

The Common Factor in Idiosyncratic Volatility

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Abstract

We show that firms' idiosyncratic volatility in returns and cash flows obeys a strong factor structure. We find that the stocks of firms with large, negative common idiosyncratic volatility (CIV) factor betas earn high average returns. The CIV beta quintile spread is 6.4% per year. To explain this spread, we develop a heterogeneous investor model with incomplete markets in which the idiosyncratic volatility of investor consumption growth inherits the factor structure of firm cash flow growth. In our model, the CIV factor is a priced state variable, because an increase in volatility represents a worsening of the investment opportunity set for the average investor. The calibrated model is able to match the high degree of comovement in idiosyncratic volatilities, the CIV beta spread, along with a host of asset price moments.

JEL: E3, E20, G1, L14, L25

Keywords: Firm volatility, Idiosyncratic risk, Cross-section of stock returns

1 Introduction

This paper presents three central findings. First, we document a strong factor structure in firm-level volatility, even after removing common variation in returns via factor models. Second, stocks that tend to lose value when common idiosyncratic volatility (CIV) rises earn high average returns. Third, to account for these findings, we develop a heterogeneous agent model with common idiosyncratic volatility in investors' consumption as well as the firms' cash flow processes, both measured by the CIV factor in stock returns. In our model, CIV is a priced state variable with a negative risk price. The calibration generates comovement in cash flow and return volatility and a spread in stock returns based on CIV exposure that are quantitatively consistent with the data.

In our first analysis, we estimate monthly realized return volatilities for over 20,000 CRSP stocks firms over the 1926-2010 sample. The first principal component explains 39% of the variation in this panel. At first glance this may not appear surprising. A wide range of finance theories model returns as linear functions of common factors¹ – if the factors themselves have time-varying volatility, then firm-level volatility will naturally inherit a factor structure as well.

More surprising is that the firm-level volatility factor structure is effectively unchanged after accounting for common factors in returns. We examine residuals from factor models that include the Fama-French (1993) three factor, as well as statistical factor decompositions using as many as 10 principal components. Stock return residuals from these models possess an extremely high degree of common variation in their second moments. Residual volatility accounts for the vast majority (over 90%) of the variation in a typical stock's volatility, thus there is little distinction between total and idiosyncratic volatility at the firm level. Total and idiosyncratic volatility possess effectively the same volatility factor structure.

To emphasize that volatility comovement does not arise from omitted common factors,

¹Prominent examples include the CAPM (Sharpe (1964)), ICAPM (Merton (1973)), APT (Ross (1976)) and the Fama and French (1993) model.

we show that return factor model residuals are virtually uncorrelated. Consider returns on the 100 Fama-French size and value portfolios. The average pairwise correlation between returns on these portfolios is 64%. But their residuals from a Fama-French three factor model or a five factor principal components model have pairwise correlations below 1% on average. However, correlations among the monthly volatilities of the 100 portfolios is 75% on average, and volatility comovement remains extremely high after removing common factors from returns. The average correlation among the idiosyncratic volatilities of the 100 portfolios is 54% based on the Fama-French factors and 59% based on a five factor principal components model. Therefore, omitted factors are not a viable explanation for comovement in volatilities.

Comovement in volatilities is not only a feature of returns, but also of the volatility of fundamentals. We estimate volatilities of firm-level sales growth using quarterly Compustat data. Despite the fact that these volatility estimates are far noisier than the return data, we again find a strong factor structure among total fundamental volatilities as well as among volatilities of cash flow factor model residuals. We thus argue that volatility patterns identified in this paper are not wholly (or even primarily) explicable with investor preferences or other pure discount rate variation. We know of no extant model in the firm growth or asset pricing literature that generates a factor structure in both fundamental and return volatilities through an economic mechanism.

To explore the asset pricing implications of the factor structure in firm-level volatility in returns and cash flows, we develop a heterogeneous investor model in which the idiosyncratic volatility of investor consumption growth inherits the same factor structure as firm cash flow growth. In this model, an increase in idiosyncratic volatility represents a deterioration of the investment opportunity set for the average investor, whose individual consumption growth is exposed to the idiosyncratic volatility factor.

In a large class of Breeden-Lucas-Rubinstein representative agent models, aggregate volatility, albeit of aggregate consumption growth or the market return, can be a priced state vari-

able provided that she has a preference for early or late resolution of uncertainty, as pointed by Campbell (1993, 1996).² An increase in aggregate volatility raises the marginal utility of wealth for the stand-in investor if she has a preference for early resolution of uncertainty. The representative agent is willing to sacrifice some portion of her expected returns in exchange for insurance against a rise in volatility. However, she does not seek to hedge against innovations in idiosyncratic volatility (even if idiosyncratic volatility has common factor) because the idiosyncratic risk can be diversified.

We create a role for the idiosyncratic volatility factor by shutting down some markets. This breaks the aggregation result that is the foundation for all representative agent models. Instead, we propose an incomplete markets model in which investors' post-trade consumption is exposed to idiosyncratic risk. Our model is motivated by the incompleteness of household consumption insurance (see, e.g. Cochrane (1991), Attanasio and Davis (1996)). This market incompleteness implies that an increase in idiosyncratic risk at the firm level will carry over to the cross-sectional distribution of household consumption growth, via increased labor income risk, job loss risk and housing price risk that cannot be insured away. The local bias in households' financial portfolios (Coval and Moskowitz (1999)) exacerbates the exposure of their consumption to local, firm-driven shocks.

The main source of common variation in idiosyncratic shocks experienced by households and investors has to be the employer, the firm, and the labor income, broadly defined, that these investors derive from the firm. While there many other sources of idiosyncratic risk (e.g., illness, divorce), these types of risks are less likely to have a factor structure in the volatility. A large literature documents an idiosyncratic volatility factor in labor income.³

Motivated by this fact, we exogenously impose the same common factor structure on the idiosyncratic volatility of consumption growth and on firm dividend growth. Indeed,

²Campbell, Giglio, and Polk (2012) extend the closed-form solutions to handle stochastic volatility.

³Storesletten, Telmer, and Yaron (2004) document evidence of counter-cyclical variation idiosyncratic labor income variance, while Guvenen, Ozkan, and Song (2012) conclude that the left-skewness is counter-cyclical.

Cogley (2002) and Brav, Constantinides, and Geczy (2002) document a counter-cyclical factor structure in the cross-sectional variance of U.S. household consumption growth.⁴

In our model, the average investor wants to hedge against an increase in idiosyncratic volatility, even if she is indifferent about the timing of uncertainty resolution. As a result, the common component in idiosyncratic volatility is a priced state variable, as in Constantinides and Duffie (1996).⁵ Stocks with positive loadings on CIV shocks provide a hedge and earn a lower risk premium in equilibrium.

In general, exposure to constant uninsurable idiosyncratic risk will not affect equilibrium risk premia, but it will merely lower the risk-free rate and increase all securities prices in a large class of incomplete market models (see. e.g. Grossman and Shiller (1982) and Krueger and Lustig (2010)). Building on Mankiw (1986)’s insight, Constantinides and Duffie (1996) showed that counter-cyclical variation in idiosyncratic volatility can increase the equity risk premium in an equilibrium model with heterogeneous agents. It is exposure to the idiosyncratic volatility factor that is priced, not idiosyncratic volatility itself.

Measuring the cross-sectional dispersion in investors’ consumption growth is hard (see Vissing-Jorgensen (2002) and Brav, Constantinides, and Geczy (2002) for recent examples), but our model gives us a license to use the CIV factor in stock returns as the priced factor, because investor consumption growth inherits its factor structure from the firms’ cash flows. Only the common variation in the dispersion of investors’ consumption growth matters for cross-sectional asset pricing. We provide empirical evidence that the common factor in idiosyncratic firm volatility is a priced state variable in the cross-section of U.S. stocks with a negative risk price, as predicted by the model. This cross-sectional evidence directly lends support to models with investor heterogeneity, and it cannot be reconciled with standard, representative agent models.

⁴Advances in the risk sharing technology could disentangle consumption from labor income risk, a theme explored by Krueger and Perri (2006), but we abstract from this.

⁵In a standard representative agent model, aggregate consumption growth volatility is only a priced factor if the agent has a preference for early or late resolution of uncertainty (Campbell (1993)). Bansal and Yaron (2004) exploit this mechanism for generating an aggregate volatility risk premium in an endowment economy.

Our paper complements the evidence in Ang, Hodrick, Xing, and Zhang (2006, 2009) who find that exposure to market volatility is priced in the cross section of stocks. They also document that high idiosyncratic volatility stocks also earn low average returns. Our finding is distinct from the cross-sectional association between stock returns and exposure to market volatility or the level of idiosyncratic firm volatility. Instead, we establish that stocks with positive exposures to CIV earn lower returns, because, according to our model, the common idiosyncratic volatility in stock returns proxies for uninsurable investor consumption risk. Our calibrated model matches moments of the investor consumption and firm dividend distribution, the commonality in idiosyncratic volatility, and the spread in returns associated with exposures to the idiosyncratic volatility factor.

Other Related Literature Idiosyncratic volatility has been studied in several asset pricing contexts. Campbell, Lettau, Malkiel, and Xu (2001a) examine secular variation in average idiosyncratic volatility over time, though do not study the cross section properties of idiosyncratic volatility. This gave rise to several papers that explore this fact in more detail, such as Bennett, Sias, and Starks (2003), Irvine and Pontiff (2009), and Brandt, Brav, Graham, and Kumar (2010), including analyzing which firm characteristics correlate with its idiosyncratic volatility. Wei and Zhang (2006) study aggregate time series variation in fundamental volatility. Bekaert, Hodrick, and Zhang (2010) find comovement in average idiosyncratic volatility across countries. We analyze comovement among volatilities at the firm-level for both returns and fundamentals. Our focus is on the joint dynamics of the entire panel of firm-level volatilities, which we document is a prominent empirical feature of returns and growth rates that is new to the literature.

Gilchrist and Zakrajsek (2010) also study the time series behavior of the average firm-level volatility of the idiosyncratic component of returns. They explore the impact of uncertainty on corporate bond prices in a structural model. Recently, Atkeson, Eisfeldt, and Weill (2013) study the distribution of volatility across financial and non-financial firms to make inference

about their financial soundness, and confirm our findings. In closely related work, Engle and Figlewski (2012) document a common factor in option-implied volatilities since 1996, and Barigozzi, Brownlees, Gallo, and Veredas (2010) and Veredas and Luciani (2012) examine the factor structure in realized volatilities of intra-daily returns since 2001. Bloom, Floetotto, Jaimovich, and Terry (2012) shows that firm-specific sales growth and productivity show cross-sectional dispersion that fluctuates with the macroeconomy.

As long as cash flow volatility is idiosyncratic, it is valued by stock market investors (see, e.g., Pastor and Veronesi (2003, 2009)), because of the convex relation between cash flow growth variance and terminal value. In our model, the stocks of firms which experience high idiosyncratic cash flow volatility will endogenously inherit a larger exposure to the CIV factor, because a positive volatility innovation increases the value of the stock more than that of other stocks. This in turn will lower the stock's equilibrium risk premium and increase its valuation, potentially helping to resolve the Ang, Hodrick, Xing, and Zhang (2006, 2009) idiosyncratic risk puzzle.

In related work, Constantinides and Ghosh (2013) explore the asset pricing implications of a factor structure not only in the cross-sectional volatility of household consumption growth, but also in the higher-order moments, but they do not explore the cross-sectional asset implications, which are the focus of our work, and they do not connect the factor in the cross-sectional moments of consumption growth to the factor structure in firm-level cash flow growth.

To account for the factor structure in idiosyncratic volatility, Kelly, Lustig, and Van-Nieuwerburgh (2013) propose a simple model in which firms are connected to other firms in a customer-supplier network. Firms' idiosyncratic growth rate shocks, which are homoskedastic, are transmitted in part to their trading partners. Differences in firms' network connections, and evolution of the network over time, impart total firm volatility with cross section and time series heteroskedasticity. Firm-level volatilities exhibit a common factor structure where the factor is firm size dispersion in the economy. In their model, each sup-

plier’s network is a random draw from the entire population of firms, so that any firm’s customer network inherits similar dispersion to that of the entire size distribution. An increase in dispersion slows down every firm’s shock diversification and increases their volatility. They show that size dispersion explains 25% of the variation in (realized) firm volatilities, as much as is explained by average volatility, a natural benchmark.⁶

The rest of the paper is organized as follows. Section 2 describes the data, section 3 describes the CIV factor in U.S. stock returns and firm-level cash flows, while section 4 establishes that CIV is a priced factor. Section 5 describes the heterogeneous agent model with CIV as priced state variable. Finally, section 6 calibrates a version of this model.

2 Data

2.1 Data Construction

To document these facts, we present evidence in the form of firm-year volatility panels. Return volatility is estimated each year for each CRSP stock as the standard deviation of the roughly 250 daily returns within the year. Fundamental volatility is estimated each year for all Compustat firms using the four quarterly year-on-year sales growth observations within the year. We also show that our return volatility results are robust to using twelve monthly returns within each year rather than daily returns to calculate volatility. Similarly, we show that our fundamental volatility results are robust to estimating volatility with a five-year rolling window of quarterly observations (rather than one year of quarterly data),

⁶The factor structure in volatilities implies strong time series correlations between moments of the size and volatility distributions. An increase in the size dispersion translates into higher average volatility among firms. It also raises the cross-sectional dispersion in volatilities. In the time series, size dispersion has a 72% correlation with mean firm volatility and 79% with the dispersion of firm volatility. Kelly, Lustig, and VanNieuwerburgh (2013) is the first paper to provide an economic explanation for the factor structure in firm-level volatility by connecting it to firm concentration. A persistent widening in the firm size dispersion should lead to a persistent rise in mean firm volatility. The data display such a widening (increase in firm concentration) between the early 1960s and the late 1990s, providing a new explanation for the trend in mean firm volatility studied by Campbell et al. (2001) Campbell, Lettau, Malkiel, and Xu (2001b).

which reduces estimation noise.

The focus of our analysis is on idiosyncratic volatility. Idiosyncratic returns are constructed within each calendar year τ by estimating a factor model using all observation within that year (we estimate it for all firms with no missing observations during the year). Our factor models are of the form

$$r_{i,t} = \gamma_{0,i} + \boldsymbol{\gamma}'_i \mathbf{F}_t + \varepsilon_{i,t} \quad (1)$$

and use all date t return observations in the year (where the frequency of t is either daily or monthly). A firm's idiosyncratic volatility is then calculated as the standard deviation of residuals $\varepsilon_{i,t}$ within the calendar year. The result of this procedure is a panel of firm-year idiosyncratic volatility estimates. The first return factor model that we consider specifies $\mathbf{F}_t$ as the 3×1 vector of Fama-French (1993) factors.⁷ The second return factor model we use is purely statistical. In this case, $\mathbf{F}_t$ contains the first $K \times 1$ principal components of returns within the year, where we allow K to range between one and ten.

We estimate idiosyncratic volatility of firm fundamentals analogously. Since there is no single predominant factor model for sales growth in the literature, we only consider principal components as factors. The approach is the same as in equation (1), with the exception that the left hand side variable is sales growth, and the frequency of t is quarterly. $\mathbf{F}_t$ contains the first $K \times 1$ principal components of growth rates within a five-year window ending in year τ , and residual volatility in year τ is estimated from the four model residuals within that year. Again, the number of principal components K ranges from one to ten.

⁷Our estimates diverge slightly from the standard Fama-French model in which returns in excess of the risk free rate are the left-hand side variables, and the excess market return is the first factor. We use gross returns on the left-hand side, and the gross market return as the first factor.

3 The Factor Structure in Volatility

3.1 The Cross Section Distribution of Volatility

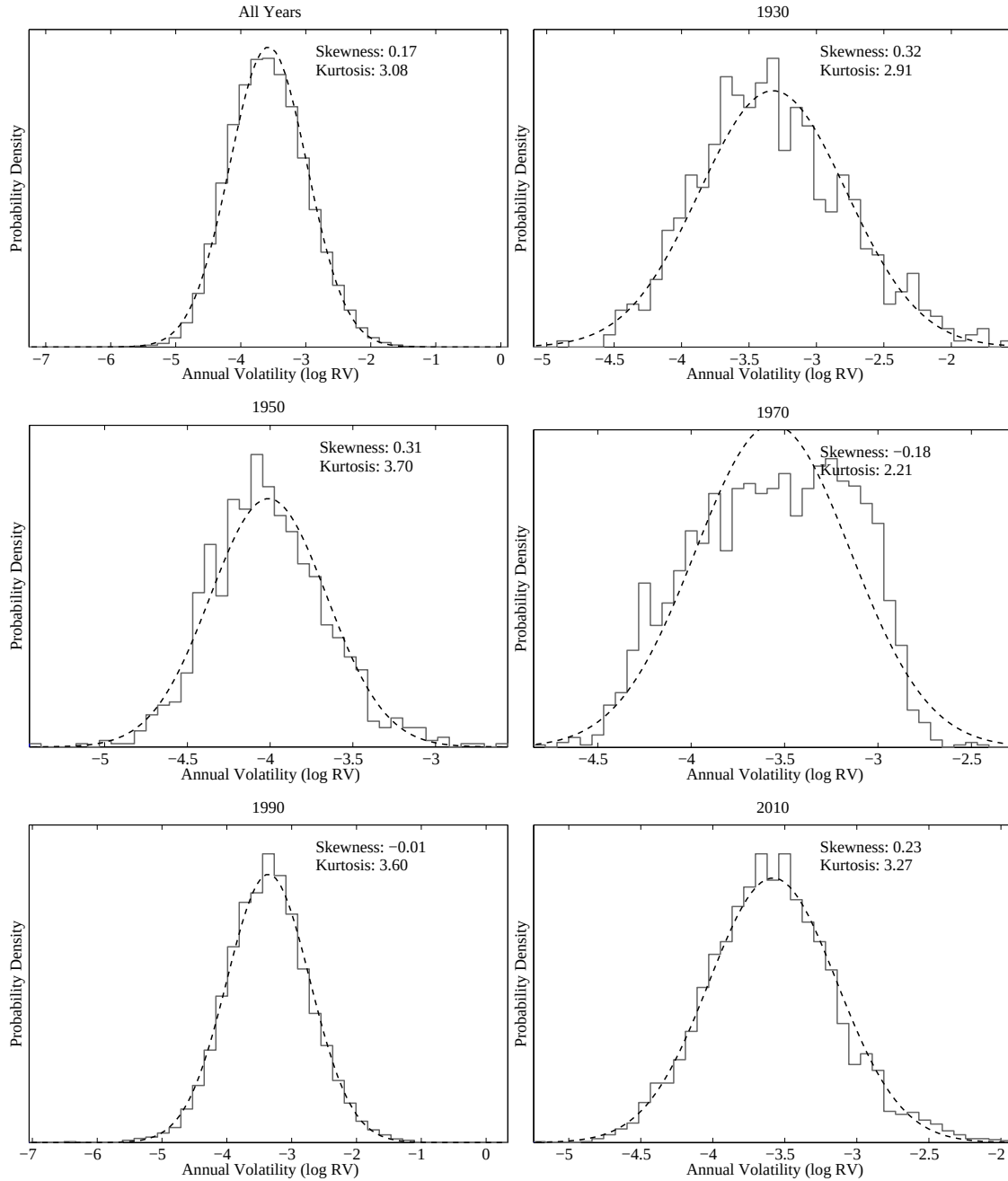
We begin by noting that the cross-sectional distributions of return volatility and fundamental volatility are lognormal to a close approximation, which motivates us to estimate our factor models using volatility in logs rather than levels. We plot histograms of the empirical cross section distribution of firm-level volatility (in logs). The upper left-hand corner of Figure 1 shows the distribution of log realized volatility pooling all firm-years from 1926-2010. The figure also shows empirical distributions for selected one-year snapshots throughout the sample (years 1930, 1950, 1970, 1990 and 2010). Overlaid on these histograms is the exact normal density with mean and variance set equal to that of the empirical distribution, and each figure reports the skewness and kurtosis of the data in the histogram. The pooled histograms and each of the snapshots (with the exception of 1970) look nearly normally distributed. They demonstrate only slight skewness (typically less than 0.3 in absolute value) and do not appear to be substantively leptokurtic. We obtain the same result for the cross-sectional distributions of yearly sales growth volatility (in logs) for all CRSP/Compustat firms. Fundamental volatility also appears to closely fit a lognormal distribution, with skewness no larger than 0.3 and kurtosis never exceeding 3.5. Figure 7 in the Appendix shows the results. Lognormality holds not only for total return and growth rate volatility, but also for residual volatility. Figure 8 in the Appendix shows the results.

3.2 Common Secular Patterns in Firm-Level Volatility

3.2.1 Return Volatility

Next, we document common time variation in volatility across stocks. Panel A of Figure 2 plots firm-level log total return volatility, averaged within size quintiles. There is a striking degree of common variation in the volatilities of the largest quintile and smallest quintile

Figure 1: TOTAL RETURN LOG VOLATILITY: EMPIRICAL DENSITY VERSUS NORMAL DENSITY



Notes: The figure plots histograms of the empirical cross section distribution of annual firm-level volatility (in logs). Within each calendar year, we calculate the standard deviation of daily returns for each stock. The upper left-hand corner histogram pools all years (1926-2010). Selected one-year snapshots of the firm volatility cross section distribution are also show (for years 1930, 1950, 1970, 1990 and 2010). Overlaid on these histograms is the exact normal density with mean and variance set equal to that of the empirical distribution. Each figure reports the skewness and kurtosis of the data in the histogram.

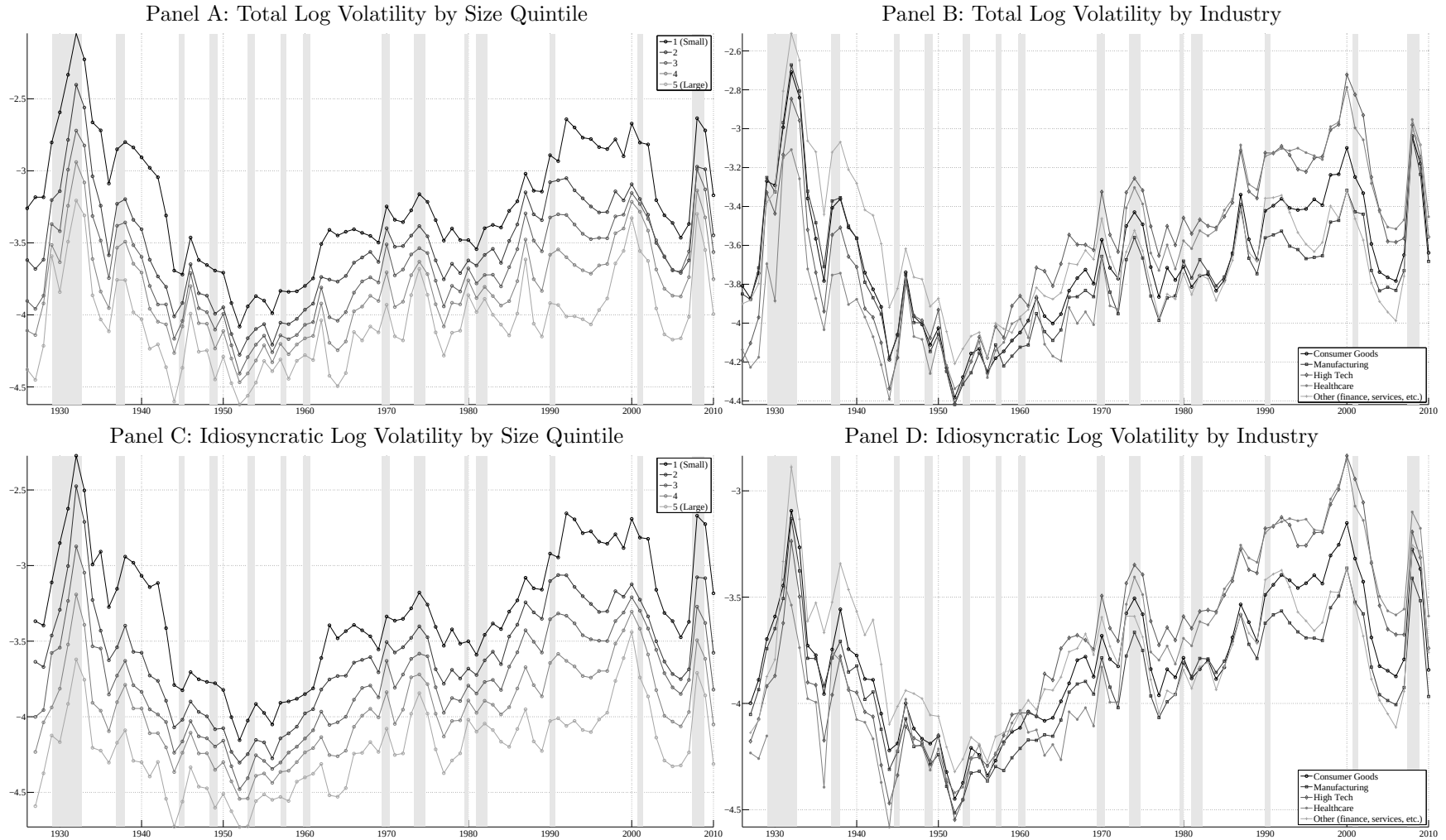
of stocks. The same is true of industry groups. Panel B reports average return volatilities among the stocks in the five-industry categorization of SIC codes provided on Ken French’s website. This is perhaps unsurprising given that firm-level returns are believed to have a substantial degree of common return variation, as evidenced by the predominance of factor-based models of individual stock returns. If returns have common factors and the volatility of those factors varies over time, then firm-level variances will also inherit a factor structure.

What is surprising is that volatilities of residuals display the same degree of common variation despite the fact that common return factors have been removed. Instead of averaging total volatility within size and industry groups, Panels C and D plot average residual volatility from a factor model that uses the first five principal components of returns as factors. The plots show that the same dynamics appear for all groups of firms when considering idiosyncratic rather than total volatility. The correlation between average log idiosyncratic volatility within size quintiles one and five is 83%. The lowest correlation among the five industry groups is 67%, which is for idiosyncratic volatilities of firms in the healthcare industry versus those in the “other” category (including construction, transportation, services, and finance).

This common variation is idiosyncratic and cannot be explained by excess comovement among factor model residuals, for instance due to omitted common factors. Figure 3 shows how firms’ idiosyncratic daily return volatility estimates are affected by using different factor models for returns. It compares raw returns to residuals from the Fama-French three factor model, as well as to residuals from a five factor principal components model. Panel A shows that raw returns share substantial common variation, with an average pairwise correlation of 13% over the 1926-2010 sample. However, the Fama-French model captures effectively all of this common variation at the daily frequency, as correlations among its residuals are less than 0.2% on average, and are never above 0.9% in a year. The same is true for the principal components model, whose residual correlation is also below 0.2% on average.

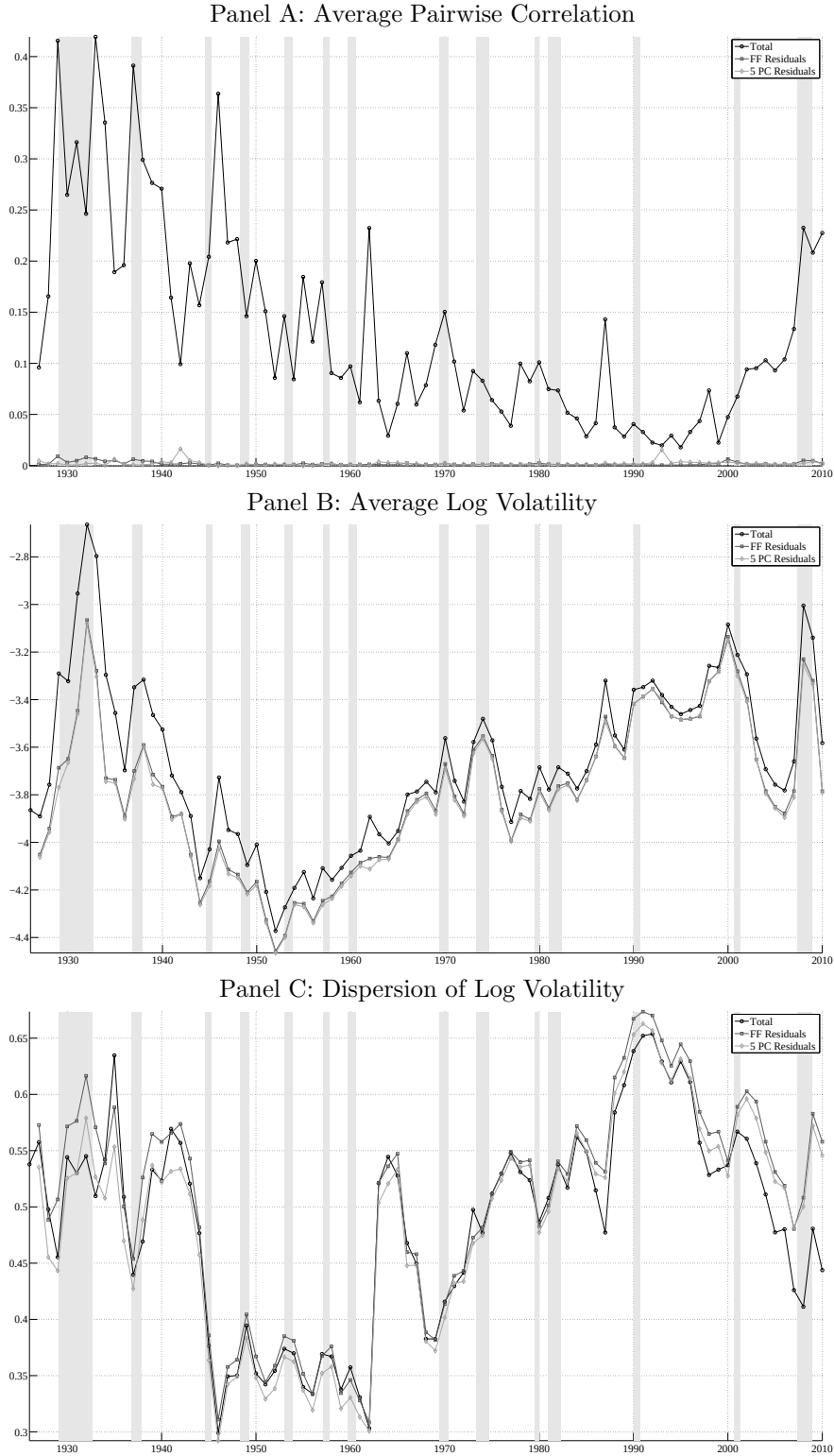
The interesting fact is that the common variation in returns, which is well accounted for

Figure 2: LOG TOTAL AND IDIOSYNCRATIC RETURN VOLATILITY BY SIZE AND INDUSTRY GROUP



Notes: The figures plot firm-level log volatility averaged within size and industry groups. Within each calendar year, volatilities are estimated as the standard deviation of daily returns for each stock. Panel A shows firm-level log total return volatility averaged within market equity quintiles. Panel B shows log total return volatility averaged within the five-industry categorization of SIC codes provided on Ken French's website. Panels C and D report the same within-group averages of firm-level log idiosyncratic volatility. Idiosyncratic volatility is the standard deviation of residuals from a five factor principal components model for daily returns.

Figure 3: VOLATILITY AND CORRELATION OF DAILY RETURNS



Notes: Panel A shows cross section average firm-level log volatility each year for total and idiosyncratic returns. Panel B shows the cross section standard deviation of firm-level log volatility and Panel C shows the average pairwise correlation for total and idiosyncratic returns within each calendar year. Idiosyncratic volatility is the standard deviation of residuals from the three factor Fama-French model or a five factor principal components model for daily returns (factor models also estimated within each calendar year).

by these factor models, is responsible for very little of the total variation in returns. Panel B of Figure 3 shows that the average log idiosyncratic volatility from the factor models is virtually the same as average volatility of total returns. In the typical year, only 4% of average log total volatility is accounted for by the five principal components factor model, while average log idiosyncratic volatility inherits 96% of the average total volatility level (the same is true for the Fama-French model).

Similarly, the dispersion in firms' log volatility is more or less unaffected by removing common factors, as shown in Panel C. Since log volatilities are approximately normally distributed, Panels B and C contain most of the relevant information about the cross section of firm volatilities over time. In short, commonalities among returns have very little influence on the commonalities in return volatilities. The cross section distribution of total volatility and idiosyncratic volatility are qualitatively identical.

The strong comovement of return volatility is similarly discernible in portfolio returns. The Appendix reports results for the 100 Fama-French size and value portfolios. Portfolio return volatilities show a strikingly similar degree of comovement across the size and book-to-market spectrum, even after accounting for common factors. Like the individual stock results above, factor models remove the vast majority of common variation in returns, thus common volatility patterns are unlikely to be driven by omitted common return factors.

One may be concerned that daily factor models miss some portion of common variation among returns due to non-synchronicity in when aggregate information is incorporated into individual stock prices. To address this, we re-estimate factor models and firm-level idiosyncratic volatilities using data at the monthly frequency, and re-plot average correlations, average volatilities, and the dispersion of volatilities in Figure 11 in the appendix. We find that, indeed, there is a higher correlation among monthly raw returns relative to daily returns. At the monthly frequency the Fama-French model continues to capture nearly all common variation, with correlations below 0.4% on average. The five factor principal components model has monthly residual correlation below 0.8% on average. At the monthly

frequency, average log idiosyncratic volatility inherits 78% of the average total volatility level. Thus, monthly return factor models do explain a larger fraction of the total return variation, but return volatility continues to be dominated by idiosyncratic rather than common variation. It is also worth noting that the bulk of the idiosyncratic volatility literature estimates volatility from daily data (e.g. Ang et al. (2006)).

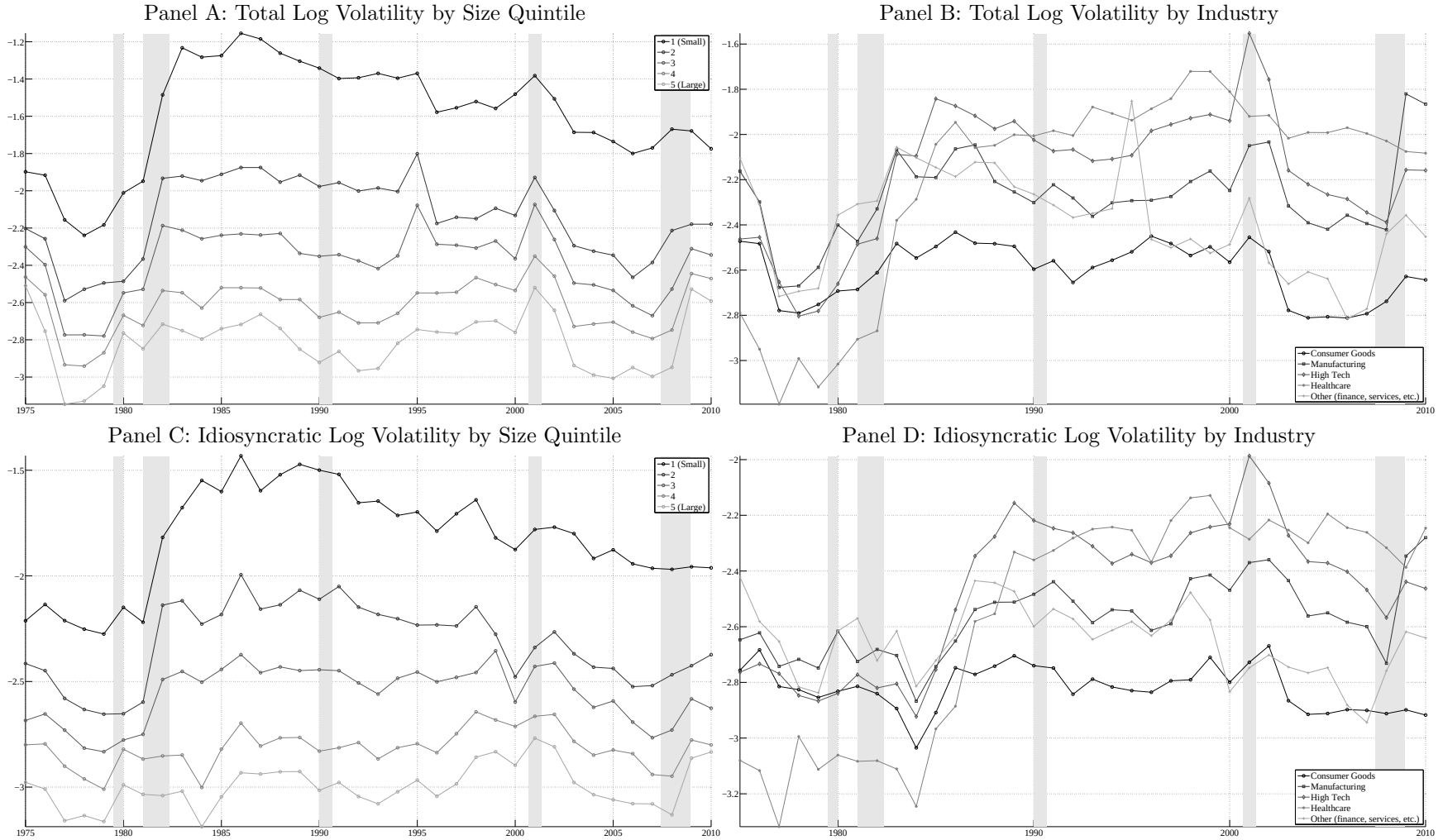
3.2.2 Fundamental Volatility

Strong comovement among volatilities is not distinct to return volatilities, but is also true for fundamental volatility. Figure 4 reports average yearly sales growth volatility (in logs) by size quintile and French's five-industry categories (Panels A and B). Despite the fact that yearly sale growth volatilities are estimated from only four observations per year, the data continues to display a high degree of volatility commonality. This is a feature of both total and residual volatility of fundamentals. Panels C and D show within-group average log idiosyncratic volatility estimated from a five factor principal components model for sales growth. These panels display the same volatility patterns as those in the top two panels.

A common factor model for firms' sales growth is perhaps less relevant than that for returns, as shown in Panel A of Figure 5. The average pairwise sales growth correlation in the 1975-2010 sample is only 2%, though it reaches as high as 17% in 2009. Accounting for common factors with a five principal component factor model lowers these correlations to below 0.3% on average, with correlations reaching a high of only 1% in 1980.

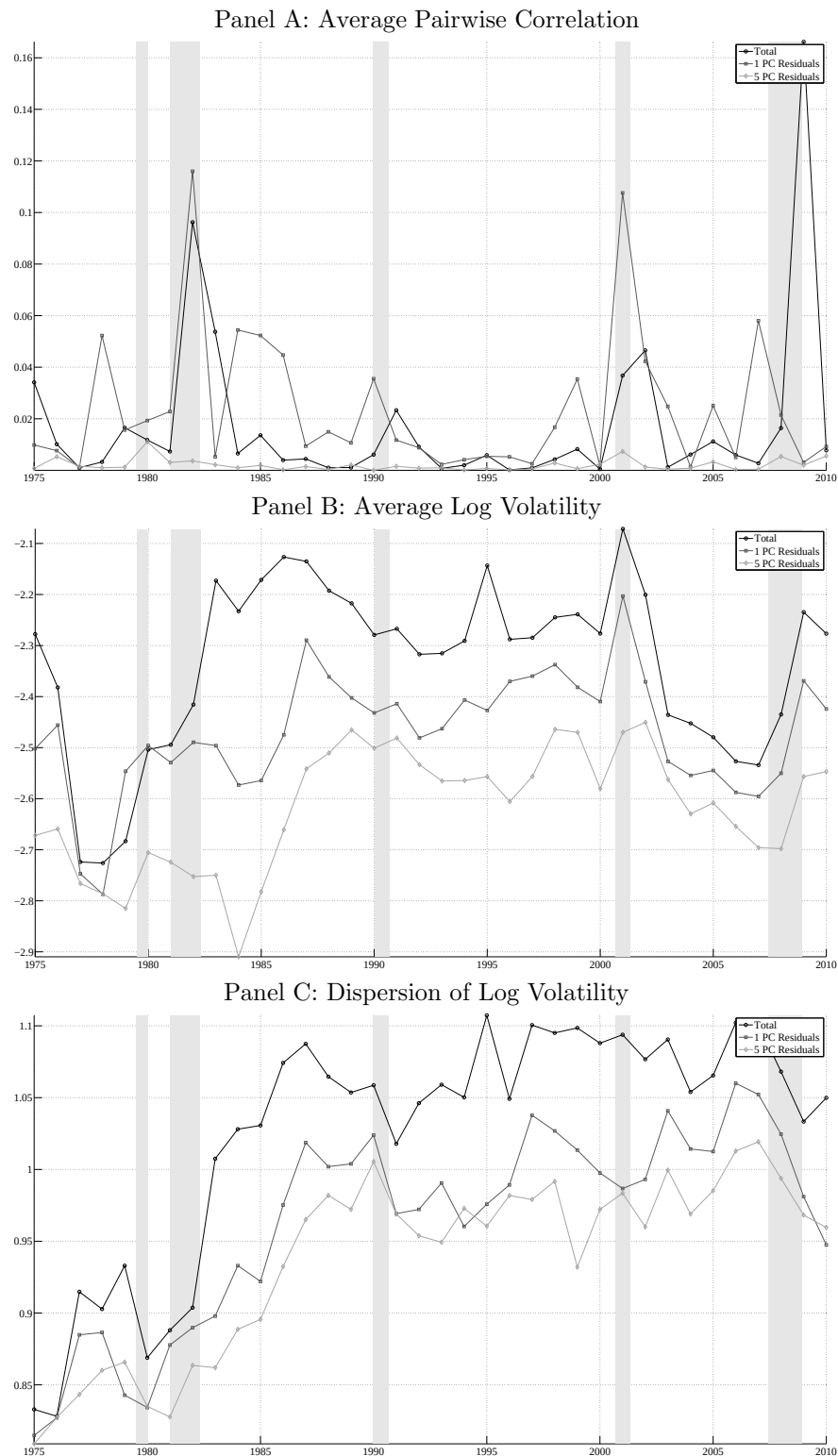
Panel B shows that, like returns, average idiosyncratic volatility of fundamentals shares the same broad pattern as total volatility (correlation of 59%), and inherits 89% of the average total volatility (11% is accounted for by the factor model). Given the near lognormality of sales growth volatility in the cross section, along with the same overall patterns between the cross section mean and standard deviation of for the total volatility and idiosyncratic volatility distribution, we conclude that idiosyncratic volatility rather than common

Figure 4: LOG TOTAL AND IDIOSYNCRATIC SALES GROWTH VOLATILITY BY SIZE AND INDUSTRY GROUP



Notes: The figures plot firm-level log volatility averaged within size and industry groups. Within each calendar year, volatilities are estimated as the standard deviation of four quarterly year-on-year sales growth observations for each stock. Panel A shows firm-level log total volatility averaged within market equity quintiles. Panel B shows log total return volatility averaged within the five-industry categorization of SIC codes provided on Ken French's website. Panels C and D report the same within-group averages of firm-level log idiosyncratic volatility. Idiosyncratic volatility is the standard deviation of residuals (four observation within each year) from a five factor principal components model for quarterly sales growth. The components are estimated in a five year rolling window ending in the year that the residual volatility is calculated.

Figure 5: VOLATILITY AND CORRELATION OF TOTAL AND IDIOSYNCRATIC SALES GROWTH



Notes: Panel A shows cross section average firm-level log volatility each year for total and idiosyncratic sales growth. Panel B shows the cross section standard deviation of firm-level log volatility and Panel C shows the average pairwise correlation for total and idiosyncratic returns within each calendar year. Idiosyncratic volatility is the standard deviation of residuals from a one or five factor principal components model for quarterly sales growth. The components are estimated in a five year rolling window ending in the year that the residual volatility is calculated.

variation drives the entire panel of firm-level fundamental volatilities.

3.3 Volatility Factor Model Estimates

Next, we estimate a one-factor model for volatility. We consider total volatility, as well as idiosyncratic volatility estimated from a Fama-French three factor model or a K factor principal component model with $K = 5$ or $K = 10$. In all cases, time series regressions are run firm-by-firm, use log volatility as the left-hand side variable, and the right-hand side factor is an equally weighted average of the left-hand side volatility measure across all firms. Our first set of results, shown in Panel A of Table 1, reports factor model results for daily return volatilities. Columns correspond to the factor model used to construct return residuals. The mean loading of an individual return volatility on the volatility factor is 0.925 for total return volatility, and is 0.920, 0.924 and 0.925 for idiosyncratic volatility based on the Fama-French, five PC and ten PC models, respectively. The average firm's intercept is between -0.181 and -0.201 . The average univariate time series R^2 is 38.5% for the total volatility model, and around 35% for idiosyncratic volatility models. Pooling all volatilities, we find a pooled R^2 34.9% and 36.3% (relative to a volatility model with only a firm-specific constant). Table 5 in the Appendix provides also the 25, 50, and 75th percentiles for the loadings and intercepts. That table shows that for portfolio rather than individual stock returns, we find even higher R^2 values. It also shows similar numbers for monthly returns than the ones we find for daily returns.

In Panel B we show volatility factor model estimates for sales growth volatility. The first three columns report total volatility, and idiosyncratic volatility from one and five principal component models in which volatility is estimated from four quarterly observations within each year. The last column reports model estimates for an annual volatility panel that uses a rolling 20 quarter window to estimate each firm-year's volatility. Due to the excessively small number of observations used to construct volatility, we might expect poorer fit in these

Table 1: LOG VOLATILITY FACTOR MODEL ESTIMATES

The table reports estimates for one factor regression models of yearly log volatility. In each panel, the single volatility factor is the equal weighted average of all firms' log volatilities within that year. Thus all estimated volatility factor models take the form: $\log \sigma_{i,t} = \text{intercept}_i + \text{loading}_i \cdot \overline{\log \sigma_{.,t}} + e_{i,t}$. Columns represent different volatility measures. For returns (Panel A), the first column represents estimates for a factor model of log total return volatility, the second column for idiosyncratic volatility based on Fama-French model residuals, and the third and fourth columns to idiosyncratic volatility from one and five factor principal component models. For sales growth volatility (Panel B), the last column reports model estimates for yearly volatilities estimated in a rolling 20 quarter window to reduce estimation noise. We report means and quantiles of the empirical distribution of firm-level intercepts and volatility factor loadings, as well as time series regression R^2 average over all firms. We also report a pooled factor model R^2 , which compares the estimated factor model to a model with only a firm-specific constant.

Panel A: Stock Returns	Total	FF	5 PCs	10 PCs
Loading (average)	0.925	0.920	0.924	0.925
Intercept (average)	-0.181	-0.201	-0.191	-0.190
R^2 (average univariate)	0.363	0.349	0.352	0.351
R^2 (pooled)	0.385	0.346	0.356	0.357
Panel B: Sales Growth	Total (1yr)	1 PC	10 PCs	Total (5yr)
Loading (average)	0.876	0.849	0.938	0.897
Intercept (average)	-0.231	-0.211	-0.096	-0.262
R^2 (average univariate)	0.140	0.229	0.127	0.144
R^2 (pooled)	0.174	0.168	0.167	0.283

regressions. Yet the results are closely in line with those for return volatility. The average firm has a volatility factor loading of between 0.849 and 0.938, with an intercept between -0.096 and -0.262 . The time series R^2 for raw and idiosyncratic growth rate volatility ranges between 12.7% and 22.9% on average. The pooled R^2 reaches as high as 28.3% when volatilities are estimated in a 20 quarter window.

4 Cross Section of Stock Returns

In this section we investigate whether pervasive fluctuations in firms' idiosyncratic volatility are associated with differences in average returns across stocks. We construct a common idiosyncratic variance factor, CIV, as the equal weighted average of CAPM residual variances

computed each month.⁸ We then calculate innovations to CIV based on a monthly AR(1) model. Finally, we orthogonalize these innovations with respect to innovations in monthly market variance. This orthogonalization disentangles our CIV exposures from the market variance exposures studied by Ang et al. (2006). Each month from 1963 until 2010, we regress monthly individual firm stock returns on orthogonalized CIV innovations using a trailing 60-month window. These CIV betas are used to sort stocks into CIV quintile portfolios, for which we construct value-weighted returns over the subsequent month.

Value-weighted average returns on CIV beta-sorted portfolios are reported in Table 2. The first row of Panel A shows that average returns are decreasing in CIV beta. Stocks in the lowest quintile (Q1) have low/negative CIV betas and tend to lose value when CIV rises. In contrast, stocks in Q5 tend to hedge CIV rises, paying off in high volatility states. The spread between highest and lowest quintiles is -6.4% per year with a *t*-statistic of -3.4. The spread in average returns is robust to controlling for market returns, SMB and HML, as shown in rows 2 and 3 of Panel A.

In Panel B we report portfolios sorted independently on CIV beta and market variance beta, where rows correspond to the market variance beta dimension. This provides a comparison with the results of Ang et al. (2006). We see that high CIV beta stocks continue to earn substantially lower average returns within each market beta quintile. The 5-1 CIV beta spread ranges between -4.2% and -8.0% per year depending on the market variance beta quintile, and is significant at the 10% level or better for all quintiles (at the 5% level for 4 quintiles out of five).

Panel C reports bivariate sorts based on CIV beta and stock-level idiosyncratic volatility. The CIV beta is between -5.3% and -6.2% per year depending on the market variance beta quintile, and has a *t*-statistic of at least two in all cases. Table 7 in the Appendix shows similar results for equal-weighted rather than value-weighted portfolio returns.

⁸We estimate the CAPM each month using all daily returns within the month. The CAPM residual variance is then estimated from the output of this regression. Results are qualitatively identical if residuals from alternative factor models, such as Fama-French or principal components, are used.

Table 2: VALUE-WEIGHTED PORTFOLIOS FORMED ON CIV BETA

The table reports results for value-weighted portfolio sorts on the basis of monthly CIV beta. Results are reported as annual percentages. Panel A shows one-way sorts using all CRSP stocks. Panel B shows independent two-way sorts on CIV beta and market variance beta. Panel B shows independent two-way sorts on CIV beta and idiosyncratic stock variance.

Panel A: Average returns and alphas in one-way sorts on CIV beta							
	CIV beta						
	1 (Low)	2	3	4	5 (High)	5-1	$t(5-1)$
$E[R]$	15.23	12.39	11.71	10.55	8.80	-6.44	-3.42
α_{CAPM}	3.38	1.47	1.14	0.27	-1.95	-5.33	-2.91
α_{FF}	2.32	0.84	0.94	0.22	-1.97	-4.28	-2.33

Panel B: Average returns in two-way sorts on CIV beta and MV beta							
	CIV beta						
MV beta	1 (Low)	2	3	4	5 (High)	5-1	$t(5-1)$
1 (Low)	16.05	14.50	11.72	11.60	9.37	-6.69	-2.55
2	14.47	13.42	11.55	11.49	10.25	-4.22	-1.91
3	16.67	12.98	13.51	11.27	10.91	-5.76	-2.48
4	17.17	11.26	10.81	9.26	9.12	-8.05	-2.95
5 (High)	14.48	12.88	10.84	10.86	8.72	-5.76	-1.96
5-1	-1.57	-1.63	-0.87	-0.73	-0.64		
$t(5-1)$	-0.54	-0.52	-0.29	-0.25	-0.22		

Panel C: Average returns in two-way sorts on CIV beta and idios. var.							
	CIV beta						
Idios. var.	1 (Low)	2	3	4	5 (High)	5-1	$t(5-1)$
1 (Low)	14.11	11.83	12.69	11.45	8.57	-5.54	-2.70
2	15.88	12.94	12.19	10.47	9.81	-6.07	-3.00
3	17.00	14.90	12.30	11.83	10.66	-6.34	-2.69
4	15.10	12.77	10.96	11.56	9.73	-5.37	-2.37
5 (High)	7.13	8.10	7.04	3.74	1.19	-5.94	-2.14
5-1	-6.98	-3.73	-5.65	-7.71	-7.38		
$t(5-1)$	-2.05	-1.08	-1.53	-2.14	-1.94		

5 Model

This section studies an equilibrium asset pricing model with heterogeneous agents in the spirit of Constantinides and Duffie (1996) and Constantinides and Ghosh (2013). The common idiosyncratic volatility factor, denoted σ_{gt}^2 , is the key state variable which drives the residual return volatility in stocks, as well the the cross-sectional volatility of investor consumption

growth. Innovations to this factor are priced, with a negative price of risk. Stocks with more negative exposure with respect to this innovation (a more negative “CIV beta”) carry a higher risk premium. To keep the model simple and to highlight the differences with the existing literature, aggregate stock market volatility and aggregate consumption growth volatility are constant over time.

5.1 Preferences

There is a unit mass of atomless agents. Each one of them has Epstein-Zin preferences. Let $U_t(C_t)$ denote the utility derived from consuming C_t . The value function of each agent takes the following recursive form:

$$U_t(C_t) = \left[(1 - \delta) C_t^{\frac{1-\gamma}{\theta}} + \delta (E_t U_{t+1}^{1-\gamma})^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}}.$$

The time discount factor is δ , the risk aversion parameter is $\gamma \geq 0$, and the inter-temporal elasticity of substitution (IES) is $\psi \geq 0$. The parameter θ is defined by $\theta \equiv (1 - \gamma)/(1 - \frac{1}{\psi})$. When $\psi > 1$ and $\gamma > 1$, then $\theta < 0$ and agents prefer early resolution of uncertainty.

Aggregate labor income is defined as I_t . There is a large number of securities in zero or positive net supply. Their combined total (and per capita) dividends are D_t . Aggregate dividend income plus aggregate labor income equals aggregate consumption: $C_t = I_t + D_t$. Individual labor income is defined by

$$I_{j,t} = S_t^j C_t - D_t$$

All agents can trade in all securities at all times and are endowed with an equal number of all securities at time zero. As in Constantinides and Ghosh (2013), given the symmetric and homogenous preferences, households choose not to trade away from their initial endowments. That is, autarky is an equilibrium and individual j ' equilibrium consumption is $C_{j,t} =$

$$I_{j,t} + D_t = S_t^j C_t.$$

5.2 Technology

On the technology side, we impose the same idiosyncratic volatility factor structure on investor consumption growth and firm dividend growth by adopting the following specification for aggregate consumption growth, consumption growth of agent j , and dividend growth of firm i :

$$\Delta c_{t+1}^a = \mu_g + \sigma_c \eta_{t+1} + \phi_c \sigma_g w_{g,t+1} \quad (2)$$

$$\Delta s_{t+1}^j = \sigma_{g,t+1} v_{t+1}^j - \frac{1}{2} \sigma_{g,t+1}^2 \quad (3)$$

$$\Delta d_{t+1}^i = \mu_i + \chi^i (\sigma_{gt}^2 - \sigma_g^2) + \varphi_i \sigma_c \eta_{t+1} + \phi_i \sigma_g w_{g,t+1} + \kappa_i \sigma_{gt} e_{t+1}^i + \zeta_i \sigma_{it} \varepsilon_{t+1}^i \quad (4)$$

$$\sigma_{g,t+1}^2 = \sigma_g^2 + \nu_g (\sigma_{gt}^2 - \sigma_g^2) + \sigma_w \sigma_g w_{g,t+1} \quad (5)$$

$$\sigma_{i,t+1}^2 = \sigma_i^2 + \nu_i (\sigma_{it}^2 - \sigma_i^2) + \sigma_{iw} w_{i,t+1} \quad (6)$$

All shocks are i.i.d standard normal and mutually uncorrelated. Lowercase letters denote logs. The cross-sectional mean and variance of the consumption share process are:

$$\begin{aligned} \mathbb{E}_j [\Delta s_{t+1}^j] &= -\frac{1}{2} \sigma_{g,t+1}^2 \\ \mathbb{V}_j [\Delta s_{t+1}^j] &= \sigma_{g,t+1}^2 \end{aligned}$$

Thus, the process $\sigma_{g,t+1}$ measures the cross-sectional standard deviation of consumption share growth. Individual consumption growth is $\Delta c_{t+1}^j = \Delta c_{t+1}^a + \Delta s_{t+1}^j$. The mean consumption share in levels is one: $\mathbb{E}_j [S_t^j] = 1$. The conditional variance of aggregate consumption growth is constant.

Dividend growth for an individual stock i contains four shocks. The first two are systematic sources of risk, while the last two are idiosyncratic sources of risk. The market portfolio

obviously only contains the former two shocks. The conditional variance of dividend growth on the market portfolio is constant since the first two shocks have constant variances. All time variation in the conditional variance of dividend growth of stock i arises from time variation in the idiosyncratic variance, $\kappa_i^2 \sigma_{gt}^2 + \zeta_i^2 \sigma_{it}^2$. As we show below, the σ_{gt}^2 process is proportional to the common idiosyncratic variance (CIV) factor. Hence the model links the CIV factor to the cross-sectional dispersion of consumption share growth. We now show that positive innovations in the CIV factor ($w_{g,t+1} > 0$) are associated with bad times and carry a negative price of risk. Thus, bad times are times with little risk sharing. Assets whose returns are low exactly when risk sharing is impaired must pay higher risk premia.

5.3 Claim to Individual Consumption Stream

We start by pricing a claim to individual consumption growth, using the individual's own intertemporal marginal rate of substitution (IMRS). We conjecture that the log wealth-consumption ratio of agent j is linear in the state variable σ_{gt}^2 , and does not depend on any agent-specific characteristics:

$$w\mathcal{C}_t^j = \mu_{wc} + W_{gs} (\sigma_{gt}^2 - \sigma_g^2)$$

We verify this conjecture by plugging in this guess into the Euler equation for the consumption claim of agent j : $E_t[SDF_{t+1}^j R_{t+1}^j] = 1$. Under symmetric preferences, this conjecture implies that the individual wealth-consumption ratio does not depend on agent-specific attributes, only on aggregate objects.

The return to agent j 's consumption claim r_{t+1}^j equals:

$$r_{t+1}^j = r_0^c + \left[W_{gs} (\nu_g - \kappa_1^c) - \frac{1}{2} \nu_g \right] (\sigma_{gt}^2 - \sigma_g^2) + \sigma_c \eta_{t+1} + \left(\phi_c + W_{gs} \sigma_w - \frac{1}{2} \sigma_w \right) \sigma_g w_{g,t+1} + \sigma_{g,t+1} v_{t+1}^j$$

where r_0^c is the unconditional mean and κ_1^c is a linearization constant slightly exceeding 1.

The intermediate steps are provided in the appendix, along with all pother derivations.

Epstein and Zin (1989) show that the log real stochastic discount factor is a function of consumption growth and the return to the consumption claim:

$$\begin{aligned}
sdf_{t+1}^j &= \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1}^j + (\theta - 1) r_{t+1}^j \\
&= \mu_s + \left[(\theta - 1) W_{gs} (\nu_g - \kappa_1^c) + \frac{1}{2} \gamma \nu_g \right] (\sigma_{gt}^2 - \sigma_g^2) \\
&\quad - \gamma \sigma_c \eta_{t+1} - \gamma \sigma_{g,t+1} v_{t+1}^j + \left[(\theta - 1) W_{gs} \sigma_w - \gamma \phi_c + \frac{1}{2} \gamma \sigma_w \right] \sigma_g w_{g,t+1}
\end{aligned}$$

where μ_s is the unconditional mean SDF.⁹

5.4 Aggregate SDF

Since all agents can invest in all risky assets, the Euler equation has to be satisfied for any two agents j and j' and for every stock i (with returns orthogonal to the agents' idiosyncratic income shocks v^j and $v^{j'}$). This also implies that the average SDF must also price all financial assets if all the individual SDFs price the return R_{t+1}^i :

$$\begin{aligned}
1 &= \mathbb{E}_t [SDF_{t+1}^j R_{t+1}^i] = \mathbb{E}_t [\mathbb{E}_j (SDF_{t+1}^j R_{t+1}^i)] = \mathbb{E}_t [\mathbb{E}_j (SDF_{t+1}^j) R_{t+1}^i] \\
&= \mathbb{E}_t [SDF_{t+1}^a R_{t+1}^i].
\end{aligned}$$

We can write the expression for the average log real stochastic discount factor as:

$$sdf_{t+1}^a = \mu_s + \frac{1}{2} \gamma^2 \sigma_g^2 + s_{gs} (\sigma_{gt}^2 - \sigma_g^2) - \lambda_\eta \sigma_c \eta_{t+1} - \lambda_w \sigma_g w_{g,t+1}$$

⁹The appendix shows that the coefficient W_{gs} is given by:

$$W_{gs} = -\frac{\gamma \nu_g \left(1 - \frac{1}{\psi}\right)}{2 (\kappa_1^c - \nu_g)} \quad (7)$$

If the IES ψ exceeds 1, then $W_{gs} < 0$. Hence, higher consumption share dispersion (less risk sharing) leads to a lower wealth-consumption ratio.

where the loadings are given by:

$$\begin{aligned} s_{gs} &\equiv \frac{1}{2}\gamma\nu_g\left(\frac{1}{\psi}+1\right), \\ \lambda_\eta &\equiv \gamma, \\ \lambda_w &\equiv -\frac{1}{2}\gamma(1+\gamma)\sigma_w + \gamma\phi_c + \frac{\gamma\nu_g\left(\frac{1}{\psi}-\gamma\right)}{2(\kappa_1^c-\nu_g)}\sigma_w, \end{aligned}$$

Hence, there are two priced sources of aggregate risk in our model: shocks to aggregate consumption growth, which carry a positive price of risk λ_η , equal to the coefficient of relative risk aversion, and shocks to the idiosyncratic volatility factor. The latter carry a negative price of risk λ_w , indicating that deterioration in risk sharing is bad news for the stand-in agent, provided that the agent has a preference for early resolution of uncertainty. All three terms in the λ_w expression are negative. The first term is always present. The second term naturally associate recessions (negative aggregate consumption growth episodes) with periods where risk sharing deteriorates ($\phi_c < 0$). The third term requires Epstein-Zin preferences. In the case of time-additive utility, the standard Mankiw (1986) result obtains and only the current volatility matters. But, in general with EZ preferences, the average investor also cares about the future dispersion of consumption growth. The size of this effect is governed by the persistence (ν_g) of the idiosyncratic volatility factor. This result will allow us to directly use the volatility factor constructed from stock returns to test our asset pricing mechanism empirically rather than measure the cross-sectional dispersion of investor consumption growth directly.

The maximum Sharpe ratio in the economy is larger, the bigger these prices of risk and the more volatile the shocks:

$$\max SR_t = Std_t[sdf_{t+1}^a] = \sqrt{\lambda_\eta^2\sigma_c^2 + \lambda_w^2\sigma_g^2}$$

It follows that the risk-free interest rate is:

$$r_t^f = -\mu_s - \frac{1}{2}\gamma^2\sigma_g^2 - \frac{1}{2}\lambda_\eta^2\sigma_c^2 - \frac{1}{2}\lambda_w^2\sigma_g^2 - s_{gs}(\sigma_{gt}^2 - \sigma_g^2)$$

Interest rates contain the usual impatience and intertemporal substitution terms, included in μ_s . The next three terms capture the precautionary savings motive: when idiosyncratic risk is high, agents increase savings, lowering interest rates. Interest rates move negatively with the state variable because $s_{gs} > 0$. The higher consumption share dispersion, the lower rates.

5.5 Firm Stock Return

Turning to the pricing of the dividend claim defined by equation (4), we guess and verify that its log price-dividend ratio is affine in the common and idiosyncratic variance terms:

$$pd_t^i = \mu_{pdi} + A_{gs}^i(\sigma_{gt}^2 - \sigma_g^2) + A_{is}^i(\sigma_{it}^2 - \sigma_i^2)$$

As usual, log returns are approximated as:

$$r_{t+1}^i = \Delta d_{t+1}^i + \kappa_0^i + \kappa_1^i pd_{t+1}^i - pd_t^i$$

Innovations in individual stock returns and the return variance reflect the additional sources of idiosyncratic risk:

$$\begin{aligned} r_{t+1}^i - \mathbb{E}_t[r_{t+1}^i] &= \beta_{\eta,i}\sigma_c\eta_{t+1} + \beta_{gs,i}\sigma_g w_{g,t+1} + \kappa_i\sigma_{gt}e_{t+1}^i + \zeta_i\sigma_{it}\varepsilon_{t+1}^i + \kappa_1^i A_{is}^i\sigma_{iw}w_{i,t+1} \\ \mathbb{V}_t[r_{t+1}^i] &= \beta_{\eta,i}^2\sigma_c^2 + \beta_{gs,i}^2\sigma_g^2 + (\kappa_1^i A_{is}^i)^2\sigma_{iw}^2 + \kappa_i^2\sigma_{gt}^2 + \zeta_i^2\sigma_{it}^2 \end{aligned}$$

where

$$\beta_{\eta,i} \equiv \varphi_i, \quad \beta_{gs,i} \equiv \kappa_1^i A_{gs}^i \sigma_w + \phi_i \quad (8)$$

Innovations in stock returns contain two sources of aggregate risk and three sources of idiosyncratic risk (equation 8). The variance of idiosyncratic stock returns are driven by the common σ_{gt} and firm-specific σ_{it} processes (equation 8). In the empirical section, we demonstrated the presence of a large first principal component in both total and residual stock returns, and showed that it was the same component in both. We also demonstrated that total and residual volatility at the firm-level were nearly identical. This model generates these features. It associates the common component in residual variance with changes in the cross-sectional dispersion of consumption growth across agents. Times of low risk sharing are times of high idiosyncratic (and total) stock return variance.

The coefficients of the price-dividend equation are obtained from the Euler equation:

$$A_{gs}^i = \frac{2s_{gs} + 2\chi_i + \kappa_i^2}{2(1 - \kappa_1^i \nu_g)} = \frac{2\chi_i + \kappa_i^2 + \left(1 + \frac{1}{\psi}\right) \gamma \nu_g}{2(1 - \kappa_1^i \nu_g)} \quad (9)$$

$$A_{is}^i = \frac{\zeta_i^2}{2(1 - \kappa_1^i \nu_i)} \quad (10)$$

The expression for the equity risk premium on an individual stock is:

$$\mathbb{E}_t \left[r_{t+1}^i - r_t^f \right] + .5 \mathbb{V}_t[r_{t+1}^i] = \beta_{\eta,i} \lambda_\eta \sigma_c^2 + \beta_{gs,i} \lambda_w \sigma_g^2. \quad (11)$$

The first term is the standard consumption CAPM term. The second term is a new term which compensates investors for movements in the cross-sectional (income and) consumption distribution, today and in the future. Stocks that have low returns exactly when risk sharing deteriorates ($\beta_{gs,i} < 0$) are risky and carry high expected returns because $\lambda_w < 0$. There are three channels that make a stock's $\beta_{gs,i} < 0$.

First, if χ^i is sufficiently negative, $A_{gs}^i < 0$ and $\beta_{gs,i} < 0$. A negative χ_i is natural because empirically, poor risk sharing are bad economic times that are associated with lower future dividend growth. We present evidence on this dividend growth predictability below.

Second, if contemporaneous deteriorations in risk sharing coincide with low dividend growth realizations ($\phi_i < 0$), then that stock has a lower (more negative) $\beta_{gs,i}$. Given the negative price of risk, it will carry a higher expected return. Hence, the contemporaneous cash-flow effects ($\phi_c < 0$ and $\phi_i < 0$) increase the equity risk premium, all else equal.

Third, a stock with lower exposure to the common idiosyncratic risk term κ_i will have a lower beta and therefore carry a higher expected return, all else equal. Intuitively, when there is a positive shock to the volatility factor, the quantity of idiosyncratic risk goes up more so for stocks with greater exposure κ_i . As a result of the convexity in the relation between growth and terminal value, also explored by Pastor and Veronesi (2003, 2009)), this in turn raises the price more of those high volatility stocks, thus increasing their beta to the vol factor and lowering their equilibrium expected return.

The following objects are useful in what follows. The idiosyncratic stock return variance is the variance of the idiosyncratic return components:

$$\mathbb{V}_t \left[r_{t+1}^{idio,i} \right] = (\kappa_1^i A_{is}^i)^2 \sigma_{iw}^2 + \kappa_i^2 \sigma_{gt}^2 + \zeta_i^2 \sigma_{it}^2$$

Define the common idiosyncratic variance (CIV) factor as the first principal component of the idiosyncratic return variance, as in section 3:

$$CIV_t \equiv \mathbb{E}_i \left[\mathbb{V}_t \left[r_{t+1}^{idio,i} \right] \right] = \bar{\kappa}^2 \sigma_{gt}^2 + \bar{\zeta}^2 \sigma_i^2,$$

where the last term is constant by virtue of the i.i.d. nature of the σ_{it} processes in a (large) cross-section of stocks, and we define $\bar{\kappa}^2 \equiv \mathbb{E}_i[\kappa_i^2]$ and $\bar{\zeta}^2 \equiv \mathbb{E}_i[\zeta_i^2]$.¹⁰ Thus, in terms of

¹⁰in computing CIV in the model, we ignore the cross-sectional average of $(\kappa_1^i A_{is}^i)^2 \sigma_{iw}^2$, which is very small in our calibration.

the dynamics, CIV is proportional to the consumption growth share dispersion process σ_{gt}^2 , where $\bar{\kappa}^2$ is the constant of proportionality.

6 Calibration and Results

Table 3 shows our parameter choices; the model is calibrated to data for the 1963-2010 period and simulated at annual frequency. Risk aversion γ is set to 15 and the inter-temporal elasticity of substitution ψ is set to 2.¹¹ The time discount factor δ is set to produce a mean real risk-free rate of 1.5% per year, given all other parameters. The model produces a risk-free rate with modest volatility of 2.4% per year. Mean consumption growth μ_g is 2% per year and σ_c is 2% per year. We set ϕ_c equal to -0.04 to capture the negative correlation between aggregate consumption growth and the degree of risk sharing. Aggregate consumption growth volatility is modest at 2.5% per year.

We set the mean of the cross-sectional dispersion in consumption growth, σ_g , to 38%. This is the value for the cross-sectional dispersion in consumption growth we measure in the Consumption Expenditure Data (CEX) for the period 1984-2011. The persistence of the cross-sectional dispersion process, ν_g , is set to 0.6 per year, a value equal to the annual persistence of the CIV factor in the data. This choice implies that our main state variable moves at business cycle rather than at much lower frequencies. We set σ_w to 0.74%. This ensures that σ_{gt}^2 remains positive. The time series standard deviation of σ_{gt} is 0.46%. The model results in a negative market price of “dispersion risk” $\lambda_w = -1.54$ and a substantial maximum conditional Sharpe ratio of 0.66.

As shown in Section 3, stocks’ whose returns have a more negative exposure to CIV innovations earn higher average returns. To represent the typical stock in each of the CIV-

¹¹The high value for γ is needed to generate a high equity risk premium for the market portfolio. It is not required to generate cross-sectional dispersion in risk premia across CIV beta-sorted portfolios. A richer model with time variation in market variance would allow us to match the equity risk premium with lower risk aversion, but at the expense of higher model complexity.

Table 3: CALIBRATION

This table lists the parameters of the model. The last panel discusses the calibration of five stock portfolios, sorted from lowest volatility (Q1) to highest volatility (Q5). The market portfolio is indicated by the letter M.

Preferences						
δ	0.19	γ	15	ψ	2	
Aggregate Consumption Growth Process						
μ_g	0.02	σ_c	0.02	ϕ_c	-0.0395	
Consumption Share Process						
σ_g	0.38	ν_g	0.6	σ_w	0.007404	
Dividend Growth Process						
σ_i	0.004	ν_i	0.15	σ_{iw}	1.5e-06	
Parameter	Q1	Q2	Q3	Q4	Q5	M
μ_i	7.42%	3.44%	5.45%	5.42%	6.35%	5.20%
φ_{di}	8.61	8.61	8.61	8.61	8.61	8.61
ϕ_i	-0.31	-0.20	-0.15	-0.11	-0.03	-0.12
χ^i	-0.42	-0.15	-0.16	-0.15	-0.00	-0.15
κ_i	1.34	1.00	0.91	0.91	1.10	$\times$
ζ_i	94.39	71.31	65.10	64.92	78.57	$\times$

beta sorted quintile portfolios, we solve our model for five assets that differ in terms of their cash-flow growth process (4). We also consider the market portfolio, which is an asset whose cash flow growth has no idiosyncratic shocks (no e_{t+1} nor ε_{t+1} terms). We set mean dividend growth μ_i equal to the values observed for the CIV beta-sorted portfolios in the data.

We set φ_{di} , a standard consumption leverage parameter, equal to 8.61 for all portfolios. By setting this parameter equal for all portfolios, we impose that all differences in risk premia across portfolios arise from differences in exposure to the $w_{g,t+1}$ shocks. The choice of 8.61 is such that the model exactly matches the equity risk premium for the market portfolio of 5.50% exactly, given all other parameters. The contribution to the equity risk premium from the η -term, the first term in equation (11), is 5.16% per year. The other parameters we hold fixed across portfolios are the parameters governing the σ_{it} process in equation (??). We set σ_i to 0.4%, ν_i to 0.15, and σ_{iw} to 1.5e-6. The persistence of σ_{it} is much lower than that of σ_{gt} , mirroring the data. σ_{iw} is chosen to prevent σ_{it} from going negative in simulations. Finally, the value for σ_i is chosen to match the mean of the observed CIV process of 0.254,

given all other parameters.

The four key parameters for each quintile portfolio are ϕ_i , χ_i , κ_i , and ζ_i . We pin down these 4 parameters to match 4 moments. The first is the CIV-beta itself, $\beta_{gs,i}$ in equation (8). The second is the slope of a regression of dividend growth on lagged CIV. The third is the slope of a regression of the idiosyncratic stock return variance on the idiosyncratic variance factor:

$$\frac{Cov\left(\mathbb{V}_t\left[r_{t+1}^{idio,i}\right], CIV_t\right)}{Var[CIV_t]} = \frac{\kappa_i^2}{\bar{\kappa}^2} \quad (12)$$

The fourth is the R^2 of that regression:

$$R^2 = 1 - \frac{\zeta_i^4 \frac{\sigma_{iw}^2}{1-\nu_i^2}}{\kappa_i^4 \frac{\sigma_w^2 \sigma_g^2}{1-\nu_g^2} + \zeta_i^4 \frac{\sigma_{iw}^2}{1-\nu_i^2}} \quad (13)$$

While these are four simultaneous equations, intuitively, χ_i mostly affects the dividend growth predictability slope, κ_i comes mostly from equation (12), ζ_i is pinned down by equation (13), while ϕ_i is chosen to match $\beta_{gs,i}$ once the three other parameters are pinned down. The last four rows of Table 3 show the chosen parameter values for these 4 parameters for each portfolio. Of course, the market portfolio in the last column has no idiosyncratic risk.

Table 4 summarizes the results. The post-formation CIV-betas in the data are reported in row 5.¹² They are monotonically increasing from very negative for Q1 (-0.18) to positive for Q5 (0.11). The spread in betas is 0.30. The model matches the betas exactly. The model also matches the moments described in equations (12) and (13), as reported in rows 14-17. The U-shaped pattern in the regression slopes in (12) translates in a U-shaped pattern for the κ_i coefficients. The model also matches exactly the dividend growth predictability slopes, reported in rows 18 and 19. The data show a monotonically increasing pattern in the dividend growth predictability slope, which the model accommodates via a negative value for χ_i for portfolio Q1, a less negative χ_i for the intermediate portfolios, and a zero slope

¹²What is reported in the table are rescaled betas, where the scaling ensures that the innovation volatility of CIV is the same in model and data.

for portfolio Q5.¹³ In addition to the monotonically increasing pattern in χ_i , matching the pattern in the CIV betas requires a monotonically increasing pattern in the ϕ_i parameter.

The main result of the calibration exercise is that the model is able to match the excess returns on the CIV beta-sorted portfolios. It generates a monotonically declining pattern in value-weighted excess stock return from Q1 to Q5 (row 2). The model exactly matches the return spread between portfolio 5 and portfolio 1 of -6.44% per year in the data (row 1). In the data, the latter has a t-statistic of -3.42 (recall Table 3). As rows 3 and 4 make clear, the common level of the equity risk premium comes from compensation for η -risk, while the entire cross-sectional slope in excess returns is due to differential exposure to the w_g -risk. The stocks in portfolio Q1 (Q5) have negative (positive) exposure to the CIV factor. Their returns fall (increase) when risk sharing opportunities deteriorate, making them risky (a hedge). As a result, they carry the highest (lowest) risk premia.

The model also does an excellent job matching total return volatilities of the CIV beta-sorted portfolios, shown in rows 7 and 8 of Table 4. Annual return volatilities (standard deviations) for the typical stock in each of the quintile portfolios range from 46% to 67%. They are highest for portfolios Q1 and Q5, displaying a U-shape. The market portfolio has a volatility of 15.7% in the data and 17.2% in the model. As in the data, most of total return volatility is idiosyncratic return volatility. In particular, the common idiosyncratic and firm-specific idiosyncratic components contribute about equally to firm volatility (rows 11 and 12). In terms of the priced components of variance, the eta shock has a larger contribution than the w_g shock. By design, the model matches the persistence of the various volatility components as well as the relative amount of variation that comes from the common and the firm-specific volatility components.

¹³These regression slopes are estimated precisely in the data for Q1, Q3, and the market portfolio, lending further credence to the cash flow channel.

Table 4: MAIN RESULTS

This tables reports moments from the model and compares them to the data. The first two rows report the average excess return in model and data. The next two rows split out the equity risk premium into a contribution representing compensation for η risk and a compensation for w_g risk. Rows 5 and 6 report the adjusted CIV-betas in data and model. Rows 7 and 8 report stock return volatilities in data and model, followed by a breakdown of volatility into its five components in rows 9-13 (see equation ??). Since the variance but not the volatility components are additive, we calculate the square root of each variance component, and then rescale all components so they sum tot total volatility. Rows 14 and 15 report the empirical object in equation (12), $\frac{Cov(\mathbb{V}_t[r_{t+1}^{idio,i}], \mathbb{V}_t[r_{t+1}^{IV}])}{Var[\mathbb{V}_t[r_{t+1}^{IV}]]}$, in data and in model, multiplied by 100. Rows 16 and 17 report the R-squared in equation (13) in data and in model, multiplied by 100. Rows 18 and 19 report the slope of a predictive regression of annual dividend growth on one-year lagged CIV. The model is simulated at annual frequency for 60,000 periods. All moments in the data are expressed as annual quantities and computed from the January 1963 to December 2010 sample.

	Moment		Q1	Q2	Q3	Q4	Q5	M
1	Excess Ret	Data	9.96	7.12	6.44	5.28	3.52	5.50
2		Model	9.15	6.78	5.69	4.67	2.71	5.50
3		η risk	5.16	5.16	5.16	5.16	5.16	5.16
4		w_g risk	3.99	1.62	0.52	-0.50	-2.45	0.34
5	Beta $\beta_{gs,i}$	Data	-0.18	-0.07	-0.02	0.02	0.11	-0.02
6		Model	-0.18	-0.07	-0.02	0.02	0.11	-0.02
7	Return Vol.	Data	67.04	52.16	47.30	46.47	54.39	15.67
8		Model	65.91	50.56	46.49	46.47	55.10	17.23
9		η risk	10.01	10.02	10.14	10.14	9.98	16.67
10		w_g risk	3.97	1.61	0.53	0.50	2.43	0.56
11		e_i risk	29.50	22.07	20.28	20.31	24.14	$\times$
12		ε_i risk	21.97	16.60	15.33	15.30	18.22	$\times$
13		w_i risk	0.46	0.26	0.22	0.22	0.32	$\times$
14	Eq. (12)	Data	1.55	0.87	0.71	0.72	1.04	$\times$
15		Model	1.58	0.89	0.73	0.73	1.07	$\times$
16	Eq. (13)	Data	17.70	17.10	16.80	17.00	16.90	$\times$
17		Model	17.70	17.10	16.80	17.00	16.90	$\times$
18	Div gr predict	Data	-0.37	-0.13	-0.14	-0.13	-0.00	-0.13
19		Model	-0.37	-0.13	-0.14	-0.13	-0.00	-0.13

7 Conclusion and Discussion

We document strong comovement of individual stock return volatilities. Removing common variation in returns has little effect on volatility comovement, as the volatility of residual returns demonstrates effectively the same factor structure as total returns, despite the fact that these residuals are uncorrelated. The distinction between stocks' total volatility and idiosyncratic volatility is tiny – almost all return variation at the stock level is idiosyncratic. Volatility comovement is not only a feature of returns, but also for volatility of firms' fundamentals. Like returns, we find a strong factor structure among sales growth volatilities, both for total growth rates as well as among idiosyncratic, uncorrelated factor model residual growth rates.

We explore the asset pricing implications of these findings in a model with heterogeneous investors whose consumption growth is subject to some of this variation in common idiosyncratic risk. In such a model, CIV becomes a priced state variable. Increases in CIV lead to a deterioration in risk sharing and are associated with high marginal utility for the average investor. Stocks with less negative exposure to positive innovations in risk sharing opportunities are less risky and carry lower returns. Sorting stocks into portfolios based on their exposure to the CIV, we find that stocks with more negative betas carry higher average returns. The model explores various channels that can generate the observed pattern in betas. It generates the observed return spread for plausible parameters.

Our work suggests a link between the cross-sectional volatility in firms' returns and cash-flow growth and the cross-sectional volatility in households' consumption growth. This link seems plausible given that firms' cashflow shocks may affect their employees' labor income and investors' financial income. A large literature documents that employees hold undiversified portfolios, often with substantial allocations to own-company stock. Furthermore, a large literature documents that households cannot or do not fully insure against their labor income shocks. Future work should try to link the dynamics of firms' cashflow distribution

more closely to the dynamics of households' income or consumption distributions.

As a first step along these lines, we explore the correlation between innovations in CIV on the one hand and innovations in the cross-sectional dispersion of income growth across households on the other hand. Figure 6 plots innovations in annual CIV alongside two measures of individual earnings growth dispersion, obtained from Guvenen, Ozkan, and Song (2014). The first is the annual innovation to the cross section standard deviation earnings growth each year, the second is the innovation in the spread between the 90th and 10th percentiles of earnings growth. The time series correlation between CIV innovations and innovations in standard deviation of earnings growth is 45.4% per year ($t=2.74$), and the correlation with innovations in the spread between the 90th and 10th percentiles is 37.4% ($t=2.17$). We also study the cross-sectional dispersion of house price growth and wage-per-job growth across metropolitan areas. House price data are from the Federal Housing Financing Agency and wage data from NIPA's Regional Economic Information System. The merged data set contains annual information from 1969 until 2009 for 386 regions. Conceptually, local house prices should reflect local labor market conditions (Van Nieuwerburgh and Weill 2010). The correlation between innovations in CIV and innovations in the cross section standard deviation of house price growth is 23.2% per quarter ($t=2.70$), and the correlation with innovations in the cross section standard deviation of per capita wage growth is 16.6% per quarter ($t=1.91$). This evidence is supportive of a link between the cross-sectional income distribution of firms and of households.

Figure 6: LINK BETWEEN CIV AND HOUSEHOLD INCOME GROWTH DISPERSION



Notes: The figure plots innovations in annual CIV alongside two measures of individual earnings growth dispersion, obtained from Guvenen, Ozkan, and Song (2014). The first is the annual innovation to the cross section standard deviation earnings growth each year, the second is the innovation in the spread between the 90th and 10th percentiles of earnings growth. For ease of comparison, all series are scaled to have mean zero and variance one.

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A Empirical Appendix

Figure 7 plots the empirical cross section distribution of annual firm-level volatility of sales growth (in logs). It shows that growth in fundamentals is log-normally distributed.

Figure 8 shows the distributions of log idiosyncratic return volatility (Panel A) and log idiosyncratic fundamental volatility (Panel B) pooling all firm-years. For both returns and sales growth, residuals are constructed from the five factor principal components model. The distributions are qualitatively identical to the empirical histograms for total volatility. They are nearly normal, with a slight amount of right skewness and mild excess kurtosis.

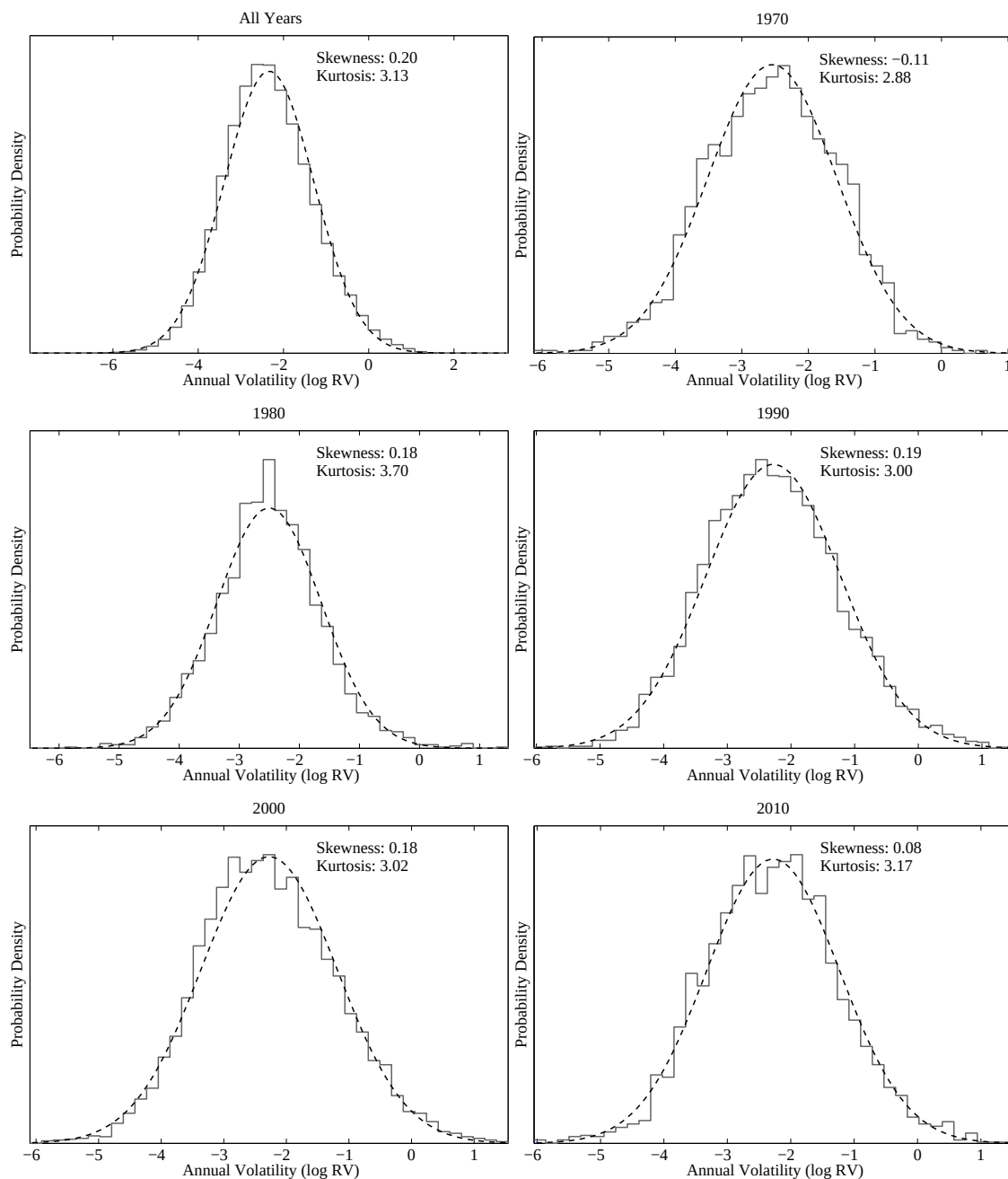
Figure 9 reports annual volatilities of the 100 Fama-French size and value portfolios. These are calculated from daily returns within the year over the 1964-2010 sample (the first available full year of Ken French's data is 1964). Panel A shows log total volatility, Panel B shows log idiosyncratic volatility using the Fama-French three factor model, and Panel C shows idiosyncratic volatilities for a five factor principal components model. Portfolio volatilities show a strikingly similar degree of comovement across the size and book-to-market spectrum, even after accounting for common factors. Like the individual stock results above, factor models remove the vast majority of common variation in returns, thus common volatility patterns are unlikely to be driven by omitted common return factors. This can be seen clearly in Figure 10. Raw portfolio returns have an average pairwise correlation of 64% between 1964 and 2010, while the correlation of factor model residuals is below 1% on average for both models. However, the average pairwise correlation between portfolio volatilities remain high whether total or idiosyncratic volatilities are analyzed. The average pairwise correlation of the volatility series in Figure 9 Panel A is 77%, falling only to 54% and 59% in Panels B and C, respectively.

Panel A of Figure 11 shows that there is a higher correlation among monthly raw returns relative to daily, with an average pairwise correlation of 23% over the sample 1926-2010. At the monthly frequency the Fama-French model continues to capture nearly all common variation, with correlations below 0.4% on average. The five factor principal components model has monthly residual correlation below 0.8% on average. At the monthly frequency, 22% of average log total volatility is accounted for by the principal components factor model, while average log idiosyncratic volatility inherits 78% of the average total volatility level. Thus, monthly return factor models do explain a larger fraction of the total return variation, but return volatility continues to be dominated by idiosyncratic rather than common variation.

Panel A of Table 5 shows that the median loadings of log volatilities on the average log volatility are similar to the average loadings reported in the main text. It also reports the interquartile ranges. In Panel B we re-estimate the volatility factor model using daily return volatility of 100 size and value portfolios. The average loadings on the volatility factor go from 0.789 to 1.116, and are between -1.172 and 0.642 for intercepts. The common variation among idiosyncratic portfolio volatility is exceptional, with average time series R^2 between 54.7% and 64.2%, and pooled R^2 between 44.1% and 49.7%. Panel C shows volatility factor model estimates for monthly (rather than daily) return volatilities. The picture is broadly similar to daily results. The average (median) firm has a loading between 0.826 and 0.926 (0.776 to 0.918) and an intercept of -0.099 to -0.276 (-0.152 to -0.603). The average time series R^2 is between 12.6% for ten factor model residuals and 26.6% for raw returns.

In Panel D we show volatility factor model estimates for sales growth volatility. The first three columns report total volatility, and idiosyncratic volatility from one and five principal component models in which

Figure 7: LOG SALES VOLATILITY: EMPIRICAL DENSITY VERSUS NORMAL DENSITY

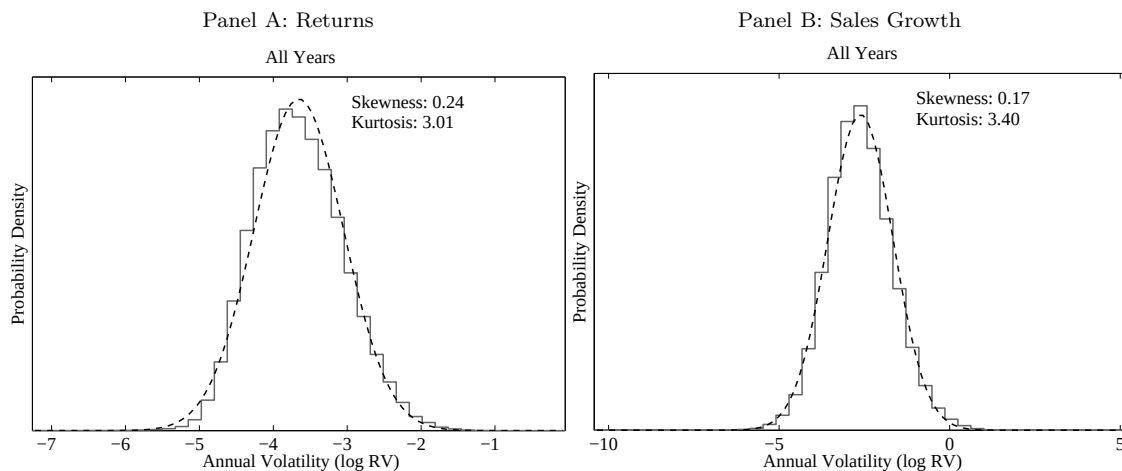


Notes: The figure plots histograms of the empirical cross section distribution of annual firm-level volatility (in logs). Within each calendar year, we calculate the standard deviation of four quarterly observations of year-on-year sales growth for each stock. The upper left-hand corner histogram pools all years (1926-2010). Selected one-year snapshots of the firm volatility cross section distribution are also show (for years 1970, 1980, 1990, 2000 and 2010). Overlaid on these histograms is the exact normal density with mean and variance set equal to that of the empirical distribution. Each figure reports the skewness and kurtosis of the data in the histogram.

Table 5: LOG VOLATILITY FACTOR MODEL ESTIMATES

Panel A: Daily Returns	Total	FF	5 PCs	10 PCs
Loading (average)	0.925	0.920	0.924	0.925
Loading (median)	0.911	0.884	0.886	0.888
Loading (25th %ile)	0.446	0.362	0.363	0.355
Loading (75th %ile)	1.387	1.400	1.405	1.412
Intercept (average)	-0.181	-0.201	-0.191	-0.190
Intercept (median)	-0.272	-0.354	-0.351	-0.344
Intercept (25th %ile)	-1.946	-2.324	-2.312	-2.361
Intercept (75th %ile)	1.491	1.584	1.585	1.617
R^2 (average univariate)	0.363	0.349	0.352	0.351
R^2 (pooled)	0.385	0.346	0.356	0.357
Panel B: Portfolio Returns	Total	FF	5 PCs	10 PCs
Loading (average)	1.000	1.000	1.000	1.000
Loading (median)	0.982	0.907	0.989	0.985
Loading (25th %ile)	0.914	0.789	0.869	0.868
Loading (75th %ile)	1.049	1.046	1.090	1.116
Intercept (average)	0.000	0.000	0.000	0.000
Intercept (median)	-0.122	-0.485	-0.055	-0.097
Intercept (25th %ile)	-0.368	-1.172	-0.777	-0.750
Intercept (75th %ile)	0.313	0.224	0.562	0.642
R^2 (average univariate)	0.760	0.547	0.597	0.606
R^2 (pooled)	0.628	0.441	0.497	0.474

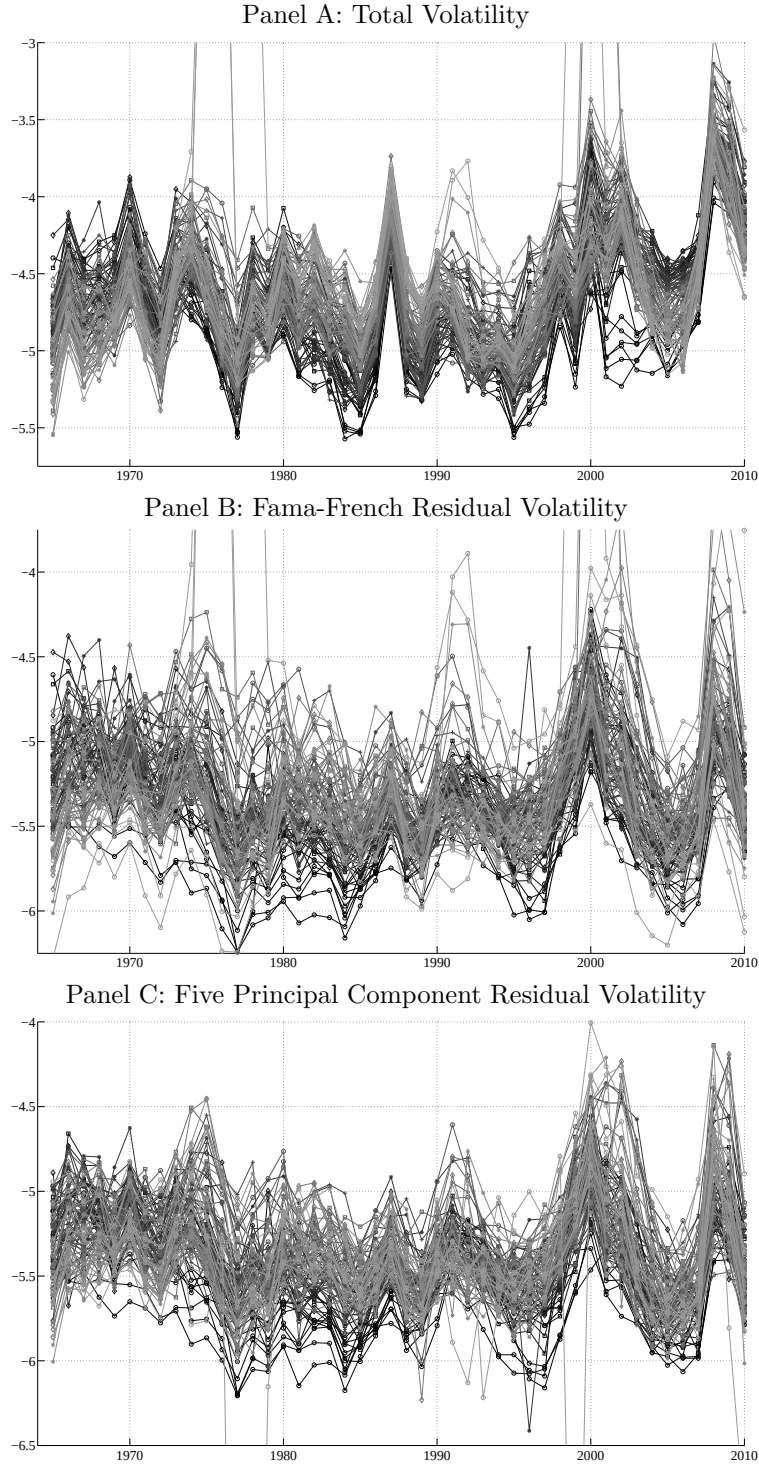
Figure 8: LOG IDIOSYNCRATIC VOLATILITY: EMPIRICAL DENSITY VERSUS NORMAL DENSITY



Notes: The figure plots histograms of the empirical cross section distribution of annual idiosyncratic firm-level volatility (in logs). Within each calendar year, we calculate the standard deviation of residuals from a factor model for daily returns for each stock (left panel) or quarterly sales growth (right panel). In both cases, residuals are constructed from a five factor principal components model. For returns, principal components are estimated from daily data within the year, while for sales growth a five year rolling window of quarterly data is used. The histograms pool all years (1926-2010 for return volatility, 1975-2010 for sales growth volatility). Overlaid on these histograms is the exact normal density with mean and variance set equal to that of the empirical distribution. Each figure reports the skewness and kurtosis of the data in the histogram.

volatility is estimated from four quarterly observations within each year. The last column reports model estimates for an annual volatility panel that uses a rolling 20 quarter window to estimate each firm-year's volatility. Due to the excessively small number of observations used to construct volatility, we might expect poorer fit in these regressions. Yet the results are closely in line with those for return volatility. The average firm has a volatility factor loading of between 0.849 and 0.938, with an intercept between -0.096 and -0.262 . The time series R^2 for raw and idiosyncratic growth rate volatility ranges between 12.7% and 22.9% on average. The pooled R^2 reaches as high as 28.3% when volatilities are estimated in a 20 quarter window.

Figure 9: VOLATILITY OF 100 SIZE AND VALUE PORTFOLIOS



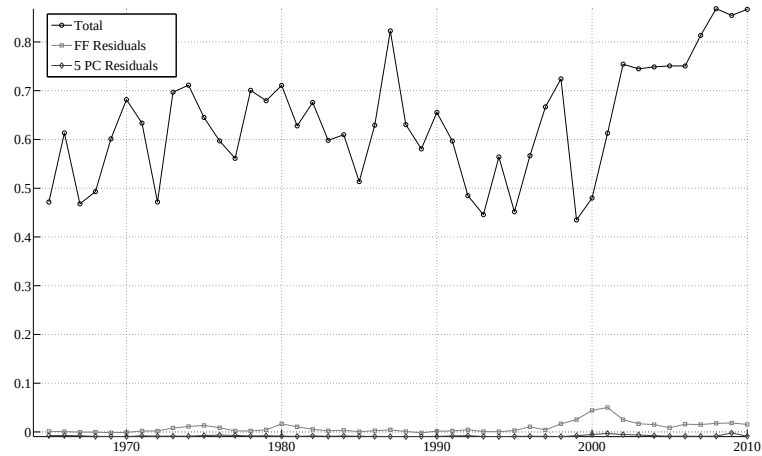
Notes: The figures plot log volatility of total and idiosyncratic returns on 100 size and value portfolios. Within each calendar year, total return volatilities are estimated from daily returns for each portfolio (Panel A), while idiosyncratic return volatility is the standard deviation of residuals from the three factor Fama-French model (Panel B) or a five factor principal components model (Panel C) for daily returns (factor models also estimated within each calendar year).

Table 1: LOG VOLATILITY FACTOR MODEL ESTIMATES, CONTINUED

Panel C: Monthly Returns	Total	FF	5 PCs	10 PCs
Loading (average)	0.918	0.888	0.872	0.926
Loading (median)	0.918	0.847	0.841	0.848
Loading (25th %ile)	0.416	0.193	0.050	-0.935
Loading (75th %ile)	1.417	1.524	1.657	2.746
Intercept (average)	-0.099	-0.195	-0.276	-0.251
Intercept (median)	-0.152	-0.348	-0.408	-0.603
Intercept (25th %ile)	-1.258	-1.998	-2.558	-8.191
Intercept (75th %ile)	1.005	1.405	1.875	7.574
R^2 (average univariate)	0.266	0.214	0.186	0.126
R^2 (pooled)	0.288	0.207	0.180	0.087
Panel D: Sales Growth	Total (1yr)	1 PC	10 PCs	Total (5yr)
Loading (average)	0.876	0.849	0.938	0.897
Loading (median)	0.777	0.776	0.889	0.864
Loading (25th %ile)	-0.711	-0.852	-0.876	-0.489
Loading (75th %ile)	2.401	2.523	2.652	2.207
Intercept (average)	-0.231	-0.211	-0.096	-0.262
Intercept (median)	-0.514	-0.432	-0.271	-0.476
Intercept (25th %ile)	-3.983	-3.379	-4.647	-4.530
Intercept (75th %ile)	3.327	2.767	4.210	3.820
R^2 (average univariate)	0.140	0.229	0.127	0.144
R^2 (pooled)	0.174	0.168	0.167	0.283

Notes: The table reports estimates for one factor regression models of yearly log volatility. In each panel, the single volatility factor is the equal weighted average of all firms' log volatilities within that year. Thus all estimated volatility factor models take the form: $\log \sigma_{i,t} = \text{intercept}_i + \text{loading}_i \cdot \log \sigma_{.,t} + e_{i,t}$. Columns represent different volatility measures. For returns (Panels A through C), the first column represents estimates for a factor model of log total return volatility, the second column for idiosyncratic volatility based on Fama-French model residuals, and the third and fourth columns to idiosyncratic volatility from one and five factor principal component models. For sales growth volatility (Panel D), the last column reports model estimates for yearly volatilities estimated in a rolling 20 quarter window to reduce estimation noise. We report means and quantiles of the empirical distribution of firm-level intercepts and volatility factor loadings, as well as time series regression R^2 average over all firms. We also report a pooled factor model R^2 , which compares the estimated factor model to a model with only a firm-specific constant.

Figure 10: AVERAGE PAIRWISE CORRELATION OF 100 SIZE AND VALUE PORTFOLIOS



Notes: The figure shows average pairwise correlation for total and idiosyncratic returns on 100 size and value portfolios within each calendar year (refer to Figure 3 for details).

Figure 11: VOLATILITY AND CORRELATION OF MONTHLY RETURNS

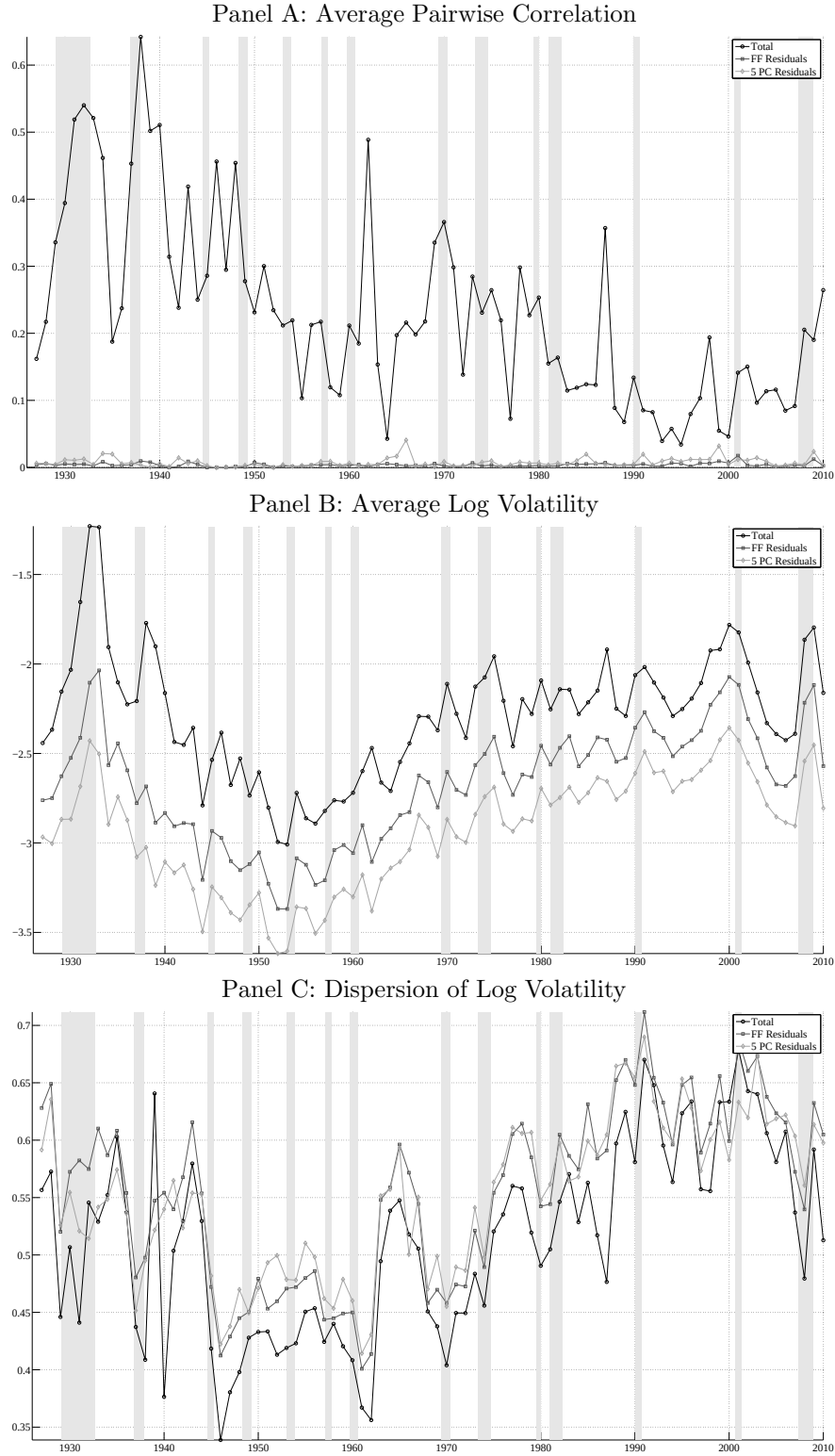


Table 7: EQUAL-WEIGHTED PORTFOLIOS FORMED ON CIV BETA

The table reports results for equal-weighted portfolio sorts on the basis of monthly CIV beta. Results are reported as annual percentages. Panel A shows one-way sorts using all CRSP stocks. Panel B shows independent two-way sorts on CIV beta and market variance beta. Panel B shows independent two-way sorts on CIV beta and idiosyncratic stock variance.

Panel A: Average returns and alphas in one-way sorts on CIV beta							
	CIV beta						
	1 (Low)	2	3	4	5 (High)	5-1	$t(5-1)$
$E[R]$	17.17	16.22	14.76	13.77	12.07	-5.10	-3.51
α_{CAPM}	5.23	5.21	4.11	3.18	0.93	-4.30	-3.02
α_{FF}	1.03	1.20	0.48	-0.18	-1.89	-2.93	-2.09

Panel B: Average returns in two-way sorts on CIV beta and MV beta							
	CIV beta						
MV beta	1 (Low)	2	3	4	5 (High)	5-1	$t(5-1)$
1 (Low)	16.46	15.58	14.45	13.72	13.16	-3.29	-1.74
2	17.12	16.58	14.64	14.54	13.38	-3.74	-2.32
3	18.01	16.32	15.87	14.32	13.08	-4.93	-3.00
4	19.92	15.69	14.17	13.41	12.48	-7.44	-3.85
5 (High)	16.48	15.99	13.62	12.87	12.40	-4.08	-1.83
5-1	0.02	0.41	-0.82	-0.85	-0.77		
$t(5-1)$	0.01	0.20	-0.39	-0.42	-0.31		

Panel C: Average returns in two-way sorts on CIV beta and idios. var.							
	CIV beta						
Idios. var.	1 (Low)	2	3	4	5 (High)	5-1	$t(5-1)$
1 (Low)	16.27	14.83	14.00	13.50	11.92	-4.34	-3.33
2	18.90	17.44	15.09	14.59	13.41	-5.49	-4.13
3	18.84	18.88	17.28	15.65	14.37	-4.46	-3.20
4	18.87	16.18	16.52	15.78	14.37	-4.49	-3.22
5 (High)	13.22	11.87	9.74	10.26	8.93	-4.29	-2.48
5-1	-3.04	-2.96	-4.26	-3.24	-2.99		
$t(5-1)$	-1.01	-1.02	-1.51	-1.11	-0.90		

B Model Appendix

Starting from the budget constraint for agent j :

$$W_{t+1}^j = R_{t+1}^j(W_t^j - C_t^j). \quad (14)$$

The beginning-of-period (or cum-dividend) total wealth W_t^j that is not spent on consumption C_t^j earns a gross return R_{t+1}^j and leads to beginning-of-next-period total wealth W_{t+1}^j . The return on a claim to consumption, the *total wealth return*, can be written as

$$R_{t+1}^j = \frac{W_{t+1}^j}{W_t^j - C_t^j} = \frac{C_{t+1}^j}{C_t^j} \frac{WC_{t+1}^j}{WC_t^j - 1}.$$

We use the Campbell (1991) approximation of the log total wealth return $r_t^j = \log(R_t^j)$ around the long-run average log wealth-consumption ratio $\mu_{wc} \equiv E[w_t^j - c_t^j]$:

$$r_{t+1}^j = \kappa_0^c + \Delta c_{t+1}^j + wc_{t+1}^j - \kappa_1^c wc_t^j,$$

where the linearization constants κ_0^c and κ_1^c are non-linear functions of the unconditional mean log wealth-consumption ratio μ_{wc} :

$$\kappa_1^c = \frac{e^{\mu_{wc}}}{e^{\mu_{wc}} - 1} > 1 \quad \text{and} \quad \kappa_0^c = -\log(e^{\mu_{wc}} - 1) + \frac{e^{\mu_{wc}}}{e^{\mu_{wc}} - 1} \mu_{wc}.$$

The return on a claim to the consumption stream of agent j , R^j , evaluated at her intertemporal marginal rate of substitution SDF^j satisfies the Euler equation:

$$\begin{aligned} 1 &= \mathbb{E}_t \left[SDF_{t+1}^j R_{t+1}^j \right] \\ 1 &= \mathbb{E}_t \left[\mathbb{E}_j \left[SDF_{t+1}^j R_{t+1}^j \right] \right] \\ 1 &= \mathbb{E}_t \left[\mathbb{E}_j \left[\exp\{sdf_{t+1}^j + r_{t+1}^j\} \right] \right] \\ 1 &= \mathbb{E}_t \left[\exp\{ \mathbb{E}_j (sdf_{t+1}^j + r_{t+1}^j) + \frac{1}{2} \mathbb{V}_j (sdf_{t+1}^j + r_{t+1}^j) \} \right] \end{aligned} \quad (15)$$

where the second equality applies the law of iterated expectations, and the last equality applies the cross-sectional normality of consumption share growth.

We combine the approximation of the log total wealth return with our conjecture for the wealth-consumption ratio of agent j :

$$wc_t^j = \mu_{wc} + W_{gs} (\sigma_{gt}^2 - \sigma_g^2)$$

We solve for the coefficients μ_{wc} and W_{gs} by imposing the Euler equation for the consumption claim.

First, we compute r_{t+1}^j :

$$\begin{aligned}
r_{t+1}^j &= \kappa_0^c + \mu_g + (1 - \kappa_1^c) \mu_{wc} + W_{gs} (\nu_g - \kappa_1^c) (\sigma_{gt}^2 - \sigma_g^2) + \sigma_c \eta_{t+1} + (\phi_c + W_{gs} \sigma_w) \sigma_g w_{g,t+1} + \Delta s_{t+1}^j \\
&= \kappa_0^c + \mu_g + (1 - \kappa_1^c) \mu_{wc} + W_{gs} (\nu_g - \kappa_1^c) (\sigma_{gt}^2 - \sigma_g^2) + \sigma_c \eta_{t+1} + (\phi_c + W_{gs} \sigma_w) \sigma_g w_{g,t+1} + \sigma_{g,t+1} v_{t+1}^j - \frac{1}{2} \sigma_{g,t+1}^2 \\
&= r_0^c + \left[W_{gs} (\nu_g - \kappa_1^c) - \frac{1}{2} \nu_g \right] (\sigma_{gt}^2 - \sigma_g^2) + \sigma_c \eta_{t+1} + (\phi_c + W_{gs} \sigma_w - \frac{1}{2} \sigma_w) \sigma_g w_{g,t+1} + \sigma_{g,t+1} v_{t+1}^j
\end{aligned}$$

where $r_0^c = \kappa_0^c + \mu_g + (1 - \kappa_1^c) \mu_{wc} - \frac{1}{2} \sigma_g^2$.

Second, Epstein and Zin (1989) show that the log real stochastic discount factor is

$$\begin{aligned}
sdf_{t+1}^j &= \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1}^j + (\theta - 1) r_{t+1}^j \\
&= \theta \log \delta - \gamma \Delta c_{t+1}^j + (\theta - 1) (\kappa_0^c + w c_{t+1}^j - \kappa_1^c w c_t^j) \\
&= \theta \log \delta - \gamma (\mu_g + \sigma_c \eta_{t+1} + \phi_c \sigma_g w_{g,t+1}) - \gamma \Delta s_{t+1}^j + (\theta - 1) (\kappa_0^c + w c_{t+1}^j - \kappa_1^c w c_t^j) \\
&= \theta \log \delta - \gamma \mu_g + (\theta - 1) [\kappa_0^c + (1 - \kappa_1^c) \mu_{wc}] + (\theta - 1) W_{gs} (\nu_g - \kappa_1^c) (\sigma_{gt}^2 - \sigma_g^2) \\
&\quad - \gamma \sigma_c \eta_{t+1} - \gamma \phi_c \sigma_g w_{g,t+1} - \gamma \Delta s_{t+1}^j + (\theta - 1) W_{gs} \sigma_w \sigma_g w_{g,t+1} \\
&= \mu_s + \left[(\theta - 1) W_{gs} (\nu_g - \kappa_1^c) + \gamma \frac{1}{2} \nu_g \right] (\sigma_{gt}^2 - \sigma_g^2) \\
&\quad - \gamma \sigma_c \eta_{t+1} - \gamma \sigma_{g,t+1} v_{t+1}^j + \left[(\theta - 1) W_{gs} \sigma_w - \gamma \phi_c + \frac{1}{2} \gamma \sigma_w \right] \sigma_g w_{g,t+1}
\end{aligned}$$

where $\mu_s = \theta \log \delta - \gamma \mu_g + (\theta - 1) [\kappa_0^c + (1 - \kappa_1^c) \mu_{wc}] + \gamma \frac{1}{2} \sigma_g^2$ is the unconditional mean log SDF.

We have that:

$$\begin{aligned}
\mathbb{E}_j (sdf_{t+1}^j) &= \mu_s + \left[(\theta - 1) W_{gs} (\nu_g - \kappa_1^c) + \gamma \frac{1}{2} \nu_g \right] (\sigma_{gt}^2 - \sigma_g^2) \\
&\quad - \gamma \sigma_c \eta_{t+1} + \left[(\theta - 1) W_{gs} \sigma_w - \gamma \phi_c + \frac{1}{2} \gamma \sigma_w \right] \sigma_g w_{g,t+1} \\
\mathbb{E}_j (r_{t+1}^j) &= r_0^c + \left[W_{gs} (\nu_g - \kappa_1^c) - \frac{1}{2} \nu_g \right] (\sigma_{gt}^2 - \sigma_g^2) \\
&\quad + \sigma_c \eta_{t+1} + (\phi_c + W_{gs} \sigma_w - \frac{1}{2} \sigma_w) \sigma_g w_{g,t+1} \\
\mathbb{V}_j [sdf_{t+1}^j + r_{t+1}^j] &= (1 - \gamma)^2 \sigma_{g,t+1}^2 \\
&= (1 - \gamma)^2 (\sigma_g^2 + \nu_g (\sigma_{gt}^2 - \sigma_g^2) + \sigma_w \sigma_g w_{g,t+1})
\end{aligned}$$

The equations above imply that:

$$\begin{aligned}
\mathbb{E}_j \left(sdf_{t+1}^j + r_{t+1}^j \right) + \frac{1}{2} \mathbb{V}_j \left[sdf_{t+1}^j + r_{t+1}^j \right] &= \mu_s + \left[(\theta - 1) W_{gs} (\nu_g - \kappa_1^c) + \gamma \frac{1}{2} \nu_g \right] (\sigma_{gt}^2 - \sigma_g^2) \\
&\quad - \gamma \sigma_c \eta_{t+1} + \left[(\theta - 1) W_{gs} \sigma_w - \gamma \phi_c + \frac{1}{2} \gamma \sigma_w \right] \sigma_g w_{g,t+1} \\
&\quad + r_0^c + \left[W_{gs} (\nu_g - \kappa_1^c) - \frac{1}{2} \nu_g \right] (\sigma_{gt}^2 - \sigma_g^2) \\
&\quad + \sigma_c \eta_{t+1} + (\phi_c + W_{gs} \sigma_w - \frac{1}{2} \sigma_w) \sigma_g w_{g,t+1} \\
&\quad + \frac{1}{2} (1 - \gamma)^2 (\sigma_g^2 + \nu_g (\sigma_{gt}^2 - \sigma_g^2) + \sigma_w \sigma_g w_{g,t+1}) \\
&= \mu_s + r_0^c + \left[\theta W_{gs} (\nu_g - \kappa_1^c) + \frac{1}{2} (\gamma - 1) \nu_g \right] (\sigma_{gt}^2 - \sigma_g^2) \\
&\quad + (1 - \gamma) \sigma_c \eta_{t+1} + \left[\theta W_{gs} \sigma_w + (1 - \gamma) \phi_c + \frac{1}{2} (\gamma - 1) \sigma_w \right] \sigma_g w_{g,t+1} \\
&\quad + \frac{1}{2} (1 - \gamma)^2 (\sigma_g^2 + \nu_g (\sigma_{gt}^2 - \sigma_g^2) + \sigma_w \sigma_g w_{g,t+1}) \\
&= \mu_s + r_0^c + \frac{1}{2} (1 - \gamma)^2 \sigma_g^2 \\
&\quad + \left[\theta W_{gs} (\nu_g - \kappa_1^c) + \frac{1}{2} \gamma (\gamma - 1) \nu_g \right] (\sigma_{gt}^2 - \sigma_g^2) \\
&\quad + (1 - \gamma) \sigma_c \eta_{t+1} + \left[\theta W_{gs} \sigma_w + (1 - \gamma) \phi_c + \frac{1}{2} \gamma (\gamma - 1) \sigma_w \right] \sigma_g w_{g,t+1}
\end{aligned}$$

Now, we can take expected value and variance conditioning on t :

$$\begin{aligned}
\mathbb{E}_t \left[\mathbb{E}_j \left(sdf_{t+1}^j + r_{t+1}^j \right) + \frac{1}{2} \mathbb{V}_j \left[sdf_{t+1}^j + r_{t+1}^j \right] \right] &= \mu_s + r_0^c + \frac{1}{2} (1 - \gamma)^2 \sigma_g^2 \\
&\quad + \left[\theta W_{gs} (\nu_g - \kappa_1^c) + \frac{1}{2} \gamma (\gamma - 1) \nu_g \right] (\sigma_{gt}^2 - \sigma_g^2) \\
\mathbb{V}_t \left[\mathbb{E}_j \left(sdf_{t+1}^j + r_{t+1}^j \right) + \frac{1}{2} \mathbb{V}_j \left[sdf_{t+1}^j + r_{t+1}^j \right] \right] &= (1 - \gamma)^2 \sigma_c^2 + \left[\theta W_{gs} \sigma_w + (1 - \gamma) \phi_c + \frac{1}{2} \gamma (\gamma - 1) \sigma_w \right]^2 \sigma_g^2
\end{aligned}$$

Plugging these different components into equation (15), and setting all the constant terms to zero yields:

$$0 = \mu_s + r_0^c + \frac{1}{2} (1 - \gamma)^2 \sigma_g^2 + \frac{1}{2} (1 - \gamma)^2 \sigma_c^2 + \frac{1}{2} \left[\theta W_{gs} \sigma_w + (1 - \gamma) \phi_c + \frac{1}{2} \gamma (\gamma - 1) \sigma_w \right]^2 \sigma_g^2 \quad (16)$$

Then setting all coefficients in $(\sigma_{gt}^2 - \sigma_g^2)$ equal to zero we obtain:

$$W_{gs} = \frac{\nu_g \gamma (\gamma - 1)}{2 \theta (\kappa_1^c - \nu_g)} = - \frac{\gamma \nu_g \left(1 - \frac{1}{\psi} \right)}{2 (\kappa_1^c - \nu_g)} \quad (17)$$

If the IES ψ exceeds 1, then $W_{gs} < 0$. Plugging these coefficients back into equation (16) implicitly defines a nonlinear equation in one unknown (μ_{wc}) , which can be solved for numerically, characterizing the average wealth-consumption ratio.

We can derive an expression for the common log real stochastic discount factor:

$$\begin{aligned}
sdf_{t+1}^a &= \mathbb{E}_j \left[sdf_{t+1}^j \right] + \frac{1}{2} \mathbb{V}_j \left[sdf_{t+1}^j \right] \\
&= \mu_s + \left[(\theta - 1) W_{gs} (\nu_g - \kappa_1^c) + \gamma \frac{1}{2} \nu_g \right] (\sigma_{gt}^2 - \sigma_g^2) \\
&\quad - \gamma \sigma_c \eta_{t+1} + \left[(\theta - 1) W_{gs} \sigma_w - \gamma \phi_c + \frac{1}{2} \gamma \sigma_w \right] \sigma_g w_{g,t+1} + \frac{1}{2} \gamma^2 \sigma_{g,t+1}^2 \\
&= \mu_s + \left[(\theta - 1) W_{gs} (\nu_g - \kappa_1^c) + \gamma \frac{1}{2} \nu_g \right] (\sigma_{gt}^2 - \sigma_g^2) \\
&\quad - \gamma \sigma_c \eta_{t+1} + \left[(\theta - 1) W_{gs} \sigma_w - \gamma \phi_c + \frac{1}{2} \gamma \sigma_w \right] \sigma_g w_{g,t+1} + \frac{1}{2} \gamma^2 (\sigma_g^2 + \nu_g (\sigma_{gt}^2 - \sigma_g^2) + \sigma_{gw} w_{g,t+1}) \\
&= \mu_s + \frac{1}{2} \gamma^2 \sigma_g^2 + \left[(\theta - 1) W_{gs} (\nu_g - \kappa_1^c) + \frac{1}{2} \gamma (1 + \gamma) \nu_g \right] (\sigma_{gt}^2 - \sigma_g^2) \\
&\quad - \gamma \sigma_c \eta_{t+1} + \left[(\theta - 1) W_{gs} \sigma_w - \gamma \phi_c + \frac{1}{2} \gamma (1 + \gamma) \sigma_w \right] \sigma_g w_{g,t+1} \\
&= \mu_s + \frac{1}{2} \gamma^2 \sigma_g^2 + s_{gs} (\sigma_{gt}^2 - \sigma_g^2) - \lambda_\eta \sigma_c \eta_{t+1} - \lambda_w \sigma_g w_{g,t+1}
\end{aligned}$$

where

$$\begin{aligned}
s_{gs} &\equiv (\theta - 1) W_{gs} (\nu_g - \kappa_1^c) + \frac{1}{2} \gamma (1 + \gamma) \nu_g = \frac{1}{2} \gamma \nu_g \left(\frac{1}{\psi} + 1 \right), \\
\lambda_\eta &\equiv \gamma, \\
\lambda_w &\equiv (1 - \theta) W_{gs} \sigma_w + \gamma \phi_c - \frac{1}{2} \gamma (1 + \gamma) \sigma_w = \frac{\gamma \nu_g \left(\frac{1}{\psi} - \gamma \right)}{2 (\kappa_1^c - \nu_g)} \sigma_w + \gamma \phi_c - \frac{1}{2} \gamma (1 + \gamma) \sigma_w,
\end{aligned}$$

The risk-free rate is:

$$\begin{aligned}
r_t^f &= -\log \left(\mathbb{E}_t [SDF_{t+1}^j] \right) = -\log \left(\mathbb{E}_t [\mathbb{E}_j [SDF_{t+1}^j]] \right) \\
&= -\mathbb{E}_t [sdf_{t+1}^a] - \frac{1}{2} \mathbb{V}_t [sdf_{t+1}^a] \\
&= -\mu_s - \frac{1}{2} \gamma^2 \sigma_g^2 - \frac{1}{2} \lambda_\eta^2 \sigma_c^2 - \frac{1}{2} \lambda_w^2 \sigma_g^2 - s_{gs} (\sigma_{gt}^2 - \sigma_g^2)
\end{aligned}$$

For individual firm's stock returns, we guess and verify that

$$pd_t^i = \mu_{pdi} + A_{gs}^i (\sigma_{gt}^2 - \sigma_g^2) + A_{is}^i (\sigma_{it}^2 - \sigma_i^2)$$

As usual, returns are approximated as:

$$r_{t+1}^i = \Delta d_{t+1}^i + \kappa_0^i + \kappa_1^i pd_{t+1}^i - pd_t^i$$

with approximation constants

$$\begin{aligned}
\kappa_1^i &= \frac{\exp(\mu_{pdi})}{1 + \exp(\mu_{pdi})}, \\
\kappa_0^i &= \log(1 + \exp(\mu_{pdi})) - \kappa_1^i \mu_{pdi}
\end{aligned}$$

Plugging in, we get:

$$\begin{aligned}
r_{t+1}^i &= \Delta d_{t+1}^i + \kappa_0^i + \mu_{pdi} (\kappa_1^i - 1) + (\sigma_{gt}^2 - \sigma_g^2) A_{gs}^i (\kappa_1^i \nu_g - 1) + (\sigma_{it}^2 - \sigma_i^2) A_{is}^i (\kappa_1^i \nu_i - 1) \\
&\quad + A_{gs}^i \kappa_1^i \sigma_w \sigma_g w_{g,t+1} + A_{is}^i \kappa_1^i \sigma_{iw} w_{i,t+1} \\
&= \mu_i + \chi_i (\sigma_{gt}^2 - \sigma_g^2) + \varphi_i \sigma_c \eta_{t+1} + \phi_i \sigma_g w_{g,t+1} + \kappa_i \sigma_{gt} e_{t+1}^i + \zeta_i \sigma_{it} \varepsilon_{t+1}^i \\
&\quad + \kappa_0^i + \mu_{pdi} (\kappa_1^i - 1) + (\sigma_{gt}^2 - \sigma_g^2) A_{gs}^i (\kappa_1^i \nu_g - 1) + (\sigma_{it}^2 - \sigma_i^2) A_{is}^i (\kappa_1^i \nu_i - 1) \\
&\quad + A_{gs}^i \kappa_1^i \sigma_w \sigma_g w_{g,t+1} + A_{is}^i \kappa_1^i \sigma_{iw} w_{i,t+1} \\
&= r_0^i + [\chi_i - A_{gs}^i (1 - \kappa_1^i \nu_g)] (\sigma_{gt}^2 - \sigma_g^2) - A_{is}^i (1 - \kappa_1^i \nu_i) (\sigma_{it}^2 - \sigma_i^2) \\
&\quad + \varphi_i \sigma_c \eta_{t+1} + (\phi_i + A_{gs}^i \kappa_1^i \sigma_w) \sigma_g w_{g,t+1} + \kappa_i \sigma_{gt} e_{t+1}^i + \zeta_i \sigma_{it} \varepsilon_{t+1}^i + A_{is}^i \kappa_1^i \sigma_{iw} w_{i,t+1}
\end{aligned}$$

where $r_0^i = \mu_i + \kappa_0^i + (\kappa_1^i - 1)\mu_{pdi}$.

Innovations in individual stock market return and individual return variance reflect the additional sources of idiosyncratic risk:

$$\begin{aligned}
r_{t+1}^i - \mathbb{E}_t[r_{t+1}^i] &= \beta_{\eta,i} \sigma_c \eta_{t+1} + \beta_{gs,i} \sigma_g w_{g,t+1} + \kappa_i \sigma_{gt} e_{t+1}^i + \zeta_i \sigma_{it} \varepsilon_{t+1}^i + \kappa_1^i A_{is}^i \sigma_{iw} w_{i,t+1} \\
\mathbb{V}_t[r_{t+1}^i] &= \beta_{\eta,i}^2 \sigma_c^2 + \beta_{gs,i}^2 \sigma_g^2 + (\kappa_1^i A_{is}^i)^2 \sigma_{iw}^2 + \kappa_i^2 \sigma_{gt}^2 + \zeta_i^2 \sigma_{it}^2
\end{aligned}$$

where

$$\begin{aligned}
\beta_{\eta,i} &\equiv \varphi_i, \\
\beta_{gs,i} &\equiv \kappa_1^i A_{gs}^i \sigma_w + \phi_i,
\end{aligned}$$

The expression for the equity risk premium on an individual stock is:

$$\mathbb{E}_t[r_{t+1}^i - r_t^f] + .5 \mathbb{V}_t[r_{t+1}^i] = \beta_{\eta,i} \lambda_\eta \sigma_c^2 + \beta_{gs,i} \lambda_w \sigma_g^2.$$

The coefficients of the price-dividend equation are obtained from the Euler equation:

$$A_{gs}^i = \frac{2s_{gs} + 2\chi_i + \kappa_i^2}{2(1 - \kappa_1^i \nu_g)} = \frac{2\chi_i + \kappa_i^2 + \left(1 + \frac{1}{\psi}\right) \gamma \nu_g}{2(1 - \kappa_1^i \nu_g)} \quad (18)$$

$$A_{is}^i = \frac{\zeta_i^2}{2(1 - \kappa_1^i \nu_i)} \quad (19)$$

and the constant μ_{pdi} is the mean log pd ratio which solves the following non-linear equation:

$$\begin{aligned}
0 &= r_0^i + \mu_s + \frac{1}{2} \gamma^2 \sigma_g^2 + \frac{1}{2} (\beta_{gs,i} - \lambda_w)^2 \sigma_g^2 + \frac{1}{2} (\beta_{\eta,i} - \lambda_\eta)^2 \sigma_c^2 \\
&\quad + \frac{1}{2} \kappa_i^2 \sigma_g^2 + \frac{1}{2} \zeta_i^2 \sigma_i^2 + \frac{1}{2} (\kappa_1^i A_{is}^i)^2 \sigma_{iw}^2
\end{aligned}$$