# Optimal insurance for time-inconsistent agents

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**Abstract :** We examine the provision of insurance against non-observable liquidity shocks for a time-inconsistent agent who can privately store resources. When the lack of self control is strong enough, hidden storage does not constrain the allocation of resources. The optimal contract is strictly concave and similar to a credit contract : It allows the agent to borrow at an increasing cost, and save at a decreasing rate of return. Extending the model to an infinite horizon, we then show that, in the presence of repeated shocks, the optimal contract leads consumers to impoverishment almost surely. By contrast, the optimal contract for a time-consistent agent only allows him to borrow at the economy's interest rate, and induces him to almost surely accumulate wealth indefinitely. Those results bring out how lack of self-control changes the nature of optimal savings and borrowing instruments, and may lead individuals and states to impoverishment. We discuss applications to consumer over-indebtedness, social security design, sovereign debt crises and Pigovian regulation.

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# 1 Introduction

As was first laid out in Diamond-Dybvig (1983)'s seminal paper, consumers exposed to future preference shocks optimally smooth their consumption through contracts with financial intermediaries. The resulting flexibility in the timing of consumption, while desirable under the time-consistent preferences assumed in the literature on optimal insurance provision, however may be costly when consumers have present-biased preferences. As pointed out by Strotz (1956), Phelps-Pollack (1968), Laibson (1997) and the subsequent literature, consumers then value constraints on their future choices. The main purpose of this paper is to analyze optimal saving and borrowing instruments for a weak-willed agent, and to understand whether such contracts are able to stabilize debt and consumption when the agent faces repeated shocks. These issues are particularly relevant when time-inconsistency may induce excessive debt accumulation, by either individual or governement (as pointed out by Persson-Svensson 1989 and Alesina-Tabellini 1990 regarding the latter).

We adress these issues for an agent with quasi-hyperbolic preferences. We first assume that, as in Diamond and Dybvig, there are two periods of consumption, 1 and 2, and an initial period 0 at which the agent contracts with a competitive financial intermediary (who therefore maximizes his expected utility). The agent receives at date 1 a private taste shock $\theta$ that acts as a multiplicative factor on his utility. The first-best contract is an insurance mechanism that allocates higher consumption to "high types", i.e. individuals facing liquidity shocks in period 1. However, under asymmetric information, the agent may report untruthfully in period 1 a higher taste shock. In other words, a "low type" may masquerade as a "high type" so as to increase his utility. This agency problem is worsened by the lack of self-control, but the agent is aware of it, i.e. sophisticated, so that he prefers a contract which counteracts at least partly his lack of self control by penalizing date 2 consumption. Besides, we assume as in Jacklin (1987) that the agent can secretly store money between periods at the same rate of return as the principal. In many situations indeed, the principal cannot perfectly control agent's savings, whereas access to credit is usually closely monitored (through "credit bureaus" for consumers, or IMF's conditionality for emerging countries) or at least limited [1]. We later extend this model to an infinite horizon.

Our work unveils two marked differences when the agent is time-inconsistent. First, there is still a role for multi-period contracts even when hidden storage is allowed. By contrast, a time-consistent agent would insure against unobservable shocks but the principal cannot provide such insurance if the agent can secretly store. Secondly, divergence

1. For instance, an individual would have to engage in a specific and costly contract in order to use his retirement account as collateral and get long-term funding from another financial intermediary.

between time-consistent and hyperbolic agents becomes even wider in the presence of repeated shocks : the optimal allocation leads almost surely a time-inconsistent agent to impoverishment whereas the time-consistent agent almost surely accumulates wealth indefinitely. Such differences arise from the fact that the lack of self-control changes the nature of the contract, that switches from an "insurance" to a "credit" contract. Indeed, a transfer of resources at the detriment of high types must be implemented : the optimal allocation is a credit contract, that allows the agent to borrow and lend at some contractual interest rate.

These results shed light on the process that leads individuals and states to overindebtedness. Regarding households [2], we show that high interest rates induce time-inconsistent consumers to accumulate unlimited debt, even if one may expect high rates to keep some of them away from temptation. In other words, high interest rates are not a strong enough commitment device to prevent impoverishment. This by itself does not suffice to reopen the case for usury laws, but it shows that such tariffs may be suboptimal from a welfare perspective. It also contributes to the understanding of behavorial patterns that may explain some empirical puzzles related to the usage of credit cards [3]. Regarding government spending, our result shows that political opportunism may prevent countries from accumulating assets, even if such strategy may relax their limited commitment problem and avoid a self-fulfilling sovereign debt crisis. In order to avoid accumulating debts, Government may instead limit or forego insurance through a fiscal rule.

Paradoxically, immerisation of time-inconsistent agents holds even though, at the aggregate level, precautionary motives may dominate and induce the population of such agents to save money at each period. More precisely, we show that this is the case when prudence is greater than twice the risk aversion or, equivalently, when the inverse of the derivative of agents' utility function is concave, a property reminiscent of Rogerson's result (1985) on repeated moral hazard.

The second-best allocation is strictly concave. Put differently, the interest rate implicit in the optimal contract increases with the amount of borrowing and decreases with savings. This result supports the idea that progressive incentive scheme may be needed in order to cope with time-inconsistency. In particular, this analysis can be applied to the optimal design of a pension system when agents suffer from the temptation to overconsume. When transfers between types are not allowed, Amador et al. (2006) show that the optimal scheme relies on imposing a minimum level of savings. Such covenants are indeed

2. Of course, household's overindebtedness has other commonly discussed causes such as financial illiteracy and risk factors (family breakdowns, unemployment, illness).

3. Namely, the co-holding of liquid assets and consumer debt pointed out by Gross and Souleles 2002 : this is partially accounted for by rational model, cf. Telyukova 2013, but may also be partly a consequence of the lack of self-control and, as showned by this work, the inefficiency of interest rates as a commitment device.

imposed by most pension schemes. However, contributions to pension schemes generally also enjoy tax relief, as with tax-favored retirement savings account with contribution limits. The principal in our approach is the financial institution, and one may expect commitment devices to be provided by the market without any State intervention. Provided that a paternalistic role of the state is justified [4], our result may support a government intervention in which state subsidies decrease with the amount of money invested in the pension plan.

The results admit reinterpretation beyond the realm of intertemporal consumption smoothing. It applies for example to situations where a continuum of agents with heterogenous preferences do not internalize a positive externality generated by the consumption of a "social good" such as education, R&D or vaccines. The classical response is to line up private incentives with social ones by using a linear Pigovian subsidy equal to the externality. Our analysis shows that, when the externality is strong enough, the solution relies on a non-linear subsidy decreasing per unit of social good. These results support for instance government subsidies for education of children whose parents have low altruism, as conjectured by Amador et al. (2006). The non-linearity means in that case that more subsidies should be given for the first schooling expenditures.

At the technical level, the corresponding optimal control problem in all its generality is complex - the existence and regularity of an extremum is not straightforward [5] , so that we make different additional restricting assumptions to solve it. We first study a two-period model with a continuum of shocks in order to describe the optimal contract, and establish its concavity property. We need to assume that the shocks are uniformly distributed, but provide robustness checks for the main findings by running simulations of the model with normally distributed shocks. We don't exhibit an explicit threshold for the short-term quasi-hyperbolic discount factor under which our results hold but, according to our simulations, this is satisfied for empirically relevant values of time-inconsistency typically measured in laboratory experiments and used as a calibration parameter in quantitative studies (such as 0.7 in Angeletos et al. 2001). Secondly, we check that our findings still hold with a two-shock version of the model, and extend it with an arbitrary horizon to derive long-run properties for agents with constant relative risk aversion. This latter assumption is quite restrictive, but it does not prevent us from bringing out the different possible scenarios previously described in order to point out the role of prudence

4. For instance in order to take into account negative externalities induced by low level retirement savings, or to cope with consumers' naivety (but such justification is outside the scope of this paper which focuses solely on sophisticated consumers).

5. Indeed, the state and the control variables are subject to joint 'mixed' constraints whereas the state variable is a convex function. Numerical simulations (cf. figure 1c) show that this last constraint is binding. Optimal control problem for which admissible functions are nondecreasing ones have been adressed in specific cases, cf. in particular Rochet & Choné 1998 and Carlier 2001, but extending those methods is outside the scope of this paper.

and time-inconsistency.

## 2 Related literature

The innovation of the present study relative to the existing literature is that it allows arbitrary resource transfers. Indeed, our two-period model is very similar to the one considered by Amador et al. (2006), except that they consider a type-by-type budget constraint while we allow resources to be transferred among types. The infinite horizon model is also very similar to that in Atkeson and Lucas (1992), except that they limit the total consumption handed out in each period to a population of time-consistent households while we allow intertemporal transfers among time-inconsistent households. Those differences are crucial when analyzing the optimal tradeoff between flexibility and commitment : whereas additional resources must be targeted at agents facing a high liquidity shock, those agents must be latter penalized in order to induce truthful reporting. Hence, the constrained optimum requires transfer of resources among types and periods.

The natural benchmark is when the agent is time-consistent. Two different strands of litterature provide important insights on that situation, whether you consider taste shocks (as in Diamond-Dybvig 1983) or revenue shocks (as in Townsend 1982). According to these works, the agent insures against unobservable shocks by holding a multiperiod contract that may, as shown by Thomas and Worrall (1990), lead him almost surely to impoverishment in the presence of repeated shocks. Somehow, if we assume, as in Jacklin (1987) or Cole-Kocherlakota (2001), that the agent can secretely store money between periods, no insurance can be provided anymore by the principal : the agent only borrows and saves using risk-free bonds and, as shown by Aiyagari (1994) and Chamberlain-Wilson (2002), almost surely accumulates precautionnary savings indefinitely. The model developed in this paper contains the case of a time-consistent agent as a special case, and extends it to take into account present-biased preferences.

Our results indicate that Thomas and Worrall's impoverishment result still hold when hidden storage is allowed, as long as the lack of control is strong enough. This reconciles the initial view that hyperbolic discounting leads to undersavings (Laibson 1997, 1998), with the result of Salanie and Treich (2006) showing that hyperbolic consumers may either oversave or undersave when knowing that they will suffer from lack of self control in the future. In this latter work, sophisticated consumers can only alleviate their time-inconsistency by saving more. When such consumers have access to commitment devices, our work shows that time-inconsistent households will on average accumulate debts or savings, depending on the same criteria of prudence identified by Salanie and Treich. Somehow, those households will nevertheless almost surely become over-indebted whatever their prudence, whereas time-consistent consumers almost surely accumulate wealth, which is consistent with the initial intuition of Laibson.

This work is related to the large literature that deals with the policy implication of time-inconsistency. In particular, regarding sovereign debt, we show that a present-biased government may forego insurance in order to avoid overindebtedness. In that situation, the optimal rule consists of a spending cap that will limit debt according to Amador, Angeletos and Werning (2006), but Halac and Yared (2014) show that the optimal ex-ante fiscal rule may still induce immerisation when shocks are persistent. Regarding private saving, Diamond and Köszegi (2003) provides another mechanism - endogeneous retirement decisions - that can also deeply change the effect of lack of self control on savings discussed in the previous paragraph. In their framework, time-inconsistent households can strategically undersave to 'force' work or oversave to encourage early retirement.

Our result matters also for a social planner who has a higher discount factor than individuals, because he values more future generations. Farhi and Werning (2007) have looked at such a problem except that, consistent with the framework of Atkeson and Lucas (1992), they consider a pure redistributional issue, and thus do not allow transfer among generations. In Atkeson and Lucas's framework, the degree of inequality continually increases, leading individual to impoverishment almost surely. By contrast, Farhi and Werning show that, when the social planner uses a more patient geometric discounting, a steady state solution exists with no one trapped at misery. Our two-period model is similar [6] to the one considered by Farhi and Werning, and our work shows that, apart when utility functions are logarithmic, transfers among generations may enhance allocative efficiency. Our results show also that the long-term properties in the presence of repeated shocks strongly differs, depending on how the principal and the agents disagree with respect to discounting (i.e. more or less patient geometric discounting, or geometric vs. quasi-hyperbolic discounting [7]).

# 3 A model of insurance for time-inconsistent agents

We analyze the optimal allocation of consumption in the presence of liquidity shocks when consumers are time-inconsistent, with a quasi-hyperbolic parameter $\beta$. Utility functions in the two periods of consumption ($t = 1, 2$) are $u$ and $w$ respectively. There is a

6. More specifically, they consider an economy populated by a continuum of individuals who live for one period and are replaced by a single descendant in the next, and assume that the social planner values future generation, so that the social discount factor $\hat{\alpha}$ is higher than the private one $\alpha$. In a two-period economy, this model is similar to ours once we identify period 1 and period 2 consumption with parent's and child's consumption, and set $\hat{\alpha} = \delta$ and $\alpha = \beta\delta$.

7. The models differ indeed on a longer time horizon since agents at period 1 will respectively put a geometric $\alpha^2 = \beta^2\delta^2$ or quasi-hyperbolic $\beta\delta^2$ weight on consumption at period 3.

single consumption good available in each period. We impose a subset of the Inada conditions [8], i.e. utility functions are smooth, increasing, strictly concave and such that the limit of their derivative at 0 and $\infty$ are respectively $\infty$ and 0. The consumer faces in the first period a liquidity shock $\theta > 0$ in a bounded interval $\Theta = [\underline{\theta}, \bar{\theta}]$, with a distribution represented by a continuous distribution function $f(\theta)$. We denote by $F(\theta)$ the corresponding cumulative distribution function. Utility functions are chosen so that the average shock is normalized to one, i.e. $\int \theta f(\theta) = 1$. The liquidity shock is privately observed at date 1 by the consumer, who contracts at date 0 with a competitive financial intermediary in order maximize his expected utility

$$\max_{c,k} \int_{\underline{\theta}}^{\bar{\theta}} (\theta u(c(\theta)) + w(k(\theta)))f(\theta) \qquad \text{(M)}$$

where $c$ and $k$ denote the first and second period's consumptions respectively, subject to incentive compatibility constraints :

$$\theta u(c(\theta)) + \beta w(k(\theta)) \geq \theta u(c(\theta^{'})) + \beta w(k(\theta^{'})) \; \forall \theta, \theta^{'} \qquad (IC)$$

and some budget constraint that remains to be specified. The financial intermediary can invest 1 at $t = 1$ to get at $t = 2$ return $\delta^{-1} \geq 1$. The allocation defines a type-contingent budget function $B(\theta) = c(\theta) + \delta k(\theta)$. In contrast with Amador et al. (2006) who consider a type-by-type budget constraint $B(\theta) \leq y$, and with Atkeson and Lucas (1992) who assume a fixed endowment in all periods (i.e. $\int c(\theta)f(\theta) \leq e$ and $\int k(\theta)f(\theta) \leq e^{'}$), we follow Diamond-Dybvig (1983) in considering a global budget constraint that allows transfer of resources between types and periods :

$$\int (c(\theta) + \delta k(\theta))f(\theta) = \int B(\theta)f(\theta) \leq y \qquad (BC)$$

We consider in the following the optimization problem (M) that maximizes the expected utility under the constraints (IC) and (BC). The first-best contract, when taste shocks are not private information, equalizes the marginal utilities, i.e.

$$\theta u^{'}(c(\theta)) = \delta^{-1} w^{'}(k(\theta)) = \text{ constant } \forall \theta. \qquad (1)$$

The first-best contract is thus an insurance mechanism that allocates higher consumption to individuals facing liquidity shocks in period 1, while consumption in period 2 is type-independent. When shocks are privately observed, consumers are tempted to report a high shock and consume more immediately. In order to prevent that, lower consumption in period 2 must be allocated to reported high types.

Regarding the constrained optimum, a useful benchmark is when the second period's utility is linear, i.e. $w(k) = wk$. In this case, reallocating period-2 consumption among

8. We do not impose that the value of $u$ and $w$ at zero is zero. In particular, all CARA utility functions satisfies our conditions.

types has no impact on the expected utility. One can then choose period-1 consumption in order to satisfy the relation (1) of the first-best contract, and pick period-2 consumption in order to respect the budget constraint (BC) subject to the incentive compatibility constraints (IC). The (IC) constraints are equivalent to $\theta \in \operatorname{argmax}_{\theta'} \left(\theta u(c(\theta')) + \beta w k(\theta')\right)$. When the relation (1) is satisfied, the first-order condition is equivalent to $c'(\theta)+\beta\delta k'(\theta) = 0$ whereas the second-order condition is satisfied (since it is then equivalent to $c$ non-decreasing). Therefore, the optimal contract is a first-best contract in which period-2 consumption is chosen such that $c(\theta) + \beta\delta k(\theta)$ is constant or, in other words, such that per-type budget is constant once the hyperbolic factor is included in the discount rate.

The optimal contract is thus like a credit contract in which additional resources used in period 1 must be paid back with interest in period 2. More precisely, time-inconsistency can be perfectly corrected by a fixed increase in interest rate from $1/\delta$ to $1/\beta\delta$. In a consumption-savings framework, with a competitive intermediary providing credit cards, the increase of interest rate must be offset by a fixed increase in consumption or, equivalently, by lowering credit card's fixed fees. We get therefore the following proposition which is a generalization of the result obtained by DellaVigna and Malmendier (2004), who assume that both $u$ and $w$ are linear.

**Proposition 1.** *When the second period's utility function is linear, time-inconsistency has no impact on welfare, as it can be perfectly corrected by an increase of interest rate i.e. a return $1/\beta\delta$, above the technological rate of return $1/\delta$.*

Somehow, as soon as the second period's utility is strictly concave, lack of self control makes it more difficult to induce truthful reporting : incentives must be strengthened and, as shown in the following proposition, the constrained optimum moves away from the first-best contract when time-inconsistency increases.

**Proposition 2.** *When $w$ is strictly concave, the expected utility provided by the optimal contract strictly decreases when time-inconsistency increases (i.e. $\beta$ decreases).*

In order to study later the dynamic with repeated shocks, we also consider a similar model with two shocks $\theta_h > \theta_l$. It is defined by the following maximization program.

$$\max \pi(\theta_h u(c_h) + w(k_h)) + (1-\pi)(\theta_l u(c_l) + w(k_l)) \qquad \text{(M2)}$$

where $\pi$ is the probability of the shock $\theta_h$, and where the first and second period's consumptions for type $x \in \{l, h\}$ are denoted respectively by $c_x$ and $k_x$. The allocations are subjected to incentive compatibility constraints

$$\begin{cases} \theta_h u(c_h) + \beta w(k_h) \geq \theta_h u(c_l) + \beta w(k_l) & (IC_h) \\ \theta_l u(c_l) + \beta w(k_l) \geq \theta_l u(c_h) + \beta w(k_h) & (IC_l) \end{cases}$$

and budget constraint

$$\pi(c_h + \delta k_h) + (1 - \pi)(c_l + \delta k_l) \leq y \qquad (BC2).$$

# 4 Optimal transfer of resources

The objective of this section is to understand how lack of self control constrains the allocation of resources among types, and in particular what type of resource transfer among types is implemented by the optimal contract. Specifically, we want to find out whether the optimal contract is an "insurance" or "credit" type contract. Following the previous section, we define this typology of contract by the monotonicity of the budget function : a contract will be considered an insurance when $B(\theta)$ is increasing, and a credit contract when it is decreasing. Indeed, an insurance contract pools risks by transfering resources to consumers in need of liquidity, whereas a credit line is another, but costly way, to provide liquidity to such consumers [9].

## 4.1 No hidden savings

We follow thereafter an intuitive approach, by using perturbations of the optimal allocation $\{u(\theta), w(\theta)\}$ that maximizes the Lagrangian associated with the budget constraint. Detailed proofs of the results are provided in the appendix. As usual, and with a slight abuse of notation, it is convenient to change variables from $c(\theta)$ and $k(\theta)$ to $\text{U}(\theta) = u(c(\theta))$ and $\text{W}(\theta) = w(k(\theta))$, and to consider the state variable $V(\theta) = V(\theta, \theta)$ where $V(\theta, \theta') = \theta\text{U}(\theta') + \beta\text{W}(\theta')$. When those functions are differentiable, the constraints (IC) are equivalent to $\text{U}(\theta)$ non-decreasing and $\theta\text{U}'(\theta) = -\beta\text{W}'(\theta)$. In the following, we drop the constraint " $\text{U}(\theta)$ non-decreasing ", and use simple perturbations that satisfy the relation

$$\theta\text{U}'(\theta) = -\beta\text{W}'(\theta). \tag{2}$$

First, we can get the so-called " transversality conditions " associated with the maximization problem by using perturbations at the two ends of the interval $\Theta$. Indeed, when $\theta = \underline{\theta}$ or $\bar{\theta}$, we can slightly change the optimal allocation in an incentive compatible way by adding $\theta\epsilon$ to $\text{W}(\theta)$ and substracting $\beta\epsilon$ from $\text{U}(\theta)$ for $\epsilon$ infinitely small. The expected utility is then increased by $(1 - \beta)\theta\epsilon$ whereas the overall budget increases by $\delta\theta\epsilon/w'(k(\theta)) - \beta\epsilon/u'(c(\theta))$. Those two terms must be equal, up to a positive multiplicative factor equal to the budget multiplier $\lambda$. More precisely, using the straightforward

9. This difference between credit and insurance becomes more obvious if we consider income shocks $u(c, \theta) = u(c - \theta)$, instead of taste shocks $u(c, \theta) = \theta u(c)$, so that the liquidity shock is an adverse rather than positive shock on utility. This alternative model is considered in section 5.2.

relation $B'(\theta)/\mathrm{U}'(\theta) = 1/u'(c(\theta)) - \delta\theta/\beta w'(k(\theta))$, the transversality conditions are :

$$\frac{B'(\theta)}{\mathrm{U}'(\theta)} = \lambda^{-1}(1-\beta^{-1})\theta \quad \text{for} \quad \theta = \underline{\theta}, \bar{\theta}.$$

Those conditions give us the marginal transfer of resources $dB(\theta)$ that must be implemented by the optimal contract at the two ends of the interval $\Theta$ in order to provide an additional unit of utility $d\mathrm{U}(\theta)$ to type $\theta + d\theta$. This transfer is zero [10] when $\beta = 1$ and strictly negative when $\beta < 1$.

We can similarly derive the Euler-Lagrange equation. We consider the perturbation starting at $\theta \in \Theta$ defined by $\mathrm{W}^\epsilon(\tilde{\theta}) = \mathrm{W}(\tilde{\theta}) + \epsilon\theta\mathbb{I}_{\tilde{\theta}\geq\theta}$ and $\mathrm{U}^\epsilon(\tilde{\theta}) = \mathrm{U}(\tilde{\theta}) - \beta\epsilon\mathbb{I}_{\tilde{\theta}\geq\theta}$. This perturbation is clearly incentive compatible, and equivalent to $k^\epsilon(\tilde{\theta}) \simeq k(\tilde{\theta}) + \epsilon\theta\mathbb{I}_{\tilde{\theta}\geq\theta}/w'(k(\tilde{\theta}))$ and $c^\epsilon(\tilde{\theta}) \simeq c(\tilde{\theta}) - \beta\epsilon\mathbb{I}_{\tilde{\theta}\geq\theta}/u'(c(\tilde{\theta}))$. The budget is raised by

$$\int_\theta^{\bar{\theta}} \left( -\frac{\beta}{u'(c(\tilde{\theta}))} + \frac{\delta\theta}{w'(k(\tilde{\theta}))} \right) \epsilon f(\tilde{\theta}) d\tilde{\theta}$$

whereas the variation of the expected utility is

$$\int_\theta^{\bar{\theta}} \left( \theta - \beta\tilde{\theta} \right) \epsilon f(\tilde{\theta}) d\tilde{\theta}.$$

Those two terms must be equal up to the budget multiplier $\lambda$. Differentiating twice this equation with respect to $\theta$ yields the Euler-Lagrange equation :

$$\frac{\partial}{\partial\theta}\left[ f(\theta)\frac{\beta B'(\theta)}{\mathrm{U}'(\theta)} \right] = \frac{\delta f(\theta)}{w'(k(\theta))} - \lambda^{-1}\frac{\partial}{\partial\theta}\left[ F(\theta) + (1-\beta)\theta f(\theta) \right].$$

By integrating this Euler-Lagrange equation, and using the tranversality conditions, we get as well the so-called inverse Euler equation (cf. appendix for a detailed proof). This relation reflects that one cannot improve the solution through a fixed transfer of utility from one period to the other [11], and can also be obtained with a perturbation that increases $w(\theta)$ by $\epsilon$ and decreases $u(\theta)$ by $\epsilon$ for all $\theta$ :

$$\lambda^{-1} = \int_{\underline{\theta}}^{\bar{\theta}} \frac{\delta}{w'(k(\theta))} f(\theta) d\theta = \int_{\underline{\theta}}^{\bar{\theta}} \frac{1}{u'(c(\theta))} f(\theta) d\theta. \tag{3}$$

10. This is quite intuitive when $\beta = 1$ : it would not be optimal to provide some insurance, i.e. slightly more budget, at the ends of the interval, i.e. for instance $B(\bar{\theta}) > B(\bar{\theta} - d\theta)$ for $d\theta$ infinitely small, because one can easily see that the incentive compatibility constraint implies the first order approximation $V(\bar{\theta}, \bar{\theta} - d\theta) \simeq V(\bar{\theta})$, so that it is more efficient to do some bunching at the top by setting $c(\bar{\theta}) = c(\bar{\theta} - d\theta)$ and $k(\bar{\theta}) = k(\bar{\theta} - d\theta)$ and use the extra budget elsewhere. When $\beta < 1$, the incentive compatibility constraints become stronger.

11. It is important to notice that this inverse Euler equation is satisfied for both the constrained optimum and the first-best optimal solution obtained if shocks were not private information. By contrast, as we will see in the last section, when no transfer of resources between types are allowed, the constrained optimum satisfies the Euler equation.

When $\beta = 1$, the Euler-Lagrange equation can be rewritten as

$$\frac{\partial}{\partial \theta}\left[f(\theta)\frac{B^{'}(\theta)}{\mathrm{U}^{'}(\theta)}\right] = f(\theta) \times \left(\frac{\delta}{w^{'}(k(\theta))} - \lambda\right)$$

whereas, when the distribution of shocks is uniform, the Euler-Lagrange equation becomes

$$\frac{\partial}{\partial \theta}\left[\beta\frac{B^{'}(\theta)}{\mathrm{U}^{'}(\theta)}\right] = \frac{\delta}{w^{'}(k(\theta))} - \lambda^{-1}(2-\beta).$$

Since $w^{'}(k(\theta))$ is increasing with excepted value equal to $\delta\lambda^{-1}$, we see that in the first case (resp. the second case) the term $f(\theta)B^{'}/\mathrm{U}^{'}$ is an increasing then decreasing (resp. concave) function of $\theta$. Besides, we know from the transversality conditions that $B^{'}(\underline{\theta})$ and $B^{'}(\bar{\theta})$ are both zero when $\beta = 1$, and both negative when $\beta < 1$. The properties (i) and (iv) of the optimal allocation presented in the following Proposition 3 are then straightforward. Properties (ii) and (iii) follow from a direct application of the previous relations once knowing that the constrained optimum tends to a bunching solution when $\beta$ tends to zero.

This proposition shows that, when consumers are time-consistent, the constrained optimum induces a positive transfer of resources in favor of types who face high shocks, and is thus an insurance contract. By contrast, when lack of self control is strong enough, a transfer of resources at the detriment of high types must be implemented : the optimal allocation is a credit contract, and only allows individuals to borrow and lend at some contractual interest rate.

**Proposition 3.** *When the distribution of shocks is uniform, there exist a unique optimal solution to the maximization problem $(A)$. This solution is smooth, and*

*(i) When $\beta = 1$, the optimum transfers resources to high types, i.e. the budget function $B(.)$ is non-decreasing (insurance contract);*

*(ii) When $\beta < 1$, the budget function is decreasing outside a strict sub-interval (where it is non-decreasing);*

*(iii) When $\beta$ is small enough, the optimum transfers resources to low types, i.e. the budget function is decreasing (credit contract).*

*A similar property hold in general for any distribution of shocks once we assume the existence of optimal solutions $\mathrm{U}(\theta)$ to the maximization problem $(A)$ that are continuous and piecewise-$C^1$. More precisely, there exist then a $\beta^*$ such that, when $\beta < \beta^*$, the optimum transfers resources to low types, i.e. the budget function is decreasing. By contrast, when $\beta = 1$, the budget function $B(.)$ is non-decreasing at least on a left sub-interval.*

*The two-type model exhibits the same properties with an explicit threshold $\beta^* = \frac{\theta_l}{\theta_h}$, that is, the budget function $B(\theta)$ is increasing when $\beta = 1$ (i.e. $B(\theta_h) > B(\theta_l)$), and decreasing when $\beta < \beta^*$.*

We refer to the appendix for detailed proofs. In short, the existence of an admissible solution $V^*$ such that $U(V^*) = U^*$ can be directly proven when we ignore the constraint " $u$ non-decreasing ", and assume that the function $u(\theta)$ belongs to a bounded control region $\mathcal{U}$. Indeed, we can use in this case functional analysis to find a sequence of admissible solutions that converges and whose limit provides $U^*$. Once we know the existence of this optimal solution, necessary conditions of optimality, derived from the Maximum Principle, imply that the function $u$ is smooth and belongs to a bounded region that is independent of our initial choice $\mathcal{U}$ (provided $\mathcal{U}$ is large enough). Besides, when the distribution is uniform, the necessary conditions imply also that $u$ is non-decreasing which proves the existence (H). A generalization of the existence of a smooth unique optimal solution to any distribution of shocks cannot be obtained with the classical tools from optimal control theory used in the mathematical appendix. Indeed, as shown on a numerical example at the end of the section, the condition " $u$ non-decreasing " can be binding even with a standard distribution of shocks such as a normal distribution. One must use more complex tools to address calculus of variations problems for which admissible functions are nondecreasing ones, which is outside the scope of this paper (existence and characterization for some specific cases have been provided, cf. in particular Rochet & Choné 1998 and Carlier 2001).

The following diagram 1 illustrates Proposition 3, and gives an estimate of the threshold value that induces a switch from insurance to credit. This threshold appears to be quite high, close to 0.95, and much higher than empirically estimated values of time-inconsistency (such as $\beta = 0.7$, cf. Angeletos et al. 2001). Those pictures are obtained by solving numerically the maximization problem when the utility of consumption is defined by the constant relative risk aversion specification $u(z) = \frac{(1+z)^{1-\gamma}}{1-\gamma}$ with $\gamma = 7$ and $\delta = 0.957$.

The next diagram 2 shows equivalent results obtained numerically when the distribution of shocks is non-uniform as described by the top-left curve (the distribution $F(\theta)$ is a truncated normal distribution). We check that we still have a budget function that is respectively increasing and decreasing for $\beta = 1$ and $\beta = 0.7$, and double-peaked curves in-between.

Lastly, the diagram 3 exhibits a numerical example with a normal distribution in which the constraint « $\mathrm{U}(\theta)$ non-decreasing » (or, equivalently, $c(\theta)$ non decreasing) appears to be binding at the left side of the interval $[\underline{\theta}, \bar{\theta}]$. As mentioned earlier, this confirms that a generalization to any distribution of shocks of the result (iii) of Proposition 3, regarding the existence of an optimal allocation, cannot be obtained with the classical tools from optimal control theory used in the mathematical appendix.

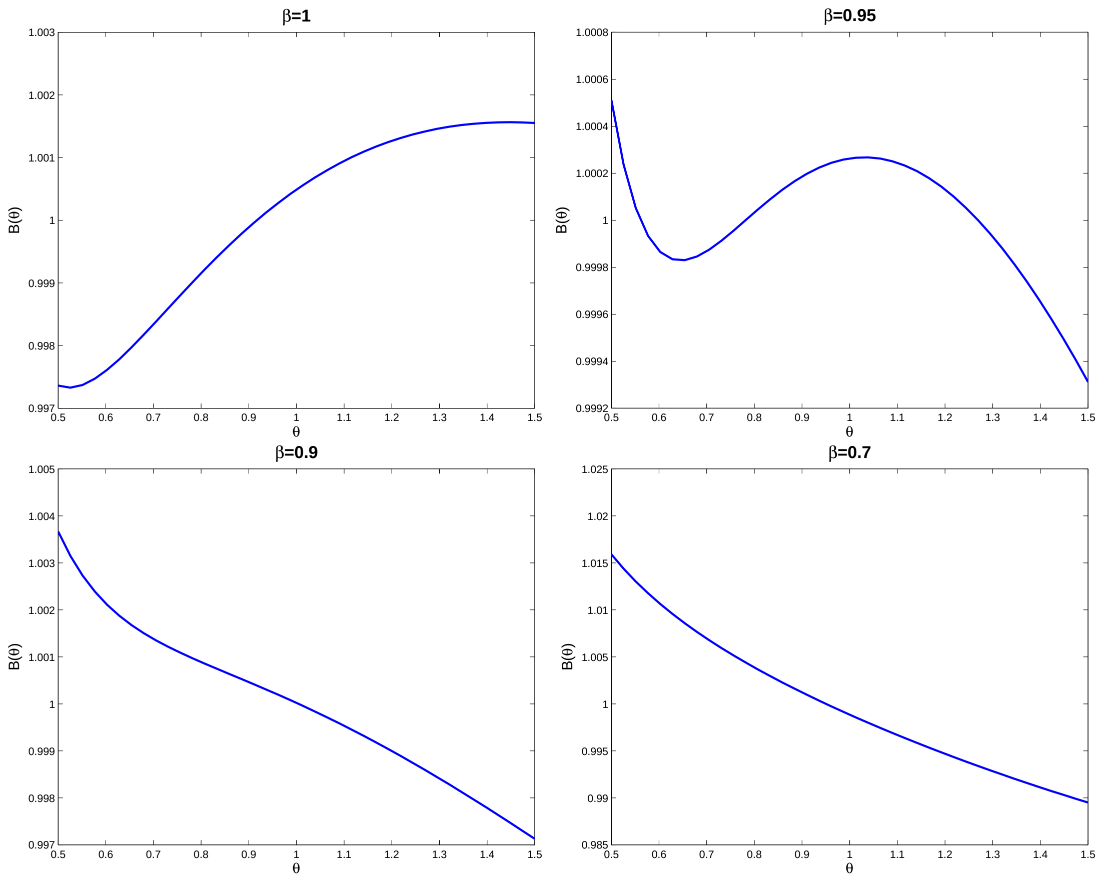


FIGURE 1 – Budget function $B(\theta)$ with a uniform distribution of shocks

## 4.2 Hidden savings

In the spirit of Jacklin (1987), it may be more realistic to assume that consumers can store money between periods at the same rate of return as the principal, i.e. to impose a more stringent incentive compatibility constraint :

$$\theta u(c(\theta)) + \beta w(k(\theta)) \geq \theta u(c(\theta') - \delta\Delta) + \beta w(k(\theta') + \Delta) \ \forall \theta', \forall \Delta \in [0, c(\theta')/\delta] \qquad \text{(HS)}$$

This new constraint implies that $B(\theta)$ is non-increasing. Indeed, it is easy to check that if $B(\theta') > B(\theta)$ for $\theta' > \theta$, then the incentive compatibility constraint does not hold : type-$\theta$ would report untruthfully type-$\theta'$ and save between periods $\Delta = (c(\theta') - c(\theta))/\delta$ (since $c(\theta)$ is non-decreasing). The contract must therefore be a credit contract, i.e. borrowing must be costly in order to prevent consumers from asking more money at period 1 and saving afterward. Similarly, we add hidden storage in the two-type model by changing the incentive compatibility constraints accordingly :

$$\begin{cases} \theta_h u(c_h) + \beta w(k_h) \geq \theta_h u(c_l - \delta\epsilon) + \beta w(k_l + \epsilon), \forall \epsilon \geq 0 & (IC_h) \\ \theta_l u(c_l) + \beta w(k_l) \geq \theta_l u(c_h - \delta\epsilon) + \beta w(k_h + \epsilon), \forall \epsilon \geq 0 & (IC_l) \end{cases}$$

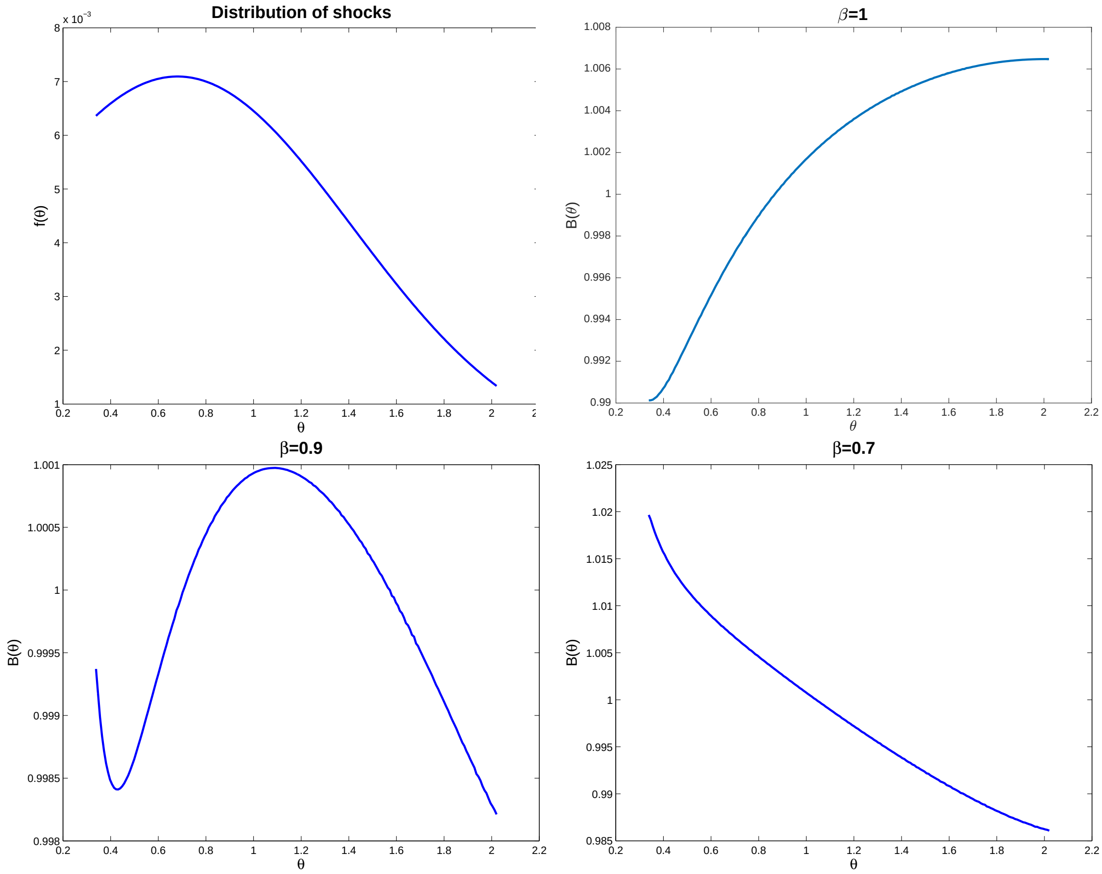


FIGURE 2 – Budget function $B(\theta)$ with a non uniform and non symmetric distribution of shocks (truncated normal)

The following proposition describes the optimal allocation for the two polar cases, time-consistent consumer and strongly hyperbolic consumer :

**Proposition 4.** *When $\beta = 1$, there is no transfer of resources among types : the optimal solution of the maximization problem $(A)$ with hidden storage (and a uniform distribution of shocks) is such that $B(\theta) = y$ for all $\theta \in \Theta$. On the other hand, when $\beta$ is small enough, the hidden storage constraint is not binding so that the solution is the optimum defined in the Proposition 3 (credit contract). In the two-type model, this latter property is true when $\beta < \frac{\theta_l}{\theta_h}$.*

The intuition behind those results is straightforward. It would be optimal to provide more resources to a consumer facing high liquidity shocks but agents' ability to secretely store prevents that. When $\beta = 1$, no resources are transferred among types at the optimal allocation, which is equivalent to a bunching solution where the principal lends the maximum amount of money that the agent would be willing to borrow, with no premium on interest rates. This result extends the *net present value* condition of Cole & Kocherlakota (2001) to the case of a continuum of liquidity shocks (instead of an arbitrary but finite

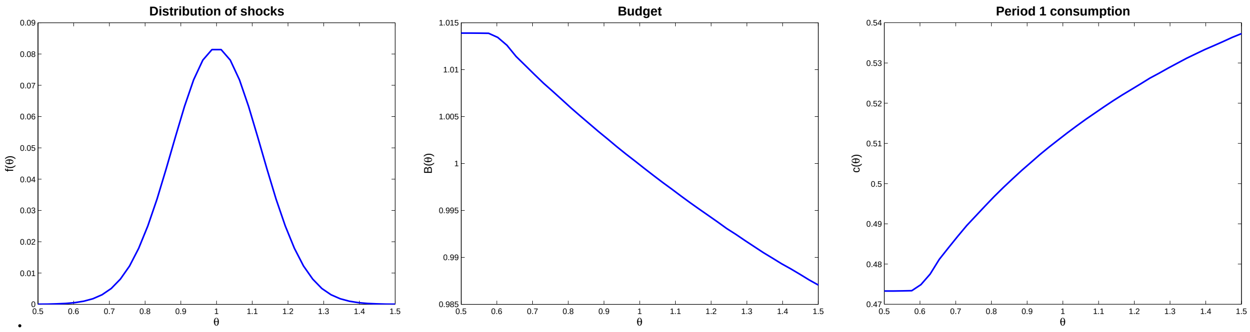


Figure 3 – Example in which the non-decreasing constraint is binding (normal distribution of shocks)

number of revenue shocks, as considered in their work). On the other hand, when $\beta$ is small enough, the optimal contract in the absence of hidden savings only allows individuals to borrow and to lend at some contractual interest rates that are high enough to deter agents to secretely store money. The following picture summarizes in a simple way those results (the middle graph is provided as an illustration, but may be more complex) :

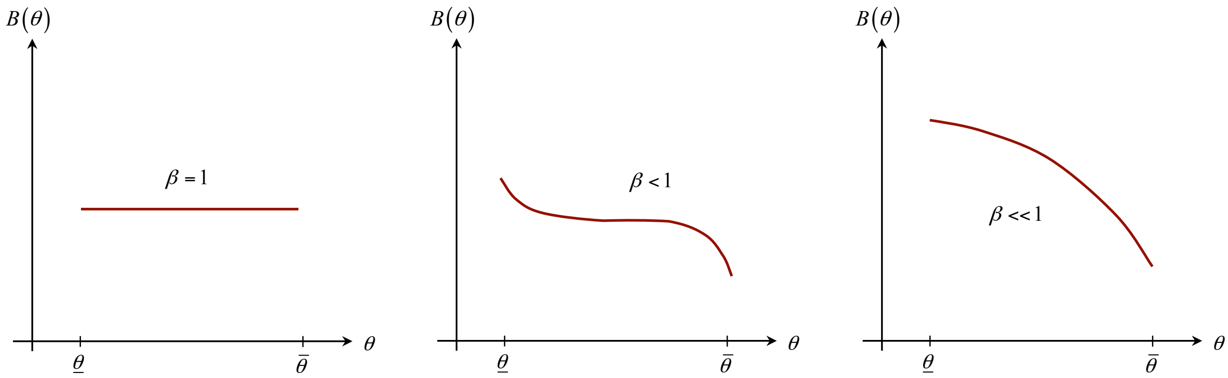


Figure 4 – Budget function $B(\theta)$ with hidden storage

# 5 Optimal taxation

## 5.1 Optimal non-linear scheme

In this section, we study the credit contract that implements the optimal allocation described in the previous section when the lack of control is strong enough so that the hidden storage constraint is not binding. Such allocation $\{c(\theta), k(\theta)\}$ is characterized by the slope of the curve $k(c)$ defined, with a slight abuse of notation, by the relation $k(.) = k(c^{-1}(.))$ on the domain spanned by $c$. In the consumption-savings framework presented above, this slope reflects the marginal cost of borrowing as well as the marginal return on saving. More precisely, if the consumer borrows and consumes at period 1 an additional $dc$, he will have to reimburse $-dk$ at the second period, and thus pay an interest rate equal to $-dk/dc - 1$.

We know from the previous section that there is no transfer of resources among types when consumers are time-consistent and can privately store money : the slope of the curve $k(c)$ is then constant and equal to $-1/\delta$. Thus, if $\delta = 1/(1+r)$ where $r$ is the interest rate on bonds in the market, the consumer benefits from a premium-free interest rate when borrowing. By contrast, when consumers are time-inconsistent with a second-period linear utility, we have seen with the Proposition 1 in section 2 that the slope of the curve $k(c)$ is also constant but steeper and equal to $-1/\beta\delta$ : the consumer pays a higher interest rate, equal to $r + \rho = 1/\beta\delta - 1$ where the premium is given by $\rho = 1/\beta\delta - 1/\delta$. Reversely, consumer's savings benefit from a higher return. Thus, this markup $\rho$ can also be interpreted as a subsidy or tax relief on savings.

In the general framework we consider, the following proposition shows that, when the lack of self control is strong enough, the curve $k(c)$ is strictly concave, i.e. the interest rate on borrowing (resp. return on savings) increases with the amount of borrowing (resp. decreases with the amount of savings). As discussed in the introduction, this argues for designing strictly concave schemes, such as pension tax reliefs at a rate that decreases with the retirement savings effort of time-inconsistent consumers.

**Proposition 5.** *The implementation curve $k(c)$ is decreasing and,*

*(i) When $\beta < 1$, the slope of the implementation curve is steeper at the upper value of $\theta$ than at its lowest value. More precisely, $k^{'}(c(\bar{\theta})) < k^{'}(c(\underline{\theta})) < -\frac{1}{\delta}$ ;*

*(ii) When $\beta$ is small enough, the implementation curve $k(c)$ is strictly concave.*

Again, we check thereafter that the property we obtained is valid once the model is calibrated with reasonable values for $\beta$. This is equivalent to check that the interest rate premium above the technological rate $\rho(c) = -k'(c) - 1/\delta$ is increasing. In the following pictures, we compare this premium with the linear utility benchmark case discussed in the previous paragraphs, that is a constant premium equal to $\rho = 1/\beta\delta - 1/\delta$. We report the curve $k(c)$ for two different values $\beta$ whereas the benchmark is given by the horizontal line. We see that the curve $k(c)$ is indeed concave for $\beta = 0.5$ and lower, and that its slope moves in the case $\beta = 0.7$ from 30% to 60%.

## 5.2 Intuition and heuristics for a more general class of shocks

Using approximation, a more intuitive sketch of these results can be given that generalizes to a larger class of shocks. Let us consider for instance utility functions $u(c, \theta)$ such that $u_{c\theta} = \partial^2 u/\partial c \partial\theta \geq 0$. This last property holds for both taste shocks $u(c, \theta) = \theta u(c)$

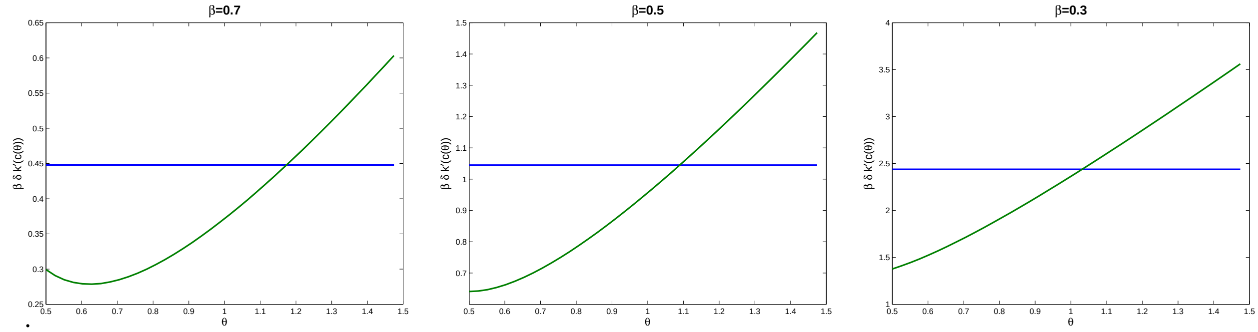


FIGURE 5 – Interest rate premium (general case and linear utility benchmark)

and income shocks $u(c, \theta) = u(c - \theta)$. In this general framework, the first-best contract equalizes marginal utilities, i.e.

$$u_c(c(\theta), \theta) = \delta^{-1} w'(k(\theta)) = \text{constant}. \tag{4}$$

By contrast, when shocks are privately observed, the agent must be induced to report truthfully his shock, i.e. the optimal allocation satisfies the incentive compatibility constraints

$$c'(\theta) u_c(c(\theta), \theta) = -\beta k'(\theta) w'(k(\theta))$$

or, equivalently,

$$k'(c) = -\frac{u_c(c(\theta), \theta)}{\beta w'(k(\theta))} \tag{5}$$

When $w$ is linear, the choice of a linear curve $k(c)$ allows to satisfy simultaneously the incentive constraint (5) and the first-best relation (4). More precisely, using similar arguments as in Proposition 1, one can show that the constrained optimum is the first-best allocation and is implemented with a linear curve of slope $-1/\beta\delta$. In this case, when $\theta$ increases, the consumption $c(\theta)$ increases so that $u_c(c(\theta), \theta)$ remains constant.

When $w$ is non-linear, to prevent low types from mimicking high types, the optimal contract cannot allocate enough consumption in period 1 to a consumer facing a high shock : the marginal utility of consumption $u_c(c(\theta), \theta)$ remains high in this case. At the extreme, when $\beta$ small enough, as seen in Proposition 6, the functions $c(\theta)$ and $k(\theta)$ exhibit relatively small variations around the optimal bunching solution. When $\theta$ increases, the main variations in the relation (5) are driven by the effect of a higher shock on the marginal utility $u_c(c, \theta)$. Therefore, since $c(\theta)$ is non-decreasing, the signs of $k''(c)$ and $u_{c\theta}$ are opposite, and the implementation curve $k(c)$ is concave. This approximation can be formalized as follows

$$k''(c) = -\frac{1}{c'(\theta)} \frac{d}{d\theta} \left( \frac{u_c(c(\theta), \theta)}{\beta w'(k(\theta))} \right) \simeq -\frac{1}{c'(\theta)} \frac{\partial}{\partial\theta} \left( \frac{u_c(c(\theta), \theta)}{\beta w'(k(\theta))} \right) = -\frac{1}{c'(\theta)} \frac{u_{c\theta}(c(\theta), \theta)}{\beta w'(k(\theta))} < 0. \tag{6}$$

Such approximations are not rigorous proofs but give an intuitive explanation of our results : it would be optimal to equalize marginal utility across all states of nature. Somehow, unless second period utility is linear, consumption is distorted from the first best level in order to prevent low type from mimicking high types. Thus, at the constrained optimum, the marginal utility is higher for consumers facing higher liquidity shocks. In order to induce truthful reports, this must be offset by a higher contractual interest rate. All things being equal, this rate is then even stronger that period-1 consumption is high (and, equivalently, $\theta$ is high provided that $u_{c\theta} \geq 0$). When lack of self control becomes significant, our result shows that this effect is strong enough so as to induce the concavity of the optimal contract.

### 5.3 A Pigovian reinterpretation

As discussed in the introduction, our model also captures the optimal Pigovian taxation of an agent who consumes two goods : an ordinary consumer good whose utility differs among agents, and a social good that generates a positive externality. Specifically, we consider a population of agents indexed by $\theta$ whose utility is given by $\theta u(c) + \beta w(k)$, whereas the externality generated by the consumption of the good $k$ is given by $(1-\beta)w(k)$. A social planner does internalize the externality, and thus maximizes the utilitarian welfare criterion $\int(\theta u(c)+w(k))f(\theta)$. The contract $k(c)$ describes the corrective taxation that will induce agents to allocate optimally their resources according to the budget constraint [12] $c + k \leq y$. If an agent consumes an additional fraction $dk$ of the social good, the optimal contract induces him to consume $-dc = -dk/k^{'}(c)$ less of the ordinary good. This reflects a marginal Pigovian subsidy equal to $-dc/dk - 1$.

Our results shows that the optimal Pigovian subsidy is non-linear, and strictly concave when the externality is strong enough. In other words, the monetary incentive should be decreasing per unit of social good. This applies to environmental public policy, but may also be applied to paternalistic regulation as suggested by Amador et al. (2006). For instance, consider the case of low-altruism parents who put a low weight $\beta$ on the education $k$ of their own children. Our result shows that a paternalistic government should provide schooling subsidies that decrease with the amount of educational expenses.

## 6 Aggregate savings

Prudent consumers generally save money when they face uncertainty (Gollier 2001, p 236). In a deterministic framework, time-inconsistency may induce under-saving (Phelps

12. In this case, there is no intertemporal transfer, hence $\delta = 1$.

and Pollak 1968). When a population of agents faces both uncertainty and inconsistency, the aggregate level of savings and debt depends on a trade-off between prudence and the demand for commitment.

We therefore investigate whether uncertainty induces a population of time-inconsistent households to save money or accumulate debts. We assume $w = \delta u$ so that, in the absence of shocks, consumption would be the same in both periods. Thus, in our framework, the intertemporal transfer $T = \int_{\underline{\theta}}^{\bar{\theta}} (c(\theta) - k(\theta)) f(\theta) d\theta$ reflects the aggregate level of net wealth (i.e. $T$ positive means that households borrow on average money in order to face uncertainty). The following proposition provides answers to the above mentioned questions when $\beta$ is small enough. This limit property is derived from the inverse Euler equation (3) : when $\beta$ is small enough, consumption varies more at the second period than at the first period, and the result follows in the limit.

**Proposition 6.** *When lack of self control is strong enough, a population of hyperbolic consumers issue aggregate debt if $1/u^{'}$ is convex, and accumulate (net) positive wealth if $1/u^{'}$ is concave.*

In other words, precautionary motives dominate at the aggregate level if $1/u^{'}$ is concave. In this case, the intertemporal transfer $T$ is negative, which reflects aggregate savings at period 1. This is reminiscent of Rogerson's result (1985) on repeated moral hazard which states that the wages paid by a risk neutral principal to a population of risk averse agents either increase or decrease over time, depending on the curvature of the inverse of agents' marginal utility of income. More generally, the condition " $1/u^{'}$ convex " can be interpreted in term of prudence. The measure of prudence $P$ follows the definition of Kimball (1990), i.e. $P(c) = -u^{'''}(c)/u^{''}(c)$. The convexity of $1/u^{'}$ is then equivalent to $P \leq 2A$, where $A$ is the risk aversion defined by $A(c) = -u^{''}(c)/u^{'}(c)$. The same criteria has been used by Salanié and Treich (2006) to determine whether time-inconsistent consumers under- or over-save when they know that they will suffer from a lack of self control [13]. This criteria is commonly used in classical risk theory, that shows that a time-inconsistent consumer facing risk accumulates money iff $P \geq 0$ when the risk cannot be avoided, or iff $P \geq 2A$ when he can adapt by choosing the amount of risk [14].

Again, we check thereafter that the property we obtained is valid once the model is calibrated with reasonable values for $\beta$. This is done for constant relative risk aversion (CRRA) utility function and different values of $\beta$ in section 7.2. According to those

13. In the three-period model of Salanié and Treich, consumers can alleviate their time-inconsistency by saving more at the first period. The previous proposition is a generalization of their results to cases where consumers can contract with a financial intermediary in order to smooth their consumption.

14. More precisely, an agent is prudent iff adding an uninsurable zero-mean risk to his future wealth raises his optimal saving (Gollier, 2001, prop 61 p 237), while the opportunity to invest in the future in a risky asset raises the aggregate saving iff $P \geq 2A$ (ibid., prop 75 p 288).

simulations, the transfer is indeed positive when the relative risk aversion $\gamma$ is greater than one, which equivalent to $1/u'$ convex, and negative otherwise.

# 7 Asset accumulation and overindebtedness

## 7.1 Infinite horizon model

The aim of this section is to understand how the wealth of a consumer facing repeated liquidity shocks changes over time. Our analysis here generalizes Thomas and Worrall's result (1990), that the utility of consumers facing unobservable income shocks (instead of liquidity shocks) becomes arbitrarily negative with probability 1. In their framework, consumers are time-consistent, and the insurer can fully control savings (no hidden storage). We show that this property is still valid when the consumer can privately store money, but only if his lack of self control is strong enough. Besides, the optimal contract leads to impoverishment almost surely even if the population of time-inconsistent consumers on average save money and accumulate a positive wealth. This situation, which occurs when consumers are prudent enough according to the previous section, was not considered by Thomas and Worrall whose assumptions rule out utility functions such as prudence is greater than twice the risk aversion [15]. Our results require other simplifying assumptions, that is we consider in the following consumers with constant relative risk aversion facing discrete shocks.

More precisely, we consider a model with repeated i.i.d. two-type shocks in $\{\theta_l, \theta_h\}$ over an infinite horizon. The felicity function is $u$ in each period. We denote by $\theta^t = (\theta_1, \theta_2, ..., \theta_t)$ the history of shocks up to time $t$. The optimal allocation maximizes the expectation of the discounted utility $V_\infty(y) = \max E[\Sigma_{t=0}^{\infty} \theta_t \delta^t u(c(\theta^t))]$ under the budget constraint $E[\Sigma_{t=0}^{\infty} \delta^t c(\theta^t)] \leq y$, and incentive compatibility constraints that must hold at each period. Since shocks are i.i.d., those latter constraints mean that we must have

$$\theta_x u(c(\theta^t, \theta_x)) + \beta\delta U^0(\theta^t, \theta_x) \geq \theta_x u(c(\theta^t, \theta_y) - \delta\epsilon) + \beta\delta U^\epsilon(\theta^t, \theta_y)$$

for any $\theta^t$, any $x, y \in \{l, h\}$ and $\epsilon \geq 0$, where $U^\epsilon(\theta^t, \theta_x)$ denotes the highest future expected utility the consumer can get after period $t+1$ by reporting $(\theta^t, \theta_x)$ and consuming an additional $\epsilon$ at period $t+1$ and/or in subsequent periods (that is, by storing money at the same rate of return as the principal). Two additional assumptions on the sophistication of the consumer must be given in order to fully define the equilibrium. First, we assume that

15. Indeed, this condition is equivalent to $1/u'$ concave. It is easy to check, for any concave utility function $u$ satisfying the above-mentioned Inada's conditions (i.e. the limits of its derivative towards 0 and $\infty$ are respectively $\infty$ and 0), that the assumption $1/u'$ concave implies that $u(z)$ tends to $\infty$ when $z$ tends to $\infty$. This assumption is thus not valid in the work of Thomas and Worrall, since they assume that the utility function $u$ has an upper bound $\sup u(z) < \infty$.

the consumer only considers future expected utilities that can be reached through hidden savings when taking into account his own lack of self-control. Secondly, in a multiperiod framework with time-inconsistency, the consumer at period $t$ may have to choose between different allocations that are indifferent to his current self, but not to his self at period $t-1$. Again, we assume thereafter that he makes always the "optimal" choice, i.e. when an incentive compatibility constraint is satisfied with an equality, self-$t$ will always choose the allocation which provides self-$(t-1)$ with the highest utility [16].

We show thereafter that the infinite horizon problem coincides with a recursive two-period model, defined by the two-period model analyzed in the previous chapter. That is, we consider the following maximization program .

$$L(v)(y) = \max \pi(\theta_h u(c_h) + \delta v(k_h)) + (1-\pi)(\theta_l u(c_l) + \delta v(k_l))$$

subject to the budget constraint $(BC2)$ and incentive compatibility constraints $(IC_l^2)$ and $(IC_h^2)$ defined previously with $w = \delta v$. We have seen that the two-period model exhibits the same property as the one with uniformly distributed types (cf. Proposition 4), with an explicit threshold for the hyperbolic discounting factor $\beta$ under which the budget function $B(\theta)$ is decreasing. The following lemma recalls those results and provides furthermore additional properties (including recursive Euler equations that will prove useful when considering repeated shocks) :

**Lemma 1.** *In the two-type case with hidden storage, and at the optimal allocation :*

*(i) When $\beta = 1$, no transfers of resources between types are needed , i.e. the budget function $B(\theta)$ is constant, and the allocation satisfies the Euler Equations $\theta_x u^{'}(c_x) = v^{'}(k_x)$, so that*

$$L^{'}(v)(y) = \pi\theta_h u^{'}(c_h) + (1-\pi)\theta_l u^{'}(c_l) = \pi v^{'}(k_h) + (1-\pi)v^{'}(k_l);$$

*(ii) When $\beta < \frac{\theta_l}{\theta_h}$, there is a transfer of resources from high to low types, i.e. the budget function $B(\theta)$ is strictly decreasing, and the allocation satisfies the Inverse Euler Equation*

$$\frac{1}{L^{'}(v)(y)} = \frac{\pi}{u^{'}(c_h)} + \frac{1-\pi}{u^{'}(c_l)} = \frac{\pi}{v^{'}(k_h)} + \frac{(1-\pi)}{v^{'}(k_l)};$$

*(iii) In both cases, there is neither money burning (i.e. $E[c_x + \delta k_x] = y$) nor bunching (i.e. $c_h > c_l$ and $k_h < k_l$). In the latter case ($\beta < \frac{\theta_l}{\theta_h}$), the incentive compatibility constraint $(IC_l)$ is binding, whereas $(IC_h)$ and the hidden storage condition ($\epsilon > 0$ in the ICs) are not binding.*

16. This assumption implies that the set of achievable utilities is closed, and does not change the results, since any such solution can be approximated arbitrarily by allocations that avoid this dilemma.

We now focus on consumers with a constant relative risk aversion, i.e. we consider utility function $u(z) = z^{1-\gamma}/(1-\gamma)$ with $\gamma \geq 0$ and $\gamma \neq 1$ [17]. It is straightforward that, for any $a > 0$, the operator $L$ transforms the function $au(z)$ in another utility function with the same constant relative risk aversion, so that we can define a function $f$ by setting $f(a)u(z) = L(au(z))$.

**Lemma 2.** *When $u$ is CARA, the infinite horizon problem coincides with the recursive two-period model, i.e. $V_\infty = L(V_\infty) = \lim_\infty L^n(u) = a^* \frac{z^{1-\gamma}}{1-\gamma}$ where $a^*$ is the unique fixed point of $f$, so that the infinite horizon optimal allocation, for a budget normalized to 1, is defined by*

$$c^*(\theta^t) = k^*_{\theta_1} k^*_{\theta_2} ... k^*_{\theta_{t-1}} c^*_{\theta_t}$$

*where $(c^*_x, k^*_x)$ is the two-period fixed point optimal allocation*

This Lemma, combined with Lemma 1, implies that the remaining budgets after $i$ periods $y^*(\theta^i) = k^*_{\theta_1} k^*_{\theta_2} ... k^*_{\theta_t}$ allocated by the optimal solution $V_\infty$ follow a martingale. More precisely, this martingale will depend on time-inconsistency in the following way. When $\beta = 1$, it is given for any occurrence $\theta^i$ by the Euler equation

$$V'_\infty(y^*(\theta^i)) = \pi V'_\infty(y^*(\theta^i, \theta_h)) + (1-\pi) V'_\infty(y^*(\theta^i, \theta_l))$$

while, when $\beta < \theta_l/\theta_h$, it follows instead the inverse Euler equation

$$\frac{1}{V'_\infty(y^*(\theta^i))} = \frac{\pi}{V'_\infty(y^*(\theta^i, \theta_h))} + \frac{1-\pi}{V'_\infty(y^*(\theta^i, \theta_l))}.$$

This martingale, according to Dobb's theorem, converges almost surely to some random variable that is necessarily equal to $\{0\}$. In the case $\beta = 1$ (respectively $\beta < \theta_l/\theta_h$), if a path $\{\theta^i\}$ exists such that $V'_\infty(y^*(\theta^i))$ (resp. $1/V'_\infty(y^*(\theta^i))$) tends to a strictly positive value, then the sequence of budget $y^*(\theta^i)$ converges to some positive $y_0$. If we omit paths where only one type of shocks occurs infinitely often (those paths have zero probability), it implies that the two continuation budgets solving the recursive problem $L(V_\infty)(y_0)$ are equal to $y_0$, which is not possible according to the last item of Lemma 1. Thus, when the consumer is time-consistent, $V'_\infty(y^*(\theta^i))$ tends a.s. (almost surely) to zero, which means that $y^*(\theta^i)$ tends a.s. to $\infty$ while, when $\beta < \theta_l/\theta_h$, the limit of $1/V'_\infty(y^*(\theta^i))$ is a.s. zero, which means that $y^*(\theta^i)$ tends a.s. to zero.

**Proposition 7.** *The wealth of a CARA consumer facing repeated binary liquidity shocks $\{\theta_l, \theta_h\}$ over an infinite horizon, becomes almost surely arbitrarily small (resp. high) if the consumer is time-inconsistent with $\beta \leq \theta_l/\theta_h$ (resp. if the consumer is time-consistent).*

17. For the sake of simplicity, we rule out the case $\gamma = 1$ (i.e. $u(z) = \ln(z)$), which is a very specific case where no intertemporal transfer occurs (that is, neither aggregate savings nor aggregate debt), and where the function $L(au(z))$ has a completely different functional form than when $\gamma \neq 1$.

Thus, although a high contractual interest rate may serve as a commitment device for restraining time-inconsistent consumers, such contract leads to poverty almost surely. This property is true even the population of consumers accumulates aggregate savings over time which is the case for CARA consumers, according to the Proposition 6, when the coefficient of relative risk aversion $\gamma$ is lower than 1 or, equivalently, when the intertemporal elasticity of substitution $1/\gamma$ is greater than 1.

## 7.2 Simulations

In this section, we provide simulation results for the two-type infinite horizon model when the utility is CRRA. That is, we find the fixed point function $v(y)$ such that $L(v) = v$. According to the previous section, we know that this is also the function $V_\infty(y) = \max E[\Sigma_{i=0}^{\infty}\theta_i\delta^i u(c(\theta^i))]$ that maximizes the allocation $y$ over an infinite period of time. Defining such a function is equivalent to define the underlying optimal allocation, i.e. consumption $c_x$ at period 1 and $k_x$ at period 2 for the two shocks $x = l$ or $h$. The consumer is time-inconsistent (with hyperbolic parameter $\beta$) and, as in section 6, we benchmark his consumption against the optimal allocation in the absence of shocks, that is $\tilde{c}_h = \tilde{c}_l = 1-\delta$ and $\tilde{k}_h = \tilde{k}_l = 1$. More precisely, we will focus of the two characteristics analyzed in the two previous sections. First, regarding aggregate savings, a population of such consumers on average aggregate savings if $Ek_x = \pi k_h + (1-\pi)k_l$ is greater than one, and pile up debts otherwise. Secondly, regarding individual wealth, this allocation leads almost surely to poverty if $E\ln(k)$ is negative, and induces almost surely individuals to accumulate wealth if it is positive [18].

Let us first consider a precise example with $\beta = 0.3$, and risk aversion $\gamma$ equal either to 0.3 or 2. We consider shocks defined by $\pi = 0.25$, $\theta_h = 1.3$ and $\theta_l = 0.9$. The budget is normalized to one and the technological discount rate is $\delta = 0.9$. The following table provides the optimal allocation (i.e. variable $c_x$ and $c_l$), the total resources during both periods ($y_x = c_x + \delta k_x$), and the two variables discussed previously (aggregate savings $Ek$ and Wealth increment $E\ln(k)$) :

18. Indeed, after $n$ shocks, the logarithm $X$ of remaining budget is the sum of i.i.d random variables $X_i$ taking value $\ln k_h$ with probability $\pi$, and $\ln k_l$ otherwise. According to Central Limit Theorem, this sums tend asymptotically to $nE\ln(k)$. Since the $X_i$ are Bernouilli distributions (up to a constant), the De Moivre Laplace theorem implies that $X$ tends almost surely to $+\infty$ (resp. $-\infty$) when $E\ln(k) > 0$ (resp. $< 0$).

| Optimal contrained resource allocation ($\beta = 0.3$) | | | | | | |
|---|---|---|---|---|---|---|
| Risk aversion | Shocks | period 1 | period 2 | total | Aggregate savings | Wealth increment |
| $\gamma = 0.3$ | High | 0.149 | 0.841 | 0.906 | 0.3% | $-0.15\%$ |
| | Low | 0.0796 | 1.06 | 1.03 | | |
| $\gamma = 2$ | High | 0.108 | 0.976 | 0.986 | - 0.01 % | - 0.02 % |
| | Low | 0.0975 | 1.008 | 1.005 | | |

We can check on this example that there is aggregate savings when $\gamma < 1$ and aggregate debt accumulation when $\gamma > 1$ (i.e. negative aggregate savings). Accordingly, we find in both cases that there is impoverishment (i.e. a negative wealth increment). Somehow, we see that these phenomena are nearly insignificant when $\gamma = 2$ : aggregate debt accumulation is only about 0.01% whereas the individual wealth trend is only 0.02%. This results seem robust to sensitivity analysis, as it is shown in the two following table where we increase the size or the frequency of the shocks (larger shock $\theta_h = 3$ with 10% probability, or monthly shocks with still $\theta_h = 1.3$ at 25% probability). On the other hand, the results are more sensitive when $\gamma = 0.3$.

| Optimal contrained resource allocation with monthly shocks | | | | | | |
|---|---|---|---|---|---|---|
| Risk aversion | Shocks | period 1 | period 2 | total | Yearly agg. savings | Year. wealth incr |
| $\gamma = 0.3$ | High | 0.018 | 0.975 | 0.984 | 0.08 % | - 0.04 % |
| | Low | 0.006 | 1.008 | 1.005 | | |
| $\gamma = 2$ | High | 0.010 | 0.997 | 0.998 | -0.002% | $-0.004\%$ |
| | Low | 0.008 | 1.001 | 1.0007 | | |

| Optimal contrained resource allocation with larger shock $\theta_h = 3$ | | | | | | |
|---|---|---|---|---|---|---|
| Risk aversion | Shocks | period 1 | period 2 | total | aggregate savings | wealth increment |
| $\gamma = 0.3$ | High | 0.53 | 0.33 | 0.84 | 3 % | - 1.7 % |
| | Low | 0.021 | 1.1 | 1.02 | | |
| $\gamma = 2$ | High | 0.16 | 0.89 | 0.96 | -0.06% | $-0.1\%$ |
| | Low | 0.094 | 1.01 | 1.004 | | |

Those simulations show that the impoverishment property strongly depends on the time inconsistency $\beta$ and the risk aversion $\gamma$. Time-consistent individuals accumulate wealth almost surely whereas time-inconsistent individuals will almost surely fall into overindebtedness but, depending on the level of risk aversion, this trend may be significant or not. Similarly, the aggregate behavior of time-inconsistent consumers is also strongly dependent on those parameters.

The following graph illustrates this heterogeneity in behavior depending on risk aversion : it shows the aggregate savings and the almost sure individual wealth trend for $\beta = 0.6$ and various value of risk aversion $\gamma$ for CRRA consumers (the shocks are the same as before).

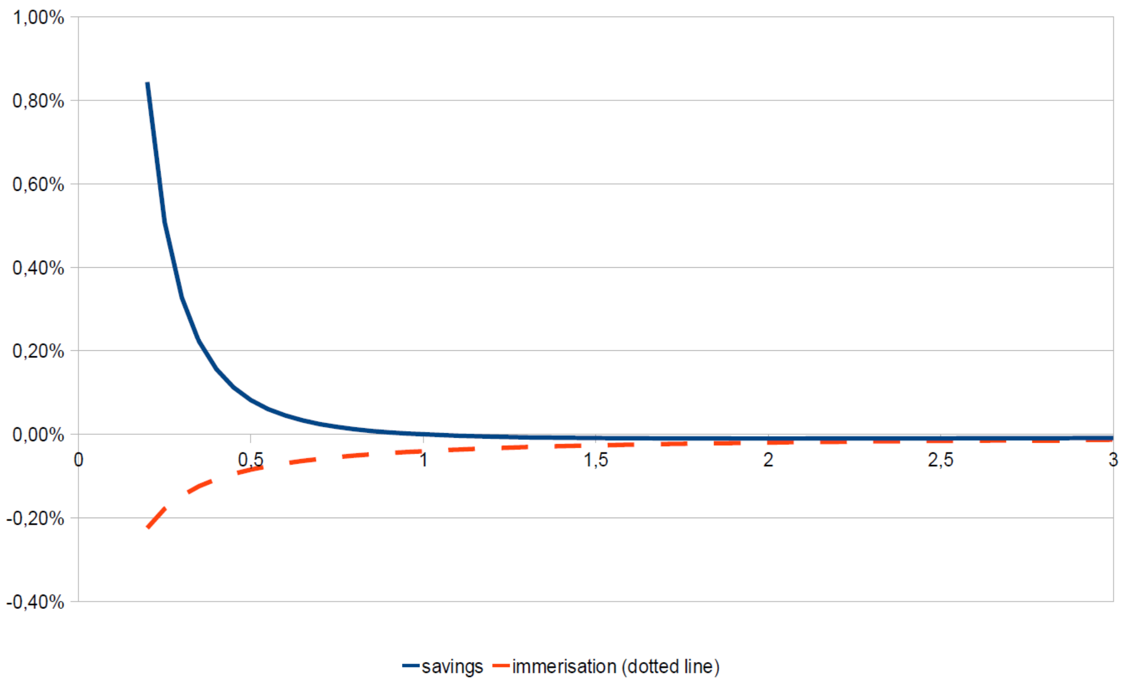


**Figure 4 : Long term behavior in terms of constant relative risk aversion**

# 8 Conclusion

This paper brings out in a clear way how lack of self-control changes the nature of optimal savings and borrowing instruments, that switch from insurance to credit contracts. It provides a detailed characterization of such commitment devices, with three significant results. First, it shows that when the agents can secretely save, Thomas and Worrall (1990)'s impoverishment results hold only when time-inconsistency is strong enough. By contrast, time-consistent agents accumulate wealth indefinitely. Secondly, and quite paradoxically, the impoverishement of time-inconsistent agents holds even though, at the aggregate level, precautionary motives may induce the population of such agents to save money. Indeed, we show that if households are prudent enough, they may accumulate aggregate savings despite their lack of self control. Lastly, we show that a progressive incentive system is always part of the optimum : the interest rate implicit in the optimal contract increases with the amount of borrowing and decreases with the amount of saving.

This analysis sheds light on the process that leads individuals and states to overindebtedness. More precisely, it shows that commitment devices (such as high interest rates on credit cards, or accumulation of precautionary reserves) may not prevent time-inconsistent households and opportunistic governments from accumulating unlimited debt. Our analysis can also be applied to the optimal design of a pension scheme : provided that a paternalistic role of the state is justified (e.g. in order to take into account negative externalities), our results may support a government intervention in which state subsidies decrease with the amount of money invested (such as tax-favored retirement savings account with contribution limits). Regarding education, our results show that, in order to cope with insufficient parental altruism, a government should provide schooling subsidies that decrease with the amount of educational expenses. While we have applied our analysis to debit-card usage, pension funds and Pigovian taxation of externalities, other interesting application such as health plans would be worth investigating. Also, an interesting direction for future research is to consider population of households who are heterogeneous with respect to time-inconsistency, risk aversion and prudence. In particular, one can wonder whether those parameters explain consumer heterogeneity with respect to credit card usage (i.e. choosing a debit-only card, keeping a credit card, co-holding credit and liquid assets..) and whether other related characteristics such as the degree of sophistication of the agent play a role.

# 9 Proof

*Proof of Proposition 1 :* If $w(z) = \delta' z$, the consumption at period 1 doesn't depend on $\beta$ and maximizes $\int_{\underline{\theta}}^{\bar{\theta}}(\theta(u(c(\theta)) - \delta^{-1}\delta' c(\theta))f(\theta)$ i.e. $u'(c(\theta)) = \delta^{-1}\delta'/\theta$. Consumption at period 2 is then equal to

$$k(\theta) = -\frac{c(\theta)}{\beta\delta} + \int_{\underline{\theta}}^{\bar{\theta}} \frac{1-\beta}{\delta\beta} c(\theta) f(\theta)$$

Therefore, when $\beta < 1$, the optimal allocation is the allocation that would be provided to a time-consistent consumer if the discount rate was distorted from $\delta$ to $\delta\beta$ (up to a fixed transfer of money balancing the global budget). $\square$

*Proof of Proposition 2 :* We show in what follows that, $\forall \beta < \beta'$, a consumer with hyperbolicity $\beta'$ enjoys more utility on average at the constrained optimum than a consumer with hyperbolicity $\beta$. Let's consider the optimal allocation $(c^*, k^*)$ for the consumer with hyperbolicity $\beta$, and define a new allocation $(c, k)$ by $c(\theta) = c^*(\theta)$ and $k(\theta) = w^{-1}(\alpha w(k^*(\theta)) + a)$ where $\alpha = \beta/\beta'$ and where the constant $a$ is chosen such that $\int(c(\theta) + \delta k(\theta))f(\theta) = 0$. This new allocation satisfies the incentive compatibility constraint $\theta u'(\theta) = -\beta' w'(\theta)$ for a consumer with hyperbolicity $\beta'$. Therefore, in order to prove the Proposition, it is enough to show that

$$\int (\theta u(c^*(\theta)) + w(k^*(\theta)))f(\theta) < \int (\theta u(c(\theta)) + w(k(\theta)))f(\theta).$$

This inequality is true iff the constant $a$ is greater than $\int(1-\alpha)w(k^*(\theta))f(\theta)$ or, equivalently,

$$\int k^*(\theta)f(\theta) > \int w^{-1}\left(\alpha w(k^*(\theta)) + \int (1-\alpha)w(k^*(\tilde{\theta}))f(\tilde{\theta})\right) f(\theta)$$

The left term of this inequality is equal to

$$\int w^{-1}\left(\int (\alpha w(k^*(\theta)) + (1-\alpha)w(k^*(\tilde{\theta})))f(\tilde{\theta})\right) f(\theta)$$

and the result follows from the fact that $w$ is concave and $w^{-1}$ is convex. Besides, this inequality is strict when $w$ is non-linear on the domain spanned by $k^*(\theta)$. $\square$

*Proof of Proposition 3 (necessary conditions)* : The necessary conditions obtained by a perturbation method in the core text are rigorously proven in the section 11.1 of the mathematical appendix by using the Maximum Principle.

*Proof of Proposition 3 (uniform distribution of shocks) :* We refer to the section 11.1 in the mathematical appendix for the proof of the existence of a smooth solution to the optimization problem once we rule out the " non-decreasing " condition on the utility function. It remains to check that those necessary conditions of optimality imply indeed that the function $\mathrm{U}$ is non-decreasing when the distribution $f(\theta)$ is uniform. The relation 2 implies that $B^{'}(\theta)/\mathrm{U}^{'}(\theta) = 1/u^{'}(c(\theta)) - \delta\theta/\beta w^{'}(k(\theta))$, and the Euler equation can be rewritten as

$$\mathrm{U}^{'}(\theta) \times \left[ -\frac{u^{''}(c(\theta))}{u^{'3}(c(\theta))} - \frac{\delta\theta^2 w^{''}(k(\theta))}{\beta^2 w^{'3}(k(\theta))} \right] = \frac{2}{\beta} \times \left[ \frac{\delta}{w^{'}(k(\theta))} - \lambda^{-1}(1 - \frac{\beta}{2}) \right] \qquad (7)$$

The left term of this equation is non negative iff $\mathrm{U}(\theta)$ is non-decreasing or, equivalently, iff the term $\delta/w^{'}(k(\theta))$ is non-increasing. The right term of this equation is non negative iff $\delta/w^{'}(k(\theta))$ has a value greater or equal to its means time $1 - \frac{\beta}{2}$. Since a function must always at least take some value greater or equal to its mean, it is easy to see that these conditions are always true and, consequently, that the control $\mathrm{U}(\theta)$ is a non-decreasing function. $\square$

*Proof of Proposition 3 ($\beta$ small) :* The inverse Euler equation 3 implies that the budget multiplier $\lambda$ belongs to a compact subset of $\mathcal{R}^{+}$. Let's consider a sequence of $\beta_n \to 0$ such that the corresponding parameter $\lambda_n$ of the optimal solution $(c_n, k_n)$ converges to a real number $\lambda$. Since, the left term in the transversality conditions 13 tends to zero when $\beta \to 0$, the decreasing function $k_n(\theta)$ tends uniformly to a constant $k$ such that $w^{'}(k) = \delta\lambda$. Besides, the relation $\theta\mathrm{U}^{'}(\theta) = -\beta\mathrm{W}^{'}(\theta)$ implies $\theta u^{'}(c_n(\theta))c_n^{'}(\theta) = -\beta_n w^{'}(k_n(\theta))k_n^{'}(\theta)$ and the function $c_n^{'}(\theta)/\beta_n$ tends uniformly to zero. Therefore $c_n$ tends to a constant $c$. Those solutions satisfy the inverse Euler equation 3 which coincides, at the limit, with the Euler equation $1/u^{'}(c) = \delta/w^{'}(k)$. In other word, the optimal solution converge, when $\beta$ tends to zero, to the optimal bunching solution.

Let's consider the function $k(c)$ defined on the domain spanned by $c$ by the relation $k(.) = k(c^{-1}(.))$. The transversality conditions can be rewritten, using the relation (2), as

$$\frac{c^{'}(\theta)}{k^{'}(\theta)} + \delta = \lambda^{-1}(1 - \beta)w^{'}(k(\theta)).$$

Therefore, when $\beta$ is small enough, the derivative of the function $k(c)$ satisfies $k^{'}(\bar{\theta}) \leq k^{'}(\underline{\theta}) < -1/\delta$. Besides, the approximation described in the relation 6 is correct since both $c^{'}(\theta)$ and $k^{'}(\theta)$ tends uniformly to zero when $\beta$ tends to zero. The function $k(c)$ is thus strictly concave when $\beta$ is small enough. Its derivative is then strictly lower than $-1/\delta$, which is equivalent to a strictly decreasing budget function $B(\theta)$. $\square$

*Proof of Proposition 3 ($\beta = 1$) :* When the distribution is uniform, the property is proved in the core text and follows from the necessary conditions obtained rigorously in the mathematical appendix. In the general case, the transversality condition are more complex. They can be formulated (lemma 4 (ii) in the mathematical appendix) as follows : there exists $\tilde{\theta}_1 < \tilde{\theta}_2$ such that the optimal solution is constant on left and right interval $[\underline{\theta}, \tilde{\theta}_1[$ and $]\tilde{\theta}_2, \bar{\theta}]$, and the term

$$\frac{B^{'}(\bar{\theta})}{\mathrm{U}^{'}(\bar{\theta})} - \lambda^{-1}(1 - \beta^{-1})\tilde{\theta}_i$$

is positive (resp. negative) for $i = 1$ (resp. $i = 2$). The Euler-Lagrange equation then implies the result. $\square$

*Proof of Proposition 3 (two-type model) :* This proof is provided in section 11.2 of the mathematical appendix.

*Proof of Proposition 4 :* We first prove that, when $\beta$ is small enough, the hidden storage constraint is not binding. Let's note $U(\theta, \theta', \epsilon) = \theta u(c(\theta') - \epsilon) + \beta w(k(\theta') + \delta^{-1}\epsilon)$. The storage condition is verified if and only if for all $\theta', \theta \in \Theta$ and $\epsilon \in [0, c(\theta')]$, we have $U(\theta, \theta', \epsilon) \leq U(\theta, \theta, 0)$. Any admissible solution satisfies the incentive compatibility constraint $U(\theta, \theta', 0) \leq U(\theta, \theta, 0)$. Therefore, a sufficient condition for the hidden storage constraint to hold is that $U_\epsilon(\theta, \theta', \epsilon) \leq U_\epsilon(\theta, \theta', 0)$. A sufficient condition is therefore that

$$\frac{\partial U}{\partial \epsilon}(\theta, \theta', \epsilon) = -\theta u'(c(\theta') - \epsilon) + \beta \delta^{-1} w'(k(\theta') + \epsilon) \leq 0$$

This is true when $\beta$ is small enough since, according to the previous paragraph, the optimal solution tends when $\beta \to 0$ to the optimal bunching solution and $w'(k(\theta))$ remains bounded.

Let's now show that, when $\beta = 1$, the hidden storage constraint prevent any transfer of resources between types. If we assume that the control function $\mathrm{U}(\theta)$ is increasing and differentiable, then the relation 11 implies that $B'(\theta)\mathrm{U}'(\theta) = -h_\theta(V(\theta), \mathrm{U}(\theta))$ with

$$h_\theta(x, \mathrm{U}) = \beta^{-1}\delta\theta K'(\beta^{-1}x - \beta^{-1}\theta\mathrm{U}) - C'(\mathrm{U})$$

Hidden storage implies that $B(\theta)$ is decreasing, which is equivalent to $h_\theta(x_1(\theta), \mathrm{U}(\theta)) \geq 0$ for all $\theta$ or equivalently $x_1(\theta) \geq l_\theta(\mathrm{U}(\theta))$ with

$$l_\theta(\mathrm{U}) = \theta\mathrm{U} + \beta K'^{-1}(\frac{\beta}{\delta\theta} C'(\mathrm{U})) \tag{8}$$

We consider thereafter the maximization problem considered previously, enriched with those additional mixed constraints. We refer to the mathematical appendix 11.1.1 for the proof of the existence of an optimal solution to this non-convex maximization problem, and to the appendix 11.1.2 for the explicit expressions of the necessary conditions. According to this appendix, if, for some $\theta' \in \Theta$, we have $h_{\theta'}(x_1^*(\theta'), \mathrm{U}^*(\theta')) \neq 0$, then there exist an interval $\Phi' = [\underline{\theta}', \bar{\theta}']$ in $\Phi$, containing $\theta'$, where the mixed constraints are not binding, the control $u$ is smooth and satisfies the necessary conditions of the initial maximization problem considered in the previous sections (i.e. without the mixed constraint) : the transversality equations, that are equivalent to $B'(\bar{\theta}') = B'(\underline{\theta}') = 0$, and the Euler-Lagrange equation that implies that the function $B'(\theta)/u'(\theta)$ is concave. Since by assumption $B' \leq 0$, we have $B'(\theta) = 0$ on $\Phi'$, which contradict our initial assumption. Therefore, the optimal allocation is such that $g = 0$ (i.e. the budget $B(\theta)$ is constant), and is the unique allocation that maximizes for each $\theta$ the utility $\theta u(c(\theta)) + w(k(\theta))$, under the type-by-type budget constraint $c(\theta) + \delta k(\theta) \leq y$. $\square$

*Proof of Proposition 5 :* For clarity of exposition, this is proved at the end of the proof of Proposition 3 ($\beta$ small). $\square$

*Proof of Proposition 6 :* We know from the previous section that the storage condition is not binding when $\beta$ is small enough, and that the inverse Euler equation is then verified, i.e.

$$\int_{\underline{\theta}}^{\bar{\theta}} \frac{\delta}{w'(k(\theta))} f(\theta) d\theta = \int_{\underline{\theta}}^{\bar{\theta}} \frac{1}{u'(c(\theta))} f(\theta) d\theta.$$

Besides, for any function twice differentiable almost everywhere, we have the general property

$$\int_{\underline{\theta}}^{\bar{\theta}} g(c(\theta))f(\theta)d\theta = g(\tilde{c}) + \int_{\underline{\theta}}^{\bar{\theta}} \int_{\tilde{c}}^{c(\theta)} g^{''}(u)(c(\theta) - u)duf(\theta)d\theta$$

where $\tilde{c} = \int_{\underline{\theta}}^{\bar{\theta}} c(\theta)f(\theta)d\theta$. Indeed, we have

$$g(x) = g(c) + \int_c^x g^{'}(u)du = g(c) + g^{'}(x)x - g^{'}(c)c - \int_c^x ug^{''}(u)du$$

or equivalently

$$g(x) = g(c)+(x-c)g^{'}(c)+(g^{'}(x)-g^{'}(c))x-\int_c^x ug^{''}(u)du = g(c)+(x-c)g^{'}(c)+\int_c^x (x-u)g^{''}(u)du$$

and the result follows by choosing $c = \tilde{c}$. If we apply this property to the function $g = 1/u^{'}$, when $w = \delta u$, we have

$$\frac{1}{u^{'}(\tilde{c})} - \frac{1}{u^{'}(\tilde{k})} = \int_{\underline{\theta}}^{\bar{\theta}} \int_{\tilde{k}}^{k(\theta)} g^{''}(u)(k(\theta) - u)duf(\theta)d\theta - \int_{\underline{\theta}}^{\bar{\theta}} \int_{\tilde{c}}^{c(\theta)} g^{''}(u)(c(\theta) - u)duf(\theta)d\theta$$

and, since $1/u^{'}$ is increasing, the transfer $T$ is positive if and only if the left term of this equation is positive. We know that, when $\beta$ tends toward zero, the optimal allocation tends to the bunching allocation $(\bar{c}, \bar{k})$ and that the function $-\frac{k^{'}(\theta)}{c^{'}(\theta)}$ tends uniformly to $\infty$. Consequently, when $\beta$ is small enough, the left term of the previous equation is positive if $g$ is convex, negative if $g$ is concave. Since the second derivative of $g = 1/u^{'}$ is $-u^{'''}/u^{'2} + 2u^{''2}/u^{'3}$, the convexity of $g$ is equivalent to $-u^{'''}/u^{''} \leq -2u^{''}/u^{'}$, i.e. the absolute prudence is lower than twice the absolute risk aversion. □

*Proof on the Lemma 2 (infinite horizon problem and two-types model) :*

We first prove two basic properties of the function $f$ that jointly imply that there exist a unique fixed point $a^*$ of $f$. We set $\pi_h = \pi$ and $\pi_l = 1 - \pi$, and we first assume that $\beta < \theta_l/\theta_h$.

- First property : the function is non-decreasing and concave when $\gamma > 1$, convex otherwise. Let's consider the optimal allocation solution to the problem $L(au)(1)$. When $\gamma > 1$, this solution is still a compatible allocation for the problem $L((a + \epsilon)u)(1)$ for $\epsilon$ positive and small enough since, according to Lemma 1, only $IC_l$ is binding and $k_l > k_h$. This implies that $L((a + \epsilon)u)(1) \geq L(au)(1)$ so that $f$ is non-decreasing. When $\gamma < 1$, we can use the same trick with $\epsilon = -\epsilon^{'} < 0$, so that $L((a - \epsilon^{'})u)(1) \geq L(au)(1)$ and thus $f(a - \epsilon^{'}) \leq f(a)$, i.e. $f$ is still non-decreasing. If we now consider two optimal allocations $(c_x, k_x)$ and $(c_x^{'}, k_x^{'})$ solving respectively $L(w)(y)$ and $L(w^{'})(y)$, then for any $\lambda \in [0, 1]$, we can build a new allocation for the problem $L(\tilde{w})(y)$, with $\tilde{w} = \lambda w + (1 - \lambda)w^{'}$, by setting $\tilde{c}_x = u^{-1}(\lambda u(c_x) + (1 - \lambda)u(c_x^{'}))$ and $\tilde{k}_x = \tilde{w}^{-1}(\lambda w(k_x) + (1 - \lambda)w^{'}(k_x^{'}))$. Indeed, this allocation is obviously incentive compatible, and since those functions $u$, $w$, $w^{'}$ and $\tilde{w}$ are concave, the budget of this allocation is lower than $y$. Lastly, this allocation provides the utility $\lambda L(w)(y) + (1 - \lambda)L(w^{'})(y) \leq L(\tilde{w})(y)$. This implies the expected curvature properties of $f$.

– Second property : the curve $y = f(x)$ cross the line $y = x$. In order to obtain this last property, we build upper and lower bound for the function $f$. When there is bunching, i.e. $c = c_h = c_l$ and $k = k_h = k_l$, the best we can get is when $u^{'}(c) = au^{'}(k)$ so that, with CARA utility function, we have $k/c = a^{1/\gamma}$ and $c = 1/(1+\delta a^{1/\gamma})$. Thus

$$L(au)(1) \geq \frac{1+\delta a^{1/\gamma}}{(1-\gamma)(1+\delta a^{1/\gamma})^{1-\gamma}} = a(\delta + a^{-1/\gamma})^{\gamma} u(1)$$

consequently we see that when $\gamma < 1$, then $f(a) > a$ when $a$ is small enough, whereas when $\gamma > 1$, we have $f(a) < a$ when $\delta + a^{-1/\gamma} < 1$, i.e. when $a$ is large enough. If we now forget about the incentive compatibility constraints, we get an upper bound on $L$ : the optimum is then given by the Euler equation $\theta_x u^{'}(c_x) = au^{'}(k)$ or equivalently $c_x = k(\theta_x/a)^{1/\gamma}$ so that

$$k = \frac{ya^{1/\gamma}}{\delta a^{1/\gamma} + \pi_h \theta_h^{1/\gamma} + \pi_l \theta_l^{1/\gamma}}$$

and the bound is given by

$$L(au)(y) \leq \frac{\delta a k^{1-\gamma} + \pi_h \theta_h c_h^{1-\gamma} + \pi_l \theta_l c_l^{1-\gamma}}{1-\gamma} = u(y)(\delta a^{1/\gamma} + \pi_h \theta_h^{1/\gamma} + \pi_l \theta_l^{1/\gamma})^{\gamma}$$

When $\gamma < 1$, this implies that $f(a) < (\delta a^{1/\gamma} + \pi_h \theta_h^{1/\gamma} + \pi_l \theta_l^{1/\gamma})^{\gamma}$ so that $f(a) < a$ when $a$ is large enough. When $\gamma > 1$, we have the inequality $f(a) > (\delta a^{1/\gamma} + \pi_h \theta_h^{1/\gamma} + \pi_l \theta_l^{1/\gamma})^{\gamma}$ so that $f(a) > a$ when $a$ is small enough. $\square$

When $\beta = 1$, those properties still hold and can be proved directly. Indeed, the Euler equation provides us with an explicit formula for $L(au)(y)$ :

$$L(au)(y) = f(a)\frac{y^{1-\gamma}}{1-\gamma} \text{ with } f(a) = E\left[(\theta_x^{1/\gamma} + \delta a^{1/\gamma})^{\gamma}\right]$$

so that

$$f^{''}(a) = \delta a^{\frac{1}{\gamma}-2}(\frac{1}{\gamma} - 1) \times E\left[\theta_x^{1/\gamma}(\theta_x^{1/\gamma} + \delta a^{1/\gamma})^{\gamma-2}\right]$$

and we see that the function $f$ is non-decreasing, concave if $\gamma > 1$ (resp. concave if $\gamma < 1$) and cross the line $y = x$.

It remains to show that this fixed point describes also the solution of the infinite horizon problem. This results from the inequality $L(V_\infty) \geq V_\infty \geq a^*u$ which implies $L^n(V_\infty) \geq V_\infty \geq a^*u$ and thus $V_\infty = L(V_\infty) = \lim_\infty L^n(u) = a^*u$.

– The inequality $V_\infty \geq a^*u$ results from the fact that the infinite allocation that replicates the two-period allocation $L(a^*u)$ is an admissible allocation for the problem $V_\infty$. Let's denote $(c_x, k_x)$ this two-period fixed point optimal allocation, and define an infinite horizon allocation by

$$c(\theta^t) = k_{\theta_1} k_{\theta_2} ... k_{\theta_t} c_{\theta_t}.$$

The only property that need to be checked is the robustness of incentive compatibility constraints to hidden storage in the case $\beta < \theta_l/\theta_h$. Since we assumed that the choice of hiding savings must be rational and that the individual is a sophisticated time-inconsistent consumer, this choice must hold at each period i.e. we have to show that there is no gain in savings $\epsilon$ by consuming $c(\theta_{t-1}) - \epsilon$ in order to consume $c_{\theta_t} + \delta^{-1}\epsilon$ at the subsequent period event for the time-inconsistent self. This is equivalent to $\theta_x u^{'}(c_x) > \beta\theta_y u^{'}(k_x c_y)$. When $u$ is CARA, this inequality is equivalent to $k_x^{\gamma} > \beta$ when $x = y$. This is true when $x = y = l$ since $k_l > 1$ (we know that $k_l > k_h$ and, from the inverse Euler equation, the expected mean of $1/u^{'}(k_x)$ is 1). When $x \neq y$, the inequality is equivalent to

$$\frac{\theta_x u^{'}(c_x)}{\theta_y u^{'}(c_y)} > \beta u^{'}(k_x)$$

and, once again, this is true when $x = l$ since $u^{'}(k_l) < 1$, by assumption $\beta\theta_x/\theta_l < 1$, and $u^{'}(c_l) > u^{'}(c_h)$ since $c_h > c_l$. This is also true when $x = h$ since we proved in the previous lemma that

$$\frac{\theta_h u^{'}(c_h)}{\theta_l u^{'}(c_l)} > \frac{u^{'}(k_h)}{u^{'}(k_l)}$$

and the right term is obviously greater than $\beta u^{'}(k_h)$. Thus, it remains to show that $k_h^{\gamma} > \beta$. According to the inverse Euler equation, we know that the expected value of $1/u^{'}(c_x)$ is $1/a^*$, so that $1/u^{'}(c_h) \geq 1/a^*$. From the previous lemma, we have the inequality $a^* u^{'}(k_h) < \theta_h u^{'}(c_h)$, so that $\beta u^{'}(k_h) < \beta\theta_h < 1$.

– In order to prove the inequality $L(V_\infty) \geq V_\infty$, let's consider an admissible allocation $\{c(\theta^t)\}$ that approximate the optimal solution $V_\infty$ of the infinite horizon problem for a budget $y$, i.e. that provides at least a utility $U^0$ greater than $V_\infty(y) - \epsilon$ for a chosen $\epsilon > 0$. We define a two period allocation by setting $c_x = c(\theta_x)$ and $k_x = E[\Sigma_{i=1}^{\infty}\delta^{i-1}c(\theta_x, \theta^i)]$. Since the allocation $c(\theta_x, \theta^i)$ is admissible, it provides an utility $U^0(\theta_x, \theta^i) = E[\Sigma_{i=1}^{\infty}\theta_i\delta^{i-1}u(c(\theta_x, \theta^i))]$ that is lower than $V_\infty(k_x)$. We can therefore choose $k_x^{'}$ lower than $k_x$ such that $V_\infty(k_x^{'}) = U^0(\theta_x, \theta^i)$. The allocation $(c_x, k_x^{'})$ provides therefore the utility $U^0$ with a budget $y^{'}$ lower than $y$. This allocation satisfies the incentive compatibility constraints of the two period problem without hidden storage. But we now from the previous lemma that hidden storage is not binding, so $L(V_\infty)(y) \geq L(V_\infty)(y^{'}) \geq U^0 \geq V_\infty(y) - \epsilon$.

□

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# 11 Mathematical Appendix (not for publication)

## 11.1 Existence and optimality conditions

We recall that the optimization problem defined in the section 3 is to maximize

$$\max_{c,k} \int_{\underline{\theta}}^{\bar{\theta}} (\theta u(c(\theta)) + w(k(\theta)))f(\theta) \qquad \text{(M)}$$

subject to a budget constraint that allows transfer of resources between types and periods :

$$\int (c(\theta) + \delta k(\theta))f(\theta) = \int B(\theta)f(\theta) \leq y \qquad (BC)$$

and to incentive compatibility constraints

$$\theta u(c(\theta)) + \beta w(k(\theta)) \geq \theta u(c(\theta^{'})) + \beta w(k(\theta^{'})) \; \forall \theta, \theta^{'} \qquad (IC)$$

that may be enriched, as in subsection 4.2, in order to allow for hidden savings

$$\theta u(c(\theta)) + \beta w(k(\theta)) \geq \theta u(c(\theta^{'}) - \delta\Delta) + \beta w(k(\theta^{'}) + \Delta) \; \forall \theta^{'}, \forall \Delta \in [0, c(\theta^{'})/\delta] \qquad \text{(HS)}$$

We note $\text{U}(\theta) = u(c(\theta))$ and $\text{W}(\theta) = w(k(\theta))$, and consider the state variable $V(\theta) = \theta\text{U}(\theta) + \beta\text{W}(\theta)$. It is easy to check that the incentive compatibility constraints implies that U is non-decreasing and W is non-increasing. Besides, those functions are bounded since they are defined over a closed interval. From the envelope property (theorem 2 and 3 of Milgrom and Segal 2002), we know that the function $\max_{\theta'} \theta\text{U}(\theta') + \beta\text{W}(\theta')$ is absolutely continuous, and left- and -right-hand differentiable with derivative $\lim_{\theta^{\pm}} \text{U}(\theta^*(\theta))$ where $\theta^*(\theta) \in$ argmax $\theta\text{U}(\theta') + \beta\text{W}(\theta')$. The conditions " $V$ absolutely continuous with left- and -right-hand derivatives such that $V^{'\pm}(\theta) = \lim_{\theta^{\pm}} \text{U}(\theta)$ " together with the condition " U non-decreasing " are then necessary conditions for incentive compatibility. It is straightfoward that these conditions are also sufficient and, when U and W are differentiable, that the first condition is equivalent to $\theta\text{U}'(\theta) = -\beta\text{W}'(\theta)$.

The maximization problem can be written as a classical constrained optimization problem on the space $X$ of convex and absolutely continuous real-valued functions $V$ on the interval $[\underline{\theta}, \bar{\theta}]$. Since $\text{W}(\theta) = \beta^{-1}V(\theta) - \beta^{-1}\theta\text{U}(\theta)$, it consists in finding the maximum $U^* = \max U(V)$ where

$$U(V) = \int_{\underline{\theta}}^{\bar{\theta}} f_0(\theta, V(\theta), V'(\theta))d\theta \tag{9}$$

and

$$f_0(\theta, x, \text{U}) = (\beta^{-1}x + (1 - \beta^{-1})\theta\text{U})f(\theta) \tag{10}$$

under the budget constraint

$$\int_{\underline{\theta}}^{\bar{\theta}} g(\theta, x, \text{U})d\theta \leq y \quad \text{with} \quad g(\theta, x, \text{U}) = B(\theta)f(\theta) = (C(u) + \delta K(\beta^{-1}x - \beta^{-1}\theta\text{U}))f(\theta) \tag{11}$$

where $C$ and $K$ are the inverse of the strictly concave utility functions $u$ and $w$. This maximization problem is convex, since $f_0$ and $g$ are respectively linear and convex in $x$ and $u$.

Instead of the hidden storage condition (HS) defined at the beginning of section 4.2, we consider here a local condition that is a consequence of (HS) when the control function $\text{U}(\theta)$ is increasing and differentiable. Hidden storage implies $B(\theta)$ non increasing, which is equivalent, under these assumptions, since $B'(\theta) = \Big(C'(\text{U}(\theta)) - \beta^{-1}\theta\delta K'(\text{W}(\theta))\Big)\text{U}'(\theta)$, $h_\theta(x_1(\theta), \text{U}(\theta)) \geq 0$ to with

$$h_\theta(x, \text{U}) = \beta^{-1}\delta\theta K'(\beta^{-1}x - \beta^{-1}\theta\text{U}) - C'(\text{U})$$

or, equivalently, $x_1(\theta) \geq l_\theta(\text{U}(\theta))$ with

$$l_\theta(\text{U}) = \theta\text{U} + \beta K'^{-1}(\frac{\beta}{\delta\theta}C'(\text{U})) \tag{12}$$

The existence of a solution $V^*$ to this problem is proved in the following section 11.1.1 once we rule out the constraint " u non-decreasing " (that is, we do not restrict ourselves to convex state variable) and we assume that the control $u$ belongs to a bounded interval $[\underline{u}, \bar{u}]$. The necessary conditions obtained by a perturbation method in the core text are rigorously proven by Lemma 4 in the subsection 11.1.2 , using the Maximum Principle when we assume that the solution is such that the utility $\text{U}(\theta)$ is continuous and piecewise continuously differentiable (so that the state variable is piecewise $C^2$). When the distribution is uniform, those necessary conditions imply that the constraint " u non-decreasing " is not binding (cf. proof of Proposition 3 in section 9).

Regarding the uniform distribution case, it remains to show that any solution is bounded by strictly positive values independent of $\beta$. This result implies that we can choose the bound $\underline{u}$ and $\bar{u}$ in such a way that they are not binding. We first prove that this is true for the budget multiplier

$\lambda$. Since $c$ is non-decreasing, the relation (3) implies that $\delta/w^{'}(k(\bar{\theta})) \leq \lambda^{-1} \leq \delta/w^{'}(k(\underline{\theta}))$ and $1/u^{'}(c(\underline{\theta})) \leq \lambda^{-1} \leq 1/u^{'}(c(\bar{\theta}))$. Besides, the transversality conditions can be written as

$$\frac{\beta}{u^{'}(c(\theta))} + \lambda^{-1}(1-\beta)\theta = \frac{\delta\theta}{w^{'}(k(\theta))} \tag{13}$$

at $\theta = \underline{\theta}, \bar{\theta}$, so that the previous inequalities implies that $1/u^{'}(c(\bar{\theta})) < \lambda^{-1}\bar{\theta}$ and that $\delta\underline{\theta}/w^{'}(k(\underline{\theta})) \leq \lambda^{-1}(\beta\bar{\theta}+(1-\beta)\underline{\theta}) \leq \lambda^{-1}\bar{\theta}$. Since there is no money burning (i.e. the budget constraint is binding, which is proven in Lemma 4), the upper values $c(\bar{\theta})$ and $k(\underline{\theta})$ of $c$ and $k$ cannot be both arbitrarily close to zero, i.e. $\min(c(\bar{\theta}), \delta k(\underline{\theta})) \geq y/(1+\delta)$. The previous inequalities imply therefore that $\lambda$ is lower than a constant independent of $\beta$. Similarly, we have

$$\lambda^{-1} \leq \frac{\delta}{w^{'}(\underline{k}(\theta))} = \frac{\beta}{\underline{\theta}u^{'}(c(\underline{\theta}))} + \lambda^{-1}(1-\beta)$$

so that $\lambda \geq \underline{\theta}u^{'}(c(\underline{\theta})) \geq \underline{\theta}u^{'}(c(y))$. It is then easy to find strictly positive upper and lower bounds for both $c(\theta)$ and $k(\theta)$ that do not depend on $\beta$. Indeed, we have already shown in the previous paragraph that $u^{'}(c(\bar{\theta})) > \lambda\bar{\theta}^{-1}$ and that $u^{'}(c(\underline{\theta})) \leq \lambda\underline{\theta}^{-1}$, which provides us with an upper and lower bound on $c(\theta)$. It is then straightforward to get from the transversality condition a lower bound on $w^{'}(k(\bar{\theta}))$ and an upper bound on $w^{'}(k(\underline{\theta}))$ that do not depend on $\beta$, and thus get simlar bounds on $k(\theta)$.

### 11.1.1 Existence of solutions to the optimal control problems

We prove the existence of a solution for both the initial maximization problem with the budget constraint $\int_\Theta g(v(\theta), v^{'}(\theta))d\theta \leq c$ (with $g$ defined by the relation 11) and the maximization problem enriched with the additional mixed constraints $v(\theta) \geq l_\theta(v^{'}(\theta))$ for all $\theta$, where $l_\theta$ is defined by the relation 12 that we derived from the hidden storage condition.

When $u$ is bounded, the state variable $V$ has bounded variations, and the budget constraint translates into an upper bound on $V$. Besides, we can restrain our analysis to admissible solutions such that $U(V) \geq U(V_0)$ where $V_0$ is the full-bunching admissible solution, and this last condition translates into a lower bound on $V$. Thus, both the control and the state variable belongs to a fixed bounded region. Lastly, since we assumed that the utility functions $u$ and $w$ are monotonous functions whose image are a closed set, the inverse function $C$ and $K$ are bounded function (i.e. the image of a bounded set is bounded), and thus the resources used at period one $C(V^{'}(\theta)$ and period two $K(\beta^{-1}V(\theta) - \beta^{-1}\theta V^{'}(\theta))$ are also bounded. Therefore, admissible solutions belong to the Sobolev space $W^{1,\infty}(\Theta)$, i.e. the variable state $v$ must be Lipschitz continuous with a bounded Lipschitz constant (and thus almost everywhere differentiable with a derivative $u = v^{'}$ that belongs to the Lebesgue space $L^\infty(\Theta)$).

The existence of an optimal solution is then a direct consequence of the following lemma :

**Lemma 3.** *Let $\tilde{W}$ be the closed unity ball of the Sobolev space $W^{1,\infty}(\Theta)$. For any sequence $(v_n)$ of elements of $\tilde{W}$, we can extract a subsequence of $(v_{n_k})$ that converges uniformly to a function $v \in \tilde{W}$, such that its derivatives $(v^{'}_{n_k})$ converge weakly to a function almost everywhere equal to $v^{'}$. Besides, there exist a convex combination $(w_k)$ that converges uniformly to to the same limit, such that its derivatives converge point-wise almost everywhere to $v^{'}$.*

Indeed, according to the previous paragraph, there exist a sequence $(v_n)$ of elements of $\tilde{W}$ such that $U(v_n)$ converges to $U^*$. The Lemma 3 shows that we can extract a subsequence $(v_{n_k})$

and build a convex combination $(w_k)$ such that we have point-wise convergence to a function $v \in \tilde{W}$. Therefore, since both the state variable and the control function belong to a bounded region, and since $f_0$ is linear, Lebesgue's dominated convergence theorem implies that

$$U^* = \lim \int_\Theta f_0(v_{n_k}, v^{'}_{n_k})d\theta = \lim \int_\Theta f_0(w_k, w^{'}_k)d\theta = \int_\Theta f_0(v, v^{'})d\theta = U(v)$$

and, since $g$ is convex, using again Lebesgue's dominated convergence theorem,

$$\int_\Theta g(v(\theta), v^{'}(\theta))d\theta = \lim \int_\Theta g(w_k(\theta), w^{'}_k(\theta))d\theta \leq \max \int_\Theta g(v_{n_k}(\theta), v^{'}_{n_k}(\theta))d\theta \leq c$$

Lastly, since the function $l_\theta$ are smooth and nondecreasing, the mixed constraints are equivalent to $v^{'}(\theta) - l_\theta^{-1}(v(\theta)) \leq 0$. The sequence $v^{'}_n - l_\theta^{-1}(v(_n\theta))$ converges weakly to $v^{'} - l_\theta^{-1}(v(\theta))$ which is therefore negative in the sense of distribution, hence almost everywhere negative.

Let's now prove the previous lemma. We consider a sequence $v_n \in \tilde{W}$ such that $f(v_n, v^{'}_n)$ converges to the maximum $U^*$. According to Arzela-Ascoli's theorem, we can extract of the sequence $v_n$ of uniformly bounded differentiable functions a subsequence $v_{n_k}$ that converges uniformly in $L^\infty(\Theta)$. Besides, since the Lipschitz constant of the functions $v_n$ is bounded, the limit is also Lipschitz continuous. Secondly, since $L^\infty(\Theta)$ is the dual of the separable space $L^1(\Theta)$, we can extract from the bounded sequence $v^{'}_n \in L^\infty(\Theta)$ a subsequence $v^{'}_{n_k}$ which converges in $L^\infty(\Theta)$ for the weak topology $*\sigma(L^\infty(\Theta), L^1(\Theta))$. Since $\Theta$ is compact, we have the inclusions $L^\infty(\Theta) \subset L^p(\Theta) \subset L^1(\Theta)$ for any real $p > 1$, and thus the sequence $v^{'}_{n_k}$ is also a sequence in $L^p(\Theta)$ that converges for the weak topology of $L^p(\Theta)$. Therefore, we can use those two extractions sequentially, so that $v_{n_k}$ converges uniformly to a function $v \in \tilde{W}$, and $v^{'}_{n_k}$ converges for the weak topology of $L^p(\Theta)$ to a function $u$. The derivative of $v$ is equal to $u$ in the sense of distribution, and hence almost everywhere (since the distribution's test space can also be used as the test spaces for the Sobolev space $W^{1,p}(\Theta)$).

Using Mazur's lemma, we can replace our sequence of admissible solutions by a sequence of convex combinations $w_k$ such that the control $w^{'}_k$ converges strongly to the same limit. We can then extract a subsequence of admissible solution $w_{k_i}$ such that $w^{'}_{k_i}$ will converge pointwise almost everywhere to the same limit (cf. Theorem IV.9, Brezis p 58). □

### 11.1.2 Necessary conditions of optimality

In the following, we present the necessary conditions required for optimality. We consider optimal solution that are continuous and piecewise continuously differentiable. Let's note $\underline{u} \in \{-\infty, \mathcal{R}\}$ and $\bar{u} \in \{\mathcal{R}, \infty\}$ the lower and upper bounds on the control $u$.

**Lemma 4.** *An optimal solution of the maximization program* $\max U(V)$*, defined by the relations* (9)*,* (10) *and* (11) *with a non-decreasing control* $\mathrm{U} \in [\underline{u}, \bar{u}]$*, satisfies the following necessary conditions :*

*(i) The utility* $\mathrm{U}$ *is smooth on any open subset* $I$ *of* $\Theta$ *where it is increasing and such that* $u(I) \subset ]\underline{u}, \bar{u}[$*, and satisfies the following Euler-Lagrange equation*

$$\frac{\partial}{\partial\theta}\left[\beta f(\theta)\frac{B^{'}(\theta)}{\mathrm{U}'(\theta)}\right] = \frac{\delta f(\theta)}{w'(k(\theta))} - \lambda^{-1}\frac{\partial}{\partial\theta}\left[F(\theta) + (1-\beta)\theta f(\theta)\right] \tag{14}$$

*(ii) Transversality conditions :*

$$\frac{B'(\underline{\theta})}{\text{U}'(\underline{\theta})} - \lambda^{-1}(1-\beta^{-1})\tilde{\theta}_1 = \beta^{-1}(\lambda^{-1} - \frac{\delta}{w'(k(\tilde{\theta}_1))})F(\tilde{\theta}_1) \geq 0 \tag{15}$$

*and*

$$\frac{B'(\bar{\theta})}{\text{U}'(\bar{\theta})} - \lambda^{-1}(1-\beta^{-1})\tilde{\theta}_2 = \beta^{-1}(\lambda^{-1} - \frac{\delta}{w'(k(\tilde{\theta}_2))})(1-F(\tilde{\theta}_2)) \leq 0 \tag{16}$$

*where $\tilde{\theta}_1$ (resp. $\tilde{\theta}_2$) is the infimum (resp. supremum) $\theta$ such that $\text{U}'(\theta) > 0$.*

*(iii) Inverse Euler equation :*

$$\lambda^{-1} = \int \frac{\delta f(\theta)}{w'(k(\theta))} d\theta \geq \int_{\underline{\theta}}^{\bar{\theta}} \frac{f(\theta)}{u'(c(\theta))} d\theta$$

*with equality when $\text{U}'(\theta) > 0$ at both end of the interval $\Theta$.*

*(iv) There is no money burning, i.e. the budget constraint* (11) *is an equality.*

First, in order to cope with the budget constraint and with the monotonicity of the utility, we add two dimensions to the state variable and consider the function $x(\theta) = (x_1(\theta), x_2(\theta), x_3(\theta))$ with $x_1(\theta) = V(\theta)$, $x_2(\theta) = \text{U}(\theta)$ and $x_3(\theta) = -\int_{\underline{\theta}}^{\theta} B(\tilde{\theta})f(\tilde{\theta})d\tilde{\theta}$. Since

$$B(\theta) = C(V'(\theta)) + \delta K(\beta^{-1}V(\theta) - \beta^{-1}\theta V'(\theta)),$$

the control function is

$$\mathcal{F}(\theta, x, u) = (x_2, u, -g(\theta, x_1, x_2)) \tag{17}$$

where $g(\theta, x_1, x_2) = f(\theta) \times (C(x_2) + \delta K(\beta^{-1}x_1 - \beta^{-1}\theta x_2)$. Thus, the optimal control problem is to maximize $\int_{\underline{\theta}}^{\bar{\theta}} f_0(\theta, x_1(\theta), x_2(\theta))d\theta$ under the control $x'(\theta) = \mathcal{F}(\theta, x(\theta), u(\theta))$, where the function $f_0$ and $\mathcal{F}$ are defined respectively by the equation 10 and 17, with initial constraint $x_3(\underline{\theta}) = 0$ and terminal constraint $x_3(\bar{\theta}) \geq -y$ (while $x_1$ and $x_3$ are free at the end-times $\underline{\theta}$ and $\bar{\theta}$)), and with $u(\theta) \geq 0$.

According to Seierstad and Sydsaeter (theorem 8 p 378 and theorem 15 p 396, 1987), there exist a number $p_0 = 0$ or 1, and a vector of absolutely continuous functions $p(\theta) = (p_1(\theta), p_2(\theta)), p_3(\theta)$, such that the generalized Lagrangian defined by

$$L(\theta, x_1, u, p) = p_0 f_0(\theta, x_1, x_2) + p_1(\theta)x_2 + p_2(\theta)u - p_3(\theta)g(\theta, x_1, x_2)$$

satisfies the Euler equations $dp_i(\theta)/d\theta = -\partial L^*/\partial x_i$ for $i = 1, 2, 3$, where $^*$ denotes evaluation at $(x^*(\theta), u^*(\theta))$. The maximum principle $p_2(\theta) = \partial L^*/\partial u = 0$ when $u \in ]\underline{u}, \bar{u}[$ and $\partial L^*/\partial u \leq 0$ (resp. $\geq 0$) If $u = \underline{u}$ (resp. $u = \bar{u}$). All those equations hold almost everywhere. The transversality conditions are $p_1(\bar{\theta}) = p_1(\underline{\theta}) = p_2(\bar{\theta}) = p_2(\underline{\theta}) = 0$, and $p_3(\bar{\theta}) \geq 0$ with equality if $x_3(\bar{\theta}) > -y$. The second Euler equation implies that $p_3(\theta)$ is a constant that we denote $p_3$. It is then easy to check that $p_0 = 0$ implies that $p_1$ is non-decreasing, and consequently that $p_1 = 0$. Then necessarily, using again the third then the second Euler equation, we must have also $p_3 = 0$ and thus $p_2 = 0$ which is ruled out. Therefore $p_0 = 1$. Lastly, if $p_3 = 0$, the first Euler equation implies that $dp_1(\theta)/d\theta = -\beta^{-1}f(\theta)$ which would contradict the transversality equations. Consequently, $p_3 > 0$ and there is no money burning, i.e. $x_3(\bar{\theta}) = -y$.

We now prove the regularity properties of the optimal solution. Let's set $\tilde{L} = f_0 - p_3 g$. The Euler equations are equivalent to $\dot{p}_1(\theta) = -\partial\tilde{L}/\partial x_1$ and $\dot{p}_2(\theta) = -p_1(\theta) - \partial\tilde{L}/\partial x_2$. On any open subset $I$ of $\Theta$ such that $u^* > 0$, we therefore have the classical Euler-Lagrange equation

$$\left(\frac{\partial}{\partial x_1} - \frac{d}{d\theta}\frac{\partial}{\partial x_2}\right)\tilde{L}^* = 0$$

which implies that $\partial\tilde{L}^*/\partial x_2$ is continuous, differentiable with respect to $\theta$, and equal up to a constant to the integrand of $\partial\tilde{L}^*/\partial x_1$. Besides, the second order Legendre condition is strictly satisfied, i.e. $\partial^2\tilde{L}/\partial^2 x_2 = -p_2 f(\theta)[C^{"}(x_2) + \delta\theta^2\beta^{-2}K^{"}(\beta^{-1}x_1 - \beta^{-1}\theta x_2)] < 0$. Thus, using the implicit function theorem, we can invert locally the function $\partial\tilde{L}/\partial x_2$, and prove by recurrence that $x_2^*(\theta)$ is smooth on $I$. For the same reason, $u$ cannot make any jump from $u \in ]\underline{u}, \bar{u}[$ to $\bar{u}$ (resp. to $\underline{u}$) since the function $\partial L^*/\partial u$ would jump from zero to a negative (resp. positive) value. Therefore, the function $u$ is continuous

Lastly, the necessary conditions can be expressed in terms of the utility functions $\mathrm{U}(\theta)$ and $\mathrm{W}(\theta)$ (we omit thereafter the superscript *) from the relation $x_1^*(\theta) = \theta\mathrm{U}(\theta) + \beta\mathrm{W}(\theta)$ and $x_2^*(\theta) = \mathrm{U}(\theta)$. Indeed, on intervals where the solution is smooth, we can use the relation $\theta\mathrm{U}^{'}(\theta) = -\beta\mathrm{W}^{'}(\theta)$, and get

$$\tilde{L}^*_{x_1} = L^*_{x_1} = \beta^{-1}f(\theta)(1 - p_3\delta K^{'}(\beta^{-1}x_1^*(\theta) - \beta^{-1}\theta x_2^*(\theta)) = \beta^{-1}f(\theta)(1 - \frac{p_3\delta}{w^{'}(k(\theta))})$$

and

$$\tilde{L}^*_{x_2} = (1-\beta^{-1})\theta f(\theta) - p_3 f(\theta)[C^{'}(x_2^*(\theta)) - \delta\beta^{-1}\theta K^{'}(\beta^{-1}x_1^*(\theta) - \beta^{-1}\theta x_2^*(\theta)]$$

or, equivalently,

$$\tilde{L}^*_{x_2} = (1-\beta^{-1})\theta f(\theta) - p_3 f(\theta)B^{'}(\theta)/\mathrm{U}^{'}(\theta)$$

The Euler-Lagrange equation $d\tilde{L}^*_{x_2}/d\theta = \tilde{L}^*_{x_1}$ is then equivalent to the relation expressed in the lemma with $\lambda = p_3$. Besides, the transversality conditions $p_1(\theta) = 0$ at $\underline{\theta}, \bar{\theta}$ imply that $\int L^*_{x_1} = 0$ or, equivalently, that $\lambda^{-1} = p_3^{-1} = \int \frac{\delta f(\theta)}{w^{'}(k(\theta))}d\theta$. If $u$ is constant at the left of the interval, and start to be strictly increasing at $\tilde{\theta}_1 > \underline{\theta}$ then, since the function $k(\theta)$ is non-increasing,

$$\tilde{L}^*_{x_2}(\tilde{\theta}_1) = -p_1(\tilde{\theta}_1) = -\int_{\underline{\theta}}^{\tilde{\theta}_1}\tilde{L}^*_{x_1} = -\beta^{-1}(1 - \frac{\delta\lambda}{w^{'}(k(\tilde{\theta}_1))})F(\tilde{\theta}_1) \le 0$$

and, similarly, if $u$ is constant at the right of the interval, starting from $\tilde{\theta}_2 < \bar{\theta}$,

$$\tilde{L}^*_{x_2}(\tilde{\theta}_2) = -p_1(\tilde{\theta}_2) = -\int_{\tilde{\theta}_2}^{\bar{\theta}}\tilde{L}^*_{x_1} = -\beta^{-1}(1 - \frac{\delta\lambda}{w^{'}(k(\tilde{\theta}_2))})(1 - F(\tilde{\theta}_2)) \ge 0.$$

Lastly, we can also rewrite the Euler-Lagrange equation, using the relation 2, as

$$\frac{\partial}{\partial\theta}\left[\frac{\beta f(\theta)}{u^{'}(c(\theta))}\right] = \frac{\partial}{\partial\theta}\left[\frac{\delta\theta f(\theta)}{w^{'}(k(\theta))}\right] + \frac{\delta f(\theta)}{w^{'}(k(\theta))} - \lambda^{-1}\frac{\partial}{\partial\theta}\left[F(\theta) + (1-\beta)\theta f(\theta)\right]$$

or, equivalently

$$\frac{\beta f(\theta)}{u^{'}(c(\theta))} = \frac{\partial}{\partial\theta}\left[\frac{\theta\beta f(\theta)}{u^{'}(c(\theta))}\right] - \frac{\partial}{\partial\theta}\left[\frac{\delta\theta^2 f(\theta)}{w^{'}(k(\theta))}\right] + \lambda^{-1}\theta\frac{\partial}{\partial\theta}\left[F(\theta) + (1-\beta)\theta f(\theta)\right]$$

we obtain similarly, by integrating the right part,

$$\int_{\underline{\theta}}^{\bar{\theta}} \frac{f(\theta)}{u'(c(\theta))} d\theta = \lambda^{-1} + [\theta\lambda^{-1}\tilde{L}_{x_2}(\theta)]_{\tilde{\theta}_1}^{\tilde{\theta}_2}.$$

The left term is negative, and equal to zero when $\tilde{\theta}_1 = \underline{\theta}$ and $\tilde{\theta}_2 = \bar{\theta}$ so that we have the inverse Euler equation. □

Let's now consider the maximization problem when $\beta = 1$, enriched with the non-convex and mixed constraints derived from the hidden storage condition, i.e. $g(\theta, x_1(\theta), u(\theta)) \geq 0$ for all $\theta$, with

$$g_\theta(x_1, u) = g(\theta, x_1, u) = \beta^{-1}\delta\theta K'(\beta^{-1}x_1 - \beta^{-1}\theta u) - C'(u) \tag{18}$$

**Lemma 5.** *Let $V^*$ be an optimal solution of the maximization problem $\max U(V)$ with the additional mixed constraints $g_\theta \geq 0$ and $\beta = 1$. If the subset $\{\theta \in \Theta, g^*_\theta > 0\}$ is non empty, then it is the union of disjoint open intervals, and on each of these intervals, the optimal solution satisfies the conditions described in the Lemma 4.*

We apply again the theorem of Seierstad and Sydsaeter mentioned above. The only difference here is that we need to consider one more multiplier, i.e. a bounded, measurable and non negative function $q(\theta)$ such that $q(\theta)g(\theta, x_1^*(\theta), u^*(\theta)) = 0$, and modify the generalized Lagrangian $L$ by adding to it the function $q(\theta)g(\theta, x_1(\theta), u(\theta))$. Since $g = \frac{\partial \mathcal{F}}{\partial u}$, the maximum principle is then equivalent to

$$p_2 g(\theta, x_1^*(\theta), u^*(\theta)) + q(\theta)\frac{\partial g}{\partial u}(\theta, x_1^*(\theta), u^*(\theta)) = -p_1(\theta)$$

According to the Euler equations, the multiplier $p_1(\theta)$ is continuous. Besides if, for some $\theta' \in \Theta$, we have $g(\theta', x_1^*(\theta'), u^*(\theta')) \neq 0$, then $q(\theta') = 0$ and $p_1(\theta') = -p_2 g(\theta', x_1^*(\theta'), u^*(\theta')) < 0$. Let $\Phi' = ]\underline{\theta}', \bar{\theta}'[$ be the largest open interval in $\Phi$ containing $\theta'$ such that $p_1(\theta) < 0$. Since $\frac{\partial g}{\partial u} < 0$ and $q^2(\theta) = -p_1(\theta)q(\theta)/\frac{\partial g}{\partial u}$, we have $q(\theta) = 0$ on $\Phi'$. Thus, the mixed constraints are not binding on $\Phi'$, and the optimal solution satisfies inside this interval the necessary conditions considered in the previous section. In particular, the Euler Lagrange equation 14 is satisfied, and the second Legendre condition hold, which implies that the control $u$ is smooth on $\Phi'$. The transversality condition is equivalent to $g = 0$ when $\beta = 1$, and is consequence of $p_1(\underline{\theta}') = 0$. This last relation is true by continuity if $\underline{\theta}' > \underline{\theta}$, and follows from the transversality condition if $\underline{\theta}' = \underline{\theta}$. □

## 11.2 Two-type case

*Proof on the Lemma 1 (two-type model) :* The assumptions of the utility function $u$ and $w$ imply that the admissible solutions belong to a bounded compact space, and thus that the maximum is an interior solution. We first consider the problem without storage, i.e. $\epsilon = 0$ in the incentive constraints, and will look at the more general problem at the end. At the optimum, since $\theta_h > \theta_l$, the $(IC)$ implies that $k_l \geq k_h$ and $c_h \geq c_l$. Those inequality are strict. Indeed, if it were not the case, both equation would be an equality in order to comply to the incentive constraints. In this case, if we increase slightly $c_h$, and decrease slightly $c_l$ while maintaining the budget constant, then the objective function will increase in proportion since $\theta_h > \theta_l$. On the contrary, if we increase slightly $k_l$, and decrease slightly $k_h$ while maintaining the budget constant, then the variation of the objective function will be dominated by a constant factor times the square of the decrease of $k_l$ (and negative if the utility function $w$ is strictly concave).

If we rewrite the incentive constraints as $\theta_h(u(c_h)-u(c_l)) \geq \beta(w(k_l)-w(k_h)) \geq \theta_l(u(c_h)-u(c_l))$, it is easy to see that we can combine those two approaches in order to maintain the incentive constraints valid.

The budget is entirely used at the optimum, i.e. (BC) is an equality. If it were not the case, it would be possible to increase the consumption at least for one agent's type (either at period 1 or 2) while maintaining the incentive constraints valid. Indeed, one of the two $(IC)$ is strict since $\theta_h > \theta_l$. Let's suppose for instance that $(IC_h)$ is a strict inequality. It would then be possible to increase consumption of agent type's $l$, while increasing the budget and the objective function, and still verifying the incentive constraints.

At the optimum, the constraint $(IC_l)$ is an equality and $(IC_h)$ is a strict inequality. Indeed, if the incentive constraint $(IC_l)$ were a strict inequality, then it would be possible to increase $k_h$ and decrease $k_l$ while maintaining constant $\pi k_h + (1-\pi)k_l$ so that both the $(BC)$ and the $(IC)$ are still valid. Since $w^{'}(k_h) > w^{'}(k_l)$, this improve the objective function. The constraint $(IC_l)$ is thus an equality and, since $\theta_h > \theta_l$, the second constraint $(IC_h)$ is slack.

Let's show that $\delta\theta_l u^{'}(c_l) \in [\beta w^{'}(k_l), w^{'}(k_l)]$ at the optimum. Since $(IC_h)$ is a strict inequality, it is possible to slightly modify $c_l$ and $k_l$ at the optimum while maintaining valid this incentive constraint. If $\delta\theta_l u^{'}(c_l) > w^{'}(k_l)$, it is possible to increase $c_l$ while maintaining constant $c_l + \delta k_l$ so that $\theta_l u(c_l) + \delta w(k_l)$ increases. This will increase also $\theta_l u(c_l) + \beta w(k_l)$, and improve the objective function while maintaining the $(IC)$ and $(BC)$. If $\delta\theta_l u^{'}(c_l) < \beta w^{'}(k_l)$, it is possible to increase $k_l$ while maintaining constant $c_l + \delta k_l$ so that $\theta_l u(c_l) + \beta w(k_l)$ increases. This will increase also $\theta_l u(c_l) + w(k_l)$, and improve the objective function while maintaining the $(IC)$ and $(BC)$. Consequently, those inequalities are false at the optimum.

The two previous results imply that

$$\frac{\beta}{\theta_l} = \frac{u(c_h) - u(c_l)}{w(k_l) - w(k_h)} \leq \frac{c_h - c_l}{k_l - k_h}\frac{u^{'}(c_l)}{w^{'}(k_l)} \leq \frac{1}{\delta\theta_l}\frac{c_h - c_l}{k_l - k_h}$$

and thus $c_l + \beta\delta k_l \leq c_h + \beta\delta k_h$.

We can then reduce the maximization problem to the following problem :

$$\max \pi(\theta_h u(c_h) + w(k_h)) + (1-\pi)(\theta_l u(c_l) + w(k_l))$$

subject to budget constraints and incentive compatibility constraint :

$$\begin{cases} \theta_l u(c_h) + \beta w(k_h) = \theta_l u(c_l) + \beta w(k_l) & (IC_l) \\ \pi(c_h + \delta k_h) + (1-\pi)(c_l + \delta k_l) = y & (BC) \end{cases}$$

Introducing lagrangian multipliers leads to the following system of equation

$$\begin{cases} \pi\theta_h u^{'}(c_h) - \lambda_2\theta_l u^{'}(c_h) - \lambda_3\pi = 0 \\ \pi w^{'}(k_h) - \lambda_2\beta w^{'}(k_h) - \delta\lambda_3\pi = 0 \\ (1-\pi)\theta_l u^{'}(c_l) + \lambda_2\theta_l u^{'}(c_l) - (1-\pi)\lambda_3 = 0 \\ (1-\pi)w^{'}(k_l) + \lambda_2\beta w^{'}(k_l) - (1-\pi)\delta\lambda_3 = 0 \end{cases}$$

which implies by combining those equation that

$$\frac{\beta}{\theta_l\delta u^{'}(c_l)} = \frac{1}{w^{'}(k_l)} - (1-\beta)\times(\frac{\pi}{w^{'}(k_h)} + \frac{1-\pi}{w^{'}(k_l)}) \tag{19}$$

and

$$\frac{\pi}{u'(c_h)} + \frac{1-\pi}{u'(c_l)} = \bar{\theta}\delta \times \{\frac{\pi}{w'(k_h)} + \frac{1-\pi}{w'(k_l)}\} \tag{20}$$

The first equation gives the previous result $\beta w'(k_l) \leq \theta_l \delta u'(c_l) \leq w'(k_l)$, which are strict inequalities except if $\beta = 1$, while those two equations can be combined in order to give period 1 consumption of type $\theta_h$ in term of period 2 consumption of both types :

$$\frac{\beta}{\delta u'(c_h)} = \frac{\theta_l}{w'(k_h)} + (\beta\theta_h - \theta_l) \times \{\frac{\pi}{w'(k_h)} + \frac{1-\pi}{w'(k_l)}\} \tag{21}$$

which implies, since $k_l > k_h$, that $\theta_h \delta u'(c_h) > w'(k_l)$ and that $\theta_h \delta u'(c_h) - \beta w'(k_h)$ has the same sign as $\theta_l - \beta\theta_h$. Combining equations 19 and 21 leads to the following equation

$$\frac{\beta}{\theta_l \delta} \times \left( \frac{1-\beta}{u'(c_h)} - \frac{1-\beta\frac{\theta_h}{\theta_l}}{u'(c_l)} \right) = \frac{1-\beta}{w'(k_h)} - \frac{1-\beta\frac{\theta_h}{\theta_l}}{w'(k_l)}.$$

When $\beta = 1$, we know already that $c_l + \delta k_l < c_h + \delta k_h$. When $\beta < \theta_l/\theta_h$, then since $k_l > k_h$, we have $\frac{\pi}{w'(k_h)} + \frac{1-\pi}{w'(k_l)} > \frac{1}{w'(k_h)}$ and the previous relation 21 implies that $\frac{1}{u'(c_h)} < \frac{\theta_h \delta}{w'(k_h)}$. The incentive constraint $(IC)_l$ implies also that

$$\frac{\beta}{\theta_l} = \frac{u(c_h) - u(c_l)}{w(k_l) - w(k_h)} > \frac{c_h - c_l}{k_l - k_h} \frac{u'(c_h)}{w'(k_h)} > \frac{c_h - c_l}{k_l - k_h} \frac{1}{\theta_h \delta}$$

and thus $c_h - c_l < \delta(k_l - k_h)$.

Other results obtained in the continuum shock's case can be easily obtained. The Lagrange multiplier for the budget constraint equals the marginal utility of budget, which is given by the following equation

$$\delta\lambda_3\pi = \pi w'(k_h) - \lambda_2 \beta w'(k_h) = \pi w'(k_h) \times \left( 1 + \beta(1-\pi) \frac{\theta_l \delta u'(c_l) - w'(k_h)}{(1-\pi)\beta w'(k_h) + \pi\theta_l \delta u'(c_l)} \right)$$

or, equivalently,

$$\frac{\lambda_3}{w'(k_h)} = \theta_l u'(c_l) \frac{\beta(1-\pi) + \pi}{(1-\pi)\beta w'(k_h) + \pi\theta_l \delta u'(c_l)}$$

and this equation combined with equation 19, becomes the Euler Inverse Equation :

$$\frac{1}{\lambda_3} = \frac{\pi\delta}{w'(k_h)} + \frac{(1-\pi)\delta}{w'(k_l)} = \bar{\theta}^{-1} \times \{\frac{\pi}{u'(c_h)} + \frac{1-\pi}{u'(c_l)}\}$$

Besides, when $\beta = 1$, we deduce from the previous section that

$$\frac{\theta_h \delta u'(c_h)}{\beta w'(k_h)} < 1 = \frac{\theta_l \delta u'(c_l)}{\beta w'(k_l)}$$

while, when $\beta < \theta_l/\theta_h$, we have $\theta_h \delta u'(c_h) > w'(k_h)$ and $\beta w'(k_l) < \theta_l \delta u'(c_l) < w'(k_l)$. Consequently,

$$\frac{\theta_h \delta u'(c_h)}{\beta w'(k_h)} > \frac{1}{\beta} > \frac{\theta_l \delta u'(c_l)}{\beta w'(k_l)} > 1$$

Thus, when $\beta$ is small enough, the slope of the indifference curve of the time-inconsistent player in period 2 is higher than 1, and greater for the type $\theta_h$ than for the type $\theta_l$. Besides, if $\beta \leq \theta_l/\theta_h$, then the function $\theta_l u(c_h - \delta\epsilon) + \beta w(k_h + \epsilon)$ is decreasing in $\epsilon$ for all $\epsilon \geq 0$. Similarly, the function $\theta_h u(c_l - \delta\epsilon) + \beta w(k_l + \epsilon)$ is decreasing in $\epsilon$ for all $\epsilon \geq 0$. The storage constraint is thus not binding.

Let's now consider the case $\beta = 1$ with the no storage condition. If we simply consider the maximization problem with the additional constraint of a non-increasing budget, i.e. $c_h + \delta k_h \geq c_l + \delta k_l$, we have seen that this constraint then is binding : the optimal solution must lies on the line $c_h + \delta k_h = c_l + \delta k_l = y$, i.e. there is no transfer, and it is easy to check that the first-best solution, defined by $\delta\theta_x u^{'}(c_h) = w^{'}(k_h)$, satisfies the incentive constraint and the no storage condition. The derivative of the utility delivered by this allocation is then equal to $U^{'}(y) = \pi\delta^{-1}w^{'}(k_h) + (1-\pi)\delta^{-1}w^{'}(k_l) = \pi u^{'}(c_h) + (1-\pi)u^{'}(c_l)$.

Lastly, we note that the incentive compatibility constraint for the high type player is never binding, and thus could be dropped in the maximization problem without any impact on the solution. We also note that we have always $c_h > c_l$ and $k_l > k_h$ both when $\beta = 1$ and when $\beta < \theta_l/\theta_h$. $\square$