

Why Do Risky Sectors Grow Fast?*

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Abstract

Why do risky sectors grow fast? Because they have good ideas. In an idea flow growth model that nests the granular hypothesis (Gabaix 2011), we show that the fatness of the firm size distribution is positively related to both growth and volatility. A fat tailed distribution enhances the diffusion of ideas and increases growth in the long run. At the same time, a fat tail reduces the extent to which firm-level disturbances average out, and thus increases volatility. We show these correlations find support in US firm-level data: on average, sectors with fat tails grow fast and display high volatility. Interestingly, the relation between sector-level tail and volatility also holds in the short run, within sector. In the US, the dispersion in estimated tails can explain about 40% of the link between growth and volatility.

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1 Introduction

Volatile assets offer high returns, because they reflect non-diversified investment in high growth, but risky technologies. While this tradeoff is a central tenet of finance, surprisingly little is known about its counterpart in the real economy: Are volatile activities growing fast because they are risky and undiversified? The question is explored theoretically, and empirically at the sector level. We construct a growth model based on ideas flow à la Lucas (2009), where the distribution of ideas also affects volatility because of a granularity argument à la Gabaix (2011). Innovation diffuses between firms that meet randomly and choose to copy other technologies if they are better, i.e., more productive. Innovation, and therefore growth both increase with the probability of copying a better firm, which in turn depends on the tail of the distribution of firm size. A fat tailed distribution also implies high volatility as the diversification of the risk inherent to each technology is hampered.

In US firm-level data, we show the fatness of firm size distribution estimated at sector level is a significant correlate for both the growth rate and the volatility of real activity, also measured at sector level. Just as in the theory, both variables are determined by the magnitude of the tail in firm size distribution, which maps with the distribution of ideas. In the data, risky sectors grow fast because they have good ideas. This is consistent with the well known trade off between risk and return, and with the cross-country evidence at sector level documented in Imbs (2007)¹.

In the model, growth at sector level is determined by the diffusion of ideas, or technologies. Innovators meet randomly, and copy each other's ideas whenever they have higher productivity, as in Lucas (2009). Along a balanced growth path the distribution of ideas is shown to be distributed following a Pareto distribution in the upper tail. With a fat-tailed distribution, the probability of meeting someone with a better idea is high, which strengthens the diffusion process, and ultimately boosts real growth.

Firms use technologies to produce, but the implementation of each technology is subject to a short-run idiosyncratic disturbance. The disturbance is independently and identically distributed across firms, so that the volatility of real sector activity increases with the granularity of the size distribution there, as in Gabaix (2011).² The fat tail nature of the firm size distribution hampers the diversification of firm-level disturbances at the sectoral level. As large firms represent an increasing

¹It is also consistent with Ramey and Ramey (1995), whose results depend on the relative magnitude of aggregate versus sector-level shocks.

²Of course, it is also possible that large firms be inherently more risky. But that is inconsistent with the view that they tend to be more diversified. Also see Gabaix (2011). This paper ascribes sector-level volatility to the relative absence of diversification between the firms of one sector, rather than within the firms that populate it.

share of sector output, firm-specific shocks matter more for real volatility at sector level.

Firm-level data on US employment, sales, and total assets are gathered from Compustat and partitioned into 4-digit SIC categories. For each category, we implement the estimator proposed by Clauset, Shalizi and Newman (2009) to estimate the tail of firm size distribution. The method is flexible and general: The fit with a Pareto distribution is optimized through an endogenous choice of the threshold value of firm size beyond which the distribution verifies the Pareto definition. Tail estimates are mapped with US data on employment, sales and total factor productivity at sector level collected by the NBER-CES Manufacturing Industry Database. These data are used to compute growth and volatility at sector level, which are then matched with tail estimates. Panel estimates of sector-level tail and volatility are obtained over rolling windows of identical length.

We show that fat-tailed sectors experiment high real growth. This result holds in the long run, in the cross-section of sectors, as in the model. Fat-tailed sectors also tend to be highly volatile. This result holds both in the cross-section and within sector: fluctuations in sector growth originate from firm idiosyncratic shocks, which, since they wash out over time, do not correlate systematically with tail estimates. But changes in sector volatility over time do depend on fluctuations in firm size distribution, and so the within-sector correlation between tail estimates and volatility should be significant. Such is the case in US data, where the dispersion in tail estimates can explain up to 40 percent of the observed link between growth and volatility.

This paper relates to three stands of the literature: models of growth based on ideas flow, models of the micro-origins of aggregate volatility, and the body of work dealing with the link between growth and volatility in real data. The growth model derived in this paper is a simple version of Lucas (2009), or equivalently a discrete time version of Alvarez Buera and Lucas (2008). The model focuses on the positive link between the tail of the ideas distribution and growth. Such relationship is also true in a larger class of model including Lucas and Moll (2014), Perla and Tonetti (2014) and Luttmer (2007, 2010, and 2012). Our paper also takes inspiration from the literature on the micro-origins of aggregate fluctuations, and specifically the seminal work of Gabaix (2011) on the “Granular Hypothesis”. Carvalho and Grassi (2014) implement this hypothesis in a quantitative dynamic framework, but focus on business cycle fluctuations rather than growth. Finally the paper contributes to the literature studying the empirical link between growth and volatility, e.g., Ramey and Ramey (1995), Aghion and Banerjee (2005), Aghion, Angeletos, Banerjee, and Manova (2011), Barlevy (2004), or Imbs (2007).

The paper is organized as follows. Section 2 describes the theoretical model that implies a positive link between the fatness of the firm size distribution, growth, and volatility. Section 3 describes the data used to investigate empirically this

relationship. Section 4 presents the main empirical results of the paper. Section 5 concludes.

2 Model

In this section, we present a framework that rationalizes the link between long term growth and short term volatility at the sectoral level. This setting is based on models of idea flows à la Alvarez, Buera and Lucas (2008), Lucas (2009), Lucas and Moll (2014), or Perla and Tonetti (2014). These models describe how technology is diffused in the economy through random meetings among agents. We add firm-level perturbations and investigate the link between tail and growth and volatility.

Firms need ideas to produce. Ideas bearers meet and copy the best ideas from their peers in their sector. Such process generates a sectoral stationary distribution of ideas whose tail is related to growth. A fatter tail implies that the probability of meeting someone with a better idea is higher and, since agents copy better ideas during such meeting, sectoral growth is higher.

Ideas are interpreted as productivity, which maps one for one with size. A fatter tail means that there exists a large share of firms with a high level of productivity, i.e., large firms. Firms are also subject to idiosyncratic perturbations, distributed independently and identically. In a fat tail environment, idiosyncratic firm-level perturbations matter for the aggregate, which implies a positive relationship between tail thickness and fundamental volatility as defined by Carvalho and Gabaix (2013).

2.1 Environment

Sectors are indexed by s . Each sector is populated by N_s firms. Each of the $j_s \in \{1, \dots, N_s\}$ firms produces y_{t,j_s} unit of sector s good. To produce, each firm in a sector needs a mass one of the same ideas. Ideas are interpreted as technology. In sector s , there is a continuum of mass N_s of heterogeneous ideas. Potential entrepreneurs, who own an idea, meet randomly with each other. When a potential entrepreneur meets one of her peers, she copies her idea if it is better than her own. Each of the N_s firms draws a mass one of ideas of the same type. Each of these ideas follows the diffusion process described above. The average of the ideas of firm j_s at date t , is denoted x_{t,j_s} ; it follows a deterministic process, since it is an average over a mass one of ideas. Time is infinite and discrete.

Firm j_s produces $y_{t,j_s} = x_{t,j_s} z_{t,j_s}$ unit of sector s good where $\frac{z_{t+1,j_s}}{z_{t,j_s}} = \varepsilon_{t+1,j_s}$ follows a log-normal distribution $\log \mathcal{N}(0, \sigma^2)$. The disturbances ε_{t,j_s} are identically and independently distributed across firms and time. The production of a given firm

is composed of a deterministic component, the average of ideas that constitutes the firm, and an idiosyncratic perturbation captured by z_{t,j_s} . We next study formally the process of idea diffusion and derive, along a sectoral balanced growth path, the stationary distribution of ideas. We next show that the idiosyncratic perturbations does not average out at the sectoral level which generate sector non-negligible fluctuations.

2.2 Selection of ideas

For clarity, this section abstracts from the sector index notation s . It is reintroduced when sectoral volatility is derived. Each potential entrepreneur meets, each period, with α of their peers drawn from the total pool of potential entrepreneurs. When a potential entrepreneur meets with one of her peers, she copies the peer's ideas if and only if it is a better idea. Let $F(x, t)$ be the cumulative distribution function (cdf) of ideas at date t in a given sector:

$$F(x, t) = \text{share of potential entrepreneurs with ideas} \leq x$$

The evolution of F is as follows:

$$\begin{aligned} F(x, t+1) &= F(x, t) \mathbb{P} \{ \text{No meeting with a better idea between } t \text{ and } t+1 \} \\ &= F(x, t) F(x, t)^\alpha \end{aligned}$$

The right hand side of the last equality is the product of the mass of firms with an idea below x and the probability that each of the α meetings is with a peer who has an idea lower than x . From this last equation, we can derive the equation that governs the evolution of the cdf of ideas across time:

$$F(x, t+1) = F(x, t)^{1+\alpha} \tag{1}$$

Definition 1 (Sectoral Balanced Growth Path) *A sectoral balanced growth path (SBGP), is a pair (Φ, ν) of a cumulative distribution function Φ and a growth rate ν such that*

$$\forall x, \Phi(x) = F((1+\nu)^t x, t) \tag{2}$$

i.e. when the x -space grows at a rate ν , the distribution of ideas is fixed at Φ .

Along such a balanced growth path, the distribution of ideas in a sector is stationary as long as the ideas space grows at the economy growth rate. To keep tractability, we assume that ideas evolve on a discrete grid $X = \{x_i\}_{i \in \mathbb{Z}}$. The grid is such that $\forall i \in \mathbb{Z}, x_i = (1+\nu)^i$ where ν is the growth rate along a given

balanced growth path. Using the definition 2 of a SBGP along (Φ, ν) , equation 1 becomes:

$$\Phi((1 + \nu)^{-t-1}x) = \Phi((1 + \nu)^{-t}x)^{1+\alpha}$$

Let i be such that $(1 + \nu)^{-t}x = (1 + \nu)^i$ and let μ_i the value of the cdf at bins i , i.e. $\mu_i := \Phi((1 + \nu)^i)$. The previous equation yields:

$$\mu_{i-1} = \mu_i^{1+\alpha}$$

This equation characterizes the stationary distribution along the balanced growth path. It gives the value of this cdf at a bin x_i as a function of the cdf at other bins. Taking logarithms gives the closed-form formula for the stationary distribution along a SBGP:

$$\Phi((1 + \nu)^i) = \mu_i = \exp\left(-\log(1/\mu_0) \left(\frac{1}{1 + \alpha}\right)^i\right)$$

Since $x_i = (1 + \nu)^i$, we have $i = \frac{\log(x_i)}{\log(1+\nu)}$. Substituting in the previous equation yields:

$$\Phi(x_i) = \mu_i = \exp(-\lambda x_i^{-\delta}) \quad (3)$$

where $\delta = \frac{\log(1+\alpha)}{\log(1+\nu)}$ and $\lambda = -\log(\mu_0) > 0$. The stationary distribution of ideas along a sectoral balanced growth path follows a discrete Fréchet with shape parameter δ and scale parameter λ .

2.3 Idea Growth and Tail of the Ideas Distribution

We first define the Pareto tail of a distribution.

Definition 2 (Tail of distribution) *A distribution characterized by a cumulative distribution function Φ over a space $X = \{x_i\}_{i \in \mathbb{Z}}$ has a Pareto tail parameter β when*

$$1 - \Phi(x_i) = \gamma x_i^{-\beta} + o(x_i^{-\beta})$$

where γ is a constant, and $a_i = o(b_i)$ if and only if $a_i/b_i \rightarrow 0$ when $i \rightarrow \infty$, where a_i and b_i are two sequences

The following proposition derives the tail of the stationary distribution along a SBGP

Proposition 1 *Along a SBGP (Φ, ν) , the tail of the stationary distribution is $\delta = \frac{\log(1+\alpha)}{\log(1+\nu)}$.*

Proof The stationary distribution along a SBGP (Φ, ν) is:

$$1 - \Phi(x_i) = 1 - \exp(-\lambda x_i^{-\delta}) = \lambda x_i^{-\delta} + o(x_i^{-\delta})$$

the last equality is obtained using a Taylor development, and using the fact that $i \rightarrow \infty$ implies $x_i = (1 + \nu)^i \rightarrow \infty$. $\square$

We now turn to growth in this framework. The average idea is defined as $I_t := \sum_{i \in \mathbb{Z}} x_i f(x_i, t)$ where $f(x_i, t)$ is the probability mass function of the distribution $F(x_i, t)$.

Proposition 2 *Along a SBGP (Φ, ν) , the average idea is equal to $I_t = (1 + \nu)^t \Xi(\delta, \lambda)$ where $\Xi(\delta, \lambda, \nu)$ is a constant independent of t . It follows that*

$$\frac{I_{t+1}}{I_t} = 1 + \nu$$

i.e. the growth rate of the average idea along a SBGP is ν .

Proof See appendix A.1. $\square$

Along a SBGP the growth rate of average idea is given by ν and the tail of the stationary distribution is $\delta = \frac{\log(1+\alpha)}{\log(1+\nu)}$, which means that $1 + \nu = (1 + \alpha)^{1/\delta}$. Sectors with a fatter tail, i.e., a smaller δ , also have a higher growth rate of ideas ν along the sectoral balanced growth path. The result is intuitive: the probability of meeting someone with a better idea increases with the tail of the distribution, since there are more potential entrepreneurs with good ideas. The mean growth of idea of a given potential entrepreneur is thus higher³.

2.4 Firms and Sectoral Volatility

Each of the $j_s \in \{1, \dots, N_s\}$ firms produces $y_{t,j_s} = x_{t,j_s} z_{t,j_s}$ units of sector s goods. Each firm in sector s draws a mass one of the same idea x_{t,j_s} from the stationary distribution of ideas along the SBGP. Each of these ideas follows the process described in the previous section. Since there is a mass one of them, the process of the average of these ideas, abusively denoted by x_{t,j_s} , is deterministic: $\text{Var}_t \frac{x_{t+1,j_s}}{x_{t,j_s}} = 0$.

³In a continuous time framework, Alvarez, Buera and Lucas (2008) show that, along a balanced growth path the stationary distribution is Fréchet with a tail parameter α/ν . In their framework, α has to be interpreted as the number of meetings per unit of time. Our framework is consistent with their finding since for small α and ν , $\frac{\log(1+\alpha)}{\log(1+\nu)} \approx \alpha/\nu$.

Output in sector s aggregates production of each firm in that sector: $Y_{t,s} = \sum_{j_s=1}^{N_s} y_{t,j_s}$. Up to a first order approximation, growth in sector s is therefore given by

$$\frac{\Delta Y_{t+1,s}}{Y_{t,s}} = \frac{1}{Y_{t,s}} \sum_{j_s=1}^{N_s} \Delta y_{t+1,j_s} = \sum_{j_s=1}^{N_s} \frac{y_{t,j_s}}{Y_{t,s}} \left(\frac{\Delta x_{t,j_s}}{x_{t,j_s}} + \frac{\Delta z_{t,j_s}}{z_{t,j_s}} \right)$$

The conditional standard deviation of output growth in sector s is then:

$$\sigma_t \left(\frac{\Delta Y_{t+1,s}}{Y_{t,s}} \right) = \sqrt{\sum_{j_s=1}^{N_s} \left(\frac{y_{t,j_s}}{Y_{t,s}} \right)^2} \sigma^2$$

where we use the approximation $(e^{\sigma^2} - 1)e^{\sigma^2} \approx \sigma^2$, which is the component of aggregate volatility due to idiosyncratic shocks ε_{t,j_s} , the “fundamental volatility” introduced by Carvalho and Gabaix(2013) . We denote it by $\sigma_{s,t}^\Gamma$.

Gabaix (2011) shows that “fundamental volatility” decays as the number of firms increases, at a rate slower than $\sqrt{N_s}$ when the tail of the distribution is fat enough. A similar result is established here, summarized in the following theorem.

Theorem 1 (Rate of Decay of Sectoral Volatility) *For a sector s along a SBGP (ν_s, Φ_s) , let δ_s be the tail parameter of the stationary distribution Φ_s .*

If $\delta_s < 2$ then

$$\sigma_{t,s}^\Gamma := \sigma_t \left(\frac{\Delta Y_{t+1,s}}{Y_{t,s}} \right) = \Omega \left(\sigma \frac{u_{t,s}^{1/2}}{N_s^{1-1/\delta_s}} \right)$$

where $\Omega()$ means that, given two sequences of positive real numbers $\{a_{N_s}\}_{N_s \in \mathbb{N}}$ and $\{b_{N_s}\}_{N_s \in \mathbb{N}}$, $a_{N_s} = \Omega(b_{N_s})$ iff $\liminf_{N_s \rightarrow \infty} a_{N_s}/b_{N_s} > 0$ and where u is a non degenerate random variable.

Proof See appendix A.2

Theorem 1 characterizes the volatility of sectoral output growth conditional on the state at time t . The key result is that for $\delta < 2$, sector volatility decays at a rate slower than $1/\sqrt{N_s}$, which would be the decaying rate of volatility implied by a conventional central limit theorem. The theorem is a simple extension of Gabaix (2011), to a framework with growth. Similar results have been also used by Carvalho and Grassi (2014) and Grassi (2014) for entry/exit mechanism and persistent shocks.

The prediction of the theorem is that $\sigma_{t,s}^\Gamma \approx K_{t,s} \sigma \frac{u_{t,s}^{1/2}}{N_s^{1-1/\delta}}$ where $K_{t,s}$ is a constant. A first order taylor approximation around a point $(\bar{N}, \bar{\delta})$ gives:

$$\sigma_{t,s}^\Gamma \approx cste - \kappa_1 \frac{1}{\bar{\delta}^2} \frac{\log(\bar{N})}{\bar{N}^{1-1/\bar{\delta}}} \delta_s + \kappa_2 \frac{(1/\bar{\delta} - 1)}{\bar{N}^{2-1/\bar{\delta}}} N_{t,s} + v_{t,s} \quad (4)$$

where κ_1, κ_2 are positive constants.

From this it is clear that a fat stationary distribution along a SBGP, i.e., low values of δ_s , implies high sectoral volatility. Intuitively, a fat-tailed distribution means large firms represent a large share of sector-level output, and thus firm-level shocks matter. The effect of an increase of the number of firms in a sector is here ambiguous since it depends on the value of the relative position of $\bar{\delta}$ and one.

3 Data Description

In the empirical part of this paper, we use two sources of information: *i*) firm-level data collected by Compustat, and *ii*) sector-level data, collected by the NBER-CES Manufacturing Industry Database. The first database is used to estimate tails at sector level; the second database is used to compute sectoral growth and volatility. The rest of this section is dedicated to the description of these data.

Firm Level Data

Compustat data are collected from the mandatory forms that each listed firm in the US fills each year. This is a firm-level yearly panel database with balance sheet information. For each 4-digit SIC category, we collect three variables between 1958 and 2009: employment, sales and total assets. The latter two are nominal. They are deflated using the price deflator given by the NBER-CES Manufacturing Industry Database for shipment (PISHIP). These deflators are computed by the Bureau of Economic Analysis, for their GDP-by-Industry data.

Sector Level Data

The NBER-CES Manufacturing Industry Database collects sector-level data. The majority of these data are extracts from the Annual Survey of Manufacturing that samples approximately 50,000 establishments selected from the approximately 330,000 establishments included in the Census of Manufacturing. The variables are only available for the manufacturing sector. The data are annual and cover the period 1958-2009⁴. For each 4-digit SIC category, we use information on sales (VSHIP), total factor productivity (TFP), and employment. Sales are deflated following the method recommended by the NBER using the provided associated deflator (PISHIP). For the TFP we use the 5-factor TFP index (TFP5) computed by the NBER.

Armed with these variables, we compute the growth rate of sales and TFP. For robustness, we also filter the data using the Hodrick-Prescott filter (with smoothing parameter 100). We compute mean growth rate and variance, on a centered rolling window of 11 years. For example, the mean growth rate in 1985 is the mean of the growth rates of years 1980 to 1991. These mean are interpreted as long-term growth rate along a SBGP.

4 Results

This section reports the main empirical results of this paper. We first describe how sectoral tails are estimated on firm-level data. Then we investigate the link between sectoral growth and sectoral tail, and perform some robustness check. Finally, we study the link between sectoral volatility and sectoral tail, also with some robustness checks.

4.1 Estimation of Tails

The tails of the sector-level size distributions are estimated following Clauset, Shalizi and Newman (2009). This uses a maximum likelihood estimator related to Hill (1975), in order to estimate the tail parameter of a distribution. Importantly, the method also allows to estimate the threshold above which the distribution can be well described by a Pareto distribution.

Traditionally, tail indexes are estimated using an OLS regression of firms' log-rank on their log-size, for firms larger than a arbitrarily chosen threshold.⁵ While convenient, the arbitrariness of the threshold is a drawback. A higher threshold

⁴Both 6-digits NAICS and 4-digit SIC level data are available. For consistency, we use the 4-digit SIC information.

⁵See Gabaix and Ibragimov (2011) for example.

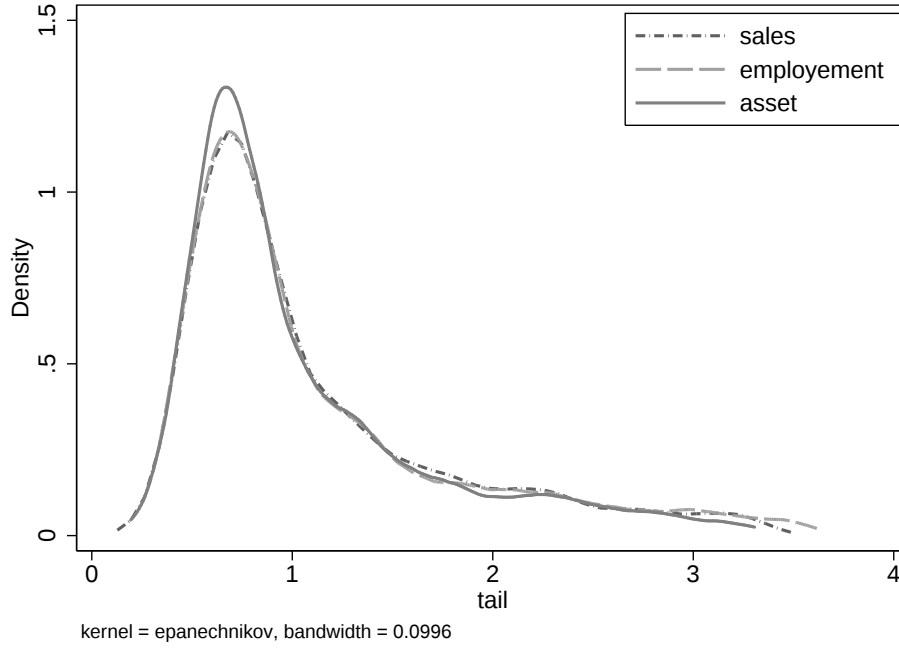


Figure 1: Density of the panel of distribution tails estimates of sector*year.

increases the fit of the truncated distribution to a Pareto, but it also reduces the sample of firms that are used in the regression. Clauset, Shalizi and Newman (2009) estimate the threshold from the data, where it is chosen optimally so that the fit of the truncated distribution to a Pareto is maximized.

Firm size is measured using sales, employment or total assets. Compustat data imply too few firms in a given sector and in a given year to perform the Clauset, Shalizi and Newman (2009) estimation. We pool firms in a given sector over a centered 11 years rolling window. For example, the tail of the firm size distribution in sector 2011 (Meat Packing Plants) in 1985 is estimated on the sample formed firms in that sector in years 1980 to 1991. The window length is the same as what was used to compute mean growth rate and volatility.

We obtain a panel formed by the estimated tails of firm size distribution, at the sector and year levels. The sample is winsorized, and we keep sector year observations with tails that are estimated on at least 20 firms, and where the estimated tails are statistically different from zero at a 5 percent confidence level.

Figure 1 displays the density of the pooled panel of estimated tails, using sales, employment and total assets as a measure of firm size. There is considerable sector-level heterogeneity in tail estimates, but the mass of estimates is below one irrespective of how firm size is measured. A tail parameter below one means that, at least at the right tail, sector-level size distributions follow a Zipf law. Therefore, a

	Sales	Employment	Assets
Mean	1.11	1.13	1.05
Std.	0.67	0.71	0.62
Min	0.23	0.25	0.19
Max	3.40	3.62	3.31
Mean # firms	105.42	123.02	112.42
Std. # firms	114.11	148.20	135.59
Min # firms	21	21	21
Max # firms	1144	1321	1398
Observations	2391	2152	2311

Table 1: Descriptive statistics on the tails estimates of a panel of sector.

majority of sectors have a size distribution fatter than Zipf law. They are highly concentrated sectors. Of course Compustat data are not representative as they consider listed firms only. They tend to be larger. But Compustat is presumably representative of large firms, i.e. firms at the right end of the firm size distribution, which is the basis for these estimations.

Table 1 displays summary statistics for the panel of estimated tails. On average, tail estimates are close to, but above one. Given figure 1, this is consistent with right skewness. The bottom panel of the table reports descriptive statistics for the number of firms in each sector for which the tail index is estimated. This number is endogenously determined by the estimation. Since observations are dropped whenever tails are estimated on fewer than 20 firms, the minimum number of firms is 21 for all measures of size. The mean number of firms goes from 105.42 to 123.02 depending on the measure of firm size considered. The number of sector year observations is above 2,100.

4.2 Tail and Growth

This section documents the empirical link between long-term growth and the tail of firm size distribution, δ , both at sector level. Theory suggests the link should be negative. Figure 2 displays a scatter of the mean growth rate of sales against the corresponding estimated tail, across sectors and years. The figure suggests a negative relationship.

To test the prediction formally, we estimate the following model:

$$\log(1 + growth_{t,s}) = \kappa + sector_s + year_t + \beta \log(1 + tail_{t,s}) + \gamma X_{t,s} + \epsilon_{t,s} \quad (5)$$

κ is a constant, $year_t$ is a time effect, $sector_s$ is a sector fixed effect and $X_{t,s}$ is a set of controls. The estimations are performed with and without time and/or

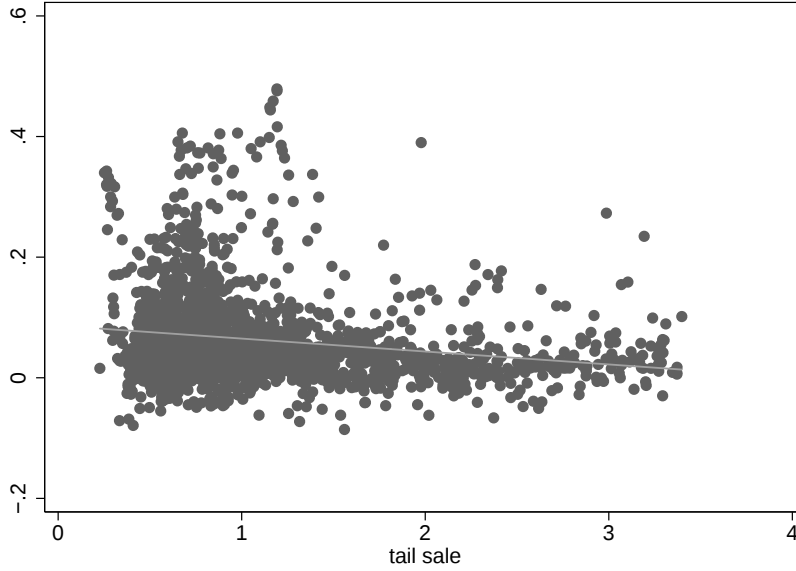


Figure 2: Scatter plot of mean growth rate of sales (y-axis) and tail estimates for sales (x-axis) for sector year observations.

sector fixed effect. The coefficient β is expected to be negative. In the baseline specification, it is the mean growth rate of sales at the 4-digit SIC level that measures $growth_{t,s}$. We also control for the size of the sector, as measured by the total number of employees there.

Table 2 reports the result. Columns (1) to (4) display the result without control at the sectoral level, columns (5) to (8) include sector-level employment. The negative relation between long-term growth and the tail index is statistically significant and negative in the absence of sector fixed effects, whether year effects are controlled for or not. Estimates of β become insignificant once sector fixed effects are added. In the cross section of sectors, fat tails are associated with high growth. But over time, fluctuations within-sector in the tail of size distribution are not associated with any difference in growth.

This result is consistent with the model described in section 2.2 and the concept of Sectoral Balanced Growth Path (SBGP). The model compares the relation between sector growth and sector tail *between* sectors. It has nothing to say about the impact of *within* variations in the sector estimates of tails on sectoral growth. In the model it is the tail of the stationary distribution of ideas, i.e the tail of the x_{t,j_s} , that affect growth. Within-sector variation in tail indexes come from the estimation of the tail of the distribution of output across firms, i.e the tail of y_{t,j_s} which have no impact on the within-sector variations in the long term growth.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
	ln(gr.y)	ln(gr.y)	ln(gr.y)	ln(gr.y)	ln(gr.y)	ln(gr.y)	ln(gr.y)	ln(gr.y)
ln(1 + δ_{sale})	-0.0467*** (0.0154)	0.0109 (0.0108)	-0.0511*** (0.0157)	0.00395 (0.0101)	-0.0485*** (0.0156)	0.00952 (0.0116)	-0.0542*** (0.0162)	-0.000873 (0.0101)
Employment					0.0000422 (0.000172)	0.0000629 (0.000113)	0.0000389 (0.000177)	0.000102 (0.000104)
Observations	2115	2115	2115	2115	1917	1917	1917	1917
Adjusted R^2	0.035	0.002	0.055	0.129	0.041	0.003	0.069	0.142
FE_year	No	No	Yes	Yes	No	No	Yes	Yes
FE_sector	No	Yes	No	Yes	No	Yes	No	Yes

Robust standard errors clustered at the sector level in parenthesis
***p<0.01, **p<0.05, *p<0.1

Table 2: Baseline specification for growth and tail relationship.

We check the robustness of the results, considering other measures of firm size, employment and total assets, as well as other measures of sector growth. Table 3 displays the results. Employment measures firm size in columns (1) to (4), and total assets do in columns (5) to (8). In columns (9) to (12), we use the sectoral mean growth rate of TFP rather than sales to measure sectoral growth. The table shows that the results are robust to other specifications. Table 4 allows for clustering in standard errors, considering the possibility of serial correlation at sector, year, or both levels. All results stand.

	(1) ln(gr.y)	(2) ln(gr.y)	(3) ln(gr.y)	(4) ln(gr.y)	(5) ln(gr.y)	(6) ln(gr.y)	(7) ln(gr.y)	(8) ln(gr.y)	(9) ln(gr.tfp)	(10) ln(gr.tfp)	(11) ln(gr.tfp)	(12) ln(gr.tfp)
ln(1 + δ_{emp})	-0.0402*** (0.0132)	-0.0000530 (0.0108)	-0.0417*** (0.0133)	0.00146 (0.00861)					-0.0150** (0.00757)	-0.000715 (0.00587)	-0.0153** (0.00754)	-0.000686 (0.00542)
ln(1 + δ_{at})					-0.0341** (0.0147)	0.0204** (0.00985)	-0.0372** (0.0148)	0.0148 (0.00948)				
Observations	1964	1964	1964	1964	2065	2065	2065	2065	1964	1964	1964	1964
Adjusted R^2	0.028	-0.001	0.047	0.134	0.017	0.010	0.035	0.144	0.010	-0.000	0.003	0.021
FE_year	No	No	Yes	Yes	No	No	Yes	Yes	No	No	Yes	Yes
FE_sector	No	Yes	No	Yes	No	Yes	No	Yes	No	Yes	No	Yes

Robust standard errors clustered at the sector level in parenthesis

***p<0.01, **p<0.05, *p<0.1

Table 3: Specification for growth and tail relationship for other measure of tail and growth.

	(1) ln(gr.y)	(2) ln(gr.y)	(3) ln(gr.y)	(4) ln(gr.y)	(5) ln(gr.y)	(6) ln(gr.y)	(7) ln(gr.y)	(8) ln(gr.y)	(9) ln(gr.y)	(10) ln(gr.y)	(11) ln(gr.y)	(12) ln(gr.y)
ln(1 + δ_{sale})	-0.0467*** (0.0154)	0.0109 (0.0112)	-0.0511*** (0.0158)	0.00395 (0.0104)	-0.0467*** (0.00287)	0.0109*** (0.00359)	-0.0511*** (0.00272)	0.00395 (0.00306)	-0.0467*** (0.0149)	0.0109 (0.0116)	-0.0511*** (0.0154)	0.00395 (0.0104)
Observations	2115	2115	2115	2115	2115	2115	2115	2115	2115	2115	2115	2115
Adjusted R^2	0.036	0.693	0.074	0.737	0.036	0.693	0.074	0.737	0.036	0.693	0.074	0.737
FE_year	No	No	Yes	Yes	No	No	Yes	Yes	No	No	Yes	Yes
FE_sector	No	Yes	No	Yes	No	Yes	No	Yes	No	Yes	No	Yes

Robust standard errors clustered at the (1-4) sector level, (5-8) year level, (9-12) sector*year level in parenthesis

***p<0.01, **p<0.05, *p<0.1

Table 4: Different clustering for growth and tail relationship

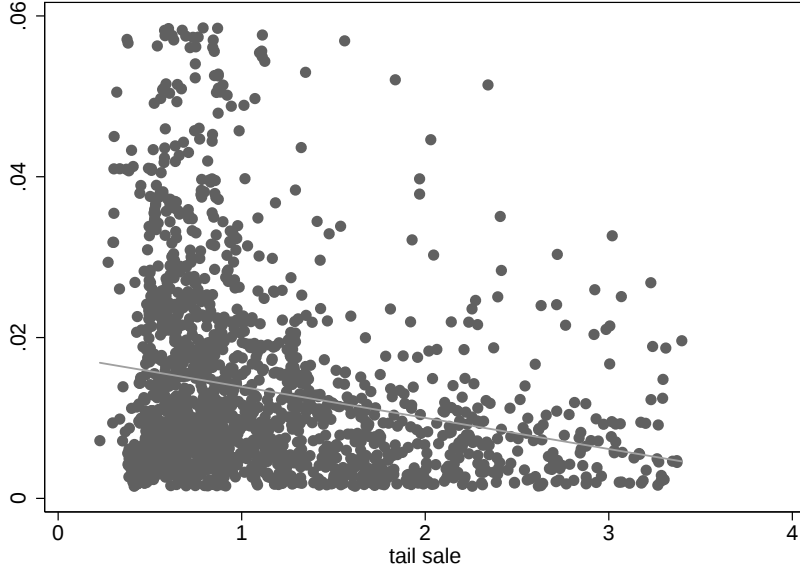


Figure 3: Scatter plot of variance of growth rate of sales (y-axis) and tail estimates (x-axis) for sales for sector year observations.

4.3 Tail and Volatility

This section analyzes the empirical relationship between volatility and tail indexes, both at sector level. Theory suggests the link should be negative.

Figure 3 displays a scatter of the variance in the growth rate of sales against the estimates of the tail index, across both sectors and years. The figure suggests a negative relation.

To test this prediction formally, we estimate the following empirical model:

$$vol_{t,s} = \kappa + sector_s + year_t + \beta(1 + tail_{t,s}) + \gamma N_{t,s} + \epsilon_{t,s} \quad (6)$$

Once again, κ is a constant, $year_t$ is a time-fixed effect, $sector_s$ is a sector fixed effect, $vol_{t,s}$ is a measure of sectoral volatility, $tail_{t,s}$ a measure of the tail index, and $N_{t,s}$ a measure of the number of firms in a given sector and a given year. We expect the estimate of β to be negative as it is in the model equation 4. In this equation the sign of γ remains indeterminate.

In the baseline specification of the model 6, we use the 11-year rolling window variance of sector-level growth rate as a measure of sectoral volatility, $vol_{t,s}$. The total number of employees is a proxy for the number of firms.

Table 5 displays the results for the baseline specification. Columns (5) to (8) include a control for the number of firms in a sector, whereas columns (1) to (4) do

not. The table shows that the negative relationship between volatility and the tail index is statistically different from zero and robust to the introduction of sectoral and time fixed effects. Furthermore, including the total number of employees barely affects estimates of β . The estimated value for γ is barely significant at 10 percent confidence level across specifications.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
	Var(gr.y)	Var(gr.y)	Var(gr.y)	Var(gr.y)	Var(gr.y)	Var(gr.y)	Var(gr.y)	Var(gr.y)
$1 + \delta_{sale}$	-0.00387*** (0.000785)	-0.00334*** (0.00101)	-0.00408*** (0.000795)	-0.00379*** (0.000865)	-0.00386*** (0.000687)	-0.00306*** (0.00116)	-0.00413*** (0.000700)	-0.00368*** (0.00106)
Employment					-0.0000350* (0.0000197)	-0.00000848 (0.0000218)	-0.0000386* (0.0000199)	-0.0000106 (0.0000251)
Observations	1817	1817	1817	1817	1649	1649	1649	1649
Adjusted R^2	0.046	0.040	0.077	0.148	0.058	0.035	0.085	0.117
FE_year	No	No	Yes	Yes	No	No	Yes	Yes
FE_sector	No	Yes	No	Yes	No	Yes	No	Yes

Robust standard errors clustered at the sector level in parenthesis

***p<0.01, **p<0.05, *p<0.1

Table 5: Baseline specification for volatility and tail relationship (model 6)

The fact that estimates of β are negative and statistically significant irrespective of fixed effects means the relation between sector volatility and tails holds both between and within sectors. The former is consistent with the model derived in section 2.4. In the model, volatility decreases with the tail of the stationary firm size distribution along a SBGP: the correlation is negative between sectors. In other words the model predict a *between* sectors negative relationship between volatility and tail index.

But the fact that the estimate of β is negative and statistically significant also with sector fixed effects also implies a negative correlation *within* sector. We suspect that an extension of our model can account for such *within* sectors correlation. The intuition is that when large firms, i.e firms with good ideas, experiment higher idiosyncratic shocks, these firms become larger. Hence the firm size distribution becomes fatter and the volatility is higher: the negative (resp. positive) relation between tail index (resp. tail fatness) and sector volatility holds also *within* sectors. Our model does not take into account such intuition since in theorem 1 only the tail of the idea distribution impact the sector volatility. We are in the process of incorporating this intuition in the model.

To ensure robustness, we considere other specifications. We use tail indexes estimated on firm employment and total assets, and use other measures of sector volatility based on TFP, or HP-filtered data instead of growth rates. Table 6 shows the negative significance of β prevails in all cases. Finally, table 7 displays the results for clustered standard errors in alternative dimensions. The results are once again robust.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
	Var(gr.y)	Var(gr.y)	Var(gr.y)	Var(gr.y)	Var(gr.tfp)	Var(gr.tfp)	Var(gr.tfp)	Var(gr.tfp)	Var(HPgr.y)	Var(HPgr.y)	Var(HPgr.y)	Var(HPgr.y)
1 + δ_{emp}	-0.00316*** (0.000741)	-0.00208* (0.00106)	-0.00335*** (0.000729)	-0.00225** (0.000987)					-0.00535*** (0.00152)	-0.00106 (0.000863)	-0.00553*** (0.00157)	-0.00142* (0.000847)
1 + δ_{asset}					-0.000759*** (0.000145)	-0.000433*** (0.000141)	-0.000682*** (0.000149)	-0.000358** (0.000159)				
Employment	-0.0000325 (0.0000204)	-0.0000153 (0.0000234)	-0.0000347* (0.0000209)	-0.0000126 (0.0000276)	-0.00000745 (0.00000469)	-0.00000413 (0.00000815)	-0.00000791* (0.00000472)	-0.00000613 (0.00000802)	-0.0000529 (0.0000346)	-0.000234** (0.0000950)	-0.0000513 (0.0000360)	-0.000207*** (0.0000767)
Observations	1491	1491	1491	1491	1607	1607	1607	1607	1784	1784	1784	1784
Adjusted R^2	0.046	0.018	0.053	0.069	0.050	0.014	0.082	0.074	0.034	0.051	0.045	0.063
FE_year	No	No	Yes	Yes	No	No	Yes	Yes	No	No	Yes	Yes
FE_sector	No	Yes	No	Yes	No	Yes	No	Yes	No	Yes	No	Yes

Robust standard errors clustered at the sector level in parenthesis
***p<0.01, **p<0.05, *p<0.1

Table 6: Other specification for sectoral volatility and tail relationship

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
	Var(gr.y)	Var(gr.y)	Var(gr.y)	Var(gr.y)	Var(gr.y)	Var(gr.y)	Var(gr.y)	Var(gr.y)	Var(gr.y)	Var(gr.y)	Var(gr.y)	Var(gr.y)
1 + δ_{sale}	-0.00386*** (0.000687)	-0.00306*** (0.00116)	-0.00413*** (0.000700)	-0.00368*** (0.00106)	-0.00386*** (0.000327)	-0.00306*** (0.000311)	-0.00413*** (0.000358)	-0.00368*** (0.000322)	-0.00386*** (0.000687)	-0.00306*** (0.00116)	-0.00413*** (0.000777)	-0.00368*** (0.00108)
Employment	-0.0000350* (0.0000197)	-0.00000848 (0.0000218)	-0.0000386* (0.0000199)	-0.0000106 (0.0000251)	-0.0000350*** (0.00000666)	-0.00000848 (0.0000121)	-0.0000386*** (0.00000713)	-0.0000106 (0.0000128)	-0.0000350* (0.0000195)	-0.00000848 (0.0000238)	-0.0000386* (0.0000231)	-0.0000106 (0.0000275)
Observations	1649	1649	1649	1649	1649	1649	1649	1649	1649	1649	1649	1649
Adjusted R^2	0.058	0.035	0.085	0.117	0.059	0.606	0.109	0.648	0.059	0.606	0.109	0.648
FE_year	No	No	Yes	Yes	No	No	Yes	Yes	No	No	Yes	Yes
FE_sector	No	Yes	No	Yes	No	Yes	No	Yes	No	Yes	No	Yes

Robust standard errors clustered at the sector level in parenthesis
***p<0.01, **p<0.05, *p<0.1

Table 7: Different clustering for sectoral volatility and tail relationship

4.4 Tails and the Growth-Volatility Relation

The framework presented in section 2 rationalizes the existence of a positive relationship between the tail indexes and both sectoral growth and volatility. This implies the dispersion in tail indexes can account for at least a fraction of the positive relation between growth and volatility. In this section, we investigate how large a fraction. We proceed in two steps. First, the cross-section of sectoral volatility is regressed on the estimated tail indexes. We then explore how much of the dispersion in sector growth rates can be explained by the fitted values of sector volatilities, explained by tail indexes only. Since the tails explain only long run differences in sector-level growth, the regressions include time effects only.

	(IV)	(non IV)
	gr.y	gr.y
Var(gr.y)	3.488***	1.806***
	(0.757)	(0.248)
Observations	1817	1817
R^2	0.120	0.297
FE_year	Yes	Yes
FE_sector	No	No

Robust standard errors clustered at the sector level in parenthesis

***p<0.01, **p<0.05, *p<0.1

Table 8: IV vs non IV estimation of Growth and Volatility relationship

Table 8 compares the result of this regression (column (IV)) with one that simply regresses sector growth on its volatility (column (non-IV)). Sectoral volatility is measured by the variance of the sales growth rates, while sectoral growth is its mean, both computed on 11-year rolling windows. The dispersion in sectoral volatility can account for about 29 percent of sectoral growth. When volatility is instrumented by tail indexes, it can still account for about 12 percent of sectoral growth. Therefore, tail indexes explain about 40% of the growth-volatility relationship between sectors.

5 Conclusion

We present a model where growth and volatility both depend on the distribution of firm size. Fat-tailed firm distributions imply the probability of learning about a more productive technology is high, and thus so is real growth. Fat-tailed firm distributions also imply that firm-specific idiosyncratic shocks do not wash out in the aggregate, and thus measured volatility is high. We provide support for both

of these links in US sector-level data. We provide sector-level estimates of the tails of firm size distribution, and show they correlate significantly with measures of real growth and real volatility. Just as in theory, tails of size distribution affect both sector-level growth and volatility in the long-run. However, in our estimations, tails of size distribution affect sector volatility in both long and short run. The dispersion in sectoral estimate of size distributions is quantitatively relevant to explain why volatile sectors tend to grow fast. Thus, risky sectors grow fast because they innovate. Whether this fact prevails in other countries than the US is an important question that we leave for further research.

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A Proofs

A.1 Proof of Proposition 2

Let us defined, the Ξ function:

$$\Xi(x, y, z) = \sum_{i=-\infty}^{\infty} (1+z)^i \left(\exp(-y((1+z)^i)^{-x}) - \exp(-y((1+z)^{i-1})^{-x}) \right)$$

Note that the probability mas function $f(x_i, t)$ of the distribution defined by the cdf $F(x_i, t)$ is such that $f(x_i, t) = F(x_i, t) - F(x_{i-1}, t)$.

We are now ready to compute the average idea along a SBGP (ν, Φ) :

$$\begin{aligned} I_t &= \sum_{i=-\infty}^{\infty} x_i f(x_i, t) && \text{by definition of } I_t \\ &= \sum_{i=-\infty}^{\infty} x_i (F(x_i, t) - F(x_{i-1}, t)) && \text{by definition of } f(x_i, t) \\ &= \sum_{i=-\infty}^{\infty} x_i (\Phi((1+\nu)^{-t} x_i) - \Phi((1+\nu)^{-t} x_{i-1})) && \text{by definition of a SBGP} \\ &= \sum_{i=-\infty}^{\infty} (1+\nu)^i (\Phi((1+\nu)^{i-t}) - \Phi((1+\nu)^{i-t-1})) && \text{by definition of } x_i \\ &= \sum_{i=-\infty}^{\infty} (1+\nu)^i (\exp(-\lambda((1+\nu)^{i-t})^{-\delta}) - \exp(-\lambda((1+\nu)^{i-t-1})^{-\delta})) && \text{using equation 3} \\ &= (1+\nu)^t \sum_{i=-\infty}^{\infty} (1+\nu)^i (\exp(-\lambda((1+\nu)^i)^{-\delta}) - \exp(-\lambda((1+\nu)^{i-1})^{-\delta})) && \text{with } i = i - t \end{aligned}$$

The last equation gives

$$I_t = (1+\nu)^t \Xi(\delta, \lambda, \nu)$$

□

A.2 Proof of Theorem 1

From the core of the paper, the variance of growth rate of the sectoral output is

$$\mathbb{V}ar_t \left(\frac{\Delta Y_{t+1,s}}{Y_{t,s}} \right) = \left(\frac{\sigma}{Y_{t,s}} \right)^2 \sum_{j_s=1}^{N_s} (y_{t,j_s})^2$$

According to the law of large number, we have:

$$(N_s)^{-1} \sum_{j_s} y_{t,j_s} = (N_s)^{-1} Y_{t,s} \rightarrow \mathbb{E} y_{t,j_s} := \bar{y}_{t,s}$$

the average of firm output in sector s at date t .

The product of two random variable has a Pareto tail equal to the smallest tail of these two random variable. It follows that the distribution of the random variable $y_{t,j_s} = x_{t,j_s} z_{t,j_s}$ has a power law tail with parameters δ , the tail parameter of the idea distribution, because the random variable z_{t,j_s} has a tail greater than two (it admit a moment of order two) and $\delta < 2$.

Since $\delta/2 < 1$, using the Lévy theorem (Gabaix 2011, Durrett 2010), we have

$$(N_s)^{-2/\delta} \sum_{j_s} (y_{t,j_s})^2 \rightarrow^d u_{t,s}$$

where $u_{t,s}$ is a standard Lévy distribution with parameters $\delta/2$.

Computing the two above results yields

$$\begin{aligned} \mathbb{V}ar_t \frac{\Delta Y_{t+1,s}}{Y_{t,s}} &= \left(\frac{\sigma}{Y_{t,s}} \right)^2 \sum_{j_s=1}^{N_s} (y_{t,j_s})^2 \\ &= \Omega \left(\left(\frac{\sigma}{Y_{t,s} N_s^{-1}} \right)^2 N_s^{-2} \sum_{j_s=1}^{N_s} (y_{t,j_s})^2 \right) \\ &= \Omega \left(\left(\frac{\sigma}{\bar{y}_{t,s}} \right)^2 N_s^{-2} (N_s)^{2/\delta} u_{t,s} \right) \\ &= \Omega \left(\left(\frac{\sigma}{\bar{y}_{t,s}} \right)^2 N_s^{-2+2/\delta} u_{t,s} \right) \\ &= \Omega \left(\sigma^2 \frac{u_{t,s}}{N_s^{2(1-1/\delta)}} \right) \end{aligned}$$

Finally, we have

$$\sigma_t \left(\frac{\Delta Y_{t+1,s}}{Y_{t,s}} \right) = \Omega \left(\sigma \frac{u_{t,s}^{1/2}}{N_s^{1-1/\delta}} \right)$$

□

B On tail index of a mixture of Pareto distribution

Let assume that in sector 1, there are M_1 firms that draw their productivity from a Pareto distribution with scale parameter $x_{min} = 1$ and shape parameter α .

Let assume that in sector 2, there are M_2 firms that draw their productivity from a Pareto distribution with scale parameter $y_{min} = 1$ and shape parameter β .

Assumption 1 $\alpha < \beta$ i.e the distribution in sector 1 is fatter than the one in sector 2.

Let us consider sector 0 as being the sector that aggregate sector 1 and sector 2's firms. There are $M = M_1 + M_2$ firms in that sector.

The counter-cumulative-distribution function (CCDF) in sector 1 is $F^1(x) = x^{1-\alpha}$, in sector 2 $F^2(x) = x^{1-\beta}$. The CCDF of the sector 0 is the mixture of the CCDF in sector 1 and 2 with weight being $\frac{M_1}{M}$ and $\frac{M_2}{M}$ respectively:

$$F^0(x) = \frac{M_1}{M}x^{1-\alpha} + \frac{M_2}{M}x^{1-\beta} \quad (7)$$

Let us show that the distribution in sector 0, stochastically dominate a Pareto distribution with shape parameter being $\frac{M_1}{M}\alpha + \frac{M_2}{M}\beta$ and that the distribution in sector 0 is stochastically dominated by the distribution in sector 1 (the fatter sector).

Proposition 3

$$F^1(x) = x^{1-\alpha} \geq F^0(x) \geq x^{1-(\frac{M_1}{M}\alpha + \frac{M_2}{M}\beta)} \quad (8)$$

Proof.

First let us show the first inequality:

$$\begin{aligned} F^0(x) &= \frac{M_1}{M}x^{1-\alpha} + \frac{M_2}{M}x^{1-\beta} \\ &\leq \frac{M_1}{M}x^{1-\alpha} + \frac{M_2}{M}x^{1-\alpha} \text{ since } \alpha < \beta \\ &\leq x^{1-\alpha} \end{aligned}$$

Second, let us look at the second inequality:

$$\begin{aligned}
\log F^0(x) &= \log \left(\frac{M_1}{M} x^{1-\alpha} + \frac{M_2}{M} x^{1-\beta} \right) \\
&\geq \frac{M_1}{M} \log x^{1-\alpha} + \frac{M_2}{M} \log x^{1-\beta} \text{ since } \log \text{ is a concave function} \\
&\geq \left[1 - \left(\frac{M_1}{M} \alpha + \frac{M_2}{M} \beta \right) \right] \log x
\end{aligned}$$

The latter inequality is equivalent to $F^0(x) \geq x^{1-(\frac{M_1}{M}\alpha + \frac{M_2}{M}\beta)}$.

The equality case is satisfied for $M_2 = 0$ for the left hand side inequality and for $M_1 = 0$ or $M_2 = 0$ for the right hand side inequality.

□

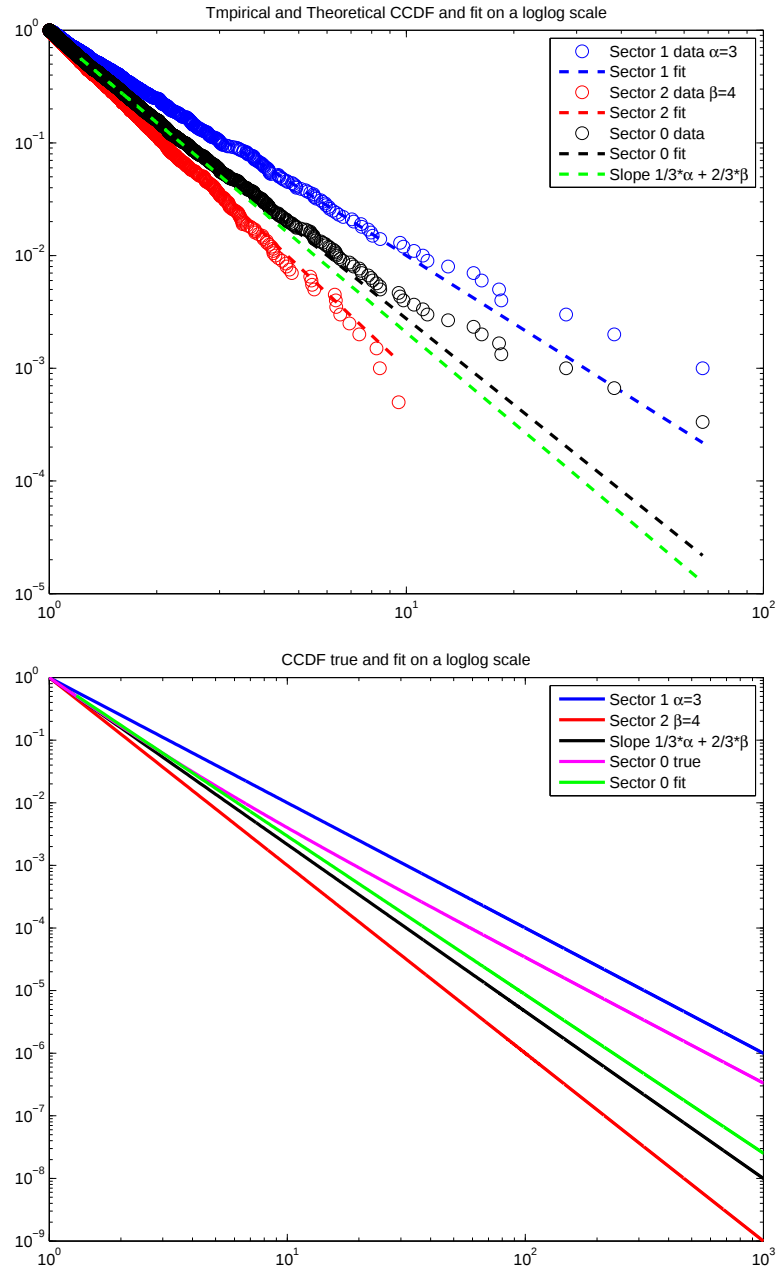


Figure 4: CCDFs on log-log scale. Top: Simulated data with Pareto fit for sector 0, 1 and 2. Bottom: True distribution for sector 0,1,2 and the Pareto fit of sector 0.