# Information Provision and Consumer Search*

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WORK IN PROGRESS

**Abstract**

Buyers often search across multiple retailers or websites to learn which product best fits their needs. We study how sellers manage these search incentives through their disclosure policies (e.g. advertisements, product trials and reviews), and ask whether competition leads sellers to provide more information. We first show that if sellers can track buyers (e.g. they can observe buyers' search history via their cookies), then in a broad range of environments, there is a unique equilibrium in which all sellers provide the "monopoly level" of information. However, if buyers are anonymous and the search cost is small, then all sellers provide full information about their products. Tracking software thus enables sellers to implicitly collude, providing a motivation for regulation.

## 1 Introduction

The internet has led to a proliferation of products, and a proliferation of sellers providing any given product. For example, eBay has around 5 billion listings from 2 million sellers in 50,000 product categories. Sophisticated consumers search across sellers to learn the properties of different products to benefit from this ever-expanding variety. Starting with Stigler (1961), there has been a large literature examining the incentives of buyers to search for better prices. This paper studies the incentives for buyers to search for better information, and how sellers manage these incentives through their disclosure policies.

Sellers have a lot of flexibility in managing buyers' information through advertisements, product trials and advice. For example, when selling books online, a retailer chooses which products to suggest, what information to provide in the descriptions, and how to present other customer reviews. As consumer tracking becomes more pervasive, such information can be increasingly tailored to individual consumers, and their shopping habits. Facebook's "Like" and Twitter's "Tweet" buttons carry code enabling the companies to track users' movements across the internet. Similarly, a slew

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of third-party tracking companies follow users from site to site, monitoring popular websites and piggybacking on data collected by other data brokers.[1]

In this paper we characterize a general model of sequential search for information in which the seller is free to choose any feasible disclosure policy, in the style of Kamenica and Gentzkow (2011). We then ask whether competition forces sellers to provide all pertinent information to consumers, and show that the answer depends on the sellers' information about consumers. When sellers can track buyers then, in a broad range of environments, there is a unique equilibrium in which all sellers provide the monopoly level of information. That is, sellers manipulate their customers to purchase the most profitable, rather than the most suitable product. In contrast, when buyers are anonymous and search costs are small, the equilibrium shifts and sellers fully reveal their information.

Tracking software thus allows sellers to implicitly collude by providing more information to a new customer than to one who came from a competitor. This finding is important since, over the last decade, internet giants such as Google and Facebook and a plethora of data brokers, have been collecting ever more information about their users, giving rise to concerns that such information makes consumers somehow "exploitable".[2] We provide a theoretical foundation for such claims and show that neither competition nor individual privacy measures – such as deleting one's cookies – can correct such distortions, providing a rationale for regulating tracking programs, or the data sharing agreements that underpin them.

In the model, we consider a prospective buyer ("he") who samples from sellers in order to learn which product best suits his needs. When sampled, a seller ("she") chooses how much information to disclose in order to encourage the buyer to purchase. On a website, this may mean letting the buyer use a trial version, have temporary access, or try a basic version for free; it may mean suggesting products to buyers, or providing them with reviews of other buyers. We suppose that the seller does not know the effect of the information on the buyer (e.g. whether the buyer likes a book when they "look inside"), and model the seller's information disclosure as a distribution of signal realizations that may increase or decrease the buyers' assessment of the product. After updating his belief, the buyer chooses to buy a product, to exit, or to pay a search cost and sample another seller. We study how much information sellers disclose, and how this depends on competition between sellers, and sellers' information about buyers.

As a benchmark, suppose there is a monopolistic seller, as in Kamenica and Gentzkow (2011), who offers a single product. If the buyer is optimistic about the product and willing to buy given his prior, then the seller should provide no information. However, if the buyer is more pessimistic, the seller should provide a binary signal such that the buyer either (i) knows for sure that he does not want the product, or (ii) is just willing to buy.

We analyze a model with many sellers, where a buyer searches randomly and pays a search cost for each new draw. In Section 3, we suppose sellers can track buyers, allowing them to observe a buyer's prior before choosing the information disclosure policy. We first provide a general characterization of equilibrium, showing that any equilibrium is locally optimal, meaning no seller can benefit from slightly increasing the support of her signal. Conversely, any locally optimal policy is

[1]To illustrate, *The Economist* reports that the 100 most widely used websites are monitored by more than 1,300 firms. See their "Special Report: Advertising and Technology" (13th September 2014).

[2]For example, see "Facebook Tries to Explain Its Privacy Settings but Advertising Still Rules," *New York Times* (13th November 2014) or "Google to Pay $17 Million to Settle Privacy Case" *New York Times* (18th November 2013).

an equilibrium if the search cost is sufficiently small.

We use this result to investigate how much information is provided by the market. If there is a single product, then there is a unique equilibrium in which sellers choose the monopoly disclosure policy. Intuitively, a seller can always provide a little less information than the market due to the search cost; this process stops when the customer is indifferent between buying and not, as in the monopoly policy. This logic is related to the Diamond paradox in the price-setting model; however our model is different because a buyer's preferences depend on the information he received from previous sellers, and the result crucially depends on the seller's ability to track buyers' past behavior. The monopoly result extends to a Hotelling model and, we conjecture, also to settings where the underlying dimension of tastes is rich relative to the number of products. Non-monopolistic equilibria may arise in situations where this assumption fails (e.g. when there are two products and one dimensional uncertainty); nevertheless, even in these cases, the monopoly policy is always an equilibrium.

In Section 4, we suppose that sellers are unable to track buyers. In other words, sellers know a buyer's initial prior but do not know who he has visited in the past. When a seller sees a buyer she conjectures what information the buyer has already received from past sellers and chooses a signal that is independent of past signals.[3] In equilibrium, sellers choose a disclosure policy such that the buyer purchases (or exits) in the first period. However, the possibility that the buyer can continue to search and pretend to be a new buyer forces sellers to provide more information. More precisely, if the equilibrium sales policy is generic, in that a buyer with any non-degenerate prior learns from the signal, then sellers provide full information as the search cost shrinks. The genericity assumption is satisfied if, for example, there are two states and the monopoly policy is nontrivial. We also show that the genericity assumption is not necessary: if there is a single product, competition leads sellers to provide enough information such that buyers obtain their full information payoff, again if the monopoly policy is nontrivial.

In Section 5 we allow buyers to avoid being tracked at a small cost. For example, they can activate the "Do Not Track" function in their browser (that most large advertising companies respect) or delete their cookies. The analysis of the previous two sections implies that becoming anonymous exerts a positive externality on other buyers, inducing sellers to provide more information to everyone. Intuitively, tracking enables firms to implicitly collude by allowing them to differentiate between new and old customers. When given this information, a seller optimally provides less information to a buyer who has visited a previous seller, lowering his outside option, and making him less likely to search. Without tracking, sellers cannot discriminate between new and old customers and the option of going elsewhere forces sellers to provide all the pertinent information to their customers.

The temptation to free-ride is a powerful effect: no matter how small the cost of becoming anonymous, there is no equilibrium where all buyers voluntarily do so. This suggests a role for the regulation of tracking software, or the data sharing agreements that allow a single seller to know whether a potential buyer has already visited her competitors.

## 1.1 Literature

There is a large literature in which firms post prices and agents pay a fixed cost for each search. In the benchmark model, Diamond (1971) shows that if all sellers sell a single identical good then,

[3] Since we are interested in large markets, the independence assumption rules out the "coordinated" signals of Gentzkow and Kamenica (2012).

in equilibrium, all firms charge the monopoly price. Intuitively, the search cost generates a holdup problem, allowing any one seller to raise her price slightly above the price set by others. This result continues to hold if firms make multiple offers to each buyer (Board and Pycia (2014)). However, it breaks down if sellers sell heterogenous products (Wolinsky (1986)) or multiple products (Zhou (2014); Rhodes (2014)), or if buyers pre-commit to a number of searches (Burdett and Judd (1983)). In our paper, sellers choose disclosure policies rather than prices. This means that a buyer's beliefs about firm $i$'s product depends on whether of not they already visited firm $j$. Moreover, in contrast to Diamond, if sellers give the same signal to all buyers then, as search costs become small, sellers choose the buyer-optimal disclosure policy.

Our work also contributes to a recent literature on information disclosure with competition, based on the Kamenica-Gentzkow framework.[4] In a general model, Gentzkow and Kamenica (2012) consider two senders who simultaneously choose a disclosure policy. They assume the senders' signals are "coordinated" in that a common random variable determines how states are mapped into signals, and show how each sender takes the information released by the other as given, choosing the optimal monopoly policy on the residual. A consequence is that two senders always provide more information than one, although Li and Norman (2014) show this depends on the signal structure. When signals are independent, adding a second sender may reduce the amount of information provided as the first sender tries to protect herself from the new information provided by the second.

In Hoffmann, Inderst, and Ottaviani (2014), heterogeneous sellers compete to sell to a single buyer, making a binary decision of whether to reveal information. When the seller's disclosure strategy is observed, they show that an increase in competition increases the amount of information provided. Intuitively, in order to make a sale, the buyer's utility at a seller has to exceed a threshold; this threshold increases in the level of competition, implying that the seller wishes to raise the variance of the buyer's posterior.[5]

Our paper contrasts by considering a buyer-seller framework in which all sellers sell the same set of goods. Moreover, while the above papers consider the simultaneous release of information by competing sellers, we study the sequential release of information, as buyers search from seller to seller.

## 2 Model

**Basics.** There is one (male) buyer and more than two (female) sellers who sell the same products. Buyers are uncertain about a finite set of states relevant for the buyers' decision which we call $S$. The set of states could be objective (e.g. whether or not 3D TV programming will take off in the industry) or subjective (e.g. whether the buyer will actually enjoy using the product in the long run). Let $\Delta S$ be the set of beliefs about $S$.

[4]This approach is also related to the single-seller models of Lewis and Sappington (1994) and Rayo and Segal (2010).

[5]There are a variety of papers concerning the revelation of information when the senders are informed. A first literature considers verifiable information ("persuasion games"). For example, Milgrom and Roberts (1986) finds conditions under which competition leads to full revelation. More recently, Bhattacharya and Mukherjee (2013) consider a model where senders are stochastically informed, characterizing the receiver's preferences over the biases of experts. A second literature supposes information is unverifiable ("cheap talk"). Here, Krishna and Morgan (2001) consider sequential communication, while Battaglini (2002) analyzes a simultaneous game, both finding conditions under which full revelation is possible when the senders have opposed preferences.

**Actions.** Buyers pick a seller at random, the seller observes the buyer's prior $p \in \Delta S$ and commits to an information disclosure policy, as described below. After the buyer sees the seller's signal, he updates his belief to form a posterior $q \in \Delta S$ and chooses whether to (1) buy a product from the seller, (2) exit the market or (3) pay a search cost $c > 0$ and pick a new seller at random. This cost could either be the time spent to drive to a different store or the hassle of signing up for a new account when making online purchases. When a buyer arrives at a seller, the seller observes the buyer's prior about the state and commits to an information disclosure policy described below.

**Buyer's strategy.** Denote $U \subset \mathbb{R}^S$ as a (finite) set of available products[6] including the exit option $0 \in U$. This means that product $u \in U$ delivers utility $u(s)$ in state $s \in S$ and expected utility $q \cdot u$ to the buyer under belief $q \in \Delta S$. Define the *acceptance set* $Q$ as the set of beliefs for which the buyer decides to stop search. If the buyer decides to accept at posterior $q$, let $u^*(q) \subset U$ denote the set of optimal choices.

**Seller's strategy.** We initially assume a seller can observe a buyer's prior $p$, which may incorporate information from other sellers. A seller chooses a *disclosure policy* $K$ such that for each buyer with prior $p$, the policy releases information that results in a distribution of buyer posteriors such that the average posterior equals the prior, i.e. $\int_{\Delta S} q \; K(p, dq) = p$ where $K$ is a continuous Markov kernel.

Let $\tilde{\pi}(u)$ be the seller's payoff given the buyer's choice $u \in U$, where exiting yields zero profits $\tilde{\pi}(0) = 0$. Assuming that ties in the buyer's actions are resolved in the seller's favor, the seller's payoff given the buyer's belief $q \in \Delta S$ is $\pi(q) := \max_{u \in u^*(q)} \tilde{\pi}(u)$. Since there are at least two sellers and search is random, sending a buyer outside of his acceptance set $Q$ is never optimal for the seller. Thus, a disclosure policy $K$ is *optimal given* $Q$ iff for all possible disclosure policies $L$

$$\int_Q \pi(q) \, K(p, dq) \geq \int_Q \pi(q) \, L(p, dq) . \tag{1}$$

Moreover, to break ties, assume that if the above holds with equality for some other $L(p)$, then $L(p)$ cannot be strictly less (Blackwell) informative than $K(p)$.

The *full information policy* is a disclosure policy such that every posterior is degenerate on some state, $q = \delta_s$. A *monopoly policy* is a disclosure policy that is optimal given $Q = \Delta S$. The monopoly policy corresponds to the static optimal information disclosure policy in Kamenica and Gentzkow (2011).

**Symmetric equilibrium.** If all sellers use policy $K$, a buyer with posterior $q$ who continues to search has continuation utility

$$V_K(q) := -c + \int_{\Delta S} \max\left\{\max_{u \in U} r \cdot u, \; V_K(r)\right\} K(q, dr)$$

A disclosure policy and acceptance set $(K, Q)$ is an *equilibrium* iff $K$ is optimal given

$$Q = \left\{ q \in \Delta S \;\middle|\; \max_{u \in U} q \cdot u \geq V_K(q) \right\}$$

[6] One could also interpret each $u \in U$ as a bundle of two products, three products, etc..

**Remarks.** We make three assumptions of note. First, following Kamenica and Gentzkow (2011), we assume that sellers commit to their signals. This means that the disclosure policy is "truthful" and does not signal qualities of the product. For example, if Amazon lets customers "look inside" a book or see their friends' reviews, Amazon does not know how the customers will interpret this information. Second, we assume the utilities (and therefore prices) of different products are exogenous. This could be because prices are chosen by an upstream firm (e.g. book prices chosen by the publisher), or because companies can target different customers with different information but restricted to charge the same price to all customers. Theoretically, this assumption allows us to isolate the effect of information disclosure. Third, we suppose that buyers start with degenerate prior. This reflects the idea that sellers have demographic data about their customers that enables the seller to predict the buyers' preferences over the product just as well as the buyer. This assumption also helps us to isolate the role of behavioral tracking. In Section 4 we describe how the derive optimal disclosure policies when buyers have heterogeneous priors.

# 3 Tracked Buyers

## 3.1 Illustrative Examples

Before discussing the general theory, we discuss three examples to illustrate the economics forces at play.

**Single product.** Suppose the seller has a single product $u$ to sell. We now show that in all equilibria, all sellers choose the monopoly disclosure policy. To illustrate what this means, suppose there are two states $\{L, H\}$, and that the buyer wishes to buy the product if state $H$ is more likely. For example, consider a consumer thinking about buying a 3D TV, but they do not know if there will be a high or low number of programs made in 3D. Suppose that $(u(L), u(H)) = (-1, 1)$, so the buyer prefers to buy if $q = \Pr(H) \geq \frac{1}{2}$, and that a sale generates profits $\tilde{\pi}(u) = 1$ for the seller (see Figure 1). The resulting monopoly policy provides just enough information to persuade the buyer to buy:

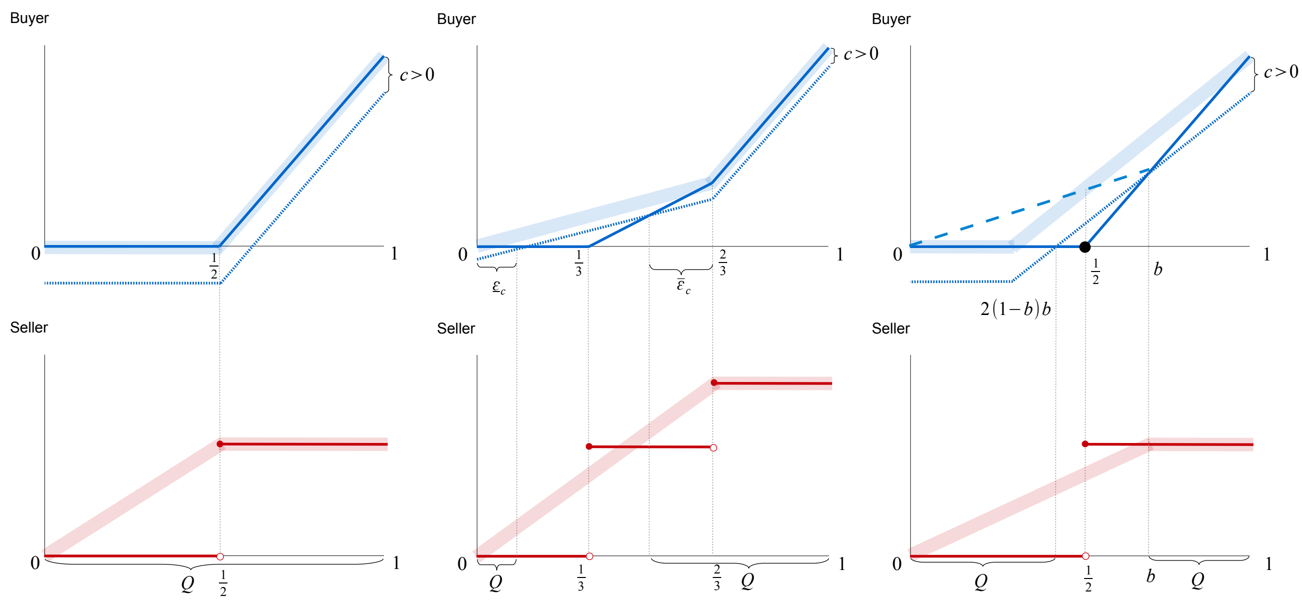
$$K(p) = \begin{cases} (1-2p)\,\delta_{\{0\}} + 2p\delta_{\left\{\frac{1}{2}\right\}} & \text{if } p \in \left[0, \frac{1}{2}\right) \\ p & \text{if } p \in \left[\frac{1}{2}, 1\right] \end{cases}$$

That is, if the buyer starts with a prior $p \geq \frac{1}{2}$, the seller provides no information and the buyer buys; if $p < \frac{1}{2}$, the seller provides just enough information to persuade the buyer to buy, taking his posterior to 0 or $\frac{1}{2}$. In the unique equilibrium of the search model, all sellers choose this disclosure policy. Moreover, since a buyer learns nothing from a second seller after leaving a first, the acceptance set is the entire state space $Q = [0, 1]$.

This result is more general, and does not require the binary state space. This yields an informational version of the Diamond paradox.

**Proposition 1.** *If there is a single product, then any equilibrium disclosure policy is the monopoly policy.*

**Figure 1: Single Product** **Figure 2: Vertical Differentiation** **Figure 3: Unobserved Priors**




The fact that the monopoly policy is an equilibrium is not surprising. If all sellers use the monopoly policy, then a buyer has no reason to continue his search and the monopoly policy is a best response. Proposition 1 says that there are no other equilibria. To understand why, consider the above two-state example and suppose all sellers take a buyer with belief $q \le \frac{2}{3}$ to either 0 or $\frac{2}{3}$. Consider a seller who deviates and takes the buyer to 0 or $\frac{2}{3} - \varepsilon$. Since there is a strictly positive search cost, the buyer will not search; moreover, this raise the profits of the seller yielding a profitable deviation. The idea is reminiscent of the classic Diamond paradox: a buyer must pay a positive cost to search, so the seller can lower his utility a little without losing the sale. The question is whether such a small deviation raises the seller's profits. In the single-product example, this is the case; however, the following example shows that small deviations may not raise the firm's profit and Proposition 1 may break down.

**Vertical differentiation.** Suppose we introduce a second product into the above example. Again, there are two states $\{L, H\}$ and $q = \Pr(H)$. The first TV is cheap and is not very good for 3D programs, yielding utility $u_1 = \left(-\frac{1}{3}, \frac{2}{3}\right)$; the second TV is expensive but great for 3D, yielding utility $u_2 = (-1, 1)$. As a result, the buyer purchases no TV if $q \in \left[0, \frac{1}{3}\right)$, buys the cheap TV if $q \in \left[\frac{1}{3}, \frac{2}{3}\right)$ and buys the expensive TV if $q \in \left[\frac{2}{3}, 1\right]$. Assume the expensive TV also has a higher margin for the seller, $\tilde{\pi}(u_1) = 1$ and $\tilde{\pi}(u_2) = \frac{3}{2}$ (see Figure 2).

The monopoly policy here is as follows:

$$K(p) = \begin{cases} (1-3p)\,\delta_{\{0\}} + 3p\delta_{\left\{\frac{1}{3}\right\}} & \text{if } p \in \left[0, \frac{1}{3}\right) \\ (2-3p)\,\delta_{\left\{\frac{1}{3}\right\}} + (3p-1)\,\delta_{\left\{\frac{2}{3}\right\}} & \text{if } p \in \left[\frac{1}{3}, \frac{2}{3}\right) \\ p & \text{if } p \in \left[\frac{2}{3}, 1\right] \end{cases}$$

However, there are equilibria where the sellers provide more information. For example, consider the following disclosure policy

$$K(p)=\begin{cases}\frac{2-3p}{2}\delta_{\{0\}}+\frac{3p}{2}\delta_{\left\{\frac{2}{3}\right\}} & \text{if } p\in\left[0,\frac{2}{3}\right)\\ p & \text{if } p\in\left[\frac{2}{3},1\right]\end{cases} \tag{2}$$

This gives rise to the acceptance set $Q=[0,\underline{\varepsilon}_c]\cup\left[\frac{2}{3}-\bar{\varepsilon}_c,1\right]$ for some $\underline{\varepsilon}_c>0$ and $\bar{\varepsilon}_c>0$. Hence a seller's profits are

$$\pi(q)\,\mathbf{1}_Q(q)=\mathbf{1}_{\left[\frac{2}{3}-\bar{\varepsilon}_c,\frac{2}{3}\right)}+\frac{3}{2}\cdot\mathbf{1}_{\left[\frac{2}{3},1\right]}$$

When $c>0$ is small enough such that $K$ is optimal for $Q$, $(K,Q)$ is an equilibrium. Note that $K$ is not a monopoly policy since it never induces the buyer to purchase the cheap TV. As shown in Figure 2, the presence of the search cost allows the seller to provide a little less information than her competitors; however, there is no local deviation that is profitable. If one seller induced a buyer to have a posterior $q=\frac{1}{3}$, as occurs in the monopoly solution, then the buyer would refuse to buy, correctly anticipating that another seller will provide significantly more information.

**Horizontal differentiation.** Finally, suppose we consider a market with two products that are horizontally differentiated as in Hotelling. Again, suppose there are two states $\{L,H\}$ and $q=\Pr(H)$. The first product is an expensive 4k TV that is great for regular programs but has no 3D capability, yielding utility $u_1=\left(\frac{1}{2},-1\right)$. The second is a good 3D TV, yielding utility $u_2=\left(-1,\frac{1}{2}\right)$. The buyer thus purchases the 4k TV if $q\in\left[0,\frac{1}{3}\right]$, buys no TV if $q\in\left(\frac{1}{3},\frac{2}{3}\right)$ and buys the 3D TV if $q\in\left[\frac{2}{3},1\right]$. Assume both TVs generate the same profits, $\tilde{\pi}(u_1)=\tilde{\pi}(u_2)=1$ (see Figure 3).

In this example, even though there are two products, the conclusion of Proposition 1 still holds and all equilibria coincides with the monopoly policy,

$$K(p)=\begin{cases}(2-3p)\,\delta_{\left\{\frac{1}{3}\right\}}+(3p-1)\,p\delta_{\left\{\frac{2}{3}\right\}} & \text{if } p\in\left(\frac{1}{3},\frac{2}{3}\right)\\ p & \text{if } p\in\left[0,\frac{1}{3}\right]\cup\left[\frac{2}{3},1\right]\end{cases}$$

That is each buyer is given just enough information so they buy some TV. Moreover, in equilibrium, the buyer never searches or $Q=[0,1]$.

## 3.2 General Theory

We now discuss the general model and demonstrate that all equilibrium disclosure policies are characterized by a local optimality condition. Given any disclosure policy, consider the set of posterior beliefs that could potentially be reached by a buyer of any prior who updates according to that policy. Call this is the *support* of the policy $K$ and denote it by $\operatorname{supp}(K)$.[7] A disclosure policy $K$ is *locally optimal* iff it generates more profits than any other policy that has support within an $\varepsilon$-ball of $K$'s support, i.e. there is some $\varepsilon>0$ where $\int_{\Delta S}\pi(q)\,K(p,dq)\geq\int_{\Delta S}\pi(q)\,L(p,dq)$ for all $L$ such that $\operatorname{supp}(L)\subset B_\varepsilon(\operatorname{supp}(K))$ along with the usual tie-breaking rule. In other words, a monopolist seller who uses a locally optimal policy has no incentive to deviate if she were able to slightly perturb the support of her policy. All equilibrium policies must be locally optimal.

[7] Formally, $\operatorname{supp}(K):=\bigcup_{p\in\Delta S}\operatorname{supp}(K(p))$ where $\operatorname{supp}(K(p))$ is the support of the measure $K(p)$.

Moreover, any disclosure policy that is locally optimal is also an equilibrium policy when the search cost becomes sufficiently small.

**Theorem 1.** *Any equilibrium policy is locally optimal. Conversely, for any locally optimal policy, there is some $\bar{c}$ such that it is an equilibrium policy for all $c \leq \bar{c}$.*

Theorem 1 provides a necessary and sufficient condition for all equilibrium policies. First, consider an equilibrium where the seller sends a buyer to a posterior $q$ in the support of the equilibrium policy $K$. Since all sellers are symmetric, the buyer with posterior $q$ will not get any additional information by continuing to search (if he did, then the original seller would have optimally disclosed that information in the first place). Hence, in equilibrium, any buyer with a posterior $q$ in $K$'s support will accept and not search. Now, since the search cost is strictly positive, a buyer with a posterior "close" to $q$ will also accept and not search. Since $K$ is optimal given the acceptance set, it must also be optimal given all beliefs sufficiently "close" to the support. In other words, small deviations around the support of $K$ cannot be profitable so $K$ is locally optimal.

For the converse, suppose that $K$ is locally optimal. As we lower the search cost $c$, buyers who originally chose to accept may now choose to continue searching. Thus, as $c$ decreases, the corresponding acceptance set $Q_c$ shrinks and converges to the support of $K$ . Hence, we can find a $\bar{c}$ small enough such $Q_{\bar{c}}$ is completely contained in the $\varepsilon$-ball around the support. Local optimality then implies that $K$ is optimal given $Q_c$ for all $c \leq \bar{c}$ so $K$ is an equilibrium policy. Note that the number of equilibria increases as search costs decrease.

Theorem 1 is useful for characterizing the equilibria of the game. Since local optimality makes no reference to the buyer's acceptance set, we can check whether any given disclosure policy is an equilibrium policy simply by looking at local deviations. For example, in the single-product example, if $K$ discloses more than the monopoly level, one can that see there is a local deviation. Conversely, it allows us to verify that the proposed equilibrium (2) is actually an equilibrium. Even though a seller would like to put weight on the posterior $q = \frac{1}{3}$, they cannot do this since it is not a local deviation.

One may wonder when the Diamond paradox attains more generally. This result holds in both the single-product and horizontal differentiation examples; in both these cases, for each product, there is a degenerate posterior such that a buyer with that posterior will buy that product for sure. For instance, in the horizontal differentiation example, a buyer with $q = 1$ will buy the 3D TV while a buyer with $q = 0$ will buy the 4K TV. We conjecture that this property is sufficient to ensure that the monopoly policy is the only equilibrium policy.

**Conjecture 1.** *Suppose that for each product, there is a degenerate belief $q = \delta_s$ such that a buyer with $q$ will buy that product. Then any equilibrium policy is a monopoly policy.*

In other words, if products are "sparse" relative to relevant states, then the Diamond paradox holds. As product offerings become increasingly rich as in the vertical differentiation example, non-monopolistic equilibria may arise. This suggests that the Diamond paradox is a robust finding when buyers' priors are observed.

# 4 Anonymous Buyers

In this section, we consider the case where buyers are anonymous, so sellers cannot provide different information to buyers with different priors. In this section, we suppose that buyers start with the same prior and show that if as search costs become small, buyers tend to obtain their full information payoff. Even though sellers have correct beliefs about the buyer's prior in equilibrium, the fact that priors cannot be observed dramatically changes the nature of equilibrium behavior.

**Model.** As before, sellers choose signal structures; a buyer sees the realization of a signal and chooses whether to buy, exit or pay the search cost and continue. Initially, it is common knowledge that a buyer's prior is $p \in \operatorname{int}(\Delta S)$. Suppose the seller chooses a signal structure on some space $\Theta$ that is independent of all previous signals received by the buyer. Any potential buyer with prior $r$ can also observe the signal and learn about the state. Let $q(\theta)$ be the posterior of a buyer with prior $p$ after observing a signal $\theta \in \Theta$, and let $q(\theta; r)$ denote the posterior of a buyer with prior $r$ who observes the same signal. The posteriors of the two buyers are related as follows

$$q(\theta; r) = \phi_r(q(\theta))$$

where $\phi_r : \Delta S \to \Delta S$ is a mapping from the posteriors of buyer $p$ to the posteriors of buyer $r$ satisfying

$$[\phi_r(q)](s) := \frac{q(s)\frac{r(s)}{p(s)}}{\sum_{s'} q(s')\frac{r(s')}{p(s')}}$$

Intuitively, for any two buyers, the proportionality of their likelihood ratios for any two states remains constant when we update via Bayes' rule. If buyer $r$ starts off twice as optimistic as buyer $p$ (i.e. $r$ has double the likelihood ratio than $p$), then $r$ will remain twice as optimistic as $p$ after any signal realization.

Since this mapping $\phi_r$ is independent of the signal structure, sellers can simply choose a *signal policy* $\mu$ that is a distribution of posteriors for an agent with prior $p$ that satisfies $\int_{\Delta S} q\, \mu(dq) = p$. Given any signal policy $\mu$, the distribution of posteriors of a buyer with prior $r$ is completely specified by the mapping $\phi_r$ and we denote this by $\mu_r$. A signal policy $\mu$ is *optimal given* the buyer's acceptance set $Q$ iff for all possible signal policies $\nu$

$$\int_Q \pi(q)\,\mu(dq) \geq \int_Q \pi(q)\,\nu(dq).$$

and the usual tie-breaking rule applies. Let $\bar{\mu}$ denote the *full information signal policy* which has full support on degenerate posteriors.

**Symmetric equilibrium.** If all sellers use signal policy $\mu$, a buyer with posterior $q$ who continues to search obtains continuation utility

$$V_\mu(q) := -c + \int_{\Delta S} \max\left\{\max_{u \in U} r \cdot u,\ V_\mu(r)\right\} \mu_q(dr)$$

A signal policy and acceptance set $(\mu, Q)$ is an *equilibrium* iff $\mu$ is optimal given

$$Q = \left\{ q \in \Delta S \;\middle|\; \max_{u \in U} q \cdot u \geq V_\mu(q) \right\}.$$

**Single product example (cont).** Let us return to the single product example but now with unobservable priors. Again, suppose there are two states $\{L, H\}$ and $q = \Pr(H)$. The seller believes the buyer's prior is $p = \frac{1}{2}$, so a monopolist would give no information. Let $b > \frac{1}{2}$ and consider a buyer strategy as follows: buy the TV if $q \in [b, 1]$, buy no TV if $q \in [0, 2(1-b)b]$ and continue to search otherwise, as shown in Figure 4. Given the seller's belief that the buyer starts out with prior $p = \frac{1}{2}$, the optimal policy is to send the buyer to posterior $b$ with probability $\frac{1}{2b}$ and posterior 0 with probability $1 - \frac{1}{2b}$. Note that

$$\phi_r(b) = \frac{br}{br + (1-b)(1-r)}$$
$$\phi_r(0) = 0$$

Thus, a buyer with prior $r$ will end up at posterior $\phi_r(b)$ with probability $\frac{r}{\phi_r(b)}$ and 0 with probability $1 - \frac{r}{\phi_r(b)}$. If $c \leq 3 - 2\sqrt{2}$, we claim that there is an equilibrium where

$$b = \frac{1}{4}\left[3 - c + \sqrt{c^2 - 6c + 1}\right] \tag{3}$$

and each buyer visits a single seller. To verify this, consider a buyer arriving with prior $p = \frac{1}{2}$. If his posterior is 0, he has no incentive to search again; if he has posterior $q = b$, then (3) implies that the buyer is indifferent between buying and searching,

$$2b - 1 = \frac{b}{\phi_b(b)}\left(2\phi_b(b) - 1\right) - c$$

Hence, a buyer with a good signal does not search in equilibrium, however if the seller discloses any less information, this will be "punished" by the buyer searching. This equilibrium illustrates an interesting property: as $c \to 0$, sellers disclose full information, i.e. $b \to 1$.[8]

The seller's equilibrium policy in this example is much more informative than the monopoly policy. When buyers are tracked, a buyer who receives information from one seller obtains no useful information from a second. However, when buyers are anonymous, the buyer can reject the first offer, pretend to be uninformed and receive more information from the next seller. Even though sellers know that all buyers have belief $p = \frac{1}{2}$ in equilibrium, the possibility of searching again enables competition to take effect.

In the example above, as the search cost becomes small, sellers release full information in equilibrium. In general, this is true as long as equilibrium policies are *generic*, that is, if a buyer with any (non-degenerate) belief learns something from the signal.[9] Under generic policies, the only equilibrium buyer payoff is equivalent to that of full information as search costs become small.

**Theorem 2.** *If equilibrium policies are generic, then $V_\mu(p) \to V_{\bar{\mu}}(p)$ as $c \to 0$.*

Generic policies imply that a buyer can always learn something more from each additional seller he visits. Hence, as search costs become small, buyers can afford to keep on searching until they

[8] In this example, the monopoly disclosure policy is zero information, and this is also an equilibrium. In addition, there is an equilibrium of the type discussed above which converges to the monopoly policy. However, these equilibria disappear if $p < \frac{1}{2}$.

[9] Formally, $\mu$ is *generic* iff $\mu_q \neq \delta_q$ for all $q \neq \delta_s$.

**Figure 4: Single Product**

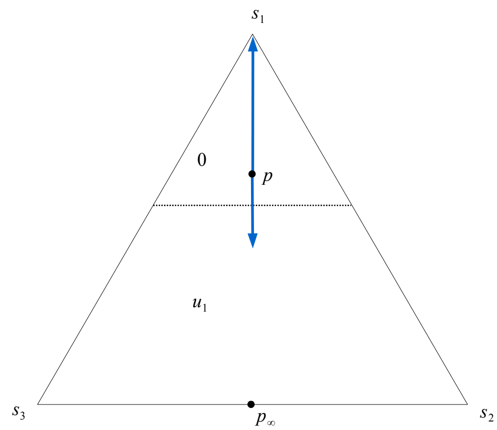
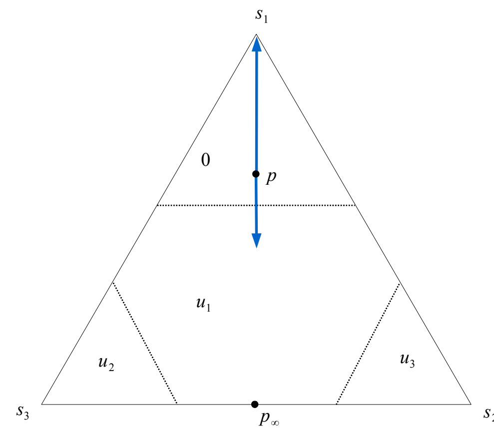



**Figure 5: Multiple Products**




approach full information. Generic policies are dense in the set of all signal policies.[10] In fact, if there are only two states, then any non-degenerate signal policy is generic. It thus follows directly from Theorem 2 that as long as buyers do not start with a prior that induces them to immediately buy a product, then all equilibrium policies approach that of full information with two states.

For the single product case, we have a stronger result. Suppose the buyer begins with a prior that does not induce him to immediately buy the product. Then as search costs become small, the buyer's equilibrium payoff approaches that of full information.

**Proposition 2.** *Suppose there is a single product and the buyer's initial prior $p$ does not induce him to buy. Then $V_\mu(p) \to V_{\bar{\mu}}(p)$ as $c \to 0$.*

Proposition 2 shows that if there is a single product then buyers' equilibrium payoffs approach that of full information as search costs become small, even if policies are not generic. The lack of genericity means that a buyer may never fully learn the true state even if they always search; however, because there is a single product, they can achieve the same payoff as that of full information.

To understand this result, suppose there are three states and consider the equilibrium policy in Figure 4. In this figure, the buyer starts with prior $p$ and the regions illustrate the buyer's action in a monopoly model. The signal illustrated is non-generic since it does not help the buyer distinguish between states $s_2$ and $s_3$. However, with a single good, the buyer achieves his full information payoff as the search cost becomes small. Intuitively, while the signal does not distinguish between all states, enough observations will convince the buyer that he should certainly purchase the good, meaning he obtains the same payoff as if they had full information. Note that this may not be true with multiple products (Figure 5). If $u_1$ is the most profitable product, the seller may only provide information along one dimension meaning that, even as search costs become arbitrarily small, buyers will never be induced to buy either $u_2$ or $u_3$ and will cot obtain their full information payoff. This example relies on the indifference line between $u_1$ and 0 being perpendicular to the non-generic signal, so we conjecture that such policies are non-generic in payoff space.

To summarize, when seller track buyers, a seller can discriminate between new and old buyers, allowing them to implicitly collude. Without tracking a seller must treat all buyers alike and markets work as one would hope: buyers receive their full information payoff with arbitrarily small costs.

[10] In fact, non-generic policies must have support in a lower-dimensional subspace containing the prior $p$.

# 5 The Choice of "Do Not Track"

We now endogenize the decision to become anonymous, supposing that buyers can pay a small cost to activate the "Do Not Track" mode on their browser. We first argue that buyers who become anonymous exert a positive externality on other buyers. We then observe that the free-riding is so severe that there is no equilibrium in which all buyers choose to become anonymous.

Consider a continuum of both buyers and sellers and suppose all agents start with prior $p_0$. At time $t = 0$, a buyer can choose to pay cost $k$ in order to "become anonymous". After this decision, buyers can be described by two dimensions: their prior $p$ before meeting a seller and their type $\theta \in \{A, T\}$, which describes whether they are anonymous ($A$) or tracked ($T$). When revealing information, a seller must treat an anonymous agent $(A, p)$ in the same way as an uninformed tracked agent $(T, p_0)$. The seller can, however, give different signals to tracked agents with different priors.

One can understand much of the economics by considering a simple $2 \times 2$ game where agents choose whether or not to be anonymous. The following table provides an illustration of this in the single-product example. This shows the pure strategy payoffs for the single product example with initial prior $p_0 = \frac{1}{3}$ and $c = 0.05$.

| | | Other Buyers | |
|---|---|---|---|
| | | Anonymous | Tracked |
| Buyer $i$ | Anonymous | $0.9 - k$ | $0.15 - k$ |
| | Tracked | 0.9 | 0 |

First, suppose all other buyers are tracked, so we are in the model from Section 3. If buyer $i$ is also tracked then a buyer receives a single "monopoly" signal from a seller and immediately buys since he knows a second seller will refuse to provide more information. In the single-product example, this means the buyer receives 0 utility. If buyer $i$ becomes anonymous then he can get the initial "monopoly" signal, delete the seller's cookie and then receive "monopoly" signals from subsequent sellers, paying cost $c$ for each search. Next, suppose all other buyers are anonymous, so we are in the model from Section 4. If buyer $i$ is tracked then he receives a single "competitive" signal that is more informative than the monopoly signal. Since he is tracked, no sellers provide any further information to him. If buyer $i$ becomes anonymous he has the option to receive a sequence of "competitive" signals. However, on the equilibrium path, the buyer will purchase at the first opportunity.

The first point to observe is that when other buyers become anonymous, this exerts a positive externality on buyer $i$. Intuitively, when faced with lots of anonymous buyers, competition is more effective and seller provide more information in equilibrium. This benefits any buyer, independent of whether they are anonymous or tracked.

Second, observe that no matter how small the cost $k$, there is no equilibrium where everyone is anonymous. To illustrate, consider the parameters in the above table. If $k > 0.15$ there is a unique equilibrium in which everyone is tracked. If $k \in (0, 0.15)$ then there is no pure strategy equilibrium. When all other buyers are anonymous then sellers provide enough information to prevent the searching; hence buyer $i$ has no incentive to become anonymous himself.

If we wish to characterize the mixed strategy equilibrium played by the buyers, we have to move beyond the heuristic $2 \times 2$ game. Since this is a difficult problem in general, let us consider the single-product example.

**Single product example (cont).** Suppose the seller has a single product to sell and that $p_0 < 1/2$, so that a buyer does not buy in the absence of information.[11] Suppose proportion $\alpha$ of buyers are initially anonymous while $1-\alpha$ are tracked. If a seller knows they are facing a tracked buyer, then each seller provides the monopoly level of information, as in Section 3. If a seller faces a buyer without tracking information, this may mean that the buyer is a new tracked buyer with prior $p_0$, or an anonymous buyer who has received a number of signals.

First, observe that the seller will use a "perfect bad news signal" with support on 0 or $p_1$. In other words, a buyer under this signal policy will end up at 0 or some posterior $p_1 > p_0$. To see this notice that a buyer purchases if his posterior is sufficiently high, so the seller's profit is given by a mixture of step functions.[12] None of these buyers will buy in the absence of information - the new buyers have $p_0 < 1/2$, while the old buyers would search again - meaning that $\pi(p_0) = 0$ and the convex hull touches the payoff function at 0 and some $p_1$. Using Bayes's rule define $\{p_0, p_1, p_2, \ldots\}$ as the sequence of beliefs that arises from repeatedly observing the good signal, so one more signal means $p_n \to \{0, p_{n+1}\}$.

Second, observe that the firm provides at least the monopoly level of information. If this were not the case, a firm could provide "two signals", taking $p_n \to \{0, p_{n+2}\}$. This would raise the seller's sales from anonymous buyers, and may even lead to sales from tracked buyers. It follows that $p_1 \geq 1/2$, and that a tracked buyer receives at most one signal.

Third, we can characterize the seller's optimal disclosure policy for buyers without tracking information in a symmetric equilibrium. Suppose anonymous buyers purchase once their posterior exceeds $q_A$, and denote $N := \min\{n : p_n \geq q_A\}$ as the minimal number of signals required for a buyer to purchase. A seller thus faces $1-\alpha$ tracked buyers with prior $p_0$, $\alpha$ anonymous buyers with prior $p_0$, $\alpha \frac{p_0}{p_1}$ anonymous buyers with prior $p_1$, and so on. Since this is a mixture of step functions, it follows that the optimal disclosure policy will be at a corner, meaning one type of buyer is indifferent between purchasing and not. Equilibrium can thus of two kinds: (a) the seller ignores the anonymous buyers and sets $p_1 = 1/2$, or (b) the seller chooses $p_1$ so that a tracked buyer with belief $p_N$ is indifferent between buying and continuing, meaning we have the equality $q_A = p_N$.

We can now establish what happens as the number of anonymous buyers $\alpha$ increases. When $\alpha = 0$, the seller chooses the monopoly information level, $p_1 = 1/2$ as in Section 3. For higher $\alpha$, the firms choose $p_1 > 1/2$. Consider the most informative equilibrium (i.e. the equilibrium with the largest $p_1$). On the equilibrium path, tracked buyers purchase at the first opportunity, while anonymous buyers obtain $N$ signals before purchasing. In the most informative equilibrium, the seller must give up on the anonymous buyers any simply sell to the tracked buyers,

$$(1-\alpha)\frac{p_0}{1/2} \leq (1-\alpha)\frac{p_0}{p_1} + \alpha\frac{p_0}{p_N}$$

An increase in $\alpha$ slackens this constraint and allows us to support a higher $p_1$. This means that, in the most informative equilibrium, sellers disclose more as $\alpha$ rises, and any one buyer is better off when when other buyers are anonymous.

[11] Without this assumption, zero information is always an equilibrium (see Proposition 2).

[12] We formally show this, and the other results in this section, in Appendix 5.

# Appendix

## A. Observed priors

Let $\mathcal{F}$ be the Borel algebra on $\Delta S$ and define an *acceptance set* as some $Q \in \mathcal{F}$. We say $\mu$ is *consistent* with $p \in \Delta S$ iff $\int_{\Delta S} q \; \mu(dq) = p$ where $\mu$ is some distribution on $\Delta S$. A *disclosure policy* is a continuous Markov kernel $K$ such that $K(p)$ is consistent with $p$ for all $p \in \Delta S$. Let $\mathcal{K}$ be the set of all disclosure policies. Define

$$E := \{q \in \Delta S \mid q(s) = 1 \text{ for some } s \in S\}$$
$$E_K := \left\{q \in \Delta S \;\middle|\; K(q) = \delta_{\{q\}}\right\}$$

for $K \in \mathcal{K}$. $E$ is the set of degenerate beliefs and $E_K$ is the set of priors for which $K$ discloses no information. Lemma A1 below shows that the restriction to continuous (in weak convergence) kernels is without loss of generality.

**Lemma** (A1)**.** *If $K \in \mathcal{K}$ is optimal given $Q \in \mathcal{F}$, then for all $p \in \Delta S$,*

$$\int_{\Delta S} \pi(q) \, K(p, dq) \geq \int_{\Delta S} \pi(q) \, \nu(dq)$$

*for all $\nu$ consistent with $p$.*

*Proof.* Suppose there is some $\nu$ consistent with some $p \in \Delta S$ such that

$$\int_{\Delta S} \pi(q) \, K(p, dq) < \int_{\Delta S} \pi(q) \, \nu(dq)$$

For each $q \in \Delta S$, define $r(q) \in \text{bd}(\Delta S)$ and $\alpha(q) \in [0,1]$ such that

$$q = \alpha(q) \, p + (1 - \alpha(q)) \, r(q)$$

where $\alpha(q) = 1$ for $q = p$ and $\alpha(q) = 0$ for $q = r(q)$. Note that both $r$ and $\alpha$ are continuous. Define the kernel $L$ such that

$$L(q) := \alpha(q) \, \nu + (1 - \alpha(q)) \, \delta_{\{r(q)\}}$$

Now,

$$\int_{\Delta S} r \; L(q, dr) = \alpha(q) \int_{\Delta S} r \; \nu(dr) + (1 - \alpha(q)) \, r(q) = q$$

so $L(q)$ is consistent with $q$ for all $q \in \Delta S$. For any bounded and continuous $\psi : \Delta S \to \mathbb{R}$,

$$\int_{\Delta S} \psi(r) \, L(q, dr) = \alpha(q) \int_{\Delta S} \psi(r) \, \nu(dr) + (1 - \alpha(q)) \, \psi(r(q))$$

which is continuous in $q$ so $L \in \mathcal{K}$. However,

$$\int_{\Delta S} \pi(q) \, L(p, dq) = \int_{\Delta S} \pi(q) \, \nu(dq) > \int_{\Delta S} \pi(q) \, K(p, dq)$$

contradicting the optimality of $K$. $\square$

**Lemma** (A2)**.** *Let $K \in \mathcal{K}$ be optimal given $Q \in \mathcal{F}$. Then*

(1) $E \subset E_K \in \mathcal{F}$.

(2) $K(p, E_K) = 1$ *for all* $p \in \Delta S$.

(3) $V_K(q) = -c + \int_{\Delta S} (\max_{u \in U} r \cdot u) K(q, dr)$ *and is continuous*

(4) *supp* $(K) = E_K$.

*Proof.* Let $K \in \mathcal{K}$ be optimal given $Q \in \mathcal{F}$. We prove the lemma in order.

(1) Note that $K(q) = \delta_{\{q\}}$ iff $K(q, \{q\}) = 1$. Hence,

$$E_K = \{q \in \Delta S \mid K(q, \{q\}) = 1\} = \left\{ q \in \Delta S \;\middle|\; \int_{\Delta S} \mathbf{1}_{\{q\}}(r) K(q, dr) = 1 \right\}$$

By Proposition I.6.9 of Cinlar (Çinlar, 2011), $E_K \in \mathcal{F}$. To see that $E \subset E_K$, let $q = 1_s \in E$ for some $s \in S$. Since $\mu := K(q)$ is consistent with $q$,

$$1 = q(s) = \int_{\Delta S} r(s) \mu(dr)$$

so $r(s) = 1$ $\mu$-a.s.. Hence, $r = 1_s = q$ $\mu$-a.s. so $K(q, \{q\}) = 1$ and $q \in E_K$ as desired.

(2) Note that since $K$ is optimal, for all $q \in \Delta S$

$$\int_Q \pi(r) K(q, dr) \geq \int_Q \pi(r) \delta_{\{q\}}(dr) = \pi(q) \mathbf{1}_Q(q)$$

Define

$$F := \left\{ q \in \Delta S \;\middle|\; \int_Q \pi(r) K(q, dr) > \pi(q) \mathbf{1}_Q(q) \right\}$$

and suppose $K(p, F) > 0$ for some $p \in \Delta S$. Let $\mu := K(p)$ and define the measure $\nu$ such that

$$\nu(B) := \mu(B \cap F^c) + \int_F K(q, B) \mu(dq)$$

for all $B \in \mathcal{F}$. Note that

$$\begin{aligned}
\int_{\Delta S} q \; \nu(dq) &= \int_{F^c} q \; \mu(dq) + \int_F \left( \int_{\Delta S} r \; K(q, dr) \right) \mu(dq) \\
&= \int_{F^c} q \; \mu(dq) + \int_F q \; \mu(dq) = p
\end{aligned}$$

so $\nu$ is consistent with $p$. However,

$$\begin{aligned}
\int_Q \pi(q) \nu(dq) &= \int_{Q \cap F^c} \pi(q) \mu(dq) + \int_F \left( \int_Q \pi(r) K(q, dr) \right) \mu(dq) \\
&> \int_{Q \cap F^c} \pi(q) \mu(dq) + \int_F \pi(q) \mathbf{1}_Q(q) \mu(dq) = \int_Q \pi(q) \mu(dq)
\end{aligned}$$

contradicting the optimality of $K$. Hence, $K(p,F)=0$ so

$$\int_Q \pi(r)\,K(q,dr)=\pi(q)\,\mathbf{1}_Q(q)=\int_Q \pi(r)\,\delta_{\{q\}}(dr)$$

$K_p$-a.s. for all $p\in\Delta S$. By the tie-breaking rule from the definition of optimality, $K(q)=\delta_{\{q\}}$ $K_p$-a.s..

(3) Consider $r\in E_K$ so

$$V_K(r)=-c+\max\left\{\max_{u\in U} r\cdot u,\ V_K(r)\right\}$$

Note that if $V_K(r)>\max_{u\in U} r\cdot u$, then $V_K(r)=-c+V_K(r)$ contradicting $c>0$. Thus, $\max_{u\in U} r\cdot u\geq V_K(r)$ for all $r\in E_K$. From (2), we have

$$\begin{aligned}V_K(q)&=-c+\int_{E_K}\max\left\{\max_{u\in U} r\cdot u,\ V_K(r)\right\}K(q,dr)\\&=-c+\int_{\Delta S}\left(\max_{u\in U} r\cdot u\right)K(q,dr)\end{aligned}$$

Continuity follows from the continuity of support functions.

(4) First note that $E_K\subset\operatorname{supp}(K)$ trivially. We prove that $E_K$ is closed. Consider $p_k\to p$ such that $p_k\in E_K$ so $K(p_k)=\delta_{\{p_k\}}$. Note that $\delta_{\{p_k\}}\to K(p)$ so for any continuous $\psi:\Delta S\to[0,1]$

$$\psi(p)=\lim_{p_k\to p}\psi(p_k)=\int_{\Delta S}\psi(q)\,K(p,dq)$$

Consider continuous $\psi^i:\Delta S\to[0,1]$ such that $\psi^i(p)=1$ and $\psi^i\to 1_{\{p\}}$. By dominated convergence,

$$K(p,\{p\})=\lim_i\int_{\Delta S}\psi^i(q)\,K(p,dq)=\lim_i\psi^i(p)=1$$

so $p\in E_K$ as desired. Thus, $E_K^c$ is open, and note that if $E_K^c\cap\operatorname{supp}(K(p))\neq\emptyset$, then by Theorem 12.14 of Aliprantis and Border (Aliprantis and Border, 2006), $K(p,E_K^c\cap\operatorname{supp}(K(p)))>0$ contradicting (2). Hence $\operatorname{supp}(K(p))\subset E_K$ for all $p\in\Delta S$ so $\operatorname{supp}(K)\subset E_K$ as desired.

□

**Lemma** (A3)**.** *Suppose $(K,Q)$ is an equilibrium for some $\bar c$. Then $(K,Q_c)$ is also an equilibrium for any $c\leq\bar c$ where*

$$Q_c:=\left\{q\in\Delta S\;\middle|\;\max_{u\in U} q\cdot u\geq -c+\int_{\Delta S}\left(\max_{u\in U} r\cdot u\right)K(q,dr)\right\}$$

*Proof.* Let $(K,Q)$ be an equilibrium for some $\bar c$. By Lemma A2, $V_K^{\bar c}(q)=-\bar c+\int_{\Delta S}\left(\max_{u\in U} r\cdot u\right)K(q,dr)$ so by the definition of equilibrium,

$$Q=\left\{q\in\Delta S\;\middle|\;\max_{u\in U} q\cdot u\geq V_K^{\bar c}(q)\right\}=Q_{\bar c}$$

Let $c\leq\bar c$ so $Q_c\subset Q_{\bar c}=Q$. If $q\in E_K$, then $q\in Q_c$ so $\operatorname{supp}(K)=E_K\subset Q_c\subset Q$ for all $c\leq\bar c$.

Since $K$ is optimal given $Q$, for any $L \in \mathcal{K}$

$$\int_{Q^c} \pi(q) K(p, dq) = \int_Q \pi(q) K(p, dq) \geq \int_Q \pi(q) L(p, dq) \geq \int_{Q^c} \pi(q) L(p, dq)$$

If $\int_{Q^c} \pi(q) K(p, dq) = \int_{Q^c} \pi(q) L(p, dq)$ for some $L(p)$ that is strictly less informative, then $\int_Q \pi(q) K(p, dq) = \int_Q \pi(q) L(p, dq)$ contradicting the optimality of $K$ given $Q$. Hence, $K$ is optimal given $Q_c$ and $(K, Q_c)$ is an equilibrium by Lemma A2. $\square$

For any $F \in \mathcal{F}$ and $\varepsilon \geq 0$, define

$$B_\varepsilon(F) := \{q \in \Delta S \mid |q - r| \leq \varepsilon \text{ for some } r \in F\}$$

Formally, $K \in \mathcal{K}$ is *locally optimal* iff there is some $\varepsilon > 0$ such that

$$\int_{\Delta S} \pi(q) K(p, dq) \geq \int_{\Delta S} \pi(q) L(p, dq)$$

for all $L \in \mathcal{K}$ such that $\mathrm{supp}(L) \subset B_\varepsilon(\mathrm{supp}(K))$ along with the usual tie-breaking rule, that is, if the above is satisfied with equality for $L(p)$, then $L(p)$ cannot be strictly less Blackwell informative than $K(p)$.

**Lemma** (A4)**.** *If $(K, Q)$ is an equilibrium, then $\mathrm{supp}(K) \subset B_\varepsilon(\mathrm{supp}(K)) \subset Q$ for some $\varepsilon > 0$.*

*Proof.* Let $(K, Q)$ be an equilibrium. From Lemma A2, if $q \in \mathrm{supp}(K) = E_K$, then $V_K(q) = -c + \max_{u \in U} q \cdot u$ so $q \in Q$ as desired. Define

$$\begin{aligned} \psi(r) &:= \max_{u \in U} r \cdot u - V_K(r) \\ I &:= \{r \in \Delta S \mid \psi(r) = 0\} \subset Q \\ \varepsilon_0 &:= \inf_{(q,r) \in E_K \times I} |q - r| \end{aligned}$$

Note that $\psi$ is continuous from Lemma A2. Suppose $\varepsilon_0 = 0$ so there are $(q_k, r_k) \in E_K \times I$ such that $|q_k - r_k| \to 0$. By completeness, they must converge but

$$\psi(q_k) = c > 0 = \psi(r_k)$$

violating the continuity of $\psi$. Hence, $\varepsilon_0 > 0$. Set $\varepsilon := \frac{\varepsilon_0}{2}$ and suppose $q \in B_\varepsilon(E_K) \backslash Q$. Hence $\psi(q) < 0$ but $|q - r| < \varepsilon_0$ for some $r \in E_K$ where $\psi(r) = c > 0$. By the continuity of $\psi$ again, we can find some $q' \in \Delta S$ such that $|q' - r| < \varepsilon_0$ and $q' \in I$ contradicting the definition of $\varepsilon_0$. Hence, $B_\varepsilon(\mathrm{supp}(K)) = B_\varepsilon(E_K) \subset Q$ as desired. $\square$

**Theorem** (A5)**.** *If $K$ is an equilibrium policy, then it is locally optimal. Conversely, if $K$ is locally optimal, then there is some $\bar{c}$ such that $K$ is an equilibrium policy for all $c \leq \bar{c}$.*

*Proof.* First, let $(K, Q)$ be an equilibrium so from Lemma A4, let $\varepsilon > 0$ and $B_\varepsilon(\mathrm{supp}(K)) \subset Q$. Consider any $L \in \mathcal{K}$ such that $\mathrm{supp}(L) \subset B_\varepsilon(\mathrm{supp}(K)) \subset Q$. By the optimality of $K$ with respect to $Q$,

$$\int_{\Delta S} \pi(q) K(p, dq) = \int_Q \pi(q) K(p, dq) \geq \int_Q \pi(q) L(p, dq) = \int_{\Delta S} \pi(q) L(p, dq)$$

as $L(p,Q)=1$ for all $p\in\Delta S$. Note that the tie-breaking property from the optimality of $K$ given $Q$ implies the tie-breaking property of local optimality.

Now, suppose $K$ is locally optimal so there is some $\varepsilon>0$ such that

$$\int_{\Delta S}\pi(q)\,K(p,dq)\geq\int_{\Delta S}\pi(q)\,L(p,dq)$$

for all $L\in\mathcal{K}$ such that $\operatorname{supp}(L)\subset B_{\varepsilon}(\operatorname{supp}(K))$ along with the tie-breaking rule. Define $Q_c$ as in Lemma A3 and recall that $\operatorname{supp}(K)=E_K\subset Q_c$ since $c>0$. Consider $c_k\to 0$ and suppose there are $q_k\in Q_{c_k}$ such that $q_k\notin B_{\varepsilon}(\operatorname{supp}(K))$. Hence, without loss of generality, we can assume that $q_k\to q$. Since $K$ is continuous,

$$\begin{gathered}\lim_k\left(\max_{u\in U}q_k\cdot u\right)\geq\lim_k\left(-c_k+\int_{\Delta S}\left(\max_{u\in U}r\cdot u\right)K(q_k,dr)\right)\\ \max_{u\in U}q\cdot u\geq\int_{\Delta S}\left(\max_{u\in U}r\cdot u\right)K(q,dr)\geq\max_{u\in U}q\cdot u\end{gathered}$$

so $\int_{\Delta S}(\max_{u\in U}r\cdot u)\,K(q,dr)=\max_{u\in U}q\cdot u$. Since support functions are convex, this implies $K(q,R_q)=1$ where $R_q:=\{r\in\Delta S\mid u^*(q)=r\cdot u\}$. By the buyer tie-breaking rule, this implies $\pi(r)=\pi(q)$ for all $r\in R_q$. Note that if $q\notin\operatorname{supp}(K)$ then we can find some $L(q)$ such that $\operatorname{supp}(L(q))\subset B_{\varepsilon}(\operatorname{supp}(K))$, $L(q)$ is strictly less informative than $K(q)$ and

$$\int_{\Delta S}\pi(q)\,K(p,dq)=\pi(q)=\int_{\Delta S}\pi(q)\,L(p,dq)$$

contradicting local optimality. Hence, $q\in\operatorname{supp}(K)$ so we can find some $\bar{c}>0$ such that $\operatorname{supp}(K)\subset Q^{\bar{c}}\subset B_{\varepsilon}(\operatorname{supp}(K))$. Suppose there is some $L\in\mathcal{K}$ such that

$$\int_{Q^{\bar{c}}}\pi(q)\,L(p,dq)\geq\int_{Q^{\bar{c}}}\pi(q)\,K(p,dq)$$

with at least one inequality strict for some $p\in\Delta S$. We can assume without loss of generality that $L(p,Q^{\bar{c}})=1$ so $\operatorname{supp}(L)\subset Q^{\bar{c}}\subset B_{\varepsilon}(\operatorname{supp}(K))$. By local optimality, for all $p\in\Delta S$

$$\int_{Q^{\bar{c}}}\pi(q)\,K(p,dq)=\int_{\Delta S}\pi(q)\,K(p,dq)\geq\int_{\Delta S}\pi(q)\,L(p,dq)\geq\int_{Q^{\bar{c}}}\pi(q)\,L(p,dq)$$

yielding a contradiction. As usual, the tie-breaking property from local optimality implies the tie-breaking property of the optimality of $K$ given $Q^{\bar{c}}$. Hence, $(K,Q^{\bar{c}})$ is an equilibrium from Lemma A2. The rest follows from Lemma A3. □

**Lemma** (A6)**.** *Let $(K,Q)$ be an equilibrium and $K^*$ be a monopoly policy. If supp $(K^*)\subset$ supp $(K)$, then $K$ is a monopoly policy.*

*Proof.* Let $(K,Q)$ be an equilibrium and $K^*$ be a monopoly policy. Suppose $\operatorname{supp}(K^*)\subset\operatorname{supp}(K)\subset$

$Q$ by Lemma A4. Hence $K(p,Q) = K^*(p,Q) = 1$ for all $p \in \Delta S$ so by optimality,

$$\begin{aligned}\int_Q \pi(q) K(p,dq) &\geq \int_Q \pi(q) K^*(p,dq) = \int_{\Delta S} \pi(q) K^*(p,dq) \\ &\geq \int_{\Delta S} \pi(q) K(p,dq) = \int_Q \pi(q) K(p,dq)\end{aligned}$$

Thus

$$\int_{\Delta S} \pi(q) K(p,dq) = \int_{\Delta S} \pi(q) K^*(p,dq) \geq \int_{\Delta S} \pi(q) L(p,dq)$$

for all $L \in \mathcal{K}$.

Suppose the above weak inequality holds with equality for some $L(p)$. Define

$$F := \left\{ q \in \Delta S \;\middle|\; \int_{\Delta S} \pi(r) K^*(q,dr) > \pi(q) \right\}$$

and suppose $L(p,F) > 0$. Let $\mu := L(p)$ and define the measure $\nu$ such that

$$\nu(B) := \mu(B \cap F^c) + \int_F K^*(q,B)\,\mu(dq)$$

for all $B \in \mathcal{F}$. Note that

$$\begin{aligned}\int_{\Delta S} q\;\nu(dq) &= \int_{F^c} q\;\mu(dq) + \int_F \left( \int_{\Delta S} r\; K^*(q,dr) \right) \mu(dq) \\ &= \int_{F^c} q\;\mu(dq) + \int_F q\;\mu(dq) = p\end{aligned}$$

so $\nu$ is consistent with $p$. However,

$$\begin{aligned}\int_{\Delta S} \pi(q)\,\nu(dq) &= \int_{F^c} \pi(q)\,\mu(dq) + \int_F \left( \int_{\Delta S} \pi(r) K^*(q,dr) \right) \mu(dq) \\ &> \int_{F^c} \pi(q)\,\mu(dq) + \int_F \pi(q)\,\mu(dq) = \int_{\Delta S} \pi(q) K^*(p,dq)\end{aligned}$$

contradicting the optimality of $K^*$. Hence, $L(p,F) = 0$. Now, $q \in F^c$ iff $\int_{\Delta S} \pi(r) K^*(q,dr) = \pi(q)$ iff $K^*(q) = \delta_q$ so $F^c = \mathrm{supp}(K^*)$ by Lemma A2. Hence, $L(p, \mathrm{supp}(K^*)) = L(p,Q) = 1$ so

$$\int_Q \pi(q) K(p,dq) = \int_{\Delta S} \pi(q) K(p,dq) = \int_{\Delta S} \pi(q) L(p,dq) = \int_Q \pi(q) L(p,dq)$$

and the tie-breaking property follows from the tie-breaking property of the optimality of $K$ given $Q$. Hence, $K$ is a monopoly strategy. □

**Proposition** (A7)**.** *Let $U = \{0,u\}$. If $K$ is an equilibrium policy, then $K$ is a monopoly policy.*

*Proof.* Let $(K,Q)$ be an equilibrium. Note that if $\tilde{\pi}(u) = 0 = \tilde{\pi}(0)$, then $K$ is a monopoly policy trivially. Hence, assume $\tilde{\pi}(u) > 0$. Define

$$H := \{ q \in \Delta S \;\mid\; q \cdot u \geq 0 \}$$

so $\pi(q) = \tilde{\pi}(u)$ if $q \in H$ and $\pi(q) = 0$ otherwise. Note that if $H = \emptyset$ or $H = \Delta S$, then again $\pi$ is constant so we assume that $H \neq \emptyset$ and $H \neq \Delta S$. Define

$$D := \operatorname{conv}(E \cap H^c) \subset H^c$$

We first show that $D \subset E_K$. Let $p \in D$ and note that since $K(p)$ is consistent with $p$, for any $s \in S$ such that $1_s \in H$,

$$\int_{\Delta S} q(s) K(p, dq) = p(s) = 0$$

Hence, $q(s) = 0$ $K(p)$-a.s. for all $1_s \in H$ so $1 = K(p, D) = K(p, H^c)$. Since $\max\{p \cdot u, 0\} = 0$ as $p \in H^c$, $p \in Q$. Now $\pi(p) = 0$ so $K(p) = \delta_{\{p\}}$ by optimality and $p \in E_K$ as desired.

We now show that $H \subset E_K$. Let $p \in H$. Note that for $\mu := \sum_{s \in S} p(s) \delta_{\{1_s\}}$, since $E \subset E_K \subset Q$ from Lemmas A2 and A4,

$$\int_Q \pi(q) K(p, dq) \geq \int_Q \pi(q) \mu(dq) = \sum_s p(s) \pi(1_s) > 0$$

so $a := K(p, H) > 0$. Suppose $a \in (0, 1)$. Define the conditional measures $\nu_1$ and $\nu_2$ where

$$\nu_1(F) := \frac{K(p, F \cap H)}{K(p, H)} \qquad \nu_2(F) := \frac{K(p, F \cap H^c)}{K(p, H^c)}$$

for all $F \in \mathcal{F}$. Let $p_i := \int_{\Delta S} q \, \nu_i(dq)$ for $i \in \{1, 2\}$ and note that

$$\begin{aligned} ap_1 + (1-a) p_2 &= K(p, H) \int_{\Delta S} q \, \nu_1(dq) + K(p, H^c) \int_{\Delta S} q \, \nu_2(dq) \\ &= \int_{\Delta S} q \, K(p, dq) = p \end{aligned}$$

For $\varepsilon > 0$, define the kernel $K_\varepsilon$ such that $K_\varepsilon(q) = K(q)$ for $q \neq p$ and

$$K_\varepsilon(p) := b_\varepsilon \left(\nu_1 \circ T_\varepsilon^{-1}\right) + (1 - b_\varepsilon) \nu_2$$

where $T_\varepsilon(q) := \varepsilon p + (1-\varepsilon) q$ and $b_\varepsilon := \frac{a}{1-(1-a)\varepsilon} > a$. Note that

$$\begin{aligned} \int_{\Delta S} q \, K_\varepsilon(p, dq) &= b_\varepsilon \int_{\Delta S} T_\varepsilon(q) \nu_1(dq) + (1 - b_\varepsilon) \int_{\Delta S} q \, \nu_2(dq) \\ &= b_\varepsilon (\varepsilon p + (1-\varepsilon) p_1) + (1 - b_\varepsilon) p_2 \\ &= b_\varepsilon \varepsilon p + \frac{(1-\varepsilon)}{1-(1-a)\varepsilon} (ap_1 + (1-a) p_2) = p \end{aligned}$$

so $K_\varepsilon \in \mathcal{K}$. From Lemma A4, we can find $\varepsilon > 0$ such that $\operatorname{supp}(K_\varepsilon) \subset Q$. However,

$$\begin{aligned} \int_Q \pi(q) K_\varepsilon(p, dq) &= \int_{\Delta S} \pi(q) K_\varepsilon(p, dq) = K_\varepsilon(p, H) \pi_0(u) \\ &= b_\varepsilon \nu_1 \left(T_\varepsilon^{-1}(H)\right) \pi_0(u) = b_\varepsilon \pi_0(u) \\ &> a\pi_0(u) = \int_{\Delta S} \pi(q) K(p, dq) = \int_Q \pi(q) K(p, dq) \end{aligned}$$

contradicting the optimality of $K$. Hence $a = 1$, so from Lemma A2,

$$V_K(p) = -c + \int_{\Delta S} \max\{q \cdot u,\ 0\} K(p, dq) = -c + \left(\int_{\Delta S} qK(p, dq)\right) \cdot u$$
$$= -c + p \cdot u < p \cdot u$$

implying $p \in Q$ and $p \in E_K$ as desired.

Hence, $H \cup D \subset E_K$. Let $K^*$ be a monopoly policy. Consider $p \in E_{K^*}$ and suppose $p \notin H \cup D$. Thus, $\pi(p) = 0$ and there is some $s \in S$ such that $p(s) > 0$ and $1_s \in H$. Let $\mu := \sum_{s \in S} p(s) \delta_{\{1_s\}}$ and note that

$$\int_{\Delta S} \pi(q) \mu(dq) = \sum_{\{s \in S |\ 1_s \in H\}} p(s) \pi_0(u) > 0 = \pi(p) = \int_{\Delta S} \pi(q) \delta_{\{p\}}(dq)$$

contradicting the optimality of $K^*$. Hence $E_{K^*} \subset H \cup D \subset E_K$ and $K$ is a monopoly policy by Lemma A6. □

**Corollary** (A8)**.** *The monopoly policy is an equilibrium policy for any $c$.*

*Proof.* Let $K$ be the monopoly policy and find some $\bar{c}$ large enough such that

$$Q_{\bar{c}} := \left\{ q \in \Delta S \;\middle|\; \max_{u \in U} q \cdot u \geq -\bar{c} + \int_{\Delta S} \left(\max_{u \in U} r \cdot u\right) K(q, dr) \right\} = \Delta S$$

Then $(K, \Delta S)$ is an equilibrium for $\bar{c}$. Note that if $c \geq \bar{c}$, then $(K, \Delta S)$ is also an equilibrium for $c$. If $c < \bar{c}$, then by Lemma A3, $(K, Q_c)$ is an equilibrium as desired. □

## B. Unobserved priors

Given $K \in \mathcal{K}$ and $n \in \mathbb{N}$, define the Markov kernel $K^n$ such that

$$K^n(q, dq_n) := K(q, dq_1) K(q_1, dq_2) \cdots K(q_{n-1}, q_n)$$

Recall that $\bar{K} \in \mathcal{K}$ is the full information disclosure policy and $\bar{\mu}$ is the full information policy.

**Lemma** (B1)**.** *$K^n \to K^*$ such that $K^*(q, E_K) = 1$ and $K^*(q)$ is consistent with $q$ for all $q \in \Delta S$.*

*Proof.* Fix $q_0 \in \Delta S$ and let $\Omega := \Delta S^{\mathbb{N}}$. By Ionescu-Tulcea, we can find a probability $\mathbb{P}$ on $\Omega$ such that for all $n \in \mathbb{N}$

$$q_n = \mathbb{E}_n[q_{n+1}] = \int_{\Delta S} q_{n+1} K(q_n, dq_{n+1})$$

Hence, beliefs follow a martingale. Since they are bounded, by the martingale convergence theorem, we have $\mathbb{P}$-a.s.

$$q^* = \lim_n q_n$$

Let $K^*(q_0) := \lim_n K^n(q_0)$ be the distribution of $q^*$. For $\varepsilon > 0$, define $X_n^\varepsilon := \mathbf{1}_{B_\varepsilon(q^*)}(q_{n+1})$ for $n \in \mathbb{N}$ and note that $\mathbb{P}$-a.s.

$$\lim_n X_n^\varepsilon = \lim_n \mathbf{1}_{B_\varepsilon(q^*)}(q_{n+1}) = 1$$

Since $K$ is continuous, by dominated convergence (Theorem V.4.22 of Cinlar (Çinlar, 2011)), we have $\mathbb{P}$-a.s.

$$K(q^*, B_\varepsilon(q^*)) = \lim_n K(q_n, B_\varepsilon(q^*)) = \lim_n \mathbb{E}_n \left[\mathbf{1}_{B_\varepsilon(q^*)}(q_{n+1})\right]$$
$$= \lim_n \mathbb{E}_n [X_n^\varepsilon] = 1$$

Since this is true for all $\varepsilon > 0$, we must have $K(q^*, \{q^*\}) = 1$ so $q^* \in E_K$ $\mathbb{P}$-a.s. or $K^*(q_0, E_K) = 1$ for all $q_0 \in \Delta S$. Note that by bounded convergence and iterated expectations,

$$\int_{\Delta S} q^* K^*(q_0, dq^*) = \mathbb{E}[q^*] = \mathbb{E}\left[\lim_n q_n\right]$$
$$= \lim_n \mathbb{E}[q_n] = \lim_n \mathbb{E}[q_1] = q_0$$

so $K^*(q_0)$ is consistent with $q_0$. □

**Lemma** (B2)**.** *If $E_K = E$, then $K^n \to \bar{K}$.*

*Proof.* By Lemma B1, $K^n \to K^*$ and $K^*(q, E) = 1$ for all $q \in \Delta S$. Moreover, since $K^*(q)$ is consistent with $q \in \Delta S$, we have

$$q = \int_E q^* K^*(q_0, dq^*) = \sum_s K^*(q, \{\delta_s\})\, \delta_s$$

so $K^* = \bar{K}$ as desired. □

**Theorem** (B3)**.** *Let $\mu_i$ be generic equilibrium policies given $c_i$. Then $V_{\mu_i}(p) \to V_{\bar{\mu}}(p)$ as $c_i \to 0$.*

*Proof.* For each $c_i$, define the kernel

$$K_i(r) := (\mu_i)_r = \mu_i \circ \phi_r^{-1}$$

Since $\phi$ is continuous, $K_i$ is continuous so $K_i \in \mathcal{K}$. Note that if $K_i(r) = \delta_{\{r\}}$, then by genericity, $r$ is degenerate. Hence, by Lemma B2, $K_i^n \to \bar{K}$ for all $i$. Since $\bar{\mu}$ is the most informative signal policy, $V_{\mu_i}(p) \le V_{\bar{\mu}}(p)$ for all $i$. Define $n_i$ to be the largest integer less than $\frac{1}{\sqrt{c_i}}$ so $n_i c_i \le \sqrt{c_i}$. Hence,

$$V_{\mu_i}(p) = -c_i + \int_{\Delta S} \max\left\{\max_{u \in U} q \cdot u,\ V_{\mu_i}(q)\right\} K_i(p, dq)$$
$$\ge -\sqrt{c_i} + \int_{\Delta S} \left(\max_{u \in U} q \cdot u\right) K_i^{n_i}(p, dq)$$

Since $c_i \to 0$ implies $n_i \to \infty$, $K_i^{n_i} \to \bar{K}$ for all $i$ so

$$V_{\bar{\mu}}(p) \ge V_{\mu_i}(p) \ge \int_{\Delta S} \left(\max_{u \in U} q \cdot u\right) \bar{K}(p, dq) = V_{\bar{\mu}}(p)$$

Hence $V_{\mu_i}(p) \to V_{\bar{\mu}}(p)$. □

**Proposition** (B4)**.** *Let $U = \{0, u\}$ and $\pi(p) < \pi(q)$ for some $q \in \Delta S$. Then $V_{\mu_i}(p) \to V_{\bar{\mu}}(p)$ as $c_i \to 0$.*

*Proof.* Note that we can assume $\tilde{\pi}(u) > 0$. Define

$$H := \{q \in \Delta S \mid q \cdot u \geq 0\}$$

so $\pi(q) = \tilde{\pi}(u)$ if $q \in H$ and $\pi(q) = 0$ otherwise. Note that $H \neq \emptyset$ and $H \neq \Delta S$ since $\pi(p) < \pi(q)$ for some $q \in \Delta S$. Define

$$K_i(r) := (\mu_i)_r = \mu_i \circ \phi_r^{-1}$$

so $K_i \in \mathcal{K}$ for all $i$.

Fix $K = K_i$ and $(\mu, Q) = (\mu_i, Q_i)$. Let

$$\begin{aligned} D_1 &:= \operatorname{conv}(E \cap H) \\ D_2 &:= \operatorname{conv}(E \cap H^c) \end{aligned}$$

so $D_1 \subset H$ and $D_2 \subset H^c$. We show that $E_K \subset D_1 \cup D_2$. Let $r \in E_K$ so $K(r) = \mu \circ \phi_r^{-1}(r) = \delta_r$. Hence

$$\mu\{q \in \Delta S \mid \phi_r(q) = r\} = 1$$

Suppose $r \notin D_1 \cup D_2$ so there are $s_1 \in E \cap H$ and $s_2 \in E \cap H^c$ such that $r(s_1) > 0$ and $r(s_2) > 0$. Now $\phi_r(q) = r$ implies

$$\frac{q(s_1)}{p(s_1)} = \sum_s q(s) \frac{r(s)}{p(s)} = \frac{q(s_2)}{p(s_2)}$$

so $\mu(R) = 1$ where

$$R := \left\{ q \in \Delta S \;\middle|\; \frac{q(s_1)}{p(s_1)} = \frac{q(s_2)}{p(s_2)} \right\}$$

Since $a := \mu(H) \in (0,1)$, we can define the conditional measures $\nu_1$ and $\nu_2$ where

$$\nu_1(F) := \frac{\mu(F \cap H)}{\mu(H)} \qquad \nu_2(F) := \frac{\mu(F \cap H^c)}{\mu(H^c)}$$

for all $F \in \mathcal{F}$. Let $r_j := \int_{\Delta S} q\ \nu_j(dq)$ so $\nu_j(R) = 1$ and $r_j \in R$ for $j \in \{1,2\}$. Define

$$\nu := a\nu_1 + (1-a) \sum_s r_2(s)\, \delta_s$$

Note that

$$\begin{aligned} \int_{\Delta S} q\ \nu(dq) &= a r_1 + (1-a) \sum_s r_2(s)\, \mathbf{1}_s \\ &= \mu(H) \int_{\Delta S} q\ \nu_1(dq) + \mu(H^c) \int_{\Delta S} q\ \nu_2(dq) = p \end{aligned}$$

so $\nu$ is consistent with $p$. However, since $r_2(s_1) > 0$, $E \subset Q$ and $\pi(q) = 0$ for all $q \in H^c$,

$$\begin{aligned}\int_Q \pi(q)\,\nu(dq) &= \mu(H)\int_Q \pi(q)\,\nu_1(dq) + \mu(H^c)\sum_s r_2(s)\,\pi(\mathbf{1}_s)\\ &> \mu(H)\int_Q \pi(q)\,\nu_1(dq) = \int_Q \pi(q)\,\mu(dq)\end{aligned}$$

contradicting the optimality of $\mu$. Thus, $E_K \subset D_1 \cup D_2$.

By Lemma B1, $(K_i)^n \to K^*$ such that $K^*(p, D_1 \cup D_2) = 1$. Again, define the conditional measures

$$\nu_j^*(F) := \frac{K^*(p, F \cap D_j)}{K^*(p, D_j)}$$

for all $F \in \mathcal{F}$ and $q_j := \int_{\Delta S} q\ \nu_j^*(dq)$ for $j \in \{1,2\}$. If we let $a^* := K^*(p, D_1)$, then since $D_1 \subset H$ and $D_2 \subset H^c$,

$$\begin{aligned}\int_{\Delta S}\left(\max_{u\in U} q\cdot u\right) K^*(p,dq) =&a^*\int_{D_1}\left(\max_{u\in U} q\cdot u\right)\nu_1^*(dq) + (1-a^*)\int_{D_2}\left(\max_{u\in U} q\cdot u\right)\nu_2^*(dq)\\ =&a^*\int_{D_1}(q\cdot u)\,\nu_1^*(dq) + (1-a^*)\int_{D_2}(q\cdot u)\,\nu_2^*(dq)\\ =&\int_{\Delta S}\left(\max_{u\in U} q\cdot u\right)\bar{\mu}(dq)\end{aligned}$$

The rest follows the same reasoning as in Theorem B3. □

## C. Choosing whether to be anonymous

Suppose $\mu^*$ is an equilibrium signal policy that generates $Q_A$ and $\eta$, the conditional distribution of posteriors of the anonymous buyers. Note that $\eta(Q_A) = 0$. The seller maximizes

$$W(\mu) := (1-\alpha)\int_{Q_T}\pi(q)\,\mu(dq) + \alpha\int_{\Delta S}\left(\int_{\phi_r^{-1}(Q_A)}\pi(\phi_r(q))\,\mu_r(dq)\right)\eta(dr)$$

where $\phi_r(q)$ is the posterior of a buyer with prior $r$ if a buyer with prior $p_0$ ends up at posterior $q$. Since buyers all observe the same signal, from Bayes' rule, it must be that

$$\begin{aligned}\mu(dq)\,\frac{q}{p_0} &= \mu_r(dq)\,\frac{\phi_r(q)}{r}\\ \mu_r(dq) &= \mu(dq)\,\frac{rq}{p_0\phi_r(q)}\end{aligned}$$

Substituting $\mu_r$ into the expression for $W$, we obtain

$$W(\mu) = (1-\alpha)\int_{Q_T}\pi(q)\,\mu(dq) + \alpha\int_{\Delta S}\left(\int_{\Delta S}\mathbf{1}_{Q_A}(\phi_r(q))\,\pi(\phi_r(q))\,\frac{rq}{p_0\phi_r(q)}\eta(dr)\right)\mu(dq)$$

Assume $u = (1, -1)$ so $\sup_{u \in U} u \cdot q = 2q - 1$ for $q \geq \frac{1}{2}$. Hence, $\pi(q) = \mathbf{1}_{\left[\frac{1}{2},1\right]}(q)$, $Q_T = [0,1]$ and $Q_A = [0, \underline{a}] \cup [\bar{a}, 1]$ for $\underline{a} \leq \frac{1}{2} \leq \bar{a}$. We can then express $W(\mu) = \int_{\Delta S} w(q)\, \mu(dq)$ where

$$w(q) := (1-\alpha)\, \mathbf{1}_{\left[\frac{1}{2},1\right]}(q) + \alpha \int_{\Delta S} \mathbf{1}_{[\bar{a},1]}(\phi_r(q)) \frac{rq}{p_0 \phi_r(q)} \eta(dr)$$

**Lemma** (C1)**.** *Let $p_0 < \frac{1}{2}$. Then $\mu\{0, p_1\} = 1$ for some $p_1 > p_0$.*

*Proof.* We show that $w(q) = 0$ for all $q \leq p_0$. Note that if $q \leq p_0$, then $\phi_r(q) \leq r$ for all $r$. Since $\eta([\bar{a}, 1]) = 0$, $r < \bar{a}$ $\eta$-a.s. so $\phi_r(q) < \bar{a}$ $\eta$-a.s.. Hence $w(q) = 0$. Thus, if $\mu$ is optimal, then $\mu\{0\} > 0$. That $\mu\{0, p_1\} = 1$ follows from the concavification of $w$ on $[0, 1]$. □

Let us assume that $p_0 \leq \frac{1}{2}$ going forward. Hence, from Lemma C1, $\mu^*\{0, p_1\} = 1$ for some $p_1 > p_0$. Suppose that $p_i \notin Q_A$ for all $i \in \{0, \ldots, k\}$ but $p_{k+1} \in Q_A$.

**Lemma** (C2)**.** *Fix $p_0$ and $p_1$, and let $p_{k+1} := \phi_{p_k}(p_1)$. Then*

$$\frac{1}{p_k} - 1 = \left(\frac{p_0}{p_1}\left(\frac{1-p_1}{1-p_0}\right)\right)^k \left(\frac{1}{p_0} - 1\right)$$

*Proof.* Given $p_0$ and $p_1$,

$$p_k := \phi_{p_{k-1}}(p_1) = \frac{p_1 \frac{p_{k-1}}{p_0}}{p_1 \frac{p_{k-1}}{p_0} + (1-p_1)\left(\frac{1-p_{k-1}}{1-p_0}\right)} = \frac{1}{1 + \frac{p_0}{p_1}\left(\frac{1-p_1}{1-p_0}\right)\left(\frac{1}{p_{k-1}} - 1\right)}$$

$$\frac{1}{p_k} - 1 = \frac{p_0}{p_1}\left(\frac{1-p_1}{1-p_0}\right)\left(\frac{1}{p_{k-1}} - 1\right)$$

Hence, by induction

$$\frac{1}{p_k} - 1 = \left(\frac{p_0}{p_1}\left(\frac{1-p_1}{1-p_0}\right)\right)^k \left(\frac{1}{p_0} - 1\right)$$

□

Note that $\frac{p_0}{p_i}$ proportion of anonymous buyers will go to $p_i$, so

$$\beta_i := \frac{\frac{p_0}{p_i}}{\sum_j \frac{p_0}{p_j}} = \frac{\frac{1}{p_i}}{\sum_j \frac{1}{p_j}}$$

is the proportion of anonymous guys who will be at $p_i$ for $i \in \{0, \ldots, k\}$. Hence $\eta = \sum_i \beta_i \delta_{p_i}$ so

$$w(q) := (1-\alpha)\, \mathbf{1}_{\left[\frac{1}{2},1\right]}(q) + \alpha \sum_i \beta_i \mathbf{1}_{[\bar{a},1]}(\phi_{p_i}(q)) \frac{p_i q}{p_0 \phi_{p_i}(q)}$$

One possibility is that $p_1 = \frac{1}{2}$ in which case the seller just serves the tracked buyers. Suppose $p_1 \neq \frac{1}{2}$ so $\phi_{p_i}(p_1) = \bar{a}$ for some $p_i$. Since $p_{k+1} \in Q_A$, it must be that $\bar{a} = p_{k+1}$.

**Lemma** (C3)**.** $\phi_{p_i}^{-1}(\bar{a}) = p_{k+1-i}$ *for all $i \geq 0$.*

*Proof.* Note that from Lemma C2,

$$\phi_{p_i}(p_{k+1-i}) = \frac{p_{k+1-i}\frac{p_i}{p_0}}{p_{k+1-i}\frac{p_i}{p_0} + (1-p_{k+1-i})\left(\frac{1-p_i}{1-p_0}\right)} = \frac{1}{1+\left(\frac{1}{p_{k+1-i}}-1\right)\left(\frac{1}{p_i}-1\right)\left(\frac{p_0}{1-p_0}\right)}$$
$$= \frac{1}{1+\left(\frac{p_0}{p_1}\left(\frac{1-p_1}{1-p_0}\right)\right)^{k+1}\left(\frac{1}{p_0}-1\right)} = p_{k+1} = \bar{a}$$

Hence, $\phi_{p_i}^{-1}(\bar{a}) = p_{k+1-i}$. □

Thus, if we define $a_i := p_{k+1-i}$, then $\bar{a} = a_0 > a_1 = p_k > \cdots > a_{k-1} = p_2 > a_k = p_1$ and $\phi_{p_i}(a_i) = \bar{a}$ for all $i$. Since $\beta_i p_i = \left(\sum_i \frac{1}{p_i}\right)^{-1} = \beta_0 p_0$, we have

$$\begin{aligned} w(q) :&= (1-\alpha)\mathbf{1}_{\left[\frac{1}{2},1\right]}(q) + \alpha\sum_i \beta_i \mathbf{1}_{[a_i,1]}(q)\frac{p_i q}{p_0\phi_{p_i}(q)} \\ &= (1-\alpha)\mathbf{1}_{\left[\frac{1}{2},1\right]}(q) + \alpha\beta_0\sum_i \mathbf{1}_{[a_i,1]}(q)\frac{q}{\phi_{p_i}(q)} \\ &= (1-\alpha)\mathbf{1}_{\left[\frac{1}{2},1\right]}(q) + \alpha\beta_0\left[\mathbf{1}_{[\bar{a},1]}(q)\frac{q}{\phi_{p_0}(q)} + \mathbf{1}_{[p_k,1]}(q)\frac{q}{\phi_{p_1}(q)} + \cdots + \mathbf{1}_{[p_1,1]}(q)\frac{q}{\phi_{p_k}(q)}\right] \end{aligned}$$

**Lemma** (C4)**.** $p_1 \geq \frac{1}{2}$.

*Proof.* Suppose $p_1 < \frac{1}{2}$ and note that

$$\begin{aligned} \frac{w(p_2)}{p_2} - \frac{w(p_1)}{p_1} &= \frac{1}{p_2}\left(\alpha\beta_0\frac{p_2}{\phi_{p_k}(p_2)} + \alpha\beta_0\frac{p_2}{\phi_{p_{k-1}}(p_2)} + (1-\alpha)\mathbf{1}_{\left[\frac{1}{2},1\right]}(p_2)\right) - \frac{1}{p_1}\alpha\beta_0\frac{p_1}{\phi_{p_k}(p_1)} \\ &= \frac{\alpha\beta_0}{\phi_{p_k}(p_2)} + \frac{\alpha\beta_0}{\bar{a}} + \left(\frac{1-\alpha}{p_2}\right)\mathbf{1}_{\left[\frac{1}{2},1\right]}(p_2) - \frac{\alpha\beta_0}{\bar{a}} \\ &= \frac{\alpha\beta_0}{\phi_{p_k}(p_2)} + \left(\frac{1-\alpha}{p_2}\right)\mathbf{1}_{\left[\frac{1}{2},1\right]}(p_2) > 0 \end{aligned}$$

Since $\frac{w(p_2)}{p_2} > \frac{w(p_1)}{p_1}$, the seller can profit by choosing $p_2$ contradicting the optimality of $p_1$. Hence, $p_1 \geq \frac{1}{2}$. □

## References


Aliprantis, C., and K. Border (2006): *Infinite Dimensional Analysis*. Springer.

Battaglini, M. (2002): "Multiple referrals and multidimensional cheap talk," *Econometrica*, 70(4), 1379–1401.

Bhattacharya, S., and A. Mukherjee (2013): "Strategic Information Revelation when Experts Compete to Influence," *RAND Journal of Economics*, 44(3), 522–544.

Board, S., and M. Pycia (2014): "Outside options and the failure of the Coase conjecture," *The American Economic Review*, 104(2), 656–671.

Burdett, K., and K. L. Judd (1983): "Equilibrium price dispersion," *Econometrica: Journal of the Econometric Society*, pp. 955–969.

Çinlar, E. (2011): *Probability and Stochastics*. Springer.

Diamond, P. A. (1971): "A Model of Price Adjustment," *Journal of Economic Theory*, 3(2), 156–168.

Gentzkow, M., and E. Kamenica (2012): "Competition in Persuasion," Discussion paper, University of Chicago.

Hoffmann, F., R. Inderst, and M. Ottaviani (2014): "Persuasion through Selective Disclosure: Implications for Marketing, Campaigning, and Privacy Regulation," Discussion paper, Bocconi.

Kamenica, E., and M. Gentzkow (2011): "Bayesian Persuasion," *American Economic Review*, 101(6), 2590–2615.

Krishna, V., and J. Morgan (2001): "A Model of Expertise," *Quarterly Journal of Economics*, 116(2), 747–775.

Lewis, T. R., and D. E. Sappington (1994): "Supplying information to facilitate price discrimination," *International Economic Review*, pp. 309–327.

Li, F., and P. Norman (2014): "On Bayesian Persuasion with Multiple Senders," Discussion paper, UNC.

Milgrom, P., and J. Roberts (1986): "Relying on the Information of Interested Parties," *The RAND Journal of Economics*, pp. 18–32.

Rayo, L., and I. Segal (2010): "Optimal Information Disclosure," *Journal of Political Economy*, 118(5), 949–987.

Rhodes, A. (2014): "Multiproduct Retailing," *Review of Economic Studies*, forthcoming.

Stigler, G. J. (1961): "The Economics of Information," *Journal of Political Economy*, 69(3), 213–225.

Wolinsky, A. (1986): "True Monopolistic Competition as a Result of Imperfect Information," *Quarterly Journal of Economics*, 101(3), 493–511.

Zhou, J. (2014): "Multiproduct search and the joint search effect," *The American Economic Review*, 104(9), 2918–2939.