

Housing Bubbles and Policy Analysis*

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Abstract

This paper provides a theory of credit-driven housing bubbles in an infinite-horizon production economy. Entrepreneurs face idiosyncratic investment tax distortions and credit constraints. Housing is an illiquid asset and also serves as collateral for borrowing. A housing bubble can form because houses command a liquidity premium. The housing bubble can provide liquidity and relax credit constraints, but can also generate inefficient overinvestment. Its net effect is to reduce welfare. Property taxes, Tobin's taxes, macroprudential policy, and credit policy can prevent the formation of a housing bubble.

Keywords: Housing Bubbles, Credit Constraints, Margins, Tax Policies, Liquidity, Multiple Equilibria

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1. Introduction

Many countries have experienced housing market bubbles. The collapse of housing bubbles is often accompanied by a financial crisis. As evidence, Figure 1 presents the real housing price indexes, the price-income ratios, and the price-rental ratios for the United States, Japan, Spain, and Greece. This figure reveals that the three series comove for each country, indicating that fluctuations in housing prices may not be driven entirely by fundamentals (i.e., incomes or rents). In particular, the US housing prices increased by 34% between 2001Q1 and 2006Q4. By 2011Q1 the price index had dropped by about 21% from its peak in 2006Q4. It is widely believed that the credit crisis resulting from the bursting of the housing bubble is the primary cause of the 2007–2009 recession in the United States. Japanese housing prices rose by 36% from 1986Q1 to 1991Q1. From the peak in 1991Q1 to 2011Q1, the housing prices declined continually by 45%. The collapse of the Japanese housing bubble contributed to the so-called “Lost Decade.” Spanish housing prices rose by 116% from 1995Q1 to 2006Q4. When the bubble bursted in 2008, the Spanish economy was severely affected. Greek housing prices increased by 93% from 1997Q1 to 2006Q4. The prices started to decline after the peak in 2006Q4 and dropped by 20% by 2011Q2. Similar patterns appeared in other European countries such as Ireland. The collapse of housing bubbles may be partly to blame for the European sovereign debt crisis.

What causes a housing bubble? What is its welfare effect? If a housing bubble reduces welfare, what policies can prevent a bubble from forming? The goal of this paper is to present a theoretical study to address these questions. We provide a model of credit-driven housing bubbles in an infinite-horizon production economy. The model economy is populated by a continuum of identical households. Each household is an extended family consisting of a continuum of entrepreneurs and a continuum of workers. Each entrepreneur runs a firm and workers work for the firms. There is no aggregate uncertainty about fundamentals.

There are three key assumptions in our model. First, entrepreneurs face borrowing constraints because of financial market imperfections. In particular, they have limited commitment and contract enforcement is imperfect. They must pledge their houses as collateral and borrow against at most a fraction of the housing value. That is, they must make down payments in order to purchase houses. This kind of borrowing constraint is often called a leverage constraint or a margin constraint. It is related to the idea put forth by Kiyotaki and Moore (1997), Aiyagari and Gertler (1999), Brunnermeier and Pederson (2009), and Brumm et al. (2013), among others.

Second, entrepreneurs face idiosyncratic policy distortions. For example, governments may offer different tax credits or subsidies to different firms financed by lump sum taxes on households. As Restuccia and Rogerson (2008) and Klenow and Hsieh (2009) argue, policy distortions can generate resource misallocations and are widespread in many developed and developing countries. In this

paper, we consider the idiosyncratic investment tax credit (ITC), which is an important policy tool to stimulate investment.¹

Third, house trading is illiquid. Following Kiyotaki and Moore (2008), we assume that entrepreneurs face a resaleability constraint, which means that they can resell at most a fraction of their existing houses. In addition, they cannot short sell houses.

Housing plays two important roles in the model. First, it is an asset that allows resources to be transferred intertemporally and generate capital gains or losses. Second, it is used as collateral to facilitate borrowing. In general, housing can also provide direct utility as consumer durable and hence can generate rents. In this paper, we abstract away from this role of housing and focus on its first two roles instead. In particular, we assume that housing is intrinsically useless so that its fundamental value is zero. We will show that housing can have a positive value in equilibrium, which represents a bubble.

In standard models with infinitely-lived agents, bubbles can typically be ruled out by transversality conditions. Why can a housing bubble exist in our model? The reason is that in our model entrepreneurs face borrowing constraints and housing can provide liquidity. Hence housing commands a liquidity premium. Consider the special case where entrepreneurs cannot borrow. Since they face idiosyncratic ITC, those with high ITC are willing to invest more. Resources should be reallocated from entrepreneurs with low ITC to those with high ITC. In the absence of a credit market, housing as an asset plays the role of transferring resources among entrepreneurs and also over time. As a result, housing is valuable just like money. In the presence of a credit market, housing also serves as collateral for borrowing and a high housing value can relax the credit constraint. Hence housing demands a collateral yield. The two benefits provided by housing constitute the liquidity premium.

Since liquidity depends at least partly on beliefs, so does the existence of a housing bubble. If no one believes that housing is valuable, then no one will trade it or use it as collateral. In this case, housing is indeed valueless in equilibrium. Thus our model features two types of equilibria: the bubbly equilibrium and the bubbleless equilibrium.² Which type is more efficient? Having discussed the good side of a housing bubble in terms of providing liquidity and relaxing credit constraints, we now turn to its bad side. Our model features idiosyncratic tax distortions. The existence of a housing bubble allows entrepreneurs with high ITC to make more investment. This creates inefficient overinvestment because the ITC must be financed by taxes on households, which reduces welfare. The overall welfare effect of a housing bubble is ambiguous. We prove that a

¹As Hassett and Hubbard (2002) point out, since 1962, the mean duration of a typical state in the United States in which an ITC is in effect has been about three and a half years, and the mean duration of the no-ITC state has been about the same length. In October 2003, China's government provided investment tax credits to six industries of the manufacturing sector in Northeastern provinces and later the tax reform was expanded to more industries in more provinces (Chen, He, and Zhang (2013)).

²There may exist a third type of equilibrium with stochastic bubbles (see, e.g., Weil (1987) and Miao and Wang (2013a)). We will not study this type in the paper.

housing bubble can reduce welfare in some special cases and provide numerical examples for more general cases.

Given that housing bubbles can reduce welfare, what policies can prevent the formation of a bubble? In the standard models of rational bubbles (e.g., Tirole (1985)), the return on the bubble is equal to the capital gains only since the bubble does not deliver any payoffs. In a deterministic model, this implies that the interest rate is equal to the growth rate of the bubble. By contrast, in our model the return on the bubble is equal to capital gains plus the liquidity premium. This asset pricing equation has important policy implications. In particular, we focus on fiscal and macroprudential policies that can reduce the liquidity premium and hence the benefit of having a housing bubble. If the benefit is sufficiently small, a housing bubble cannot exist. We study four types of policies: (i) limit the loan-to-value (LTV) ratio to a sufficiently low level or raise the downpayment to a sufficiently high level; (ii) raise property taxes to a sufficiently high level and transfer the tax revenue to households; (iii) raise property transaction taxes (or Tobin's taxes) to a sufficiently high level and transfer the tax revenue to households; and (iv) the government purchases private bonds financed by lump sum taxes.

We show that the interest rate in the bubbleless steady state is lower than that in the bubbly steady state. The reason is that the housing bubble crowds out the bond demand, thereby reducing the bond price and raising the interest rate. This implies that all four policies will reduce the interest rate in the long run after the bubble is eliminated. This seems to contradict the conventional wisdom that a low interest rate may cause a housing bubble because a low interest rate encourages excessive mortgage borrowing.³ But this can be reconciled by noting that the conventional wisdom ignores the general equilibrium effect of the housing bubble.

Our paper is related to a growing literature on rational bubbles.⁴ Most models of rational bubbles adopt the overlapping generations framework (Tirole (1985) and Weil (1987)). Introducing rational bubbles into an infinite-horizon model is generally nontrivial due to the transversality conditions (Santos and Woodford (1997)). As Kocherlakota (1992) points out, infinite-horizon models with trading frictions or borrowing constraints can generate bubbles. Kocherlakota (2008) and Hellwig and Lorenzoni (2009) provide infinite-horizon endowment economies with such features. Recently, there has been a growing interest in introducing rational bubbles into production economies with borrowing constraints. Examples include Caballero and Krishnamurthy (2006), Farhi and Tirole (2012), and Martin and Venture (2012) in the overlapping generations framework and Kocherlakota (2009), Wang and Wen (2012), Miao and Wang (2012, 2013a,b,c), Miao, Wang, and Xu (2013), Miao, Wang, and Xu (2013), and Hirano and Yanagawa (2013) in the infinite-horizon

³Glaeser, Gottlieb, and Gyourko (2010) document evidence that low interest rates cannot explain the housing bubble between 2001 and the end of 2005 in the US.

⁴See Brunnermeyer and Oehmke (2013) for a survey of the literature on rational and irrational bubbles.

growth framework. In particular, Miao and Wang (2013a) study how a variety of endogenous credit constraints with limited commitment can generate bubbles.⁵ They show that bubbles can relax credit constraints and generate dividend/collateral yields. These yields also represent the liquidity premium.

Fiat money is a pure bubble. Kiyotaki and Moore (2008) provide a model in which money is valued due to its liquidity. Our idea is similar to theirs. But housing is different from money because housing is illiquid and serves as collateral and also because housing is privately provided.

Our paper is more closely related to the literature on housing bubbles.⁶ Kocherlakota (2009) provides a model of housing bubbles based on Kiyotaki and Moore (2008).⁷ In his model, firms face idiosyncratic productivity shocks and collateral constraints. Land is intrinsically useless, but serves as collateral as in Kiyotaki and Moore (1997). He, Wright, and Zhu (2013) build a model of housing bubbles in a monetary economics framework based on Lagos and Wright (2005). Their model does not incorporate real investment and the credit constraint applies to households instead of firms. As in our paper, the existence of a housing bubble is due to the liquidity premium. Unlike our model with two steady states, their model delivers a unique steady state. Housing in their model can also provide direct utility. Arce and Lopez-Salido (2011) study housing bubbles in an overlapping generations framework with credit constraints. The interest rate is equal to the growth of the bubble in their model. In contrast to our result, they show that the interest rate in the bubbly steady state is lower than that in the bubbleless steady state. They also incorporate utility from housing and show that the housing price in the bubbly equilibrium is less than the discounted value of the utility flow (or dividends).

Bubbles must provide some benefits to economic agents, or else, they could not exist in the first place. However, policymakers and researchers are more concerned about the welfare costs of bubbles. Potential costs include volatility and fire sales after the collapse of bubbles (Caballero and Krishnamurthy (2006) and Miao and Wang (2013c)) and misallocation of resources in the presence of market distortions such as externality (Grossman and Yanagawa (1993) and Miao and Wang (2013b)). In this paper, we focus on the cost generated by resource misallocation in the presence of idiosyncratic tax policy distortions. Most papers in the literature discuss the role of monetary policy in preventing bubbles. In an overlapping generations model, Galí (2013) studies how monetary policy can affect the fluctuations of bubbles. But monetary policy cannot eliminate bubbles. He shows that a systematic increase in interest rates in response to a growing bubble

⁵See Kehoe and Levine (1993), Alvarez and Jermann (2000), and Albuquerque and Hopenhayn (2004) for related modeling of endogenous credit constraints.

⁶Brunnermeier and Julliard (2008) and Burnside, Eichenbaum, and Rebelo (2013) present models of housing bubbles based on heterogeneous beliefs or irrational behaviors. There are also many studies of housing prices in the standard macroeconomic models without bubbles, e.g., Ioviallo (2005), Kiyotaki, Michaelides, and Nikolov (2011), Landvoigt, Schneider, and Piazzesi (2013), Liu, Wang, and Zha (2013), Liu, Miao, and Zha (2013), among others.

⁷Following Kocherlakota (2009), we may refer to housing as land and use these two terms interchangeably.

enhances the fluctuations in the latter because the expected growth rate of the bubble is equal to the interest rate in his model. Thus the leaning against the wind monetary policy lacks a theoretical foundation. We agree with his view since the interest rate in the bubbleless equilibrium is lower in our model. However, because the asset pricing equation for the bubble includes the liquidity premium in our model, we argue that other policy tools can be used to lower this premium and eliminate the housing bubble.

Property tax policy and LTV policy are often discussed by the policymakers and the general public. For example, the Chinese government has implemented these policies to curb the growth of housing prices and to prevent housing bubbles.⁸ Our analysis provides a theoretical foundation for these policies. The asset purchase policy proposed in our paper is related to those in Kocherlakota (2009), Hirano, Inaba, and Yanagawa (2013), and Miao and Wang (2013a). Kocherlakota (2009) discusses credit policy to restore the bubbly equilibrium. Miao and Wang (2013a) provide a credit policy to achieve the first-best allocation. Hirano, Inaba, and Yanagawa (2013) study bailout policy and welfare implications for workers who are taxpayers.

The key difference between our paper and some of the aforementioned papers is that our model adopts the infinite-horizon growth framework, which is amenable to quantitative studies (see, e.g., Miao, Wang, and Xu (2013) and Miao, Wang, and Zha (2014)). In addition, the borrowing constraint in our model differs from those often used in the literature on housing prices. Many papers adopt the Kiyotaki and Moore (1997) collateral constraint, which ensures that the debt repayment does not exceed the collateral value so that the borrower will never default. The borrowing constraint in this paper is a type of margin constraint, consistent with the institutional feature in the mortgage market. Similar constraints are imposed in the study of stock trading (see Brumm et al. (2013) and the references therein). We show that given the margin constraint, the Kiyotaki-Moore collateral constraint is always satisfied so that default never occurs in our model. The margin constraint is also adopted in Arce and Lopez-Salido (2011) and some references therein.

The rest of the paper proceeds as follows. Section 2 presents a baseline model. Section 3 provides the model solution. Section 4 analyzes the welfare implications of housing bubbles. Section 5 studies four different policies to prevent bubbles. Section 6 concludes. Data description and technical proofs are relegated to the appendices.

⁸For example, a new nationwide real estate sales tax was introduced in late 2009. Families purchasing a second home were required to make at least a 40% downpayment in 2010, and legislation for a property tax was passed in November 2013.

2. The Baseline Model

To preserve the tractability of the representative agent framework and also allow for firm heterogeneity, we consider an economy populated by a continuum of identical households of unit mass. Each household is an extended family consisting of a continuum of ex ante identical entrepreneurs of unit mass and a continuum of identical workers also of unit mass. Each entrepreneur runs a firm. There is a government that subsidizes entrepreneurial investment financed by lump-sum taxes on households. Following Restuccia and Rogerson (2008), we assume that this policy distortion is idiosyncratic and take it as a given institutional feature throughout the analysis. For simplicity, we do not consider government spending.

Time is discrete and denoted by $t = 0, 1, \dots$. There is no aggregate uncertainty about fundamentals. Assume that a law of large numbers holds so that aggregate variables are deterministic.

2.1 Entrepreneurs

An entrepreneur is indexed by $j \in [0, 1]$. Each entrepreneur j runs a firm using a constant-returns-to-scale technology to produce output according to

$$Y_{jt} = K_{jt}^\alpha N_{jt}^{1-\alpha}, \quad \alpha \in (0, 1)$$

where K_{jt} and N_{jt} represent capital and labor inputs, respectively. Entrepreneurs can borrow and lend among themselves by trading one-period riskless bonds. They can also trade houses. Normalize housing supply to one. In Appendix C, we introduce endogenous housing supply and show that this will not change our insights and results. For simplicity, assume that housing is intrinsically useless in that it does not deliver any payoff or direct utility.⁹ One may follow Kocherlakota (2009) and refer to housing as land, thereby using these two terms interchangeably.¹⁰ Houses can be used by entrepreneurs as collateral for borrowing. Assume that each entrepreneur is initially endowed with zero bond and one unit of house, i.e., $B_{j0} = 0$ and $H_{j0} = 1$ for all j . Assume that houses do not depreciate.

Solving the static labor choice problem,

$$R_{kt}K_{jt} \equiv \max_{N_{jt}} K_{jt}^\alpha N_{jt}^{1-\alpha} - W_t N_{jt},$$

⁹See Tirole (1985), Kocherlakota (2009), Miao and Wang (2013a) for the discussions of how to introduce rents into the asset with bubbles. One way is to introduce economic growth and assume that rents grow at a rate lower than economic growth. See Miao, Wang and Zha (2014) for another way.

¹⁰Davis and Heathcote (2007) document that fluctuations of housing prices are largely driven by those of land prices. We reinterpret the bubble in housing prices as reflecting the one in land prices.

gives

$$N_{jt} = \left(\frac{1 - \alpha}{W_t} \right)^{\frac{1}{\alpha}} K_{jt}, \quad (1)$$

and

$$R_{kt} = \alpha \left(\frac{1 - \alpha}{W_t} \right)^{\frac{1 - \alpha}{\alpha}},$$

where W_t denotes the wage rate. We will show later that R_{kt} is equal to the rental rate of capital.

Entrepreneur j 's dividends are given by

$$D_{jt} = R_{kt}K_{jt} - \tau_{jt}I_{jt} - P_t(H_{jt+1} - H_{jt}) + \frac{B_{jt+1}}{R_{ft}} - B_{jt}, \quad (2)$$

where I_{jt} , P_t , H_{jt} , and R_{ft} denote the investment level, the house price, house holdings, and the (gross) interest rate, respectively. In addition, B_{jt} represents the debt level if it is positive; and savings, otherwise. Note that $\tau_{jt} > 0$ represents policy distortions and is an important variable in the model. We interpret $1 - \tau_{jt}$ as a subsidy to investment, e.g., ITC, if it is positive, and as a tax on investment if it is negative. For simplicity, suppose that τ_{jt} is independently and identically distributed across firms and over time, and is drawn from a fixed distribution with the density function f on $[\tau_{\min}, \tau_{\max}]$.

Entrepreneur j 's capital accumulation equation is given by

$$K_{j,t+1} = (1 - \delta)K_{jt} + I_{jt}, \quad K_{j0} \text{ given}, \quad (3)$$

where $\delta \in (0, 1)$ denotes the depreciation rate.

Entrepreneurs face several constraints due to real and financial frictions. First, there is empirical evidence that equity financing is more costly than debt financing. For simplicity, we assume that equity financing is so costly that entrepreneurs cannot raise new equity. We thus impose the constraint,¹¹

$$D_{jt} \geq 0. \quad (4)$$

Second, due to imperfect contract enforcement, there is a down payment restriction or margin requirement on housing purchases:

$$\frac{B_{jt+1}}{R_{ft}} \leq \theta P_t H_{jt+1}, \quad (5)$$

where $\theta \in (0, 1)$ represents the LTV ratio and $1 - \theta$ represents the down payment or margin requirement. Land is used as collateral. To ensure that entrepreneur j will not default in the next

¹¹We can allow limited equity financing in the sense that $D_{jt} \geq -\zeta K_{jt}$ for some $\zeta > 0$. This modeling will not change our key results.

period, we require that

$$B_{jt+1} \leq P_{t+1}H_{jt+1}. \quad (6)$$

This constraint ensures that debt repayments do not exceed the collateral value in the next period. Kiyotaki and Moore (1997) introduce this constraint, but ignore the margin constraint (5).

Third, house trading is illiquid. Following Kiyotaki and Moore (2008), we impose the following resaleability constraint

$$H_{jt+1} \geq \omega H_{jt}, \quad (7)$$

where $\omega > 0$ represents the liquidity of house trading. This constraint means that entrepreneurs can sell at most a fraction $1 - \omega$ of their existing houses. We also rule out short sales of houses so that $H_{jt} \geq 0$.

The last constraint is that investment is irreversible at the firm level, i.e.,

$$I_{jt} \geq 0. \quad (8)$$

As will become clear later, this assumption is useful for deriving optimal investment given constant-returns-to-scale technology.

Now we describe entrepreneur j 's decision problem by dynamic programming. We use $V_t(\tau_{jt}, K_{jt}, H_{jt}, B_{jt})$ to denote entrepreneur j 's value function, where we suppress aggregate state variables as arguments in the value function. The dynamic programming problem is given by

$$V_t(\tau_{jt}, K_{jt}, H_{jt}, B_{jt}) = \max_{N_{jt}, I_{jt}, B_{jt+1}, H_{jt+1}} D_{jt} + \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} V_{t+1}(\tau_{jt+1}, K_{jt+1}, H_{jt+1}, B_{jt+1}), \quad (9)$$

subject to (2), (3), (4), (5), (6), (7), and (8). Here E_t represents the conditional expectation operator with respect to the idiosyncratic shock and Λ_t is the representative household's marginal utility.

2.2 Households

Assume that labor supply is inelastic and normalized to one. Entrepreneurs and workers hand over their dividends and wages to their family. The family pool their income and distribute it equally among family members. A representative household chooses family consumption C_t to maximize its life-time expected utility,

$$\max_{\{C_t\}} \sum_{t=0}^{\infty} \beta^t \ln(C_t),$$

subject to

$$C_t = W_t N_t + D_t - \Gamma_t,$$

where $N_t = 1$, D_t denotes the total dividends from all firms, and Γ_t denotes lump-sum taxes satisfying

$$\Gamma_t = \int (1 - \tau_{jt}) I_{jt} dj.$$

Given the above utility function, we can derive the marginal utility $\Lambda_t = 1/C_t$. For simplicity, we have assumed that households do not borrow or save as in Kocherlakota (2009) and Kiyotaki and Moore (2008). We can relax this assumption and suppose that households can save, but cannot borrow against their future incomes. Then households will optimally choose not to save because, as we will show later, the equilibrium interest rate is too low, i.e., $R_{ft} < \Lambda_t / (\beta \Lambda_{t+1})$.¹² Consequently, none of results will change.

2.3 Competitive equilibrium

A competitive equilibrium consists of sequences of individual quantities $\{I_{jt}, N_{jt}, K_{jt+1}, Y_{jt}, H_{jt+1}\}$ and aggregate quantities $\{C_t, I_t, N_t, Y_t\}$ and prices $\{W_t, R_{kt}, R_{ft}, P_t\}$ such that (i) households optimize; (ii) workers and entrepreneurs optimize; and (iii) the markets for labor, houses, bonds, and consumption goods all clear, i.e.,

$$\begin{aligned} C_t + I_t &= Y_t, \\ N_t &= \int_0^1 N_{jt} dj = 1, \quad \int_0^1 H_{jt} dj = 1, \quad \int_0^1 B_{jt} dj = 0, \end{aligned}$$

where $I_t = \int I_{jt} dj$ and $Y_t = \int Y_{jt} dj$.

3. Model Solution

We first solve entrepreneurs' decision problem and then characterize the equilibrium system. Finally, we analyze the steady state and local dynamics of the system.

3.1 Entrepreneurs' Decision Problem

We conjecture entrepreneur j 's value function takes the form:

$$V_t(\tau_{jt}, K_{jt}, H_{jt}, B_{jt}) = v_t(\tau_{jt})K_{jt} + p_t(\tau_{jt})H_{jt} - \varphi_t(\tau_{jt})B_{jt},$$

where $v_t(\tau_{jt})$, $p_t(\tau_{jt})$, and $\varphi_t(\tau_{jt})$ are to be determined and satisfy the following restrictions:

$$P_t = \beta \frac{\Lambda_{t+1}}{\Lambda_t} \int p_{t+1}(\tau) f(\tau) d\tau, \tag{10}$$

¹²The model in Kiyotaki and Moore (2008) has a similar feature.

$$\frac{1}{R_{ft}} = \beta \frac{\Lambda_{t+1}}{\Lambda_t} \int \varphi_{t+1}(\tau) f(\tau) d\tau. \quad (11)$$

Define $Q_t \equiv \beta \frac{\Lambda_{t+1}}{\Lambda_t} \int v_{t+1}(\tau) f(\tau) d\tau$ as Tobin's marginal Q or the marginal value of one additional unit of installed capital. We substitute the conjecture above into the Bellman equation and write it as

$$\begin{aligned} & v_t(\tau_{jt})K_{jt} + p_t(\tau_{jt})H_{jt} - \varphi_t(\tau_{jt})B_{jt} \\ = & \max_{H_{jt+1}, I_{jt}, B_{jt+1}} R_{kt}K_{jt} + P_t H_{jt} - B_{jt} - \tau_{jt}I_{jt} + Q_t I_{jt} + Q_t(1 - \delta)K_{jt}, \end{aligned} \quad (12)$$

subject to (5), (7), and

$$0 \leq \tau_{jt}I_{jt} \leq R_{kt}K_{jt} + \frac{B_{jt+1}}{R_{ft}} - B_{jt} - P_t(H_{jt+1} - H_{jt}), \quad (13)$$

where we have substituted (2) into the objective function and combined (2), (4), and (8) to obtain (13). In addition, H_{jt+1} and B_{jt+1} are canceled out in (12) given (10) and (11). Note that we have temporarily ignored constraint (6). In Section 3.2 we will verify that this constraint is always satisfied in any bubbly equilibrium.

Solving the above problem yields the following:

Proposition 1 (i) For $\tau_{jt} \leq Q_t$,

$$\begin{aligned} I_{jt} &= \frac{1}{\tau_{jt}} [R_{kt}K_{jt} + (1 - \omega + \theta\omega) P_t H_{jt} - B_{jt}], \\ \frac{B_{jt+1}}{R_{ft}} &= \theta P_t H_{jt+1}, \quad H_{jt+1} = \omega H_{jt}. \end{aligned}$$

For $\tau_{jt} > Q_t$, $I_{jt} = 0$, and entrepreneur j is indifferent among any choices of H_{jt+1} and B_{jt+1} satisfying (5), (7) and

$$0 \leq R_{kt}K_{jt} + \frac{B_{jt+1}}{R_{ft}} - B_{jt} - P_t(H_{jt+1} - H_{jt}).$$

(ii) The house price, Tobin's Q , and the interest rate satisfy

$$P_t = \beta \frac{\Lambda_{t+1}}{\Lambda_t} P_{t+1} \left\{ 1 + \underbrace{(1 - \omega + \omega\theta) \int_{\tau \leq Q_{t+1}} \frac{Q_{t+1} - \tau}{\tau} f(\tau) d\tau}_{\text{liquidity premium}} \right\}, \quad (14)$$

$$Q_t = \beta \frac{\Lambda_{t+1}}{\Lambda_t} \left[(1 - \delta)Q_{t+1} + R_{kt+1} + \underbrace{R_{kt+1} \int_{\tau \leq Q_{t+1}} \frac{Q_{t+1} - \tau}{\tau} f(\tau) d\tau}_{\text{liquidity premium}} \right], \quad (15)$$

$$\frac{1}{R_{ft}} = \beta \frac{\Lambda_{t+1}}{\Lambda_t} \left[1 + \underbrace{\int_{\tau \leq Q_{t+1}} \frac{Q_{t+1} - \tau}{\tau} f(\tau) d\tau}_{\text{liquidity premium}} \right], \quad (16)$$

and the transversality conditions hold

$$\lim_{i \rightarrow +\infty} \beta^i \frac{\Lambda_{t+i}}{\Lambda_t} Q_{t+i} K_{jt+i+1} = 0, \quad (17)$$

$$\lim_{i \rightarrow +\infty} \beta^i \frac{\Lambda_{t+i}}{\Lambda_t} \frac{1}{R_{ft+i}} B_{jt+i+1} = 0, \quad (18)$$

$$\lim_{i \rightarrow +\infty} \beta^i \frac{\Lambda_{t+i}}{\Lambda_t} P_{t+i} H_{jt+i+1} = 0. \quad (19)$$

We first discuss the intuition behind the optimal investment policy given in part (i) of the proposition. Due to idiosyncratic policy distortions, one dollar of investment costs τ_{jt} dollars. Its benefit is given by Tobin's marginal Q . Thus, when $\tau_{jt} \leq Q_t$, investing is profitable and optimal investment reaches the upper limit. In addition, entrepreneur j borrows as much as possible to finance investment so that the credit constraint (5) binds. Because $\theta \in (0, 1)$, he also wants to sell houses as much as possible to finance investment so that (7) binds. When $\tau_{jt} > Q_t$, investing is not profitable so that $I_{jt} = 0$. Because B_{jt+1} and H_{jt+1} are canceled out in (12), they need to satisfy the feasibility constraints (5), (7) and (13) only. The entrepreneur is indifferent among any choices of B_{jt+1} and H_{jt+1} in the set of the feasibility constraints.

Next we consider part (ii) of the proposition, which gives the asset pricing equations for the house price, Tobin's Q and the interest rate. The left-hand side of equation (14) represents the cost of buying one unit of house. The right-hand side of this equation represents the benefit of holding this unit of house. It consists of two components. The first component is the usual resale value. The second component is a special feature of our model. It represents the role of liquidity and collateral played by the house as an asset. Specifically, to finance investment, the entrepreneur can sell $(1 - \omega)$ units of house and borrow against the value of $\omega\theta$ units of house. The entrepreneur makes investment if and only if $\tau \leq Q_{t+1}$. The expected return from one dollar of the investment is given by $\int_{\tau \leq Q_{t+1}} \frac{Q_{t+1} - \tau}{\tau} f(\tau) d\tau$. Thus the total expected return from the investment is given by the second component on the right-hand side of (14). We call this component the "liquidity premium" in the house price.

Note that it is straightforward to show that the Lagrange multiplier associated with the dividend

constraint (4) is equal to $(Q_t - \tau_{jt})/\tau_{jt}$ if $Q_t > \tau_{jt}$; and 0, otherwise. This Lagrange multiplier is also equal to that associated with the borrowing constraint (5). Thus the liquidity premium essentially reflects the shadow value of relaxing external financing constraints by an additional dollar. The house has liquidity value because it can relax these constraints.¹³

Alternatively we may interpret (14) when $P_t > 0$ as a standard Euler equation,

$$1 = \beta \frac{\Lambda_{t+1}}{\Lambda_t} R_{t+1}^H,$$

where R_{t+1}^H denotes the return on the house. This return consists of two components: capital gains P_{t+1}/P_t and the liquidity premium in returns defined as the liquidity premium in the house price multiplied by P_{t+1}/P_t . Note that houses are intrinsically useless and do not deliver any rent. The liquidity premium is generated from the belief about the future value of the house $P_{t+1} > 0$.

In traditional literature on bubbles (e.g., Tirole (1985)), there is no liquidity premium so that the return on the house is equal to the capital gains or the growth rate of the house price (or the housing bubble). The transversality condition (19) for infinitely-lived agents then rules out the existence of a bubble.¹⁴ Because of the liquidity premium, the transversality condition cannot rule out bubbles in our model.

Equation (15) is the asset price equation for Tobin's Q. The dividend generated from capital consists of rents R_{kt+1} and a liquidity premium for capital. Due to the credit constraint, a unit of capital generates R_{kt+1} units of internal funds (or liquidity) which can be used to finance investment. The investment generates expected return given by the last component in (15).

Equation (16) shows that the bond price also carries a liquidity premium due to credit constraints. The liquidity premium causes the equilibrium interest rate to be lower than the implicit interest rate $\Lambda_t/(\beta\Lambda_{t+1})$ in an economy without any frictions. This result proves our previous claim in Section 2.2.

Note that liquidity premium has three different expressions in (14), (15), and (16). They reflect different degrees of liquidity provided by houses, capital, and bonds. Two special cases merit discussions. First, when $\omega = 0$, house trading is liquid. Land as an asset is a perfect substitute for bonds and they earn the same liquidity premium. Second, when $\theta = 1$, entrepreneurs can borrow against the full value of the non-resaleable house. Even though house trading may be illiquid, the non-resaleable house is effectively traded through bond trading. Thus house trading is effectively liquid. In this case, houses and bonds are also perfect substitutes and earn the same liquidity premium.

¹³See He, Wright, and Zhu (2013) for a similar discussion.

¹⁴The transversality conditions are necessary for infinite-horizon optimization problems with discounting and finite value functions (see, e.g., Ekeland and Scheinkman (1986)).

3.2 Equilibrium System

We now aggregate individual decision rules and impose market-clearing conditions. Define aggregate capital as $K_t \equiv \int K_{jt} dj$. We can characterize the equilibrium system as follows:

Proposition 2 *The equilibrium system is given by the following nine equations: (14), (15), (16), and*

$$I_t = [R_{kt}K_t + (1 - \omega + \theta\omega)P_t] \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau, \quad (20)$$

$$C_t + I_t = Y_t, \quad (21)$$

$$K_{t+1} = (1 - \delta)K_t + I_t, \quad (22)$$

$$Y_t = K_t^\alpha, \quad (23)$$

$$W_t = (1 - \alpha)K_t^\alpha, \quad R_{kt} = \alpha K_t^{\alpha-1}, \quad (24)$$

for nine variables $\{C_t, I_t, Y_t, K_{t+1}, W_t, R_{kt}, R_{ft}, Q_t, P_t\}$. The usual transversality conditions hold.

Equation (20) shows that only firms with tax distortions $\tau \leq Q_t$ contribute to aggregate investment. Other firms do not invest. Aggregate investment is financed by internal funds $R_{kt}K_t$, house sales $(1 - \omega)P_t$, and external borrowing $\theta\omega P_t$. Equations (21)-(24) are standard as in the literature.

We have already explained the three asset pricing equations (14), (15), (16). We observe that $P_t = 0$ for all t always satisfies equation (14). We call such an equilibrium a bubbleless equilibrium. Later we will show that there can exist an equilibrium in which $P_t > 0$ for all t . We call such an equilibrium a bubbly equilibrium.

Note that in any bubbly equilibrium, the Kiyotaki-Moore type collateral constraint (6) is always satisfied and hence our omission of this constraint in Section 3.1 is without loss of generality. To verify this claim, we derive the following:

$$\begin{aligned} B_{jt+1} &= \frac{B_{jt+1}}{R_{ft}} R_{ft} \leq \theta P_t H_{jt+1} R_{ft} \\ &= \theta H_{jt+1} \frac{\beta \frac{\Lambda_{t+1}}{\Lambda_t} P_{t+1} \left\{ 1 + [1 - \omega(1 - \theta)] \left[\int_{\tau \leq Q_{t+1}} \frac{Q_{t+1}-\tau}{\tau} f(\tau) d\tau \right] \right\}}{\beta \frac{\Lambda_{t+1}}{\Lambda_t} \left[1 + \int_{\tau \leq Q_{t+1}} \frac{Q_{t+1}-\tau}{\tau} f(\tau) d\tau \right]} \\ &\leq \theta P_{t+1} H_{jt+1} \leq P_{t+1} H_{jt+1}, \text{ for } \omega \geq 0, \theta \in [0, 1], \end{aligned}$$

where the first inequality follows from (5) and the second equality follows from (14) and (16). The other equalities and inequalities are straightforward to derive.

We now describe how the two types of equilibria work in the model. In a bubbleless equilibrium, houses have no value and will not be traded. The credit market is essentially shut down because no collateral is available. For highly subsidized entrepreneurs with $\tau_{jt} \leq Q_t$, investment is profitable. These entrepreneurs use internal funds to finance investment. For entrepreneurs with $\tau_{jt} > Q_t$, investment is not profitable and hence they do not invest.

In a bubbly equilibrium, entrepreneurs with $\tau_{jt} \leq Q_t$ borrow and sell houses to finance investment as much as possible until both the borrowing and resaleability constraints bind. Entrepreneurs with $\tau_{jt} > Q_t$ do not invest. They are indifferent between saving and borrowing, and between buying and selling houses. To clear the bond and housing markets, their aggregate behavior is to save and lend to highly subsidized entrepreneurs and also buy houses from them. In the special case of $\omega = 0$, house trading is liquid. Highly subsidized entrepreneurs with $\tau_{jt} \leq Q_t$ sell all their houses to finance investment. They will not borrow because they have no house collateral. To clear the housing market, entrepreneurs with $\tau_{jt} > Q_t$ must purchase houses. Borrowing and saving take place within these firms.

3.3 Bubbleless Steady State

We use a subscript f to denote a variable in an equilibrium without bubble. We also remove the time subscript for any variable in the steady state. Using the steady-state version of equations (20) and (22), we can show that

$$R_{kf} = \frac{\delta}{\int_{\tau \leq Q_f} \frac{1}{\tau} f(\tau) d\tau}. \quad (25)$$

Substituting R_{kf} into the steady-state version of equation (15) yields an equation for Q_f .

Proposition 3 *The equation*

$$1 - \beta(1 - \delta) = \beta\delta \frac{\int \max\left(\frac{1}{\tau}, \frac{1}{Q_f}\right) f(\tau) d\tau}{\int_{\tau \leq Q_f} \frac{1}{\tau} f(\tau) d\tau} \quad (26)$$

has a unique solution for $Q_f \in (\tau_{\min}, \tau_{\max})$. If R_{kf} in (25) satisfies

$$R_{kf} > \alpha\delta, \quad (27)$$

then Q_f is equal to Tobin's Q in the bubbleless steady state.

Given Q_f , we can derive the steady-state rental rate of capital R_{kf} from equation (25). We then use (24) to determine the steady-state capital stock K_f . The steady-state investment, output, and consumption are given by $I_f = \delta K_f$, $Y_f = K_f^\alpha$, and $C_f = Y_f - I_f$, respectively. Condition (27)

ensures that $C_f > 0$. A sufficient condition for it in terms of primitives is given by $\alpha \int \frac{1}{\tau} f(\tau) d\tau < 1$, since $Q_f \in (\tau_{\min}, \tau_{\max})$.

3.4 Bubbly Steady State

In this subsection we study the bubbly steady state in which $P_t = P > 0$ for all t . We remove the time subscript and use a subscript b to indicate a bubbly steady state. Equation (14) implies that the bubbly steady-state Tobin's Q , denoted by Q_b , satisfies the equation,

$$\frac{\beta^{-1} - 1}{1 - \omega(1 - \theta)} = \int_{\tau \leq Q_b} \frac{Q_b - \tau}{\tau} f(\tau) d\tau. \quad (28)$$

By the Intermediate Value Theorem, if

$$\frac{\beta^{-1} - 1}{1 - \omega(1 - \theta)} < \tau_{\max} \int \frac{1}{\tau} f(\tau) d\tau - 1, \quad (29)$$

then (28) has a unique solution $Q_b \in (\tau_{\min}, \tau_{\max})$. We can then derive the steady-state rental rate of capital R_{kb} using equation (15),

$$R_{kb} = \frac{1 - \beta(1 - \delta)}{\beta \int \max\left(\frac{1}{\tau}, \frac{1}{Q_b}\right) f(\tau) d\tau}. \quad (30)$$

We then use (24) to determine the steady-state capital stock K_b . The bubbly steady-state investment, output, and consumption are given by $I_b = \delta K_b$, $Y_b = K_b^\alpha$, and $C_b = Y_b - I_b$, respectively. We use equation (20) to determine the steady-state house price P ,

$$\frac{P}{Y_b} = \frac{1}{1 - \omega(1 - \theta)} \left[\frac{\delta \alpha \beta}{1 - \beta(1 - \delta)} \frac{\int \max\left(\frac{1}{\tau}, \frac{1}{Q_b}\right) f(\tau) d\tau}{\int_{\tau \leq Q_b} \frac{1}{\tau} f(\tau) d\tau} - \alpha \right]. \quad (31)$$

We need $P > 0$ and $C_b > 0$ for the existence of a bubbly steady state. The following proposition provides a characterization.

Proposition 4 *Suppose that condition (29) holds and that R_{kb} in (30) satisfies*

$$R_{kb} > \alpha \delta. \quad (32)$$

Then the bubbly and bubbleless steady states coexist if and only if

$$1 < \beta \left[1 + (1 - \omega + \theta \omega) \int_{\tau \leq Q_f} \frac{Q_f - \tau}{\tau} f(\tau) d\tau \right], \quad (33)$$

where Q_f is determined by (26).

Condition (32) ensures that $C_b > 0$. If this condition is satisfied, then (27) also holds since we can show that $R_{kb} < R_{kf}$. A sufficient condition for (32) in terms of primitives when $\tau_{\min} > 0$ is given by

$$\tau_{\min} [1 - \beta(1 - \delta)] > \alpha\beta\delta. \quad (34)$$

Condition (33) ensures that $P > 0$. To interpret this condition, we recall the discussion following Proposition 1. The right-hand side of (33) represents the steady-state benefit of purchasing one unit of house when Tobin's Q is equal to the bubbleless steady state value Q_f . When this benefit is larger than the unit cost, a housing bubble can exist.

The following proposition compares the two steady states.

Proposition 5 *If the bubbleless and bubbly steady states coexist, then $Q_f > Q_b$, $R_{kb} < R_{kf}$, $R_{fb} > R_{ff}$, $K_b > K_f$, $I_b > I_f$, and $Y_b > Y_f$.*

This proposition shows that the existence of a housing bubble in the steady state allows entrepreneurs to finance more investment and accumulate more capital stock. This causes the rental rate of capital and Tobin's marginal Q to be lower and output to be higher in the bubbly steady state than in the bubbleless steady state. However, it is not necessarily true that consumption is higher in the bubbly steady state than in the bubbleless steady state. The intuition is that a housing bubble may cause entrepreneurs to overinvest, causing fewer resources to be allocated to consumption. Thus a housing bubble may reduce welfare. We will study this issue in Section 4.

Note that when $\omega = 0$, house trading is liquid. Equations (14) and (16) imply that $R_{fb} = 1$ in the bubbly steady state and $R_{ff} < 1$ in the bubbleless steady state. But when $\omega > 0$, we must have $R_{fb} < 1$ by (14) and (16). The intuition is that when $\omega = 0$, houses and bonds are perfect substitutes. Since houses are intrinsically useless in the model, the net interest rate of bonds must be zero. But when $\omega > 0$, houses are an illiquid asset. For houses and bonds to coexist in a bubbly equilibrium, the net interest rate of bonds must be negative. To generate a positive steady-state net interest rate, we can introduce economic growth. Specifically, we can assume that aggregate productivity grows at a constant rate. See Miao and Wang (2013a) for a related analysis.

Proposition 5 shows that the interest rate in the bubbleless steady state is lower than that in the bubbly steady state.¹⁵ The reason is that the housing bubble crowds out the bond demand, thereby reducing the bond price and raising the interest rate. This result has an important policy implication as we will show in Section 5.

¹⁵Tirole's (1985) model also implies this result.

3.5 Local Dynamics

We now study local dynamics around the bubbly and bubbleless steady states. Because of the complexity of the model, we are unable to provide a full characterization. The following proposition characterizes the bubbleless steady state for general distribution functions.

Proposition 6 *When both the bubbly and bubbleless steady states exist, then the local equilibrium around the bubbleless steady state is indeterminate of degree one. When only the bubbleless steady state exists, then it is a saddle point and there is a unique bubbleless equilibrium converging to it.*

The idea behind the proof of this proposition is the following. We use Proposition 2 to simplify the equilibrium system to a system of four nonlinear difference equations for four unknown variables C_t , K_t , Q_t , and P_t . Only K_t is a predetermined variable. The other three variables are nonpredetermined. We then linearize the equilibrium system around the bubbleless steady state to obtain a linear system $MX_{t+1} = X_t$, where $X_t = (\hat{C}_t, \hat{K}_t, \hat{Q}_t, P_t)'$ and a hatted variable denotes log deviation from the steady state. P_t is the deviation from 0. We check properties of the eigenvalues of the coefficient matrix M . We can show that when both the bubbly and bubbleless steady states exist, there are two eigenvalues outside the unit circle and two eigenvalues inside the unit circle. This means that the local equilibrium around the bubbleless steady state is indeterminate of degree one. In particular, given K_0 and for any initial value $P_0 > 0$ in the neighborhood of the bubbleless steady state, there is a unique equilibrium path (C_t, K_t, Q_t, P_t) converging to the bubbleless steady state. That is, the housing bubble eventually bursts. However, when only the bubbleless steady state exists, the matrix M has three eigenvalues inside the unit circle and one eigenvalue outside the unit circle. This means that the bubbleless steady state is determinate and there is a unique equilibrium converging to this steady state. Since $P_t = 0$ always satisfies equation (14), this equilibrium must be bubbleless.

We now turn to the bubbly steady state. We are able to derive the following theoretical result for a general distribution in the special case of $\omega(1 - \theta) = 0$ and $\delta = 1$.

Proposition 7 *Let $\omega(1 - \theta) = 0$ and $\delta = 1$. Suppose that the bubbly steady state exists. Then there is a unique local bubbly equilibrium converging to the bubbly steady state.*

The idea of the proof is similar to that for Proposition 6. For general distributions and parameter values, we are unable to derive theoretical results. However, we have verified numerically that the results in Propositions 6 and 7 hold for a wide range of parameter values and for many different types of distributions for the idiosyncratic shock. Note that Tirole (1985) and Miao and Wang (2013a) prove similar results in other models of bubbles.

4. Welfare Analysis

In this section we study the welfare implications of the bubbleless and bubbly equilibria. Both equilibria are inefficient due to idiosyncratic policy distortions and credit constraints. We will take these distortions as a given institutional feature and compare welfare between the bubbleless and bubbly equilibria.

4.1 Welfare Comparison

Let $U_f(K_0)$ and $U_b(K_0)$ denote the household life-time utility level in the bubbleless equilibrium and in the bubbly equilibrium, respectively, given the economy starts at the aggregate capital stock K_0 . Then the bubbleless and bubbly steady-state life-time utility levels are given by $U_f(K_f)$ and $U_b(K_b)$, respectively.

We first provide a theoretical result for a special case.

Proposition 8 *Let $\delta = 1$ and $\omega(1 - \theta) = 0$. Suppose that both bubbly and bubbleless steady states exist. If $\int_{\tau \leq Q_f} \frac{1}{\tau} f(\tau) d\tau > 1$, then $U_f(K_f) > U_b(K_b)$ and $U_f(K_0) > U_b(K_0)$.¹⁶*

Here we sketch the key idea of the proof. We show that the saving rate $s_t \equiv I_t/Y_t = s$ must be at the respective constant steady state value for all t in both the bubbly and bubbleless equilibria. In addition, Tobin's Q must be at the constant steady state value during the transition. Importantly, the bubble-to-output ratio is also constant over time in the bubbly equilibrium. However, the capital stock, investment, consumption, and output change over time. In particular, the law of motion for capital satisfies $K_{t+1} = I_t = s_t Y_t = s K_t^\alpha$ and consumption is given by $C_t = (1 - s) K_t^\alpha$. We can then write the life-time utility level as

$$\sum_{t=0}^{\infty} \beta^t \ln(C_t) = \frac{\ln(1-s)}{1-\beta} + \frac{\alpha}{1-\alpha\beta} \left[\frac{\beta}{1-\beta} \ln(s) + \ln(K_0) \right]. \quad (35)$$

To compare welfare, we only need to compare the equilibrium saving rate. It turns out that the saving rate in the bubbly equilibrium is too high, generating too much investment. This causes welfare to be lower.

For any given non-steady-state value K_0 , the life-time utility level in (35) is concave in s and maximized at $s = \alpha\beta$. The assumption in Proposition 8 ensures that $s_f > \alpha$ and hence $s_b > s_f > \alpha\beta$. It follows that $U_f(K_0) > U_b(K_0)$.

In any steady state with $\delta = 1$, $K = I = sK^\alpha$. It follows that the steady-state capital stock

¹⁶In the appendix, we prove that a sufficient condition for $U_f(K_0) > U_b(K_0)$ is that the support of the distribution for τ is $[0, 1]$.

satisfies $K = s^{1/(1-\alpha)}$. Using (35), we can then compute the steady-state welfare as

$$U = \frac{1}{1-\beta} \left[\ln(1-s) + \frac{\alpha}{1-\alpha} \ln(s) \right].$$

It is a concave function of s and is maximized at $s = \alpha$. Since $s_b > s_f > \alpha$, it follows that $U_f(K_f) > U_b(K_b)$.

4.2 Examples

We first provide an explicitly solved example to illustrate Proposition 8 as well as Propositions 3-7. Let $\delta = 1$, $\omega(1-\theta) = 0$, and $f(\tau) = \eta\tau^{\eta-1}$, $\eta > 1$, for $0 \leq \tau \leq 1$. By Proposition 3, we can compute the following values in the bubbleless steady state:

$$Q_f = \left[\frac{\beta(\eta-1)}{\eta-\beta} \right]^{\frac{1}{\eta}}, \quad R_{kf} = \frac{\eta-1}{\eta} \left(\frac{\eta/\beta-1}{\eta-1} \right)^{\frac{\eta-1}{\eta}}.$$

Along the bubbleless equilibrium path, $Q_t = Q_f$ and $R_{kt} = R_{kf}$ for all t and they are the unique local equilibrium solutions. In addition, the saving rate is also constant and given by

$$s_f = \frac{I_t}{Y_t} = \frac{I_f}{Y_f} = \frac{K_f}{Y_f} = \frac{\alpha}{R_{kf}} = \frac{\alpha\eta}{\eta-1} \left(\frac{\eta-1}{\eta/\beta-1} \right)^{\frac{\eta-1}{\eta}}. \quad (36)$$

Consider next the bubbly equilibrium. By equations (28) and (30), the steady-state Tobin's Q and the rental rate of capital are given by

$$Q_b = [(1/\beta-1)(\eta-1)]^{\frac{1}{\eta}} = R_{kb}.$$

Along the bubbly equilibrium path, $Q_t = R_{kt} = Q_b$ for all t and this is the unique local equilibrium solution. In addition, the saving rate is given by

$$s_b = \frac{I_t}{Y_t} = \frac{I_b}{Y_b} = \frac{K_b}{Y_b} = \frac{\alpha}{R_{kb}} = \alpha(1/\beta-1)^{-\frac{1}{\eta}}(\eta-1)^{-\frac{1}{\eta}}.$$

By (31), the steady-state bubble to output ratio is given by

$$\frac{P}{Y_b} = \frac{\alpha}{\eta(1/\beta-1)} - \alpha.$$

Along the transition path, the bubble-to-output ratio is also equal to the above constant value.

We now check the conditions in Proposition 4. For (29) to hold, we need $\eta < 1/(1-\beta)$. For (32) to hold, we need

$$\left(\frac{1}{\beta} - 1 \right)^{\frac{1}{\eta}} (\eta-1)^{\frac{1}{\eta}} > \alpha. \quad (37)$$

This condition ensures that $C_b > 0$. To ensure $P/Y_b > 0$, i.e. for (33) to hold, we need

$$\eta < \frac{\beta}{1 - \beta}. \quad (38)$$

Thus, if conditions (37) and (38) are satisfied, then the bubbly and bubbleless steady states coexist. It can be verified numerically that these two conditions hold for a wide range of parameter values.

We can verify Proposition 5 and show that $Q_f > Q_b$, $R_{kb} < R_{kf}$, and $s_b > s_f$. We can also verify the assumption in Proposition 8 holds so that $s_f > \alpha$.¹⁷ Thus Proposition 8 is valid.

Finally, we provide some numerical examples for general values of ω and θ . Suppose that one period corresponds to a quarter. We set $\alpha = 0.3$, $\beta = 0.99$, $\delta = 0.025$, $\omega = 0.2$, and $\theta = 0.75$. We set $\eta = 5.7$ so that the bubbleless steady-state capital to output ratio is equal to 10 as in the US data. We find the following numerical results: in the bubbleless steady state, $K_f = 28.67$, $Y_f = 2.737$, $I_f = 0.7166$, $C_f = 2.020$, $s_f = 0.2619$, $Q_f = 0.9324$, $R_{kf} = 0.02864$, $R_{ff} = 0.8839$, and $U_f(K_f) = 70.31$; in the bubbly steady state, $K_b = 46.11$, $Y_b = 3.156$, $I_b = 1.153$, $C_b = 2.003$, $s_b = 0.3653$, $Q_b = 0.5912$, $R_{kb} = 0.02053$, $R_{fb} = 0.9995$, $P = 10.30$, and $U_b(K_b) = 69.48$. Clearly the saving rate in the bubbly steady state is 39% higher than that in the bubbleless steady state. But the life-time utility level in the bubbly steady state is about 1.2% lower than that in the bubbleless steady state. We can measure the welfare cost as a proportional compensation for consumption in the bubbly equilibrium such that the household is indifferent between the bubbly and bubbleless equilibria.¹⁸ We find that the steady-state welfare cost is 0.83% of consumption.

The welfare cost is even larger during the transition period. Figure 2 plots the paths of life-time utility levels in the bubbly and bubbleless equilibria for two initial values of the capital stock, $K_0 = 1.05K_b$ and $K_0 = 0.95K_f$. The initial utility gap is large and then gradually shrinks over time. In the long run, the difference in utility is still significant. When measured in terms of consumption compensation, the initial welfare cost is equal to 7.45% and 7.48% for $K_0 = 1.05K_b$ and $K_0 = 0.95K_f$, respectively.

5. Policy Analysis

In the previous section we have shown that housing bubbles generate excessive investment and reduce welfare. In this section we will study the policies that can eliminate housing bubbles and

¹⁷Using (36), one can verify that s_f is decreasing in η when $\eta > 1$ by checking derivatives with respect to η . The result then follows from (37) and (38).

¹⁸Formally, define the welfare cost as Δ such that

$$\sum_{t=0}^{\infty} \beta^t \ln(C_{t,f}) = \sum_{t=0}^{\infty} \beta^t \ln((1 + \Delta) C_{t,b}),$$

where $\{C_{t,f}\}$ and $\{C_{t,b}\}$ denote the consumption streams in the bubbleless and bubbly equilibria, respectively.

allow the economy to achieve the bubbleless equilibrium. We will introduce one policy at a time in the baseline model presented in Section 2. We emphasize that both the bubbly and bubbleless equilibria are inefficient because of the presence of idiosyncratic tax distortions and credit market imperfections. We take these distortions as a given institutional feature. To achieve the first-best allocation, one has to remove the idiosyncratic policy distortions and credit market imperfections. Study of such policies is beyond the scope of this paper.

5.1 Loan-to-Value Ratio

Recently some countries, such as Hungary, Norway, Sweden and the UK, have adopted maximum loan-to-value (LTV) ratios for mortgages as a macroprudential instrument to regulate the housing market. The intuition is that the LTV ratio can control the credit limit and hence stabilize the credit market. In our model, lowering the LTV ratio θ reduces the credit limit and hence reduces the collateral yield generated by the housing bubble when house trading is illiquid $\omega > 0$. This can reduce the benefit of holding houses. When θ is sufficiently small, the benefit is sufficiently small so that the expression on the right-hand side of (33) is smaller than 1,¹⁹ causing the existence condition for a bubbly equilibrium to be violated. In this case, a bubbly equilibrium cannot exist.

This result is consistent with the general view that one important cause of the housing bubble is excessive credit. If the policymaker can adequately control credit, a housing bubble cannot exist.

Note that this result depends on the assumption that $\omega > 0$. When house trading is liquid (i.e., $\omega = 0$), entrepreneurs can sell all of their house holdings to finance investment and be left with no collateral for borrowing. In this case, controlling the LTV ratio is an ineffective way to eliminate a bubble.

5.2 Property Tax

Next we consider the impact of the property tax. Suppose that the government taxes the property and transfers the tax revenue to households in a lump-sum manner. Then the entrepreneur's flow-of-funds constraint becomes

$$D_{jt} = R_{kt}K_{jt} - \tau_{jt}I_{jt} - P_t(H_{jt+1} - H_{jt}) + \frac{B_{jt+1}}{R_{ft}} - B_{jt} - \tau_H P_t H_{jt}, \quad (39)$$

where τ_H represents the tax rate on the property. In this case, we can characterize the equilibrium system as in Proposition 2 with equations (14) and (20) replaced by

$$P_t = \beta \frac{\Lambda_{t+1}}{\Lambda_t} P_{t+1} \left[1 - \tau_H + (1 - \omega + \omega\theta - \tau_H) \int_{\tau \leq Q_{t+1}} \frac{Q_{t+1} - \tau}{\tau} f(\tau) d\tau \right],$$

¹⁹Note that Q_f is determined in (26) and is independent of θ and ω .

and

$$I_t = [R_{kt}K_t + (1 - \omega + \theta\omega - \tau_H) P_t] \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau.$$

Other equations are the same. The interpretation of the preceding two equations is straightforward. Holding one unit of house means one has to pay τ_H units of property taxes. This will lower the liquidity premium because it reduces the entrepreneur's net worth and investment. As in Proposition 4, we can show that the bubbly and bubbleless steady states coexist if and only if²⁰

$$1 < \beta \left[1 - \tau_H + (1 - \omega + \theta\omega - \tau_H) \int_{\tau \leq Q_f} \frac{Q_f - \tau}{\tau} f(\tau) d\tau \right].$$

Since Q_f is independent of τ_H , this condition will be violated when τ_H is sufficiently large. In this case a bubbly equilibrium cannot exist.

5.3 Property Transaction Tax

It is often argued that the Tobin tax on financial transactions can stabilize the financial market. We now consider the impact of the Tobin tax or the property transaction tax in the housing market. Suppose that the transaction of houses is taxed at the rate $\phi \in (0, 1)$ and that the tax revenue is rebated to households in a lump-sum manner. Then entrepreneur j 's flow-of-funds constraint becomes

$$D_{jt} = R_{kt}K_{jt} - \tau_{jt}I_{jt} - P_t(H_{jt+1} - H_{jt}) + \frac{B_{jt+1}}{R_{ft}} - B_{jt} - \phi P_t |H_{jt+1} - H_{jt}|. \quad (40)$$

His decision problem is to solve (9) subject to (3), (4), (5), (6), (7), (8), and (40).

Proposition 9 *The equilibrium system is given by the following equations:*

$$\begin{aligned} (1 + \phi) P_t = & \beta \frac{\Lambda_{t+1}}{\Lambda_t} P_{t+1} \left\{ 1 + \phi + \theta \int_{\frac{1-\phi-\theta}{1+\phi-\theta} Q_{t+1} < \tau \leq Q_{t+1}} \frac{Q_{t+1} - \tau}{\tau} f(\tau) d\tau \right. \\ & \left. + \int_{\tau \leq \frac{1-\phi-\theta}{1+\phi-\theta} Q_{t+1}} \left[[\theta\omega + (1 - \omega)(1 - \phi)] \frac{Q_{t+1} - \tau}{\tau} - 2\phi(1 - \omega) \right] f(\tau) d\tau \right\}, \end{aligned} \quad (41)$$

²⁰Note that we need conditions (32) and

$$\frac{\beta^{-1} - 1 + \tau_H}{1 - \omega + \omega\theta - \tau_H} < \tau_{\max} \int \frac{1}{\tau} f(\tau) d\tau - 1$$

to hold. The latter condition is analogous to (29).

$$\begin{aligned}
I_t = & R_{kt}K_t \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau + P_t H_t \theta \int_{\frac{1-\phi-\theta}{1+\phi+\theta} Q_t < \tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau \\
& + P_t H_t [\omega\theta + (1-\omega)(1-\phi)] \int_{\tau \leq \frac{1-\phi-\theta}{1+\phi-\theta} Q_t} \frac{1}{\tau} f(\tau) d\tau,
\end{aligned} \tag{42}$$

and equations (15), (16), (21), (22), (23) and (24) for nine variables $\{C_t, I_t, Y_t, K_{t+1}, W_t, R_{kt}, R_{ft}, Q_t, P_t\}$. The usual transversality conditions hold.

The presence of the property transaction tax implies a different calculation of the cost and benefit arising from house trading. Selling one dollar's worth of a house gives the seller only $1 - \phi$ dollars and buying one dollar's worth of a house costs the buyer $1 + \phi$ dollars. Thus there may exist cases where entrepreneurs do not buy or sell houses in order to avoid transaction taxes.

We now describe the equilibrium in the presence of property transaction taxes as follows. For sufficiently subsidized entrepreneurs with $\tau_{jt} \leq \frac{1-\phi-\theta}{1+\phi-\theta} Q_t$, they invest as much as possible until the dividend constraint (4) binds.²¹ They finance investment by selling as many houses as possible until the resaleability constraint (7) binds, and by borrowing as much as possible until the collateral constraint (5) binds. For entrepreneurs in the middle with $\frac{1-\phi-\theta}{1+\phi-\theta} Q_t < \tau_{jt} \leq Q_t$, it is profitable for them to invest. But the investment subsidy is not high enough to compensate for the property transaction tax. Thus these entrepreneurs do not buy or sell houses. They finance investment exclusively by borrowing and with internal funds. Finally, for less subsidized entrepreneurs with $\tau_{jt} \geq Q_t$, investment is not profitable. They do not invest and are indifferent among any feasible choices of house and bond holdings. In order to clear the housing market and the bond market, they buy houses from highly subsidized entrepreneurs and also buy bonds from all other entrepreneurs.

We can interpret the asset pricing equation (41) as follows. The expression on the left-hand side represents the purchase cost of a unit of house. The expression on the right-hand represents the discounted benefit. The future benefit consists of the holding value $(1 + \phi) P_{t+1}$ and the liquidity premium given by the two integral terms. The liquidity premium in the first line of equation (41) describes the case where the entrepreneur does not buy or sell any house. He uses borrowing to finance investment. The liquidity premium in the second line of equation (41) reflects the fact that investment is financed through both borrowing and sale of houses, $\theta\omega + (1 - \omega)(1 - \phi)$. The last negative term represents the transaction tax incurred from buying and selling of the fraction $1 - \omega$ of houses.

As in the baseline model presented in Section 2, there is a bubbleless equilibrium in which $P_t = 0$ for all t . In this equilibrium, houses are not traded and the credit market is shut down because no house collateral is available. There may exist many bubbly equilibria in which $P_t > 0$ for all t . But, by adapting the proof of Proposition 4, we can show that there is a unique bubbly

²¹If $\phi > 1 - \theta$, this case cannot happen since τ_{jt} must be nonnegative.

steady state in which $P_t = P > 0$ for all t if and only if²²

$$1 < \beta \left\{ 1 + \theta \int_{\frac{1-\phi-\theta}{1+\phi-\theta} Q_f < \tau \leq Q_f} \frac{Q_f - \tau}{\tau(1+\phi)} f(\tau) d\tau \right. \\ \left. + \int_{\tau \leq \frac{1-\phi-\theta}{1+\phi-\theta} Q_f} \left[[\theta\omega + (1-\omega)(1-\phi)] \frac{Q_f - \tau}{\tau(1+\phi)} - \frac{2\phi(1-\omega)}{1+\phi} \right] f(\tau) d\tau \right\},$$

where Q_f is determined by (26). Since the right-hand side of the above inequality is decreasing in ϕ , when ϕ is sufficiently high the inequality is violated. As a result, a housing bubble cannot exist.

5.4 Asset Purchases

The government can manipulate the interest rate by intervening in the private bond market. When the government participates in trading in the private bond market, the bond market-clearing condition is given by

$$\int_0^1 B_{jt} dj = B_{gt}, \quad (43)$$

where $B_{gt} > 0$ represents the government's bond holdings. The government budget constraint is given by

$$\int (1 - \tau_{jt}) I_{jt} dj + \frac{B_{gt+1}}{R_{ft}} = B_{gt} + \Gamma_t,$$

where Γ_t represents lump-sum taxes net of government spending.

By Proposition 1 and the bond market clearing condition (43), we can derive aggregate investment

$$I_t = [R_{kt} Y_t + (1 - \omega + \omega\theta) P_t - B_{gt}] \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau.$$

Other equilibrium conditions described in Proposition 2 remain unchanged.

Note that the government's trading in the private bond market does not affect the borrowing constraint (5) faced by the entrepreneurs. But it does affect the entrepreneurs' debt liabilities. When setting $B_{gt} = (1 - \omega + \omega\theta) P_t$, then the debt liabilities to the government will offset the liquidity benefit provided by the house. In this case investment I_t is effectively financed by internal funds only. Hence, a housing bubble does not provide any liquidity to entrepreneurs and hence it cannot exist. The economy will reach the bubbleless equilibrium. Once the bubbleless equilibrium is reached, the government does not need to purchase any private bonds since $B_{gt} = 0$.

²²Note that we need conditions (32) and

$$\beta^{-1} - 1 < \theta \int_{\frac{1-\phi-\theta}{1+\phi-\theta} \tau_{\max} < \tau \leq \tau_{\max}} \frac{\tau_{\max} - \tau}{\tau(1+\phi)} f(\tau) d\tau + \int_{\tau \leq \frac{1-\phi-\theta}{1+\phi-\theta} \tau_{\max}} \left[[\theta\omega + (1-\omega)(1-\phi)] \frac{\tau_{\max} - \tau}{\tau(1+\phi)} - \frac{2\phi(1+\omega)}{1+\phi} \right] f(\tau) d\tau$$

to hold. The latter condition is analogous to (29).

Notice that the interest rate in the bubbleless steady state is lower than that in the bubbly steady state by Proposition 5. The intuition is that the government purchase of private bonds raises the demand for private bonds and hence the bond price. This means that the asset purchase policy will not only eliminate housing bubbles, but also lower the interest rate in the long run. This result contradicts the usual view that the central bank should increase interest rates in response to a growing bubble. Thus the leaning against the wind monetary policy lacks a theoretical foundation. Galí (2013) also makes this point in an overlapping generations model of bubbles.

6. Conclusion

In this paper we have presented a theory of credit-driven housing bubbles in an infinite-horizon production economy, in which entrepreneurs face idiosyncratic investment tax distortions and credit constraints. To focus on the speculative nature of rational bubbles, we assume that housing is an intrinsically useless asset. Housing also serves as collateral for borrowing. A housing bubble can form because it commands a liquidity premium. The housing bubble can provide liquidity and relax credit constraint, but it can also generate inefficient overinvestment. Property taxes, Tobin's taxes, macroprudential policy, and credit policy can prevent the housing bubble.

For future research, it would be interesting to introduce housing rents and study the disconnect between housing prices and rents. Miao, Wang, and Zha (2014) have provided such a study. For simplicity, we have ignored aggregate uncertainty and the volatility generated by housing bubbles. Excessive volatility is also a potential cost of housing bubbles. In addition, asset bubbles may contribute to business cycles. Introducing bubbles into the dynamic stochastic general equilibrium framework and studying their quantitative implications should be an exciting research topic. Miao, Wang, and Xu (2013) and Miao, Wang, and Zha (2014) have initiated such research.

Appendix

A Data Description

We download the data from the Department of Economics of Queen's University via the link [www.econ.queensu.ca/files/other/House_Price_indices%20\(OECD\).xls](http://www.econ.queensu.ca/files/other/House_Price_indices%20(OECD).xls). All series are quarterly and seasonally adjusted. The data are defined as follows.

1. The nominal house price index of the US is the all-transaction index (estimated using sales price and appraisal data) from Federal Housing Finance Agency (FHFA).
2. The nominal house price index of Japan is the nationwide urban land price index from the Japan Real Estate Institute.
3. The nominal house price index of Spain is the average price per square meter of private housing (more than one year old) from the Bank of Spain.
4. The nominal house price index of Greece is the price per square meter of residential properties (all flats) in urban areas from the Bank of Greece.
5. The real house price index used in Figure 1 is the above nominal house price index deflated by the private consumption deflator. The average real index in 2000 is normalized to 100.
6. The price-income ratio used in Figure 1 is the ratio of the nominal house price index to the nominal per capita disposable income. The sample average is normalized to 100.
7. The price-rental ratio used in Figure 1 is the ratio of the nominal house price index to the rent component of the consumer price index. The sample average is normalized to 100.

B Proofs

Proof of Proposition 1: (i) By (12) and (13), when $\tau_{jt} \leq Q_t$, we must have

$$\tau_{jt} I_{jt} = R_{kt} K_{jt} + \frac{B_{jt+1}}{R_{ft}} - B_{jt} - P_t (H_{jt+1} - H_{jt}). \quad (\text{B.1})$$

In addition, it follows from (5) and (7) that both the borrowing and resaleability constraints bind. When $\tau_{jt} > Q_t$, it follows from (8) that $I_{jt} = 0$. Because B_{jt+1} and H_{jt+1} are canceled out in the objective of (12), the entrepreneur is indifferent among the feasible choices of B_{jt+1} and H_{jt+1} .

(ii) Substituting the decision rules in part (i) into (12) and matching coefficients, we obtain

$$v_t(\tau_{jt}) = \begin{cases} \frac{Q_t}{\tau_{jt}} R_{kt} + (1 - \delta) Q_t & \text{if } \tau_{jt} \leq Q_t, \\ R_{kt} + (1 - \delta) Q_t & \text{if } \tau_{jt} > Q_t \end{cases}, \quad (\text{B.2})$$

$$p_t(\tau_{jt}) = \begin{cases} P_t + (1 - \omega + \omega\theta) \left(\frac{Q_t}{\tau_{jt}} - 1 \right) P_t & \text{if } \tau_{jt} \leq Q_t, \\ P_t & \text{if } \tau_{jt} > Q_t \end{cases}, \quad (\text{B.3})$$

$$\varphi_t(\tau_{jt}) = \begin{cases} \frac{Q_t}{\tau_{jt}} & \text{if } \tau_{jt} \leq Q_t, \\ 1 & \text{if } \tau_{jt} > Q_t \end{cases}. \quad (\text{B.4})$$

Using equations (10) and (11), and the definition of Q_t , we can derive equations (14), (15), and (16). The transversality conditions follow from the infinite-horizon dynamic optimization problem, e.g., Ekeland and Scheinkman (1986). Q.E.D.

Proof of Proposition 2: By part (i) of Proposition 1, we can derive aggregate investment

$$I_t = \int_{\tau_{jt} \leq Q_t} \frac{1}{\tau_{jt}} [R_{kt} K_{jt} + (1 - \omega + \theta\omega) P_t H_{jt} - B_{jt}] dj.$$

Since τ_{jt} is independently and identically distributed and since K_{jt} , B_{jt} , and H_{jt} are predetermined, τ_{jt} is independent of these variables. By a law of large numbers, we obtain

$$\begin{aligned} I_t &= \frac{\int_{\tau_{jt} \leq Q_t} \frac{1}{\tau_{jt}} [R_{kt} K_{jt} + (1 - \omega + \theta\omega) P_t H_{jt} - B_{jt}] dj}{\int_{\tau_{jt} \leq Q_t} dj} \int_{\tau_{jt} \leq Q_t} dj \\ &= \left[\int_{\tau_{jt} \leq Q_t} dj \right] \left[\frac{\int_{\tau_{jt} \leq Q_t} \frac{1}{\tau_{jt}} dj}{\int_{\tau_{jt} \leq Q_t} dj} \right] \left[\frac{\int_{\tau_{jt} \leq Q_t} [R_{kt} K_{jt} + (1 - \omega + \theta\omega) P_t H_{jt} - B_{jt}] dj}{\int_{\tau_{jt} \leq Q_t} dj} \right] \\ &= \int_{\tau_{jt} \leq Q_t} \frac{1}{\tau_{jt}} dj \left[R_{kt} \int K_{jt} dj + (1 - \omega + \theta\omega) P_t \int H_{jt} dj - \int B_{jt} dj \right] \\ &= [R_{kt} K_{jt} + (1 - \omega + \theta\omega) P_t] \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau, \end{aligned}$$

where we have used the market clearing conditions to derive the last equality.

By (1) and the labor market-clearing condition,

$$1 = N_t = \int N_{jt} dj = \left(\frac{1-\alpha}{W_t} \right)^{\frac{1}{\alpha}} \int K_{jt} dj = \left(\frac{1-\alpha}{W_t} \right)^{\frac{1}{\alpha}} K_t.$$

From this equation, we can derive other equations in the proposition. Q.E.D.

Proof of Proposition 3: The right hand side of (26) is strictly decreasing in Q_f . When Q_f approaches the lower support of the distribution for τ_{jt} , the right-hand side approaches infinite. When Q_f approaches the upper support of the distribution, the right-hand side approaches $\beta\delta < 1 - \beta(1 - \delta)$. Thus, by the Intermediate Value Theorem, there is a unique solution $Q_f \in (\tau_{\min}, \tau_{\max})$ to equation (26). Condition (27) ensures that $C_f > 0$. Q.E.D.

Proof of Propositions 4 and 5: The proof consists of two parts.

Part I. Suppose that the bubbly and bubbly steady states coexist. We then prove Proposition 5 and the necessity of condition (33).

Step 1. We prove $Q_b < Q_f$. By equation (26),

$$1 - \beta(1 - \delta) = \beta\delta \frac{\int \max\left(\frac{1}{\tau}, \frac{1}{Q_f}\right) f(\tau) d\tau}{\int_{\tau \leq Q_f} \frac{1}{\tau} f(\tau) d\tau}. \quad (\text{B.5})$$

Equation (31) and $P > 0$ imply that

$$R_{kb} < \frac{\delta}{\int_{\tau \leq Q_b} \frac{1}{\tau} f(\tau) d\tau}.$$

Combining equation (30) and the preceding inequality, we can derive that

$$\begin{aligned} 1 - \beta(1 - \delta) &= \beta R_{kb} \int \max\left(\frac{1}{\tau}, \frac{1}{Q_b}\right) f(\tau) d\tau \\ &< \beta\delta \left[1 + \frac{1 - F(Q_b)}{Q_b \int_{\tau \leq Q_b} \frac{1}{\tau} f(\tau) d\tau} \right]. \end{aligned}$$

Combining the preceding inequality with (B.5) yields

$$\beta\delta \frac{\int \max\left(\frac{1}{\tau}, \frac{1}{Q_f}\right) f(\tau) d\tau}{\int_{\tau \leq Q_f} \frac{1}{\tau} f(\tau) d\tau} < \beta\delta \left[1 + \frac{1 - F(Q_b)}{Q_b \int_{\tau \leq Q_b} \frac{1}{\tau} f(\tau) d\tau} \right].$$

This inequality is equivalent to the following inequality:

$$\frac{1 - F(Q_b)}{Q_b \int_{\tau \leq Q_b} \frac{1}{\tau} f(\tau) d\tau} > \frac{1 - F(Q_f)}{Q_f \int_{\tau \leq Q_f} \frac{1}{\tau} f(\tau) d\tau}.$$

Thus $Q_b < Q_f$.

Step 2. We prove $R_{kf} > R_{kb}$. The steady-state version of equation (15) is given by

$$1 = \beta \left[(1 - \delta) + R_k \int \max\left(\frac{1}{\tau}, \frac{1}{Q}\right) f(\tau) d\tau \right]. \quad (\text{B.6})$$

The above equation implies that $R_{kf} > R_{kb}$ since $Q_b < Q_f$.

Step 3. Because $R_{kb} = \alpha K_b^{\alpha-1} < R_{kf} = \alpha K_f^{\alpha-1}$, we have $K_b > K_f$. Hence, $Y_b = K_b^\alpha > Y_f = K_f^\alpha$, $I_b = \delta K_b > I_f = \delta K_f$. In addition, equation (16) implies that $1/R_{fb} = \beta \int \max(\frac{Q_b}{\tau}, 1) f(\tau) d\tau < \beta \int \max(\frac{Q_f}{\tau}, 1) f(\tau) d\tau = 1/R_{ff}$, i.e., $R_{fb} > R_{ff}$.

Step 4. In the bubbly steady state, equation (14) implies that

$$1 = \beta \left[1 + (1 - \omega + \omega\theta) \int_{\tau \leq Q_b} \frac{Q_b - \tau}{\tau} f(\tau) d\tau \right]. \quad (\text{B.7})$$

Since $Q_b < Q_f$, condition (33) must hold. This proves the necessity of (33) as well as Proposition 5.

Part II. Now, we suppose that conditions (29), (32), and (33) hold. We then prove that the bubbly and bubbleless steady states coexist.

Step 1. The right-hand side of (28) is strictly increasing in Q_b . It is equal to 0 when $Q_b = \tau_{\min}$ and equal to $\tau_{\max} \int \frac{1}{\tau} f(\tau) d\tau - 1$ when $Q_b = \tau_{\max}$. If condition (29) holds, then (28) has a unique solution $Q_b \in (\tau_{\min}, \tau_{\max})$ by the Intermediate Value Theorem.

Step 2. By (B.7) and condition (33), $Q_b < Q_f$, where Q_f is given by equation (26). By Step 2 of Part I, $R_{kb} < R_{kf}$. Condition (32) implies that $R_{kf} > R_{kb} > \alpha\delta$. By Proposition 3, a bubbleless steady state exists.

Step 3. To show the existence of a bubbly steady state, we must show $C_b > 0$ and $P > 0$.

Condition (32) ensures that $C_b > 0$ holds. We now check $P > 0$. By (31),

$$\begin{aligned}
\frac{P}{Y_b} &= \frac{1}{1 - \omega(1 - \theta)} \left[\frac{\delta\alpha\beta}{1 - \beta(1 - \delta)} \frac{\int \max\left(\frac{1}{\tau}, \frac{1}{Q_b}\right) f(\tau) d\tau}{\int_{\tau \leq Q_b} \frac{1}{\tau} f(\tau) d\tau} - \alpha \right] \\
&> \frac{1}{1 - \omega(1 - \theta)} \left[\frac{\delta\alpha\beta}{1 - \beta(1 - \delta)} \frac{\int \max\left(\frac{1}{\tau}, \frac{1}{Q_f}\right) f(\tau) d\tau}{\int_{\tau \leq Q_f} \frac{1}{\tau} f(\tau) d\tau} - \alpha \right] \\
&= \frac{1}{1 - \omega(1 - \theta)} \left[\frac{\delta\alpha\beta}{1 - \beta(1 - \delta)} \frac{1 - \beta(1 - \delta)}{\beta\delta} - \alpha \right] \\
&= 0,
\end{aligned} \tag{B.8}$$

where the first inequality follows from $Q_f > Q_b$ by Step 2 of Part II and the second equality follows from equation (26). Q.E.D.

Proof of Proposition 6: Denote by F the cumulative distribution function of τ and define $J(Q_t) = \int_{\tau_{\min}}^{Q_t} \frac{1}{\tau} f(\tau) d\tau$. We can use Proposition 2 to show that the equilibrium system can be described by the following four difference equations:

$$C_t + K_{t+1} - (1 - \delta)K_t = K_t^\alpha, \tag{B.9}$$

$$\frac{Q_t}{C_t} = \beta \frac{1}{C_{t+1}} \{ (1 - \delta)Q_{t+1} + \alpha K_{t+1}^{\alpha-1} [Q_{t+1}J(Q_{t+1}) + 1 - F(Q_{t+1})] \}, \tag{B.10}$$

$$\frac{P_t}{C_t} = \beta \frac{P_{t+1}}{C_{t+1}} \{ 1 + (1 - \omega + \omega\theta)[Q_{t+1}J(Q_{t+1}) - F(Q_{t+1})] \}, \tag{B.11}$$

$$K_{t+1} - (1 - \delta)K_t = [\alpha K_t^\alpha + (1 - \omega + \omega\theta)P_t]J(Q_t), \tag{B.12}$$

for four unknowns $\{K_t, C_t, Q_t, P_t\}$. Only K_t is predetermined. The other three variables are nonpredetermined.

Linearizing P_t around zero and log-linearizing Q_t , K_t and C_t around their bubbleless steady state values, we obtain

$$\begin{aligned}
\frac{C_f}{K_f^\alpha} \hat{C}_t + \frac{K_f}{K_f^\alpha} \hat{K}_{t+1} - \frac{(1 - \delta)K_f}{K_f^\alpha} \hat{K}_t &= \alpha \hat{K}_t, \\
\hat{Q}_t - \hat{C}_t &= -\hat{C}_{t+1} - (1 - \alpha)[1 - \beta(1 - \delta)]\hat{K}_{t+1} + \beta \hat{Q}_{t+1}, \\
P_t &= \beta \{ 1 + (1 - \omega + \omega\theta)[Q_f J(Q_f) - F(Q_f)] \} P_{t+1}, \\
\hat{K}_{t+1} - (1 - \delta)\hat{K}_t &= \delta \frac{f(Q_f)}{J(Q_f)} \hat{Q}_t + \alpha \delta \hat{K}_t + (1 - \omega + \omega\theta) \frac{J(Q_f)}{K_f} P_t,
\end{aligned} \tag{B.13}$$

where $\hat{C}_t$, $\hat{K}_t$, and $\hat{Q}_t$ denote log-deviation from the steady state. We rewrite the system in the

following matrix form:

$$B \begin{bmatrix} \hat{C}_{t+1} \\ \hat{K}_{t+1} \\ \hat{Q}_{t+1} \\ P_{t+1} \end{bmatrix} = G \begin{bmatrix} \hat{C}_t \\ \hat{K}_t \\ \hat{Q}_t \\ P_t \end{bmatrix},$$

where

$$B = \begin{bmatrix} 0 & -\frac{K_f}{K_f^\alpha} & 0 & 0 \\ -1 & -(1-\alpha)(1-\beta+\beta\delta) & \beta & 0 \\ 0 & 0 & 0 & B_{34} \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} \frac{C_f}{K_f^\alpha} & -\frac{(1-\delta)K_f}{K_f^\alpha} - \alpha & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 - \delta + \alpha\delta & \delta \frac{f(Q_f)}{J(Q_f)} & (1 - \omega + \omega\theta) \frac{J(Q_f)}{K_f} \end{bmatrix},$$

with

$$B_{34} \equiv \beta\{1 + (1 - \omega + \omega\theta)[Q_f J(Q_f) - F(Q_f)]\}.$$

It is straightforward to check that G is invertible.

To study the local dynamics around the bubbleless steady state, we study the eigenvalues of the matrix $M \equiv G^{-1}B$.

First, we check that 0 must be an eigenvalue of matrix M . Note that matrix B is singular because its columns 1 and 3 are linearly dependent. Thus $\det(M) = \det(M - 0 \cdot I) = 0$, implying that 0 is an eigenvalue.

Second, note that

$$\begin{aligned} \det(M - B_{34} \cdot I) &= \det[G^{-1}B - B_{34} \cdot G^{-1}G] \\ &= \det(G^{-1}) \cdot \det(B - B_{34}G) \\ &= 0. \end{aligned}$$

Thus $B_{34} = \beta\{1 + (1 - \omega + \omega\theta)[Q_f J(Q_f) - F(Q_f)]\}$ is an eigenvalue of matrix M . Let $\lambda_1 \equiv B_{34}$ denote this eigenvalue.

Third, we can show that the other two eigenvalues are positive real numbers, with one greater than 1 and the other smaller than 1. Let λ_2 and λ_3 denote these two eigenvalues. We can then write

$$\det(M - \lambda I) = \lambda(\lambda_1 - \lambda)(-\lambda^2 + b\lambda + c),$$

where

$$\begin{aligned}
b &\equiv \frac{1}{d} \left\{ [1 + \beta(1 - \delta + \alpha\delta)] \frac{C_f}{K_f^\alpha} + (1 - \alpha)(1 - \beta + \beta\delta) \delta \frac{f(Q_f)}{J(Q_f)} \frac{C_f}{K_f^\alpha} \right. \\
&\quad \left. + (2 - \delta)\alpha f(Q_f) + \alpha\delta \frac{f(Q_f)}{J(Q_f)} \right\}, \\
c &\equiv -\frac{1}{d} \left[\beta \frac{C_f}{K_f^\alpha} + \delta K_f^{1-\alpha} \frac{f(Q_f)}{J(Q_f)} \right], \\
d &\equiv (1 - \delta + \alpha\delta) \frac{C_f}{K_f^\alpha} + \alpha(1 - \delta) f(Q_f) + \alpha\delta \frac{f(Q_f)}{J(Q_f)} > 0.
\end{aligned}$$

Since $c < 0$, it follows that $\lambda_2 \lambda_3 > 0$. We can also show that

$$-1 + b + c = \frac{\delta(1 - \alpha)}{d} \frac{C_f}{Y_f} \left[(1 - \beta) + (1 - \beta + \beta\delta) \frac{f(Q_f)}{J(Q_f)} \right] > 0.$$

Thus the quadratic equation always has two real solutions, with one larger than 1 and the other smaller than 1.

Without loss of generality, we suppose that $\lambda_2 < 1 < \lambda_3$. We then have two eigenvalues (0 and λ_2) inside the unit circle and one (λ_3) outside the unit circle. Whether the local dynamic around the bubbleless steady state is determinate depends on whether $\lambda_1 = \beta \{1 + (1 - \omega + \omega\theta) [Q_f J(Q_f) - F(Q_f)]\}$ is smaller than 1. By Proposition 4, when both the bubbly and bubbleless steady states exist, condition (33) must hold, i.e.,

$$\lambda_1 = \beta \{1 + (1 - \omega + \omega\theta) [Q_f J(Q_f) - F(Q_f)]\} > 1,$$

implying that the matrix M has two eigenvalues outside the unit circle and two eigenvalues inside the unit circle. Since there are three nonpredetermined variables, this means that the bubbleless steady state is a saddle with indeterminacy of degree 1.

When only the bubbleless steady state exists, we must have

$$\lambda_1 = \beta \{1 + (1 - \omega + \omega\theta) [Q_f J(Q_f) - F(Q_f)]\} \leq 1.$$

If $\lambda_1 < 1$, then the matrix M has three eigenvalues inside the unit circles and one eigenvalue outside the unit circle, implying that the local dynamic is determinate. If $\lambda_1 = 1$, then (B.13) implies that $P_t = P_{t+1}$. Since $\lim_{t \rightarrow +\infty} P_t = 0$, it follows that $P_t = 0$ for all t . In both cases, there is a unique equilibrium, which is bubbleless. Q.E.D.

Proof of Proposition 7: See the proofs of Proposition 8 and Lemma 2. Q.E.D.

Proof of Proposition 8: Consider the equilibrium without bubble first. Let $\delta = 1$ and $\omega(1 - \theta) = 0$. Equation (15) becomes

$$\frac{Q_t}{(1 - s_t)Y_t} = \frac{\beta}{(1 - s_{t+1})Y_{t+1}} \left[\frac{\alpha Y_{t+1}}{s_t Y_t} \int \max\left(\frac{Q_{t+1}}{\tau}, 1\right) f(\tau) d\tau \right].$$

We can further reduce the above equation to

$$\frac{Q_t}{1 - s_t} = \frac{\beta}{1 - s_{t+1}} \frac{\alpha}{s_t} \int \max\left(\frac{Q_{t+1}}{\tau}, 1\right) f(\tau) d\tau. \quad (\text{B.14})$$

Equation (20) implies that

$$K_{t+1} = I_t = \alpha Y_t \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau = s_t Y_t,$$

or

$$s_t = \alpha \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau. \quad (\text{B.15})$$

Hence the system of two difference equations (B.14) and (B.15) determine the bubbleless equilibrium trajectories for Q_t and s_t . Clearly a constant steady state is a solution to the system. The following lemma shows that this is the unique local solution.

Lemma 1 *There is a unique local solution to the system of two equations (B.14) and (B.15), which is the bubbleless steady state.*

Proof. We use F to denote the cumulative distribution function of τ and define $H(Q_t) \equiv \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau$. In the steady state, equations (B.14) and (B.15) imply that

$$Q_f = \frac{\beta\alpha}{s_f} [Q_f H(Q_f) + 1 - F(Q_f)],$$

$$s_f = \alpha H(Q_f).$$

Then log-linearizing the system around this steady state, we obtain

$$\hat{Q}_t + \frac{s_f}{1 - s_f} \hat{s}_t = \frac{s_f}{1 - s_f} \hat{s}_{t+1} - \hat{s}_t + \beta \hat{Q}_{t+1},$$

$$\hat{s}_t = \frac{f(Q_f)}{H(Q_f)} \hat{Q}_t.$$

We rewrite the above two equations as

$$B_f \begin{bmatrix} \hat{s}_{t+1} \\ \hat{Q}_{t+1} \end{bmatrix} = G_f \begin{bmatrix} \hat{s}_t \\ \hat{Q}_t \end{bmatrix},$$

where

$$B_f = \begin{bmatrix} \frac{s_f}{1-s_f} & \beta \\ 0 & 0 \end{bmatrix},$$

$$G_f = \begin{bmatrix} \frac{1}{1-s_f} & 1 \\ 1 & -\frac{f(Q_f)}{H(Q_f)} \end{bmatrix}.$$

In order to understand the local dynamics around this steady state, we need to study the two eigenvalues of the matrix $G_f^{-1}B_f$. They are 0 and λ_f where

$$\lambda_f = \frac{\frac{s_f}{1-s_f} \frac{f(Q_f)}{H(Q_f)} + \beta}{\frac{1}{1-s_f} \frac{f(Q_f)}{H(Q_f)} + 1}.$$

It is straightforward that $0 < \lambda_f < 1$ since $0 < \frac{f(Q_f)}{H(Q_f)} \frac{s_f}{1-s_f} < \frac{1}{1-s_f} \frac{f(Q_f)}{H(Q_f)}$ and $0 < \beta < 1$. This means the two eigenvalues are both inside the unit circle. Therefore there is a unique local solution to the system since both s_t and Q_t are nonpredetermined. ■

The above lemma shows that the steady state is the unique solution to the system of two equations (B.14) and (B.15) for s_t and Q_t in the neighborhood of the bubbleless steady state. We then use these two equations to determine Q_f by

$$\frac{1}{\beta} - 1 = \frac{1 - F(Q_f)}{Q_f \int_{\tau \leq Q_f} \frac{1}{\tau} f(\tau) d\tau}. \quad (\text{B.16})$$

Since $\lim_{Q_f \rightarrow \tau_{\min}} \frac{1-F(Q_f)}{Q_f \int_{\tau \leq Q_f} \frac{1}{\tau} f(\tau) d\tau} = +\infty$ and $\lim_{Q_f \rightarrow \tau_{\max}} \frac{1-F(Q_f)}{Q_f \int_{\tau \leq Q_f} \frac{1}{\tau} f(\tau) d\tau} = 0$, by the Intermediate Value Theorem, there is a unique solution in $(\tau_{\min}, \tau_{\max})$. Once Q_f is determined, then the saving rate is given by

$$s_f = \alpha\beta \int \max\left(\frac{1}{\tau}, \frac{1}{Q_f}\right) f(\tau) d\tau. \quad (\text{B.17})$$

from equation (B.14).

We need $s_f \in (0, 1)$ for a bubbleless equilibrium to exist. This condition is equivalent to (27). This is because (27) implies that $\frac{1}{\int_{\tau \leq Q_f} \frac{1}{\tau} f(\tau) d\tau} > \alpha$. By equation (B.15), $s_f = \alpha \int_{\tau \leq Q_f} \frac{1}{\tau} f(\tau) d\tau < 1$.

We compute the life-time utility as

$$U_f(K_0) = \sum_{t=0}^{\infty} \beta^t [\ln(1-s_f) + \ln(Y_t)] = \frac{\ln(1-s_f)}{1-\beta} + \alpha \sum_{t=0}^{\infty} \beta^t \ln(K_t),$$

where s_f is given by equation (B.17) and

$$\ln(K_{t+1}) = \ln(s_f Y_t) = \ln(s_f) + \alpha \ln(K_t).$$

Hence, the welfare for any given K_0 is given by

$$U_f(K_0) = \frac{\ln(1-s_f)}{1-\beta} + \frac{\alpha}{1-\alpha\beta} \left[\frac{\beta}{1-\beta} \ln(s_f) + \ln(K_0) \right]. \quad (\text{B.18})$$

Next, consider the equilibrium path to the bubbly steady state. Equation (B.14) still holds. Equation (20) becomes

$$K_{t+1} = I_t = (\alpha Y_t + P_t) \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau = s_t Y_t.$$

Dividing by Y_t on the two sides of this equation yields

$$s_t = (\alpha + p_t) \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau, \quad (\text{B.19})$$

where $p_t = P_t/Y_t$. Using $C_t = (1 - s_t) Y_t$, we rewrite equation (14) as

$$\frac{p_t}{1 - s_t} = \beta \frac{p_{t+1}}{1 - s_{t+1}} \int \max\left(\frac{Q_{t+1}}{\tau}, 1\right) f(\tau) d\tau. \quad (\text{B.20})$$

Therefore, the system of three difference equations (B.14), (B.19) and (B.20) determine three sequences for s_t , p_t , and Q_t . Clearly, the steady state is a solution to this system. The following lemma shows that it is a unique local solution.

Lemma 2 *There exists a unique solution $s_t = s_b$, $p_t = p_b$ and $Q_t = Q_b$ for all t to the system of three equations (B.14), (B.19) and (B.20) in the neighborhood of the bubbly steady state.*

Proof. Substituting (B.19) into (B.14) and (B.20), we obtain

$$\begin{aligned} \frac{Q_t}{1 - (\alpha + p_t) \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau} &= \frac{\beta \int \max\left(\frac{Q_{t+1}}{\tau}, 1\right) f(\tau) d\tau}{1 - (\alpha + p_{t+1}) \int_{\tau \leq Q_{t+1}} \frac{1}{\tau} f(\tau) d\tau} \frac{\alpha}{(\alpha + p_t) \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau}, \\ \frac{p_t}{1 - (\alpha + p_t) \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau} &= \frac{\beta p_{t+1} \int \max\left(\frac{Q_{t+1}}{\tau}, 1\right) f(\tau) d\tau}{1 - (\alpha + p_{t+1}) \int_{\tau \leq Q_{t+1}} \frac{1}{\tau} f(\tau) d\tau}. \end{aligned}$$

As before, denote $J(Q_t) = \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau$. Then $\int \max\left(\frac{Q_{t+1}}{\tau}, 1\right) f(\tau) d\tau = Q_{t+1} J(Q_{t+1}) + 1 - F(Q_{t+1})$. At the bubbly steady state, the two equations above imply that

$$1 = \beta [Q_b J(Q_b) + 1 - F(Q_b)],$$

$$\alpha = Q_b (\alpha + p_b) J(Q_b).$$

We log-linearize these two difference equations around the bubbly steady state and obtain

$$\begin{aligned}
& \left[1 + \frac{s_b}{1-s_b} \frac{p_b}{\alpha + p_b} \right] \hat{p}_t + \frac{s_b}{1-s_b} \frac{f(Q_b)}{J(Q_b)} \hat{Q}_t \\
= & \left[1 + \frac{s_b}{1-s_b} \frac{p_b}{\alpha + p_b} \right] \hat{p}_{t+1} + \left[\beta \frac{\alpha}{\alpha + p_b} + \frac{s_b}{1-s_b} \frac{f(Q_b)}{J(Q_b)} \right] \hat{Q}_{t+1}, \\
& \frac{1}{1-s_b} \frac{p_b}{\alpha + p_b} \hat{p}_t + \left[1 + \frac{1}{1-s_b} \frac{f(Q_b)}{J(Q_b)} \right] \hat{Q}_t \\
= & \frac{s_b}{1-s_b} \frac{p_b}{\alpha + p_b} \hat{p}_{t+1} + \left[\beta \frac{\alpha}{\alpha + p_b} + \frac{s_b}{1-s_b} \frac{f(Q_b)}{J(Q_b)} \right] \hat{Q}_{t+1}.
\end{aligned}$$

We rewrite the two equations above in the following form

$$B_b \begin{bmatrix} \hat{p}_{t+1} \\ \hat{Q}_{t+1} \end{bmatrix} = G_b \begin{bmatrix} \hat{p}_t \\ \hat{Q}_t \end{bmatrix},$$

where

$$B_b = \begin{bmatrix} 1 + \frac{s_b}{1-s_b} \frac{p_b}{\alpha + p_b} & \beta \frac{\alpha}{\alpha + p_b} + \frac{s_b}{1-s_b} \frac{f(Q_b)}{J(Q_b)} \\ \frac{s_b}{1-s_b} \frac{p_b}{\alpha + p_b} & \beta \frac{\alpha}{\alpha + p_b} + \frac{s_b}{1-s_b} \frac{f(Q_b)}{J(Q_b)} \end{bmatrix}$$

and

$$G_b = \begin{bmatrix} 1 + \frac{s_b}{1-s_b} \frac{p_b}{\alpha + p_b} & \frac{s_b}{1-s_b} \frac{f(Q_b)}{J(Q_b)} \\ \frac{1}{1-s_b} \frac{p_b}{\alpha + p_b} & 1 + \frac{1}{1-s_b} \frac{f(Q_b)}{J(Q_b)} \end{bmatrix}.$$

As before, we need to check the eigenvalues of the matrix $G_b^{-1}B_b$. The characteristic function of the matrix $G_b^{-1}B_b$ is $\lambda^2 + b\lambda + c$ where

$$b \equiv -\frac{1}{d} \left[\beta \left(\frac{\alpha}{\alpha + p_b} \right)^2 + \frac{s_b}{1-s_b} \frac{p_b}{\alpha + p_b} \frac{f(Q_b)}{J(Q_b)} + 1 + \frac{1}{1-s_b} \frac{f(Q_b)}{J(Q_b)} + \frac{s_b}{1-s_b} \frac{p_b}{\alpha + p_b} \right] < 0,$$

$$c \equiv \frac{1}{d} \left[\beta \frac{\alpha}{\alpha + p_b} + \frac{s_b}{1-s_b} \frac{f(Q_b)}{J(Q_b)} \right] > 0,$$

$$d \equiv 1 + \frac{1}{1-s_b} \frac{f(Q_b)}{J(Q_b)} + \frac{s_b}{1-s_b} \frac{p_b}{\alpha + p_b} > 0.$$

We then prove the following two facts: (1) $0 < c < 1$; (2) $1 + b + c > 0$.

(1) Claim $0 < c < 1$.

Since

$$\begin{aligned}
0 & \leq \frac{s_b}{1-s_b} \frac{f(Q_b)}{J(Q_b)} \leq \frac{1}{1-s_b} \frac{f(Q_b)}{J(Q_b)}, \\
0 & < \beta \frac{\alpha}{\alpha + p_b} < 1,
\end{aligned}$$

then

$$\begin{aligned}
0 &< \frac{s_b}{1-s_b} \frac{f(Q_b)}{J(Q_b)} + \beta \frac{\alpha}{\alpha + p_b} \\
&< 1 + \frac{1}{1-s_b} \frac{f(Q_b)}{J(Q_b)} \\
&< 1 + \frac{1}{1-s_b} \frac{f(Q_b)}{J(Q_b)} + \frac{s_b}{1-s_b} \frac{p_b}{\alpha + p_b},
\end{aligned}$$

which implies $0 < c < 1$.

(2) Claim $1 + b + c > 0$.

We use the definition of b and c to compute

$$1 + b + c = \frac{\beta}{d} \frac{\alpha}{\alpha + p_b} \frac{p_b}{\alpha + p_b} > 0.$$

Given these four facts, if the two roots (denoted by λ_1 and λ_2) are real numbers, they must both be positive because $0 < \lambda_1 \lambda_2 = c < 1$ and $\lambda_1 + \lambda_2 = -b > 0$. Since $1 + b + c > 0$, the two roots must be smaller than 1, otherwise $\lambda_1 \lambda_2 > 1$. If the two eigenvalues are complex numbers, it follows from $0 < c = \lambda_1 \lambda_2 < 1$ that they must be inside the unit circle. We conclude that, in both cases, there is a unique local solution for p_t and Q_t since both are nonpredetermined variables. The solution is the bubbly steady state. We then use (B.19) to determine the solution for s_t , which is also the steady state value. ■

Now, we compute the bubbly equilibrium welfare for any given initial non-steady-state capital stock K_0 :

$$U_b(K_0) = \frac{\ln(1-s_b)}{1-\beta} + \frac{\alpha}{1-\alpha\beta} \left[\frac{\beta}{1-\beta} \ln(s_b) + \ln(K_0) \right]. \quad (\text{B.21})$$

We then compare $U_f(K_0)$ and $U_b(K_0)$. Note that the life-time utility in (35) as a function of the saving rate s is concave and has a maximum at $s = \alpha\beta$. Using (25) and $\delta = 1$, $s_f = \alpha/R_{kf} = \alpha \int_{\tau \leq Q_f} \frac{1}{\tau} f(\tau) d\tau$. By assumption, $s_f > \alpha$. By Proposition 5, $R_{kb} < R_{kf}$. Thus $s_b = \alpha/R_{kb} > \alpha/R_{kf} = s_f > \alpha > \alpha\beta$. Hence, $U_f(K_0) > U_b(K_0)$.

Another sufficient condition for $U_f(K_0) > U_b(K_0)$ is $\tau \in [0, 1]$. Under this condition,

$$\int \max\left(\frac{1}{\tau}, \frac{1}{Q_f}\right) f(\tau) d\tau > 1,$$

so that $s_b > s_f = \alpha\beta \int \max\left(\frac{1}{\tau}, \frac{1}{Q_f}\right) f(\tau) d\tau > \alpha\beta$, where the equation for s_f follows from (B.17).

Finally, we compare $U_f(K_f)$ and $U_b(K_b)$. In any steady state, $K = I = sK^\alpha$. It follows that the steady-state capital stock satisfies $K = s^{1/(1-\alpha)}$. Using (35), we can then compute the steady-state

welfare as

$$U = \frac{1}{1-\beta} \left[\ln(1-s) + \frac{\alpha}{1-\alpha} \ln(s) \right].$$

It is a concave function of s and is maximized at $s = \alpha$. Since $s_b > s_f > \alpha$, it follows that $U_f(K_f) > U_b(K_b)$. Q.E.D.

Proof of Proposition 9: We conjecture that entrepreneur j 's value function takes the form, $V_t(\tau_{jt}, K_{jt}, H_{jt}, B_{jt}) = v_t(\tau_{jt})K_{jt} + p_t(\tau_{jt})H_{jt} - \varphi_t(\tau_{jt})B_{jt}$, where v_t , p_t and φ_t are functions to be determined and satisfy

$$(1 + \phi)P_t = \beta \frac{\Lambda_{t+1}}{\Lambda_t} \int p_{t+1}(\tau_{jt+1})dj, \quad (\text{B.22})$$

$$\frac{1}{R_{ft}} = \beta \frac{\Lambda_{t+1}}{\Lambda_t} \int \varphi_{t+1}(\tau_{jt+1})dj. \quad (\text{B.23})$$

Denote $Q_t = \beta \frac{\Lambda_{t+1}}{\Lambda_t} \int v_{t+1}(\tau_{jt+1})dj$ as Tobin's marginal Q . Given the preceding conjecture, we can rewrite the Bellman equation as

$$\begin{aligned} & v_t(\tau_{jt})K_{jt} + p_t(\tau_{jt})H_{jt} - \varphi_t(\tau_{jt})B_{jt} \\ &= \max_{I_{jt}, H_{jt+1}} R_{kt}K_{jt} - B_{jt} + (Q_t - \tau_{jt})I_{jt} + Q_t(1 - \delta)K_{jt} \\ & \quad - P_t(H_{jt+1} - H_{jt}) - P_t\phi|H_{jt+1} - H_{jt}| + (1 + \phi)P_tH_{jt+1}, \end{aligned} \quad (\text{B.24})$$

subject to (4), (5), (7) and (8). Note that terms related to B_{jt+1} are canceled out in the Bellman equation. We have also ignored constraint (6) temporarily. We can easily check that this constraint is satisfied in equilibrium.

We first consider a low- τ entrepreneur with $\tau_{jt} \leq Q_t$. This entrepreneur would like to invest as much as possible until (4) and (5) bind. Thus,

$$\tau_{jt}I_{jt} = R_{kt}K_{jt} + \frac{B_{jt+1}}{R_{ft}} - B_{jt} - P_t(H_{jt+1} - H_{jt}) - \phi P_t|H_{jt+1} - H_{jt}|,$$

and

$$\frac{B_{jt+1}}{R_{ft}} = \theta P_t H_{jt+1}.$$

Substituting this investment rule into the preceding Bellman equation yields

$$\begin{aligned} & v_t(\tau_{jt})K_{jt} + p_t(\tau_{jt})H_{jt} - \varphi_t(\tau_{jt})B_{jt} \\ &= \max_{H_{jt+1} \geq 0} R_{kt}K_{jt} - B_{jt} - P_t(H_{jt+1} - H_{jt}) - P_t\phi|H_{jt+1} - H_{jt}| + Q_t(1 - \delta)K_{jt} \\ & \quad + \left(\frac{Q_t}{\tau_{jt}} - 1 \right) [R_{kt}K_{jt} - B_{jt} + \theta P_t H_{jt+1} - P_t(H_{jt+1} - H_{jt}) - P_t\phi|H_{jt+1} - H_{jt}|] \\ & \quad + (1 + \phi)P_t H_{jt+1}, \end{aligned} \quad (\text{B.25})$$

subject to (7).

Now consider the choice of H_{jt+1} . We claim that entrepreneur j will never buy houses (i.e. $H_{jt+1} > H_{jt}$) because this would imply that the marginal benefit of holding one more unit of house is negative, i.e., $-(1 + \phi - \theta) \left(\frac{Q_t}{\tau_{jt}} - 1 \right) P_t < 0$. It must be the case that $H_{jt+1} \leq H_{jt}$. We can then compute the marginal benefit of holding one more unit of house as

$$\left[2\phi - (1 - \phi - \theta) \left(\frac{Q_t}{\tau_{jt}} - 1 \right) \right] P_t.$$

This expression is positive when $\tau_{jt} > \frac{1-\phi-\theta}{1+\phi-\theta} Q_t$. In this case, entrepreneur j will keep buying until $H_{jt+1} = H_{jt}$. However, when $\tau_{jt} < \frac{1-\phi-\theta}{1+\phi-\theta} Q_t$, the marginal benefit of holding one more unit of house is negative so that entrepreneur j prefers to sell as many houses as possible until $H_{jt+1} = \omega H_{jt}$. In sum, optimal house holdings are given by

$$H_{jt+1} = \begin{cases} H_{jt} & \text{when } \frac{1-\phi-\theta}{1+\phi-\theta} Q_t < \tau_{jt} \leq Q_t \\ \omega H_{jt} & \text{when } \tau_{jt} \leq \frac{1-\phi-\theta}{1+\phi-\theta} Q_t \end{cases}.$$

Substituting the decision rule for H_{jt+1} above into the Bellman equation in (B.25), we can simplify the Bellman equation. In particular, for $\tau_{jt} \leq \frac{1-\phi-\theta}{1+\phi-\theta} Q_t$, the value function satisfies

$$\begin{aligned} & v_t(\tau_{jt})K_{jt} + p_t(\tau_{jt})H_{jt} - \varphi_t(\tau_{jt})B_{jt} \\ = & \frac{Q_t}{\tau_{jt}} (R_{kt}K_{jt} - B_{jt}) + Q_t(1 - \delta)K_{jt} \\ & + \left[(1 - \phi)(1 - \omega) + \left(\frac{Q_t}{\tau_{jt}} - 1 \right) [\theta\omega + (1 - \omega)(1 - \phi)] + \omega(1 + \phi) \right] P_t H_{jt}, \end{aligned} \tag{B.26}$$

where $\left[(1 - \phi)(1 - \omega) + \left(\frac{Q_t}{\tau_{jt}} - 1 \right) (1 - \omega)(1 - \phi) \right] P_t H_{jt}$ is the investment financed by selling a fraction $(1 - \omega)$ of the current house holdings net of transaction tax, $\left(\frac{Q_t}{\tau_{jt}} - 1 \right) \theta\omega P_t H_{jt}$ is the investment financed by borrowing using a fraction ω of the current house holdings as collateral, and $\omega(1 + \phi) P_t H_{jt}$ is the shadow value of the house.

For $\frac{1-\phi-\theta}{1+\phi-\theta} Q_t < \tau_{jt} \leq Q_t$, the value function satisfies

$$\begin{aligned} & v_t(\tau_{jt})K_{jt} + p_t(\tau_{jt})H_{jt} - \varphi_t(\tau_{jt})B_{jt} \\ = & \frac{Q_t}{\tau_{jt}} (R_{kt}K_{jt} - B_{jt}) + Q_t(1 - \delta)K_{jt} + \left[\left(\frac{Q_t}{\tau_{jt}} - 1 \right) \theta + (1 + \phi) \right] P_t H_{jt}, \end{aligned} \tag{B.27}$$

where $\left(\frac{Q_t}{\tau_{jt}} - 1 \right) \theta P_t H_{jt}$ is the investment financed by borrowing with the current house holdings as collateral, and $(1 + \phi) P_t H_{jt}$ is the shadow value of the house.

Next, consider a high- τ entrepreneur with $\tau_{jt} > Q_t$. In this case, investing is unprofitable so

that $I_{jt} = 0$. The Bellman equation in (B.24) becomes

$$\begin{aligned} & v_t(\tau_{jt})K_{jt} + p_t(\tau_{jt})H_{jt} - \varphi_t(\tau_{jt})B_{jt} \\ &= \max_{H_{jt+1} \geq 0} R_{kt}K_{jt} - B_{jt} - P_t(H_{jt+1} - H_{jt}) - P_t\phi|H_{jt+1} - H_{jt}| \\ & \quad + Q_t(1 - \delta)K_{jt} + (1 + \phi)P_tH_{jt+1}, \end{aligned} \quad (\text{B.28})$$

subject to (7). If $H_{jt+1} \leq H_{jt}$, then the marginal benefit of holding one more unit of house is $2\phi P_t > 0$. In this case, the entrepreneur will increase house holdings until $H_{jt+1} = H_{jt}$. If $H_{jt+1} \geq H_{jt}$, then the terms related to H_{jt+1} are canceled out in the preceding Bellman equation. This means that the entrepreneur is indifferent among any feasible choices of $H_{jt+1} \geq H_{jt}$. We can then rewrite (B.28) as

$$v_t(\tau_{jt})K_{jt} + p_t(\tau_{jt})H_{jt} - \varphi_t(\tau_{jt})B_{jt} = R_{kt}K_{jt} + (1 + \phi)P_tH_{jt} - B_{jt} + Q_t(1 - \delta)K_{jt}. \quad (\text{B.29})$$

Matching coefficients of K_{jt} , H_{jt} and B_{jt} on the two sides of equations (B.26), (B.27), and (B.29), respectively, we can derive expressions for $v_t(\tau_{jt})$, $p_t(\tau_{jt})$, and $\varphi_t(\tau_{jt})$. Substituting these expressions into (B.22), (B.23) and using the definition of Q_t , we obtain equations (41), (15), and (16) after some manipulation.

Using a law of large numbers, we compute aggregate investment as

$$\begin{aligned} I_t &= \int_{\tau_{jt} \leq \frac{1-\phi-\theta}{1+\phi-\theta}Q_t} \frac{1}{\tau_{jt}} \left[R_{kt}K_{jt} - B_{jt} + [\theta\omega + (1 - \omega)(1 - \phi)]P_tH_{jt} \right] dj \\ & \quad + \int_{\frac{1-\phi-\theta}{1+\phi-\theta}Q_t < \tau_{jt} \leq Q_t} \frac{1}{\tau_{jt}} \left[R_{kt}K_{jt} - B_{jt} + \theta P_tH_{jt} \right] dj \\ &= \int_{\tau_{jt} \leq \frac{1-\phi-\theta}{1+\phi-\theta}Q_t} \frac{1}{\tau_{jt}} dj \left[R_{kt} \int K_{jt} dj + [\theta\omega + (1 - \omega)(1 - \phi)]P_t \int H_{jt} dj - \int B_{jt} dj \right] \\ & \quad + \int_{\frac{1-\phi-\theta}{1+\phi-\theta}Q_t < \tau_{jt} \leq Q_t} \frac{1}{\tau_{jt}} dj \left[R_{kt} \int K_{jt} dj + \theta P_t \int H_{jt} dj - \int B_{jt} dj \right] \\ &= R_{kt}K_t \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau \\ & \quad + P_tH_t \left[[\omega\theta + (1 - \omega)(1 - \phi)] \int_{\tau \leq \frac{1-\phi-\theta}{1+\phi-\theta}Q_t} \frac{1}{\tau} f(\tau) d\tau + \theta \int_{\frac{1-\phi-\theta}{1+\phi-\theta}Q_t < \tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau \right]. \end{aligned}$$

We can also derive equations (21), (22), (23) and (24) as before. It is known from the literature that the transversality conditions are part of the necessary and sufficient conditions for optimality in infinite-horizon problems. Q.E.D.

C Endogenous Housing Supply

Following Poterba (1991) and He, Wright, and Zhu (2013), we introduce endogenous housing supply. Suppose that the cost function of producing new houses $H_{t+1} - H_t$ is given by $c(H_{t+1} - H_t)$, where $c' > 0$, $c'' > 0$, and $c'(0) > 0$. Profit maximization gives the housing supply function $P_t = c'(H_{t+1} - H_t)$. Suppose that housing suppliers also belong to the extended family and hand over their profits to the family. If the housing price is zero, no new house is produced. Aggregating the investment equation in Proposition 1, we can rewrite equation (20) as

$$I_t = [R_{kt}K_t + (1 - \omega + \theta\omega) P_t H_t] \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau. \quad (\text{C.1})$$

In the bubbleless equilibrium, $P_t = 0$ for all t . No new houses are produced and the solution is the same as in the baseline model. When there is a housing bubble, the housing supply equation gives the steady-state housing price $P = c'(0)$. We then use equation (C.1) to determine the steady-state housing stock $H > 0$. All the existence conditions and policy analysis in the main text still apply to this extension.

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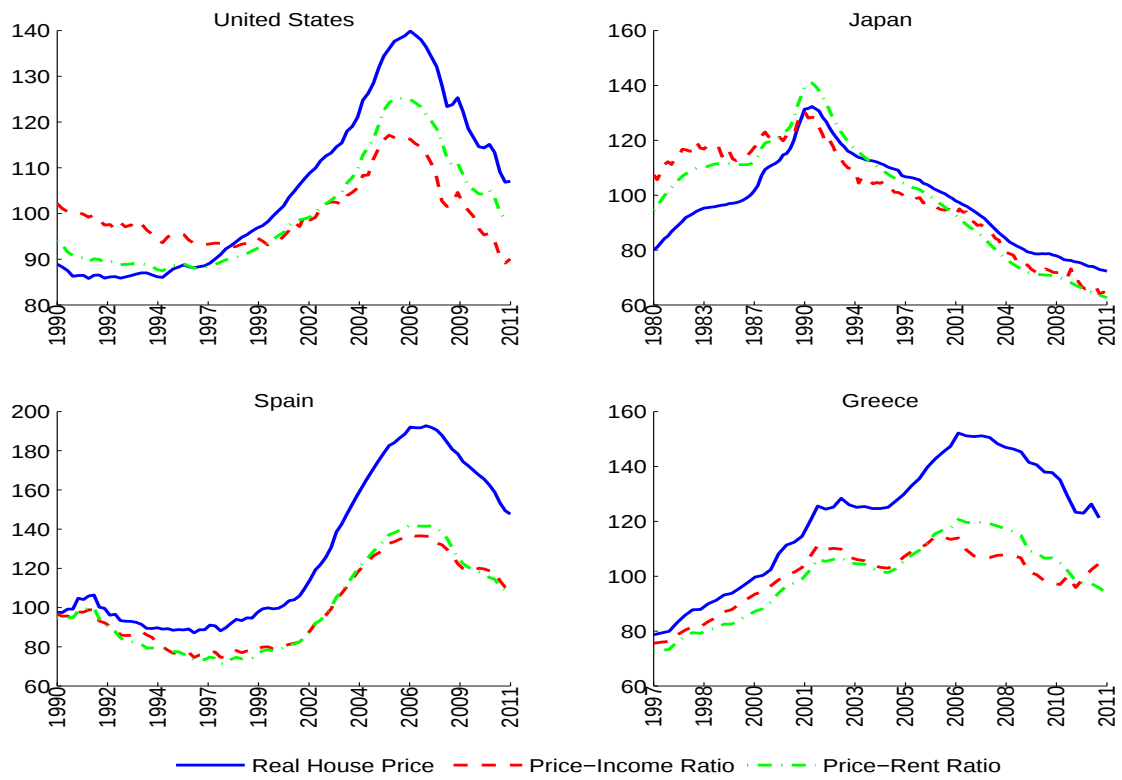


Figure 1: Real housing price indexes, price-income ratios, and price-rental ratios. See Appendix A for the data description.

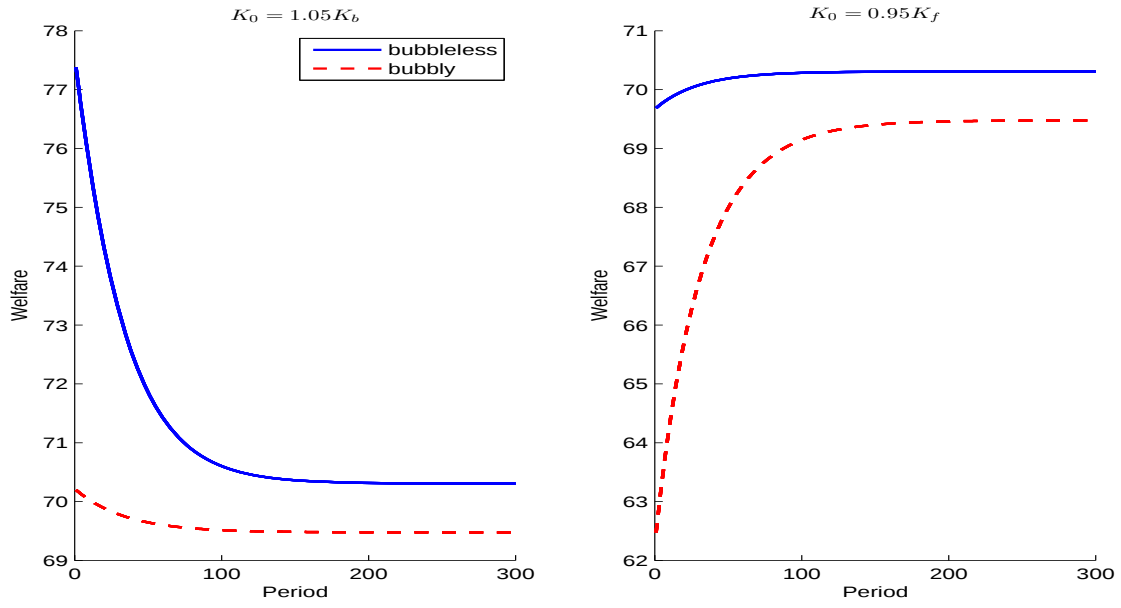


Figure 2: Transition paths of life-time utility levels. Parameter values are given by $\alpha = 0.3$, $\beta = 0.99$, $\delta = 0.025$, $\omega = 0.2$, $\theta = 0.75$, and $\eta = 5.7$.