IG}$ is an inverse-gamma distribution. The generalized distributions are parameterized by a location parameter, scale parameter, lower bound, and upper bound. The columns titled L.B. and U.B. represent the lower and upper bounds defining the support of these distributions. The upper bound for the measurement error is equal to $\sqrt{0.10} \times \text{std}(\Delta \ln(Y_t^d))$, so that the variance of this error is at most ten percent of the observed variance in GDP growth.

	Prior	Posterior				
Parameter	Mean	Mean	Mode	10th	90th	Description
γ	5	10.50	10.37	9.18	11.86	risk aversion
λ	2	2.66	2.70	2.22	3.05	equity leverage
ν	0.5	0.06	0.06	0.05	0.07	wealth elasticity of labor
ϕ	0.5	0.68	0.68	0.65	0.70	habit persistence
ψ	2	1.20	1.20	1.15	1.25	labor supply elasticity
κ	4	13.9	13.5	12.27	15.6	investment adj costs
δ_2/δ_1	0.5	0.13	0.13	0.11	0.15	utilization cost elasticity
θ_W	0.66	0.03	0.03	0.02	0.04	wage no reset probability
θ_P	0.66	0.08	0.08	0.06	0.10	price no reset probability
χ_W	0.5	0.50	0.51	0.31	0.68	wage indexation
χ_P	0.5	0.26	0.20	0.13	0.41	price indexation
φ_Π	1.7	4.99	4.77	4.36	5.67	feedback rule inflation
φ_{Y_d}	0.125	0.23	0.22	0.15	0.31	feedback rule output growth
ρ_{ZG}	0.6	0.63	0.69	0.40	0.82	smooth trend govt spending

Table 4: Estimated structural parameters. The mean, 10th, and 90th percentiles come from 100,000 draws from a Markov chain approximating the posterior distribution of the parameters. These draws are taken from 10 Markov chains of length 10,000 after removing a burn-in sample of 1,000 draws.

	Prior	Posterior					
Parameter	Mean	Mean	Mode	10th	90th	Type	Fundamental
ρ_{z,ζ_C}	0.6	0.75	0.75	0.62	0.86	current	preference
ρ_{z,ζ_I}	0.6	0.03	0.02	-0.05	0.11	current	inv. specific
ρ_{z,ζ_Y}	0.6	0.98	0.98	0.98	0.99	current	temp. productivity
ρ_{z,ζ_W}	0.6	0.99	0.99	0.99	1.00	current	union marginal cost
ρ_{z,ζ_P}	0.6	1.00	1.00	1.00	1.00	current	retailer marginal cost
ρ_{z,ζ_R}	0.6	0.18	0.18	0.14	0.23	current	monetary policy
$\rho_{z,g}$	0.6	0.96	0.97	0.95	0.98	current	gov. spending
ρ_{z,μ_Z}	0.2	0.69	0.72	0.58	0.78	current	permanent productivity
ρ_{z,μ_X}	0.2	0.46	0.47	0.40	0.50	current	rel. price of inv.
ρ_{a,ζ_C}	0	0.05	0.03	-0.20	0.31	advance	preference
ρ_{a,ζ_I}	0	0.73	0.73	0.69	0.77	advance	inv. specific
ρ_{a,ζ_Y}	0	0.29	0.28	0.08	0.49	advance	temp. productivity
ρ_{a,ζ_W}	0	0.98	0.98	0.98	0.99	advance	union marginal cost
ρ_{a,ζ_P}	0	0.29	0.28	0.08	0.50	advance	retailer marginal cost
ρ_{a,ζ_R}	0	0.90	0.90	0.86	0.92	advance	monetary policy
$\rho_{a,g}$	0	0.01	0.00	-0.24	0.27	advance	gov. spending
ρ_{a,μ_Z}	0	0.06	0.04	-0.16	0.27	advance	permanent productivity
ρ_{a,μ_X}	0	-0.03	-0.01	-0.30	0.24	advance	rel. price of inv.
ρ_{b,ζ_C}	0	0.04	0.04	-0.22	0.30	distortions	preference
ρ_{b,ζ_I}	0	-0.00	-0.00	-0.27	0.26	distortions	inv. specific
ρ_{b,ζ_Y}	0	0.29	0.28	0.08	0.50	distortions	temp. productivity
ρ_{b,ζ_W}	0	0.20	0.19	-0.02	0.42	distortions	union marginal cost
ρ_{b,ζ_P}	0	0.99	0.99	0.99	1.00	distortions	retailer marginal cost
ρ_{b,ζ_R}	0	0.85	0.85	0.79	0.89	distortions	monetary policy
$\rho_{b,g}$	0	0.72	0.73	0.66	0.78	distortions	gov. spending
ρ_{b,μ_Z}	0	0.55	0.52	0.47	0.65	distortions	permanent productivity
ρ_{b,μ_X}	0	0.81	0.81	0.79	0.84	distortions	rel. price of inv.

Table 5: Estimated persistence parameters of exogenous processes. "Type" denotes the type of exogenous process, either: current fundamentals (current), advance information (advance), or belief distortions (distortions). The column titled "Fundamental" provides a brief description of the fundamental process associated with each type of disturbance. The mean, 10th, and 90th percentiles come from 100,000 draws from a Markov chain approximating the posterior distribution of the parameters. These draws are taken from 10 Markov chains of length 10,000 after removing a burn-in sample of 1,000 draws.

	Prior	Posterior					
Parameter	Mean	Mean	Mode	10th	90th	Type	Fundamental
σ_{z,ζ_C}	0.5	0.41	0.41	0.22	0.60	current	preference
σ_{z,ζ_I}	0.5	21.22	20.91	18.60	24.02	current	inv. specific
σ_{z,ζ_Y}	0.5	0.84	0.83	0.78	0.89	current	temp. productivity
σ_{z,ζ_W}	0.5	0.30	0.23	0.18	0.46	current	union marginal cost
σ_{z,ζ_P}	0.5	0.75	0.75	0.71	0.79	current	retailer marginal cost
σ_{z,ζ_R}	0.5	1.10	1.04	0.94	1.27	current	monetary policy
$\sigma_{z,g}$	0.5	1.05	1.05	0.98	1.11	current	gov. spending
σ_{z,μ_Z}	0.5	0.14	0.13	0.12	0.17	current	permanent productivity
σ_{z,μ_X}	0.5	0.30	0.31	0.28	0.33	current	rel. price of inv.
σ_{a,ζ_C}	0.1	0.05	0.04	0.03	0.07	advance	preference
σ_{a,ζ_I}	0.1	6.91	6.89	5.65	8.24	advance	inv. specific
σ_{a,ζ_Y}	0.1	0.03	0.03	0.02	0.04	advance	temp. productivity
σ_{a,ζ_W}	0.1	0.10	0.10	0.08	0.12	advance	union marginal cost
σ_{a,ζ_P}	0.1	0.03	0.03	0.02	0.04	advance	retailer marginal cost
σ_{a,ζ_R}	0.1	0.50	0.47	0.41	0.60	advance	monetary policy
$\sigma_{a,g}$	0.1	0.07	0.05	0.04	0.13	advance	gov. spending
σ_{a,μ_Z}	0.1	0.04	0.04	0.03	0.05	advance	permanent productivity
σ_{a,μ_X}	0.1	0.07	0.05	0.04	0.13	advance	rel. price of inv.
σ_{b,ζ_C}	0.1	0.05	0.04	0.03	0.07	distortions	preference
σ_{b,ζ_I}	0.1	0.09	0.05	0.04	0.16	distortions	inv. specific
σ_{b,ζ_Y}	0.1	0.03	0.03	0.02	0.04	distortions	temp. productivity
σ_{b,ζ_W}	0.1	0.03	0.03	0.03	0.04	distortions	union marginal cost
σ_{b,ζ_P}	0.1	0.04	0.04	0.03	0.05	distortions	retailer marginal cost
σ_{b,ζ_R}	0.1	0.44	0.41	0.35	0.53	distortions	monetary policy
$\sigma_{b,g}$	0.1	1.55	1.52	1.34	1.77	distortions	gov. spending
σ_{b,μ_Z}	0.1	0.15	0.14	0.12	0.17	distortions	permanent productivity
σ_{b,μ_X}	0.1	0.57	0.56	0.51	0.62	distortions	rel. price of inv.
σ_{u,Y_d}	0.1	0.29	0.29	0.29	0.29	m.e.	output growth

Table 6: Estimated volatility parameters of exogenous processes. “Type” denotes the type of exogenous process, either: current fundamentals (current), advance information (advance), belief distortions (distortions), or measurement error (m.e.). The column titled “Fundamental” provides a brief description of the fundamental process associated with each type of disturbance. The mean, 10th, and 90th percentiles come from 100,000 draws from a Markov chain approximating the posterior distribution of the parameters. These draws are taken from 10 Markov chains of length 10,000 after removing a burn-in sample of 1,000 draws.

Variable	Mean	Std Dev	AC(1)	Corr GDP
$\Delta\ln(Y_t^d)$	1.76	2.25	0.60	1.00
	(1.76)	(1.85)	(0.34)	(1.00)
$\Delta\ln(C_t)$	1.76	1.69	0.66	0.67
	(2.01)	(1.09)	(0.29)	(0.55)
$\Delta\ln(X_t I_t)$	1.76	6.25	0.63	0.87
	(2.01)	(4.56)	(0.44)	(0.77)
$\Delta\ln(H_t)$	0.00	2.70	0.25	0.53
	(-0.15)	(1.75)	(0.61)	(0.73)
$\Delta\ln(W_t)$	1.76	2.82	-0.02	0.29
	(1.64)	(1.28)	(0.01)	(0.14)
$\Delta\ln(\Pi_t)$	3.43	0.92	0.75	-0.25
	(3.43)	(1.16)	(0.86)	(-0.25)
$\ln(R_t^e)$	4.75	23.28	-0.04	0.16
	(5.82)	(17.04)	(0.09)	(0.13)
$\ln(R_t)$	4.11	2.03	0.87	-0.30
	(5.20)	(1.43)	(0.96)	(-0.10)
$\tilde{E}_t[\Delta\ln(Y_{t+1}^d)]$	1.75	1.84	0.82	0.67
	(1.15)	(0.92)	(0.70)	(0.53)
$\tilde{E}_t[\ln(\Pi_{t+1})]$	3.43	0.88	0.89	-0.20
	(3.82)	(0.99)	(0.96)	(-0.15)
$\tilde{E}_t[\ln(R_{t+1})]$	4.11	1.81	0.88	-0.30
	(5.41)	(1.36)	(0.97)	(0.10)

Table 7: Selected moments in the model and data. Moments from the data are listed in parentheses below their model-implied counterparts. Means and standard deviations are computed at the quarterly frequency and then annualized by multiplying by 4 (means) or 2 (standard deviations). Units are in percentage points.

Variable	Current	Advance	Distortions
$\Delta \ln(Y_t^d)$	0.51	0.27	0.22
	(0.47,0.55)	(0.22,0.31)	(0.19,0.25)
$\Delta \ln(C_t)$	0.23	0.14	0.62
	(0.19,0.28)	(0.12,0.17)	(0.58,0.67)
$\Delta \ln(X_t I_t)$	0.50	0.44	0.06
	(0.45,0.56)	(0.37,0.50)	(0.05,0.07)
$\Delta \ln(H_t)$	0.41	0.24	0.34
	(0.38,0.44)	(0.21,0.28)	(0.32,0.37)
$\Delta \ln(W_t)$	0.61	0.05	0.34
	(0.58,0.64)	(0.04,0.06)	(0.31,0.38)
$\ln(\Pi_t)$	0.33	0.55	0.12
	(0.28,0.38)	(0.48,0.62)	(0.09,0.15)
$\ln(R_t^e)$	0.16	0.11	0.72
	(0.13,0.20)	(0.07,0.16)	(0.65,0.80)
$\ln(R_t)$	0.29	0.20	0.50
	(0.25,0.33)	(0.18,0.23)	(0.46,0.55)
$\tilde{E}_t[\Delta \ln(Y_{t+1}^d)]$	0.36	0.31	0.34
	(0.32,0.39)	(0.26,0.36)	(0.29,0.39)
$\tilde{E}_t[\ln(\Pi_{t+1})]$	0.08	0.61	0.31
	(0.06,0.11)	(0.53,0.69)	(0.23,0.38)
$\tilde{E}_t[\ln(R_{t+1})]$	0.27	0.21	0.51
	(0.24,0.31)	(0.18,0.25)	(0.47,0.56)
$\Delta \ln(G_t)$	0.99	0.01	0
	(0.98,1.00)	(0.00,0.02)	(–)
$\Delta \ln(\mathrm{TFP}_t)$	1.00	0.00	0
	(0.99,1.00)	(0.00,0.01)	(–)
$\Delta \ln(X_t)$	0.93	0.07	0
	(0.84,0.99)	(0.01,0.16)	(–)

Table 8: Variance decomposition. Each column contains the contribution from a different type of disturbance. "Current" denotes current fundamentals, "Advance" denotes advance information, and "Distortions" denotes belief distortions. Decompositions are computed based on a random sample of 1000 parameters from the posterior distribution. The main entry represents the mean value, and the numbers in parentheses below are the 10th and 90th percentiles, respectively.

Variable	Current ζ_I	Current ζ_Y	Current ζ_P	Advance ζ_I	Advance ζ_W	Advance ζ_R	Distortions ζ_P	Distortions g	Distortions μ_X
$\Delta \ln(Y_t^d)$	0.14	0.22	0.05	0.21	0.05	0.00	0.00	0.07	0.10
$\Delta \ln(C_t)$	0.01	0.11	0.02	0.08	0.06	0.00	0.02	0.17	0.36
$\Delta \ln(X_t I_t)$	0.21	0.21	0.06	0.40	0.03	0.00	0.01	0.02	0.01
$\Delta \ln(H_t)$	0.10	0.23	0.03	0.15	0.09	0.00	0.01	0.10	0.20
$\Delta \ln(W_t)$	0.00	0.29	0.30	0.02	0.03	0.00	0.01	0.02	0.32
$\ln(\Pi_t)$	0.00	0.07	0.01	0.01	0.01	0.53	0.06	0.01	0.03
$\ln(R_t^e)$	0.00	0.09	0.06	0.02	0.09	0.00	0.51	0.00	0.19
$\ln(R_t)$	0.00	0.22	0.04	0.03	0.06	0.11	0.27	0.01	0.12
$\tilde{E}_t[\Delta \ln(Y_{t+1}^d)]$	0.01	0.22	0.06	0.23	0.07	0.00	0.01	0.20	0.07
$\tilde{E}_t[\ln(\Pi_{t+1})]$	0.00	0.06	0.01	0.01	0.01	0.59	0.05	0.01	0.02
$\tilde{E}_t[\ln(R_{t+1})]$	0.00	0.21	0.03	0.03	0.06	0.11	0.26	0.01	0.12

Table 9: Variance decomposition for selected exogenous processes. Each column contains the contribution from a different exogenous process. The first set of three columns contain the contributions from the three current disturbances that are most important for explaining fluctuations in the eleven endogenous variables: investment-specific technology ζ_I, temporary neutral productivity ζ_Y, and product market marginal costs ζ_P. The second set of three columns contain the contributions from the three exogenous processes with the largest advance information component: investment-specific technology ζ_I, labor market marginal costs ζ_W, and monetary policy ζ_R. The last set of three columns contain the contributions from the three exogenous processes with the largest belief distortion component: product market marginal costs ζ_P, government spending g, and the relative price of investment μ_X. Each entry in the table represents the mean value based on a random sample of 1000 parameters from the posterior distribution (the same random sample used in Table 6).

Variable	Current	Advance	Distortions
$\Delta \ln(Y_t^d)$	0.92	0.04	0.03
$\Delta \ln(C_t)$	0.91	0.05	0.04
$\Delta \ln(X_t I_t)$	0.94	0.04	0.02
$\Delta \ln(H_t)$	0.79	0.12	0.09
$\Delta \ln(W_t)$	0.94	0.04	0.02
$\ln(\Pi_t)$	0.94	0.05	0.01
$\ln(R_t^e)$	0.94	0.02	0.04
$\ln(R_t)$	0.95	0.03	0.01
$\tilde{E}_t[\Delta \ln(Y_{t+1}^d)]$	0.92	0.05	0.03
$\tilde{E}_t[\ln(\Pi_{t+1})]$	0.92	0.06	0.02
$\tilde{E}_t[\ln(R_{t+1})]$	0.91	0.06	0.04
$\Delta \ln(G_t)$	0.96	0.04	0
$\Delta \ln(\mathrm{TFP}_t)$	0.96	0.04	0
$\Delta \ln(X_t)$	0.96	0.04	0

Table 10: Variance decomposition at the prior mean. Each column contains the contribution from a different type of disturbance. The decomposition is computed at the prior mean values listed in Table 4. Column headings are the same as those in Table 6.

Variable	Current	Advance	Distortions
$\Delta \ln(Y_t^d)$	0.51	0.27	0.22
$\Delta \ln(C_t)$	0.22	0.12	0.67
$\Delta \ln(X_t I_t)$	0.48	0.47	0.06
$\Delta \ln(H_t)$	0.49	0.18	0.33
$\Delta \ln(W_t)$	0.66	0.03	0.31
$\ln(\Pi_t)$	0.34	0.58	0.07
$\ln(R_t^e)$	0.35	0.09	0.57
$\ln(R_t)$	0.40	0.20	0.40
$\tilde{E}_t[\Delta \ln(Y_{t+1}^d)]$	0.38	0.30	0.32
$\tilde{E}_t[\ln(\Pi_{t+1})]$	0.10	0.63	0.28
$\tilde{E}_t[\ln(R_{t+1})]$	0.37	0.20	0.42
$\Delta \ln(G_t)$	1.00	0.00	0
$\Delta \ln(\mathrm{TFP}_t)$	1.00	0.00	0
$\Delta \ln(X_t)$	0.98	0.02	0

Table 11: Variance decomposition without stock returns. The decomposition is computed at the posterior mode. Column headings are the same as those in Table 6.

Variable	Current	Advance	Distortions
$\Delta\ln(Y_t^d)$	0.52	0.28	0.20
$\Delta\ln(C_t)$	0.24	0.13	0.63
$\Delta\ln(X_t I_t)$	0.50	0.45	0.05
$\Delta\ln(H_t)$	0.42	0.23	0.35
$\Delta\ln(W_t)$	0.62	0.04	0.34
$\ln(\Pi_t)$	0.42	0.42	0.15
$\ln(R_t^e)$	0.13	0.08	0.79
$\ln(R_t)$	0.33	0.18	0.50
$\tilde{E}_t[\Delta\ln(Y_{t+1}^d)]$	0.39	0.31	0.30
$\tilde{E}_t[\ln(\Pi_{t+1})]$	0.16	0.56	0.28
$\tilde{E}_t[\ln(R_{t+1})]$	0.30	0.19	0.51
$\Delta\ln(G_t)$	1.00	0.00	0
$\Delta\ln(\mathrm{TFP}_t)$	1.00	0.00	0
$\Delta\ln(X_t)$	0.98	0.02	0

Table 12: Variance decomposition with classical measurement error in survey forecasts. The decomposition is computed at the posterior mode. Column headings are the same as those in Table 6.

Variable	Current	Advance	Distortions
$\Delta \ln(Y_t^d)$	0.54	0.46	0
$\Delta \ln(C_t)$	0.57	0.43	0
$\Delta \ln(X_t I_t)$	0.51	0.49	0
$\Delta \ln(H_t)$	0.56	0.44	0
$\Delta \ln(W_t)$	0.59	0.41	0
$\ln(\Pi_t)$	0.45	0.55	0
$\ln(R_t^e)$	0.60	0.40	0
$\ln(R_t)$	0.50	0.50	0
$\tilde{E}_t[\Delta \ln(Y_{t+1}^d)]$	0.36	0.64	0
$\tilde{E}_t[\ln(\Pi_{t+1})]$	0.17	0.83	0
$\tilde{E}_t[\ln(R_{t+1})]$	0.51	0.49	0
$\Delta \ln(G_t)$	0.62	0.38	0
$\Delta \ln(\mathrm{TFP}_t)$	0.89	0.11	0
$\Delta \ln(X_t)$	0.73	0.27	0

Table 13: Variance decomposition without distorted beliefs. The decomposition is computed at the posterior mode. Column headings are the same as those in Table 6.

Variable	Current	Advance	Distortions
$\Delta\ln(Y_t^d)$	0.67	0.07	0.26
$\Delta\ln(C_t)$	0.48	0.02	0.50
$\Delta\ln(X_t I_t)$	0.88	0.10	0.02
$\Delta\ln(H_t)$	0.65	0.10	0.25
$\Delta\ln(W_t)$	0.43	0.28	0.29
$\ln(\Pi_t)$	0.60	0.23	0.17
$\ln(R_t^e)$	0.32	0.37	0.31
$\ln(R_t)$	0.67	0.16	0.17
$\tilde{E}_t[\Delta\ln(Y_{t+1}^d)]$	0.49	0.09	0.43
$\tilde{E}_t[\ln(\Pi_{t+1})]$	0.57	0.24	0.19
$\tilde{E}_t[\ln(R_{t+1})]$	0.62	0.18	0.21
$\Delta\ln(G_t)$	1.00	0.00	0
$\Delta\ln(\text{TFP}_t)$	1.00	0.00	0
$\Delta\ln(X_t)$	0.98	0.02	0

Table 14: Variance decomposition with more nominal rigidity. The two parameters governing the frequency of (optimal) price adjustment are set to $\theta_W = \theta_P = 0.66$. The decomposition is computed at the posterior mode. Column headings are the same as those in Table 6.

A Data Description

All data series used in estimation and their sources are listed here. All series from the BEA were downloaded from `www.bea.gov`, and those from the BLS from `www.bls.gov`, and all those from the Federal Reserve Bank of Philadelphia from `www.philadelphiafed.org`.

A.1 Raw Data

(1) Gross Domestic Product: NIPA table 1.1.5, line 1.

(2) Personal Consumption Expenditure on Durable Goods: NIPA table 1.1.5, line 4.

(3) Personal Consumption Expenditure on Nondurable Goods: NIPA table 1.1.5, line 5.

(4) Personal Consumption Expenditure on Services: NIPA table 1.1.5, line 6.

(5) Nonresidential Gross Private Domestic Fixed Investment: NIPA table 1.1.5, line 9.

(6) Residential Gross Private Domestic Fixed Investment: NIPA table 1.1.5, line 13.

(7) Government Consumption Expenditure: NIPA table 3.9.5, line 2.

(8) Government Gross Investment: NIPA table 3.9.5, line 3.

(9) Implicit Deflator for Gross Domestic Product: NIPA table 1.1.9, line 1.

(10) Implicit Deflator for Personal Consumption Expenditure on Durable Goods: NIPA table 1.1.9, line 4.

(11) Implicit Deflator for Nonresidential Gross Private Domestic Fixed Investment: NIPA table 1.1.9, line 9.

(12) Implicit Deflator for Residential Gross Private Domestic Fixed Investment: NIPA table 1.1.9, line 13.

(13) Civilian Noninstitutional Population Over 16: BLS series LNU00000000Q.

(14) Nonfarm Business Hours Worked: BLS series PRS85006033.

(15) Nonfarm Business Hourly Compensation: BLS series PRS85006103.

(14) Nonfarm Business Utilization-Adjusted Total Factor Productivity: from John Fernald. Used in Basu, Fernald, and Kimball (2006). Downloaded from `www.frbsf.org`.

(15) 3 Month Treasury Bill Secondary Market Rate: FRB H.15 release. Downloaded from `www.federalreserve.gov`.

(16) Stock Return Index Including Distributions: CRSP NYSE/NASDAQ/AMEX monthly value-weighted stock return index, series name VWRETD. Downloaded from CRSP through WRDS at `wrds-web.wharton.upenn.edu/wrds`.

(17) Mean Forecasts of Real Gross Domestic Product for the current quarter: Federal Reserve Bank of Philadelphia Survey of Professional Forecasters.

(18) Mean Forecasts of Real Gross Domestic Product one quarter ahead: Federal Reserve Bank of Philadelphia Survey of Professional Forecasters.

(19) Mean Forecasts of Implicit Deflator for Gross Domestic Product for the current quarter: Federal Reserve Bank of Philadelphia Survey of Professional Forecasters.

(20) Mean Forecasts of Implicit Deflator for Gross Domestic Product one quarter ahead: Federal Reserve Bank of Philadelphia Survey of Professional Forecasters.

(21) Mean Forecasts of 3-Month Treasury Bill Rate one quarter ahead: Federal Reserve Bank of Philadelphia Survey of Professional Forecasters.

A.2 Data Transformations

(22) Real per capita GDP: $Y_t^d = (1)/(9)/(13)$.

(23) Real per capita consumption: $C_t = [(3) + (4)]/(9)/(13)$.

(24) Real per capita investment: $X_t I_t = [(2) + (5) + (6)]/(9)/(13)$.

(25) Per capita hours worked: $H_t = (14)/(13)$.

(26) Real wage: $W_t = (15)/(9)$.

(27) Inflation: $\Pi_t = (9)_t/(9)_{t-1}$.

(28) Real stock market return: $R_t^e = (1 + (16))/(27)$.

(29) Nominal risk-free return: $R_t = [1 - \frac{91}{36000}(15)]^{-1}$.

(30) Expected per capita real GDP growth: $\tilde{E}_t[\Delta \ln(Y_{t+1}^d)] = \ln((18)/(17)/(13))$.

(31) Expected inflation: $\tilde{E}_t[\Delta\Pi_{t+1}] = \ln((20)/(19))$.

(32) Expected nominal risk-free return: $\tilde{E}_t[\ln(R_{t+1})] = -\ln([1 - \frac{91}{36000}(21)])$.

(33) Real per capita government consumption: $G_t = [(7) + (8)]/(9)/(13)$.

(34) Total factor productivity: $\mathrm{TFP}_t = 1 + (14)/100$.

(35) Real price of investment $X_t =$ Fisher index of (10),(11),(12) using (2),(5), and (6).

B Related Types of Disturbances

This section presents three different sources of non-fundamental variation that have recently been explored in the literature and clarify how those are related to the type of belief distortions modeled in this paper. These approaches are all similar and highly complementary; nevertheless, they are theoretically distinct. I show that to empirically differentiate them, it is important to incorporate observations of agents' subjective forecasts.

B.1 Ambiguity

Recent work by Ilut and Schneider (2014) has emphasized the importance of time-varying ambiguity together with ambiguity-averse agents for explaining business cycle fluctuations. To describe a situation similar to the one in their paper, consider a simple economic model that relates an agent's action x to his expectation of a fundamental random variable z:

$$x = \tilde{E}[z].$$

Here $\tilde{E}$ represents the agent's subjective expectation. For the purpose of this section, the subjective expectation may not correspond to any underlying well-defined probability measure.

At the time of his action, the agent believes that z will be drawn from an interval $[b - \xi, b + \xi]$ with $\xi \geq 0$. For simplicity, he believes z will be drawn according to a symmetric probability distribution. However, he is averse to ambiguity, and bases his decisions only on the worst possible outcome in this set. So he chooses

$$x = b - \xi.$$

If instead the agent were to act based on his average belief, he would choose

$$x = b.$$

With only observations of x, it is not possible to tell these alternative hypotheses apart. However, with additional observations of the agent's average or worst-case forecast (or both), it is possible. This is because the actions of an ambiguity-averse agent should be more highly correlated with his worst-case forecast than his average forecast. The focus of this paper is on belief distortions of the second type. This is consistent with the way that the average survey forecasts are interpreted and used in the empirical analysis.

B.2 Confidence

Research in progress by Angeletos, Collard, and Dellas (2014) introduces exogenous variation in agents' higher-order beliefs. In settings with strategic interaction, these types of disturbances affect endogenous outcomes. For example, consider a simple economic model that relates an agent's action x_i to his expectation of a fundamental random variable z and his expectation of the aggregate action taken in the economy, $x \equiv \int x_i di$:

$$x_i = \alpha \tilde{E}_i[z] + (1-\alpha)\tilde{E}_i[x].$$

Suppose that each atomistic agent believes that z has a mean of b. But at the same time, he maintains that all other agents in the economy believe that z has mean $b+\xi$. All agents agree to disagree about these beliefs. The random variable ξ is what the authors interpret as "confidence." By integrating the equilibrium condition above and iterating across all higher-order beliefs, it is possible to show that the optimal aggregate action satisfies

$$x = b + \frac{(1-\alpha)^2}{\alpha}\xi.$$

With only observations of x it is not possible to distinguish b from ξ. However, with additional observations of the average (first-order) forecast, it is possible. This is because distortions in higher-order beliefs do not affect first-order beliefs. The focus of this paper is on belief distortions of the second type. Again, this is consistent with the way that average survey forecasts are interpreted and used in the empirical analysis.

B.3 Noise Traders

Hassan and Mertens (2014) explore the role of exogenous non-fundamental variation driven by "noise traders:" agents who randomly demand an exogenous quantity of real resources each period. As a simple example of how noise traders can look like belief distortions, consider an economy with two agents. The first has mean-variance preferences with a unit coefficient of absolute risk aversion and maximizes his expected utility of terminal wealth. The other is a noise trader who exogenously demands an amount ξ of the resource. The resource is traded competitively at price x and pays of a random amount z in the next period. The demand function of the utlity-maximizing agent depends on his expectation of the terminal payout of that resource and its price. By market clearing:

$$1 = \frac{\tilde{E}[z] - x}{\tilde{\text{var}}[z]} + \xi.$$

Finally, suppose that the utility-maximizing agent believes that z has a mean of $(1 + b)$ and a variance of one. Then solving for the equilibrium price,

$$x = b + \xi.$$

With only observations of x it is not possible to distinguish b from ξ. However, with additional observations of the average forecast $\tilde{E}[z]$, it is possible. This is because the additional noise introduced by noise traders does not affect agents' beliefs about the mean of z. Again, this highlights how direct observations of subjective forecasts can be helpful for disentangling different hypotheses that are mathematically closely related.

One important caveat here (as in all the examples of this section) is that the subjective mean variable b is unrelated to ξ. In addition to incorporating noise traders, Hassan and Mertens also assume that agents are subject to belief distortions. Namely, that agents have imperfect information about z and use observations of x to compute their forecasts. This type of signal-extraction has the result of making b a function of ξ. In that case, it becomes more difficult to differentiate these hypotheses using forecasts alone, because even average forecasts would depend on the disturbance introduced by noise traders.

C Belief Distortions from Noisy Signals

Instead of working directly with distorting martingales, another common approach to introduce belief distortions is to assume that agents receive noisy signals of fundamentals, which they use to generate forecasts according to Bayes' rule. This section considers two such signal extraction problems, and discusses the types of belief distortions they imply. Two differences relative to the empirical specification in Section 3.4

are that signal extraction with Bayesian updating generates (i) distortions in conditional variances as well as conditional means, and (ii) (possibly dynamic) correlations between current fundamental innovations and innovations to beliefs.

C.1 Signals about the Future

As in Section 3, a scalar fundamental process follows the law of motion

$$z_t = \rho_z z_{t-1} + e_t^z, \quad e_t^z \overset{\mathbb{P}}{\sim} \mathcal{N}(0, \sigma_z^2).$$

In addition to observing this process, an economic agent also receives a subjective, noisy signal about future fundamentals:

$$s_t = z_{t+1} + e_t^s, \quad e_t^s \overset{\mathbb{P}}{\sim} \mathcal{N}(0, \sigma_s^2).$$

The innovation e_t^s is the source of exogenous variation in the agent's beliefs. It is completely independent of fundamentals. The agent incorporates the information from this signal, and forecasts future fundamentals according to the conditional distribution

$$z_{t+1}|z_t, s_t \overset{\mathbb{P}}{\sim} \mathcal{N}((1-k)\rho_z z_t + k s_t, (1-k)\sigma_z^2),$$

where $k \equiv \sigma_z^2/(\sigma_z^2 + \sigma_s^2)$ controls how much weight is given to the signal. When the signal is completely uninformative ($\sigma_s \to \infty$), this weight is zero. When the signal is perfectly revealing ($\sigma_s \to 0$), this weight is one.

Define the distorting martingale M_t in the following way: $M_0 = 1$ and

$$M_{t+1} = M_t \frac{p(z_{t+1}|z_t, s_t)}{p(z_{t+1}|z_t)}.$$

Some algebraic manipulation shows that

$$M_{t+1} = M_t \left(\frac{\sigma_z}{\sigma}\right) \exp\left(\frac{1}{2}\left[\frac{1}{\sigma_z^2} - \frac{1}{\sigma^2}\right] e_{t+1}^z - \frac{1}{2}\frac{b_t^2}{\sigma^2} + \frac{b_t}{\sigma^2} e_{t+1}^z\right),$$

where $b_t \equiv k(s_t - \rho_z z_t)$ and $\sigma \equiv (1-k)\sigma_z^2$. Using this martingale to define a subjective probability measure $\tilde{\mathbb{P}}$ as in Section 3, it follows that the evolution of fundamentals under both objective and subjective probabilities can be written in the form

$$\begin{aligned} z_t &= \rho_z z_{t-1} + e_t^z, \quad e_t^z \overset{\mathbb{P}}{\sim} \mathcal{N}(0, \sigma_z^2) \overset{\tilde{\mathbb{P}}}{\sim} \mathcal{N}(b_{t-1}, \sigma^2) \\ b_t &= \rho_b b_{t-1} + \tau e_{t+1}^z + e_t^b, \quad e_t^b \overset{\mathbb{P},\tilde{\mathbb{P}}}{\sim} \mathcal{N}(0, \sigma_b^2), \end{aligned}$$

with the restrictions $\sigma_b = k\sigma_s$, $\tau = k$, and $\rho_b = 0$. Note that there are two main differences relative to the dynamics of belief distortions in (5). First, there is a distortion to the conditional variance of z_t — by a constant factor $(1-k)$ — and second, the current innovation to z_t is (intertemporally) correlated with b_t. The reason for the first difference is that the signal provides additional information concerning future fundamentals, which reduces the subjective uncertainty of the agent in proportion to its precision. The reason for the second difference is that the signal is informative about future productivity; if the agent sees a high signal, he increases his conditional mean estimate of future productivity in proportion to the signal's precision.

C.2 Limited Information

In the previous subsection, the agent could observe current and past fundamentals, and received *additional* information about future fundamentals in the form of a noisy signal. In this subsection, I suppose instead that the agent has *less* information; he never observes the fundamental process and must base his forecasts exclusively on a sequence of noisy signals.

In each period, the agent observes a noisy signal of the current realization of z_t,

$$s_t = z_t + e_t^s, \quad e_t^s \overset{\mathbb{P}}{\sim} \mathcal{N}(0, \sigma_s^2).$$

He incorporates the information from this signal (and all past signals) to forecast future fundamentals. Let $\{\mathfrak{F}_t^s\}_{t\geq 0}$ denote the natural filtration generated by these signals. Note that $\mathfrak{F}_t^s \subseteq \mathfrak{F}_t$, with equality only in the limiting case that the signal is perfectly informative ($\sigma_s = 0$). His forecasting distribution is

$$z_{t+1} | \mathfrak{F}_t^s \overset{\mathbb{P}}{\sim} \mathcal{N}(\rho_z \mu_t, \sigma^2),$$

where $\mu_t = (1-k)\rho_z \mu_{t-1} + k s_t$, $k \equiv \sigma^2/(\sigma^2 + \sigma_s^2)$, and σ^2 is the solution to the equation[39]

$$\sigma^2 = \left(\frac{\sigma_s^2}{\sigma^2 + \sigma_s^2} \right) \rho_z^2 \sigma^2 + \sigma_z^2.$$

Define the distorting martingale M_t in the following way: $M_0 = 1$ and

$$M_{t+1} = M_t \frac{p(z_{t+1} | \mathfrak{F}_t^s)}{p(z_{t+1} | \mathfrak{F}_t)}.$$

[39]Here I assume for simplicity that prediction is taking place under "steady-state" learning.

As before, some algebraic manipulation shows that

$$M_{t+1} = M_t \left(\frac{\sigma_z}{\sigma}\right) \exp\left(\frac{1}{2}\left[\frac{1}{\sigma_z^2} - \frac{1}{\sigma^2}\right] e_{t+1}^z - \frac{1}{2}\frac{b_t^2}{\sigma^2} + \frac{b_t}{\sigma^2} e_{t+1}^z\right),$$

where $b_t \equiv \rho_z(\mu_t - z_t)$. Using this martingale to define a subjective probability measure $\tilde{\mathbb{P}}$ as in Section 3, it follows that the evolution of fundamentals under both objective and subjective probabilities can be written in the form

$$\begin{aligned} z_t &= \rho_z z_{t-1} + e_t^z, \quad e_t^z \overset{\mathbb{P}}{\sim} \mathcal{N}(0, \sigma_z^2) \overset{\tilde{\mathbb{P}}}{\sim} \mathcal{N}(b_{t-1}, \sigma^2) \\ b_t &= \rho_b b_{t-1} + \tau e_t^z + e_t^b, \quad e_t^b \overset{\mathbb{P},\tilde{\mathbb{P}}}{\sim} \mathcal{N}(0, \sigma_b^2), \end{aligned} \tag{50}$$

with the restrictions $\sigma_b = \rho_z k \sigma_s$, $\tau = -\rho_z(1-k)$, and $\rho_b = \rho_z(1-k)$. As before, the two main differences relative to the dynamics in (5) are that the conditional variance of z_t is distorted, and the current innovation to z_t is correlated with b_t.

There is one important caveat in this case, however. The martingale M_t captures distortions about *future* fundamentals, but not about current or past fundamentals. But, due to the hypothesis that the agent never observes the fundamental process, his beliefs about current and past fundamentals are distorted as well. That means, for example, that

$$E[z_{t-j}|\mathfrak{F}_t^s] \neq z_{t-j} \quad \text{for any } j \geq 0.$$

This presents a difficulty because it amounts to a violation of absolute continuity between subjective and objective beliefs. Because z_t is never observed, it is always viewed as a random variable by the agent. From the objective (full-information) perspective of the model, however, all randomness associated with z_t is resolved once date t is reached. For example, if the realization of z_t is z^*, the objective probability that z_t is not equal to z^* is zero for any $t \geq 0$, but the subjective probability is always non-zero.

Fortunately, in the case that only a finite number of lags of the fundamental process are relevant for the agent's decisions, it is still mechanically possible to proceed in the same spirit. For illustration, suppose it is necessary to determine the joint distribution of z_{t+1} and z_t. This is the most common case, but additional lags can be accommodated in the same way. Define the vector $x_t \equiv (z_t, z_{t-1})'$, and write the observation and state equations as

$$\begin{aligned} s_t &= A x_t + D e_t^s \\ x_t &= B x_{t-1} + C e_t^z, \end{aligned}$$

where $A \equiv (1,0)$, $D \equiv (1,0)'$, $C \equiv (1,0)'$, and

$$B \equiv \begin{bmatrix} \rho_z & 0 \\ 1 & 0 \end{bmatrix}.$$

The agent's forecasting distribution is

$$x_{t+1}|\mathfrak{F}_t^s \overset{\mathbb{P}}{\sim} \mathcal{N}(B\mu_t, \Sigma),$$

where $\mu_t = (I_2 - KA)B\mu_{t-1} + Ks_t$, $K \equiv \Sigma A'(A\Sigma A' + \sigma_s^2 DD')^{-1}$, and Σ is the solution to the discrete algebraic Ricatti equation

$$\Sigma = B[I - \Sigma A'(A\Sigma A' + \sigma_s^2 DD')^{-1}A]\Sigma B' + \sigma_z^2 CC'.$$

As before, it is possible to write the subjective dynamics of x_t "as if" it were observable to the agent but subject to stochastic distortions given by $b_t \equiv B(\mu_t - x_t)$. Specifically, the dynamic system is of the form[40]

$$\begin{aligned} x_t &= Bx_{t-1} + e_t^x, \quad e_t^x \overset{\mathbb{P}}{\sim} \mathcal{N}(0, \sigma_z^2 DD') \overset{\tilde{\mathbb{P}}}{\sim} \mathcal{N}(b_{t-1}, \Sigma) \\ b_t &= Rb_{t-1} + Te_t^x + e_t^b, \quad e_t^b \overset{\mathbb{P},\tilde{\mathbb{P}}}{\sim} \mathcal{N}(0, \Sigma_b), \end{aligned}$$

with the restrictions $\Sigma_b = BKD$, $T = -B(I - KA)$, and $R = B(I - KA)$. In this case, in addition to having distortions of the conditional covariance of fundamentals and correlation between fundamentals and the degree of distortions, the distortion process b_t is now a two-dimensional vector.

D An Optimal Policy Example

This section presents a simple example to illustrate how the optimal policy response to advance information and distorted beliefs about future fundamentals are different. Specifically, I show that policy is more accomodative to exogenous innovations reflecting advance information about future cost-push shocks than it is to innovations that reflect distorted beliefs. This is because advance information generates adverse cost-push effects both through agents' anticipation of future conditions, and through

[40]Whenever the notation $\mathcal{N}$ is used with a singular covariance matrix, what is meant is a "singular" normal distribution, in the sense of Anderson (1984).

subsequent realizations of those conditions. Distortions in agents' beliefs only engender a policy response through the first channel.

The example takes place in the context of the New Keynesian model presented in Adam and Woodford (2012). I will omit the details of the model and jump directly to the optimal monetary policy problem. The policymaker is charged with choosing maximizing the level of expected utility of a representative household,

$$E_0 \sum_{t=0}^{\infty} \beta^t U(Y_t, \Delta_t; \xi_t),$$

where Y_t is aggregate output, Δ_t is an index of price dispersion arising from staggered, monopolistic price-setting in product markets, and ξ_t is a vector of exogenous disturbances. His maximization is subject to the equilibrium conditions characterizing equilibrium in the private sector,

$$\begin{aligned} Z_t &= z(Y_t; \xi_t) + \alpha\beta \tilde{E}_t[\Phi(Z_{t+1})] \\ \Delta_t &= \tilde{h}(\Delta_{t-1}, K_t/F_t), \end{aligned}$$

together with a pre-commitment constraint necessary to deliver a time-invariant solution. Here the notation exactly follows Adam and Woodford (2012), with the exception that the probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are explicitly labeled. As in their paper, the objective is a paternalistic one. The policymaker understands that private sector beliefs may be different from his own, but still operates under the assumption that those beliefs are nevertheless computed according to a well-defined probability measure.

Let $\tilde{M}_t$ denote a distorting martingale with unit expectation under $\tilde{\mathbb{P}}$ that governs the change of measure from $\mathbb{P}$ to $\tilde{\mathbb{P}}$. That is, for any $\mathcal{A} \in \mathfrak{F}_t$, the following relation holds:

$$\mathbb{P}(\mathcal{A}) = \tilde{E}[1_{\mathcal{A}} \tilde{M}_t]$$

Using this martingale process, the Lagrangian for the policy problem can be written under the distorted probability measure:

$$\tilde{E}_0 \sum_{t=0}^{\infty} \beta^t \tilde{M}_t \Big\{ U(Y_t, \Delta_t; \xi_t) + \gamma_t [\tilde{h}(\Delta_{t-1}, K_t/F_t) - \Delta_t] + \Gamma_t' [z(Y_t; \xi_t) + \alpha\beta \Phi(Z_{t+1}) - Z_t] \Big\} \\ + \alpha \Gamma_{-1}' \Phi(Z_0),$$

The associated first order conditions are:

$$U_t(Y_t, \Delta_t; \xi_t) + \Gamma_t' z_Y(Y_t; \xi_t) = 0 \tag{51}$$

$$-\gamma_t \tilde{h}_2(\Delta_{t-1}, K_t/F_t)\frac{K_t}{F_t^2} - \Gamma_{1t} + \alpha\Gamma_{t-1}' D_1(K_t/F_t) = 0 \tag{52}$$

$$\gamma_t \tilde{h}_2(\Delta_{t-1}, K_t/F_t)\frac{1}{F_t} - \Gamma_{2t} + \alpha\Gamma_{t-1}' D_2(K_t/F_t) = 0 \tag{53}$$

$$U_\Delta(Y_t, \Delta_t; \xi_t) - \gamma_t + \beta\tilde{E}_t[(\tilde{M}_{t+1}/\tilde{M}_t)\gamma_{t+1}\tilde{h}_1(\Delta_t, K_{t+1}/F_{t+1})] = 0 \tag{54}$$

Computing a log-linear approximation of these conditions around the deterministic steady-state (still under $\tilde{\mathbb{P}}$), optimal policy ensures that the following system of equations in inflation (π_t) and the output gap (x_t) is always satisfied:

$$\begin{aligned} \pi_t &= \kappa x_t + \beta\tilde{E}_t[\pi_{t+1}] + u_t \\ \xi_\pi \pi_t &+ \lambda_x(x_t - x_{t-1}) = 0. \end{aligned} \tag{55}$$

In the first equation, u_t is a composite cost-push disturbance defined as a function of the exogenous fundamentals in the economy. The parameters $\kappa, \xi_\pi, \lambda_x > 0$ are reduced-form parameters that are functions of the underlying deep parameters of the model. The only differences in this example from Adam and Woodford (2012) are that the log-linear approximation has been taken under $\tilde{\mathbb{P}}$ rather than $\mathbb{P}$, and the policymaker does not have a concern for robustness against the possibility that private sector beliefs can change in ways that represent *worst-case* outcomes from the perspective of his policy objective.

Suppose that the exogenous process u_t follows dynamics like those in system (5), but for the purspose of this example, without any persistence:

$$\begin{aligned} u_t &= a_{t-1} + e_t^u, \quad e_t^u \overset{\mathbb{P}}{\sim} \mathcal{N}(0, \sigma_u^2) \overset{\tilde{\mathbb{P}}}{\sim} \mathcal{N}(b_{t-1}, \sigma_u^2) \\ a_t &= e_t^a, \quad e_t^a \overset{\mathbb{P},\tilde{\mathbb{P}}}{\sim} \mathcal{N}(0, \sigma_a^2) \\ b_t &= e_t^b, \quad e_t^b \overset{\mathbb{P},\tilde{\mathbb{P}}}{\sim} \mathcal{N}(0, \sigma_b^2) \end{aligned}$$

Solving system (55) using these dynamics for u_t under $\tilde{\mathbb{P}}$, the optimal evolution of inflation and the output gap are

$$\begin{aligned} \pi_t &= \gamma u_t + \beta\gamma^2(a_t + b_t) + \frac{\lambda_x}{\xi_\pi}(1-\gamma)x_{t-1} \\ x_t &= -\frac{\xi_\pi}{\lambda_x}\gamma u_t - \beta\frac{\xi_\pi}{\lambda_x}\gamma^2(a_t + b_t) + \gamma x_{t-1}, \end{aligned} \tag{56}$$

where $0, \gamma < 1$ is the smaller of the two real roots of

$$\gamma^2 - \left(1 + \frac{1}{\beta} + \frac{\kappa \xi_\pi}{\lambda_x \beta}\right) \gamma + \frac{1}{\beta} = 0.$$

Let $R_t[\pi_{t+\tau} | e_t = e^*] \equiv E_{t-1}[\pi_{t+\tau} | e_t = e^*] - E_{t-1}[\pi_{t+\tau}]$ denote the impulse response function of inflation in period $\tau \geq 0$ to an innovation $e_t = e^*$ in period t. Now consider the difference between the response of inflation to an exogenous unit innovation in e_t^a and e_t^b. The relations in (56) and the exogenous dynamics of u_t imply that

$$R_t[\pi_{t+\tau} | e_t^a = 1] - R_t[\pi_{t+\tau} | e_t^b = 1] = \gamma 1_{\{\tau=0\}} + \gamma^\tau \frac{\lambda_x}{\xi_\pi}(1 - \gamma) 1_{\{\tau>0\}}.$$

Because $\lambda, \xi_\pi > 0$ and $0 < \gamma < 1$, this difference is strictly positive for all horizons $\tau > 0$. This shows that it is optimal for the policymaker to allow inflation to respond more to innovations that reflect advance information than to innovations that reflect distortions in agents' beliefs. The same logic shows that the output gap x_t exhibits a greater negative response to advance information compared to distorted beliefs.