

Life-Cycle Asset Allocation with Ambiguity Aversion and Learning^{*}

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Abstract

I show that ambiguity (Knightian uncertainty) and learning about the equity premium can simultaneously explain the low fraction of financial wealth allocated to stocks over the life cycle as well as the stock market participation puzzle. I assume that individuals are ambiguous about the size of the equity premium and are averse with respect to this ambiguity, which results in a lower optimal allocation to stocks over the life cycle. As agents get older, they learn about the equity premium and increase their allocation to stocks. Furthermore, I find that ambiguity aversion leads to higher saving rates.

Keywords: Life-cycle portfolio choice, savings, ambiguity aversion, learning, behavioral finance
JEL classification: D14, D8, D91, G11

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Key inputs of a life-cycle model, such as the equity risk premium, variance of stock returns, and labor income risk, are generally assumed to be known by the agent. Optimal portfolio allocations, consumption, and savings are calculated as if the agent takes these parameters as given, and the resulting optimal policies are subsequently compared to the empirically observed life-cycle patterns. However, the predictions of most life-cycle models do not match well with some of the empirical findings. For instance, the overall low stock market participation rates are ill understood. Furthermore, the allocated fraction of financial wealth to stocks, conditional on participation in the stock market, appears difficult to align with the predictions from life-cycle models. I propose a standard life-cycle model, taking into account that agents are ambiguous about the equity risk premium and are averse to this ambiguity (in contrast to the ambiguity neutral approach). During their lifetime individuals learn about the equity premium. With this parsimonious adjustment to the standard framework I can explain both the life-cycle pattern of participation in the stock market and the conditional fraction of financial wealth allocated to equity. Furthermore, I show that saving rates are substantially higher for ambiguity averse people.

I assume that agents not only face risk, but are uncertain about the true parameters describing this risk (Knight (1921)).¹ A common way to deal with parameter uncertainty is the ambiguity-neutral approach, where the decision maker treats the unknown parameters as random variables and combines his prior belief about the parameter with observed signals, which forms the predictive distribution. He then evaluates his expected utility with respect to this predictive distribution. In this case the agent *is ambiguous* but *is not ambiguity averse*. However, there is substantial evidence that agents are not neutral with respect to this parameter uncertainty (see for instance the classical by Ellsberg (1961), who argues that people are ambiguity averse using an urn experiment). Therefore, I assume that agents are not ambiguity neutral, but ambiguity averse. Ambiguity about the equity risk premium is included, but no assumptions are made regarding the origin of this ambiguity. It can arise from, for instance, a lack of statistical evidence, a lack of theoretical

¹The difference between risk and uncertainty is that when agents face risk they are able to attach probabilities to random events, while when facing uncertainty they do not know the probabilities. In the context of this paper, the agent faces risk because the return on stocks is stochastic, but the agent is also uncertain because he does not know the expected stock return.

evidence, unsophistication of investors, and so on. Focussing on, for example, statistical ambiguity, even when every agent possesses all the historical stock return data over the past 100 years and uses these to estimate the equity premium, the confidence interval will still be sizeable: $[4\% - 2 * 20\% / \sqrt{100} : 4\% + 2 * 20\% / \sqrt{100}] = [+0\% : +8\%]$.

A short note on terminology is in order. As Guidolin and Rinaldi (2013) point out, in the literature ambiguity and uncertainty are not always clearly defined. Throughout the paper I use the terms uncertainty and ambiguity interchangeably, and I define ambiguity/uncertainty as a random event where the probabilities are not known (as opposed to a coin toss), but agents have a distribution of priors over the uncertain parameter.

I use maxmin preferences to model ambiguity aversion and individuals learn about the equity premium. Gilboa and Schmeidler (1989) propose that agents have maxmin preferences in a multiple priors framework, which entails that agents evaluate policies by maximizing utility according to the worst case belief. This atemporal framework is generalized by Epstein and Schneider (2003) to a dynamic setup. I do not assume that agents learn about the equity risk premium in a rational manner; agents weigh realized stock returns during life with a prior belief about the equity risk premium, putting no weight on returns before the year of birth. Malmendier and Nagel (2011) find that agents' "experienced return" has a larger influence on beliefs about the equity risk premium than stock return realizations before the year of birth. I assume agents learn independently of stock market participation and I employ Bayes' rule as the updating rule for the beliefs about the equity risk premium. Furthermore, agents have CRRA preferences and their labor income is risky.

The contributions of this paper are threefold. First, I find that ambiguity with respect to the equity risk premium can have a substantial effect on the optimal stock allocations. Stock market participation is substantially lower as well as the conditional allocation to equity. Both effects decline with age due to learning about the equity premium, since learning results in older agents being less ambiguous about the equity premium compared to younger agents.

Second, ambiguity about the equity premium influences wealth levels of individuals as well as their savings rates. The optimal wealth profile is lower for ambiguity averse individuals, which

reflects the lower investment returns due to lower optimal stock allocations. However, savings rates are higher when taking into account ambiguity about the equity premium. Keeping up wealth levels by saving to compensate for lower investment returns is quantitatively more important than lower savings rates induced by the relatively less attractive investment opportunities perceived by ambiguity averse investors. This is the first paper, to my knowledge, to explore the impact of ambiguity on life-cycle savings choices and wealth levels.

Third, when comparing the optimal fraction allocated to stocks to the empirical levels, I find a close match at all ages. On average over the life cycle the model predicts 50% allocated to stocks while the empirical average is about 45%. A comparable good fit is found when examining the participation in the stock market. Hence by extending the often used life-cycle model with ambiguity aversion and learning, I can simultaneously explain the low stock market participation as well as the low conditional fraction of financial wealth allocated to stocks over the life cycle. Similar results cannot be obtained by assuming no ambiguity aversion and instead high risk aversion, because higher risk aversion actually increases participation levels due to much higher precautionary savings.

In their seminal works, Merton (1969) and Samuelson (1969) find that agents should hold a constant fraction in risky assets over the life cycle in the absence of labor income and complete markets. More recent work by Benzoni, Collin-Dufresne, and Goldstein (2007), Cocco, Gomes, and Maenhout (2005), Heaton and Lucas (2000), Polkovnichenko (2007), and Viceira (2001) examines the effect of (risky) labor income on the optimal portfolio choice. If human capital is riskless, young agents have a substantial investment in this “bond-like” asset and, as a result, invest a large fraction of their liquid wealth in risky assets. This is in contrast to the empirically observed low allocation to stocks, especially early in the life cycle. In contrast to other papers, I do not need to include several additional features in the model to be able to explain low stock participation, such as participation costs (Vissing-Jorgenson (2002)), Epstein-Zin preferences, bequests, housing, cointegration between labor income and dividends, and minimum investment requirements. The intuitive modification with ambiguity aversion alone can explain the empirical evidence very

closely. Similar to this paper, Gomes and Michaelides (2005) try to match the empirically observed allocation to stocks by assuming a bequest motive, fixed entry costs of 2.5% of income, preference heterogeneity, and Epstein-Zin preferences. The participation levels match closely, except after retirement, however the predictions about the conditional allocation to equity differ about 40% from the empirically observed levels at younger ages. I can match both low participation levels as well as the allocation to equity conditional on participation in the stock market very well, especially at young ages. Benzoni, Collin-Dufresne, and Goldstein (2007) assume cointegration between stock and labor markets and find a hump-shaped allocation to equity, however the absolute differences with empirical levels are substantially larger than the findings in this paper.

Two other papers include ambiguity and learning about the parameters in a life-cycle framework and address similar questions as in this paper. Campanale (2011) assumes agents have maxmin preferences and are uncertain about the probability of a high stock return. The return on stocks can take on two values, high or low. Learning occurs when agents invest in the stock market, and only with a percentage lower than 100% if they do not participate. In contrast to my paper, this simplified stock return process prevents bringing this model to the data. Furthermore, I examine a broader set of choices, namely savings and wealth levels, and I use a more parsimonious model.² Linnainmaa (2007) examines the influence of ambiguity in a life-cycle framework, but maximizes over financial wealth and stock prices follow a binomial tree.

1 The model

I extend the standard life-cycle framework by including ambiguity aversion and learning. I use the most common model for ambiguity averse preferences, namely maxmin preferences.³ Agent's update beliefs about the equity premium using Bayes' rule.

²Campanale (2011) includes a fixed annual stock market participation cost, a bequest motive, minimum stock investment of 4% of average annual earnings in the economy (about \$1,400), and a more complicated learning process.

³In the robustness section an alternative way to include ambiguity aversion, smooth recursive preferences, is used.

1.1 Ambiguity about the equity premium

I assume the agent is uncertain about the equity premium and updates his beliefs according to realized stock market returns, which can be either actively or passively observed. The updating of beliefs about the equity risk premium follow from Bayes' rule, which is described in Section 1.6. The mean belief about the equity risk premium is denoted by λ_t^B and the standard deviation by σ_t^B . λ_t^B and σ_t^B describe the set of priors, which are normally distributed. The domain of equity premiums that the agents thinks are possible at time t , Λ_t , is given by $[\lambda_t^B - 2\sigma_t^B, \lambda_t^B + 2\sigma_t^B]$. Hence the mean of the belief about the equity premium, λ_t^B , is not the only value the agent considers, but he expects the true mean to lie within the 95% confidence interval of beliefs about the equity premium. Garlappi, Uppal, and Wang (2007) make a related assumption when incorporating ambiguity by stating that the expected return of an asset lies within a specified confidence interval of its estimated value, and the agent behaves as if the worst case belief in the confidence interval is the true belief. I assume similar preferences, maxmin expected utility, which will be described in the next section.

I do not make assumptions about the source of ambiguity about the equity risk premium. Uncertainty could stem from lack of statistical evidence, since stock market returns are so volatile which makes it difficult to measure the expected return.⁴ Furthermore, ambiguity about the equity premium could also result from inconsistent theoretical evidence or unsophistication of investors.

1.2 The individuals preferences

I consider a life-cycle investor of age $t = 1, \dots, T$, where t is the adult age, T is the maximum age possible, and K is the retirement age. Individuals maximize utility over consumption and preferences are represented by a time-separable utility function over consumption. The agent's decision variables at time t are consumption C_t and fraction invested in stocks w_t . I assume investors' preferences are described by maxmin expected utility, which essentially means that agents maximize

⁴Even if we use all the stock market returns from the past 100 years to estimate the equity premium, we would still end up with a large confidence interval: $[4\% - 2 * 20\% / \sqrt{100} : 4\% + 2 * 20\% / \sqrt{100}] = [+0\% : +8\%]$.

expected utility according to the belief which generates the lowest utility. Gilboa and Schmeidler (1989) axiomatize this behavior in a static setting and Epstein and Schneider (2003) in a dynamic framework.

As described above, the agent is uncertain about the equity premium and tries to maximize the value function at each period t ,

$$V_t = \max_{w_t, C_t} \min_{\lambda \in \Lambda_t} [u(C_t) + \beta p_{t+1} \mathbb{E}_t^\lambda \{V_{t+1}(W_{t+1})\}] , \text{ with} \quad (1)$$

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}, \quad (2)$$

$$(3)$$

where β is the time preference discount factor, u is the utility function, and C_t is the amount of wealth consumed at the beginning of period t . The optimal fraction allocated to stocks is denoted by w_t , which is implicit in W_{t+1} . The probability of surviving to age $t + 1$, conditional on having lived to period t , is indicated by p_{t+1} . $\mathbb{E}_t^\lambda$ is the expectation calculated as if λ is the true equity premium. I assume a CRRA utility function, u , and γ is the risk aversion coefficient. In effect, the agent maximizes expected utility as if $\lambda_t^B - 2\sigma_t^B$ is the equity premium. Note that I assume that the agent limits his beliefs to a range of possible equity premium, a confidence interval. An interval of beliefs instead of the entire distribution is not only intuitive but also necessary, because the beliefs are normally distributed and hence the worst belief is infinitely negative.

1.3 The individuals constraints

The individual faces a number of constraints on the consumption and investment decisions. First, I assume that the agent faces short-sales and borrowing constraints

$$w_t \geq 0 \text{ and } \iota' w_t \leq 1. \quad (4)$$

Second, I impose that the investor is liquidity constrained

$$C_t \leq W_t + Y_t, \quad (5)$$

which implies that the individual cannot borrow against future income to increase consumption today. W_t denotes financial wealth and Y_t is income. The intertemporal budget constraint equals:

$$W_{t+1} = (W_t - C_t + Y_t)(1 + R^f + w_t(R_{t+1} - R^f)). \quad (6)$$

The portfolio return is given as

$$R_{t+1}^P = 1 + R^f + (R_{t+1} - R^f)w_t. \quad (7)$$

1.4 Financial market

I consider a simple financial market with a constant interest rate R^f and stocks with i.i.d. returns R_{t+1} . The stock returns, R_{t+1} , are normally distributed with an annual mean equity return $R_f + \lambda^R$ and a standard deviation σ_R , where λ^R is the “correct” equity risk premium. The agent is uncertain about the value of the equity premium. At time t , the agent has a distribution of beliefs over the equity premium, λ_t . This distribution of beliefs changes over time because of learning about the equity risk premium. The distribution of the equity risk premium, given the information at time t , is itself characterized by a state-variable, containing the learned mean λ_t^B , and its variance $(\sigma_t^B)^2$. The parameters used are described in Section 1.7.

1.5 Labor income process

I assume that labor income is uncertain and given by

$$Y_t = \exp(f_t + v_t + \epsilon_t) \text{ for } t < K, \quad (8)$$

where

$$v_t = v_{t-1} + u_t. \quad (9)$$

After retirement age K income is riskless and equals a fraction of the labor income at age 65 (the replacement rate). Labor income exhibits a hump-shaped profile over the life cycle which is accommodated by f_t , where f_t is a deterministic function of age. The error term consist of a transitory component and a permanent component. ϵ_t is a transitory shock and is distributed as $N(0, \sigma_\epsilon^2)$. u_t presents a permanent shock, where $u_t \sim N(0, \sigma_u^2)$. This representation follows Cocco, Gomes, and Maenhout (2005) and I calibrate the labor income process according to their estimates. The function f_t is modeled by a third order polynomial in age,

$$f_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2/10 + \alpha_3 t^3/100. \quad (10)$$

1.6 Learning and updating of beliefs

Agents learn about the equity risk premium throughout their lifetime and become less uncertain with age because they have received more information. The updating process for the set of priors follows Bayes' rule.⁵ The agent is uncertain about the equity risk premium, λ^R . Before observing any signals, the set of priors are normally distributed with mean λ^B and variance $(\sigma^B)^2$. An individual of age t ($t = 1$ corresponds to age 20) has received $t + 19$ independent signals about λ^R , $R_t = R^f + \lambda^R + \epsilon_t$, where ϵ_t is normally distributed with mean zero and a known variance σ_R^2 . These signals, the realized excess returns, are observed annually. The updated priors about λ^R are

⁵Other updating rules for beliefs are explored in Epstein, Noor, and Sandroni (2010), Epstein and Schneider (2007), and Hanany and Klibanoff (2009).

normally distributed with mean λ_t^B and variance $(\sigma_t^B)^2$, where

$$\lambda_t^B = \lambda^B \underbrace{\frac{\frac{1}{(\sigma^B)^2}}{\frac{1}{(\sigma^B)^2} + \frac{t+19}{\sigma_R^2}}}_{\text{weight mean prior}} + \frac{1}{t+19} \sum_{\tau=-19}^{t-1} (R_t - R^f) \underbrace{\frac{\frac{t+19}{\sigma_R^2}}{\frac{1}{(\sigma^B)^2} + \frac{t+19}{\sigma_R^2}}}_{\text{weight returns}} \quad (11)$$

$$(\sigma_t^B)^2 = \frac{1}{\frac{1}{(\sigma^B)^2} + \frac{t+19}{\sigma_R^2}}. \quad (12)$$

The posterior mean λ_t^B is a precision weighted average of the prior mean and the average signal. At time $t = 1$, representing age 20, the agent has observed twenty stock market returns. Unlike λ_t^B , the posterior variance $(\sigma_t^B)^2$ does not depend on the specific realizations of the signals, only the number of signals. This variance, which measures the uncertainty/ambiguity about λ_R , decreases as the number of signals t increases (learning reduces uncertainty), hence $(\sigma_{t+1}^B)^2 < (\sigma_t^B)^2$.

I assume agents update their beliefs irrespective of whether or not they participate in the stock market, e.g., since everyone receives similar information via newspapers, television, and other media. People start with prior beliefs about the equity risk premium when born and update those beliefs according to the realized returns from their birth year onwards. The updating rule puts no explicit weight on stock returns before the year of birth and thus only takes into account realizations during lifetime. The priors at birth could be thought of as containing to some extent the realized stock returns before birth, but I do not assume that prior beliefs at birth are equal to the confidence interval from the stock return data available. I choose this specific starting age for updating beliefs instead of, for instance, adult age or long before birth, because Malmendier and Nagel (2011) find that stock returns experienced receive a much larger weight when forming beliefs about expected stock market returns compared to stock returns before birth. In their baseline model they use experienced returns dating back to the year of birth. In contrast to Malmendier and Nagel (2011) I apply equal weights to all experienced returns and update beliefs according to Bayes' rule.

Two additional underlying assumption are that: (1) the level of ambiguity, σ_1^B , is the same for every person at birth, independently of birth year and (2) the mean of the beliefs about the equity risk premium, λ_1^B , at birth is independent of birth year and hence independent of stock return

realizations before birth. In regard to assumption (1), the reason why I assume that the level of ambiguity (standard deviation of belief) about the equity risk premium is the same in 1970 and 2000, is that data going back more than for instance 70 years may, according to the agent, not be that relevant for estimating the equity premium today, due to, for instance, structural changes (Pastor and Veronesi (2009)). Structural changes, induced by for example technologic innovations, might permanently change the equity risk premium. Hence the amount of uncertainty does not disappear with time and is thus irrespective of the year in which the agent is born.

Regarding assumption (2), the mean of the belief is the same for every person at birth and does not depend on birth year. Different priors at birth could generate additional cohort effects, however I assume that the prior is independent of birth year, because agents incorporate realized stock returns during their life more heavily into beliefs than returns before birth, see Malmendier and Nagel (2011). In Section 4 I explore the impact of starting updating from age 20 onwards and different levels of initial ambiguity on the main findings.

As an application of the life-cycle model with ambiguity aversion, I will show how cohort effects can arise due to learning from realized stock returns. For instance, a 25 year old in 2010 has faced 25 realized stock returns, and the mean of these excess returns ($R_t - R_f$) induces a higher or lower beliefs compared to the prior mean belief about the equity premium, λ_1^B . If the updated belief, λ_6^B , has increased due to high realized stock returns, this can result in higher allocations to stocks. I calculate the pattern of stock allocations over the life cycle taking into account these cohort effects.

1.7 Benchmark parameters for the life-cycle model with ambiguity

I set the risk aversion coefficient (γ) equal to 5, which is the same as used in Benzoni, Collin-Dufresne, and Goldstein (2007) and Gomes and Michaelides (2005). Time ranges from $t = 1$ to time T , which corresponds to age 20 and 100 respectively. Agents retire at time $K = 45$, corresponding to age 65. The survival probabilities are the current male survival probabilities in

the US which are obtained from the Human Mortality Database.⁶ I assume a certain death at age 100.

The true equity premium λ^R is assumed to be normally distributed with a annual mean of 4% and an annual standard deviation σ_R of 16%, which is in accordance with historical stock returns. The risk free rate is 2%, hence the expected stock return is 6%. The mean of the priors about the equity premium at birth is equal to the correct equity premium; $\lambda_1^B = 4\%$. The standard deviation of the beliefs at time $t = 1$, σ_1^B , is 2%.

I take the parameters for the labor income process estimated in Cocco, Gomes, and Maenhout (2005). The deterministic hump-shaped profile of income is generated by the parameters, $\alpha_1 = 0.1682$, $\alpha_2 = -0.0323$, and $\alpha_3 = 0.002$. I choose the constant, α_0 to accommodate different income levels at time $t=1$. The benchmark income level at age 20 is \$15,000. The variance of the transitory shock to labor income, σ_u^2 , is 7.38% and the variance of the permanent shock, σ_ϵ^2 , is 1.06%. The replacement rate of the labor income at age 65 is 68% of the wage at age 65. The income during retirement is riskless. These numbers are for a high school graduate which are estimated in Cocco, Gomes, and Maenhout (2005) and used as the benchmark parameters in their analysis.

1.8 The individuals optimization problem and numerical method

The timing, during one year, is as follows, first an individual receives his labor or retirement income after which he consumes. Subsequently the remaining wealth is invested. The optimization problem is solved via dynamic programming and I proceed backwards to find the optimal investment and consumption strategy. In the last period the individual consumes all his remaining wealth, hence his utility from terminal wealth is known.

Due to the richness and complexity of this model it cannot be solved analytically, so I employ numerical techniques following Brandt, Goyal, Santa-Clara, and Stroud (2005) and Carroll (2006) with several extensions by Koijen, Nijman, and Werker (2010). Brandt, Goyal, Santa-Clara, and

⁶I refer for further information to the website, www.mortality.org.

Stroud (2005) adopt a simulation-based method which can deal with many exogenous state variables. In this model, the mean of the priors about the equity premium, λ_t^B , and income, Y_t , are the relevant exogenous state variables. Wealth acts as an endogenous state variable. For this reason, following Carroll (2006), I specify a grid for wealth *after* income and consumption. As a result, I do not need numerical rootfinding to obtain the optimal consumption decision.

In each period I find the optimal asset weights by setting the first order condition equal to zero

$$\mathbb{E}_t^{\lambda_t^{min}}(C_{t+1}^{*- \gamma}(R_{t+1} - R^f)) = 0, \quad (13)$$

where λ_t^{min} is the lowest equity premium in Λ_t . C_{t+1}^* denotes the optimal consumption level. The optimal consumption follows from

$$C_t^{*- \gamma} = \beta p_{t+1} \mathbb{E}_t^{\lambda_t^{min}}(C_{t+1}^{*- \gamma} R_{t+1}^{P*}). \quad (14)$$

The details of the numerical method I use to solve the life-cycle problem with maxmin preferences is described in Appendix A.

1.9 Data

When comparing the predicted stock allocation to the data I use the 2010 Survey of Consumer Finances which is the most comprehensive dataset on households assets and liabilities in the United States. High income household are over-sampled to obtain a sufficient number of wealthy households in the study. I employ a measure for financial wealth and stock investment according to the method suggested by the Survey of Consumer Finances. The same measures are used in Gomes and Michaelides (2005). Financial wealth consists of both retirement and non-retirement wealth and stock investment is calculated as the sum of direct investment in stock and stock mutual funds as well as stock investment of pension wealth. More details on the data from the Survey of Consumer Finances can be found in Appendix C.

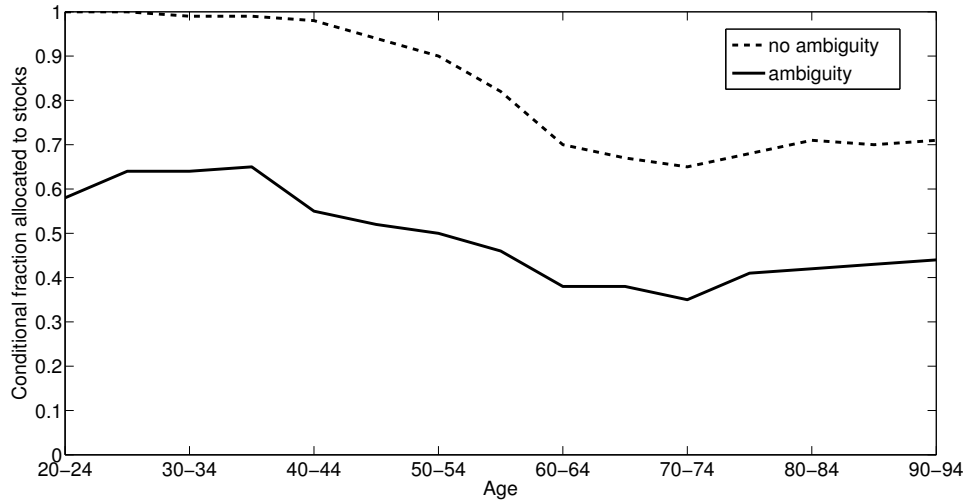
2 Effect of ambiguity aversion and learning on optimal allocations

We determine the optimal life-cycle choices using simulations and analyze the importance of ambiguity about the equity premium on optimal stock allocations and savings decisions.

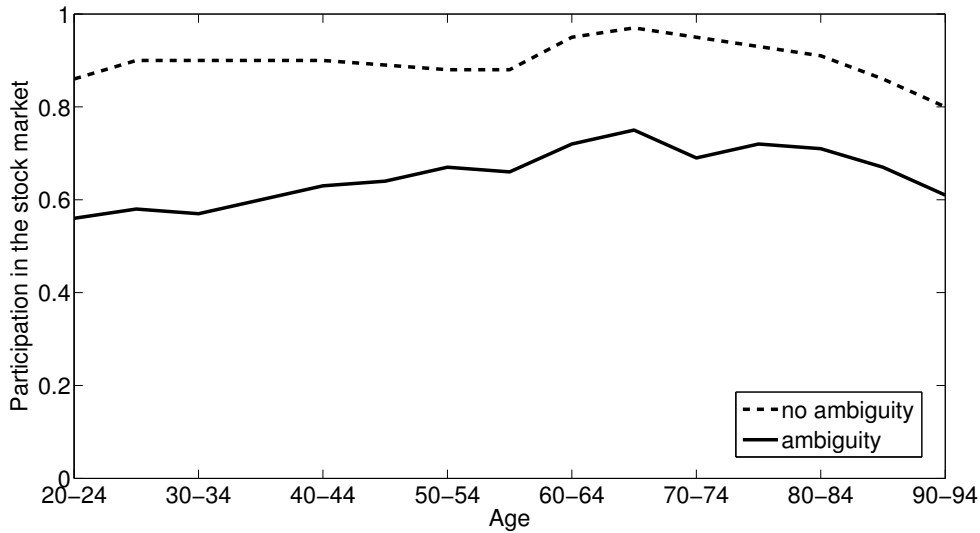
2.1 Effect of ambiguity aversion on optimal portfolio choice

The optimal fraction allocated to stocks, conditional on participation in the stock market is plotted in Figure 1a. Comparing optimal allocations including ambiguity (solid line) with no ambiguity (dashed line), shows that the allocation to stocks when agents are ambiguity averse is much lower. At all ages the fraction allocated to stocks is around 50%. The impact of ambiguity aversion is substantial at young ages, but this effect declines slightly with age as the level of ambiguity about the equity risk premium decreases over time when agents learn.

Focussing on the analysis without ambiguity, I find that if agents are fully certain about the values of all the parameters in the model, they allocate 100% of financial wealth to stocks before age 40. Similar results are found in Cocco, Gomes, and Maenhout (2005). The reason for this high fraction is that young agents have only a small amount of financial wealth compared to a high level of human capital. Since human capital is like an implicit investment in a riskless asset, an agent allocates his entire financial wealth to equity. Between age 40 and 65 the conditional allocation to the risky asset decreases. At those ages (retirement) savings are high while at the same time the net present value of labor income decreases, hence the fraction of financial wealth to human capital increases. This results in a decline of the relative allocation to the riskless asset “human capital” and, as a consequence, the optimal fraction of financial wealth invested in stocks decreases to maintain a similar risk-profile. After retirement the allocation to stocks increases slightly, as in Cocco, Gomes, and Maenhout (2005). At that time the agent depletes his financial wealth more rapidly due to the additional implicit discount factor, survival probabilities, and hence the fraction of financial wealth to human capital decreases, which induces a higher fraction of financial wealth



(a) Fraction allocated to stocks, conditional on participation



(b) Stock market participation

Figure 1:

Optimal fraction allocated to stocks and optimal participation in the stock market.

These figures show the optimal conditional fraction of financial wealth allocated to stocks and optimal participation in the stock market for agents who are (1) ambiguous about the equity risk premium, averse to this ambiguity, and learn about this parameter and who are (2) not ambiguous about the equity premium. The upper panel shows the fraction of financial wealth allocated to stocks, conditional on stock market participation. The lower panel shows the optimal participation level. In case an agent has a near zero financial wealth level (below \$100), the optimal participation is assumed to be zero.

invested in stocks.

Figure 1b displays optimal participation levels in the stock market. The effect of ambiguity is substantial, the participation level drops by 25% on average over the life cycle. When agents are not ambiguous about the equity premium, the participation levels in the stock market are high. Since labor income is not correlated with returns on the stock market, it is optimal for all agents, even with low financial wealth, to allocate at least a small fraction of financial wealth to stocks. The reason for less than 100% participation is that I assume that agents with financial wealth less than \$100 do not invest in stocks. Taking these agents with near zero wealth into account would distort the subsequent comparison of the model predictions to the data since in reality people with less than \$100 of wealth would not invest, due to participation costs and minimum balance requirements.⁷ As before, the impact of ambiguity aversion decreases with age since the ambiguity about the equity risk premium declines, as agents learn by observing the realized stock returns. In a non-life-cycle framework, Cao, Wang, and Zhang (2005) and Easley and O'Hara (2009) confirm that ambiguity aversion can limit the participation levels.

Vissing-Jorgenson (2002) examines the implications of fixed participation costs on optimal participation levels and find that it can explain why less wealthy household do not participate, but not the low participation levels of the wealthy. I find that ambiguity about the equity risk premium can provide an explanation for the low participation levels also of wealthy individuals. Furthermore, ambiguity about the equity premium can simultaneously explain the low participation levels as well as the low conditional fraction allocated to stocks, while fixed participation costs only impact participation levels.

In the previous paragraphs, the optimal allocations are explored for agents who are ambiguous *and* are averse to this ambiguity. In contrast, in the more standard ambiguity neutral framework agents are only uncertain about the parameters, but not averse with respect to this uncertainty. When this is the case, the optimal allocations almost do not change. In the benchmark model, the agents' beliefs about the equity risk premium are normally distributed with a mean of 4%

⁷In addition, the simulation inaccuracy of optimal stock allocations is higher for these low wealth levels, since the difference in utility of the agent when he invest 100% or 0% in stocks is negligible.

and a standard deviation equal to 2%. If agents are ambiguity neutral, their behavior is induced by the so called predictive distribution. The standard deviation for the compound distribution of the volatility of the return on equity, σ_R and the volatility of the belief, σ_t^B , can be reduced to the predictive volatility $\sqrt{\sigma_R^2 + (\sigma_t^B)^2}$. For the benchmark parameters this results in a standard deviation of 16.1% (note that σ_R is 16%). Hence uncertainty about the equity risk premium will have (almost) no effect on optimal portfolio choices when assuming uncertainty neutrality.

2.2 Effect of ambiguity aversion on savings

The optimal mean wealth levels are plotted in Figure 2. Agents who face ambiguity about the equity risk premium have lower amounts of wealth compared to individuals not facing ambiguity. For instance, at age 65 the mean wealth level when facing ambiguity is \$225,000 compared to \$250,000 when exposed to only risk. Note that the size of these wealth levels are comparable to the findings in Cocco, Gomes, and Maenhout (2005).

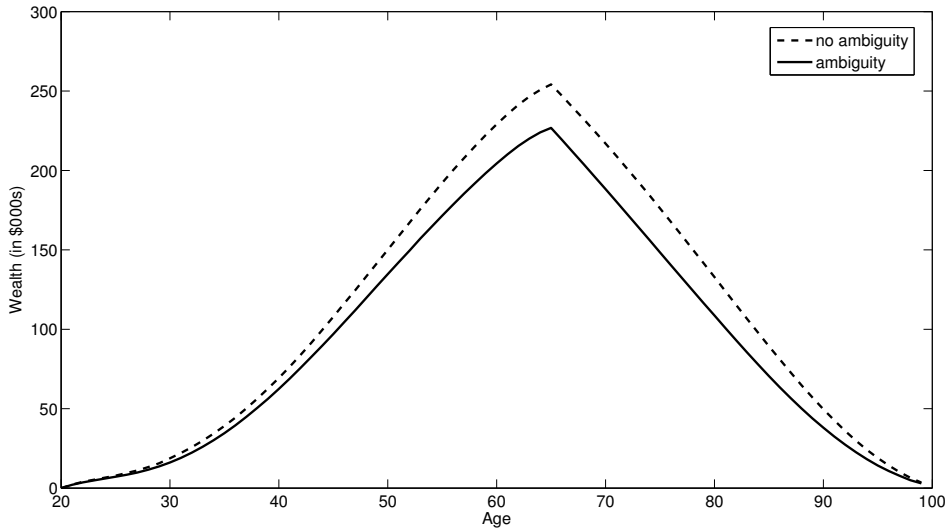


Figure 2:
Optimal wealth levels

The figure shows the average optimal wealth for agents who are (1) ambiguous about the equity risk premium, averse to this ambiguity, and learn about this parameter and who are (2) not ambiguous about the equity premium.

These lower wealth levels for ambiguity averse agents does not necessarily imply lower savings

out of income and wealth. Several factors are resulting in differential wealth levels. First, these lower wealth levels are (partly) an automatic result of ambiguity averse agents investing less in equity, and thus having less wealth (savings plus investment return) accumulated. An additional rational for lower wealth levels could be that individuals have less incentives to save, because they make investment decisions based on the worst case equity premium which does not generate sufficient investment income on their savings. Going in the opposite direction are ambiguity averse agents saving more to compensate for the lower wealth levels due to lower investment returns. Figure 3 shows the difference in savings levels between individuals facing ambiguity and not facing ambiguity with similar wealth and income levels. If the lower wealth levels of ambiguity averse agents are merely a reflection of lower investment returns, then the savings out of income and wealth should be the same for both and the difference in Figure 3 should be zero. However, the savings out of income and wealth are higher for ambiguity averse agents. Thus keeping up wealth levels by saving extra is quantitatively more important than lower savings rates induced by the relatively less attractive investment opportunities perceived by ambiguity averse investors.⁸

3 Comparing the optimal stock allocations to the empirical evidence

In this section I compare the predictions from the life-cycle model with ambiguity aversion and learning with the stock allocations data from the Survey of Consumer Finances in 2010.

3.1 Fraction of financial wealth allocated to stocks

Figure 4 compares the optimal fraction allocated to stocks to the empirical levels in 2010. We can see that the fraction of financial wealth allocated to stocks for agents facing ambiguity is much closer to the empirical levels. The model without ambiguity predicts an optimal average fraction

⁸Sutter, Kocher, Gltzle-Retzler, and Trautmann (2013) finds that more ambiguity averse adolescents save less, however, these results are insignificant.

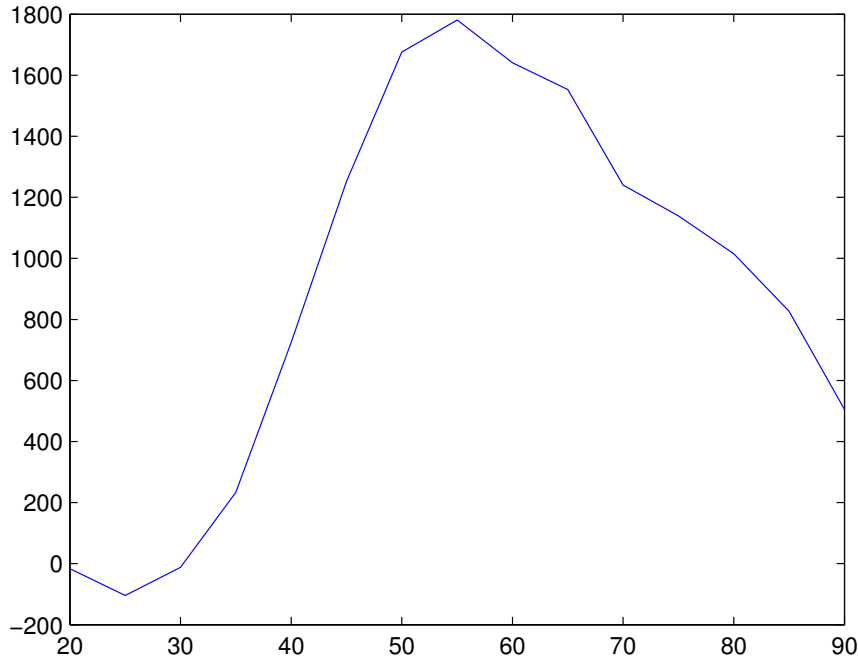


Figure 3:

Difference in savings between ambiguity averse and not ambiguity averse individuals

The figure shows the difference in the average savings out of wealth and income between agents who are ambiguity averse and not ambiguity averse. For a given wealth and income level (median wealth and income at each age for the non-ambiguous agent), savings are calculated for both types of agents: ambiguity averse and not ambiguity averse. The optimal savings for the ambiguity averse agent minus the optimal savings for the not-ambiguity averse agent are displayed.

over the life cycle of about 85% versus 50% when taking ambiguity into account. The empirical average is slightly less than 45%.

In the previous graph I focussed on the means of the conditional allocation to stocks, not examining other moments. Table 1 displays the stock allocations predicted by the model and empirical estimates for different percentiles. Focussing on the age group 25-84, the median matches well, 40% of financial wealth allocated to stocks in the data compared to 44% according to the model with ambiguity. Furthermore, for instance the 10th percentile is 7% in the data, 12% according to the model that includes ambiguity, and 67% for the model without ambiguity. Furthermore, when splitting the fraction invested in stocks up into different age groups, the fit to the data is similar.

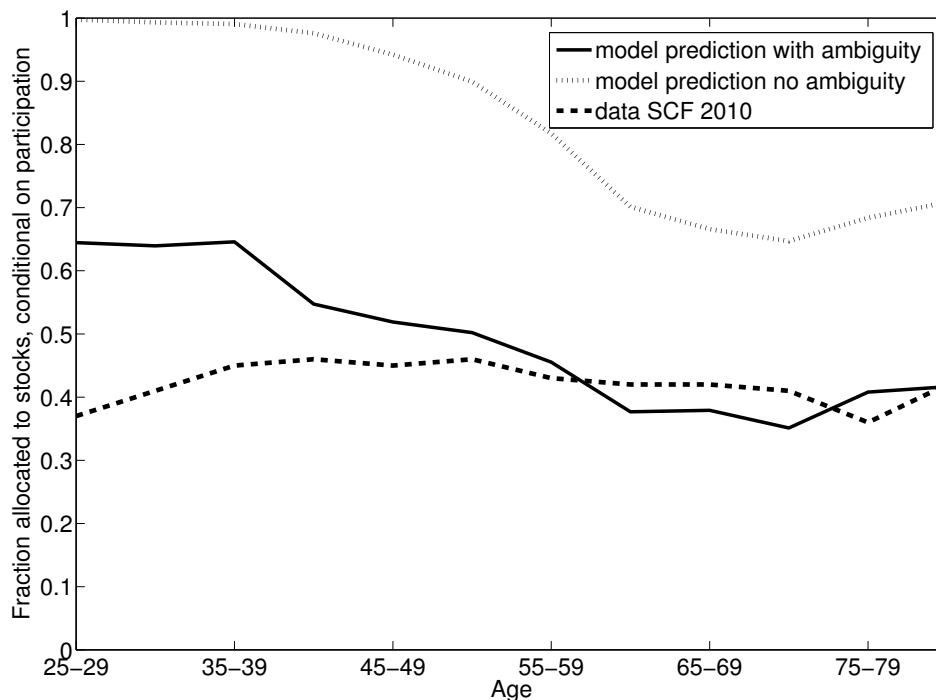


Figure 4:

Conditional fraction allocated to stocks: Data and optimal allocations

The figure shows the empirical fraction of financial wealth allocated to stocks, conditional on participation, and the optimal fraction invested in the stock market for agents who are (1) ambiguous about the equity risk premium, averse to this ambiguity, and learn about this parameter and who are (2) not ambiguous about the equity premium. The data is from the Survey of Consumer Finances in 2010.

3.2 Participation in the stock market

In Figure 5 optimal stock market participation and empirical participation levels are plotted. Comparing the empirical participation levels to the optimal participation levels when taking into account ambiguity, we find a very close match. In 2010, averaging over the entire life cycle, about 50% of people invest in the stock market and the model predicts about 65%. To compare, Gomes and Michaelides (2005) find optimal allocation levels of almost 100% at young ages, while the model with ambiguity predicts optimal levels less than 60%. Note that it is not insightful to present the percentiles for the participation levels, since this is a binary variable and all the information is already contained in Figure 5.

Data	10th percentile	25th percentile	50th percentile	75th percentile	90th percentile
age 25-84	7	19	40	65	86
age 25-34	6	15	35	62	84
age 35-44	9	22	42	69	89
age 45-54	9	21	43	67	89
age 55-64	6	19	41	62	84
age 65-74	7	18	37	65	81
age 75-84	5	12	36	58	80
Model with ambiguity	10th percentile	25th percentile	50th percentile	75th percentile	90th percentile
age 25-84	12	25	44	72	98
age 25-34	17	36	67	99	100
age 35-44	17	35	59	90	100
age 45-54	13	26	47	76	100
age 55-64	9	19	35	59	99
age 65-74	7	15	27	51	91
age 75-84	10	20	33	57	98
Model without ambiguity	10th percentile	25th percentile	50th percentile	75th percentile	90th percentile
age 25-84	67	75	84	97	100
age 25-34	100	100	100	100	100
age 35-44	98	100	100	100	100
age 45-54	71	91	100	100	100
age 55-64	49	59	79	98	100
age 65-74	40	47	61	91	100
age 75-84	42	50	67	96	100

Table 1:

Percentiles of the optimal and empirical conditional fraction of financial wealth allocated to stocks. The conditional fraction allocated to stocks in 2010 are calculated using the Survey of Consumer Finances. The optimal fraction is calculated using the simulations result of the benchmark life-cycle model for agents who are (1) ambiguous about the equity risk premium, averse to this ambiguity, and learn about this parameter and who are (2) not ambiguous about the equity premium. Both the optimal and the empirical fractions are conditional on stock market participation.

3.3 Application of the model: age, cohort, and time effects in stock allocations

In the previous section age patterns in stock allocations induced by ambiguity, learning, and human capital are explored. However, this structural model with ambiguity aversion and learning can also be used to examine cohort and time effects in life-cycle portfolio choices. In a reduced form model age patterns of stock allocation cannot be identified separately from cohort effects and time effects, since time, age, and cohort do not vary independently.⁹ Cohort effects relate to individuals' experiences during life, common to those growing up at the same time, which may influence behavior

⁹The identification problem arises because age equals time minus cohort (birth year). Ameriks and Zeldes (2004) and Poterba and Samwick (1997) try to uncover specific age patterns, but reach inconclusive results.

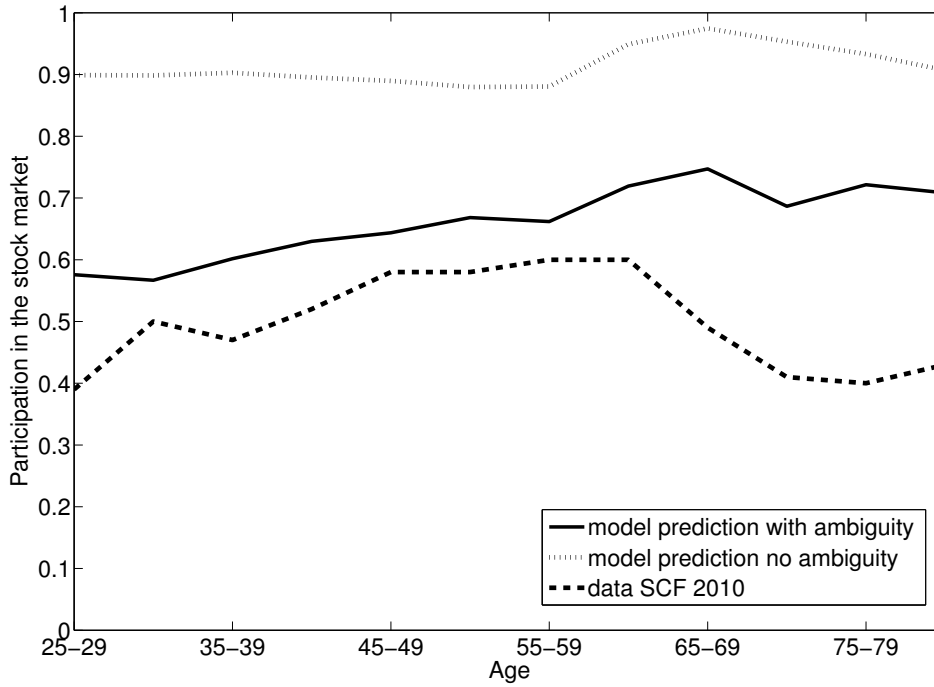


Figure 5:

Participation in the stock market: Empirical and optimal allocations

I display the empirical participation levels and the optimal participation levels in the stock market for agents who are (1) ambiguous about the equity risk premium, averse to this ambiguity, and learn about this parameter and who are (2) not ambiguous about the equity premium. In case an agent has near zero financial wealth level (below \$100), the optimal participation is assumed to be zero.

and beliefs. For instance high stock returns can lead to upward revisions in expectations about future stock returns (see Malmendier and Nagel (2011)). Cohort effects in my model can be explored via learning about the expected stock return. This learning results in differences between cohorts in their optimal beliefs and allocation to stocks; for instance a 25-year old in 2010 has a different mean belief about the equity premium compared to a 25-year old in 1990 due to differences in the realized stock returns in the preceding years. Time effects can arise for a variety of reasons, for instance due to differential stock market realizations, decreasing fees, or lower costs of obtaining information over time. In this application of the model time effects will be assumed by adding a low transaction fee in 2010 (0.5%), reflecting that in earlier years probably the costs of investing where higher.¹⁰

¹⁰Fees have decreased over time which is a possible way to model time effects. The results presented in this section are also performed for the years 1998 and 1989 assuming a proportional fee of respectively 1% and 1.5%. These

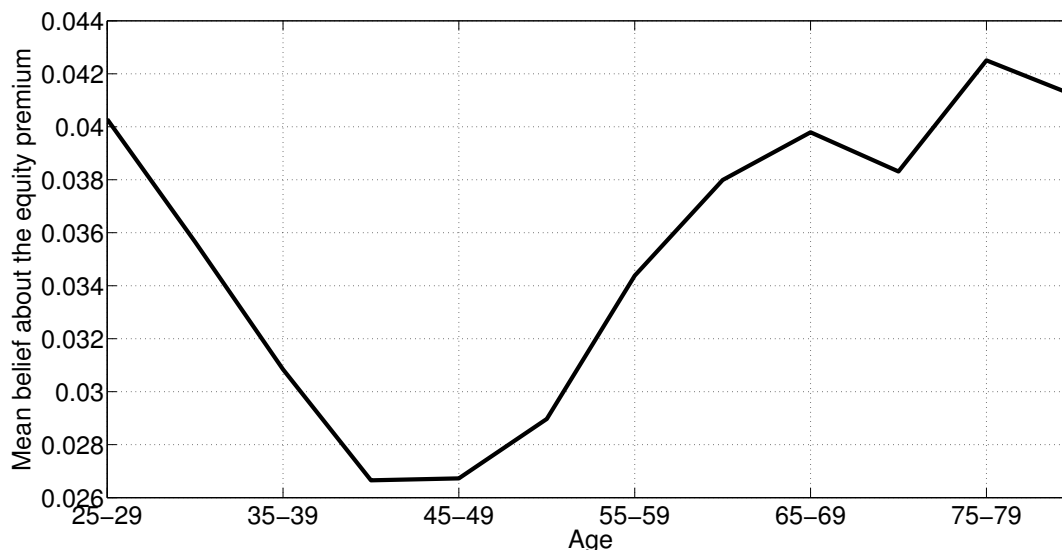


Figure 6:
Beliefs about the equity risk premium in the year 2010.

The graph reports the mean beliefs about the equity risk premium for different ages in 2010. The mean of the priors about the equity risk premium, λ^B is displayed, not the worst case prior. To obtain these figures I used the realized stock returns downloaded from Robert Shillers' website (<http://www.econ.yale.edu/shiller/data.htm>), which contains US stock market data from 1871 onwards. To calculate the belief for a 25 year old in 2010 I combine the prior belief with the average of returns in 1986 to 2009. The mean of the belief for age 25-29 is the average of the beliefs for a 25-year old agent, 26-year old, and so on.

Figure 6 displays the mean beliefs about the equity premium in 2010 using realized returns from 1871 to 2009. People learn from the year of birth onwards hence a 30-year old in 2010 incorporates stock return realizations from 1980-2009 when shaping beliefs about the stock market. In that period the average annual equity premium on the stock market was less than the prior belief of 4%, hence the mean belief is revised downwards, which can be seen in Figure 6. The deviations of the mean belief from 4% generates the cohort effects.

I assume agents are heterogenous with respect to their perceived level of ambiguity about the equity premium. One-third has a standard deviation of beliefs about the equity premium of 1%, one-third of 2%, and one-third of 3%. This heterogeneity is introduced in part because whether agents participate in the stock market depends only on the worst case belief about the equity risk premium. If the worst case belief is zero or negative, the agent does not participate in the stock market, while if the worst case equity risk premium is positive, the agent participates. Even if the

results are available upon request.

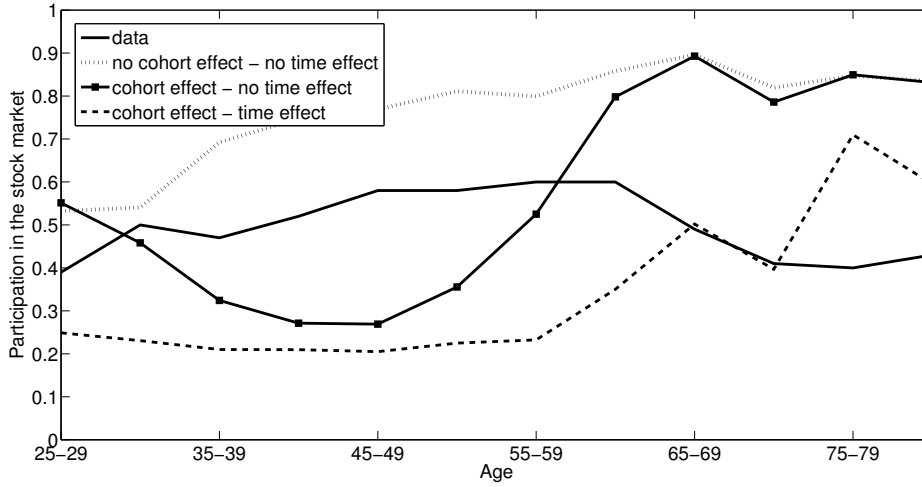
agent has little wealth, if the worst case belief is positive, the agent optimally invests a positive fraction of his wealth in stocks. Furthermore, in reality there will be heterogeneity in the level of ambiguity about the equity premium.

Figure 7 shows portfolio allocations taking into account age, cohort, and time effects. First, note that optimal participation levels are higher than the participation levels observed in the data when not taking into account cohort and time effects.¹¹ Given that we have seen that in 2010 for almost all ages the mean belief about the equity premium was below 4%, adding cohort effects could help overcome this problem. The solid line with squares presents the optimal participation levels when cohort effects are included, and indeed, we find a closer match with the data. Adding time effects, which is represented by the dashed line, doesn't improve the fit to the data much. Focussing on the conditional fraction allocated to stocks in Figure 7b, first of all note that the quantitative impact of cohort and/or time effects on allocations is smaller compared to the influence on participation levels. This is intuitive, since differential beliefs about the equity premium induces some people to not participate due to low previously realized stock returns. However, once the hurdle is overcome (the mean belief is high enough) then the fraction allocated to stocks does not fluctuate as much. The fit to the data does not improve substantially by adding cohort and time effects.

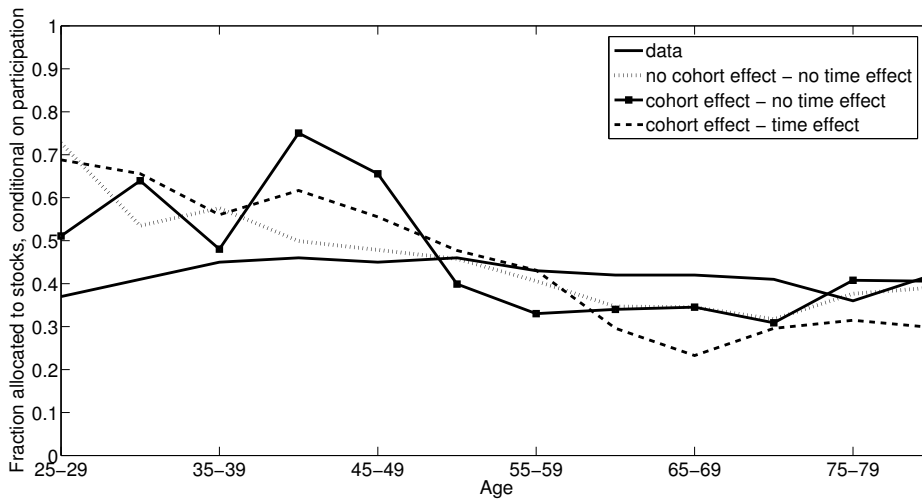
4 Sensitivity analysis and an alternative ambiguity model

In this section the importance of several features of the model are tested. I explore whether increasing risk aversion can substitute for ambiguity aversion and thereby change the optimal stock allocation in a similar way. Furthermore, the impact of at which age the agent starts learning as well as the influence of the initial level of ambiguity about the equity premium is shown. Finally, an alternative ambiguity model, smooth recursive preferences, will be examined.

¹¹In contrast to the previous graphs, the beliefs about the initial level of ambiguity at the year of birth is heterogenous among agents.



(a) Stock market participation



(b) Fraction allocated to stocks, conditional on participation

Figure 7:

Stock portfolio choices with age, cohort, and time effects.

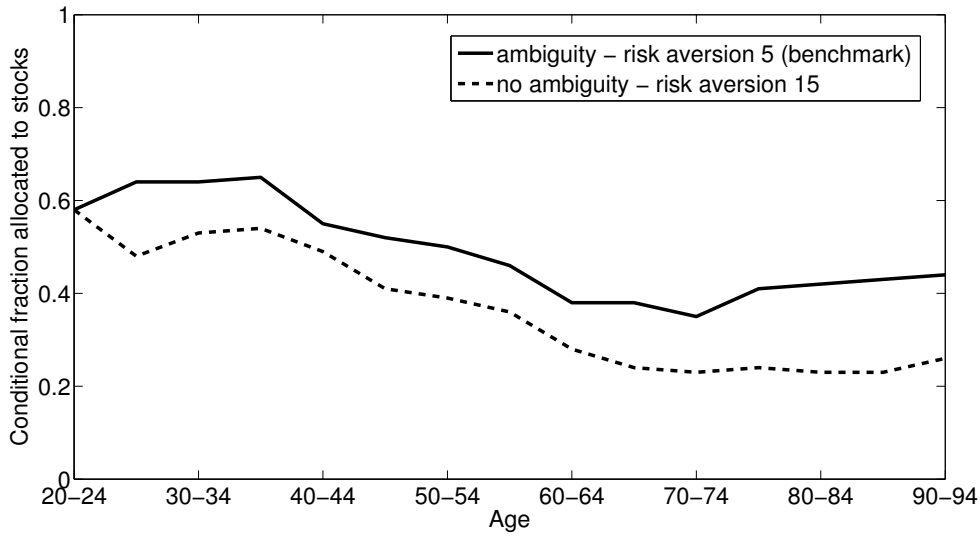
I display the optimal conditional fraction of financial wealth allocated to stocks and optimal participation in the stock market for agents who are (1) ambiguous about the equity risk premium, averse to this ambiguity, and learn about this parameter and who are (2) not ambiguous about the equity premium. The upper panel shows the optimal participation level, which is unconditional on having positive financial wealth, and thus includes all the simulation paths. The lower panel shows the fraction of financial wealth allocated to stocks, conditional on stock market participation. In case an agent has a near zero financial wealth level (below \$100), the optimal participation is assumed to be zero. Agents learn about the equity risk premium. Cohort and/or time effects in stock allocations are taken into account. Cohort effects are assumed to stem from differences in stock market realizations for different cohorts. Time effects are modeled via fees for stock market investments. Agents have heterogeneous beliefs about the initial level of ambiguity about the equity premium. One-third has a standard deviation of beliefs about the equity premium of 1%, one-third of 2%, and one-third of 3%.

4.1 Can risk aversion substitute for ambiguity aversion?

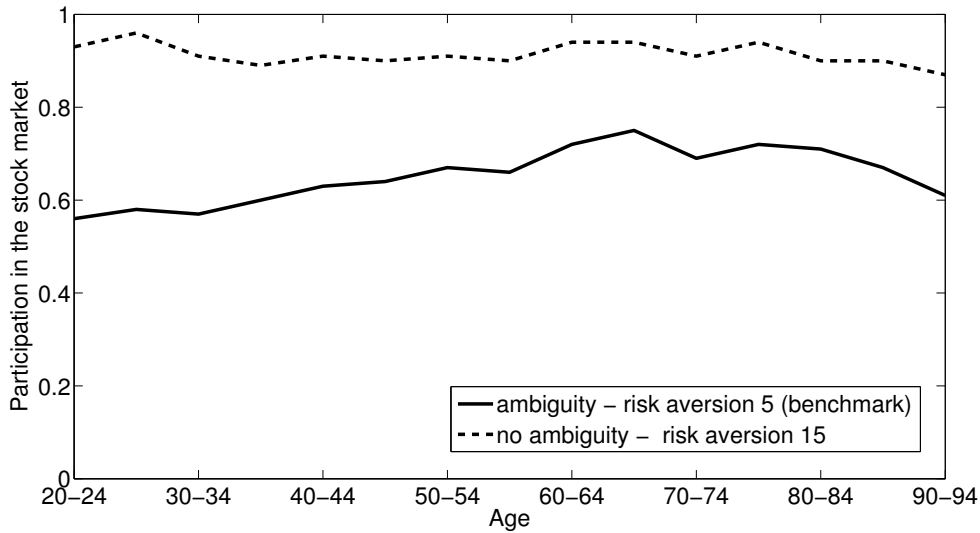
This paper shows that ambiguity about the equity risk premium can help solve the participation puzzle and explain the low fraction of financial wealth allocated to stocks over the life cycle. In this section I show that similar findings cannot be obtained by assuming higher risk aversion. The results are presented in Figure 8. First of all, risk aversion has almost no influence on optimal participation levels while ambiguity aversion has a large impact. Higher risk aversion actually increases participation, since it increases precautionary savings. Focussing on the optimal fraction, when agents have a risk aversion of 15 and are not ambiguity averse, the optimal fraction until age 65 is similar for the baseline risk aversion of 5 combined with ambiguity aversion. After age 65, the difference is substantial. Hence risk aversion does not act as a substitute for ambiguity aversion and I do not obtain the same results with increased risk aversion compared to including ambiguity aversion.

4.2 Impact of starting learning at a later age

In the benchmark model, I assume that people incorporate in their beliefs returns realized from the year of birth onwards, putting equal weight on each stock market realization. In this section I will explore the impact of a different approach for updating beliefs about the equity premium. Specifically, I assume that agents learn from age 20 onwards instead of birth and the results are plotted in Figure 9. Both the optimal participation levels as well as the fraction of financial wealth allocated to stocks is lower if agents update beliefs from age 20 onwards (compare to Figure 1). The reason is that the level of ambiguity, the standard deviation of the beliefs about the equity premium, is higher when updating starts at a later age. When assuming that individuals incorporate the stock market realizations from age 20 onwards, they will have seen only one stock market realization at age 21, compared to 21 realizations when incorporating all realizations after birth. Hence assuming learning starts from age 20 makes the main results in this paper stronger; the impact of ambiguity about the equity premium on optimal stock allocations is larger.



(a) Fraction allocated to stocks, conditional on participation

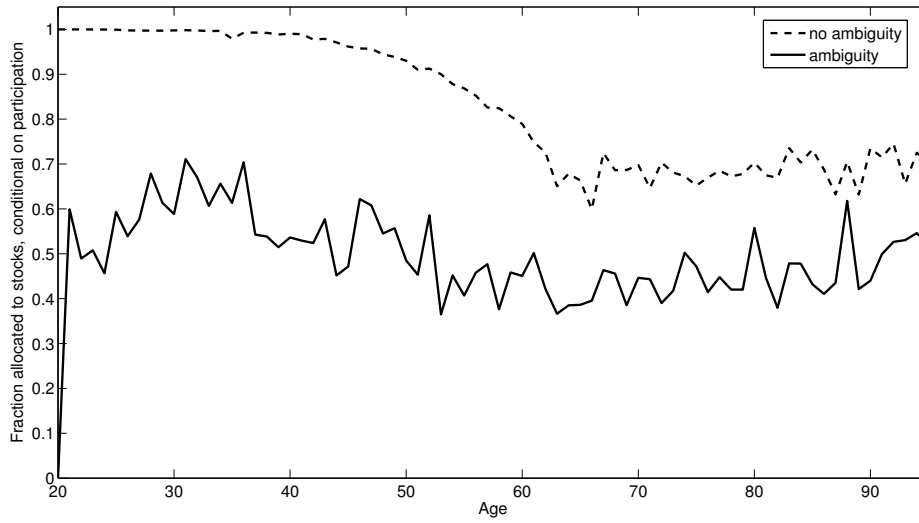


(b) Stock market participation

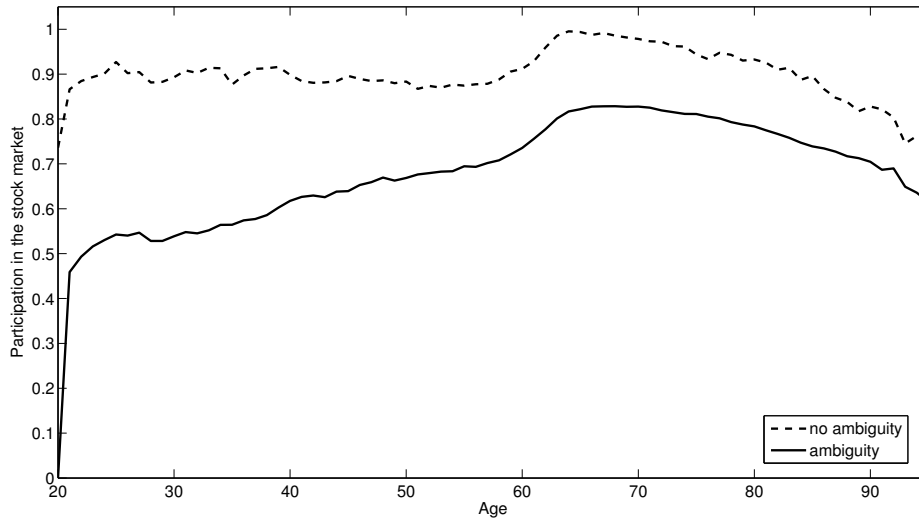
Figure 8:

Stock allocations; can risk aversion substitute for ambiguity aversion?

The figures show the optimal conditional fraction of financial wealth allocated to stocks and optimal participation in the stock market for agents who are (1) ambiguous about the equity risk premium, averse to this ambiguity, and learn about the parameters and who are (2) highly risk averse ($\gamma=15$) and not ambiguous about the equity premium. The upper panel shows the fraction of financial wealth allocated to stocks, conditional on stock market participation. The lower panel shows the optimal participation level. In case an agent has near zero financial wealth level (below \$100), the optimal participation is assumed to be zero.



(a) Fraction allocated to stocks, conditional on participation



(b) Stock market participation

Figure 9:

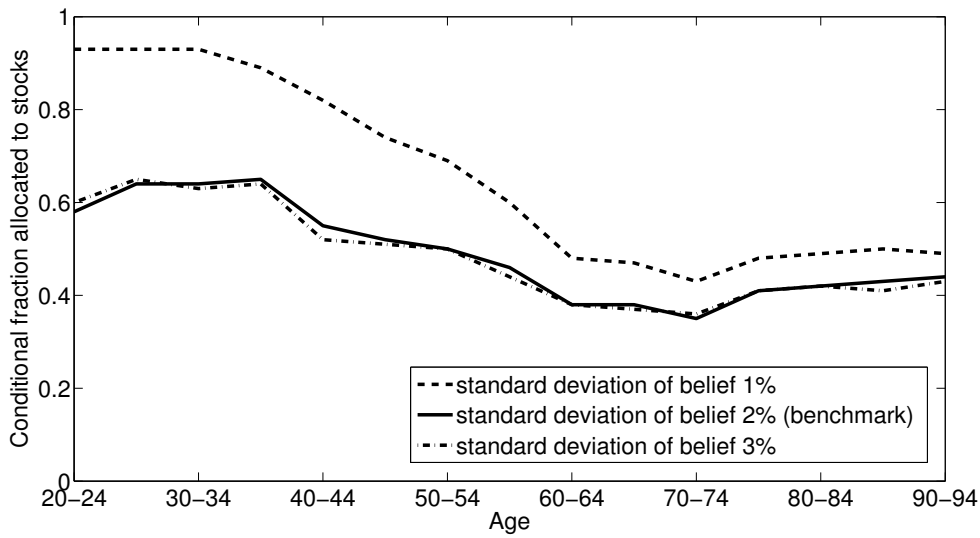
Stock allocations: learning from age 20 onwards

These figures show the impact of learning from age 20 onwards instead of the year of birth. I display the optimal conditional fraction of financial wealth allocated to stocks and optimal participation in the stock market for agents who are (1) ambiguous about the equity risk premium, averse to this ambiguity, and learn about this parameter and who are (2) not ambiguous about the equity premium. The upper panel shows the fraction of financial wealth allocated to stocks, conditional on stock market participation. The lower panel shows the optimal participation level. In case an agent has near zero financial wealth level (below \$100), the optimal participation is assumed to be zero.

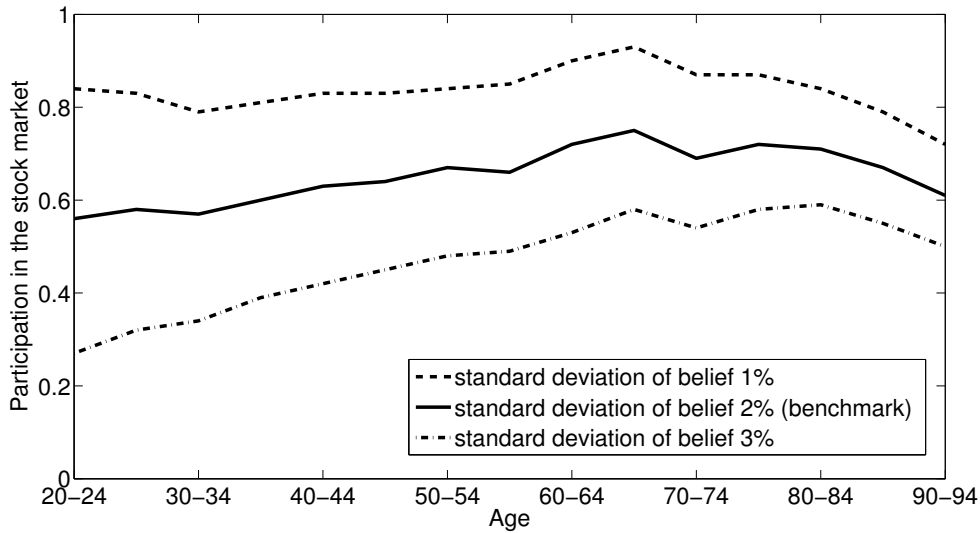
4.3 Impact of initial ambiguity about the equity risk premium

The level of initial ambiguity, i.e. the standard deviation of the belief about the equity risk premium, is set to 2% in the benchmark case. There is no direct evidence on which to base this level of ambiguity on, so I perform sensitivity analysis with respect to this parameter. Intuitively, a standard deviation of 2% seems reasonable, since this ensures that the 95% confidence interval of the equity risk premium that the agent beliefs is possible is between 0% and 8% at birth. Compelling evidence that this is not overstating the degree of ambiguity can be derived from the financial literacy literature. When answering questions to establish financial literacy levels, Rooij van, Lusardi, and Alessie (2011) find that 22% of survey respondents answer that they do not know whether "considering a long time period, stocks, bonds, or savings accounts give the highest return". Furthermore 30% gives the wrong answer and less than half gives the correct answer. This at least indicates that it is a valid assumptions that a large fraction of agents is ambiguous about the equity premium, and in general about financial market parameters. Furthermore, even assuming that agents look up all previous stock market returns, the confidence interval about the equity premium would still be large. But since I have no means to determine the range of equity risk premium that agents deem possible, this section examines the influence of the initial ambiguity level on the results. The results from this section can also be viewed in light of agents having different levels of ambiguity and how this influences the optimal fraction allocated to stocks and optimal participation levels.

Figure 10 displays the conditional allocation to equity and participation in the stock market for three different levels of initial ambiguity. The optimal fraction allocated to stocks is approximately the same if the standard deviation of beliefs is 3% or 2%. The reason is that only if agents have a positive worst case belief, they participate in the stock market. However, once they start participating, the optimal fraction when this hurdle is overcome is the same. In other words, conditional on participation, the average belief about the equity premium is the same for agents starting out with a standard deviation of beliefs of 2% and 3%. The fraction allocated to stocks if agents have a standard deviation of beliefs of 1% is much higher at young ages, because the average worst case belief, conditional on having a positive worst case belief, is higher. Participation levels differ sub-



(a) Fraction allocated to stocks, conditional on participation



(b) Stock market participation

Figure 10:

Stock allocations: three different initial levels of ambiguity.

These figures show the impact of the initial level of ambiguity. I display the optimal conditional fraction of financial wealth allocated to stocks and optimal participation in the stock market for agents who are (1) ambiguous about the equity risk premium, averse to this ambiguity, and learn about this parameter and who are (2) not ambiguous about the equity premium. The upper panel shows the fraction of financial wealth allocated to stocks, conditional on stock market participation. The lower panel shows the optimal participation level. In case an agent has a near zero financial wealth level (below \$100), the optimal participation is assumed to be zero.

stantially since the agent with a standard deviation of 3% needs a much larger positive update to have a worst case belief higher than 0% and participate, compared to an agent with a 2% standard deviation.

4.4 Alternative ambiguity model: *smooth recursive preferences*

In the previous section I show that for agents having maxmin preferences, ambiguity aversion has a large effect on optimal portfolio choices. However, there is no consensus on whether agents exhibit smooth recursive preferences or maxmin preferences hence in this section I examine the influence of ambiguity aversion when agents behave according to smooth recursive preferences with moderate ambiguity aversion. I assume preferences as specified in Klibanoff, Marinacci, and Mukerji (2005), which include an ambiguity function ϕ and total optimal lifetime utility equals

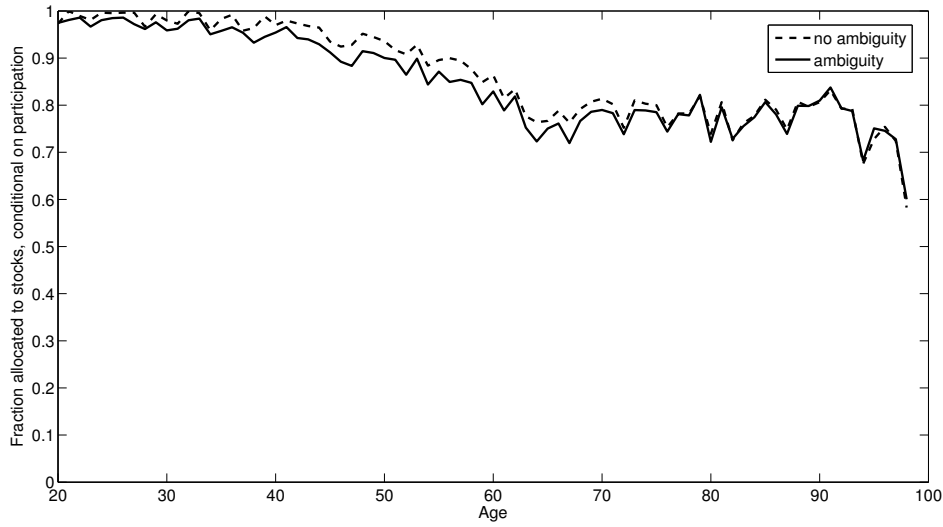
$$V_t = \max_{w_t, C_t} u(C_t) + \beta p_{t+1} \phi^{-1} \left(\int_{\Lambda_t} \phi \left(\mathbb{E}_t^\lambda \{V_{t+1}(W_{t+1})\} \right) p_t(\lambda) d\lambda \right), \quad (15)$$

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}, \quad (16)$$

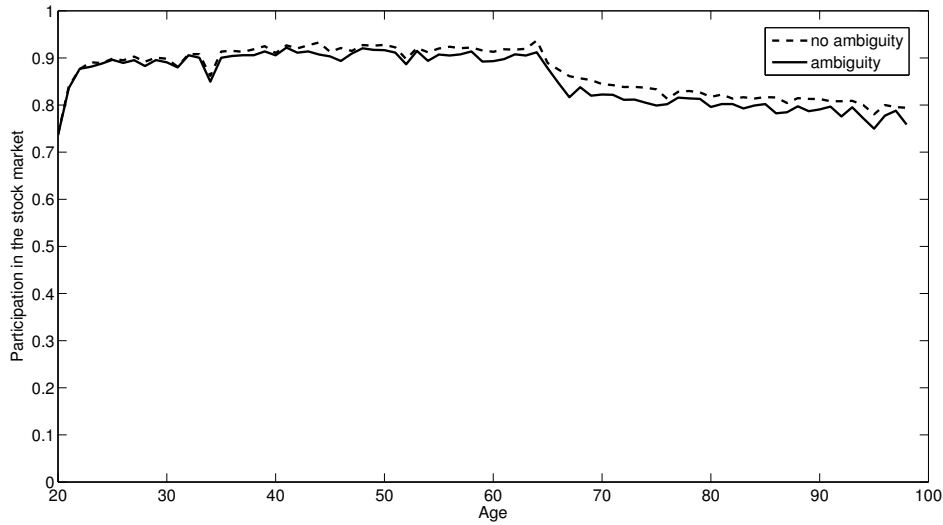
$$\phi(x) = -\exp(-\alpha x), \quad (17)$$

where ϕ is the constant relative ambiguity aversion function (CRAA) and α is the ambiguity aversion coefficient. Think of each prior $\lambda_t \in \Lambda_t$ as describing a possible scenario (a possible equity risk premium) and $p_t(\lambda)$ as the probabilistic belief over the different scenarios. This utility function can be interpreted as being solved in two stages. First, the expected utility for all the priors in Λ_t are calculated, to get a set of expected utilities. Maxmin would then take the minimum of these expected utilities, while smooth preferences takes an expectation over the distorted probabilities. The ambiguity aversion function ϕ distorts the probabilities, giving a higher weight to lower expected utilities, reflecting ambiguity aversion.

Figure 11a shows the optimal fraction of financial wealth allocated to stocks, conditional on participation in the case that (1) the parameters are not ambiguous and (2) the parameters are



(a) Fraction allocated to stocks, conditional on participation



(b) Stock market participation

Figure 11: Stock allocations: alternative ambiguity aversion model

These figures show the impact of an alternative model for ambiguity aversion: smooth recursive preferences. I display the optimal conditional fraction of financial wealth allocated to stocks and optimal participation in the stock market for agents who are (1) ambiguous about the equity risk premium and learn about this parameter and who are (2) not ambiguous about the equity premium. The upper panel shows the fraction of financial wealth allocated to stocks, conditional on stock market participation. The lower panel shows the optimal participation level, which is unconditional on having positive financial wealth, and thus includes all the simulation paths. In case an agent has a near zero financial wealth level (below \$100), the optimal participation is assumed to be zero.

ambiguous and the agent is moderately averse to this ambiguity. There is a small decrease in the allocation to stocks. When comparing the lines in Figure 11b, I find that the participation levels do not vary much with the level of ambiguity. Independent of whether the agent is uncertain, the participation levels are high. Overall, for the benchmark parameters the effect of ambiguity aversion on the optimal portfolio allocation is negligible. Note that smooth ambiguity preferences with infinite ambiguity aversion equals maximin preferences. Hence in the limit when ambiguity aversion goes to infinity the effect of ambiguity about the equity risk premium is sizeable and can help explain the empirically observed low allocation to stocks. Consistent with the findings in this paper, experimental evidence suggests that agents behave more according to kinked (maxmin) preferences than smooth ambiguity preferences (see Ahn, Choi, Gale, and Kariv (2013)).

5 Conclusion

In this paper I develop a realistically calibrated life-cycle model with ambiguity aversion and learning to explore the impact of ambiguity about the equity risk premium on optimal portfolio allocations. Taking into account ambiguity about the equity premium reduces the optimal participation levels on average by 25% and the optimal conditional fraction of financial wealth allocated to stocks by 40%. Furthermore, wealth levels over the life cycle are reduced substantially while the savings out of income and wealth are increased. I compare the model predictions with data from the Survey of Consumer Finances. Two important empirical facts are matched, the low participation levels in the stock market over the life cycle and the low fraction of financial wealth allocated to equity, conditional on participation.

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A Numerical method to solve the life-cycle model with maxmin preferences

A.1 Short summary life-cycle problem with maxmin preferences

Investors preferences are described by maxmin expected utility, which in effect means that the agent maximizes his utility with respect to the worst case belief. The agent is uncertain about the equity risk premium. I solve the following Bellman equation:

$$V_t = \max_{w_t, C_t} \min_{\lambda \in \Lambda_t} [u(C_t) + \beta p_{t+1} E_t^\lambda \{V_{t+1}(W_{t+1})\}], \text{ with} \quad (18)$$

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}. \quad (19)$$

$$(20)$$

As described in Section 1.1 I restrict the domain of beliefs about the equity risk premium to lie between $[\lambda_t^B - 2\sigma_t^B, \lambda_t^B + 2\sigma_t^B]$. This is necessary to put a bound on the worst case belief, which would else be minus infinity since the beliefs are normally distributed.

A.2 The first order conditions - maxmin preferences

In each period I find the optimal asset weights by setting the first order condition equal to zero

$$E_t^{\lambda_t^{min}} (C_{t+1}^{*- \gamma} (R_{t+1} - R^f)) = 0, \quad (21)$$

where λ_t^{min} is the lowest equity premium in Λ_t . C_{t+1}^* denotes the optimal real consumption level.

The optimal consumption follows from

$$C_t^{*- \gamma} = \beta p_{t+1} E_t^{\lambda_t^{min}} (C_{t+1}^{*- \gamma} R_{t+1}^{P*}). \quad (22)$$

A.3 Optimization procedure for the optimal asset weights and consumption - maxmin preferences

As described in Section 1.8, I calculate the realization of the Euler condition and regress these on a polynomial expansion in the state variables to obtain an approximation of the conditional expectation of the Euler condition

$$\lambda_t^{min} (C_{t+1}^{*- \gamma} (R_{t+1} - R^f)) \simeq \rho' f(Y_t, \lambda_t^B). \quad (23)$$

In addition I employ a further extension, introduced in Koijen, Nijman, and Werker (2010). They found that the regression coefficients ρ are smooth functions of the asset weights and, consequently, I approximate the regression coefficients ρ by projecting them further on polynomial expansion in the asset weights:

$$\rho(x) \simeq \Psi g(x). \quad (24)$$

The Euler condition must be set to zero to find the optimal asset weights:

$$f(Y_t, \lambda_t^B) \Psi g(w)' = 0. \quad (25)$$

Similarly, I approximate the Euler condition for optimal consumption via regressing the realization of the Euler conditions on a polynomial expansion in the state variables.

B Asset allocation with alternative ambiguity preferences

B.1 Short summary life-cycle problem with smooth ambiguity preferences

The investor solves the following Bellman equation at time $t \neq T$

$$V_t(W_t, Y_t, \lambda_t^B) = \max_{w_t, C_t} u(C_t) + \beta p_{t+1} \phi^{-1} \left(\int_{\Lambda_t} \{ \phi(\mathbb{E}_t^\lambda \{ V_{t+1}(W_{t+1}, Y_{t+1}, \lambda_{t+1}^B) \}) \} \right), \quad (26)$$

where C_t is consumption. Agents employ an uncertainty aversion function ϕ . The exogenous state variables are income (Y_t) and the mean of the belief about the expected equity premium (λ_t^B). Wealth (W_t) is an endogenous state variable. At time T the investor consumes all wealth, hence the value function equals:

$$V_T(W_T, Y_T, \lambda_T^B) = u(W_T). \quad (27)$$

The dynamics of financial wealth are given by

$$W_{t+1} = (W_t - C_t + Y_t)(1 + R^f + w_t(R_{t+1} - R^f)). \quad (28)$$

I assume a constant relative risk aversion utility function (CRRA) and a constant absolute ambiguity aversion utility function (CAAA):

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma} \quad (29)$$

$$\phi(x) = -\exp(-\alpha x). \quad (30)$$

The individual faces a number of constraints on the consumption and investment decisions. First, I assume that the agent faces borrowing and short-sales constraints

$$w_t \geq 0 \text{ and } l'w_t \leq 1. \quad (31)$$

Second, I impose that the investor is liquidity constrained

$$C_t \leq W_t, \quad (32)$$

which implies that the individual cannot borrow against future income to increase consumption today.

The timing is as follows, first an individual receives his income, after which he consumes. Subsequently he invests the remaining wealth, either in equity or a riskless asset.

B.2 Beliefs about the equity risk premium

The agent has beliefs about the equity risk premium and the mean of his beliefs is λ_t^B and the standard deviation of his belief about the equity risk premium is σ_t^B . I limit the set of beliefs that the agent thinks are viable to be bounded by a 95% confidence interval. Hence the beliefs on the equity premium lies in the range of $[\lambda_t^B - 2\sigma_t^B; \lambda_t^B + 2\sigma_t^B]$. I make a grid for the possible equity premiums by dividing the confidence interval in K equal probability areas. Subsequently I calculate the average of the outer bound of each area separately and the probability that the agent attaches to this equity premium is $1/K$.

B.3 The first order conditions - smooth recursive preferences

In period T the optimal policies are easily determined. Namely the agent consumes the entire wealth level and no optimal investment strategy need to be made. In all other time periods optimal decisions on consumption and investment are calculated by setting the first order conditions equal to zero. The optimization problem is solved via dynamic programming and I proceed backwards.

I define the portfolio return as:

$$R_{t+1}^P = 1 + R^f + (R_{t+1} - R^f)w_t. \quad (33)$$

Furthermore I denote the wealth level after income and consumption as:

$$A_t = W_t + Y_t - C_t. \quad (34)$$

Consider that the agent is at time t , after having consumed C_t , and he/she has to choose w_t so as to maximize the bellman equation. The first-order condition with respect to w_t for this problem is.

$$\begin{aligned} \frac{\partial V_t}{\partial w_t} = & \beta p_{t+1} \left(\frac{1}{\phi'(\phi^{-1}(.))} \right) \\ & \int_{\Lambda_t} \left\{ \phi' \left(\mathbb{E}_t^\lambda \{V_{t+1}(W_{t+1})\} \right) \mathbb{E}_t^\lambda \left\{ \frac{\partial V_{t+1}}{\partial W_{t+1}} (W_t + Y_t - C_t)(R_{t+1} - R^f) \right\} \right\}, \end{aligned} \quad (35)$$

where

$$. = \int_{\Lambda_t} \{ \phi \left(\mathbb{E}_t^\lambda \{V_{t+1}(W_{t+1})\} \right) \}. \quad (36)$$

So similar but written differently:

$$\begin{aligned} \frac{\partial V_t}{\partial w_t} = & \beta p_{t+1} (\phi^{-1})' \left(\int_{\Lambda_t} \{ \phi \left(\mathbb{E}_t^\lambda \{V_{t+1}(W_{t+1})\} \right) \} \right) \\ & \int_{\Lambda_t} \left\{ \phi' \left(\mathbb{E}_t^\lambda \{V_{t+1}(W_{t+1})\} \right) \mathbb{E}_t^\lambda \left\{ \frac{\partial V_{t+1}}{\partial W_{t+1}} (W_t + Y_t - C_t)(R_{t+1} - R^f) \right\} \right\}. \end{aligned} \quad (37)$$

The first order condition with respect to C_t equals:

$$\begin{aligned} \frac{\partial V_t}{\partial C_t} = & C_t^{-\gamma} - \beta p_{t+1} (\phi^{-1})' \left(\int_{\Lambda_t} \{ \phi \left(\mathbb{E}_t^\lambda \{V_{t+1}(W_{t+1})\} \right) \} \right) \\ & \int_{\Lambda_t} \left\{ \phi' \left(\mathbb{E}_t^\lambda \{V_{t+1}(W_{t+1})\} \right) \mathbb{E}_t^\lambda \left\{ \frac{\partial V_{t+1}}{\partial W_{t+1}} (1 + R^f + w_t(R_{t+1} - R^f)) \right\} \right\}. \end{aligned} \quad (38)$$

Next I take the total derivative with respect to W_t

$$\begin{aligned} \frac{\partial V_t}{\partial W_t} = & \beta p_{t+1}(\phi^{-1})' \left(\int_{\Lambda_t} \{ \phi (\mathbb{E}_t^\lambda \{ V_{t+1}(W_{t+1}) \}) \} \right) \\ & \int_{\Lambda_t} \left\{ \phi' (\mathbb{E}_t^\lambda \{ V_{t+1}(W_{t+1}) \}) \mathbb{E}_t^\lambda \left\{ \frac{\partial V_{t+1}}{\partial W_{t+1}} (1 + R^f + w_t(R_{t+1} - R^f)) \right\} \right\}. \end{aligned} \quad (39)$$

Substitute equation (38) into equation (39)

$$\frac{\partial V_t}{\partial W_t} = C_t^{-\gamma} \quad (40)$$

$$\frac{\partial V_{t+1}}{\partial W_{t+1}} = C_{t+1}^{-\gamma}. \quad (41)$$

Substitute equation (41) into equation (37) to obtain the first order condition for the asset allocation:

$$\begin{aligned} \frac{\partial V_t}{\partial w_t} = & \beta p_{t+1}(\phi^{-1})' \left(\int_{\Lambda_t} \{ \phi (\mathbb{E}_t^\lambda \{ V_{t+1}(W_{t+1}) \}) \} \right) \\ & \int_{\Lambda_t} \{ \phi' (\mathbb{E}_t^\lambda \{ V_{t+1}(W_{t+1}) \}) \mathbb{E}_t^\lambda \{ C_{t+1}^{*- \gamma} (R_{t+1} - R^f) \} \}. \end{aligned} \quad (42)$$

To solve for the optimal consumption, substitute equation (41) into equation (38) to get the following first order condition

$$\begin{aligned} C_t^{-\gamma} = & \beta p_{t+1}(\phi^{-1})' \left(\int_{\Lambda_t} \{ \phi (\mathbb{E}_t^\lambda \{ V_{t+1}(W_{t+1}) \}) \} \right) \\ & \int_{\Lambda_t} \{ \phi' (\mathbb{E}_t^\lambda \{ V_{t+1}(W_{t+1}) \}) \mathbb{E}_t^\lambda \{ C_{t+1}^{*- \gamma} (1 + R^f + w_t(R_{t+1} - R^f)) \} \}. \end{aligned} \quad (43)$$

In addition I use:

$$\phi^{-1} = -\frac{1}{\alpha} \ln(-y), \quad (44)$$

$$(\phi^{-1})' = \frac{-1}{\alpha y}, \quad (45)$$

$$\phi' = \alpha \exp(-\alpha x). \quad (46)$$

B.4 Optimization procedure for the optimal asset weights - smooth recursive preferences

Due to the complexity of the model it cannot be solved analytically. Instead I use numerical optimization techniques to solve the problem. In this section I will explain this procedure, which combines the methods of Brandt, Goyal, Santa-Clara, and Stroud (2005) and Carroll (2006), with several extensions added by Koijen, Nijman, and Werker (2010). Brandt, Goyal, Santa-Clara, and Stroud (2005) propose to approximate the conditional expectations by regressing the realizations of the Euler conditions on a polynomial expansion of the state variables. All state variables except for wealth can be simulated, since only wealth is endogenous. To deal with this endogenous state variable I follow Carroll (2006) who proposes a grid for wealth *after* consumption, A_t , instead of a grid for wealth, W_t . This choice allows us to solve the Euler conditions analytically instead of numerically and I form a M-dimensional grid for wealth after consumption. Additionally, I use extensions by Koijen, Nijman, and Werker (2010) to increase the optimization speed. I construct H test portfolios and let the weight invested in the risky asset run from 0% to +100%, with steps of 5%, hence H is 21. The return on the test portfolios is defined as R_{t+1}^{test} . Furthermore we simulate N trajectories of T periods for every state variable

The problem is solved via backwards recursion and to solve the optimal policies at time t , I have available the endogenous wealth grid at time $t + 1$ and the optimal consumption at time $t + 1$.

First I need to determine the two conditional expectations in equation (42):

$$\mathbb{E}_t^\lambda \{ C_{t+1}^{*- \gamma} (R_{t+1} - R^f) \} \quad (47)$$

$$\mathbb{E}_t^\lambda \{ V_{t+1}(W_{t+1}) \} \quad (48)$$

The conditional expectation in equation (47) is straightforward to calculate. I have the optimal consumption at period $t + 1$, since I solve via backwards recursion. To obtain C_{t+1}^* I interpolate linearly to make sure it is the optimal consumption next period that belongs to the grid point for after consumption wealth at time t , A_t . I approximate the conditional expectation with a polynomial

expansions in the state variables:

$$\mathbb{E}_t^\lambda \{C_{t+1}^{*- \gamma}(R_{t+1} - R^f)\} \simeq \rho f(Y_t, \lambda_t^B). \quad (49)$$

This is done for each simulation path and MxK grid points.

The second conditional expectation (48) requires some more steps. The goal is to determine the realizations of V_{t+1} regress these on the state variables at time t to obtain the conditional expectation. The value function at time $t + 1$ is

$$V_{t+1} = u(C_{t+1}^*) + \beta p_{t+1} \phi^{-1} \left(\int_{\Lambda_{t+1}} \{ \phi \left(\mathbb{E}_{t+1}^{\lambda_{t+1}} \{ V_{t+2}(W_{t+2}, w_{t+2}^*) \} \} \right) \right). \quad (50)$$

A star '*' denotes the optimal policies which I already calculated. Again I use interpolation to obtain the intermediate consumption levels. Furthermore I need V_{t+2} which belongs to the grid points for after consumption wealth at time t , and not the grid point at time $t + 1$, so similarly I use interpolation. The value of the Bellman equation at time $t + 2$, V_{t+2} , is saved at the end of every time period since I solve via backwards recursion. As before, to obtain $\mathbb{E}_{t+1}^{\lambda_{t+1}}(V_{t+2})$ I regress V_{t+2} on the state variables at time $t + 1$. Note that when determining the optimal policies at time $T - 1$, $V_{T+1} = 0$ and $V_T = u(W_T)$.

Agents are uncertain about the equity risk premium hence the optimization problem requires several additional steps. Namely in this setup $\mathbb{E}_{t+1}(V_{t+2})$ is random. The beliefs are distributed in such a way that there is a $1/K$ probability that the true equity risk premium lies between two of the grid points for the equity premium. Hence to calculate $\left(\int_{\Lambda_{t+1}} \{ \phi \left(\mathbb{E}_{t+1}^{\lambda_{t+1}} \{ V_{t+2}(W_{t+2}, w_{t+2}^*) \} \} \right) \right)$, I need to take the (weighted) average of $\phi \mathbb{E}_{t+1}^{\lambda_{t+1}}(V_{t+2})$ over the grid of beliefs about the equity risk premium. Next I plug all these calculated numbers in equation (50). Following Brandt, Goyal, Santa-Clara, and Stroud (2005) I regress the realizations of the Euler condition ($V_{t+1}(W_{t+1})$) on the state variables to obtain the conditional expectation, $\mathbb{E}_t^\lambda \{ V_{t+1}(W_{t+1}) \}$.

Recall the first order condition for the optimal asset weights:

$$\begin{aligned} \frac{\partial V_t}{\partial w_t} = & \beta p_{t+1}(\phi^{-1})' \left(\int_{\Lambda_t} \{ \phi \left(\underline{\mathbb{E}_t^\lambda \{ V_{t+1}(W_{t+1}) \}} \right) \} \right) \\ & \int_{\Lambda_t} \left\{ \phi' \left(\underline{\mathbb{E}_t^\lambda \{ V_{t+1}(W_{t+1}) \}} \right) \underline{\mathbb{E}_t^\lambda \{ C_{t+1}^{*- \gamma}(R_{t+1} - R^f) \}} \right\}. \end{aligned} \quad (51)$$

Note that the steps to calculate the underlined parts of the equations are already explained. $\mathbb{E}_t^\lambda \{ V_{t+1}(W_{t+1}) \}$ is a $N \times M \times K \times H$ matrix and I plug these numbers in the ambiguity aversion function ϕ . Subsequently the weighted average is taken and the K -dimension falls out. Analogue the entire equation (51) is calculated.

Following Koijen, Nijman, and Werker (2010) the optimal asset weights are determined in two steps. First I approximate the conditional expectation with a polynomial state variables.

$$\begin{aligned} & \beta p_{t+1}(\phi^{-1})' \left(\int_{\Lambda_t} \{ \phi \left(\mathbb{E}_t^\lambda \{ V_{t+1}(W_{t+1}) \} \right) \} \right) \int_{\Lambda_t} \{ \phi' \left(\mathbb{E}_t^\lambda \{ V_{t+1}(W_{t+1}) \} \right) \mathbb{E}_t^\lambda \{ C_{t+1}^{*- \gamma}(R_{t+1} - R^f) \} \} \\ & = \rho' f(Y_t, \lambda_t^B) \end{aligned} \quad (52)$$

Subsequently the projection coefficients, ρ , are parameterized in the asset weights. I let the projection coefficients depend on the "test" asset weights (x), which I previously made a H -dimensional grid over. Hence for every simulated path I calculate H test portfolio returns. Since ρ is a smooth function of the asset weights I can obtain:

$$\rho(x) \simeq \Psi g(x), \quad (53)$$

where $g(x)$ is a polynomial expansion in the asset weights. This implies that the conditional expectation of the Euler condition is approximated via

$$\begin{aligned} & \beta p_{t+1}(\phi^{-1})' \left(\int_{\Lambda_t} \{ \phi \left(\mathbb{E}_t^\lambda \{ V_{t+1}(W_{t+1}) \} \right) \} \right) \int_{\Lambda_t} \{ \phi' \left(\mathbb{E}_t^\lambda \{ V_{t+1}(W_{t+1}) \} \right) \mathbb{E}_t^\lambda \{ C_{t+1}^{*- \gamma}(R_{t+1} - R^f) \} \} \\ & = g(x)' \Psi' f(Y_t, \lambda_t^B). \end{aligned} \quad (54)$$

A polynomial expansion of order one is sufficient for this life-cycle problem, hence for every simulation path I solve:

$$0 = \begin{pmatrix} 1 \\ w^* \end{pmatrix} \Psi' f(Y_t, \lambda_t^B, \sigma_t^B), \quad (55)$$

which can be solved analytically, taking into account the portfolio constraints.

B.5 Optimization procedure for the optimal consumption - smooth recursive preferences

The derivative of the value function with respect to C_t is equal to:

$$C_t^{-\gamma} = \beta p_{t+1} (\phi^{-1})' \left(\int_{\Lambda_t} \{ \phi (\mathbb{E}_t^\lambda \{ V_{t+1}(W_{t+1}) \}) \} \right) \int_{\Lambda_t} \{ \phi' (\mathbb{E}_t^\lambda \{ V_{t+1}(W_{t+1}) \}) \mathbb{E}_t^\lambda \{ C_{t+1}^{*- \gamma} (1 + R^f + w_t^* (R_{t+1} - R^f)) \} \}. \quad (56)$$

The timing is as follows, first the agent consumes and afterwards the investment is made, so because I solve this problem via backward recursion I already found the optimal asset weights at time t , hence I have R_{t+1}^{P*} . I proceed as before, first I calculate the inner conditional expectations, if necessary plug them into the appropriate functions, take the weighted averages to get the K-dimension out, and finally plug parts of the calculations into other functions. The optimal consumption strategy then follows analytically.

Note however that the conditional expectation $\mathbb{E}_t^\lambda \{ C_{t+1}^{*- \gamma} (1 + R^f + w_t^* (R_{t+1} - R^f)) \}$ needs to be strictly positive, otherwise the optimal consumption will be negative. Hence following Koijen, Nijman, and Werker (2010) I approximate the logarithm of this conditional expectation:

$$\mathbb{E}_t^\lambda \{ C_{t+1}^{*- \gamma} (1 + R^f + w_t^* (R_{t+1} - R^f)) \} \simeq \exp(\rho_0 + \rho' f(Y_t, \lambda_t^B)). \quad (57)$$

After all these steps I obtain the optimal consumption and investment decisions at all (N x

M) grid points at each point in time. Finally I start from the initial states and simulate forward. Depending on the realized wealth levels at each time period (the endogenous state variable), I use the corresponding optimal investment and consumption strategies. This results in the optimal policies for every simulation path.

C Survey of Consumer Finances and allocation to stocks

The Survey of Consumer Finances is a triennial survey on the financial assets of the household. It provides information on assets on the balance sheet, pensions, income, and demographics of the household. Participation in the survey is strictly voluntary and about 4500 families are interviewed. It is a repeated cross-section and only the years 1983 to 1989 are partly a panel study. The median length of an interview is about 75 minutes, but an interview with a family with complex finances can take up to several hours. High income households are over-sampled to measure asset holdings more accurately, since wealth in the US is highly concentrated among a relatively small number of households. About two thirds of the sample, 3000 households, is drawn from a national area probability sample which represents the entire population. The remaining one third, 1500 households, is drawn from tax records to get the list of high income households. Weights are used to account for both nonresponse and the difference between the initial sample design and to the actual distribution of population characteristics. In the case of missing data, multiple imputation is used to solve this problem.

Financial wealth (FIN) is the sum of liquid assets (checking, savings, money market, and call accounts), certificates of deposit, directly held mutual funds, stocks, bonds, quasi-liquid retirement accounts which consists of IRAs/Keoghs, thrift accounts, and future pensions, savings bonds, cash value of whole life insurance, other managed assets (trusts, annuities, and managed investment accounts), and other financial assets (loans from the household to someone else, future proceeds, royalties, futures, non-public stock, deferred compensation, oil/gas/mineral investment). The part of financial assets invested in stocks (EQUITY) consists of directly held stock, stock mutual funds,

and retirement assets invested in stocks. I follow the Survey of Consumer Finances in calculating this. The stock investment includes the entire directly held stock, entire stock mutual funds, half of the value of the combination mutual funds, and the fraction of the value of IRAs/Keoghs that is invested in stocks. Similarly the fraction of the value of other managed assets invested in stocks is added and the part of the value of the thrift account that is allocated to stocks.

The fraction of agents participating in the stock market is determined by calculating which weighted fraction in the total sample has a stock investment larger than zero. Furthermore the conditional allocation to equity is the fraction allocated to stocks, conditional on participation in the stock market. Note that I use weights to calculate the participation rate and the conditional allocation to stocks to adjust for nonresponse and the non-equal probability design of the survey.