Commercial Revolutions, Search, and Development

Maurizio Iacopetta*
OFCE (Sciences-Po) and Skema Business School

Abstract

A society can reap more benefits generated by an improvement in communication and transportation when the gap between privileged and less-privileged individuals is small. This hypothesis is investigated in a Kiyotaki-Wright search environment where people enjoy different rental positions. Historical evidence about the rise of Italian seaport cities during the Middle Ages, of the Netherlands in the 17th century, and about the French nobility's resistance to trade during the pre-revolutionary period is used to support the hypothesis.

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JEL codes: O11, N13, C61, C63, D63.

1 Introduction

It has been argued that when an economy stagnates it is often because the most capable people choose to be occupied in rent-seeking activities over alternatives that would improve the society's welfare (Krueger, 1974, Baumol, 1990, 1993, 2002, and Murphy, Schliefer and Vishny, 1991, 1993). Murphy, Shleifer and Vishny (1991) (henceforth MSV) suggest that good communications and transportation that facilitate trade can direct people towards more socially useful occupations (see their Table 1, p. 519).

While this suggestion is sounds, it lacks a formal analysis of how and under what conditions an improvement of contacts among individuals can induce some to devote more effort into socially useful activities. This paper provides such an analysis using a variant of the Kiyotaki and Wright (1989) (KW henceforth) search model. There are two aspects of the KW environment that are useful for evaluating MSV's conjecture. First, since exchanges occur in decentralized markets, it is possible to be explicit about shocks that affect the intensity of

*Correspondence to: maurizio.iacopetta@sciencespo.fr. I would like to thank Nobuyuki Hanaki for helpful suggestions. All errors are mine.

communication among individuals, as this is measured by a matching rate. Second, because the population is heterogenous with respect to rental income, it is possible to evaluate how the society's stratification affects its ability to reap the benefits of a communication shock.

That a large gap between privileged and unprivileged individuals can slow down economic progress has been widely discussed in the literature. Landes (1998) argues, among others, that people in Europe did not show the same dynamism as the settlers in North America because the presence of a rich landed aristocracy in Europe abetted inaction. France, where the stratification of society was particularly pronounced before the Revolution (Wickham, 2009, pp. 145-146), missed the chance of rivaling the Genoese and the Venetians during the medieval trade revolution, and watched with preoccupation the formidable rise of the Dutch maritime power in the 17th century.

Conversely, in America "Equality bred self-esteem, ambition, a readiness to enter and compete in the marketplace, a spirit of individualism and contentiousness" (Landes, 1998, p. 297). Similarly, in his seminal summary of the medieval European commercial revolution, Lopez recounts:

> The business fever, when it came, left almost no one untouched. Perhaps the most telling change occurred among those noble families that grew too large to live comfortably off their inherited land. Outside Italy the supernumerary children tried to escape poverty and boredom through an ecclesiastic career, marriage to a noble heiress, or military service at somebody's hire. In the Italian towns they more often found the same opportunities and thrills by pooling their capital in business ventures, which involved a chance of pick up along the way a fight with pirates, brigands, unfriendly lords, and possibly Infidels. ...Indeed it is usually difficult, in the scanty documents of the tenth and eleventh centuries, to tell apart merchants who had bought real estates with the profits of trade, and were called "honorable" or "noble", from the noblemen who had sold their estate, invested the proceed in trade, and married merchants' daughters. (1976, p. 66)

To formalize these arguments, I consider a KW economy consisting of three groups of people with different privileges, measured by their flows of rents. An individual's only choice is whether or not to engage in (indirect) trade. For people at the bottom of the ladder of privileges, the commoners, trading is always the best option. Conversely, for those at the top, that will be identified with the higher nobility, it never is. The choice of individuals in the middle can, however, go either way. They abandon the existing rent-seeking practices if the lure of commercial activities is strong enough to compensate them for giving up the comfort of receiving a generous regular rent. When the middle group chooses to trade, a new trade link is created and all groups trade at higher frequencies. The consequences of this network externality effect are illustrated by a quantitative experiment that captures the main stylized facts of a commercial revolution.

Despite the policy debate spurred by Baumol (1990) and Murphy et al. (1991) on how to reduce unproductive occupations, formal analysis of why trade could be an effective instrument in pursuing such an objective is lacking. A notable exception is Holmes et al. (2001). In a growth model with vertical innovation, they evaluate the effects of regulatory entry restrictions imposed on out-of-state firms. Their analysis and the type of shock they consider are, nevertheless, different than mine. I focus on a pre-industrial economy, where sudden booms in income are typically driven by trade (Lopez, 1976, p. 85-91). As for the communication shock, I increase the matching rate; they instead reduce the level of tariffs.

Another related work is Acemoglu (1995) who used evidence about the medieval commercial revolution to argue that the reward system is not exogenous, as, for instance, in Baumol (1990), but adjusts over time as people move from unproductive to productive occupations. A similar result emerges in my analysis: When the middle group switches to trade the ratio of rental over production income declines for all individuals.

This paper contributes also to the literature that sees growth spurts as the result of qualitative changes in the structure of the economy. Peretto (2015) highlights that an important condition for the onset of modern growth is market expansion. Likewise, Kelly (1997) points out that market expansion was the main driver of the medieval Chinese economic growth. In this paper, a larger market, generated by an acceleration of the matching rate, leads, as in Kelly, to a temporary growth spurt – in the long run the economy converges to a higher but constant level of production.

The rest of the paper is organized as follows. The next section briefly describes the economic environment, illustrates the evolution of the distribution of inventories under a given profile of strategies, and defines the best response functions of three types of representative agents. The section that follows studies the dynamics of the system. Section (4) proposes two quantitative experiments to illustrate the propagation mechanism of a shock to the matching rate. Section (5) discusses key stylized facts of the medieval and Atlantic trade revolutions and comments on the resistance of the French aristocracy to trade. Section (6) deals with multiple steady states. Section (7) concludes.

2 The Model Economy

There are only minor differences with respect to the decentralized economy described in KW: time is continuous; the ranking on the returns across assets is allowed to change; and agents are not necessarily equally distributed across types. In the brief description that follows, I emphasize aspects of the model that are useful to give an historical interpretation to the quantitative experiments detailed in section (6).[1] The overall size of the population is N.

[1] See also Lagos et al. (2015), section two, for a summary of the model.

There are three types of infinitely-lived agents, measured by N_i, for $i = 1, 2, 3$. The fraction of each type is $\mu_i = N_i/N$. Type i agent [2] consumes only good i and can produce only good $i + 1$ (mod. 3). Production takes place immediately after consumption. Agent i's instantaneous utility from consumption and the disutility of producing good $i+1$ (the terms good, commodity, inventory, and asset will be used interchangeably) are denoted by U_i and D_i, respectively, and their difference is $u_i = U_i - D_i$. Without loss of generality, I assume that $D_i = 0$ and $U_i = 1$, for all i. There is a capacity constraint: at each instant of time, an individual can hold only one unit of asset i that offers a return r_i, measured in units of utility. The discount rate is denoted by $\rho > 0$. A pair of agents is randomly and uniformly chosen from the population to meet for a possible trade. After a pair is formed, the waiting time for the next pair to be called is governed by a Poisson process with intensity α. A bilateral trade occurs if, and only if, it is mutually agreeable. Agent i always accepts good i but never holds it because there is immediate consumption. Therefore, agent i always enters the market either with one unit of good $i + 1$, or with one unit of good $i + 2$.

The proportion of type i agents that hold good j at time t is denoted by $p_{i,j}(t)$. Then, the vector $\tilde{\mathbf{p}}(t) = \{p_{i,j}(t)\}$ for $i = 1, 2, 3$ and $j = 1, 2, 3$ describes the state of the economy at time t (from now on, it is understood that i and j go from 1 to 3). But since $p_{i,i}(t) = 0$,

$$p_{i,i+1}(t) + p_{i,i+2}(t) = \mu_i. \tag{1}$$

for any $t > 0$, the state of the economy can be represented in a more parsimonious way by $\mathbf{p}(t) = \{p_{1,2}(t), p_{2,3}(t), p_{3,1}(t)\}$. To simplify the notation, sometimes $p_{i,i+1}$becomes p_i. An individual i has only to decide whether to exchange his production good for the other type of good. Agent i's choice in favor of indirect trade is denoted with $\tau_i(t) = 1$ and against it with $\tau_i(t) = 0$. Agent i has to select a time path $\tau_i(t)$ that maximizes her expected stream of present and future net utility, given other agents' paths of strategies, $\boldsymbol{\theta}(t) = [\theta_1(t), \theta_2(t), \theta_3(t)]$, and $\mathbf{p}(t)$, for any $t > 0$.

2.1 Distribution of Assets and Value Functions

For a given profile of strategies $\boldsymbol{\theta}(t)$, the evolution in the stock of good $i + 1$ held by agents of type i is given by[3]

$$\dot{p}_{i,i+1} = \alpha\{p_{i,i+2}[p_{i+1,i}(1 - \theta_{i+1}) + p_{i+2,i} + p_{i+2,i+1}(1 - \theta_i)] - p_{i,i+1}[p_{i+1,i+2}\theta_i]\}. \tag{2}$$

The terms inside the brackets before the minus sign calculate the probability that a type i agent is called for a match while holding good $i + 2$, and ends up in the position of carrying

[2] When no confusion arises, I will use the loose language of calling an agent of type i simply as agent or individual i.

[3] Duffie and Sun (2012) show that in a similar matching environment frequency coincides with probability. See in particular their Theorem 1.

good $i+1$. Such an event materializes either because of barter or because the agent leaves the meeting with good i, consumes it, and then immediately produces good $i+1$. The following expression accounts for the probability that an agent of type i who holds good $i+1$ ends up with good $i+2$. The behavior of $p_{i,i+2}$ is derived through (1). The ensemble of the system that describes the evolution of the inventories' distribution is denoted by $F(\mathbf{p}(t))$.

Consider now a representative agent of type i that has to compute her best profile of strategies, given a pattern of inventories $\mathbf{p}(t)$ and a pattern of strategies for other agents $\boldsymbol{\theta}(t)$ – including those of her own type. Let $V_{i,j}(t)$ be the value function when carrying good j at time t. When $j = i+1$, we have that

$$\begin{aligned} V_{i,i+1}(t) = \max_{\{\tau_i(s)\}_{s\geq t}} \int_t^{\infty} & \alpha e^{-\alpha(s-t)} \{ e^{-r(s-t)} ([p_{i,i+2}\tau_i(1-\theta_i) + p_{i+1,i+2}\tau_i] V_{i,i+2} + \\ & + [1 - p_{i,i+2}\tau_i(1-\theta_i) - p_{i+1,i+2}\tau_i] V_{i,i+1} + [p_{i+1,i} + p_{i+2,i}\theta_{i+2}] u_i) + \\ & + \frac{1 - e^{-(s-t)r}}{r} r_{i+1} \} ds, \end{aligned} \tag{3}$$

where the term $\alpha e^{-\alpha(s-t)} ds$ measures the probability that an agent of type i is called to form a match for the first time after time t in the time interval $(s, s+ds)$. The term after the discount factor is the probability that this agent goes through indirect trading – in which case, she is left with $V_{i,i+2}$ as continuation value. Else, she ends up with good $i+1$ either because no trade takes place or because she acquired her consumption good – an event that occurs with probability $p_{i+1,i} + p_{i+2,i}\theta_{i+2}$ – and then produces good $i+1$. The last term is the return for holding the asset $i+1$ from time t to time s. An additional equation for $V_{i,i+2}(t)$, reported in the Appendix, completes the description of the optimization problem.

Let $\Delta_i(s) \equiv V_{i,i+1}(s) - V_{i,i+2}(s)$, and let $\tilde{\tau}_i(s; \boldsymbol{\theta}(s), \mathbf{p}(s))$ denote the optimal (or best) response profile of strategies of representative agent i to other players' strategies $\boldsymbol{\theta}(s)$ along the pattern of inventories $\mathbf{p}(s)$ for $s > t$. Then, it must be that

$$\tilde{\tau}_i(s; \boldsymbol{\theta}(s), \mathbf{p}(s)) = \begin{cases} 1 & \text{if } \Delta_i(s) < 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

for any $s \geq t$. Hence, the way the problem has been formulated corresponds to a Markov decision process in which the representative agent optimizes over a sequence of functions $\tilde{\tau}_i(t)$ that allows the ex-post decision on indirect trading to vary with the current state of the inventory distribution, and the pattern of strategies of other agents.

2.2 Definition of Dynamic Nash Equilibrium

Given an initial distribution of inventories $\mathbf{p}(0) = \mathbf{p}_0$, a *Dynamic Nash Equilibrium (DNE)* is a path of strategies $\boldsymbol{\theta}^*(t)$ together with a distribution of inventories $\mathbf{p}^*(t)$ such that for all

$t > 0$:

i. $\mathbf{p}^*(t)$ and $\boldsymbol{\theta}^*(t)$ satisfy the dynamics equations (2) with the initial condition $\mathbf{p}^*(0) = \mathbf{p}_0$, and subject to the constraint (1);

ii. For all $t > 0$, every agent maximizes his or her expected utility given the profile strategies of the rest of the population;

iii $\tilde{\tau}_i(t; \boldsymbol{\theta}^*(t), \mathbf{p}^*(t)) = \theta_i^*(t)$ for all $t > 0$.

2.3 Rent Seeking and Production

Following Lagos et al. (2016) I interpret a good as a Lucas tree producing a fruit. Let R_i and Q_i be the rental income and the level of production of the ensemble of type i agents. It is easy to show that

$$R_i = (r_{i+1} - r_{i+2})p_{i,i+1} + r_{i+2}\mu_i, \tag{5}$$

and that

$$Q_i = \alpha\{p_{i,i+1}(p_{i+1,i} + p_{i+2,i}\theta_{i+2}) + p_{i,i+2}[p_{i+1,i}(1 - \theta_{i+1}) + p_{i+2,i}]\}. \tag{6}$$

Let

$$Y_i = Q_i + R_i$$

be the income of type i agents. Therefore, the aggregate level or rent, production, and income are $R = \sum_1^3 R_i$, $Q = \sum_1^3 Q_i$, and $Y = \sum_1^3 Y_i$, respectively. The rental ratio, r_i, is defined as

$$\tilde{r}_i \equiv \frac{R_i}{Y_i} = \frac{1}{1 + Q_i/R_i}$$

Hence the rental ratio for the overall economy is

$$\tilde{r} \equiv \frac{R}{Y} = \frac{1}{1 + Q/R}. \tag{7}$$

A first insight of equation (6) is that the flow of production of group i, Q_i, depends on the trade strategies of the other two groups of agents, θ_{i+1} and θ_{i+2}. A second insight is that the set of strategies of the three types of agents affects Q_i through the distribution of assets $\mathbf{p}$. Section (2.5) characterizes analytically the dependence of production from the strategies for some interesting equilibria.

Finally, it is also useful to define the frequency of trade, $t_i(s)$. This measures the number of times a good is traded in a unit of time. It can be shown that

Table 1: Steady State Equilibria and Strategies

Returns	F	S	Assets (F)	Assets (S)
$r_3 < r_2 < r_1$	(0,1,0)	(1,1,0)	$\frac{1}{3}[1, \frac{1}{2}, 1]$	$\frac{1}{3}[a, b, 1]$

- Note: The F and S columns contain the triplet $(\theta_1, \theta_{2,}\theta_3)$ that describes the fundamental (F) and the speculative (S) steady state strategy, respectively. The two columns that follow are the assets' stationary distributions $[p_{1,2}, p_{2,3}, p_{3,1}]$, with $a = \frac{1}{2}\sqrt{2}$ and $b = \sqrt{2} - 1$, in the two equilibria.

$$t_i(s) = \alpha\{p_{i+2}(\mu_i - p_i(1 - \theta_{i+2}) + \theta_{i+1}p_{i+1}) + (\mu_{i+1} - p_{i+1})[p_i+ \\ (\mu_i - p_i)(1 - \theta_{i+1}) + (\mu_{i+2} - p_{i+2})(1 - \theta_{i+2})]\}.$$

Then the average trade frequency is

$$\bar{t}(s) = x_1(s)t_1(s) + x_2(s)t_2(s) + x_3(s)t_3(s)$$

where $x_i(s) = \sum_{j=1}^{3} p_{j,i}$, is the outstanding stock of type i good.

2.4 Privileges

I turn now to the description of economic privileges in an environment similar to Model A of KW where $r_3 < r_2 < r_1$. Because $u_i = 1$ for $i = 1, 2, 3$, the order of privileges can be defined through the returns r_i. An agent i benefits of two types of advantages when his production good fetches a higher return, r_{i+1}: he receives a greater flow of dividends while waiting for a successful trade; he can market more easily his production good, as this becomes more attractive for the buyer. Clearly, the most privileged individuals are type 3 agents, as they produce the good that yields the highest rent. Next come type 1 agents. Type 2 agents, who produce the good with the lowest return, are at the bottom. A variety of circumstances lead to the creation of rents (Baumol (1990) and MSV). Here, however, I am concerned with their consequences on production. Baumol remarks that: "If the rules are such as to impede the earning of much wealth via activity A, or are such as to impose social disgrace on those who engage in it, then, other things being equal, entrepreneurs' efforts will tend to be channeled to other activities, call them B. But if B contributes less to production or welfare than A, the consequences for society may be considerable." (1990, p. 898). Next I characterize this statement in the KW environment.

2.5 Steady States

There are six possible return rankings. When $r_3 < r_2 < r_1$ the economy has a fundamental steady state equilibrium, that is $\boldsymbol{\theta} = [0, 1, 0]$ and $\mathbf{p} = \frac{1}{3}[1, 0.5, 1]$, if $\frac{r_2 - r_3}{\alpha} > \frac{1}{6}$. It has a speculative steady state, that is $\boldsymbol{\theta} = [1, 1, 0]$ and $\mathbf{p} = \frac{1}{3}[a, b, 1]$ where $a = \frac{1}{2}\sqrt{2}$ and $b = \sqrt{2} - 1$ if $\frac{r_2 - r_3}{\alpha} < \frac{1}{3}(\sqrt{2} - 1)$. I will refer to these two equilibria as (0,1,0) and (1,1,0) equilibrium, respectively, for short. The return ranking $r_3 < r_2 < r_1$ corresponds to Model A in KW. Two other rankings, $r_2 < r_1 < r_3$ and $r_1 < r_3 < r_2$, are just relabeling of the same economy (see columns R2 and R3 of Table (8) in the Appendix). The three remaining rankings are more than relabeling exercises, as they can give rise to multiple equilibria. They will be discussed in Section (6).

Returning now to the baseline case $r_3 < r_2 < r_1$, notice that the level of production is different in the two equilibria (0,1,0) and (1,1,0). Let Q_i^J and Q^J be the steady state equilibrium quantities of agents i and of the overall economy, respectively, when the equilibrium is fundamental ($J = F$) or speculative ($J = S$). From Eq. (6) it follows that $Q_i^F = \frac{1}{18}\alpha$ for $i = 1, 2, 3$ and $Q^F = \frac{1}{6}\alpha$, and that $Q_1^S = \alpha\frac{2}{18}(2 - \sqrt{2})$, and $Q_2^S = Q_3 = \alpha\frac{\sqrt{2}}{18}$. Clearly, each group of agents produces more on the speculative than on the fundamental equilibrium. At an aggregate level the relationship between the two equilibria is $Q^S = \frac{4}{3}Q^F$.

Conversely, the level of rent is higher in the fundamental equilibrium. Let R_i^J be flow of rent earned by the group of agents of type i in the steady state J, for $J = \{F, S\}$. One can verify that $R_1^F = \frac{1}{3}r_2$, $R_2^F = \frac{1}{3}\frac{1}{2}(r_3 + r_1)$ and $R_3^F = \frac{1}{3}r_1$ and that

$$
\begin{aligned}
R_1^S &= \frac{1}{3}[ar_2 + (1 - a)r_3] \\
R_2^S &= \frac{1}{3}[br_3 + (1 - b)r_1] \\
R_3^S &= \frac{1}{3}r_1
\end{aligned}
$$

Hence $R_3^F = R_3^S$, but $R_1^S > R_1^F$ and $R_2^S > R_2^F$. This result and the previous one that established a higher production in the speculative equilibrium, imply that the rental ratio $\tilde{r}$ defined in (7) is higher in the fundamental than in the speculative equilibrium.

What are the conditions that favor the emergence of a speculative equilibrium? As noticed, for such an equilibrium to exist $\frac{r_2 - r_3}{\alpha} < \frac{1}{3}(\sqrt{2} - 1)$, meaning that the gap between the rents generated by the goods produced by the middle and bottom group (the lower nobility and the bourgeoisie) must be smaller than the threshold $\frac{\alpha}{3}(\sqrt{2} - 1)$. Hence, policies that level the playing fields can favor the emergence of a speculative equilibrium. But the condition also suggests that a change in matching rate can achieve the same result.

Proposition 1 *Let α^H and α^L be the matching rate of the α^H-economy and the α^L-economy* *with $\alpha^H > \alpha^L$. Assume that both economies have the same r_i with $r_3 < r_2 < r_1$ and that*

$$\frac{1}{6} < \frac{r_2 - r_3}{\alpha^L} \tag{8}$$

and

$$\frac{r_2 - r_3}{\alpha^H} < \frac{1}{3}(\sqrt{2} - 1) \tag{9}$$

Then the α^L- and α^H-economy are characterized by a fundamental and a speculative steady state equilibrium.

Proof. See Technical Appendix. ■

Intuitively, condition (8) says that type 1 agents adopt fundamental strategies when they enjoy a high rent from their own production good, or when the state of communication is poor (low α). Conversely, in an economy where type 1 agents' privileges are modest or where it is relatively easy to meet potential trade partners, type 1 agents choose speculative strategies.

Next, I turn to the dynamics of the model and then provide an illustration of the adjustment process of an economy hit by a positive shock to the matching rate.

3 Dynamics

KW provide a characterization of the steady state equilibrium properties of their model, but do not attempt to describe the process by which the economy converges to a steady state equilibrium, nor have given an account of how a new trading strategy may emerge along the transition. Several researchers have contributed to fill the gap along different methodological approaches. Marimon et al. (1990) and Başçı (1999), ask if artificially-intelligent agents can learn to play equilibrium strategies. A similar question is tested in a number of controlled laboratory experiments with real people (Brown (1996) and Duffy and Ochs (1999)). Matsuyama et al. (1993), Wright (1995), Luo (1999) and Sethi (1999) approach the issue through evolutionary dynamics.

The dynamics of this class of models pose difficulties not encountered in economies where individuals trade their goods and services in centralized markets, because the evolution of the distribution of goods across individuals depends on the history of exchanges, that is, on agents' trading strategies. These, in turn, are affected by the expected evolution of the distribution of goods.

I obtain the dynamics building directly on the concept of open-loop Nash equilibrium with many players as in Fudenberg and Levine (1988). The idea is to find an equilibrium such that, given the actions of all other players, no player can make any gain by changing her action. The analysis of this section is does not require any specific ordering of the rents.

The law of motion of $\mathbf{p}(t)$ is a function of the profile of strategies $\boldsymbol{\theta}(t)$ that can change over time in discrete steps. Specifically, Eq. (2) says that the system of the distribution of assets is given by (the time index is dropped):

$$\dot{p}_{1,2} = \alpha\{p_{1,3}[p_{2,1}(1-\theta_2) + p_{3,1} + p_{3,2}(1-\theta_1)] - p_{1,2}p_{2,3}\theta_1\}, \tag{10}$$

$$\dot{p}_{2,3} = \alpha\{p_{2,1}[p_{3,2}(1-\theta_3) + p_{1,2} + p_{1,3}(1-\theta_2)] - p_{2,3}p_{3,1}\theta_2\}, \tag{11}$$

$$\dot{p}_{3,1} = \alpha\{p_{3,2}[p_{1,3}(1-\theta_1) + p_{2,3} + p_{2,1}(1-\theta_3)] - p_{3,1}p_{1,2}\theta_3\}. \tag{12}$$

The evolution of the system, can be followed for any arbitrary pattern of strategies $\boldsymbol{\theta}(t)$ and for any set of parameters. Nevertheless, to maintain analytical tractability, I restrict the attention to a time constant set of strategy. In section () I will relax the time-constant restriction through a numerical algorithm that can compute any Nash switch along the dynamics.

Proposition 2 *Except for the set of strategy (1,1,1), under any other time-constant pure strategies (τ_1,τ_2,τ_3), with $\tau_i = \{0,1\}$,for $i = 1,2,3$, $\mathbf{p}(t)$ converges to a stationary distribution, from any initial position.*

Proof. See Appendix

This proposition guarantees global convergence of the system for seven out of eight possible set of strategies. These, however, are not necessarily Nash strategies. The following proposition specifies the paths that lead to a Nash steady state equilibrium.

Proposition 3 *When the population is equally split across the three types, the following six steady state Nash Equilibria (NE) exist:*

Strategies	Assets Distribution	Strategies	Assets Distribution
(0,1,0)	$\frac{1}{3}[1, \frac{1}{2}, 1]$	(1,1,0)	$\frac{1}{3}[a, b, 1]$
(1,0,0)	$\frac{1}{3}[\frac{1}{2}, 1, 1]$	(1,0,1)	$\frac{1}{3}[b, 1, a]$
(0,0,1)	$\frac{1}{3}[1, 1, \frac{1}{2}]$	(0,1,1)	$\frac{1}{3}[1, a, b]$

where $a = \frac{1}{2}\sqrt{2}$ and $b = \sqrt{2} - 1$. In some cases a pair of NE coexist.

Proof. To prove that a given steady state distribution is a NE, one needs to verify that the sign of Δ_i is consistent with the profile of strategies assumed for that particular steady state distribution. See Technical Appendix for details.

Notice that the (1,1,1) set of strategies for which it is more difficult to establish analytically the convergence of the distribution $\mathbf{p}(t)$ does not support a steady state Nash equilibrium. When the economy is outside of the steady state the same principle of verifying the consistency of Δ_i with respect to τ_i applies. Nevertheless, obtaining an analytical solution is rather challenging, for it depends on the overall forecasted evolution of the asset distribution. Instead, I solve it through a numerical algorithm. Next I briefly illustrate the working of the algorithm. The appendix contains a more technical description. The algorithm is quite general, for it allows also switch of strategies along the dynamics.

The value functions $V_{i,j}$ are the criteria that agents follow to decide their optimal patterns of strategies. On a steady state $V_{i,j}$ can be determined analytically. To obtain their values also outside of the steady state, I exploit two properties of the system. First, for any interesting profile of strategies, the state variable, $\mathbf{p}(t)$, converges towards a fixed point (which is not necessarily an Nash equilibrium). Second, along a given pattern of $\mathbf{p}(t)$, the numerical value functions converge to their theoretical values when integrated backward in time. It is then possible to verify whether the value function of a representative agent along a specific trajectory $\mathbf{p}(t)$ is consistent with the profile of strategies that are used to obtain such a trajectory $\mathbf{p}(t)$ – that is no agent has an incentive to deviate at any point in time from the designated profile of strategies. The algorithm generates a sequence of rounds that seeks convergence towards an open-loop Nash equilibrium for $\mathbf{p}(t)$ and $\boldsymbol{\theta}(t)$. To prove that a given steady state distribution is a NE, one needs to verify that the sign of Δ_i is consistent with the profile of strategies assumed for that particular steady state distribution. The section continues by describing the convergence of the system to a stationary distribution. The Appendix provides further details on the design of the algorithm and the technique to verify the guess about the agents' trading strategies.

4 Quantitative Experiments

I now turn to a quantitative illustration of the central hypothesis of this paper: that the arrival of a communication shock can fuel an economic boom by inducing type 1 individuals, who are in the middle on the ladder of privileges, to switch from fundamental to speculative strategies.

4.1 The Commercial Revolution

Consider an economy characterized by a set of parameters (see Table 2) that satisfies the condition for a fundamental steady state equilibrium (0,1,0), that is, $\frac{r_2 - r_3}{\alpha} > \frac{1}{6}$. The economy is in such an equilibrium when it is hit by a communication shock. The magnitude of the shock is large enough that with the new level of the matching rate, α', the condition for a speculative steady state is verified, that is, $\frac{r_2 - r_3}{\alpha'} < \frac{1}{3}(\sqrt{2}-1)$. This shock captures, for instance, the initiatives of the Italian seaport cities that broke the trade freeze in the Mediterranean, or the Colombian voyages.

After the shock the three types of agents adopt the (1,1,0) constant-profile of Nash strategies as the economy converges to the speculative steady state $\mathbf{p} = \frac{1}{3}[a, b, 1]$ with $a = \frac{1}{2}\sqrt{2}$ and $b = \sqrt{2}-1$. The convergence is guaranteed by proposition (2). Fig. (1) depicts the evolution of the distribution of assets and that of trade frequencies for the three goods triggered by a 50% increase in α. Type 1 individuals, by switching from $\tau_2 = 0$ to $\tau_2 = 1$, that is, by abandoning the rent-seeking behavior to embracing trade, fuel an aggregate boom in trade and production for all goods. Specifically, the frequency of trade more than doubles for good 2 and good 3 and it increases by about 40% for good one. Each good exhibits a peculiar adjustment process. After an initial trade boom, the frequency of trade of good 1 increases even further, that of good 2 declines, and that of good 3 has a hump-shaped behavior. But all goods are traded more frequently than in the fundamental steady state equilibrium.

Fig. (2) shows the implications of the same shock (a 50% increase in α) on agents' income and on the rent-income ratios. In the short run type 2 individuals (commoners) benefit the most favored by the sudden increase of the market for their production good 3. A second important consequence of the shock is the convergence of income between the middle and bottom group, that is type 1 and type 2 individuals. After the shock the adjustment is so large that type 2 individuals (commoners) overtake type 1 individuals – the lower nobility, now turned into commercial nobility. As noted in the introduction historical evidence suggests that in places where the commercial revolution occurred, the dividing line between merchants and nobles was quite blurred.

A second effect of the communication shock, already emphasized in Proposition 1, is a reduction of the fraction of income accounted for by rents. Said it differently, the relative importance of rent-seeking is diminished as a result of the communication shock – the MSV's conjecture discussed in the introduction. After an initial drop across the three types, the rental ratio $\tilde{r}_i$, keeps declining for type 1 and type 3 individuals, but increases for type 3 individuals, because the trade frequency of their consumption good goes down (see Panel B, Fig. (1)). The long run effects of the shock on Y_i, Y, $\tilde{r}_i$, and $\tilde{r}$, are summarized in Tables (3) and (4)

4.2 Missing the Commercial Revolution

The quoting passage of Lopez reported in the introduction suggests that outside Italy societies have been less eager to reap the gains of the 13th century commercial revolution. One

Table 2: Parameters

	Population	**Discount**	**Matching**	**Utility**	**Returns**		
	μ_i	δ	α	u_i	r_1	r_2	r_3
Baseline	$\frac{1}{3}$	0.03	1	1	0.21	0.2	0
Higher Privileges	$\frac{1}{3}$	0.03	1	1	0.31	0.3	0

Table 3: Income Changes – Long Run

	Lower Nobility	Commoners/ Merchants	Higher Nobility	Population
	Y_1	Y_2	Y_3	Y
Baseline	30.01	56.20	40.29	41.37
Higher Privileges	16.43	23.04	16.11	18.03

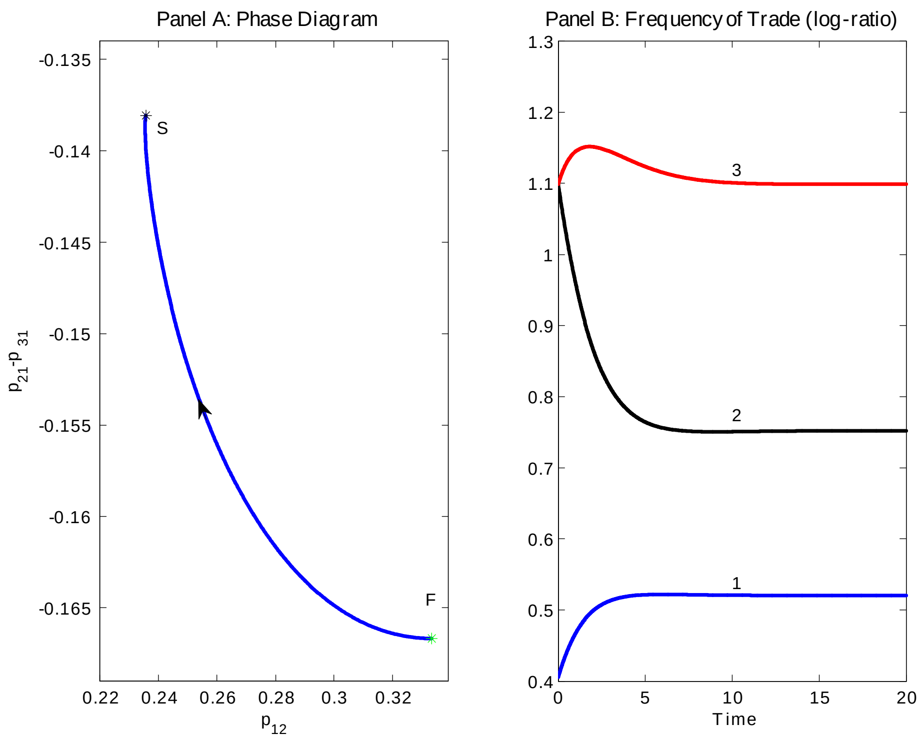
- Note: The table shows the steady state effects of a 50% increase in α on Type i average income, and on the population average income. The underlying parameters of the two economies are specified in Table (2). The figures are in percentages.

Table 4: Changes in Rent-Income Ratios – Long Run

	Lower Nobility	Commoners/ Merchants	Higher Nobility	Population
	$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_3$	$\tilde{r}$
Baseline	-64.66	-40.37	-40.29	-49.57
Higher Privileges	-16.43	-23.04	-16.11	-18.03

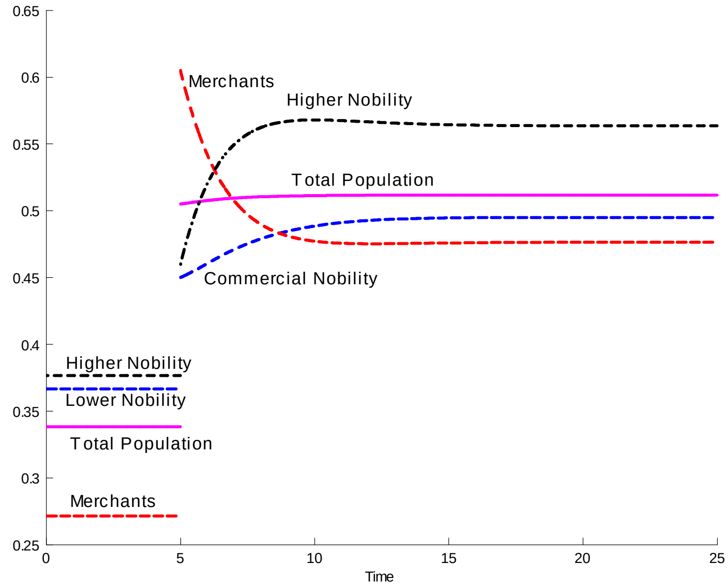
- Note: The table shows the steady state effects of a 50% increase in α on $\tilde{r}_i$ and $\tilde{r}$. The underlying parameters of the two economies are specified in Table (2). The figures are in percentages.

Figure 1: The Commercial Revolution, Distribution of Goods, and Trade

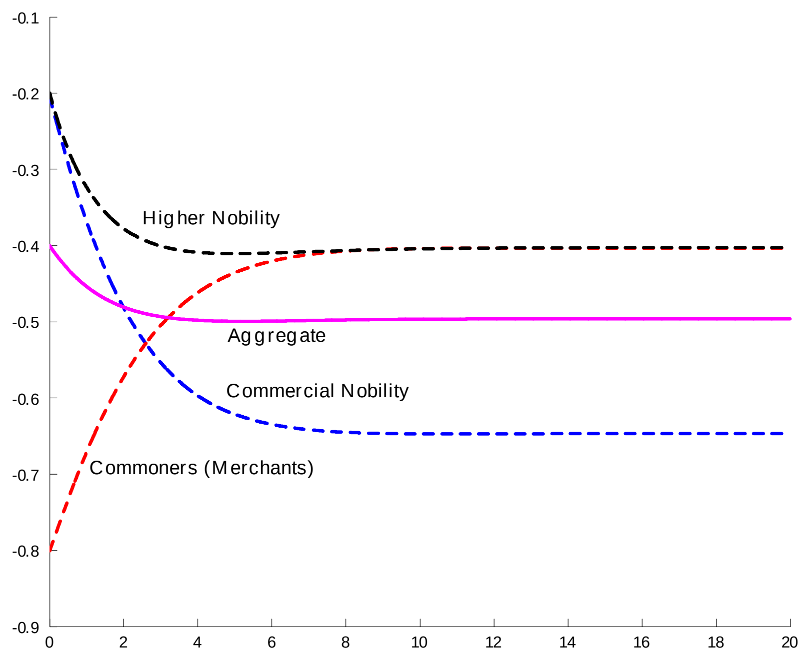


- Note: The baseline economy (see Table (2), first row) is on the fundamental steady state equilibrium when it is hit by a shock that raises the matching parameter permanently by 50 percent. The ratios and differences are calculated with respect to the pre-shock state. The average frequency of trade goes up by about 70%.

Figure 2: Communication Shock
Panel A: Income



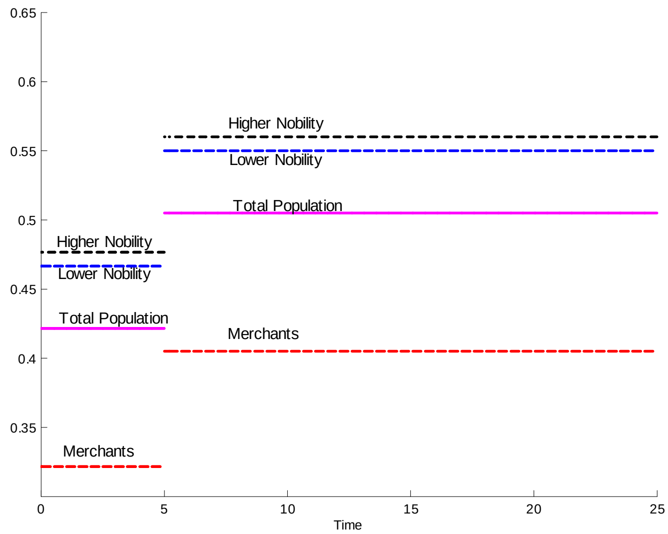
Panel B: Rent-Income Ratio



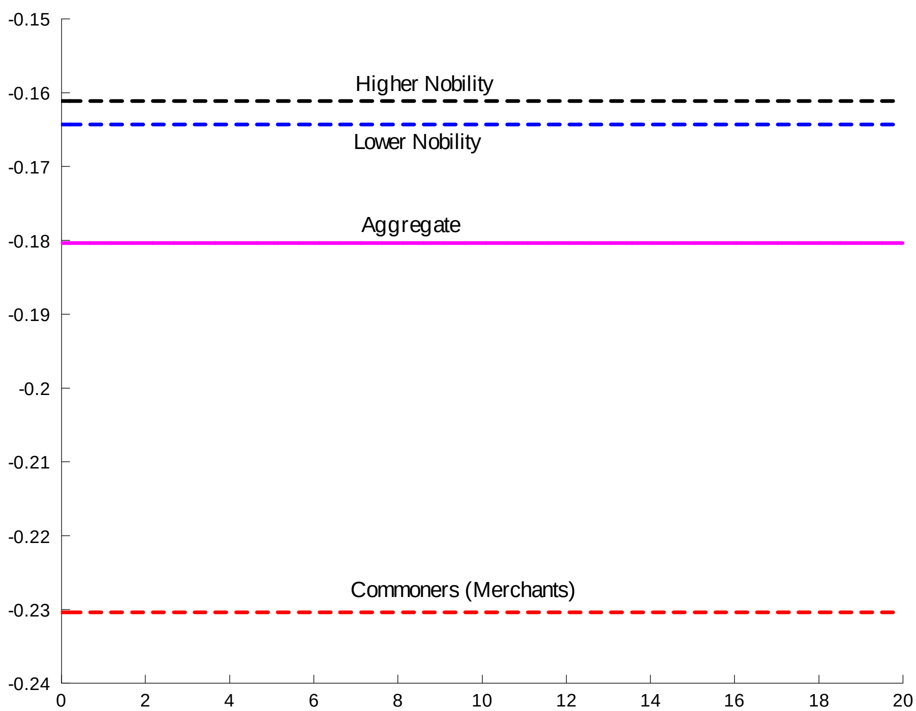
- Note: Panel A plots the average income of type 3 (higher nobility), type 2 (lower nobility), and type 1 (merchants) agents, respectively, before and after a shock that increases the matching rate by 50 percent. Initially the economy is fundamental steady state equilibrium. Panel B shows the rent-output ratio $\frac{R_i}{Y_i}$ and $\tilde{r}$ (see eqs. (5 and (7), after the shock relative to the ratios before the shock (in log-ratio). Table (3) reports the baseline parameters value.

Figure 3: Communication Shock

Panel A: Income



Panel B: Rent-Income Ratio



- Note: Panel A plots the average income of type 3 (higher nobility), type 2 (lower nobility), and type 1 (merchants) agents, respectively, before and after a shock that increases the matching rate by 50 percent. Initially the economy is fundamental steady state equilibrium. Panel B plots the rent-output ratio $\frac{R_i}{Y_i}$ and $\tilde{r}$ (see eqs. (5 and (7), respectively resulting from the same experiment. The trade frequency of any good goes up by 41%. Table (3) reports the baseline parameters value.

factor that could have impeded the spread of the benefits of commerce was the relatively high levels of rents enjoyed by the nobility. As will be discussed in the next section, during the middle ages the economic gap between the aristocracy and commoners was substantially higher in France than in Northern Italy (Lopez, 1976, Wickham, 2009).

In the KW environment a larger gap between the privileged and the less privileged groups reduces the odds that a communication shock generates an economic boom. Recall that when $\frac{r_2 - r_3}{\alpha} > \frac{1}{6}$ the economy's long run equilibrium is (0,1,0). Hence, if the difference between the rent fetched by goods 2 and 3, which are produced by the lower nobility and the commoners, respectively, is large, a shock to the matching parameter α may fail to convince the lower nobility to adopt a speculative strategy. This event is reproduced in Fig. (3) that shows the effects of a 50% increase in the matching rate in an economy similar to the one studied in the previous experiment, except that the goods produced by the two privileged classes offers more generous returns ($r_1 = 0.31$ and $r_2 = 0.3$, see Table 2). The higher matching rate makes everyone better off, but it stop shorts from triggering the economic boom observed in the previous experiment. Indeed the economy remains in the same pre-shock fundamental steady state equilibrium, although the trading and production activities are scaled up because of the higher meeting frequency. In the current experiment, average income increases by 18 percent, compared to a 41 percent rise in the baseline experiment and the average rent-income ratio declines by about 18% (all driven by the rise in production income), while in the baseline experiment it drops by about 50% (see Table 3).

5 Discussion

In this section I verify the plausibility of central results of the model by reviewing historical evidence on the effects of a communication shock. The baseline experiment demonstrated that if the gap between the privileged classes and the commoners is not too large, an improvement of the communication favors: (1) the emergence of a commercial nobility; (2) the convergence of income between commoners and commercial nobility; (3) a substantial increase in per capita income. Conversely, the alternative experiment showed that if the gap is relatively large, the nobility will resist trade, the social and economic gap will persist, and per capita income will increase only moderately, reflecting the more intensive frequency of existing patterns of trade.

To match the phenomena emerging in the two experiments, I will review key facts about the onset and the dissemination of the 13th commercial revolution and about the Atlantic trade. I will also discuss evidence that leads support to the idea that the resistance to trade of the landed aristocracy prevented France to experience an economic boom comparable to the one observed in Northern and Central Italy during the Middle Ages and in the Lower

Countries in the 16th century.

5.1 The Middle Age Communication Shock

With the disintegration of the Roman Empire, the regions on the shores of the Mediterranean broke their communication ties. In the period roughly going from the year 700 to 950, trade had reached a low point in Europe, with the exception of those regions under the economic influence of the Arabs and of Byzantium, most notably Spain and Sicily.

"[B]y the time of Charlemagne the sea that had functioned as a central highway for the Greco-Roman community became the border between three different communities, which in the lack of an appropriate economic term may be identified through their predominant religions: Islamic, Orthodox, and Catholic." (Lopez, 1976, p. 22).

A tenuous link between these three separate communities was initially provided by the Jews. Their success in trade, however, was often halted by the reaction of the majority of the population that would drive them out of the community and forfeit their capital (Lopez 1976, p. 62). The positive and lasting shock that improved significantly the communication networks in the Mediterranean came from through the actions of small, poor communities such as the one that took refuge on the Venetian islands. With the rise of Italian seaport cities the Mediterranean became again a commercial highway. Cipolla (1974) cites as evidence of the intensification of communication network in Europe and in the Mediterranean the rapid spread of the Black Death, an hypothesis later supported by several studies in economic history, demography, and epidemiology (see Boerner and Severgnini (2012))

The communication shock was not the result of a policy of a centralized government as in Sung China (Kelly, 1997). Quite to the contrary it occurred in the most politically fragmented part of Europe. Lopez states that:

Venice was then [829] practically independent, but honored her allegiance to Byzantium by supplying naval assistance, and used her eastern connections to unlock the gates of the Western Empire. She also maintained with Muslim Africa and the Levant as good relations as the sudden transitions from cold to hot war permitted. Thus she gradually build up a thriving triangular trade. (1976, p. 63).

The new fervor for commerce was not associated with any major developments in naval engineering either. The most common Venetian vessel was an improved version of the Roman and later Byzantine galley. Venetian trade was mostly done in short haul navigation with relatively small galleys.

5.2 Privileges: The Italian Commercial Nobility and the French Aristocracy

In the baseline experiment, the communication shock triggered a major transformation in the model economy. Type 1 individuals (the lower nobility) abandon the rent-seeking activity and turn to trade. As a result income of the type 1 and type 2 individuals (the commoners) converge towards a higher level (see Panel A, Figure (2)).

It has been said that the presence of the king with his power to distribute privileges absorbed the attention of the most resourceful of the society (Baumol, 2002). This seem to have been the case in France (Acemoglu et al., 2005). Conversely, the Venetian nobility was too far away from Constantinople and in conflict with the Frankish and German kings to have developed any expectations of earning substantial rents out of traditional lordships. It is revealing, for instance, that Venetian nobility was not determined by land but by wealth accumulated through commercial activities (Madden, 2003). On this subject Lopez states that:

As the progress of Venice and Amalfi was shifting the center of economic and naval power from the Byzantine and Muslim to the Catholic shore of the Mediterranean, two seaports from the barbarian part of Italy joined the race. Pisa and Genoa, too, were driven into a predominately commercial way of life by an original handicap. In the tenth century their territory had been frequently laid waste by Muslim raiders looking for slaves an other booty; not even the towns had been spared. Led by their urban nobility, the Pisans and the Genoese fought back." (1976, p. 66).

Puga and Trefler (2014) also documented high social mobility and low income inequality in Venice up to the end of the 13th century, that is, up to the establishment of entry restrictions into the lucrative long-distance trade (the *Serrata*). Similarly in Genoa, the dividing line between nobles and merchants was rather blurred. Lopez recounts that:

The habit of sharing risk on the decks of the ships and rubbing shoulders in the narrow alleys and shops of the business center created a strong equalitarian tradition. Ever since the origins of the Genoese Commune, the law admitted no privileges of birth and ignored the very word "nobility." Naturally, this does not mean that nobility was unknown, for aristocrats turned businessmen played a leading roles in town and at sea; but the first recognition it got was through fourteenth-century decrees that excluded noblemen from public offices. The informal term bonitas, good or solid citizens, described the Genoese upper class more accurately (1964, p. 447- 48).

The second experiment clarified that the switch to a speculative strategy does not occur (see Figure (3)) when the upper classes' rents are above a certain threshold. Medieval France was a region dominated by a rich landed aristocracy: "Aristocracy varied substantially in their wealth across western societies. In Merovingian Francia, there were some really rich landowners, with dozens of landed estates each, and a highly militarized factional politics. Bavaria was like Francia, although probably on a smaller scale; only a handful of families (apart from the ruling dues) seem to have been important owners. In Lombard Italy, however, the wealth of the aristocratic strata was much more modest." (Wickham, 2009, pp. 204-205).

More specifically on the situation of aristocracy in Northern Italy Wickham states that "Most Lombard aristocrats were fairly restricted in their wealth. Almost none of our documents show any of them with more than between five and ten estates, which is close to a minimum for aristocrats in Francia. The king and the ruling dukes of the south [Italy] had immense lands, of course, and a small number of powerful ducal families, particularly the north-east were rich, but the bulk of the elite owned only a handful of properties, usually only in the city territory they lived in, plus perhaps its immediate neighbors, with, quite often, a house in Pavia." (Wickham 2009, pp.145-146).

The accumulation of conspicuous wealth of the French landed aristocracy continued in the Carolingian and post-Carolingian period when more land came to be under aristocratic control than before, and less under the control of non-aristocrats. This change was particularly evident in Saxony, where Charlemagne's conquest resulted in a rapid takeover of land previously under peasant ownership (Wickham (2009)

5.3 The Middle Age Growth Spurt

In the baseline experiment there is a more pronounced acceleration of trade (70%) than in the alternative high-rent scenario (40%) because the communication shock triggers the emergence of a new trade link. A a result, in the baseline experiment income also goes up substantially more (41%) than in a high-rent scenario (18%) (see Tables (3) and (4)).

Some data, admittedly of a rather rough variety, are available to verify the responses of trade and income to a communication shock in places that differed for the level of privileges of the upper classes. Table (5), shows the impressive population rise in Genoa and Venice, the two Italian cities that were at the core of the commercial revolution. Bairoch (1988) estimates that the population in Venice went from about 45 thousands in the year 1000 to 75 thousands in the year 1200 and reached a Medieval pick of 110 thousands in the year 1300. The population explosion in Genoa is even more staggering. The corresponding figures are 15, 30, and 100 thousands, respectively. Bairoch's data concord with Lopez's remark: "by the end of the thirteenth century, Genoa proper may have reached the 100,000 mark, with probably another half million people in the territory of the state, which included several smaller urban centers. These figures are no more than very rough approximations, but they would indicate that the capital city was the fifth largest in Europe (larger than Paris and London, but smaller than Constantinople, Venice, Milan, and Florence) and the entire state was more thickly settled than Pennsylvania today." (1964, p. 448).

In contrast, during the 13th century the population of Marseille increased only moderately and even declined in the following century: In the year 1200 Marseille was about of the same size of Genoa but by the year 1400 it had a population of only a fifth of that of Genoa or Venice.

The dynamism of northern Italian cities with maritime trade is associated with an increase in income. Angus Maddison (2009) data indicate that between the year 1000 and 1500 per capita income more than doubled in Italy. Furthermore, Bold and van Zanden (2014) data indicate in the year 1500 income in northern Italy was 40 percent higher in than Italy's average reported by Maddison (2009).

The use of the GDP per capita statistics in contemporary prices for earlier centuries inevitably relies on strong assumptions. Nevertheless they do agree with Lopez's description of the 13th century economic situation in northern and central Italy: "[N]ever before had such a large proportion of the population been free from want, or such a variety and abundance of goods been constantly available – not in ancient Rome at is peak, not in Byzantine and Islamic countries at their best." (1976, p. 106)

5.4 Atlantic Trade

As observed in section (4.2) despite France being in a favorable geographical position it did not benefit from the trade revolutions as the nearby Lower Countries and northern Italy.

That pre-revolutionary France did not catch the Atlantic trade wave appears evident from the modest change in population on its Atlantic seaport cities compared to that observed in the Dutch Republics. In the year 1700 the population of Bordeaux, the most important French eighteenth century port, was about the same as that of two centuries earlier with a population of only a quarter of that of Amsterdam. Furthermore, the share of French among the merchants population of Bordeaux in the 18th century was rather small: Sephardi, Germans, Dutch and Irish made up a third of Bordeaux merchants and half of the French firms were run by Huguenots linked to the Protestant merchants of northern Europe. Only after 1793 new pattern of trades were established in Bordeaux, most notably with the United States (Marzagalli, 2005).

By contrast, Amsterdam between the year 1600 and 1700 went through explosive growth going from 54 to 200 thousands people. Income also jumped up considerably in the Netherlands. In the year 1500 per capita income in France and the Netherlands was approximately the same. In the year 1600, however, income in the Netherlands was almost double of that observed a century earlier, whereas in France it had increase by less than 20 percent. In the following century the divergence between the two regions became more pronounced, with a gap factor of 2.3 in the year 1700 (see Table (7)).

5.5 More on the Resistance of the French Aristocracy to Commerce

In the experiment of section (4.2) the lower nobility did not switch to speculative strategies because $r_2 - r_3 > \frac{1}{6}\alpha'$, where α' is the matching rate after the positive communication shock. Clearly, the larger the gap $r_3 - r_2$, the higher the expected loss in terms of rent associated with a switch to a speculative strategy.

A number of historians argue that the resistance of the landed French aristocracy in embracing commerce were fueled by concerns that such a move would cost them privileges – especially property tax exemptions. Such a believe was well grounded. In 1295, when Philip the Fair exempted the nobles from paying him a certain aid, he specifically excluded those in trade. In 1435, Charles VII refused to exempt from taxation nobles who sold their wines in taverns. Earlier in the sixteen century, Francis I voided the tax exemptions of nobles who meddled in the commercial activities.

The position of the French monarchy with respect to having a commercial nobility was ambiguous as two conflicting interests were at stake. On the one side the monarch had a clear interest in developing trade as this was an important source of revenues to finance warfare. For instance, Lopez (1964) reports that at the end of the 13th century the taxes raised by Genoa from her sea trade was roughly ten times the receipts of the French royal treasury. On the other side the king could not afford to loose the support of the nobility that for a long time opposed the idea of being mangled with merchants.

It is revealing that, Louis XI, who was very familiar with the state of affairs in the Italian cities, took some steps at establishing commercial routes to compete with Venice and Genoa. In the second half of the 15th century, the monarch tried prepared legislation to break the separation between nobility and merchants, in imitation to what was already in place in Italian cities. But the law was never put into effect. Instead, a few decades later, in 1560, former legislation and customs that sanctioned the nobility engaged in commerce was codified (the Code of Ordinances of Orleans, see Zeller, 1946). It was not until 1629, under Richelieu, that nobles were allowed to engage in maritime commerce, either directly or through proxy, without fear of loosing their status. Although some nobles seized the opportunities in overseas trade by participating in trading companies (Chaussinand-Nogaret, 1975), by the middle of the 18th century the debate among French intellectuals on whether or not to have a commercial nobility was still lively. An example of the controversy stirred by the pamphlet "Commercial Nobility" (*La Noblesse Commerçante*), that appeared in 1756. The author, who was an influential writer at that time, rebutted Montesquieu's position, elaborated in *The Spirit of the Laws* (1748, Book XX, Ch. 21), that opposed the idea of having a commercial nobility (see Adam, 2003, and Grassby, 1960, for details on the controversy).

Table 5: Population in French and Italian Seaport Cities

	Year			
	1000	1200	1300	1400
Venice	45	70	110	100
Genoa	15	30	100	100
Marseille	7	25	31	21
Bordeaux	n.a.	15	30	30

- Source: Bairoch (1988)

Table 6: Population in French and Dutch Seaport Cities

	Year		
	1500	1600	1700
Amsterdam	15	54	200
Bordeaux	50	40	45

- Source: Bairoch (1988).

6 Multiple Equilibria

So far I considered economies with unique equilibria. It is well known that KW environment may generate multiple steady states. Any of the ranking in Panel B Table (8), which correspond to the Model B of KW, may exhibit multiple steady states. Do these configurations of parameters add insight on the arrival of the commercial revolution? The short answer is no. In Model A the commercial revolution was triggered by the emergence of a new trade link that magnified the trade frequencies and caused a boom in income. In model B, however, it is not possible to generate the emergence of an expanded trade network when going from a fundamental to a speculative steady state equilibrium. Consider, for instance the case in which $r_3 < r_1 < r_2$ (see R5, Panel B, Table 8). The higher nobility, lower nobility, and commoners, now correspond to type 2, type 3, and type 1 agents, respectively. A (0,1,1) steady state equilibrium always exists. A (1,1,0) steady state exists (namely, type 1 and type 3 agents follow a speculative behavior) if the following two conditions are satisfied (see Appendix for derivations):

$$\frac{\sqrt{2}-1}{3} > \frac{r_2 - r_3}{\alpha},$$

Table 7: Average per Capita GDP in 1990 International GK Dollars

	Year			
	1000	1500	1600	1700
Italy	450	1100	1100	1100
Italy (Center-North)	n.a.	1533	1363	1476
Netherlands	425	751	1381	2105
France	473	727	841	910

- Source: The figures for Center-North Italy are from Bold and van Zanden (2014) The remaining data are drawn from Maddison (2003).

and

$$\frac{1}{\sqrt{2}}\frac{\sqrt{2}-1}{3} > \frac{r_2 - r_1}{\alpha}.$$

Imagine that initially α is low so that at least one of the two above conditions is violated. A positive shock to α can bring up a (0,1,1) equilibrium which would coexist with the (1,1,0) equilibrium. Because there are multiple equilibria agents can either maintain their current strategies or type 1 and type 3 agents switch to a speculative strategies. (Expectations, not history, determine the outcome). Nevertheless neither of the two outcomes is able to capture the commercial revolution. Even if type 1 and type 3 agents react to the communication shock by turning to speculative strategies (the numerical algorithm suggest that this can happen for any reasonable discount rate), the frequency of aggregate trade is hardly affected. Indeed, in the fundamental equilibrium one of the two upper classes is already engaged in indirect trade. If agents coordinate on the (1,1,0) simply the two upper classes switch strategies with type 1 agents flipping from $\tau_1 = 0$ to $\tau_1 = 1$ and type 3 agents from $\tau_3 = 1$ to $\tau_3 = 0$.

References

Acemoglu, D., S. Johnson, and J. Robinson (2005). "The Rise of Europe: Atlantic Trade, Institutional Change, and Economic Growth." American Economic Review, 95 (3): 546-579.

Acemoglu, D. (1995), Reward Structures and the Allocation of Talent, European Economic Review 39, 17-33.

Adam, U. (2003): "Nobility and Modern Monarchy." *History of European Ideas*, 29, pp. 141-157.

Bairoch, P. (1988). Cities and Economic Development: From the Dawn of History to the Present. The University of Chicago Press.

Başçı, E. (1999): "Learning by Imitation," *Journal of Economic Dynamics and Control*, 23, 1569-85.

Baumol, W. J. (1990) "Entrepreneurship: Productive, Unproductive, and Destructive," Journal of Political Economy, 98 (5) pp. 893-921.

Baumol, W. J. (1993), Entrepreneurship, Management and the Structure of Payoffs, MIT press.

Baumol, W. J. (2002), The Free-Market Innovation Machine: Analyzing the Growth Miracle of Capitalism, Princeton University Press.

Boerner, L. and B. Severgnini (2012): "Epidemic Trade," EHES Working Papers in Economic History, n. 24.

Bolt, J., and J. L. van Zanden (2014). The Maddison Project: Collaborative Research on Historical National Accounts. *The Economic History Review*, 67 (3): 627–651.

Brown, P. M. (1996): Experimental Evidence on Money as a Medium of Exchange, *Journal of Economic Dynamics and Control*, 20 (4), 583-600.

Cipolla (1974): "The Plague and the pre-Malthus Malthusians," *The Journal of European Economic History*, 3, 277-284.

Chaussinand-Nogaret, G. (1975). Une histoire des élites: 1700-1848. La Haye, Mouton, 1975.

Duffie, D., and Y. Sun (2012): "The Exact Law of Large Numbers for Independent Random Matching," *Journal of Economic Theory*, 147, 1105–1139.

Duffy, J., and Jack Ochs (1999): "Emergence of Money as a Medium of Exchange: An Experimental Study," *American Economic Review*, 89 (4), 847-877.

Fudenberg, D., and D. Levine (1988): "Open-Loop and Closed-Loop Equilibria in Dynamic Games with Many Players," *Journal of Economic Theory*, 44 (1), 1-18.

Grassby, R. B. (1960):"Social Status and Commercial Enterprise under Louis XIV," *The Economic History Review*, 13 (1), 19-38.

Holmes, T. J. and J. A. Schmitz (2001), "A Gain from Trade: From Unproductive to Productive Entrepreneurship," Journal of Monetary Economics, 47, pp. 417–446.

Kelly, M. (1997), The Dynamics of Smithian Growth, Quarterly Journal of Economics 112 (3), 939-964.

Kiyotaki, N., and R. Wright (1989): "On Money as a Medium of Exchange," *Journal of Political Economy*, 97, 927–954.

Krueger, A. (1974). The Political Economy of the Rent-Seeking Society, The American Economic Review, 64 (3) pp. 291-303.

Lagos, R., G. Rocheteau, and R. Wright (2016): "Liquidity: A New Monetarist Perspective," *Journal of Economic Literature*, forthcoming.

Landes, D. S. (1998). *The Wealth and Poverty of Nations: Why Some Are So Rich and Some So Poor.* W. W. Norton & Company.

Lopez, R. S. (1964): "Market Expansion: The Case of Genoa," *Journal of Economic History*, 24 (4), 445-464.

Lopez, R. S. (1976): *The Commercial Revolution of the Middle Ages, 950 1350.* Cambridge University Press, Oxford.

Luo, G. (1999): "The Evolution of Money as a Medium of Exchange," *Journal of Economic Dynamics and Control*, 23, 415-58.

Maddison, A. (2009). The World Economy, OECD Development Center.

Marimon, R., E. McGrattan, and T. Sargent (1990): "Money as a Medium of Exchange in an Economy with Artificially Intelligent Agents," *Journal of Economic Dynamics and Control,* 14, 329-73.

Marzagalli, S. (2005). Establishing Transatlantic Trade Networks in Time of War: Bordeaux and the United States, 1793-1815, The Business History Review, 79 (4): 811-844

Matsuyama, K., N. Kiyotaki and A. Matsui (1993). Toward a Theory of International Currency, The Review of Economic Studies, 60 (2), 283-307

Murphy, K. M., A. Shleifer, and R. W. Vishny (1991): "The Allocation of Talents: Implications for Growth," The Quarterly Journal of Economics, 106 (2) pp. 503-530.

Murphy, K. M., A. Shleifer, and R. W. Vishny (1993): "Why Is Rent-Seeking So Costly to Growth?" American Economic Review Papers and Proceedings 83 (2): 409-414.

Peretto, P. F. (2015). "From Smith to Schumpeter: A theory of take-off and convergence to sustained growth," European Economic Review, 78, 1-26.

Puga, D., and D. Trefler, (2014): "International Trade and Institutional Change: Medieval Venice's Response to Globalization," The Quarterly Journal of Economics, 129 (2), 753-821.

Sethi, R. (1999): "Evolutionary Stability and Media of Exchange," *Journal of Economic Behavior and Organization*, 40, 233-54.

Wickham, C. (2009), The Inheritance of Rome: Illuminating the Dark Ages 400-1000, Penguin Publishing Group.

Wright, R. (1995): "Search, Evolution, and Money," *Journal of Economic Dynamics and Control*, 19, 181-206.

Zeller, G. (1946): "Procès à reviser? Louis XI, la noblesse et la marchandise." *Annales. Économies, Sociétés, Civilisations*, 4, 331-341.

Table 8: Steady State Equilibria, Strategies, and Money

	Returns	F	S	Assets (F)	Assets (S)
		Panel A			
R1	$r_3 < r_2 < r_1$	(0,1,0)	(1,1,0)	$\frac{1}{3}[1, \frac{1}{2}, 1]$	$\frac{1}{3}[a, b, 1]$
R2	$r_2 < r_1 < r_3$	(1,0,0)	(1,0,1)	$\frac{1}{3}[\frac{1}{2}, 1, 1]$	$\frac{1}{3}[b, 1, a]$
R3	$r_1 < r_3 < r_2$	(0,0,1)	(0,1,1)	$\frac{1}{3}[1, 1, \frac{1}{2}]$	$\frac{1}{3}[1, a, b]$
		Panel B			
R4	$r_2 < r_3 < r_1$	(1,1,0)	(1,0,1)	$\frac{1}{3}[a, b, 1]$	$\frac{1}{3}[b, 1, a]$
R5	$r_3 < r_1 < r_2$	(0,1,1)	(1,1,0)	$\frac{1}{3}[1, a, b]$	$\frac{1}{3}[a, b, 1]$
R6	$r_1 < r_2 < r_3$	(1,0,1)	(0,1,1)	$\frac{1}{3}[b, 1, a]$	$\frac{1}{3}[1, a, b]$

- Note: $a = \frac{1}{2}\sqrt{2}$ and $b = \sqrt{2}-1$. The F and S columns contain the triplet $(\theta_1, \theta_{2,}\theta_3)$ that describes the fundamental and the speculative steady state strategy, respectively. The following two columns are the assets' stationary distributions: $[p_{1,2}, p_{2,3}, p_{3,1}]$. The last two columns indicate which asset is traded indirectly – or acts as 'money'– in the fundamental and speculative equilibrium, respectively. In rows R1 through R3 the equilibria are unique; only one type of agent plays speculative strategies in the S equilibrium. In the R4-R6 rows the two equilibria may coexist; two types of agents play speculative strategies in the S equilibrium.

Appendix

Technical Appendix [Not intended for publication]

Distribution of Inventories (Proposition 1)

For a given profile of strategies the system (10)-(12) converges globally to the unique steady state reported in the following table:

Strategies	Assets Distribution	*Strategies*	Assets Distribution
(0,1,0)	$[\mu_1, \frac{\mu_2\mu_1}{\mu_3+\mu_1}, \mu_3]$	*(1,0,1)*	$[p^{\#}_{1,2}, \mu_2, p^{\#}_{3,1}]$
(1,0,0)	$[\frac{\mu_1\mu_3}{\mu_2+\mu_3}, \mu_2, \mu_3]$	*(0,1,1)*	$[\mu_1, p^{\boxtimes}_{2,3}, p^{\boxtimes}_{3,2}]$
(0,0,1)	$[\mu_1, \mu_2, \frac{\mu_1\mu_3}{\mu_1+\mu_2}]$	*(0,0,0)*	$[\mu_1, \mu_2, \mu_3]$
(1,1,0)	$[p^*_{1,2}, p^*_{2,3}, \mu_3]$		

Case (0,1,0). Eq. (10) reduces to $\dot{p}_{1,2} = \alpha p_{1,3}\mu_3$, implying that the line $p_{1,2} = \mu_1$, is globally attractive (henceforth g.a.). Similarly because Eq. (12) $\dot{p}_{3,1} = \alpha p_{3,2}[p_{1,3} + \mu_2]$, $p_{3,1} = \mu_3$ is g.a.. Finally, along these lines the system collapses to

$$\dot{p}_{2,3} = (\mu_2 - p_{2,3})\mu_1 - p_{2,3}\mu_3,$$

which clearly converges globally to $\frac{\mu_2\mu_1}{\mu_3+\mu_1}$. In brief, under the profile of strategies (0,1,0) the distribution of inventories converges globally to the stationary distribution $[\mu_1, \frac{\mu_2\mu_1}{\mu_3+\mu_1}, \mu_3]$.

Case (0,0,1) and *Case (1,0,0).* One can verify that the stationary distribution converges to $[\mu_1, \mu_2, \frac{\mu_1\mu_3}{\mu_1+\mu_2}]$ and $[\frac{\mu_1\mu_3}{\mu_2+\mu_3}, \mu_2, \mu_3]$, respectively, using the same observations as in the previous case.

Case (1,1,0). Eq. (12) becomes $\dot{p}_{3,1} = \mu_2(\mu_3 - p_{3,1})$. Consequently, $\mu_3 = p_{3,1}$ is an invariant set. The Jacobian, J, of the system of the two remaining equations (10) and (11) along the line $\mu_3 = p_{3,1}$ is

$$J = \alpha \begin{bmatrix} -(\mu_3 + p_{2,3}) & -p_{1,2} \\ (\mu_2 - p_{2,3}) & -(\mu_3 + p_{1,2}) \end{bmatrix}.$$

The determinant is positive and the trace is negative; therefore, both eigenvalues are negative and the system is globally stable. To find the stationary distribution, set (10) and (11) to zero. They yield $p_{1,2} = \mu_1\mu_3/(\mu_3 + p_{2,3})$ and $p_{1,2} = \frac{\mu_3}{\mu_2/p_{2,3}-1}$, respectively. The two lines necessarily cross once and only once for $p_{2,3}$ in the interval $[0,\mu_3]$. The fixed point is $[p^*_{1,2}, p^*_{2,3}, \mu_3]$ where $p^*_{2,3} = \frac{1}{2}[-(\mu_1 + \mu_3) + \sqrt{(\mu_1 + \mu_3)^2 + 4\mu_1\mu_2}]$ and $p^*_{1,2} = \frac{\mu_1\mu_3}{\mu_3+p^*_{2,3}}$.

Cases (1,0,1) and (0,1,1). A Jacobian with similar properties can be obtained when the profiles of strategies are (1,0,1) or (0,1,1). The fixed point with (1,0,1) is $[p^{\#}_{1,2}, \mu_2, p^{\#}_{3,1}]$, where $p^{\#}_{1,2} = \frac{1}{2}[-(\mu_3 + \mu_2) + \sqrt{(\mu_3 + \mu_2)^2 + 4\mu_3\mu_1}]$ and $p^{\#}_{3,1} = \frac{\mu_1\mu_3}{p^{\#}_{1,2}+\mu_2}$. Similarly, under (0,1,1) the fixed point is $[\mu_1, p^{\boxtimes}_{2,3}, p^{\boxtimes}_{3,2}]$ where $p^{\boxtimes}_{3,1} = \frac{1}{2}[-(\mu_2 + \mu_1) + \sqrt{(\mu_2 + \mu_1)^2 + 4\mu_2\mu_3}]$ and $p^{\boxtimes}_{2,3} = \frac{\mu_1\mu_2}{p^{\boxtimes}_{3,1}+\mu_1}$.

Case (0,0,0). The system converges globally $\mathbf{p} = [\mu_1,\mu_2,\mu]$. In this stationary state agents keep their production goods.

When the profile of strategies is (1,1,1) it is more difficult to characterize the properties of the Jacobian. This turns out not to be a Nash equilibrium, at least when the population is equally split across types.

Eq. (3) describes $V_{i,i+1}(t)$. The following does the same for $V_{i,i+2}(t)$

$$\begin{aligned} V_{i,i+2}(t) = \max_{\{\tau_i(s)\}_{s\geq t}} \int_t^\infty \alpha e^{-\alpha(s-t)}\{e^{-\rho(s-t)}([p_{i+1,i}(1-\theta_{i+1}) + p_{i+2,i}](V_{i,i+1}+u_i)+ \\ [p_{i,i+1}\theta_i + p_{i+2,i+1}](1-\tau_i)V_{i,i+1} \\ [1-p_{i+1,i}(1-\theta_{i+1}) - p_{i+2,i} - (p_{i,i+1}\theta_i + p_{i+2,i+1})(1-\tau_i)]V_{i,i+2}+ \\ \frac{1-e^{-(s-t)\rho}}{\rho} r_{i+2}\}ds, \end{aligned} \tag{13}$$

Time derivatives of eqs. (3) and (13) give

$$\begin{aligned} \dot{V}_{i,i+1} = &-\alpha([p_{i,i+2}\tau_i(1-\theta_i) + p_{i+1,i+2}\tau_i]V_{i,i+2}+ \\ &p + [1-p_{i,i+2}\tau_i(1-\theta_i) - p_{i+1,i+2}\tau_i]V_{i,i+1}+ \\ &[p_{i+1,i} + p_{i+2,i}\theta_{i+2}]u_i) - r_{i+1} + (\alpha+\rho)V_{i,i+1}(s), \end{aligned} \tag{14}$$

$$\begin{aligned} \dot{V}_{i,i+2} = &-\alpha([p_{i+1,i}(1-\theta_{i+1}) + p_{i+2,i}](V_{i,i+1}+u_i) + [p_{i,i+1}\theta_i + p_{i+2,i+1}](1-\tau_i)V_{i,i+1} \\ &+ [p_{i,i+1}\theta_i + p_{i+2,i+1}](1-\tau_i)V_{i,i+1}+ \\ &+ [1-p_{i+1,i}(1-\theta_{i+1}) - p_{i+2,i} - (p_{i,i+1}\theta_i + p_{i+2,i+1})(1-\tau_i)]V_{i,i+2})+ \\ &- r_{i+2} + (\alpha+\rho)V_{i,i+2}, \end{aligned}$$

respectively.

The last two expressions can also be written as

$$\begin{aligned} \dot{V}_{i,i+1} = &-\alpha([p_{i,i+2}\tau_i(1-\theta_i) + p_{i+1,i+2}\tau_i](-\Delta_i) + V_{i,i+1}+ \\ &[p_{i+1,i} + p_{i+2,i}\theta_{i+2}]u_i) - r_{i+1} + (\alpha+\rho)V_{i,i+1}(s), \end{aligned}$$

$$\begin{aligned} \dot{V}_{i,i+2} = &-\alpha([p_{i+1,i}(1-\theta_{i+1}) + p_{i+2,i} + [p_{i,i+1}\theta_i + p_{i+2,i+1}](1-\tau_i)]\Delta_i + [p_{i+1,i}(1-\theta_{i+1}) + p_{i+2,i}]u_i \\ &+ V_{i,i+2}) - r_{i+2} + (\alpha+\rho)V_{i,i+2}, \end{aligned}$$

Subtracting side-by-side we obtain

$$\dot{\Delta}_i = -\alpha([p_{i,i+2}\tau_i(1-\theta_i) + p_{i+1,i+2}\tau_i + p_{i+1,i}(1-\theta_{i+1}) + p_{i+2,i} + (p_{i,i+1}\theta_i + p_{i+2,i+1})(1-\tau_i)](-\Delta_i)+ \\ + \Delta_i + [p_{i+1,i}\theta_{i+1} - p_{i+2,i}(1-\theta_{i+2})]u_i) - r_{i+1} + r_{i+2} + (\alpha+\rho)\Delta_i$$

Let

$$\chi_i \equiv p_{i,i+2}\tau_i(1-\theta_i) + p_{i+1,i+2}\tau_i + p_{i+1,i}(1-\theta_{i+1}) + p_{i+2,i} + (p_{i,i+1}\theta_i + p_{i+2,i+1})(1-\tau_i)$$

where $\phi_i \equiv [p_{i,i+2}\tau_i(1-\theta_i) + p_{i+1,i+2}\tau_i]$. Then the last two differential equations reduce to:

$$\dot{\Delta}_i = -\alpha((1-\chi_i)\Delta_i + [p_{i+1,i}\theta_{i+1} - p_{i+2,i}(1-\theta_{i+2})]u_i) - r_{i+1} + r_{i+2} + (\alpha+\rho)\Delta_i, \qquad (15)$$

Since $(\alpha+\rho) > 0$, for any given pattern of asset distribution, the system is unstable. The Technical Appendix explains how this equation is used to verify whether a stationary distribution of assets is a Nash equilibrium.

Stationary Nash Equilibria (Proposition 2)

The stationary distribution of inventories are derived from (2) under the assumption that $\mu_1 = \mu_2 = \mu_3 = \frac{1}{3}$. The key condition to determine whether a stationary distribution is a NE is the sign of Δ_i. From (15) it follows that $\Delta_i > 0$ if

$$p_{i+1,i}\theta_{i+1} - p_{i+2,i}(1-\theta_{i+2}) > \frac{r_{i+2} - r_{i+1}}{\alpha u_i} \qquad (16)$$

Consistency requires that $\theta_i = 0$ (1) with $\Delta_i > 0$ (< 0). This section reviews the consistency conditions (16) for the rankings listed in Table 8.A.

Model A ($r_3 < r_2 < r_1$). There are two unique NE: (0,1,0) and (1,1,0). The (0,1,0) equilibrium requires that

$$p_{2,1} - p_{3,1} > \frac{r_3 - r_2}{\alpha u_1} \qquad (17)$$

$$-p_{1,2} < \frac{r_2 - r_1}{\alpha u_2}, \qquad (18)$$

and

$$0 > \frac{r_2 - r_1}{\alpha u_3}, \qquad (19)$$

with $p_{2,1} = \frac{1}{3} - p_{2,3}$, $p_{1,2} = \frac{1}{3}$, $p_{2,3} = \frac{1}{6}$, and $p_{3,1} = \frac{1}{3}$. Conditions (18) and (19) are clearly verified. From (17) it follows that the (0,1,0) exists if $\frac{1}{6} > \frac{r_2 - r_3}{\alpha u_1}$. For the (1,1,0) equilibrium the stationary distribution is $\mathbf{p} = \frac{1}{3}[a, b, 1]$. The above three conditions are replaced by

$$p_{2,1} - p_{3,1} < \frac{r_3 - r_2}{\alpha u_1},$$

$$0 < \frac{r_1 - r_3}{\alpha u_2},$$

and

$$p_{1,3} > \frac{r_2 - r_1}{\alpha u_3},$$

respectively, with $p_{2,1} = \frac{1}{3} - p_{2,3}$, $p_{1,2} = \frac{a}{3}$, $p_{2,3} = \frac{b}{6}$, and $p_{3,1} = \frac{1}{3}$. Again, the last two conditions are obviously satisfied. The first condition says that in a NE agents 1 play speculative if

$$-\frac{b}{6} < \frac{r_3 - r_2}{\alpha u_1}.$$

There are no other NE.

Model B. One fundamental NE always exist. This could coexist with another equilibrium in which two types of agents play speculative strategies (multiple equilibria).

When $r_3 < r_1 < r_2$. The (0,1,1) is the fundamental NE. It exists for any set of parameters (for which the value functions are non negative). Under the following two conditions it also exists the (1,1,0) NE:

$$p_{2,1} - p_{3,1} < \frac{r_3 - r_2}{\alpha u_1}$$

$$p_{1,3} > \frac{r_2 - r_1}{\alpha u_3}$$

with $p_{2,1} = \frac{1}{3}(1-b)$, $p_{3,1} = \frac{1}{3}$ and $p_{1,3} = \frac{1}{3}(1-a)$. The top one requires $-\frac{b}{3} > \frac{r_3 - r_2}{\alpha u_1}$ and the bottom one that $\frac{1}{3}(1-a) > \frac{r_2 - r_1}{\alpha u_3}$.

Algorithm

The algorithm sets up an iteration on the profile of strategies $\boldsymbol{\theta}(t)$ and on the distribution of assets $\mathbf{p}(t)$. The value function $V_{i,j}(t)$ of the representative agents i holding good j serves as device to update the guess on the profile of strategies, and to determine when the algorithm has converged. Since only pure strategies are considered,[4] a representative agent i has a binary choice at each point in time. The algorithm seeks for the convergence of the representative agent i's best response, $\tilde{\tau}_i(t; \boldsymbol{\theta}(t), \mathbf{p}_0)$, to the profile followed by the rest of individuals of her type $\theta_i(t)$. When the representative individual i does not have an interest in deviating from a strategy that coincides with that followed by the rest of type i agents, a Dynamic Nash Equilibrium is found. The algorithm works as follows.

Step 1. The distribution of inventories $F(\mathbf{p}(t))$ is integrated forward in time starting from some $\mathbf{p}_0$, under a guess $\boldsymbol{\theta}^{(0)}(t)$. The integration is stopped at some time T large enough so that $|F(\mathbf{p}(T))| < 10^{-6}$. An obvious initial guess is $\boldsymbol{\theta}^{(0)}(t) = \boldsymbol{\theta}^{ss}$, where $\boldsymbol{\theta}^{ss}$ is the steady

[4] Kehoe et al. (1993) build cyclycal equilibria in a similar environment under sets of parameters that do not admit pure strategies equilibria.

state Nash profile of strategies. (For some $s > \bar{s}$ with $\bar{s}$ sufficiently large, one can expect that $\tilde{\tau}_i(s) = \boldsymbol{\theta}_i^{(0)}(s)$ for $s > \bar{s}$). Let $\mathbf{p}^{(0)}(t)$ be the inventory solution under such a guess.

Step 2. The algorithm computes the best response of a representative agent i, on the trajectory $\mathbf{p}^{(0)}(t)$. His Δ_i is computed integrating (20) backward in time, starting from the initial condition $(\Delta_i(\boldsymbol{\theta}^{ss}, \mathbf{p}^{(0)}(T)), \mathbf{p}^{(0)}(T))$.[5] At the end of this step, one obtains a trajectory $\Delta^{(0)}(t)$, and, more importantly, the corresponding best response $\tilde{\tau}_i^{(0)}(t)$ of the representative agent i.

Step 3. The consistency between $\theta_i^{(0)}(t)$ and $\tilde{\tau}_i^{(0)}(t)$ is verified. If these are different, $\tilde{\tau}_i^{(0)}(t)$ becomes the new guess on the next round, namely $\theta_i^{(1)}(t) = \tilde{\tau}_i^{(0)}(t)$, and the procedure restarts from step one. The method allows the profile of strategies to change at any point in time.

The iteration is repeated until convergence between $\theta_i^{(n+1)}(t)$ and $\tilde{\tau}_i^{(n)}(t)$ is achieved, or until a maximum number of iterations is reached. If the iteration converges to a fixed point, say $\mathbf{p}^*(t)$ and $\boldsymbol{\theta}^*(t)$, then $\mathbf{p}^*(t)$ and $\boldsymbol{\theta}^*(t)$ are the distribution of assets and the trading strategies, respectively, of a Markov-perfect Nash equilibrium. The procedure checks that at such fixed point the value function of any agent is at its maximum value, given the action of the rest of the agents.

Forward and Backward Integration

This section examines the dynamic properties of the value functions of the representative agent i. These properties are important because they will be used by the algorithm to verify whether a particular distribution of assets and of strategies is an open-loop Nash equilibrium. According to (4), what matters for representative i's decision is only the *sign* of Δ_i. After some algebra, one obtains that

$$\dot{\Delta}_i = (\alpha\chi_i + r)\Delta_i + \omega_i, \tag{20}$$

where $\chi_i \equiv p_{i,i+2}\tau_i(1-\theta_i) + p_{i+1,i+2}\tau_i + p_{i+1,i}(1-\theta_{i+1}) + p_{i+2,i} + (p_{i,i+1}\theta_i + p_{i+2,i+1})(1-\tau_i) > 0$ and $\omega_i \equiv -\alpha[p_{i+1,i}\theta_{i+1} - p_{i+2,i}(1-\theta_{i+2})]u_i + (r_{i+2} - r_{i+1})$.

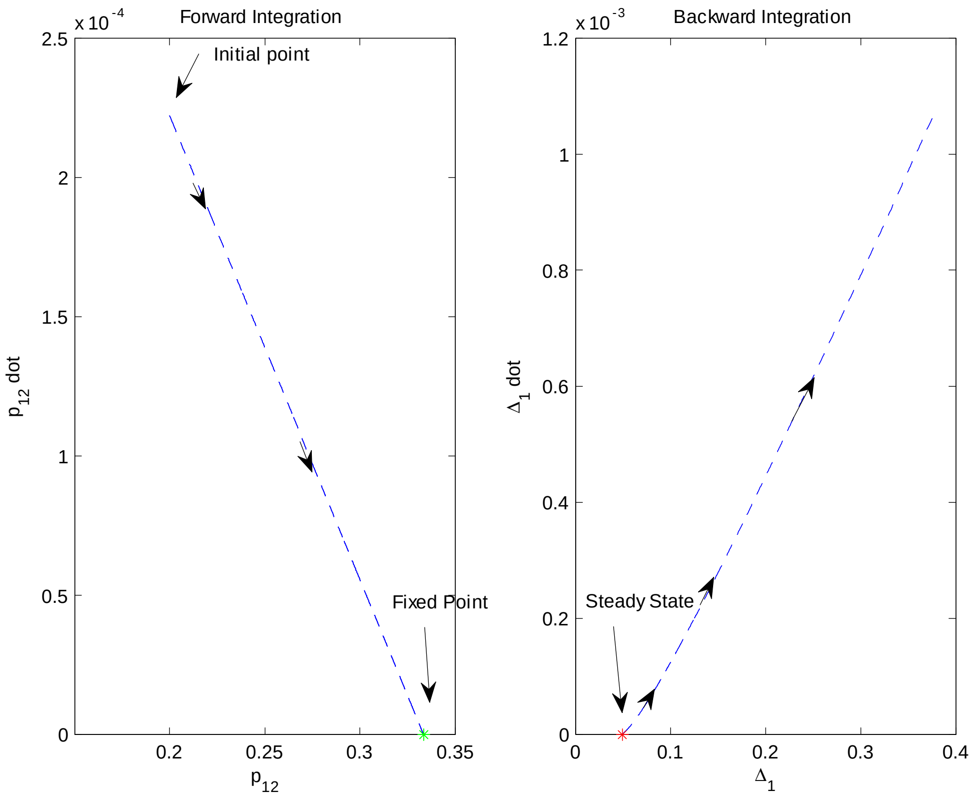
For a given pattern of χ_i the solution of Δ_i can be obtained numerically by integrating (20) backward in time,[6] starting from a neighborhood of the steady state Δ_i^*, where this satisfies $(\alpha\chi_i + \delta)\Delta_i^* + \omega_i = 0$.

It is important to recognize that the distribution of inventories $\mathbf{p}(t)$ and the value functions in differences $\Delta_i(t)$ could be studied together as a whole system. In particular, it is possible to generate equilibrium trajectories by simply applying the principle of backward induction as long as the profile of strategies of the population are kept consistent with the

[5] $\Delta_i(\boldsymbol{\theta}^{ss}, \mathbf{p}^{ss})$ could also be used as initial point. In principle on $\Delta_i(\boldsymbol{\theta}^{ss}, \mathbf{p}^{ss})$ the system stays still, but if computed numerically, there is always a small machine error which allows the integration to start. In the experiments the difference – in norm – between the two points is smaller than 10^{-5} when $|F(\mathbf{p}(t)| < 10^{-6}$.

[6] The mechanism is illustrated in a figure contained in the Technical Appendix.

Figure 4: Combining Forward and Backward Integration



Note - For parameters see Table (2), with $r_2 = 0.1$. The initial value of $p_{1,2}$ is 60 percent of its steady state value. The slope of the right plot is $\alpha\chi_1(t) + \delta$ with $\tilde{\tau}_1(t) = 0$ and $\boldsymbol{\theta}(t) = [0, 1, 0]$. The path of $\mathbf{p}(t)$ in $\chi_1(t)$ is obtained by integrating the system (10)-(12) forward in time (left plot).

sign of $\Delta_i(t)$ at each point in time. The problem with this procedure is that it is hard to build a trajectory that goes through a designated point in the space of the assets' distribution. In models with unidimensional state variable, backward induction works well– at least when the dynamics are not cyclical or chaotic – because it can be stopped when the state variable reaches a desired level. But as the dimension of the manifold expands, guiding the system towards a particular point on the state space (that is on the initial condition) becomes a hurdle. In fact, if the system has Liapunov exponents of a different order of magnitude, some regions of the manifold cannot even be reached. Conversely, the method proposed here gives total control on the initial condition, a feature that turns out to be essential for most interesting macroeconomic experiments.

The right plot of Fig. (4) illustrates a solution of (20) for $i = 1$ for a particular pattern of χ_1. This pattern is obtained by integrating forward in time (10)-(12) starting from some arbitrary initial $\mathbf{p}(0)$, under an exogenous profile of strategies.

Liquidity

This section constructs the indices of market thickness, frequency of trade, and acceptability. The market thickness captured by the stock of commodity i on the market at a given time is

$$x_i(s) = p_{i+2} + (\mu_{i+1} - p_{i+1}),$$

for all i. Let $o_i(s)ds$ be the probability that good i is offered (but not necessarily traded) on the market between time s and $s + ds$. Then

$$o_i(s) = \alpha p_{i+2}[p_{i+1} + (\mu_i - p_i) + (p_i + \mu_{i+2} - p_{i+2})\theta_{i+2}]+ \\ \alpha(\mu_{i+1} - p_{i+1})[p_i + (\mu_{i+2} - p_{i+2}) + (p_{i+1} + \mu_i - p_i)(1 - \theta_{i+1})];$$

for $i = 1, 2, 3$.

Let $t_i(s)ds$ the probability that good i is traded on the market between time s and $s+ds$. Then

$$t_i(s) = \alpha\{p_{i+2}(\mu_i - p_i(1 - \theta_{i+2}) + \theta_{i+1}p_{i+1}) + (\mu_{i+1} - p_{i+1})[p_i+ \\ (\mu_i - p_i)(1 - \theta_{i+1}) + (\mu_{i+2} - p_{i+2})(1 - \theta_{i+2})]\}.$$

The 'velocity' of circulation of good i is $v_i(s) = \frac{t_i(s)}{x_i(s)}$. This quantity is not a good indicator of the moneyness of an object, because, paradoxically, the velocity can be very high even if it is rarely traded.

Finally, the acceptability of commodity i is

$$a_i(s) = \frac{t_i(s)}{o_i(s)}.$$

This indicates how willing people are to accept commodity i, once it is being offered.

Numerical Methods

All the programming is done in Matlab. The main running file, called 'dynamics_KW', sets the parameters, specifies the initial distribution of inventories and the initial guess of the strategies, launches the iteration solution procedure, and generates the plots. The iteration procedure is based on the interaction between two files, triggered within the file 'dynamics_KW': 'backward_ode_kw89' and 'forward_ode_kw89'. The 'backward' file gives instructions to integrate the distribution of inventories forward. The resulting distribution of inventories (reversed with respect to time) is saved and used as an input for in the 'backward' file. This integrates the value functions using, as initial condition, their steady state values. The 'backward' file delivers the three the value functions (in differences) of the representative agents $i = 1, 2, 3$ as well as a profile of strategies. The 'dynamics_KW' file saves this profile and uses it as new initial guess for the overall population. After each iteration, the 'dynamics_KW' file computes the differences between vector θ_i $(t)$and of $\tau_i(t)$. When no discrepancy is noticed, the iteration is stopped, the resulting trajectory of strategies and of the assets distribution is recorded as an equilibrium, and all remaining variables (acceptability, consumption, etc...) are computed along such a trajectory. In both in the backward and in the forward files the ordinary differential equations are computed at a fixed 0.0001 time-step of a year. The Runge-Kutta approximation method with adjustable steps readily available in Matlab, is not used because it would require an additional layer of coding to synchronize the timing in the 'backward_ode_kw89' and 'forward_ode_kw89' file.