

Transitional Dynamics and Long-Run Optimal Taxation under Incomplete Markets*

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Abstract

Aiyagari (1995) showed that long-run optimal fiscal policy features a positive tax rate on capital income in Bewley-type economies with heterogeneous agents and incomplete markets. However, determining the magnitude of the optimal capital income tax rate was considered to be prohibitively difficult due to the need to compute the optimal tax rates along the transition path. This paper shows that, in this class of models, long-run optimal fiscal policy and the corresponding allocation can be studied *independently* of the initial conditions and the transition path. Numerical methods based on this finding are used on a model calibrated to the U.S. economy. I find that the observed average capital income tax rate in the U.S. is *too high*, the average labor income tax rate and the debt-to-GDP ratio are *too low*, compared to the long-run optimal levels. The implications of these findings for existing literature on the optimal quantity of debt and constrained efficiency are also addressed.

1 Introduction

Aiyagari (1994a, 1995) showed that the optimal capital tax rate is positive even in the long run in Bewley-type models, where labor income risk is not perfectly insurable.¹ This result is in stark contrast with the earlier results of Chamley (1986) and Judd (1985) for economies with complete financial markets.² However, the analysis of the Ramsey problem in Bewley-type models has remained theoretical. Quantitative studies in the Ramsey tradition have provided only limited characterizations of optimal taxation, for instance, by choosing tax rates that maximize long-run welfare in the economy, or by imposing non-trivial restrictions on the policy tools available to the government. A quantitative analysis of long-run optimal taxation was

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¹See Hubbard and Judd (1986), Imrohoroglu (1998), Erosa and Gervais (2002), Domeij and Heathcote (2004), Conesa, Kitao, and Krueger (2009), Gottardi, Kajii, and Nakajima (2011), among others, for quantitative work and extensions in other environments.

²See Jones, Manuelli, and Rossi (1997), Atkeson, Chari, and Kehoe (1999), Chari and Kehoe (1999), and others for extensions.

considered to be a daunting task, because it was believed that the long-run allocation was dependent on the initial conditions of the economy as well as the transition path, as in models with complete markets.³

The main contribution of this paper is to show that this premise is premature in an infinite-horizon model with incomplete markets and heterogeneous agents. Using the necessary conditions for optimality, I illustrate that the long-run income tax rates, level of government debt, and the distribution of wealth and consumption can be investigated independently of the transition path, and without taking a stand on the initial conditions of an economy.⁴ I develop and apply numerical methods that rely on this observation to compute these levels for the U.S. economy. A simple comparison of the quantitative results with the corresponding values for the U.S. economy suggests that, (i) the government debt-to-GDP ratio is too low and ought to be *much* higher, (ii) the average labor income tax rate in the U.S. economy ought to be higher, and (iii) the average capital income tax rate ought to be *much* lower than the currently observed levels.

The disconnect between long-run optimal allocation and initial conditions arises from a characteristic feature of Bewley-type models. As long as households are impatient relative to the after-tax interest rate, the long-run allocation in the economy is characterized by a stationary invariant distribution that is independent of the initial conditions. In addition, if there are tight borrowing constraints, so that some of the agents are credit-constrained in each period, the distortions induced by future tax rates on earlier periods are irrelevant asymptotically. Although the Ramsey problem is non-stationary in general, in a Bewley-type environment, the solution to the problem resembles that of a stationary problem. In the long run, fiscal policy and the induced allocation depend only on the “deep parameters” of the model along with the underlying income process.

The seemingly extreme quantitative results for the U.S. economy are not surprising once we understand how the policy tools available to the government interact with the main frictions in a Bewley-type economy. By issuing debt, the government effectively relaxes the borrowing constraints of private agents.⁵ The government finances the interest payments on steady-state debt by taxing labor income heavily, reducing the share of income that is stochastic, and effectively attenuating the income risk the private agents face. The quantitative results suggest that in the long run, it is efficient for the government to maintain the *maximal debt* that can be supported under the modified golden rule, i.e. when capital-labor ratio (or capital intensity) is equal

³In Aiyagari (1994a)’s words (pg. 21-22), “[The question of whether actual tax policy is long-run optimal] can only be resolved by computing the solution path for the optimal tax problem and the associated limiting values. This computational problem is very hard,... because the consumer’s problem is non-stationary and one of the state variables for the economy is the cross-section distribution of asset holdings, which is an infinite dimensional variable.”

⁴This is in sharp contrast to models with certainty or complete markets. Surprisingly, in these models, without making a reference to the initial conditions, we can say very little aside from optimal zero capital income tax in the long run. See Lucas (1990) and Auerbach and Kotlikoff (1987) for further details.

⁵See Aiyagari and McGrattan (1998), Altig and Davis (1989), Hayashi (1985), Heathcote (2005), Holmstrom and Tirole (1998), Hubbard and Judd (1986), Kocherlakota (2007), Vissing-Jorgensen and Krishnamurthy (2008), Shapiro and Slemrod (2003), Woodford (1990), and Yotsuzuka (1987).

to the first-best level. Along with high labor income tax rates, the government achieves a close-to-minimal average labor income share in the economy. In the long run, instead of imposing high capital tax rates to suppress precautionary motives for asset accumulation, the government acts to alleviate the *source* of the problem that leads to inefficient level of savings. Interestingly, the need to finance a stream of government expenditures has nothing to do with the qualitative features of long-run optimal fiscal policy, supporting the view that inefficiencies induced by incomplete markets drive these results.

The quantitative exercise in this paper also suggests that the Ramsey planner delivers a flow welfare level in the long run that is *significantly lower* than those that can be achieved by alternative tax rates. This result warns us about the quantitative literature that relies exclusively on maximizing the flow welfare in the long run. There are at least two reasons why such limited results are misleading. First, given the initial state of the economy, transition to a candidate steady state (in particular, one that maximizes welfare in the long run) can potentially require extreme policy measures to be implemented over any feasible transition path. In this case, a benevolent tax authority would not implement a policy that leads to the prescribed steady state since the policy would not be ex-ante optimal. Such fiscal policy reform might not even be politically feasible due to the heavy burden it imposes on the initial periods/generations. Second, a pure steady-state analysis disregards the intertemporal incentives of the tax authority. By contrast, using the necessary conditions for optimality, I look specifically for long-run optimal allocations that could be achieved following an optimal transition path. In the quantitative section, I illustrate that a social planner who maximizes the flow welfare in the long run would choose a very high capital income tax rate, a very low labor income tax rate, and a *negative* government debt for the U.S. economy, qualitatively the complete opposite of the long-run optimal fiscal policy.

The steady-state results I provide downplay the appeal of providing a complete solution to the Ramsey problem for the sole purpose of determining long-run optimal fiscal policy. Therefore, an optimal transition analysis is left for future research. On the other hand, the results highlight the importance of the transition path in evaluating the *source* of the welfare gains. Since, by construction, the Ramsey planner maximizes average discounted welfare over the entire time horizon, and settles with a relatively low steady-state welfare, studying the optimal transition path is *the only way* to understand where the welfare gains come from. To gain some insight on these issues, I provide a constrained transition analysis where I illustrate the existence of feasible and welfare-improving dynamic paths leading to the long-run optimal allocation, even when the economy starts from an allocation that features the highest possible steady-state welfare. In all of these transition paths, welfare gains come from consumption front-loading, and/or a significant reduction in consumption inequality.

The following section presents the benchmark model and the notions of equilibrium in detail. Section

3 is a steady-state analysis of the Ramsey problem under incomplete markets, illustrating the main result of the paper. This is followed by calibration of the benchmark model in Section 4, and quantitative results for the U.S. economy in Section 5. Section 6 provides a constrained transition analysis of the optimal fiscal policy. Section 7 contains a literature review and a discussion of the main findings in relation to the earlier results. Section 8 concludes the paper.

2 The Model

In this section, I provide the details of the optimal fiscal policy problem in a Bewley-type model with heterogeneous agents and incomplete markets. To facilitate a comparative discussion, I consider a setup that is very similar to the one used by Aiyagari (1994a).

2.1 The Environment

Time $t \in \{0, 1, \dots, \infty\}$ is discrete. There is a continuum of ex-ante identical households of measure one, a representative competitive firm, and a benevolent government that has access to a commitment technology. There are no aggregate shocks.

The government taxes or subsidizes capital income at rate τ_{kt} , market labor income at rate τ_{nt} . It also issues debt B_t , and finances an exogenous and constant stream of government expenditures $G_t = G$.⁶ Without loss of generality, I assume that tax burden is on the households. Let r_t and w_t represent the interest rate and the wage at time t and let $\bar{r}_t = (1 - \tau_{kt})r_t \geq 0$ and $\bar{w}_t = (1 - \tau_{nt})w_t \geq 0$ represent factor prices net of taxes.

Households

In every period, each household is subject to an idiosyncratic labor productivity shock e_t that follows a discrete, first-order Markov process with transition matrix M and support $E = \{e_1, \dots, e_k\}$. I assume that there is a unique non-degenerate stationary distribution π . Let $(E, \mathcal{E})$ denote the measurable space of productivity where $\mathcal{E}$ denotes all subsets of E . Let $(E^t, \mathcal{E}^t)$ denote the product space of labor efficiency shocks up to and including period t . Let $h^t = \{e_0, e_1, e_2, \dots, e_t\} \in E^t$ represent a particular realization of idiosyncratic productivities up until time t and define $\Pi : \mathcal{E}^t \rightarrow [0, 1]$ to denote the probability measure over the product space of labor productivities. With some abuse of notation, I use $\Pi(h^t)$ to denote the date-0 probability of realization of the history of shocks h^t . Assuming that a law of large numbers holds, $\Pi(h^t)$ also represents

⁶None of the steady-state results depend on the assumption that government expenditures are constant, as long as the expenditures converge to some fixed level G in the long run.

the mass of agents with this particular realization of history at time t . At period 0, agents draw from the unconditional distribution π .

Financial markets are incomplete and agents only have access to a single risk-free asset that represents claims to physical capital. In each period, agents are subject to an exogenous borrowing constraint $a_{t+1} \geq -\underline{a}$ that is tighter than the *natural borrowing limit*. For illustrative purposes and to rationalize an egalitarian objective, I assume that all agents enter period-0 with assets a_0 . I relax this assumption later for the quantitative results. Let $\mathcal{A} = [-\underline{a}, \infty) \subset \mathbb{R}$ denote the space for assets.

Households have access to two productive technologies. They can either work in the market (using a fraction $0 \leq n_t \leq 1$ of total time) and earn the market wage net of taxes, $e_t \bar{w}_t n_t$, or use the alternative tax-free home-production technology $H(1 - n_t)$ which satisfies $H'(1 - n) > 0$, $H''(1 - n) < 0$, and $H(0) = 0$. Every period, given a post-tax wage level $\bar{w}_t$ and labor efficiency e_t , a household divides time optimally between the two production technologies. It is clear that there is no income effect on labor supply in this model.⁷ The total labor income of a household, $y_t(h^t, \bar{w}_t)$, and supply of market hours, $n_t(h^t, \bar{w}_t)$, satisfy

$$y_t(h^t, \bar{w}_t) = y(e_t, \bar{w}_t) = \max_{0 \leq n_t \leq 1} H(1 - n_t) + e_t n_t \bar{w}_t \text{ for each } t, h^t, \quad (1)$$

$$n_t(h^t, \bar{w}_t) = n(e_t, \bar{w}_t) = \arg \max_{0 \leq n_t \leq 1} H(1 - n_t) + e_t n_t \bar{w}_t \text{ for each } t, h^t. \quad (2)$$

The budget constraint of a household is

$$\begin{aligned} c_t(h^t) + a_{t+1}(h^t) &\leq a_t(h^{t-1})(1 + \bar{r}_t) + y(h^t, \bar{w}_t) \text{ for each } t, h^t \\ a_{t+1}(h^t) &\geq -\underline{a} \text{ for each } t, h^t \end{aligned} \quad (3)$$

Households derive utility from consumption goods and their objective is to solve

$$V^H(a_0; \bar{\mathbf{r}}, \bar{\mathbf{w}}) = \max_{\{a_{t+1}(h^t), c_t(h^t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{h^t \in E^t} \Pi(h^t) u(c_t(h^t)) \quad (4)$$

subject to (1) and (3), given the sequence of post-tax prices $\bar{\mathbf{r}} = \{\bar{r}_0, \bar{r}_1, \dots\}$, $\bar{\mathbf{w}} = \{\bar{w}_0, \bar{w}_1, \dots\}$, and initial condition $a_0(h^{-1}) = a_0$.

I assume that the per-period utility function $u(c)$ satisfies the standard assumptions $u'(c) > 0$, $u''(c) < 0$,

⁷The absence of income effect is important for obtaining cleaner results on the steady-state properties of the Ramsey problem and the distinction drawn between a complete markets and an incomplete markets economy in the later sections does not rely on this particular assumption. However, it is not clear whether the qualitative features of the model hinge on this assumption. Based on the quantitative findings, it is reasonable to expect that similar results hold for an alternative model that features limited income effect on labor supply.

$\lim_{c \rightarrow 0} u'(c) = \infty$. In addition, following the literature on incomplete markets, I assume that there exist constants $\bar{c}, \sigma > 0$ such that $-u''(c)c/u'(c) \leq \sigma$ for all $c \geq \bar{c}$. The last assumption ensures (when the shocks are i.i.d.) that the level of assets remain bounded in the long run for each agent, provided that the long-run return on assets is lower than the inverse of the discount rate.⁸

The policy functions solve the following system of necessary conditions:

$$u'(c_t(h^t)) \geq \beta(1 + \bar{r}_{t+1}) \sum_{h^{t+1} \in E^{t+1}} \Pi(h^{t+1}|h^t) u'(c_{t+1}(h^{t+1})) \text{ for each } t, h^t \quad (5)$$

$$(a_{t+1}(h^t) + \underline{a}) \left(u'(c_t(h^t)) - \beta(1 + \bar{r}_{t+1}) \sum_{h^{t+1} \in E^{t+1}} \Pi(h^{t+1}|h^t) u'(c_{t+1}(h^{t+1})) \right) = 0 \text{ for each } t, h^t \quad (6)$$

$$a_{t+1}(h^t) + \underline{a} \geq 0 \text{ for each } t, h^t \quad (7)$$

I also impose the following transversality condition:

$$\lim_{t \rightarrow \infty} \beta^t \mathbb{E}[u'(c_t(h^t))(a_{t+1}(h^t) + \underline{a})] = 0 \quad (8)$$

Let $g_t : E^t \rightarrow A$ denote the policy function for assets; g_t is measurable with respect to $(E^t, \mathcal{E}^t)$.

Representative Firm

The output net of depreciation, $F(K, N)$ is constant-returns-to-scale, satisfying the usual neoclassical assumptions $F_K > -\delta, F_N > 0, F_{KK} < 0, F_{NN} < 0, \lim_{K \rightarrow 0} F_K = \infty, \lim_{N \rightarrow 0} F_N = \infty$.

The firm's objective is to maximize profits in each period t . Therefore the following conditions are satisfied:

$$r_t = F_K(K_t, N_t) \quad (9)$$

$$w_t = F_N(K_t, N_t).$$

The Government

The government has the following budget constraint in each period:

⁸See Aiyagari (1994b) for a proof of existence of an upper bound on assets for the case in which the shocks are i.i.d. To the best of my knowledge, there is no general proof of assets being bounded in the long run when the shocks follow a Markov process. Miao (2002) excludes the natural borrowing limit case and obtains the boundedness result for a restrictive class of first-order Markov processes. Due to absence of general theoretical results, following the literature, I verify this property quantitatively. On the other hand, assets being bounded is a *sufficient* condition for the Ramsey problem to be well-defined. A stationary invariant distribution for assets might exist and per-capita levels of all quantities can be bounded even when the relevant state space is not compact. See Szeidl (2013). Unfortunately, theoretical results in this direction are limited.

$$r_t \tau_{kt} A_t + w_t \tau_{nt} N_t + B_{t+1} \geq G_t + (1 + r_t) B_t$$

where $A_t = K_t - B_t$ is the aggregate assets held by the households in the economy.

Using the CRS assumption for the market technology, adding $F(K_t, N_t)$ to both sides, one can express this constraint in terms of post-tax prices:

$$G_t + (1 + \bar{r}_t) B_t + \bar{r}_t K_t + \bar{w}_t N_t \leq F(K_t, N_t) + B_{t+1} \quad (10)$$

2.2 Competitive Equilibrium and the Ramsey Problem

The competitive equilibrium in this economy can be defined in the standard way.

Definition 1 For given initial conditions (a_0, B_0) and time paths $\bar{r}$, $\bar{w}$, $\mathbf{B}$, a **competitive equilibrium with fiscal policy** consists of a household value function $V^H(\cdot)$; household policy functions $g_t(\cdot)$, $n_t(\cdot)$; and sequences C_t , K_t , N_t , H_t , A_t , r_t , and w_t such that the following are satisfied:

1. The policy function $g_t(\cdot)$ and value function $V^H(\cdot)$ solve problem (4), $n_t(\cdot)$ satisfies equation (2),
2. Given the sequence of factor prices r_t, w_t , the representative firm maximizes profits: K_t and N_t satisfy (9),
3. The government resource constraint (10) is satisfied.
4. All markets clear:

Asset market clearing

$$K_t = A_t - B_t \text{ for each } t \quad (11)$$

Goods market clearing

$$C_t + G_t + K_{t+1} = F(K_t, N_t) + H_t + K_t \text{ for each } t$$

Labor market clearing

$$N_t = \sum_{h^t \in E^t} \Pi(h^t) e_t n_t(h^t, \bar{w}_t) \text{ for each } t \quad (12)$$

Sequences A_{t+1} and H_t are generated by household policy functions:

$$A_{t+1} = \sum_{h^t \in E^t} \Pi(h^t) g_t(h^t; \bar{\mathbf{r}}, \bar{\mathbf{w}}) \text{ for each } t \quad (13)$$

$$H_t = \sum_{h^t \in E^t} \Pi(h^t) H(1 - n_t(h^t, \bar{w}_t)) \text{ for each } t.$$

In period 0, government chooses a sequence of prices $\bar{\mathbf{r}}$, $\bar{\mathbf{w}}$, and debt $\mathbf{B}$ in order to maximize a utilitarian aggregate of sum of discounted utilities for all households, subject to market clearing and government resource constraints. This defines the Ramsey problem in this environment.

Definition 2 Given the initial level of assets a_0 and government debt B_0 , household policy functions $g_t(\cdot)$ and $n_t(\cdot)$, and the exogenous government expenditure process G_t , the **Ramsey Problem** consists of the choice of sequences of post-tax factor prices $\bar{\mathbf{r}} = \{\bar{r}_0, \bar{r}_1, \dots\}$, $\bar{\mathbf{w}} = \{\bar{w}_0, \bar{w}_1, \dots\}$, and government debt $\mathbf{B} = \{B_1, B_2, \dots\}$ that solve

$$V(a_0, B_0) = \max_{\bar{\mathbf{r}}, \bar{\mathbf{w}}, \mathbf{B}} V^H(a_0, \bar{\mathbf{r}}, \bar{\mathbf{w}}) \quad (14)$$

subject to (10), (11), (12), and (13), given a_0 , B_0 .

It is clear that every solution to the Ramsey Problem is a competitive equilibrium allocation with taxes. Since household policy functions depend on prices in all periods, a marginal change in date- t post-tax prices would in general alter consumption and savings decisions for all periods.⁹ One standard way to simplify this problem is to use the first-order necessary conditions for the household's problem as implementability constraints for the planner's problem. This approach is valid since, given any sequence of prices, the household's problem is convex. The first-order conditions of the household, (5), (6) and (7), along with the transversality condition (8), are necessary and sufficient for an optimum.

$$V(a_0, B_0) = \max_{\{\bar{r}_t, \bar{w}_t, B_{t+1}, a_{t+1}(h^t), c_t(h^t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{h^t \in E^t} \Pi(h^t) u(c_t(h^t)) \quad (15)$$

subject to (5), (6), (7), (3), (10), and

$$K_{t+1} = \sum_{h^t \in E^t} \Pi(h^t) a_{t+1}(h^t) - B_{t+1} \text{ for each } t$$

$$N_t = \sum_{h^t \in E^t} \Pi(h^t) e_t n_t(h^t, \bar{w}_t) \text{ for each } t$$

⁹Due to absence of wealth effect on labor supply, this is not true for labor supply decisions. Labor supply at time t is only affected by the post-tax wage level at time t .

given (1), (2), and initial conditions $a_0(h^{-1}) = a_0$ and B_0 .

Next, I write a Lagrangian for problem (15). Let $\beta^t \Pi(h^t) \theta_{t+1}(h^t)$ and $\beta^t \Pi(h^t) \eta_{t+1}(h^t)$ represent the Lagrange multipliers for (5) and (6) respectively. For convenience, define the following auxiliary variable:

$$\lambda_{t+1}(h^t) \equiv \eta_{t+1}(h^t)(a_{t+1}(h^t) + \underline{a}) - \theta_{t+1}(h^t). \quad (16)$$

By carrying terms across time periods, I obtain the following formulation, incorporating only the dynamic implementability constraints into the objective function:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \sum_{h^t \in E^t} \Pi(h^t) \left[u(c_t(h^t)) + u'(c_t(h^t)) [\lambda_t(h^{t-1})(1 + \bar{r}_t) - \lambda_{t+1}(h^t)] \right] \quad (17)$$

subject to (3), (16), (10), and

$$K_{t+1} = \sum_{h^t \in E^t} \Pi(h^t) a_{t+1}(h^t) - B_{t+1} \text{ for each } t$$

$$N_t = \sum_{h^t \in E^t} \Pi(h^t) e_t n_t(h^t, \bar{w}_t) \text{ for each } t$$

given (1), (2), and initial conditions $a_0(h^{-1}) = a_0$, B_0 and $\lambda_0(h^{-1}) = 0$.

Observe that agents start with $\lambda_t(h^t)$ equal to zero since at period-0, there are no “Euler equation promises” to be kept, therefore $\theta_0(h^{-1}), \eta_0(h^{-1}) = 0$.

Problem (15) is not stationary and it involves constraints that are forward-looking. For instance, choice variable $\bar{r}_{t+1}$ shows up as part of Euler equation constraints that belong to period t . Therefore, recursive methods cannot be applied directly. Following Marcet and Marimon (2011), I expand the state space of the problem to include Lagrange multipliers of the dynamic implementability constraints to recover stationarity. To be more precise, although the primary problem (15) does not admit a recursive structure, the Lagrangian (17) does, once we keep track of the auxiliary variables (16) along with the usual state variables, assets and labor efficiency. Since there is a continuum of households, the relevant state variable for the Ramsey planner is the joint distribution of these three variables.¹⁰

Let $\mathcal{L} = \mathbb{R}$ represent the space for λ . For the recursive representation, I index all households by $(s, e) \equiv (a, \lambda, e) \in \mathcal{A} \times \mathcal{L} \times E$. Let Σ_s represent the Borel σ -algebra on $\mathcal{A} \times \mathcal{L}$, and Σ represent the product σ -algebra on $\mathcal{A} \times \mathcal{L} \times E$. $\mathbb{P}$ represents the set of all probability measures over Σ with typical elements $\mu, \mu' \in \mathbb{P}$.

¹⁰Marcet and Marimon (2011) construct the recursive Lagrangian by “dualizing” the dynamic incentive constraints (equations (5) and (6), in this case) period by period, assuming that the solution to the primal problem is a saddle-point of the corresponding Lagrangian. An earlier draft of their paper (1994) features the Ramsey problem under complete markets as an example whose formulation looks very similar to this model.

Let μ_0 be the distribution with all probability mass on $a = a_0$ and $\lambda = 0$. Then we have $V(a_0, B_0) = W(\mu_0, B_0)$ where $W : \mathbb{P} \times \mathbb{R} \rightarrow \mathbb{R}$ solves:

$$W(\mu, B) = \min_{\theta'(\cdot), \eta'(\cdot) \geq 0} \max_{\bar{r}, \bar{w}, B', a'(\cdot), c(\cdot)} \sum_e \int u(c(\cdot)) + u'(c(\cdot)) [\lambda(1 + \bar{r}) - \lambda'(\cdot)] \mu(ds, e) + \beta W(\mu', B') \quad (18)$$

subject to

$$c(\cdot) + a'(\cdot) \leq a(1 + \bar{r}) + y(e, \bar{w}) \quad \text{a.e. } \mu \quad (19)$$

$$a' + \underline{a} \geq 0 \quad \text{a.e. } \mu$$

$$G + (1 + \bar{r})B + \bar{r}K + \bar{w}N = F(K, N) + B'$$

$$K = \sum_e \int a \mu(ds, e) - B$$

$$N = \sum_e \int en(e, \bar{w}) \mu(ds, e) = \sum_e \pi_e en(e, \bar{w})$$

$$\mu'(S', e') = \sum_e \pi_{ee'} \int I[(a'(\cdot), \lambda'(\cdot)) \in S'] \mu(ds, e) \text{ for each } S' \in \Sigma_s \text{ and each } e' \in E$$

given (1) and (2), where $a'(\mu, B, s, e)$, and $c(\mu, B, s, e)$ denote the choice of assets and consumption respectively, and

$$\lambda'(\mu, B, s, e) \equiv \eta'(\mu, B, s, e)(a'(\mu, B, s, e) + \underline{a}) - \theta'(\mu, B, s, e).$$

$I[\cdot]$ is the indicator function taking a value of 1 if the condition in the brackets is true, and 0 otherwise.

For the rest of the exposition, I will denote the policy functions for assets and the induced auxiliary variables by $a' = g(\mu, B, s, e)$ and $\lambda' = h(\mu, B, s, e)$, respectively. In the Appendix, I show that the following first-order conditions are necessary at an interior (with respect to policy variables $\bar{r}$, $\bar{w}$, and B) solution to the Ramsey problem.

Proposition 1 *An interior solution to the Ramsey problem satisfies the following conditions:*

$$\begin{aligned}
\lambda' = h(\cdot) : \quad & u'(c) \geq \beta(1 + \bar{r}')\mathbb{E}[u'(c')|e] \text{ with equality if } a' > -\underline{a}, \quad \text{a.e. } \mu \\
a' = g(\cdot) : \quad & u'(c) + u''(c)[\lambda(1 + \bar{r}) - \lambda'] = \beta(1 + \bar{r}')\mathbb{E}\left[u'(c') + u''(c')[\lambda'(1 + \bar{r}') - \lambda'']|e\right] \\
& + \beta\gamma'(F_K(K', N') - \bar{r}') \text{ if } a' > -\underline{a}, \text{ otherwise } \lambda' = 0, \quad \text{a.e. } \mu \\
B'(\mu, B) : \quad & \gamma = \beta(1 + F_K(K', N'))\gamma' \\
\bar{r}(\mu, B) : \quad & \gamma A = \sum_e \int u'(c)\lambda\mu(ds, e) + \sum_e \int a\left(u'(c) + u''(c)[\lambda(1 + \bar{r}) - \lambda']\right)\mu(ds, e) \\
\bar{w}(\mu, B) : \quad & \gamma N = \gamma(F_N(K, N) - \bar{w})N'(\bar{w}) + \sum_e \int en(e, \bar{w})\left(u'(c) + u''(c)[\lambda(1 + \bar{r}) - \lambda']\right)\mu(ds, e)
\end{aligned}$$

where γ is the multiplier for the government budget constraint.

Observe that at an optimal solution, a functional household Euler equation must be satisfied. Due to the particular structure of this problem, the household-specific multiplier λ does *not* appear directly in this equation. This property will allow me to conjecture that the policy function for assets satisfies $a' = g(\mu, B, a, \lambda, e) = g(\mu, B, a, \tilde{\lambda}, e)$, for all $\lambda, \tilde{\lambda} \in \mathcal{L}$. Note that if this property does not hold over a set of agents with positive measure, the planner effectively chooses different consumption and savings for two types of households with the same asset and labor efficiency levels, but with different histories (hence λ). Clearly, this cannot hold over an optimal path since this would violate the sequences of implementability constraints for at least one of those households.¹¹ This is *not* equivalent to stating that these multipliers are irrelevant. The *distribution* of multipliers is a component of the planner's state variable μ , and matters for the planner's choice of post-tax prices; this can be seen in the first-order conditions for $\bar{r}$ and $\bar{w}$. These post-tax prices, in turn, show up in the household's Euler equation and so affect consumption and saving. However, conditional on the distribution of multipliers, two households with the same assets and labor efficiency will have the same consumption and saving, independent of their particular values of λ . I will henceforth assume that the policy function for assets takes the form $a' = g(\mu, B, a, e)$, which simplifies the problem significantly.

3 Steady-State Analysis of the Ramsey Problem

Having provided the details of the model, I next seek the answer to the following question: Which steady states, if any, could be optimal in the long run? This is not merely an investigation of a steady state that

¹¹Assuming that the solution to the recursive problem is equivalent to the solution to the sequential problem, the time-series generated by the recursive problem must be consistent with the consumption-savings plans of households who take the sequence of prices as given, since these are internalized by the implementability conditions.

maximizes flow welfare *at the steady state*. By contrast, this is an investigation of the limiting values of the policy variables of an optimal transition. In particular, government's discounting over the entire planning horizon is taken into account. For the discussion that follows, I will refer to these limiting values and the induced allocation as the “long-run optimal steady states”, or “Ramsey steady states.”

This “steady state refinement” turns out to be quite powerful. For many parameterizations of the model, quantitative results show that a unique steady state survives this refinement. This property is particularly surprising in this environment. It is well known that the complete markets benchmark can accommodate multiple steady states, each of which can potentially be optimal for a different set of initial conditions. Indeed, in the Chamley-Judd benchmark, there is no way of figuring out the long-run optimal labor taxes, government debt, private assets, and consumption levels without providing a complete solution to the problem, and without taking a stand on the initial conditions of the economy.¹²

To understand why we have an indeterminacy in the complete markets benchmark, it is illuminating to take a closer look at the first-order necessary conditions of the planner for a version of the benchmark model without uncertainty. To this end, I seek the steady state of the Ramsey problem with no idiosyncratic risk in which consumption C , private assets A , post-tax prices $\bar{r}$, $\bar{w}$, government debt B , and exogenous government expenditure G are constant. Below, I provide only a heuristic characterization of the steady state under complete markets since the detailed proofs are available in Chamley (1986) for a slightly different version of the same model. The following equations constitute the first-order necessary conditions of a planner when there are no idiosyncratic shocks:

$$\lambda_{t+1} : \quad u'(C_t) = \beta(1 + \bar{r}_{t+1})u'(C_{t+1}) \quad (20)$$

$$A_{t+1} : \quad u'(C_t) + u''(C_t)[\lambda_t(1 + \bar{r}_t) - \lambda_{t+1}] = \beta(1 + \bar{r}_{t+1}) \left[u'(C_{t+1}) + u''(C_{t+1})[\lambda_{t+1}(1 + \bar{r}_{t+1}) - \lambda_{t+2}] \right] \quad (21)$$

$$B_{t+1} : \quad \gamma_t = \beta(1 + F_K(K_{t+1}, N_{t+1}))\gamma_{t+1} \quad (22)$$

$$\bar{r}_t : \quad \gamma_t A_t = u'(C_t)\lambda_t + A_t \left(u'(C_t) + u''(C_t)[\lambda_t(1 + \bar{r}_t) - \lambda_{t+1}] \right) \quad (23)$$

$$\bar{w}_t : \quad \gamma_t N_t = \gamma_t (F_N(K_t, N_t) - \bar{w}_t)N'(\bar{w}_t) + N_t(\bar{w}_t) \left(u'(C_t) + u''(C_t)[\lambda_t(1 + \bar{r}_t) - \lambda_{t+1}] \right) \quad (24)$$

¹²See Lucas (1990) for a discussion of this point for an infinite-horizon model. Auerbach and Kotlikoff (1987) make the same point for a life-cycle model.

Absent idiosyncratic risk, since $C = C_t = C_{t+1}$ at a steady state, the household's Euler equation (20) reads $\beta(1 + \bar{r}) = 1$. The government's Euler equation (22) implies the modified golden rule holds, i.e. $\beta(1 + F_K(K, N)) = 1$ at a steady state. Combining these two equations, as one would expect, I obtain the Chamley-Judd zero capital tax result $\bar{r} = r = F_K(K, N)$.¹³ It is straightforward to show that imposing the zero-tax result, the first-order condition for assets (equation (21)) becomes redundant. Now, observe that the steady-state versions of the first-order conditions for $\bar{r}$ and $\bar{w}$ (equations (23) and (24)), the modified golden rule expression $\beta(1 + F_K(K, N)) = 1$, along with steady-state household and government budget constraints constitute five independent equations with six unknowns $\bar{w}$, B , A , C , λ , and γ (treating labor supply $N(\bar{w})$ as a function of $\bar{w}$).

What causes this system to be underdetermined in the Chamley-Judd benchmark? Since capital income tax is non-distortionary in the initial period, it is efficient to impose confiscatory capital income taxes. Following the initial period, optimal capital tax rate converges to zero very rapidly, and in a large class of models, this rate is independent of the initial conditions.¹⁴ The optimal policy, under complete markets, is to front-load all intertemporal distortions, which is shown rigorously in a more general framework by Albanesi and Armenter (2012). However, the fact that capital income should not be taxed in the long run is not sufficient to define the optimal long-run policy completely. Given that capital tax rate ought to be zero except for the few initial periods, the present value of labor income tax revenue is determined, to a large extent, by the initial debt and present value of government expenditures net of initial capital levy available to the government. The level of debt to be serviced in the long run, and the long-run labor income tax are jointly determined by the optimal way to smooth these tax distortions over time.

Another way to see this point is to look at the structure of the representative household's problem. With no idiosyncratic risk and no borrowing constraints, given any sequence of prices $(\bar{r}, \bar{w})$ that converge to steady-state values, the representative household's Euler equation holds with equality in every period t . Therefore, the initial level of assets (along with a transversality condition) determines the consumption and asset levels at the steady state. It is clear that there is no hope of studying the steady-state levels of these variables without making a reference to the initial conditions. Indeed, since $\beta(1 + \bar{r}) = \beta(1 + F_K) = 1$ holds, the steady-state version of the household's Euler equation provides us no information about the level of consumption in the long run.

¹³Both Chamley (1986) and Judd (1985) pointed out that government having access to debt is not responsible for the zero capital tax result. In this economy, absent debt, household Euler equation implies positive capital tax ($\beta(1 + \bar{r}) < 1$) leads to $u'(C_t) \rightarrow \infty$, and a capital subsidy ($\beta(1 + \bar{r}) > 1$) leads to $u'(C_t) \rightarrow 0$. In this sense, there are actually *two* independent reasons why capital tax rate is zero in the long run for a complete markets economy.

¹⁴For instance, when the utility function is separable in consumption and leisure, and consumption utility function is of the CRRA type, optimal capital tax rate is zero for all $t \geq 1$. See Chari and Kehoe (1999). Obviously, this result holds true for more general environments if the initial capital levy equals the present value of all government expenditures. This is equivalent to having access to a lump-sum tax, since the government does not need to resort to distortionary taxation.

This observation led many economists to conjecture that an analogous result holds in economies with incomplete markets and heterogeneous agents, i.e. the premise that long-run optimal fiscal policy and allocation depend on initial conditions of the economy. I argue below that the problem of indeterminacy, a characteristic of complete markets economy, does not arise in a Bewley-type economy. For the discussion that follows, I provide the steady-state versions of the optimality conditions in Proposition 1, where I drop the distribution as a state variable due to stationarity.¹⁵

$$\lambda'(a, \lambda, e) : \quad u'(c) \geq \beta(1 + \bar{r})\mathbb{E}[u'(c')|e] \text{ with equality if } a' > -a \quad (25)$$

$$a'(a, \lambda, e) : \quad u''(c)[\lambda(1 + \bar{r}) - \lambda'] = \beta(1 + \bar{r})\mathbb{E}\left[u''(c')[\lambda'(1 + \bar{r}) - \lambda']|e\right] \\ + \beta\gamma(F_K(K, N) - \bar{r}) \text{ if } a' > -a, \text{ otherwise } \lambda' = 0 \quad (26)$$

$$\bar{r} : \quad \gamma A = \sum_e \int u'(c) \lambda \mu(ds, e) + \sum_e \int a \left(u'(c) + u''(c)[\lambda(1 + \bar{r}) - \lambda'] \right) \mu(ds, e) \quad (27)$$

$$\bar{w} : \quad \gamma N = \gamma(F_N(K, N) - \bar{w})N'(\bar{w}) + \sum_e \int en(e, \bar{w}) \left(u'(c) + u''(c)[\lambda(1 + \bar{r}) - \lambda'] \right) \mu(ds, e) \quad (28)$$

My argument hinges critically on the optimal positive capital income tax result provided by Aiyagari (1995). He proved, as an intermediate step, that the modified golden rule property still holds in this environment; this follows from the steady-state version of the government's Euler equation.¹⁶ It is well-known that, as long as there is idiosyncratic risk, $\beta(1 + \bar{r}) < 1$ is necessary for the stationarity of the joint distribution of assets and labor efficiency in the long run.¹⁷ Aiyagari's (1995) long-run optimal positive capital income tax result, $\bar{r} < F_K$, follows immediately from these two observations. Also note that at an optimal solution to the Ramsey problem, households are more impatient than the government in the long run.¹⁸

Observe that, given any optimal steady-state candidate post-tax prices $(\bar{r}, \bar{w})$ such that $\beta(1 + \bar{r}) < 1$ holds, standard methods in Aiyagari (1994b) and Huggett (1993) can be used to solve the household Euler equation

¹⁵Observe that I assume implicitly that the joint distribution of (a, λ, e) converges to a stationary invariant distribution. A potential problem arises from the fact that under the given assumptions, there is no guarantee that λ is bounded or that there is an ergodic set for λ . The numerical algorithm explained in the Appendix actually *does not* impose any boundedness assumptions on the multipliers. As long as the relevant moments of the distribution are finite, the problem is still well-defined. Although I do not provide a proof in this paper, it should not be surprising that λ remains bounded in this weaker sense since it follows a process which resets whenever assets hit the lower bound; this happens with probability one for any household.

¹⁶Aiyagari (1995) assumes that government spending is endogenous. This makes his proof more transparent because the multiplier γ on the government's budget constraint is equal to the marginal utility of government spending. However, he noted that the modified golden rule result still holds even when government expenditure is exogenous. See footnote 15 in Aiyagari (1995).

¹⁷See Schechtman and Escudero (1977).

¹⁸Households are impatient in the sense that they would deplete all assets given the post-tax prices in the hypothetical case of no uncertainty. Since there is no uncertainty at the aggregate level, government is "patient," i.e. $\beta(1 + F_K(K, N)) = 1$ holds.

for the saving policy in the long run. Equilibrium saving policy, under standard technical conditions, provides us the stationary distribution of assets and consumption. All of this is possible thanks to a particular feature of Bewley-type models: Given that $\beta(1 + \bar{r}) < 1$ holds and the households face idiosyncratic uncertainty, the tension between impatience and the incentives to engage in precautionary saving ultimately defines a stable invariant distribution, whose moments pin down the steady-state aggregate levels of household variables.¹⁹ In a nutshell, *any* candidate long-run optimal price pair $(\bar{r}, \bar{w})$ determines the average levels of private assets and consumption under incomplete markets. In the complete markets benchmark, *none* of them do! I exploit this feature to solve the system of necessary conditions to characterize long-run optimal fiscal policy and allocation *without* having to solve for the transition. This is possible because the relevant system of equations is *not* underdetermined.

Why is it the case that we can study long-run optimal fiscal policy in a Bewley-type economy independent of the initial conditions? So far, the discussion has pointed out this is a feasible procedure, but I have yet to examine why there is a disconnect between the initial period and the state of the economy in the long run. A rigorous proof of the irrelevance of initial conditions is beyond the scope of this paper. The difficulty arises from the fact that there are no theoretical results even on the convergence of the optimal Ramsey solution to a steady state. However, *assuming* that the solution converges to a steady state, we can suggest a particular characteristic of Bewley-type models to be responsible for this result: As long as the borrowing constraints are tighter than the natural borrowing limits, and $\beta(1 + \bar{r}) < 1$ holds, every agent hits the borrowing constraint infinitely often.²⁰ Every time an agent faces the borrowing constraint, the process that determines the intertemporal allocation of resources is “reset,” resulting in history-independence. From a date-0 planner’s perspective, marginal change in the tax rate in period t sufficiently far into the future has no effect on an agent’s saving decision for earlier time periods due to the wedge in the Euler equation for periods in which the agent hits the borrowing constraint.²¹

To further motivate this point, it is useful to consider some of the established results in the life-cycle literature. Erosa and Gervais (2002) show that initial conditions play no role in the determination of long-run optimal fiscal policy, as long as there are no wealth transfers across generations. The relevance of their result

¹⁹See Deaton (1991), Carroll (1997), Szeidl (2013), and others for an extensive discussion.

²⁰An interesting case is one in which the agents are subject to the natural borrowing limit. In the current model, when long-run prices are $(\bar{r}, \bar{w})$, this limit would be $-\frac{y(e_1, \bar{w})}{\bar{r}}$ where e_1 is the lowest idiosyncratic shock realization. However, under the natural borrowing limit, some of the crucial moments of the stationary distribution that appear in the first-order necessary conditions are not defined. In particular, marginal utility $u'(c)$ is not integrable with respect to the measure μ , and neither is $u'(c) + u''(c)[\lambda(1 + \bar{r}) - \lambda']$. For example, if $u'(c)$ were integrable, integrating both sides of household’s Euler equation (which holds with equality almost surely) (25) and cancelling out integrals, we would get $\beta(1 + \bar{r}) = 1$, which cannot hold in the long run. Therefore, it is not clear whether the Ramsey problem is well-defined for this case.

²¹To understand the intuition, suppose heuristically that for a sequence of prices $(\bar{r}, \bar{w})$, an agent hits the borrowing constraint with probability one before period $T < \infty$. Then a marginal increase in $\bar{r}_t$ where $t > T$ will have no effect on savings choice in period 0. This is due to the fact that with probability one, there exists period $k \leq T$ at which the Euler equation between periods k and $k + 1$ is slack. If this property held true for all agents in the economy, there would be a disconnect between $t = 0$ and $t = T$.

stems from the fact that Bewley-type models feature agents who have a sequence of finite planning horizons of uncertain length, similar to overlapping-generations models, as described by Aiyagari (1994a) and Aiyagari and McGrattan (1998). The long-run stationary distribution of the Bewley-type economy resembles the cross-sectional distribution in a typical life-cycle model. The borrowing-constrained agents in the former model act like the newborn generation in the latter model, who start their lives with assets $a = -\underline{a}$.

Fortunately, whether initial conditions matter for the long-run policy can be resolved quantitatively by figuring out whether a unique policy and corresponding allocation solve the system of necessary equations for optimality. Since optimal steady states constitute a subset of the steady states that satisfy the modified golden rule, the first step is to identify the latter set. For this, I follow Aiyagari (1994a) and show that $\bar{w}$ can be used as an index for all such steady states. To see this, I can write the steady-state version of government budget constraint as

$$A = K + B = \frac{F(K, N) - \bar{w}N - G}{\bar{r}}$$

Due to assumption of constant returns to scale, $\frac{K}{N}$ is equal across all steady states that satisfy the modified golden rule. Since aggregate labor supply depends only on $\bar{w}$, given $\bar{w}$ (and G), $F(K, N) - \bar{w}N - G$ is completely determined. Therefore the right-hand side of the above expression is strictly decreasing in $\bar{r}$ (for $\bar{r} > 0$). On the supply side, for a fixed $\bar{w}$, the aggregate steady-state private assets A is strictly increasing in $\bar{r}$ without bound.²² This implies that there is at most one $\bar{r}$ that solves the above equation for any given $\bar{w}$. Therefore, in practice, I can compute steady-state values of all variables and associate them with the post-tax market wage. I let $\bar{r}(\bar{w})$ denote the associated post-tax interest rate and $m(a, e; \bar{w})$ denote the long-run distribution of (a, e) for given prices $(\bar{r}(\bar{w}), \bar{w})$.

Following this step, for each such steady-state allocation, using equations (26) and (27), I solve for the supporting multiplier policy $\lambda' = h(a, \lambda, e; \bar{w})$ and for γ . Observe that this is possible since, given any candidate value for γ and the policy function $a' = g(a, e; \bar{w})$, the functional equation (26) can be solved for $\lambda'(a, \lambda, e; \bar{w})$. The policy functions can then be used to compute the stationary joint distribution $\mu(a, \lambda, e; \bar{w})$. The relevant moments of the distribution are next used to pin down the value of γ using equation (27). As a final step, I check whether the last necessary condition (28) is satisfied. In my quantitative analysis, for all parameter values I used, there was a unique policy that satisfied the last necessary condition. The numerical procedure is explained in detail in the Computational Appendix.

²²One caveat is that even when $\bar{w}$ is fixed, the relationship between $\bar{r}$ and A might not be monotone due to potentially prevalent wealth effects. On the other hand, in this class of models, it is very difficult to generate an example where this is the case. Not surprisingly, for all my quantitative results, $A(\bar{r})$ is monotonically increasing in $\bar{r}$, given $\bar{w}$.

4 Model Specification and Calibration

The quantitative exercise involves the thought experiment of comparing the current system in the U.S. with those that would be chosen by a Ramsey planner in the long run. Some of the key parameters in the model, such as those related to the home-production technology, and the subjective discount rate, are identified from the steady-state labor supply and savings choices of the households, who respond to the current tax system optimally. The remaining parameters are either obtained from previous studies, or directly matched to their counterparts in the data.

I use a CRRA-type utility function and a home-production function that induces a constant elasticity of labor supply

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}$$

$$H(1-n) = \theta^{-\frac{1}{\gamma}} \frac{1 - n^{1+\frac{1}{\gamma}}}{1 + \frac{1}{\gamma}}$$

With this functional form, for an agent with efficiency e , labor supply satisfies

$$n(e, \bar{w}) = \min\{\theta(e\bar{w})^\gamma, 1\}$$

where $\gamma > 0$ represents the labor supply elasticity at an interior solution.

The production function is assumed to be of the Cobb-Douglas type:

$$F(K, N) = K^{1-\alpha} N^\alpha - \delta K$$

The stochastic process for labor efficiency e_t follows the AR(1) process

$$\log(e_{t+1}) = \text{const} + \rho \log(e_t) + \sqrt{\sigma_e^2(1-\rho^2)}\epsilon_t$$

where $\epsilon_t \sim N(0, 1)$.

The model period is assumed to be a year. I use $\delta = 0.08$ for the depreciation rate and $\alpha = 0.64$ for the labor share in the production function, these are standard in the literature.

I use a coefficient of risk aversion of $\sigma = 2$ for the benchmark calibration and use a range of values $\sigma \in [1.0, 4.0]$ for comparison. The quantitative results depend critically on the labor supply elasticity, therefore I report results for a range of values $\gamma \in [0.5, 2.0]$, using $\gamma = 1.0$ for the benchmark calibration. These different

values capture a wide range of estimates provided in the literature. I assume that agents cannot borrow for all quantitative results, i.e. $\underline{a} = 0$.

I use the annual estimates by Chang and Kim (2006) for the labor efficiency process, $\rho = 0.818$ and $\sigma_e = 0.506$. These are obtained from PSID data for 1971-1992. The constant term is chosen such that $\mathbb{E}(e) = 1.0$. I discretize this continuous process using 9 grid points $\{e_1, \dots, e_9\}$ following Tauchen (1986).²³

I use a capital-to-market production ratio of $K/Y = 3$, which is a reasonable value for the U.S. economy. Using the constant-returns-to-scale property of the production function, this value implies a pre-tax real interest rate and wage level of $r = 0.04$ and $w = 1.19$ respectively.

Domeij and Heathcote (2004) report average tax rates in the U.S. to be around 39.7% for capital income and 26.9% for labor income. Following these results, I assume a 40% capital tax rate, ($\tau_k = 0.40$) which implies a post-tax rate of $\bar{r} = 0.024$. I use a 27% labor tax rate, ($\tau_n = 0.27$) which implies $\bar{w} = 0.87$. Given $\bar{w}$, I assign $\theta = 0.39$ to match per-capita market hours equal to $1/3$. Given the parameter values and labor supply N , other steady-state aggregates, in particular, capital K , market production Y , and home-production H are uniquely pinned down.

Government expenditure $G = 0.14$ is set such that at the initial steady state, it constitutes 17% of the market production, the annual average in post-war U.S. data.

Given post-tax prices $\bar{w}, \bar{r}$, using the household's problem, the discount rate uniquely determines the steady-state demand for private assets A . Using the monotonicity of the asset demand function with respect to discount rate, a value of $\beta = 0.9749$ is chosen such that the government budget constraint balances at the steady state.²⁴ The steady-state government debt level is then determined from $B = A - K$. All parameter values used in the benchmark calibration are summarized in table 1.

²³Continuity of the *aggregate* labor elasticity is critical in establishing existence of a solution to the Ramsey problem. With discretization, even though $N(\bar{w})$ is continuous throughout the entire range of $\bar{w}$, $N'(\bar{w})$ exhibits jumps at points where certain types of agents hit the $n = 1$ limit. This problem is overcome by increasing the number of grid points until $N'(\bar{w})$ becomes continuous over the relevant range for $\bar{w}$.

²⁴Comparative-statics analysis of this class of models is a challenging task, but some results are available in the recent literature. Acemoglu and Jensen (2012) prove monotonicity of the supply of assets with respect to the discount rate.

Table 1: Parameter Values-Benchmark Calibration

Parameters	Value	Target Moments/Source
<i>Exogenously Calibrated Parameters</i>		
Coefficient of Risk Aversion (σ)	2.00	
Labor Supply Elasticity (γ)	1.00	
Labor Share in Production (α)	0.64	
Depreciation Rate (δ)	0.08	
Borrowing Constraint (a)	0.00	
Persistence of Labor Productivity (ρ)	0.82	Chang and Kim (2006)
Standard Deviation of Labor Productivity (σ_e)	0.51	Chang and Kim (2006)
Labor income Tax Rate (τ_n)	0.27	Domeij and Heathcote (2004)
Capital income Tax Rate (τ_k)	0.40	Domeij and Heathcote (2004)
<i>Jointly Calibrated Parameters</i>		
Labor Supply Constant (θ)	0.39	Market Hours=1/3
Discount Rate (β)	0.97	$K/Y = 3.00$
Government Expenditure (G)	0.14	$G/Y = 0.17$

In the next section, I present and discuss the results for the benchmark calibration as well as alternative parameterizations of the model.

5 Results and Discussion of Findings

For all calibrations of the model, I numerically compute the optimal steady state using the methods discussed in Section 3 and in more detail in the Appendix. To make a comparison with other steady states, I also solved for all steady states that satisfy the modified golden rule over a grid-point of labor tax rates.²⁵

5.1 Long-Run Optimal Fiscal Policy

Table 2 summarizes the long-run optimal tax rates. Not surprisingly, labor income tax rate is very sensitive to labor supply elasticity. What is quite striking is that the magnitude of the labor tax rate is quite large for all parameter values. Under benchmark calibration with $\gamma = 1.0$, labor tax rate is about 48%, about 22 percentage points higher than the observed average labor tax rates based on the figures reported by Domeij and Heathcote (2004). Low labor elasticity ($\gamma = 0.5$) and high elasticity ($\gamma = 2.0$) cases generate long-run labor tax rates of 65% and 36% respectively, both of which are significantly above the U.S. average.

²⁵As discussed in section 3, I can index all steady states by post-tax wage rate $\bar{w}$. Since pre-tax wage is fixed for all steady states that satisfy the modified golden rule, I can also use labor tax rate as an index.

Table 2: Long-Run Optimal Steady-States

Variables	$\gamma = 1.0, \sigma = 2.0$	$\gamma = 0.5, \sigma = 2.0$	$\gamma = 2.0, \sigma = 2.0$	$\gamma = 1.0, \sigma = 1.0$	$\gamma = 1.0, \sigma = 4.0$
Labor Tax Rate	48.30%	65.08%	35.90%	48.30%	48.30%
Capital Tax Rate	2.04%	0.33%	5.74%	0.95%	4.29%
Debt-to-GDP	4.19	6.83	3.34	4.14	4.40
Market Hours	0.25	0.25	0.30	0.25	0.25

An equally interesting observation is that long-run optimal capital income tax rate is *very low* for all parameter values. Under benchmark calibration, the capital tax rate is about 2%, *much lower* than the U.S. average of 40%. The corresponding figures for low labor elasticity ($\gamma = 0.5$) and high elasticity ($\gamma = 2.0$) cases are 0.3% and 5.7% respectively. It is obvious that labor elasticity and steady-state capital tax rate are positively related. In this model, high elasticity of labor supply corresponds to a high labor income risk. Consequently, the precautionary savings motive is more pronounced under high labor elasticity, leading to inefficiently high level of savings. The Ramsey planner optimally responds to suppress these incentives by increasing capital taxes accordingly. As intertemporal elasticity of substitution goes up from 0.25 to 1.00 (under benchmark labor supply elasticity of $\gamma = 1.0$), the long-run optimal capital tax rate goes down from 4.3% to 1.0%, both of these rates are much lower than the current U.S. average.

For all parameter values used in the quantitative exercise, the optimal debt-to-GDP ratio (more precisely, debt-to-market production in this model) is *much higher* than the observed values in the U.S.²⁶ The benchmark calibration suggests that optimal debt-to-GDP ratio for the U.S. ought to be 4.19. Alternative parameterizations yield values ranging from 3.34 ($\gamma = 0.5$ and $\sigma = 1.0$) to 6.83 ($\gamma = 2.0$ and $\sigma = 1.0$). Figures 1 and 2 exhibit long-run optimal tax rates and debt-to-GDP ratios for the entire range of labor supply elasticities and intertemporal elasticities of substitution.

²⁶U.S. Debt-to-GDP ratio averaged 0.60 over the period 1990-2008 and has been increasing since 2008, up to around 1.00 in 2012.

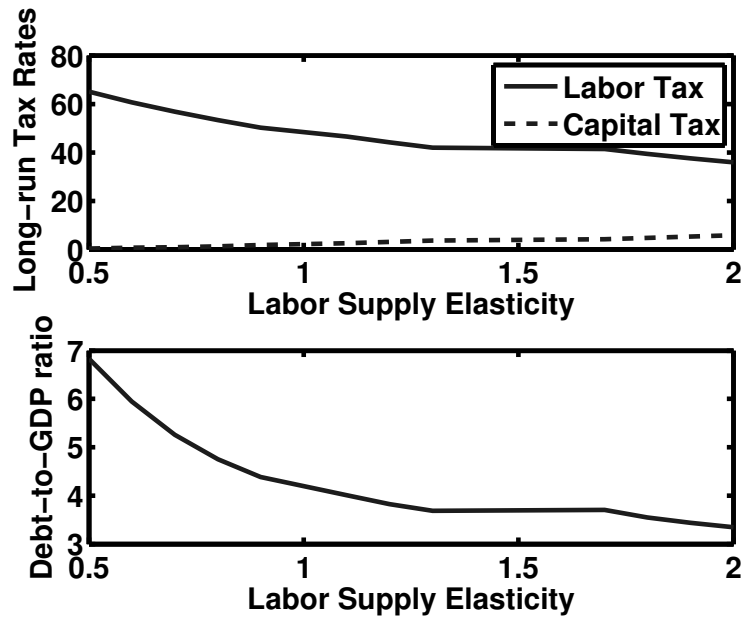


Figure 1: Long-run Optimal Taxes and Debt-to-GDP Ratio for a range of labor supply elasticities.

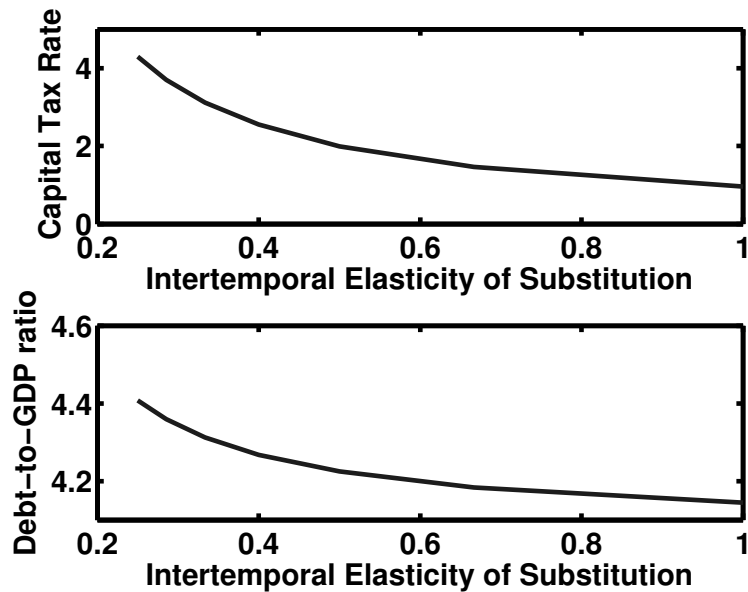


Figure 2: Long-Run Optimal Capital Tax and Debt-to-GDP Ratio for a range of intertemporal elasticities of substitution.

To get a better sense of why the model delivers quite extreme values, I can compare the optimal steady state to all other steady states that satisfy the modified golden rule property. Figures 3-7 summarize all numerical findings. For all figures, the dashed vertical line marks the labor income tax rate for the optimal

steady state.

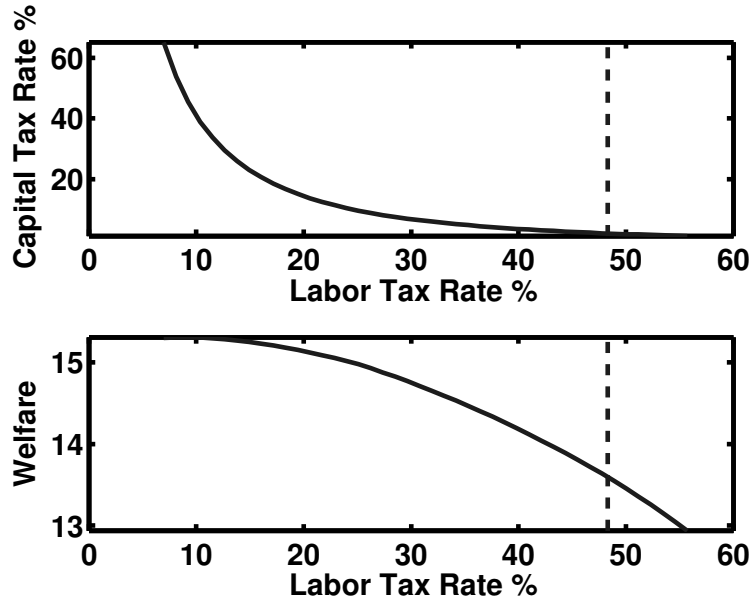


Figure 3: Long-Run Optimal Tax Rates and Welfare for Benchmark Calibration

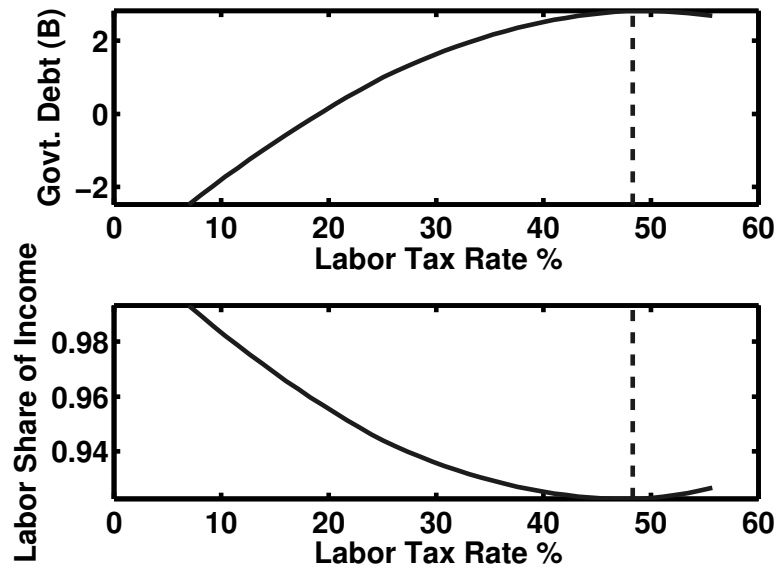


Figure 4: Long-Run Optimal Government Debt and Average Labor Share of Income for Benchmark Calibration

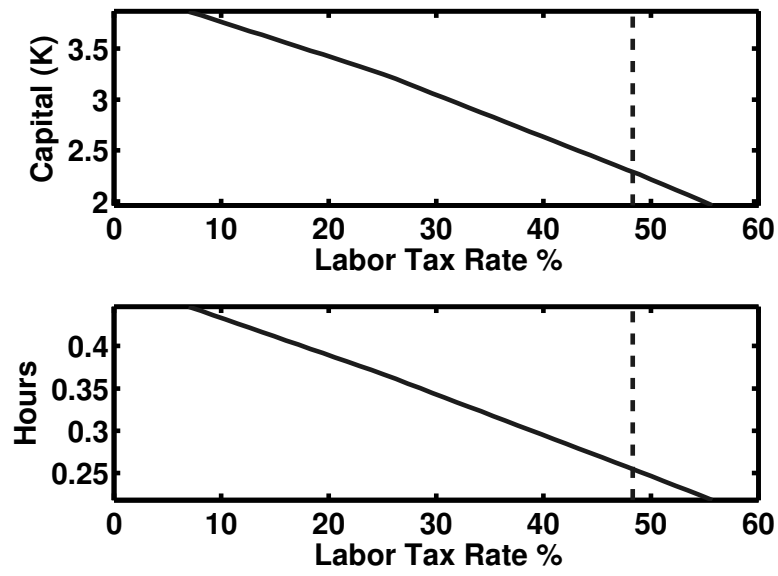


Figure 5: Long-Run Optimal Capital Stock and Hours for Benchmark Calibration

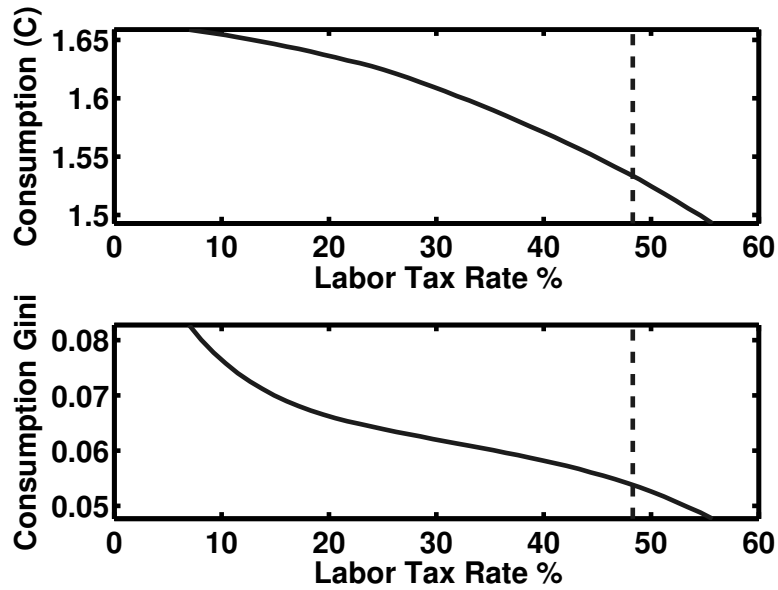


Figure 6: Long-Run Per-Capita Consumption and Consumption Inequality for Benchmark Calibration

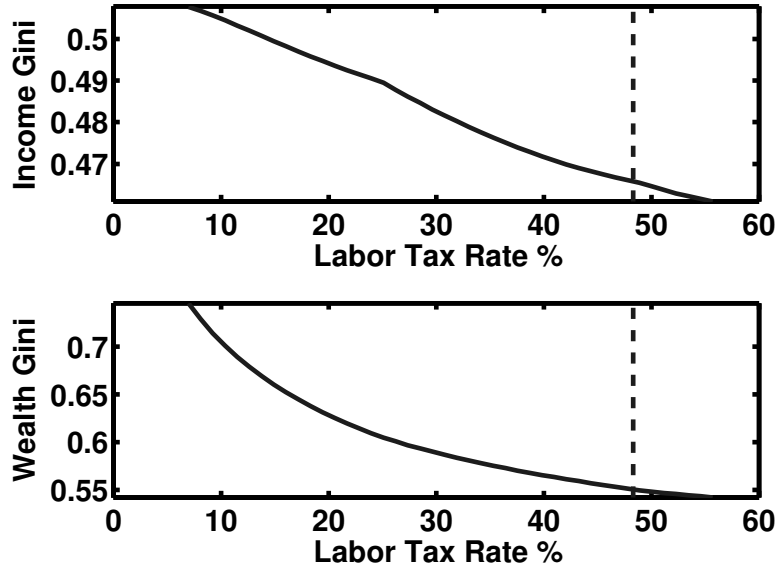


Figure 7: Long-Run Income and Wealth Inequality for Benchmark Calibration

Figure 4 gives us a clear picture of what the government would like to achieve in the long run. First, the level of government debt is *maximized* at the optimal steady state, conditional on satisfying the modified golden rule. With the given parameter values, there is a unique such point. Second, the average labor share of income is close to being *minimized*.²⁷ To gain some more insight on these incentives of the planner, we can look at the steady-state version of the budget constraint of a typical household. Defining $k \equiv a - B$, adding and subtracting relevant terms to equation (19), I obtain the following equivalent formulation at the steady state:

$$c + k' \leq k(1 + \bar{r}) + y(e, \bar{w}) + \bar{r}B$$

$$k' \geq -B$$

Issuing debt which is financed through labor income taxes plays a dual role. First, the borrowing constraints are effectively relaxed, an observation made very early on in the literature.²⁸ Second, the households

²⁷Average labor share of income is calculated as follows:

$$ALS(\bar{w}) = \int \frac{y(e, \bar{w})}{\bar{r}(\bar{w})a + y(e, \bar{w})} \mu(ds; \bar{w})$$

where $\mu(\cdot; \bar{w})$ is the steady-state distribution and $\bar{r}(\bar{w})$ is the post-tax interest rate that clears all markets, satisfying the modified golden rule when the post-tax wage level is $\bar{w}$.

²⁸This argument is *exact* when the government has access to lump-sum taxes, i.e. a marginal increase in debt is observationally equivalent to a marginal relaxation of borrowing constraints. In this model, all taxes are distortionary, therefore this is only suggestive.

are exposed to less risk since the riskless part of income goes up through the $\bar{r}B$ term and the risky component $y(e, \bar{w})$ goes down through taxes on labor income. The government can neither absorb all the risk, nor relax the borrowing constraints all the way by issuing debt excessively, because tax revenues are bounded in this model. The efficient level of debt should induce optimal smoothing of tax burden over time which implies the modified golden rule property holds in the long run.

Motivated purely by the prediction that agents save “too much” in this model vis-à-vis the complete markets benchmark, one would expect the government to tax capital somewhat heavily depending on the degree of market incompleteness. It is clear, however, that the scope of intervention through capital taxes remains fairly limited quantitatively. Instead, the results point to the direction that the government provides the necessary adjustment by suppressing the source of the problem, rather than correcting its consequences. This is achieved by (i) alleviating the labor income risk through reduction of labor share of income, and (ii) effectively relaxing borrowing constraints by issuing debt. The quantitatively small capital income tax also suggests that the Chamley-Judd zero-capital-tax result is not too far from optimal even under incomplete markets.

The features of the long-run fiscal policy are independent of whether the government needs to finance expenditures, supporting the premise that inefficiencies induced by incomplete markets drive the main results. Figures 8 and 9 represent an economy with $G = 0$; the same qualitative features are apparent.

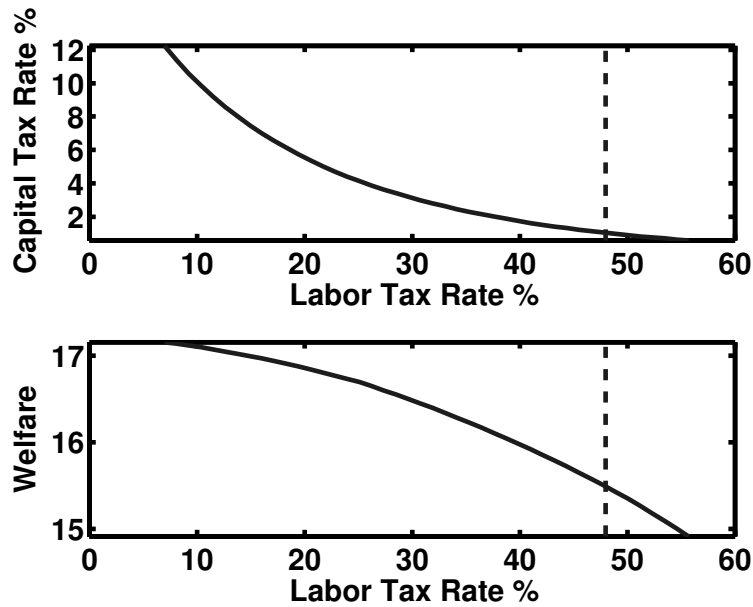


Figure 8: Long-Run Tax Rates and Welfare for Benchmark Calibration

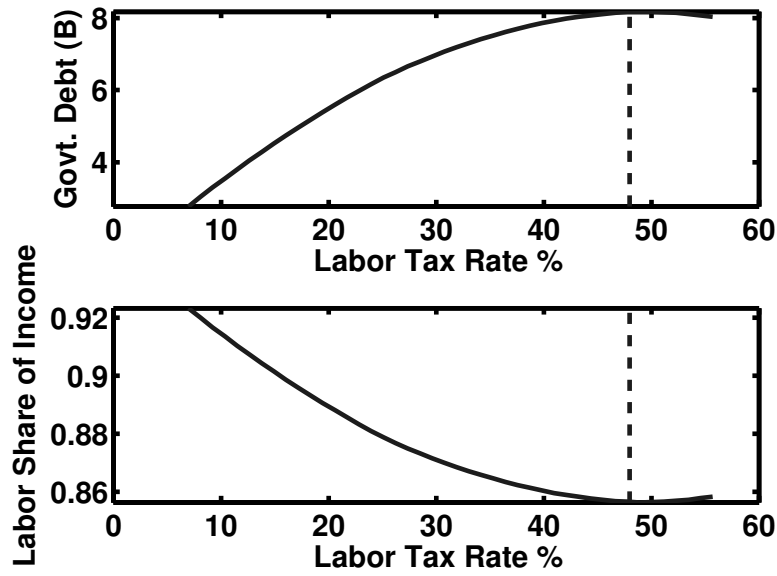


Figure 9: Long-Run Government Debt and Average Labor Share of Income for Benchmark Calibration

Figures 10 and 11 clearly demonstrate that the qualitative results discussed above for the benchmark calibration are robust to different values for labor elasticity. In all cases, the optimal steady state satisfies maximum debt and close to minimum labor share of income.

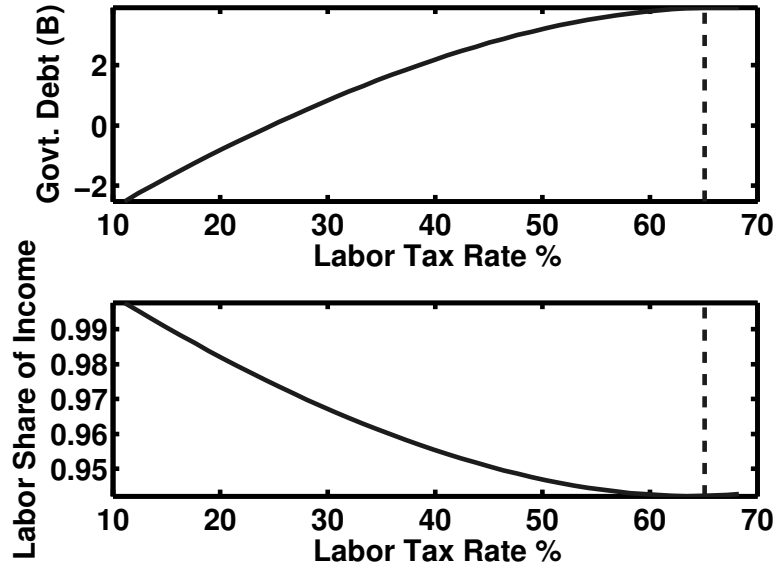


Figure 10: Long-Run Government Debt and Average Labor Share of Income for $\gamma = 0.5$, $\sigma = 2.0$

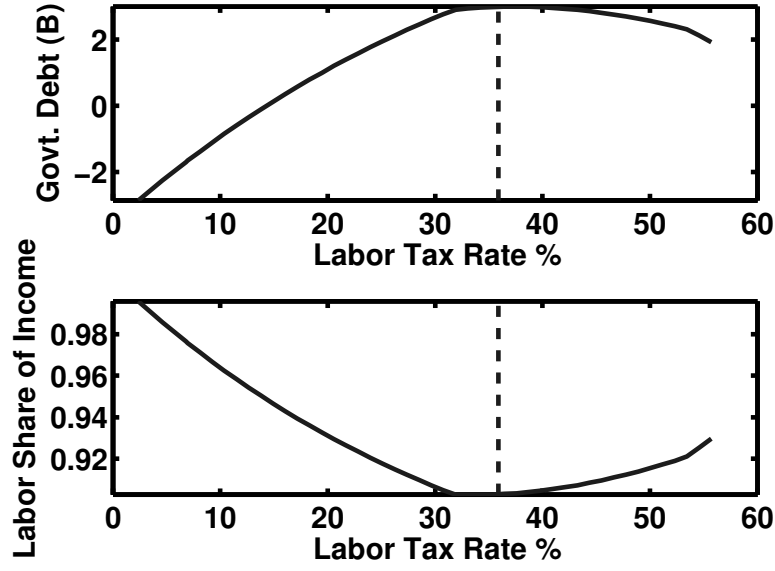


Figure 11: Long-Run Government Debt and Average Labor Share of Income for $\gamma = 2.0$, $\sigma = 2.0$

5.2 Welfare and Inequality in the Long Run

Figure 3 illustrates a very striking fact about the steady-state welfare level. The steady state which is reached following an optimal plan delivers a (flow) welfare level that is *significantly lower* than those that could be achieved using alternative tax rates.²⁹ This is also clear from the levels of per-capita consumption in Figure 6. Indeed, if we were inclined to accept the view that the steady state that maximizes flow welfare is close to being long-run optimal, we would be led to prescribe a labor income tax rate of 9%, a capital income tax rate of 45% and a *negative* government debt that is 176% of GDP! This observation makes it clear that the results provided in the quantitative literature that relies exclusively on maximizing the steady-state welfare are potentially misleading.

Figures 12 and 13 demonstrate the flow welfare levels for low and high labor supply elasticity cases. Welfare levels share the same comparative features with the benchmark calibration.

²⁹I calculate steady-state welfare in utility terms as

$$Welfare(\bar{w}) = \int u(c)\mu(ds; \bar{w})$$

where $\mu(\cdot; \bar{w})$ is the steady-state distribution when the post-tax wage level is $\bar{w}$.

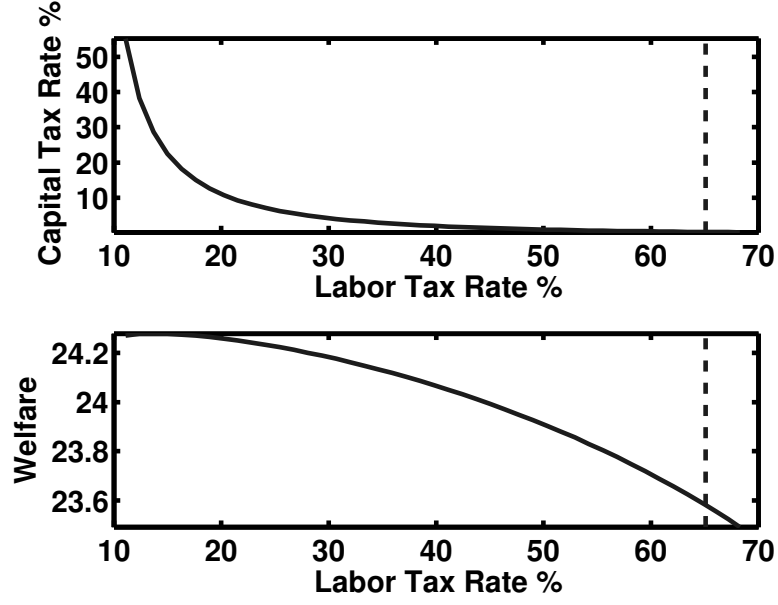


Figure 12: Long-Run Tax Rates and Welfare for $\gamma = 0.5, \sigma = 2.0$

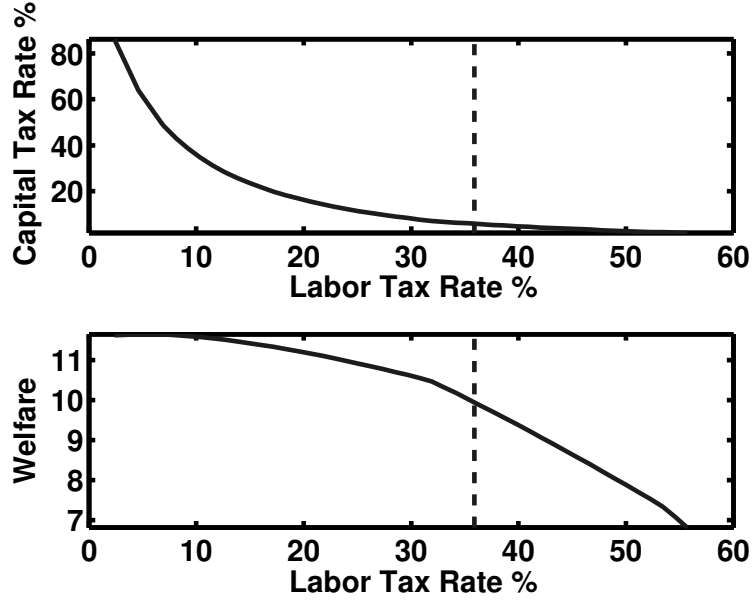


Figure 13: Long-Run Tax Rates and Welfare for $\gamma = 2.0, \sigma = 2.0$

The model economy features non-market production as an alternative activity, therefore, to facilitate comparison with the data, I use taxable *market* income Gini coefficient as a measure of income inequality.³⁰

Figures 6 and 7 exhibit consumption, income and wealth inequalities across all steady states that fea-

³⁰Market income Gini coefficient measures the inequality in $ra + wn(e, \bar{w})e$.

ture the modified golden rule property. All long-run optimal inequality measures are slightly lower than the model-generated numbers for the U.S. economy. The long-run optimal income Gini, wealth Gini, and consumption Gini coefficients are 0.466, 0.551, and 0.053 respectively. The corresponding model-generated numbers for the U.S. economy (initial steady state) are 0.474, 0.597, and 0.058.³¹

6 Transitional Dynamics

The steady-state analysis sheds light on the incentives of the Ramsey planner in the long run but it is completely silent on the magnitude and the source of welfare gains from following an optimal plan. Since initial conditions play no role in determining the long-run optimal fiscal policy, providing a complete solution to the Ramsey problem, i.e. characterizing the optimal transition path from any given initial condition, is left for future research. On the other hand, it is essential that we get some insight on the way in which an optimal plan improves welfare, especially given the result that the Ramsey planner is far from maximizing flow welfare in the long run.

This controversial finding naturally begs the following hypothetical question: If the *status quo* in the economy features fiscal policy that provides the highest achievable flow welfare level, why would the Ramsey planner propose a reform that provides a significantly lower flow welfare in the long run? Provided that there is convergence to the steady state, there is only one possibility: The welfare gains over the transition are large enough to compensate for the losses in the long run.

It turns out it is straightforward to construct feasible, but not necessarily optimal, fiscal policy paths, where there are overall welfare gains moving from the steady state with the highest flow welfare to the optimal steady state. In all of these examples, government achieves a welfare improvement by front-loading consumption and/or by reducing consumption inequality significantly over the transition path. This reform necessitates a significant increase in debt over the transition.

To keep things simple, for the numerical results, I constrain the transition paths to feature an initial jump in the post-tax wage to $\bar{w}_0$, followed by a linear transition to the long-run optimal level for $T = 300$ periods. I also assume that the government cannot confiscate assets by imposing the constraint that $\bar{r}_t \geq 0$. The government then optimally imposes a 100% capital income tax in period 0, that is, $\bar{r}_0 = 0$. Following another jump in period 1 to a given $\bar{r}_1$, $\bar{r}_t$ converges linearly to the long-run optimal level, just like the sequence $\bar{w}_t$. Using the optimal savings and consumption responses of the households that follow the tax reform announced in period 0, I compute per-capita consumption and private assets over the transition. I use these levels to back out aggregate capital and government debt using the government resource constraint.

³¹Diaz-Gimenez, Glover, and Rios-Rull (2011) report income and wealth Gini of 0.58 and 0.82 using SCF data.

With all these restrictions in place, given a value for $\bar{r}_1$, $\bar{w}_0$ is pinned down by the initial conditions of the economy. Therefore, in this constrained transition analysis, the only free choice variable is $\bar{r}_1$.

Figures 14 and 15 represent such a transition, where the economy starts from the steady state that features the highest welfare under the benchmark calibration. In all figures, the horizontal dashed line represents the initial steady-state values for comparison. In this particular example, fiscal policy reform effectively features a net income transfer initially, since both $\bar{r}$ and $\bar{w}$ go up significantly after period 1. After an initial decline in consumption level due to 100% tax on capital income in period 0, a dominant income effect leads to a gradual increase in consumption which surpasses the pre-reform level after a few periods. Consumption level starts declining around period 50, as $\bar{w}$ goes below the pre-reform level, and eventually converges to the long-run optimal level. Due to discounting, the initial increase in consumption that lasts for about 90 periods offsets, at least in part, the large decline in the long run.

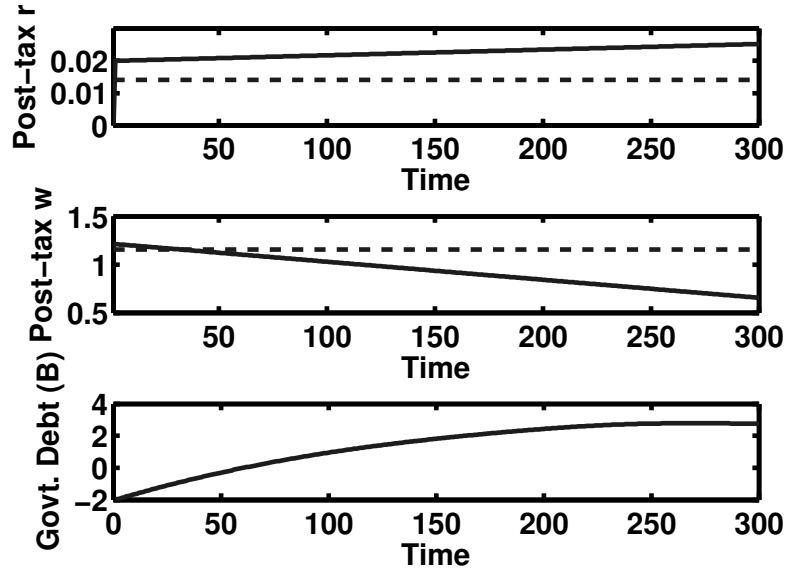


Figure 14: Change in Fiscal Policy over Transition

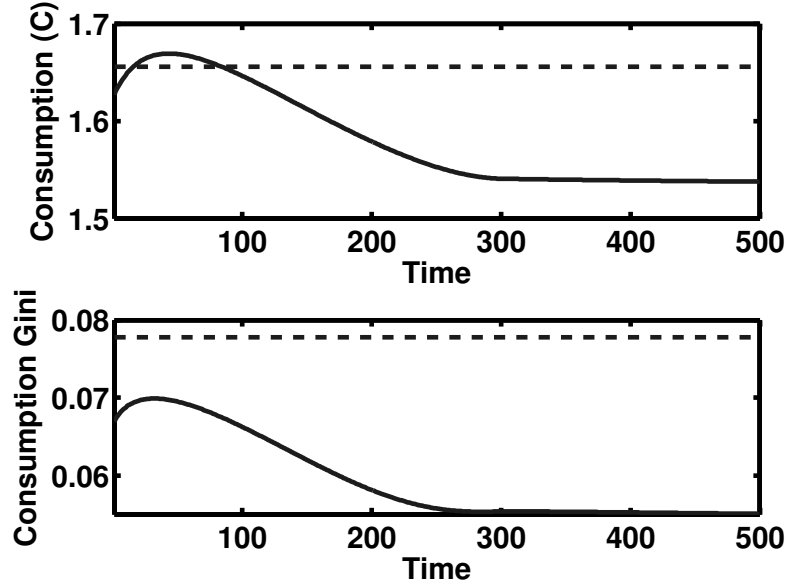


Figure 15: Change in Consumption and Consumption Gini Coefficient over Transition

A significant second source of welfare gains in this reform comes from the decline in consumption inequality. Since $\bar{r}$ is significantly higher than the pre-reform level, the households respond by increasing their holdings of assets. This leads to a large decline in labor share of income for all households, leading to lower income risk. As the severity of market incompleteness diminishes through this channel, households enjoy more consistent consumption levels, both over the transition, and in the long run. Naturally, this shows up in the cross-section through a low Gini coefficient. Figure 15 exhibits that consumption Gini goes down permanently in period 0 and stays lower than the pre-reform level throughout the entire transition.

7 Literature Review and Further Discussion

This research is closely related to the branch of literature that emphasizes the importance of the transition path for long-run policy. Domeij and Heathcote (2004) rank alternative fiscal policies by quantifying welfare gains or losses associated with the transition to a steady state. However, they restrict taxes to be constant over the transition path. While this is a clear improvement over any approach that relies on a steady-state welfare comparison, their analysis would provide a biased estimate of the welfare gains and the steady state might not be long-run optimal.³²

³²See Krueger and Ludwig (2013) and Bakis, Kaymak, and Poschke (2012) for more recent applications of the same approach. Bakis, Kaymak, and Poschke (2012) report quantitative results that support the main message of this paper. In their analysis of optimal progressivity of taxes, they show that a social planner who maximizes the steady-state welfare would choose a *regressive* tax scheme, whereas a constrained transition analysis yields a *progressive* tax scheme instead.

Gottardi, Kajii, and Nakajima (2011) provide a complete characterization of the optimal dynamic fiscal policy under incomplete markets in a highly stylized model of human-capital accumulation. Sacrificing the generality of the problem in favor of a complete solution, their approach allows for elegant closed-form solutions. By contrast, this paper answers quantitative questions within the confines of a workhorse model used in the study of heterogeneity in macroeconomics, keeping the environment as general as possible.

This paper is also related to the recent literature on constrained efficiency. It is well known that an equilibrium allocation in a model with incomplete financial markets can be improved on, since welfare theorems do not hold in this environment. Therefore, fiscal policy has a role that goes beyond the need to finance government spending. The widespread view that saving is inefficiently high in Bewley-type models was recently challenged. Gottardi, Kajii, and Nakajima (2013) draw a striking conclusion based on their constrained inefficiency analysis: whether capital income should be taxed or not has nothing to do with agents saving “too much” relative to a complete markets economy.

For a reasonably calibrated incomplete markets model, Davila et al. (2012) find that the capital stock is too low in the U.S. economy from a constrained efficiency perspective, and it *ought to be* higher, a result that crucially depends on the income composition of the consumption-poor. A higher capital stock leads to an increase in marginal product of labor. If the consumption-poor have labor-intensive income, this change, in fact, leads to an improvement in an egalitarian sense, since consumption-poor have *de facto* higher Pareto weights. As a consequence, their decentralization exercise prescribes an optimal fiscal policy that involves a capital income *subsidy*.

On close inspection, we can point out three reasons why the long-run optimal policy in the model outlined in this paper leads to a completely different outcome: First, the notion of constrained efficiency in Davila et al. (2012) necessitates that all proceeds from taxes be rebated back to the households. There are no explicit transfers between households. Therefore, the improvement in their economy *has* to come from price effects. Second, due to the specific structure of their model, the planner cannot smooth tax distortions over time, whereas the government in this model can issue debt, allowing the planner to achieve dynamic efficiency.³³ Third, labor supply is inelastic in Davila et al. (2012). Although “capital intensity ought to be higher in the U.S.” is a common finding, in the current model, the government achieves this goal by lowering labor supply through labor income taxes. By contrast, the only way it can be achieved in Davila et al. (2012) is through a capital subsidy.

An important and somewhat confusing distinction between the analysis by Davila et al. (2012) and Aiyagari (1994a, 1995) (and this paper) is that the former argues that whether capital income should be

³³Dynamic efficiency only in the sense of achieving the first-best capital intensity. This should not be confused with the notion of dynamic inefficiency that is standard in the overlapping-generations literature.

taxed or subsidized is a quantitative question, whereas the latter argues that optimal capital income tax rate is always positive, regardless of the calibration of the model. I would like to point out that Aiyagari's (1995) positive capital tax result hinges on the assumption that the government can issue debt. Whether or not the government can issue debt, $\beta(1 + \bar{r}) < 1$ holds in the long run for an economy with heterogeneous agents and incomplete markets. However, it is the access to debt that puts discipline on the level of capital intensity K/N in the economy. Since the marginal rate of transformation of the tax burden across time is the tax-free interest rate $1 + F_K$, the modified golden rule level of capital intensity holds in the long run. Absent government debt, the rate at which the tax burden is transferred across periods is *irrelevant* for the planner and there is no reason to expect *a priori* that the modified golden rule holds. It is reasonable to conjecture that a model without debt delivers either $\bar{r} < F_K$ or $\bar{r} > F_K$, depending on the parameterization. In this paper, due to its normative appeal, I only provide a quantitative analysis of the model with debt.

Aiyagari's (1995) intuition for positive capital income tax rate was that the households in a Bewley economy save "too much" vis-à-vis the complete markets benchmark. However, as pointed out by Davila et al. (2012) and Gottardi, Kajii, and Nakajima (2013), as long as the planner does not have the policy tools to "complete" the financial markets, there is no good reason why one should take the complete markets benchmark as a point of reference. Indeed, the positive capital income tax result is most likely an artifact of the availability of debt as an instrument for the planner.

8 Conclusion

Quantitative analysis of optimal dynamic fiscal policy is a difficult task since the problem is time-inconsistent and non-stationary. The main contribution of this paper is to reveal that this problem is much easier to solve than previously thought in Bewley-type models with idiosyncratic income risk and incomplete markets. As illustrated, the dependence of long-run optimal fiscal policy on the initial conditions disappears asymptotically in this environment, much like life-cycle models in which there are no private wealth transfers across generations. This leads to a long-run optimal policy that depends only on the "deep parameters" of the model and the underlying income process. The emphasis in this paper was on the quantitative implications. Since this property is likely to hold in a broader class of optimal fiscal policy problems, a theoretical study of minimal modeling assumptions that deliver this property is a promising next step.

Although a constrained transition analysis is provided in this paper, for the sake of preserving a unified theme, the study of optimal transition path is left out for future research. However, as pointed out earlier, a complete solution to the Ramsey problem is necessary to fully understand the source of welfare gains from a fiscal policy reform. The recursive version of the Ramsey problem introduced in Section 2 can be

conveniently used for this task. The real challenge, however, comes from the dimensionality of the state variable. An adaptation of the the “approximate aggregation” method of Krusell and Smith (1998) might render this analysis feasible.

The quantitative results in this paper provide a striking counter to the claims in the literature in favor of high capital taxation when markets are incomplete. In this widely-used framework for analyzing optimal capital taxation, this paper shows that a very high debt level that is financed by taxes on the source of income that is stochastic could improve efficiency by suppressing the consequences of missing financial markets. Moreover, distortionary capital income tax is largely redundant once such a public debt management is implemented. A detailed policy recommendation for the U.S. economy requires further research on the robustness of these results with respect to alternative tax instruments and different specifications of the income process. The quantitative methods used in this paper can be adapted in a straightforward manner to models that feature tax instruments that resemble the U.S. tax code more closely. For instance, a study of optimal progressivity of income taxes is a promising extension of this kind.

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A Technical Appendix

A.1 Proof of Proposition 1

Proposition 1 *An interior solution to the Ramsey problem satisfies the following conditions:*

$$\lambda' = h(.) : \quad u'(c) \geq \beta(1 + \bar{r}')\mathbb{E}[u'(c')|e] \text{ with equality if } a' > -\underline{a}, \quad \text{a.e. } \mu \quad (29)$$

$$a' = g(.) : \quad u'(c) + u''(c)[\lambda(1 + \bar{r}') - \lambda'] = \beta(1 + \bar{r}')\mathbb{E}\left[u'(c') + u''(c')[\lambda'(1 + \bar{r}') - \lambda']|e\right] \quad (30)$$

$$+ \beta\gamma'(F_K(K', N') - \bar{r}') \text{ if } a' > -\underline{a}, \text{ otherwise } \lambda' = 0, \quad \text{a.e. } \mu$$

$$B'(\mu, B) : \quad \gamma = \beta(1 + F_K(K', N'))\gamma' \quad (31)$$

$$\bar{r}(\mu, B) : \quad \gamma A = \sum_e \int u'(c)\lambda\mu(ds, e) + \sum_e \int a \left(u'(c) + u''(c)[\lambda(1 + \bar{r}) - \lambda'] \right) \mu(ds, e) \quad (32)$$

$$\bar{w}(\mu, B) : \quad \gamma N = \gamma(F_N(K, N) - \bar{w})N'(\bar{w}) + \sum_e \int en(e, \bar{w}) \left(u'(c) + u''(c)[\lambda(1 + \bar{r}) - \lambda'] \right) \mu(ds, e) \quad (33)$$

where γ is the multiplier for the government budget constraint.

Proof.

Let $y(e, \bar{w}) = \max_n \bar{w}en + H(1 - n)$ be the total labor income of a household with state e and let $n(e, \bar{w}) = \arg \max_{n \in [0, 1]} \bar{w}en + H(1 - n)$. Applying envelope theorem, I have $y'(e, \bar{w}) = n(e, \bar{w})e$. I can then write the aggregate labor supply as a function of $\bar{w}$ only: $N(\bar{w}) = \sum_e \int en(e, \bar{w})\mu(ds, e) = \sum_e \int y'(e, \bar{w})\mu(ds, e)$.

I assume further that the sequential formulation of the Ramsey problem admits a stationary solution so that the policy functions $a' = g(\mu, B, s, e)$ and $\lambda' = h(\mu, B, s, e)$ are well-defined. Also define $\eta' = \eta^*(\mu, B, s, e)$ and $\theta' = \theta^*(\mu, B, s, e)$ to represent the solution for the multipliers. By definition, $h(.) = \eta^*(.)(g(.)) + \underline{a} - \theta^*(.)$ holds.

Exploiting the induced Markov structure, it is straightforward to show that the sequential problem (17) is equivalent to the following problem:

$$\min_{\eta_{t+1}^*, \theta_{t+1}^* \geq 0} \max_{g_{t+1}, \bar{r}_t, \bar{w}_t, B_{t+1}} \sum_{t=0}^{\infty} \beta^t \sum_e \int \left[u(c_t) + u'(c_t)[\lambda(1 + \bar{r}_t(\mu_t, B_t)) - h_{t+1}(\mu_t, B_t, s, e)] \right] \mu_t(ds, e) \quad (34)$$

subject to

$$G_t + (1 + \bar{r}_t(\mu_t, B_t))B_t + \bar{r}_t(\mu_t, B_t)K_t + \bar{w}_t(\mu_t, B_t)N(\bar{w}_t(\mu_t, B_t)) \leq F(K_t, N(\bar{w}_t(\mu_t, B_t))) + B_{t+1}(\mu_t, B_t)$$

$$h_{t+1}(\mu_t, B_t, s, e) \equiv \eta_{t+1}^*(\cdot)(g_{t+1}(\cdot) + \underline{a}) - \theta_{t+1}^*(\cdot)$$

where

$$c_t = a(1 + \bar{r}_t(\mu_t, B_t)) + y(e, \bar{w}_t(\mu_t, B_t)) - g_{t+1}(\mu_t, B_t, s, e)$$

$$K_t = \sum_e \int a\mu_t(ds, e) - B_t$$

$$\mu_{t+1}(S', e') = \sum_e \pi_{ee'} \int I[(g_{t+1}(\cdot), h_{t+1}(\cdot)) \in S'] \mu_t(ds, e) \text{ for each } S' \in \Sigma_s, \text{ each } e' \in E \quad (35)$$

given a_0, B_0 .

Let $\beta^t \gamma_t(\mu_t, B_t)$ represent the Lagrange multiplier for the government budget constraint at period t . By taking the derivative of the function (34) with respect to B_{t+1} , $\bar{r}_t$ and $\bar{w}_t$, I obtain the last three necessary conditions (31), (32), (33).

$$\gamma_t = \beta \gamma_{t+1}(1 + F_K(K_{t+1}, N_{t+1}))$$

$$\gamma_t(K_t + B_t) = \gamma_t A_t = \sum_e \int u'(c_t) \lambda \mu_t(ds, e) + \sum_e \int a \left(u'(c_t) + u''(c_t) [\lambda(1 + \bar{r}_t) - h_{t+1}] \right) \mu_t(ds, e)$$

$$\gamma_t N_t = \gamma_t (F_N(K_t, N_t) - \bar{w}_t) N'_t(\bar{w}_t) + \sum_e \int e n(e, \bar{w}_t) \left(u'(c_t) + u''(c_t) [\lambda(1 + \bar{r}_t) - h_{t+1}] \right) \mu_t(ds, e)$$

For what is to follow, I use calculus of variations to prove the first-order condition (30). Consider the part of the objective function and government budget constraints that correspond to two subsequent periods. For clarity of exposition, I eliminate time subscripts. Given the optimal policies for next period, a'', λ'' , optimal fiscal policy variables $\bar{r}, \bar{r}', \bar{w}, \bar{w}', B, B'$, and the government multipliers γ, γ' , the optimal policies $g(\mu, B, s, e)$, $\eta^*(\mu, B, s, e)$ and $\theta^*(\mu, B, s, e)$ would solve the following saddle-point problem:

$$\begin{aligned}
& \min_{\eta', \theta' \geq 0} \max_{a' \geq -\underline{a}} \sum_e \int u(a(1 + \bar{r}) + y(e, \bar{w}) - a') + u'(a(1 + \bar{r}) + y(e, \bar{w}) - a')(\lambda(1 + \bar{r}) - \eta'(a' + \underline{a}) + \theta')\mu(ds, e) \\
& \quad + \gamma \left[F(K, N) + B' - G - (1 + \bar{r})B - \bar{r}K - \bar{w}N \right] \\
& \quad + \beta \sum_{e'} \int u(a'(1 + \bar{r}') + y(e', \bar{w}') - a'') + u'(a'(1 + \bar{r}') + y(e', \bar{w}') - a'')((\eta'(a' + \underline{a}) - \theta')(1 + \bar{r}') - \lambda'')\mu'(ds', e') \\
& \quad + \beta \gamma' \left[F(K', N') + B'' - G' - (1 + \bar{r}')B' - \bar{r}'K' - \bar{w}'N' \right]
\end{aligned}$$

where $\mu'(ds', e') = \mu_{t+1}(ds', e')$ is given by (35) and

$$K = \sum_e \int a\mu(ds, e) \quad K' = \sum \int a'\mu'(ds', e')$$

Using the transition function induced by (35) and applying Theorem 8.3 (pg. 216) in Stokey, Lucas, and Prescott (1989), I express the above expression in a single integral:

$$\begin{aligned}
& \sum_e \int \left[u[a(1 + \bar{r}) + y(e, \bar{w}) - g(\cdot)] + u'[a(1 + \bar{r}) + y(e, \bar{w}) - g(\cdot)](\lambda(1 + \bar{r}) - h(\cdot)) \right. \\
& \quad \left. + \beta \sum_{e'} \pi_{ee'} u[g(\cdot)(1 + \bar{r}') + y(e', \bar{w}') - a''] + u'[g(\cdot)(1 + \bar{r}') + y(e', \bar{w}') - a''] (h(\cdot)(1 + \bar{r}') - \lambda'') \right] \mu(ds, e) \\
& \quad + \gamma \left[F(K, N) + B' - G - (1 + \bar{r})B - \bar{r}K - \bar{w}N \right] \\
& \quad + \beta \gamma' \left[F(K', N') + B'' - G' - (1 + \bar{r}')B' - \bar{r}'K' - \bar{w}'N' \right]
\end{aligned}$$

where

$$\begin{aligned}
K &= \sum_e \int a\mu(ds, e) & K' &= \sum_e \int g(\mu, B, s, e)\mu(ds, e) \\
h(\cdot) &= \eta^*(\cdot)(g(\cdot) + \underline{a}) - \theta^*(\cdot)
\end{aligned}$$

For ease of exposition, I assume that measure μ admits density $m(a, \lambda, e)$ so that $\sum_e \int m(a, \lambda, e)dad\lambda = 1$; this assumption is not necessary for the result. Also assume for the rest of the proof that the borrowing constraint does not bind, so that $a' + \underline{a} > 0$ is optimal.³⁴

For a fixed efficiency e_0 and λ_0 , let $g^\epsilon(\mu, a, \lambda, e) = g(\mu, a, \lambda, e) + \epsilon I[e = e_0, \lambda = \lambda_0]\delta_g(a)$ be an ϵ -perturbation of the policy function for assets at (λ_0, e_0) with an arbitrary measurable function $\delta_g(a)$. For the policy function to be optimal, $g^\epsilon(\mu, a, \lambda, e)$ must be suboptimal for any $\epsilon > 0$. Define the perturbed objective

³⁴The case in which the borrowing constraint binds is trivial, and it is therefore omitted.

$$\begin{aligned}
\kappa(\epsilon) = & \sum_e \int \left[u[a(1+\bar{r}) + y(e, \bar{w}) - g^\epsilon(\cdot)] + u'[a(1+\bar{r}) + y(e, \bar{w}) - g^\epsilon(\cdot)](\lambda(1+\bar{r}) - h^\epsilon(\cdot)) \right. \\
& \left. + \beta \sum_{e'} \pi_{ee'} u[g^\epsilon(\cdot)(1+\bar{r}') + y(e', \bar{w}') - a''] + u'[g^\epsilon(\cdot)(1+\bar{r}') + y(e', \bar{w}') - a''] (h^\epsilon(\cdot)(1+\bar{r}') - \lambda'') \right] m(a, \lambda, e) da d\lambda \\
& + \gamma \left[F(K, N) + B' - G - (1+\bar{r})B - \bar{r}K - \bar{w}N \right] \\
& + \beta \gamma' \left[F(K'^\epsilon, N') + B'' - G' - (1+\bar{r}')B' - \bar{r}'K'^\epsilon - \bar{w}'N' \right]
\end{aligned}$$

where $K'^\epsilon = \sum_e \int g^\epsilon(\cdot) m(a, \lambda, e) da d\lambda$ and

$$h^\epsilon(\cdot) = \eta^*(\cdot)(g^\epsilon(\cdot) + \underline{a}) - \theta^*(\cdot).$$

Taking the derivative of $\kappa(\epsilon)$ and evaluating at $\epsilon = 0$ should equal 0, since policy function is optimal.

$$\begin{aligned}
\kappa'(0) = & \int \left[- \left(u'(c) + u''(c)(\lambda_0(1+\bar{r}) - h(\cdot)) \right) \right. \\
& + \beta(1+\bar{r}') \sum_{e'} \pi_{e_0 e'} \left(u'(c') + u''(c')(h(\cdot)(1+\bar{r}) - \lambda'') \right) \\
& \left. - \eta^*(\cdot) \left(u'(c) - \beta(1+\bar{r}) \sum_{e'} \pi_{e_0 e'} u'(c') \right) \right] \delta_g(a) m(a, \lambda_0, e_0) da \\
& + \beta \gamma' \left[(F_K(K', N') - \bar{r}) \int \delta_g(\tilde{a}) m(\tilde{a}, \lambda_0, e_0) d\tilde{a} \right] = 0
\end{aligned}$$

Since this is true for an arbitrary function $\delta_g(a)$, let $\delta_g(a) = I[a \geq a_0]$ for some a_0 . Substituting this function, I get

$$\begin{aligned}
& \int_{a_0}^{\infty} \left[- \left(u'(c) + u''(c)(\lambda_0(1+\bar{r}) - h(\cdot)) \right) \right. \\
& + \beta(1+\bar{r}') \sum_{e'} \pi_{e_0 e'} \left(u'(c') + u''(c')(h(\cdot)(1+\bar{r}) - \lambda'') \right) \\
& \left. - \eta^*(\cdot) \left(u'(c) - \beta(1+\bar{r}) \sum_{e'} \pi_{e_0 e'} u'(c') \right) \right] m(a, \lambda_0, e_0) da \\
& + \beta \gamma' \left[(F_K(K', N') - \bar{r}) \int_{a_0}^{\infty} m(\tilde{a}, \lambda_0, e_0) d\tilde{a} \right] = 0
\end{aligned}$$

I picked a_0 arbitrarily, therefore the above expression is identically zero for all a_0 . Taking the derivative

with respect to a_0 , I get

$$\begin{aligned} & - (u'(c) + u''(c)(\lambda_0(1 + \bar{r}) - h(.))) - \eta^*(.) \left(u'(c) - \beta(1 + \bar{r}) \sum_{e'} \pi_{e_0 e'} u'(c') \right) \\ & + \beta(1 + \bar{r}') \sum_{e'} \pi_{e_0 e'} (u'(c') + u''(c')(h(.)(1 + \bar{r}') - \lambda'')) + \beta\gamma'(F_K(K', N') - \bar{r}') = 0 \end{aligned}$$

Therefore for all (a, λ, e) , it must be the case that

$$\begin{aligned} & - (u'(c) + u''(c)(\lambda(1 + \bar{r}) - h(.))) - \eta^*(.) \left(u'(c) - \beta(1 + \bar{r}) \sum_{e'} \pi_{ee'} u'(c') \right) \\ & + \beta(1 + \bar{r}') \sum_{e'} \pi_{ee'} (u'(c') + u''(c')(h(.)(1 + \bar{r}') - \lambda'')) + \beta\gamma'(F_K(K', N') - \bar{r}') = 0 \end{aligned} \quad (36)$$

Following the identical steps above, but for the policy functions $\eta^*(.)$ and $\theta^*(.)$, I obtain the following conditions.

$$(g(.) + \underline{a}) \left(u'(c) - \beta(1 + \bar{r}) \sum_{e'} \pi_{ee'} u'(c') \right) \leq 0 \text{ with equality if } \eta^*(.) > 0 \quad (37)$$

$$u'(c) - \beta(1 + \bar{r}) \sum_{e'} \pi_{ee'} u'(c') \geq 0 \text{ with equality if } \theta^*(.) > 0 \quad (38)$$

Using our initial interiority assumption on assets, that $g(.) + \underline{a} > 0$, equations (37) and (38) jointly imply the functional Euler equation (38) holds with equality. The necessary condition (29) in the proposition follows immediately. In addition, (36) simplifies to

$$-(u'(c) + u''(c)(\lambda(1 + \bar{r}) - h(.))) + \beta(1 + \bar{r}') \sum_{e'} \pi_{ee'} (u'(c') + u''(c')(h(.)(1 + \bar{r}') - \lambda'')) + \beta\gamma'(F_K(K', N') - \bar{r}') = 0 \quad (39)$$

This is identical to equation (30) in the proposition. This completes the proof for the case in which $g(.) + \underline{a} > 0$.

Q.E.D.

B Computational Appendix

B.1 Computing the Optimal Steady States

B.1.1 Policy Functions

As described in the main text, using standard techniques described in Aiyagari (1994b), for any given $\bar{w}$, I can solve for $\bar{r}(\bar{w})$ that clears all markets and satisfies the modified golden rule. This provides the policy functions $c(a, e; \bar{w})$ and $a'(a, e; \bar{w})$ along with the long-run stationary distribution $m(a, e; \bar{w})$.

For ease of exposition, for the entire numerical exercise I will use an auxiliary variable $q \equiv \lambda/\gamma$ and eliminate λ from the system. In particular, all policy functions and the steady-state distribution will be defined over (a, q, e) space instead.

Using the fact that the K/N ratio satisfies the modified golden rule property, I can write $\beta(F_K - \bar{r}) = 1 - \beta(1 + \bar{r})$. Using this result and dividing (26) by Lagrange multiplier γ , I obtain

$$u''(c)[q(1 + \bar{r}) - q'] = \beta(1 + \bar{r})\mathbb{E}\left[u''(c')[q'(1 + \bar{r}) - q'']|e\right] + 1 - \beta(1 + \bar{r}) \quad (40)$$

Since $\gamma > 0$ at the steady state, $q = 0$ if and only if $\lambda = 0$.

Since equation (40) is linear in q , I conjecture that the policy function for q' is also linear in q .

$$q'(a, q, e; \bar{w}) = \alpha_0(a, e; \bar{w})q + \frac{\alpha_1(a, e; \bar{w})}{(1 + \bar{r})u''(c)} \quad (41)$$

where $\alpha_0(a, e; \bar{w})$ and $\alpha_1(a, e; \bar{w})$ are potentially non-linear functions in (a, e) .

Substituting this functional form into (40), I get the following expression at a steady state:

$$u''(c)\left[q(1 + \bar{r} - \alpha_0) - \frac{\alpha_1}{(1 + \bar{r})u''(c)}\right] = \beta(1 + \bar{r})\mathbb{E}\left[u''(c')\left[(1 + \bar{r} - \alpha'_0)\left(\alpha_0 q + \frac{\alpha_1}{(1 + \bar{r})u''(c)}\right) - \frac{\alpha'_1}{(1 + \bar{r})u''(c')}\right]|e\right] + 1 - \beta(1 + \bar{r})$$

where $\alpha_0 = \alpha_0(a, e; \bar{w})$, $\alpha_1 = \alpha_1(a, e; \bar{w})$, $\alpha'_0 = \alpha_0(a', e'; \bar{w})$, $\alpha'_1 = \alpha_1(a', e'; \bar{w})$. For the states in which $a'(a, e) = -\underline{a}$, $\alpha_0 = \alpha_1 = 0$.

Equating the coefficients of q on each side of this equality, I obtain the following functional equation for $\alpha_0(a, e; \bar{w})$.

$$u''(c)(1 + \bar{r} - \alpha_0) = \alpha_0\beta(1 + \bar{r})\mathbb{E}\left[u''(c')(1 + \bar{r} - \alpha'_0)|e\right] \quad (42)$$

Similarly, equating the constants on both sides of the equation, I obtain a functional equation in α_1 .

$$-\frac{\alpha_1}{1+\bar{r}} = \frac{\alpha_1}{u''(c)} \beta \mathbb{E} \left[u''(c')(1+\bar{r}-\alpha'_0)|e \right] - \beta \mathbb{E}(\alpha'_1|e) + 1 - \beta(1+\bar{r}) \quad (43)$$

Next, I solve for $\beta \mathbb{E} \left[u''(c')(1+\bar{r}-\alpha'_0)|e \right]$ in equation (42) and substitute into equation (43) to get

$$\alpha_1 = \alpha_0 [\beta \mathbb{E}(\alpha'_1|e) - [1 - \beta(1+\bar{r})]] \quad (44)$$

Using (42), define the mapping

$$T_0 \alpha_0 = \begin{cases} \frac{u''(c)(1+\bar{r})}{u''(c) + \beta(1+\bar{r}) \mathbb{E} [u''(c)(1+\bar{r}-\alpha'_0)]} & : a'(a, e) > -\underline{a} \\ 0 & : a'(a, e) = -\underline{a} \end{cases}$$

Mapping T_0 satisfies monotonicity but does not satisfy discounting unless I impose strict assumptions on the domain. Therefore, contraction mapping theorem might not apply. It can be shown analytically that the derivative of the policy function $a'(a, e; \bar{w})$ solves the functional equation (42).³⁵ Since policy function is continuous and monotone, the derivative exists *almost everywhere* in $(-\underline{a}, \infty)$.³⁶ (See Carroll (2012) for an extensive discussion and characterization of points at which the derivative does not exist.) I verified quantitatively that starting from any initial function, iterating on α_0 converged to the numerical derivative of $a'(a, e; \bar{w})$.

Suppose $\alpha_0^*(a, e; \bar{w})$, solves (42). Define the functional mapping

$$T_1 \alpha_1 = \begin{cases} \alpha_0^* [\beta \mathbb{E}(\alpha'_1|e) - (1 - \beta(1+\bar{r}))] & : a'(a, e) > -\underline{a} \\ 0 & : a'(a, e) = -\underline{a} \end{cases}$$

Provided that the sufficiency condition $\sup \alpha_0^* < \frac{1}{\beta}$ holds, it is easy to show that mapping T_1 defined over bounded functions satisfies Blackwell's Sufficiency Conditions, therefore iterating over α_1 will yield to a unique function α_1^* from any starting point.³⁷ Under the given assumptions on the utility function, the savings policy function is convex and has a right-derivative with an upper bound strictly less than one.

³⁵This is just obtained by differentiating both sides of the functional Euler equation with respect to a . Consider a state (a, e) in which the agent is not borrowing constrained and the policy function is differentiable. The functional Euler equation reads

$$u'(a(1+\bar{r}) + y(e, \bar{w}) - a'(a, e; \bar{w})) = \beta(1+\bar{r}) \mathbb{E} \left[u' [a'(a, e; \bar{w})(1+\bar{r}) + y(e, \bar{w}) - a'(a'(a, e; \bar{w}), e')] \right]$$

Differentiating both sides with respect to a , I obtain

$$u''(c)(1+\bar{r}) - \frac{\partial a'}{\partial a}(a, e) = \beta(1+\bar{r}) \frac{\partial a'}{\partial a}(a, e) \mathbb{E} \left[u''(c')(1+\bar{r}) - \frac{\partial a'}{\partial a}(a', e') \right]$$

Observe that this equation has precisely the same structure as equation (42).

³⁶We also need μ to be absolutely continuous with respect to Lebesgue measure to be able to use this result, more precisely, the requirement is that the set of states in which the policy function is not differentiable has measure zero with respect to μ .

³⁷It is easy to show that this is the set of functions that take constrained values in $[-(1+\bar{r}), 0]$.

Convexity of savings follows from concavity of consumption function (Carroll and Kimball (1996)) and the unit upper bound on the derivative follows from a variation of the arguments provided in Carroll (2012) for the consumption policy function under CRRA preferences.³⁸ The sufficiency condition for contraction holds, therefore there exists a unique fixed point α_1^* .

Functions α_0^* and α_1^* provide us the policy function $q'(a, q, e; \bar{w})$.

B.1.2 Moments of the Distribution

Solving for the steady-state policy and allocation requires evaluation of certain moments of the distribution in (27) and (28). Dividing these equations by γ and substituting the auxiliary variable $q = \lambda/\gamma$, I obtain the following expressions:

$$A = \sum_e \int u'(c)q\mu(ds, e) + \frac{1}{\gamma} \sum_e \int au'(c)\mu(ds, e) + \sum_e \int au''(c)[q(1 + \bar{r}) - q']\mu(ds, e) \quad (45)$$

$$N = (F_N(K, N) - \bar{w})N'(\bar{w}) + \frac{1}{\gamma} \sum_e \int en(e, \bar{w})u'(c)\mu(ds, e) + \sum_e \int en(e, \bar{w})u''(c)[q(1 + \bar{r}) - q']\mu(ds, e) \quad (46)$$

One way to obtain the moments above is to use policy functions $a'(a, e; \bar{w})$ and $q'(a, q, e; \bar{w})$ to find the limiting distribution $\mu(a, q, e; \bar{w})$ and evaluating these moments. Since this is computationally demanding due to two continuous variables, I do not take this route: Due to the special structure of this problem and the fact that I know the distribution of (a, e) at the steady state, I can iterate on the moments *directly* without computing the limiting distribution.

Observe that taking the conditional expectation of (41) with respect to (a, e) and multiplying by the pdf $m(a, e; \bar{w})$, I have

$$\mathbb{E}(q'|a, e)m(a, e; \bar{w}) = \alpha_0 \mathbb{E}(q|a, e)m(a, e; \bar{w}) + \frac{\alpha_1 m(a, e; \bar{w})}{u''(c)(1 + \bar{r})} \quad (47)$$

Due to the Markov structure, I can also show that

$$\mathbb{E}(q'|a', e')m(a', e') = \sum_e \pi_{ee'} \frac{\mathbb{E}(q'|a'^{-1}(a', e), e)m(a'^{-1}(a', e), e)}{\frac{d}{da} a'(a'^{-1}(a', e), e)} \quad (48)$$

Starting from any initial guess $f_0(a, e)$ for $\mathbb{E}(q|a, e)m(a, e; \bar{w})$, I can apply (47) and (48) to revise the guess, i.e. by letting $f_1(a', e') = \mathbb{E}(q'|a', e')m(a', e'; \bar{w})$. This is repeated until convergence.

Now observe that once I have the expectation of q conditional on any (a, e) , all moments in (45) and (46) can be computed by straightforward numerical integration.

³⁸To be more precise, this upper bound is $[\beta(1 + \bar{r})]^{-1/\sigma}$ where σ is the bound on the coefficient of risk aversion. Since $\beta(1 + \bar{r}) < 1$ holds in the long run, the result follows.

B.1.3 Main Algorithm

1. Given $\bar{w}^j$, solve for $\bar{r}^j$ and policy functions $c(a, e; \bar{w}^j)$, $a'(a, e; \bar{w}^j)$ such that at the steady state, the modified golden rule and government budget constraint are satisfied. There is at most one such equilibrium.
2. Solve (42) and (44) using the method described above for the policy function $q'(a, q, e; \bar{w}^j)$.
3. Using the moment iteration method described above, calculate all the moments in (45) and (46).
4. Use (45) to solve for the unique γ that satisfies the equality.
5. Check if (46) is satisfied at this γ . If it is, stop. If not, update $\bar{w}^j$ to $\bar{w}^{j+1}$. Let $j := j + 1$ and move to step 1.