Reputation Cycles

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Abstract

This paper shows that two-period cycles may arise endogenously when products are experience goods. Then firms invest in the quality of their output in order to establish a good reputation. Multiple equilibria arise because investment in reputation has a pecuniary external effect that works through the aggregate discount factor. Cycles are more likely to occur when information diffuses slowly and when agents are patient.

Keywords: Endogenous Fluctuations, Reputation, Intangible Capital.

1 Motivation

Economic growth is achieved through a continuous stream of innovations. Each one of them carries its own risk, as new products and processes vary in their quality, and quality is gradually revealed over time. Since most transactions entail an up-front payment without an ex-post quality guarantee, the clearing price reflects the buyer's expectation of the quality of the good or service that he is likely to get. A reputable seller is a firm that is thought to be selling high quality products and services. If a firm has a good reputation, it fetches higher prices.

The quest to profit from a good reputation drives firms to invest in the quality of their output so as to raise the spot-market prices they will be paid in the future. This is why a

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fraction of a firm's market value originates in a form of capital that accountants labeled as "intangible", and some of this capital is related to the firm's reputation. The most obvious manifestation of reputational capital is the premium that consumers are ready to pay in order to acquire goods from established brands.

Whereas a large literature studies how reputations arise and evolve over time, the macro implications remain largely unexplored.[1] Instead, we propose to study how investment in reputation affects aggregate outcomes. We find that multiple equilibria arise because production has a pecuniary external effect that works through the discount factor. Although the model has no direct externalities, some of the equilibria are cyclical even if there are no shocks.

Shleifer (1986) also generates business cycles via a pecuniary external effect that arises because consumers spend the profits of one innovator on buying output produced by other innovators. This leads firms to implement their innovations simultaneously. High output today causes interest rise, however, and if rates rise too much consumers will prefer to save, thereby removing firms' incentives to implement today. Cycles require preferences to be less curved than log whereas our model requires no such restriction. Rather, the amplitude of the cycle rises with the curvature of utility.

Learning-by-doing (LBD hereafter) shares with our model the feature that higher effort today raises an agent's income in the future. By contrast, however, LBD means that higher output today also raises productivity and output tomorrow and it cannot lead to as large a volatility in the discount factor. At any rate if the general equilibrium model with LBD (Chang, Gomes and Schorfheide, 2002; Qureshi, 2009, and Gunn and Johri, 2011) has an endogenous cycle, this has yet to be shown.

Section 2 lays out the set-up of the model. Endogenous cycles are characterized in Section 3. Section 4 explains how the model can be parametrized, and shows that cycles may arise for realistic parameters values.

[1] Atkeson, Hellwig and Ordonez (2012) is an exception. There, firms have a hidden reputational concern only at entry with no further hidden actions thereafter. There is a literature on the reputation of central bankers for pursuing tight-money policies (Backus and Driffill, 1985) that deals with a set of issues quite different from the ones I will raise here.

2 The Model

Consider a version of Lucas' (1978) model where the only assets are a set of finitely-lived trees. Aggregate output equals the fruit of the trees which cannot be stored but is used for consumption as well as investment. The model differs from Lucas' in two important dimensions. First, trees are heterogeneous in their productivity. Second, the actual utility of a tree's output cannot be observed ex-ante nor contracted upon. The combination of these two features give rise to reputational concerns.

Technology.—Each tree's output is given by

$$y_t^i = \theta_t^i + a_t^i + \varepsilon_t^i,$$

where the subscript $i \in [0,1]$ varies across trees. To ease notation, we hereafter drop the subscript i when not necessary. The variable θ is the tree's efficiency which may vary over time, a is the investment in terms of the consumption good and ε is a normally distributed idiosyncratic shock with mean zero and variance σ_ε^2. The simple model described in this section features no variation in aggregate productivity, only idiosyncratic shocks.

Fruits are "experience goods" whose quality is only revealed *ex post*. Payments are made through spot market transactions in which the period-t payment cannot be contingent on y_t; in other words the buyer pays the seller up front. Because investment a is hidden, the firm's only incentive is so as to raise prices it will receive in the future.

Output has a persistent effect on prices because it affects the market posterior about the tree's productivity. Trees differ in their efficiencies whose initial values are drawn from a normal distribution $\theta_0 \sim \mathcal{N}(m_0, \sigma_\theta^2)$. Then "firm-type heterogeneity" at age 0 is σ_θ^2. A tree's productivity fluctuates over time

$$\theta_t = \theta_{t-1} + \nu_t,$$

with $\nu \sim \mathcal{N}(0, \sigma_\nu^2)$. The normality of the shocks ν preserves the normality of the cross-sectional distribution of θ.

Learning.—Within any given period, events unfold as follows:

1. Sellers auction their services to the highest bidder, with R_t denoting the winning bid;

2. The seller chooses a_t, and it is not observed by anyone else, ever;

3. Output y_t is realized, everyone observes it, and it becomes part of the public history.

At the time that the winning bid R_t is determined, everyone knows the history $\{y_\tau\}_{\tau=0}^{t-1}$ up to date $t-1$. In a Nash equilibrium in which a_t^* denotes the seller's action, a sufficient statistic for the information revealed about θ is the sequence $x^t \equiv (x_0, ..., x_{t-1})$, where $x_t \equiv y_t - a_t^*$. x_t is normally distributed as is the posteriors about θ, more precisely $\theta_t \sim \mathcal{N}\left(m_t, \sigma_{\theta,t}^2\right)$ where

$$\sigma_{\theta,t}^2 = \frac{1}{\sigma_{\theta,t-1}^{-2} + \sigma_\varepsilon^{-2}} + \sigma_\nu^2. \tag{1}$$

In order to form its expectation about θ, the firm does not have to keep track of the whole output path as

$$m_{t+1} = E\left[\theta_{t+1} \middle| m_t, x_t\right] = \lambda_t m_t + (1-\lambda_t)\, x_t, \text{ where } \lambda_t \equiv \frac{\sigma_{\theta,t}^{-2}}{\sigma_{\theta,t}^{-2} + \sigma_\varepsilon^{-2}}. \tag{2}$$

In the long-run, posterior precision converges to its stationary level $\bar{\sigma}_\theta^{-2}$, whose value follows setting $\sigma_{\theta,t} = \sigma_{\theta,t-1}$ in (1),

$$\bar{\sigma}_\theta^{-2} = \frac{1}{2}\left(\sqrt{\frac{1}{\sigma_\varepsilon^4} + \frac{4}{\sigma_\varepsilon^2\sigma_\nu^2}} - \frac{1}{\sigma_\varepsilon^2}\right). \tag{3}$$

Observe that $\partial\bar{\sigma}_\theta^{-2}/\partial\sigma_\varepsilon < 0$ and $\partial\bar{\sigma}_\theta^{-2}/\partial\sigma_\theta < 0$ as both output noise and fundamental fluctuations lower the stationary precision. Hereafter we assume that firms are born with the stationary level of prior, i.e., $\sigma_{\theta,t} = \bar{\sigma}_\theta$ for all firms of vintage t.

A firm's individual state and its aggregate distribution.—Assume that at birth a firm's θ is drawn from $\mathcal{N}(m_0, \bar{\sigma}_\theta^2)$. Then $\sigma_\theta^2 = \bar{\sigma}_\theta$ does not change as the firm ages, and firm age is not a state. The firm's individual state is then just m, which evolves as $m' = m + (1-\lambda)\varepsilon$. Hence, instead of i, we shall denote a firm by m.

A firm's lifetime is a geometrically distributed random variable. An additional period of life adds $(1-\lambda)^2\sigma_\varepsilon^2$ to the variance of m, and therefore the stationary variance of m is

$$(1-\lambda)^2\sigma_\varepsilon^2 \sum_{t=1}^{\infty} t\delta(1-\delta)^t = (1-\lambda)^2\sigma_\varepsilon^2\frac{1-\delta}{\delta},$$

(because the mean of a geometric distribution is $\sum_t^\infty t\delta(1-\delta)^{t-1} = 1/\delta$), and the stationary CDF of m is

$$\Psi(m) = \Phi\left(\frac{m - m_0}{(1-\lambda)\,\sigma_\varepsilon\sqrt{\frac{1-\delta}{\delta}}}\right) \tag{4}$$

where $\Phi(.)$ is the standard normal CDF.

Product prices.—The risk-averse household fully diversifies its purchases of goods over firms so as to eliminate the ε risk. Any firm's product is marginal to the family, and it pays up front the expected value of output for it. Since all market participants observe y^t, and the sequence of equilibrium actions a_t is common knowledge, firm m has revenue $R^m = m + a$, and profit

$$\pi^m = R^m - g(a), \tag{5}$$

where $g(\cdot)$ is a convex cost function. The firm pays its profits out as dividends to its shareholders. We show below that the normality of beliefs ensures that, as in Holmström's model, a does not depend on m.

Preferences and assets.—The only store of value are shares of the firms. Then the representative family chooses at each t a sequence of share vectors $(s^m)_{m\in\mathbb{R}}$ to maximize its expected utility

$$E\left[\sum_{t=0}^{\infty}\beta^t U(c_t) \mid s_0^m = 1\right], \tag{6}$$

where c_t is consumption and β the discount factor. Let p^m denote the price of shares of firms of type m. The period budget constraint, assuming the family starts with a representative portfolio consisting of unit ownership of all firms, and that it buys s^m shares of firm m, reads

$$c = \int_{\mathbb{R}} (\pi^m - (1-\delta)\, p^m [s^m - 1])\, d\Psi(m) + \delta p^{m_0}. \tag{7}$$

On the RHS of (7), the first term is the dividend. The last term is an exogenous endowment of δ new firms that all have the prior mean m_0; these replace the fraction δ that are about to die before the next period starts, and they are sold to the public at the IPO price of p^{m_0}. The middle term is the net outlay on the $1-\delta$ firms that it owns from the previous period. The market for securities is in equilibrium when $s^m = 1$ for all m, which means that the family buys the market portfolio. From the FOCs to the problem defined by (6) and (7) we get the market value, p^m, of each firm.

Dividends.—The family takes the dividend process as given. Firm m's dividends, given in (5), depend on a, and the latter is chosen by firms' managers who maximize the market value of the firm. The value of the firm is

$$p^m = \beta (1 - \delta) E \left[\frac{U'(c')}{U'(c)} \left([m' + a'] - g(a') + p^{m'} \right) \mid m \right]. \tag{8}$$

It presumes that the firm's manager takes the equilibrium action a. One can show that a manager who maximizes $p^m - g(a)$ chooses a that solves

$$g'(a_t) = \frac{1 - \lambda}{\lambda} \sum_{s=t+1}^{\infty} (\rho\lambda)^{s-t} E_t \left[\frac{U'(c_s)}{U'(c_t)} \right], \tag{9}$$

where $\rho \equiv (1 - \delta) \beta$ is the discount rate adjusted for the risk of market exit. The LHS is the marginal cost, and the RHS is the discounted benefit because a deviation from a to $a + 1$ at date t would raise the posterior mean at date s by $\partial m_s / \partial a_t = (1 - \lambda) \lambda^{s-t-1}$ for all $s > t$.[2]

Substituting in the expression of $g'(a_{t+1})$ on the right hand side of (9), we obtain a recursive FOC

$$\frac{dg(a)}{da} = \rho \frac{U'(c')}{U'(c)} \left(1 - \lambda + \lambda \frac{dg(a')}{da} \right), \tag{10}$$

which is independent of the firm's idiosyncratic quality m, so that all firms choose the same level of investment because of the linearity of the technology. An additional unit of effort today has two benefits: (i) It raises next period discounted earnings by $\rho(1 - \lambda)$, and (ii) it enables the agent to reduce its investment in period $t + 1$ by λ units. Both benefits are transformed into today's utils[3] through multiplication by the discount factor $U'(c_{t+1}) / U'(c_t)$.

Definition of equilibrium.—Equilibrium consists of three functions (a, p^m, R^m), where a solves (9) and (p, R) clear the asset and goods markets.

Definition 1 *An equilibrium is a pair $\{a, P\}$ that solves the Incentive Constraint and Asset Pricing equations*

$$\begin{aligned} (IC) \quad &: \quad \frac{dg(a)}{da} = \rho \frac{U'(c')}{U'(c)} \left(1 - \lambda + \lambda \frac{dg(a')}{da} \right), \\ (AP) \quad &: \quad P = \rho \frac{U'(c')}{U'(c)} (c' + P'), \end{aligned}$$

subject to the resource constraint $c = a - g(a)$.

[2] Eq. (9) is the counterpart of Holmstrom's eq. (22). The essential difference is the term $E_t \left[\frac{U'(c_s)}{U'(c_t)} \right]$.

[3] Recall that effort is measured in output units.

3 Endogenous Cycles

In this section, we explain why the basic model described above gives rise to endogenous cycles. In order to obtain an explicit solution, we assume that (i) costs are quadratic in investment, and (ii) the family's utility function is CRRA:

$$\begin{aligned}
[H1] &: \; g(a) = a^2/2, \\
[H2] &: \; U(c) = \frac{C^{1-\gamma}}{1-\gamma}, \text{ with } \gamma > 0.
\end{aligned}$$

Then the incentive constraint (10) has a stationary solution for a_t in which c_t is also constant and positive[4]

$$a^S \equiv \frac{\rho(1-\lambda)}{1-\rho\lambda}. \tag{11}$$

The rest point a^S is the unique solution when the model is, as in Holmström (1999), devised in partial equilibrium. Then the incentive constraint boils down to $a = \rho(1-\lambda+\lambda a')$. It is easily verified that the rest point a^S solves this simplified condition, whereas any other levels of initial investment lead to diverging paths. Uniqueness of the partial equilibrium solution therefore follows from the exclusion of exploding paths. We now explain why general equilibrium feedbacks can generate multiple equilibria.

Bifurcation.— In general equilibrium, the rest point a^S is not always unstable. To the contrary, when the discount factor ρ is low enough, a^S becomes stable. The model exhibits a flip bifurcation so that an unstable period 2 cycle coexists with the steady state solution as ρ gets close enough to the critical value $\tilde{\rho} < 1$. At the bifurcation point $\tilde{\rho}$, the rest point becomes unstable and there are no other stable solutions in its neighborhood.

Claim 2 *Provided that $\gamma > 1/2$, there exists a unique bifurcation point $\tilde{\rho} \in (0,1)$ such that $da_{t+1}/da_t = -1$ when $a_t = a^S$.*

In the limit case where $\lambda = 1$, the bifurcation point $\tilde{\rho} = 2\gamma - 1$, showing that the set admissible discount factors where the rest point a^S is stable gets larger with γ. This confirms the original intuition that you need curvature in utility to generate cycles. The existence of a flip bifurcation implies that the model generates 2 period cycles in the neighborhood of $\tilde{\rho}$. One can show that this behavior holds globally.

[4] The general equilibrium model also admits a stationary solution as c approaches zero.

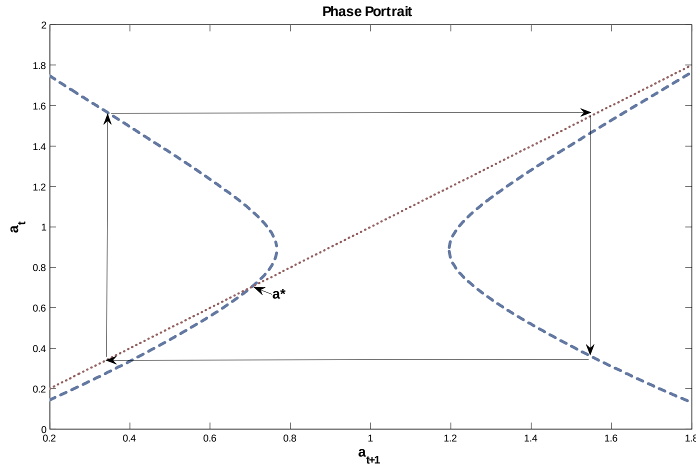

Figure 1: PHASE PORTRAIT

Proposition 3 *Provided that $\gamma > 1/2$, the model exhibits 2 period cycles whenever $\rho \in (0, \tilde{\rho})$.*

The phase portrait of the dynamic system is reported in Figure 1. For any level a_t of investment today, two values of a_{t+1} are consistent. Cycles arise when expectations oscillate between the upper and lower branch of the phase portrait.[5]

The emergence of 2-period cycles is due to the pro-cyclical fluctuations of the discount factor. Starting from a period t where investment and thus consumption are inefficiently low, an increase in next period investment raises future consumption. This lowers the discount factor, thereby making it more attractive to forgo a unit of consumption tomorrow. This substitution effect across periods compensates for the lower cost of investment today. The reverse mechanism holds when current investment is high.

Stability of cycles.—Simulations show that cycles are stable when expectations are formed according to the mechanism described above.When we raise the curvature of utility to $\gamma = 6$, the cycles become more volatile but otherwise retain the qualitative properties of the $\gamma = 3$ case. Thus the amplitude of the cycle rises with the curvature of utility.

Comparative statics.—According to Proposition 3, cycles may arise when firms are sufficiently impatient. Corollary 4 describes how the threshold discount factor depends on the model's exogenous parameters.

[5] By contrast, when expectations only select the lower branch, the economy converges to the rest point.

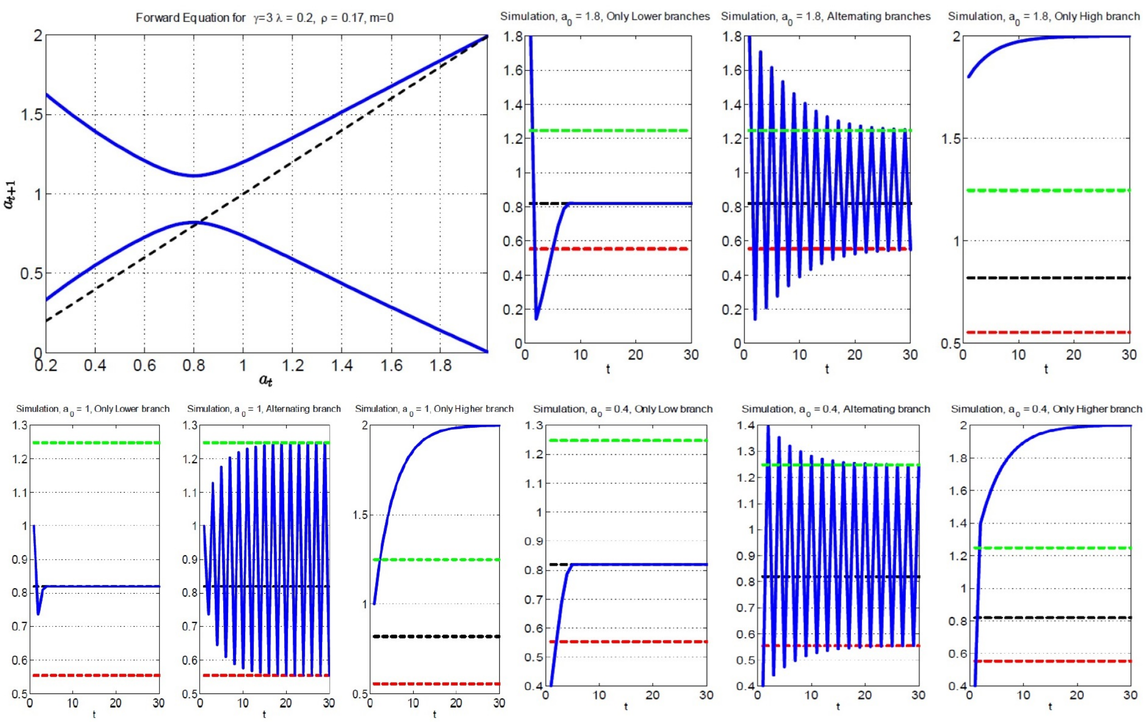

Figure 2: Cycles in a_t when $\gamma = 3$

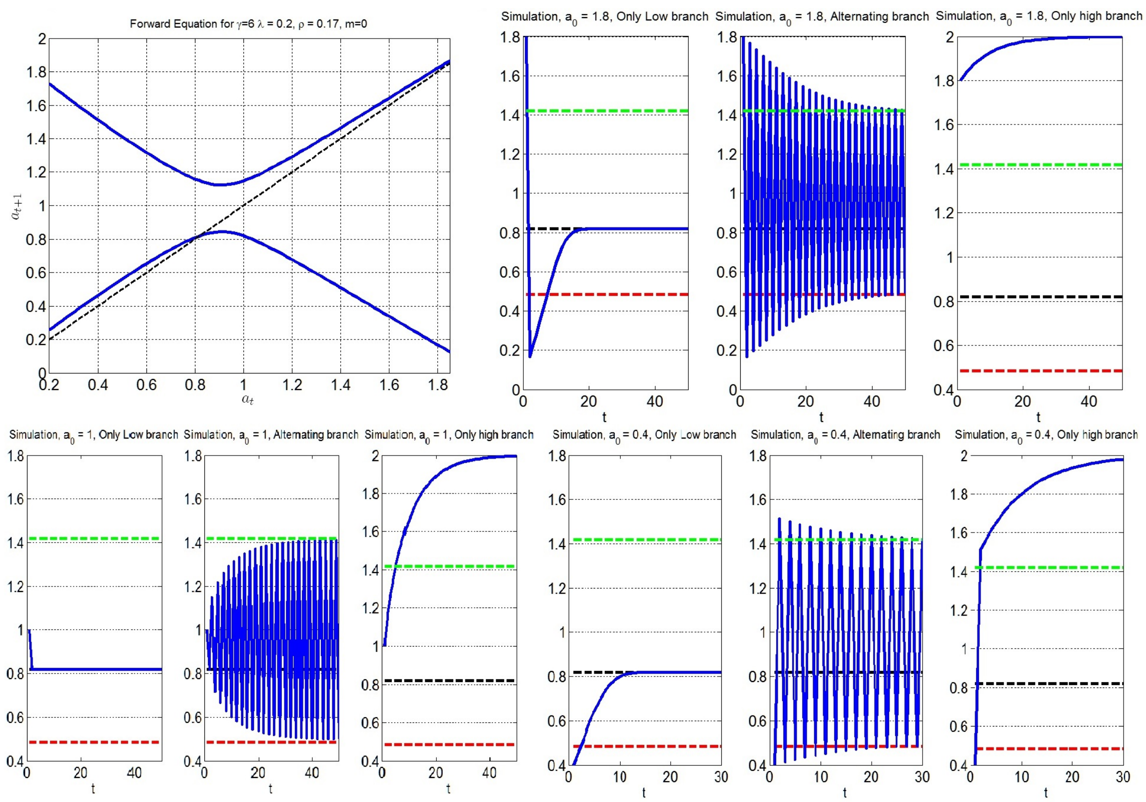

Figure 3: Cycles in a_t when $\gamma = 6$

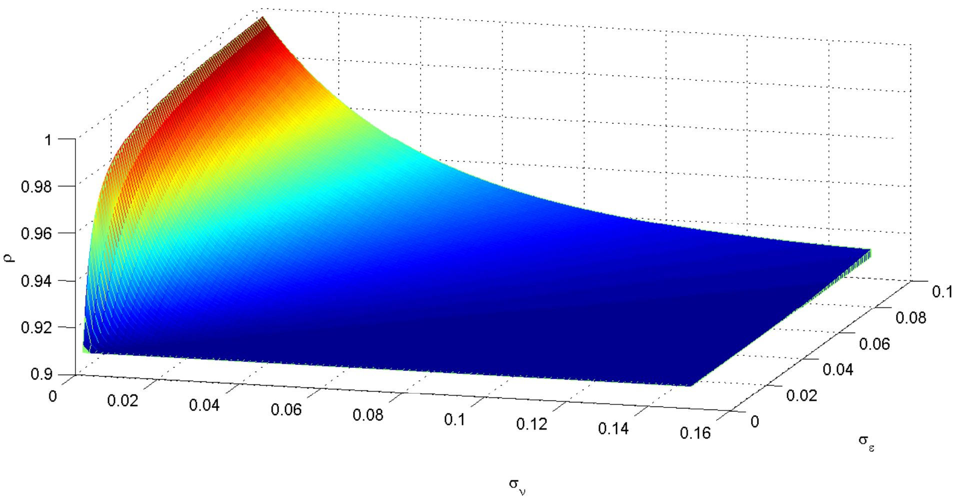

Figure 4: BIFURCATION POINT $\tilde{\rho}$ AS A FUNCTION OF σ_ε AND σ_ν.

Corollary 4 *The fundamental volatility, σ_ν, and output volatility, σ_ε, have opposite effects on the bifurcation point $\tilde{\rho}$. Moreover, $\tilde{\rho}$ is increasing in the coefficient of risk aversion γ.*

Whether or not σ_ε, and thus σ_ν, increase $\tilde{\rho}$ depends on the degree of risk aversion. Yet, for most parameter values and as shown in Figure 4, σ_ε widens the region where cycles arise. The two types of volatility have opposite effects on $\tilde{\rho}$ because σ_ε raises the persistence parameter λ, whereas σ_ν decreases it. The impact of σ_ν is rather intuitive: the more volatile θ, the more weight consumers put on new information. The influence of σ_ε is subtler because it depends on two opposite channels. On the one hand, output becomes noisier when σ_ε increases, which makes it more difficult to infer information from realized output. On the other hand, the dispersion of beliefs $\bar{\sigma}_\theta$ is raised by σ_ε and so consumers are more likely to revise their priors. However, the first effect always dominates so that σ_ε reinforces the inertia of the updating process.

4 Illustration

Endogenous cycles can be viewed as mere curiosities when they require unrealistic parameter values to arise. We parametrize the model to assess whether such a criticism might be levied against its mechanism. The two key parameters are the volatility coefficients σ_ε and σ_ν. From an empirical standpoint, they determine the volatility of firms' sales. However, since in our model all firms use the same amount of input a_t^*, firms actually correspond to production units and changes in sales are observationally equivalent to changes in revenues based Total Factor Productivity (TFP).

Estimates of the volatility of TFP are readily available in the empirical literature on industry dynamics. We use the recent study by Castro et al. (2011) who rely on the Annual Survey and Census of Manufactures, for the years 1972 through 1997, to estimate the following equation

$$z_{i,t+1} = \rho z_{i,t} + \mu_i + \boldsymbol{\gamma_t} \mathbf{X}_{i,t} + \epsilon_{i,t}, \tag{12}$$

where $z_{i,t}$ is the log-TFP for plant i at time t as estimated from a first stage regression of real sales on capital, labor and materials. $\mathbf{X}_{i,t}$ is a vector of observables that are systematically related to innovations in TFP.[6] Equation (12) is the empirical counterpart of the theoretical relationship[7]

$$m_{t+1}^i = \lambda m_t^i + (1-\lambda)\,\theta_\tau^i + (1-\lambda)\left[\sum_{s=\tau+1}^{t} \nu_s^i + \varepsilon_t^i\right], \tag{13}$$

where τ is the vintage of firm i.[8] Since the survival probability of firms follows a geometric distribution, the cross-sectional variance of $\epsilon_{i,t}$ is given by

$$\sigma_\epsilon^2 = (1-\lambda)^2\left[\sigma_\nu^2 \sum_{t=1}^{\infty} t\delta\,(1-\delta)^t + \sigma_\varepsilon^2\right] = (1-\lambda)^2\left[\sigma_\nu^2 \frac{1-\delta}{\delta} + \sigma_\varepsilon^2\right] .$$

As λ is a function of σ_ε and σ_ν only, we can, for any exit rate δ, use the autocorrelation coefficient ρ and the cross-sectional dispersion of TFP to identify both volatility coefficients. According to Bartelsman et al. (2004), the rate of market exit of US firms in the 1990's was close to 8%. Castro et al. (2011) find that the conditional standard deviation of TFP across all manufacturing plants is equal to 20.53% while the autocorrelation coefficient is centered around 0.5. Combining these three moments, we find that $\sigma_\varepsilon = .195$ and $\sigma_\nu = .137$.

[6] More specifically, Castro et al. (2011) control for the industry in which firms operate as well as their sizes and ages.

[7] Equation (13) follows reinserting $x_t^i = \theta_t^i + \varepsilon_t^i$ and $\theta_t^i = \theta_{t-1}^i + \nu_t^i$ into the law of motions of beliefs (2).

[8] The regression (12) being specified in log, it should be interpreted as a log-linearization of the steady-state economy. The approximation $ln(R_t^i) = ln(m_t^i + a_t^*) \approx m_t^i$ is accurate for small enough m when a_t^* is close to one, that is when the discount factor $\beta(1-\delta)$ is high as the rest point a^S converges to the first-best level $a^{FB} = 1$.

We also relax the normalization of the average firm efficiency $m_0 = 0$. Adding this extra degree of freedom allows us to match the value of intangible capital as a share of GDP reported in Corrado et al. (2009). Our choice of parameters is summarized in Table 4.

Parameter	Interpretation	Moment/Source
$\gamma = 3$	Relative risk aversion	Standard
$m_0 = .078$	Average firm efficiency	Intangible capital/GDP=14%
$\sigma_\varepsilon = .506$	Volatility of output	Volatility of TFP=20%
$\sigma_\nu = .358$	Volatility of firm's efficiency	Autocorrelation of TFP=0.5
$\delta = .08$	Exit rate of firms	Bartelsman et al. (2004)

For this set of parameters, the value of the bifurcation point $\tilde{\rho} = 0.91$. This may appear to be an unreasonably low discount factor, but one has to remember that $\rho = \beta(1-\delta)$ as investors discount the future at a higher rate than their preference for the present because they also face a risk of bankruptcy. Combining Bartelsman et al. (2004) estimates for δ with a standard discount factor $\beta = 0.96$, we obtain an effective discount rate ρ of .88 that falls well within the endogenous cycles region.

The effect of ρ on the amplitude of cycles is described in Figure 5. It reports consumption in the low and high phases of the endogenous cycle along with the steady state value. When the discount factor is close to $\tilde{\rho}$, consumption in booms is higher than in steady-state. However, as ρ decreases, investment in the busy year starts yielding negative profits by exceeding its first best level. This is why consumption in both phases of the cycle eventually becomes smaller than in the steady-state. Then the stable solution unambiguously dominates the cycling one. We have also included a vertical line in Figure 5 at the empirically plausible discount factor of 0.88, showing that it is indeed consistent with endogenous fluctuations.

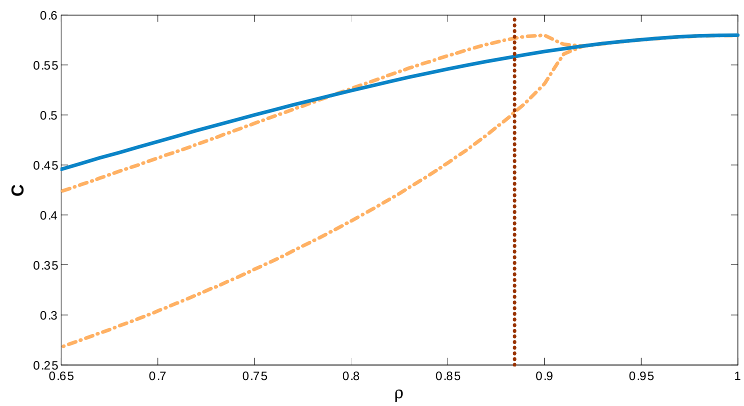

Figure 5: Cycles and steady-state as a function ρ.

APPENDIX

Proof. Claim 2: With quadratic cost and CRRA utility, the incentive constraint is satisfied if

$$c_t^{-\gamma} a_t = \rho \lambda c_{t+1}^{-\gamma} \left(\frac{1-\lambda}{\lambda} + a_{t+1} \right).$$

Taking logs on both side and differentiating with respect to a_t yields

$$-\frac{\gamma}{c_t}(1-a_t) + \frac{1}{a_t} = -\frac{\gamma}{c_{t+1}}(1-a_{t+1})\frac{da_{t+1}}{da_t} + \frac{1}{\frac{1-\lambda}{\lambda} + a_{t+1}} \frac{da_{t+1}}{da_t}.$$

Evaluated at the rest point $\left(a^S, c^S\right)$, this condition reads

$$\left.\frac{da_{t+1}}{da_t}\right|_{a^S} = \frac{\frac{1}{a^S} - \frac{\gamma}{c^S}\left(1-a^S\right)}{\frac{1}{\frac{1-\lambda}{\lambda}+a^S} - \frac{\gamma}{c^S}\left(1-a^S\right)}.$$

Reinserting the rest-point solution $a^S = \rho\lambda(1-\lambda)/[\lambda(1-\rho\lambda)]$, we find that

$$\left.\frac{da_{t+1}}{da_t}\right|_{a^S} = \frac{\frac{1}{a^S} - \frac{\gamma}{c^S}\left(1-a^S\right)}{\frac{\rho\lambda}{a^S} - \frac{\gamma}{c^S}\left(1-a^S\right)}.$$

Hence $\tilde{\rho}\lambda$ solves

$$\left.\frac{da_{t+1}}{da_t}\right|_{a^S} = -1 \Rightarrow \frac{1+\tilde{\rho}\lambda}{a^S} = \frac{2\gamma\left(1-a^S\right)}{c^S}.$$

Replacing the expression of a^S into this condition yields

$$\frac{\gamma}{1+\tilde{\rho}\lambda}=\frac{1}{4}\left[1+\frac{1-\tilde{\rho}\lambda}{1-\tilde{\rho}}\right]. \tag{14}$$

The left hand side is decreasing in $\tilde{\rho}$ and goes from γ to $\gamma/(1+\lambda)$ as $\tilde{\rho}$ increases from 0 to 1. By contrast, the right hand side is increasing and goes from 1/2 to infinity. Thus there exists a unique $\tilde{\rho}$ solving (14) whenever $\gamma > 1/2$. ■

To prove Proposition 3, we use a direct approach by changing variable and defining a new fixed point problem.

Lemma 5 *The model exhibits 2 period cycles when the fixed point problem*

$$s=\psi(s)\equiv\left(\frac{\frac{\rho(1-\lambda)}{1-(\rho\lambda)^2}(\rho\lambda+s)-\frac{1}{2}\left[\frac{\rho(1-\lambda)}{1-(\rho\lambda)^2}(\rho\lambda+s)\right]^2}{\frac{\rho(1-\lambda)}{1-(\rho\lambda)^2}(\rho\lambda+s^{-1})-\frac{1}{2}\left[\frac{\rho(1-\lambda)}{1-(\rho\lambda)^2}(\rho\lambda+s^{-1})\right]^2}\right)^{\gamma},$$

admits a solution $s^*\in\left(1,2\frac{1-(\rho\lambda)^2}{\rho(1-\lambda)}-\rho\lambda\right)$.

Proof. : Denoting this period by a and next period by a', the period after by a, and so on. In other words, we have the SDFs from today til tomorrow and from tomorrow til the day after, respectively,

$$S \equiv S(a,a')=\left(\frac{a-g(a)}{a'-g(a')}\right)^{\gamma} \tag{15}$$

$$S' \equiv S(a',a)=1/S(a,a') \tag{16}$$

Therefore, if we start at $t=0$, so that a is the action at $t=0,2,4,6...$ and a' the action for $t=1,3,5,7,...$, then (a,a') solve the following two equations

$$a = \frac{1-\lambda}{\lambda}\left(\sum_{\mathrm{t=1,3,5,7,...}}^{\infty}(\rho\lambda)^t S+\sum_{\mathrm{t=2,4,6,8,...}}^{\infty}(\rho\lambda)^t\right)=\frac{1-\lambda}{\lambda}\left(\frac{\rho\lambda S}{1-(\rho\lambda)^2}+\frac{(\rho\lambda)^2}{1-(\rho\lambda)^2}\right),$$

$$a' = \frac{1-\lambda}{\lambda}\left(\sum_{\mathrm{t=1,3,5,7,...}}^{\infty}(\rho\lambda)^t S'+\sum_{\mathrm{t=2,4,6,8,...}}^{\infty}(\rho\lambda)^t\right)=\frac{1-\lambda}{\lambda}\left(\frac{\rho\lambda S'}{1-(\rho\lambda)^2}+\frac{(\rho\lambda)^2}{1-(\rho\lambda)^2}\right).$$

These simplify to[9]

$$a = \frac{1-\lambda}{1-(\rho\lambda)^2}\rho\left(\rho\lambda+S\right), \tag{17}$$

$$a' = \frac{1-\lambda}{1-(\rho\lambda)^2}\rho\left(\rho\lambda+S'\right). \tag{18}$$

Thus there are 4 equations (15), (16), (17), and (18), and 4 unknowns, (a,a',S,S'). One solution is $(a,A,s,S)=(1,1,1,1)$, which is a version of Holmström's Proposition 1. Now let's treat s as a parameter to begin with, and then close the model later. As a function of s, we have

$$a(s)=\frac{\rho(1-\lambda)}{1-(\rho\lambda)^2}(\rho\lambda+s), \text{ and } A(s)=\frac{\rho(1-\lambda)}{1-(\rho\lambda)^2}\left(\rho\lambda+s^{-1}\right).$$

We look for a fixed point in s of the function

$$s=\psi(s)\equiv\left(\frac{a(s)-\frac{a(s)^2}{2}}{a(1/s)-\frac{a(1/s)^2}{2}}\right)^{\gamma}.$$

The solution is well defined when consumption is positive. Since $c(a(s))>0$ when $a(s)\in\left(0,1+\sqrt{1+2m}\right)$, the requirement is satisfied if $s\in\left(1,2\left[\frac{1-(\rho\lambda)^2}{\rho(1-\lambda)}\right]-\rho\lambda\right)$. ■

Proof. Proposition 3: It follows from the definition of $\psi(\cdot)$ in Lemma 5 that $\psi(1)=1$. Let $\bar{s}\equiv 2\frac{1-(\rho\lambda)^2}{\rho(1-\lambda)}-\rho\lambda$, since $c(\bar{s})=0$ and $c(\bar{s}^{-1})>0$, we have $\psi(\bar{s})=0$. By continuity of the mapping $\psi(\cdot)$, there will be a fixed point $s\in(1,\bar{s})$ if $\psi'(1)>1$. Differentiating $\psi(\cdot)$, we obtain

$$\psi'(s)=\gamma\left(\frac{c(s)}{c(s^{-1})}\right)^{\gamma-1}\frac{\left[(1-a(s))\,c(s^{-1})-c(s)\left(1-a(s^{-1})\right)(-1/s^2)\right]}{c(s^{-1})^2}a'(s).$$

Since $a'(s)=\rho(1-\lambda)/\left[1-(\rho\lambda)^2\right]$, we have

$$\psi'(1)>1\Leftrightarrow\frac{2\gamma\left[1-a(1)\right]}{c(1)}\frac{\rho(1-\lambda)}{1-(\rho\lambda)^2}>1.$$

[9] The solutions (17), and (18) are consistent with the incentive constraint since

$$\begin{aligned} a &= \frac{1-\lambda}{\lambda}\left(\frac{\rho\lambda S}{1-(\rho\lambda)^2}+\frac{(\rho\lambda)^2}{1-(\rho\lambda)^2}\right)=\frac{1-\lambda}{\lambda}\rho\lambda S\left(\frac{1}{1-(\rho\lambda)^2}+\frac{\rho\lambda S'}{1-(\rho\lambda)^2}\right) \\ &= \rho\lambda S\left[\frac{1-\lambda}{\lambda}+\frac{1-\lambda}{\lambda}\left(\frac{(\rho\lambda)^2}{1-(\rho\lambda)^2}+\frac{\rho\lambda S'}{1-(\rho\lambda)^2}\right)\right]=\rho\lambda S\left[\frac{1-\lambda}{\lambda}+a'\right] \\ &= \rho\frac{U'(a')}{U'(a)}\left[1-\lambda+\lambda a'\right]. \end{aligned}$$

But since

$$\frac{\rho\left(1-\lambda\right)}{1-\left(\rho\lambda\right)^{2}}=\frac{a(1)}{1+\rho\lambda}\ ,$$

we get

$$\psi'\left(1\right)>1\Leftrightarrow\frac{2\gamma\left[1-a(1)\right]}{c\left(1\right)}>\frac{1+\rho\lambda}{a(1)}\ ,$$

or

$$\left.\frac{da_{t+1}}{da_{t}}\right|_{a(1)}=\frac{\frac{1}{a(1)}-\frac{\gamma}{c(1)}\left(1-a(1)\right)}{\frac{\rho\lambda}{a(1)}-\frac{\gamma}{c(1)}\left(1-a(1)\right)}>-1\ .$$

The claim immediately follows because we have shown that the condition above holds true whenever $\rho<\tilde{\rho}$. ■

Proof. Corollary 4: By definition of λ and $\bar{\sigma}_{\theta}$, we have

$$\lambda\equiv1-\frac{\sigma_{\varepsilon}^{-2}}{\bar{\sigma}_{\theta}^{-2}+\sigma_{\varepsilon}^{-2}}=1-\frac{2\sigma_{\varepsilon}^{-2}}{\sqrt{\frac{1}{\sigma_{\varepsilon}^{4}}+\frac{4}{\sigma_{\varepsilon}^{2}\sigma_{\nu}^{2}}}+\frac{1}{\sigma_{\varepsilon}^{2}}}=1-\frac{2}{\sqrt{1+4\frac{\sigma_{\varepsilon}^{2}}{\sigma_{\nu}^{2}}}+1}\ .$$

and so $\partial\lambda/\partial\sigma_{\varepsilon}>0$ while $\partial\lambda/\partial\sigma_{\nu}<0$. Hence they have opposite effect on the implicit equation (14) defining $\tilde{\rho}$ since it only depends on λ. As for the effect of γ, totally differentiating (14), we find that

$$\frac{\partial\tilde{\rho}}{\partial\gamma}=\frac{\frac{1}{1+\tilde{\rho}\lambda}}{\frac{\gamma\lambda}{(1+\tilde{\rho}\lambda)^{2}}+\frac{1}{4}\left[\frac{1-\lambda}{(1-\tilde{\rho})^{2}}\right]}>0.$$

■

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