

Portfolio Choice and Partial Default in Emerging Markets: a quantitative analysis

(Job Market Paper)*

Kieran Walsh[†]

Yale University

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Abstract

What are the determinants and economic consequences of cross-border asset positions? I develop a new quantitative portfolio choice model and apply it to emerging market international finance. The model allows for partial default and accommodates trade in a rich set of assets. The latter means I am able to draw distinctions both between debt and equity finance and between gross and net debt. The main contribution is in developing portfolio choice techniques to analyze capital flows and default in an international finance context. I calibrate the pricing kernel of the model to match properties of U.S. stock returns and yield curves. I then analyze optimal emerging market portfolio and default behavior in response to realistic international financial fluctuations. My calibrated model jointly captures four empirical regularities that have been difficult to produce in the quantitative international finance literature: (1) Gross capital inflow and outflow are pro-cyclical. My model generates this as well as pro-cyclicality in equity liabilities and short-term debt. This is important because recent empirical work emphasizes that the level and composition of gross capital flows are at least as important as current accounts in understanding risk and predicting crises. (2) Most external defaults are partial. (3) Levels of gross external debt in excess of 50% of GNI are common. (4) Usually, borrowers default in bad economic times.

*For newer versions, visit my website: <https://sites.google.com/site/kieranjameswalsh/>

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1 Introduction

What are the determinants and economic consequences of cross-border asset positions? In this paper, I develop a new quantitative portfolio choice model and apply it to emerging market international finance. The model allows for partial default¹ and accommodates trade in a rich set of assets. The main contribution is in developing portfolio choice techniques to analyze capital flows and default in an international finance context. The set of assets I consider includes emerging market equity (claims to GDP), long- and short- term emerging market bonds, which are defaultable, and risk-free bonds issued outside of the emerging market (say, U.S. bonds). Therefore, in contrast with most other quantitative international macro/finance studies, I am able to draw distinctions both between debt and equity finance and between gross and net debt.² Another important feature of my model is state prices (or a pricing kernel) that represent(s) international investors. I calibrate the state prices to match historical properties of U.S. stock returns and yield curves. As portfolios and default behavior respond to movements in the international yield curve and equity premium, a rich asset set necessitates this kind of realistic pricing kernel.³

Quantitatively, my model reconciles four empirical regularities that have been elusive in the quantitative international finance literature.⁴ (1) Gross capital inflow and outflow are pro-cyclical. My model generates this as well as pro-cyclicality in equity liabilities and short-term debt. This is important because recent empirical work emphasizes that the level and composition of gross capital flows are at least as important as current accounts in understanding risk and predicting crises.⁵ (2) Most external defaults are partial. (3) Levels of gross external debt in excess of 50% of GNI are common. (4) Usually, borrowers default in bad economic times.

Without a unified framework that captures these facts, it is difficult to evaluate many of the policies and counterfactuals of interest to policymakers and investors. Is gross external debt too high? Are emerging markets using too much short-term debt

¹My model of default and spreads is based on Dubey, Geanakoplos, and Shubik (2005).

²The classic quantitative external debt and default papers model one or two assets. See, for example, Eaton and Gersovitz (1981), Aguiar and Gopinath (2006), and Arellano (2008).

³The sovereign debt literatures assumes a constant risk-free rate and a flat risk-free yield curve. See, for example, Borri and Verdelhan (2012) and Arellano and Ramanarayanan (2012). My pricing kernel estimation is related to recent work on explaining asset pricing puzzles in consumption-based models. See Wachter (2006), for example.

⁴See Tomz and Wright (2012) for a survey of empirical work.

⁵See, for example, Forbes and Warnock (2012a), Forbes and Warnock (2012b), Obstfeld (2012), Broner, Didier, Erce, and Schmukler (2013), Pavlova and Rigobon (2012), and Alfaro, Kalemli-Ozcan, and Volosovych (2013).

and too little equity finance? How big are haircuts, and how are they distributed across assets? What drives emerging market demand for U.S. risk-free assets? While these questions are at the center of international finance policy work and empirical research, they are difficult to answer in current models, which usually include just one asset and assume full default.

First, assuming CRRA utility, I theoretically characterize the solution to a stochastic, infinite horizon portfolio choice problem with a rich asset structure and the possibility of default. The punishment for default is utility loss, which is proportional to the amount defaulted. Closed-form characterizations that I derive enable rapid and robust computation of the solution to the portfolio problem with default. In particular, I show that consumption and default are proportional to wealth. An implication is that bond haircuts and thus spreads do not depend on the agents' wealth. Then, I prove that default increases as market prospects deteriorate. When the future looks bleak, the marginal utility of consumption rises, and agents are willing to pay the proportional cost of default. This is not a general property of standard Eaton-Gersovitz models in which the punishment for default is capital market exclusion: with growth persistence, the exclusion penalty may be least effective in good times, leading to boom-time default. I also derive a proposition showing that debt increases in the maturity length of available bonds.

Next, I estimate a joint Markov process for emerging market and world GDP growth and calibrate state prices to match historical properties of price-dividend ratios and risk-free yield curves in the U.S. Using my model I study the implications of these processes and state prices for emerging market default, portfolios, and capital flows. The probability of default is about 18%, which is consistent with Tomz and Wright (2012). As in the data, there is substantial variation in haircuts. Depending on bond maturity and the state of the economy, haircuts range from less than 1% to 74%. While this 18% probability of being in default may seem high, there is only a 1% chance (in the model) of a haircut in excess of 8%. Consistent with recent empirical evidence, haircuts are higher on short-term debt than on long-term debt. These periods of default coincide with low emerging market growth, high international stock prices, and high risk-free rates: in these states, market prospects are very grim for emerging market agents.

My main quantitative results concern the composition and cyclicity of portfolios. First, consistent with empirical evidence, my model generates pro-cyclicality in both equity liabilities and the share of short-term debt (short-term debt divided

by total debt).⁶ For example, there is a correlation of .2 between equity inflow and emerging market GDP growth. This number is on the order of empirical counterparts for Argentina, Brazil, Colombia, and Mexico. International equity premium and yield curve fluctuations, not emerging market GDP, predominantly drive these relationships. The pro-cyclicality arises from the correlation of emerging market GDP with World GDP, which is what determines state prices. In general I find that state price fluctuations drive portfolio volatility. For example, pro-cyclical upward pressure on equity prices leads agents to sell equity in good times. *Simply put, the pro-cyclicality of portfolio quantities in my model is a natural consequence of pro-cyclicality in world stock prices and risk-free rates.*

I also find that gross capital inflow (sales of emerging market assets) and gross capital outflow (purchases of international bonds) are pro-cyclical, consistent with Broner, Didier, Erce, and Schmukler (2013). The respective correlations with GDP growth are .12 and .13, which are comparable to empirical counterparts for Latin America. With respect to inflow, a major contributor is the pro-cyclicality in equity liabilities explained above. With respect to outflow, a contributor is pro-cyclicality in risk-free rates, which makes international bonds attractive investments in good times, on average.

In section 2, I describe the four motivating empirical regularities and discuss related literature. In section 3, I build the model and analyze its theoretical properties. Section 4 explains my calibration and presents my quantitative findings. In section 5, I provide some concluding remarks.

2 Literature Review and Empirical Regularities

I argue there are four sets of facts that are both of central economic interest and difficult to capture in existing models:

1. **The level and composition of gross capital flows are at least as important as current accounts in understanding risk and predicting crises:** A long-standing view amongst many economists and pundits is that trade deficits

⁶Throughout, when I say that a variable is pro-cyclical, I mean that it is positively correlated with GDP growth in the corresponding country. For example, pro-cyclicality in the short-term debt share means that short-term debt divided by total debt is positively correlated with emerging market GDP growth. When I say U.S. stock prices are pro-cyclical, however, I mean they are positively correlated with U.S. GDP growth.

or current account deficits⁷ are harbingers of economic distress.⁸ Current account deficits for a country often indicate declines in the net foreign asset position (NFA), that is, a deepening of net liabilities. The concern is that persistent or growing current account deficits may be symptomatic of unsustainable debt that will lead to default, crisis, and declines in the consumption of goods and services. However, a growing literature⁹ argues that whether or not a country's NFA is dangerous or unsustainable is as much about its composition as its level. *In short, gross flows and portfolio composition are potentially more useful than net levels and flows in evaluating current and future economic conditions.* For example, Broner, Didier, Erce, and Schmukler (2013) show that gross capital inflow and gross capital outflow are pro-cyclical and collapse in crises. Also, in the appendix, I establish that for the seven largest Latin American economies, both equity liabilities (or equity inflow) and the short-term debt share are pro-cyclical. With respect to the effects of debt, Reinhart and Rogoff (2009) and others emphasize *gross* quantities. As Reinhart, Rogoff, and Savastano (2003) comment,¹⁰ sovereign default is on *gross* debt. For example, Venezuela missed debt service payments in 2004 as a net lender with a positive trade balance.¹¹ Many if not most quantitative international macro/finance models only include one asset and thus generate *net* quantities.¹² This is not without loss of generality: a current account deficit of \$1 billion could reflect either \$1 billion of new gross debt or \$10 billion of new debt financing \$9 billion in asset purchases.

2. **Partial Default:** The majority of external defaults are only partial. Tomz and Wright (2012) report that average haircuts (percentage investor losses)¹³ range from 37% to 87%, depending on the sample of default episodes and how one measures haircuts. Furthermore, Sturzenegger and Zettelmeyer (2008) find a quite strong and negative correlation between remaining maturity and present value

⁷The current account is, roughly, net exports plus net foreign asset income, like net dividends or interest.

⁸See Obstfeld (2012) or Bernanke (2005).

⁹See, for example, Johnson (2009), Forbes and Warnock (2012a), Forbes and Warnock (2012b), Obstfeld (2012), Shin (2012), Bai (2013), and Alfaro, Kalemli-Ozcan, and Volosovych (2013).

¹⁰See footnote 12 of Reinhart, Rogoff, and Savastano (2003).

¹¹Sources: Lane and Milesi-Ferretti (2007), World Development Indicators (World Bank), "Venezuela Debt Rating Cut to Selective Default by S&P (Update2)," *Bloomberg News*, 2005, and my calculations.

¹²Bianchi, Hatchondo, and Martinez (2013), Mendoza and Smith (2013), Arellano and Ramanarayanan (2012), Evans and Hnatkovska (2012), Devereux and Sutherland (2011), Tille and van Wincoop (2010), Mendoza, Quadrini, and Ríos-Rull (2009), and Pavlova and Rigobon (2012) allow for international trade in more than one asset.

¹³I will explicitly define "haircut" below.

haircuts.¹⁴ Zettelmeyer, Trebesch, and Gulati (2013) show that this relationship was particularly strong in the case of the Greek crisis: Long-term lenders (>15 years remaining duration) received a haircut of 20-40%, while short-term lenders (<2 years remaining duration) received a haircut of 70-80%. In general, there is substantial variation in haircuts across official sovereign default episodes. For example, Benjamin and Wright (2009) estimate average haircuts of 63% and ~0% respectively for the 2001 Argentine and 2004 Venezuelan defaults. However, in the standard Eaton-Gersovitz model of international debt,¹⁵ default entails full debt repudiation: the borrower, knowing the punishment is not proportional to the amount defaulted, optimizes by reneging completely. Indeed, whether he defaults a lot or a little, the punishment is a period of market exclusion coordinated by foreigners.¹⁶ As I argue below in section 3.2.1., there is little evidence supporting such coordinated penalties. See also Tomz (2007).

3. **High Levels of Gross External Debt:** Gross external debt levels in excess of 50% of GNI are common. For example, in *each year* from 1983 to 1990 and from 2002-2003, the *average* Debt/GNI across Latin America's seven biggest economies exceeded 50%.¹⁷ For sovereign debt, Reinhart, Rogoff, and Savastano (2003), Mendoza and Yue (2012), and Tomz and Wright (2012) all report cross-country averages in excess of 70% surrounding crises. However, recent quantitative international macro/finance studies have had difficulty matching these debt levels. Many models of sovereign debt in the Eaton-Gersovitz tradition produce mean Debt/GDP levels less than around 10%¹⁸. Moreover, as Hatchondo and Martinez (2009) observe, there is a disconnect between debt in

¹⁴Some defaults have only affected short-term lenders. This was the case in the 1998-99 Ukrainian restructuring. Later, however, in 2000 there was principal reduction affecting long-term lenders. See Sturzenegger and Zettelmeyer (2008).

¹⁵See, for example, Arellano (2008), Aguiar and Gopinath (2006), and Chatterjee and Eyigungor (2012).

¹⁶Forced market exclusion only lasts one period in the models of Benjamin and Wright (2009) and D'Erasmus (2011), which introduce a renegotiation period and thus endogenous haircuts into the Eaton-Gersovitz framework. In a contemporaneous working paper, Arellano, Mateos-Planas, and Ríos-Rull (2013) consider a one asset, small open economy model with a proportional output cost of default.

¹⁷Sources: Lane and Milesi-Ferretti (2007), World Development Indicators (World Bank), and my calculations.

Note: Calculation include public and private external debt. The countries are Brazil, Mexico, Argentina, Colombia, Venezuela, Peru, and Chile.

¹⁸See Lizarazo (2013), Arellano (2008), and Aguiar and Gopinath (2006). D'Erasmus (2011) produces an average Debt/GDP of 45%. Chatterjee and Eyigungor (2012) and Benjamin and Wright (2009) produce average debt/GDP levels in excess of 50.

the Eaton-Gersovitz class of models and external debt measured and studied in the empirical literature: with only one traded asset, the standard Eaton-Gersovitz models calculate *net* debt.

4. **Default Occurs in “Bad Times:”** In most instances, default occurs when economic conditions are adverse for the borrower. For sovereign default, for example, Tomz and Wright (2012) report that for a sample covering 1820 to 2005 annual GDP was below trend in 60% of cases. For the seven largest South American economies, which are the primary targets of quantitative macro/finance studies, the relationship is even stronger. Since 1980, *all* sovereign defaults (with an average haircut in excess of 2%) in these countries have coincided with low output. For the three largest economies, Brazil, Mexico, and Argentina, all sovereign defaults since 1970 have occurred in years with negative real GDP growth.¹⁹ However, in the Eaton-Gersovitz class and in market exclusion-based models in general, the temptation to default is often strongest when output is high: with persistence in output, access to financial markets may be least-needed in boom times. Most studies do not report the correlation between default and the state of the economy. The ones that do, for example Benjamin and Wright (2009) and Mendoza and Yue (2012), generate quite frequent boom-time default.

Two studies that are closely related to mine are Bianchi, Hatchondo, and Martinez (2013) and Toda (2013). Bianchi, Hatchondo, and Martinez (2013) introduce a risk-free foreign bond (giving a total of two bonds) into a model similar to that of Chatterjee and Eyigungor (2012).²⁰ This allows the authors to differentiate between gross and net debt. Their model generates pro-cyclicality in borrowing and lending, but as Bai (2013) observes, they abstract from equity, which is an empirically important component of capital flows. Furthermore, they abstract from international price and interest rate shocks, partial default, the maturity structure of debt, and the debt/equity decision. In that these are the elements of my model which are key in explaining portfolio structure and gross flows, my study is quite different from Bianchi, Hatchondo, and Martinez (2013). In general, while a number of quantitative international macro/finance studies consider two asset models, computational limi-

¹⁹Sources: World Development Indicators (World Bank), Reinhart and Rogoff (2009), and my calculations.

²⁰The model of Chatterjee and Eyigungor (2012) is essentially the standard Eaton-Gersovitz model but with long-term instead of 1-period bonds and with a quadratic, asymmetric output cost of default (in addition to exclusion).

tations prevent the inclusion of many more assets in the standard framework. My method of solving the portfolio problem extends Toda (2013) to include default and risk spreads. Toda (2013) generalizes Samuelson (1969) to the case with many assets and Markov shocks, and he provides a solution algorithm.

My study is also related to the growing literature on incomplete markets general equilibrium models of international portfolios. Evans and Hnatkovska (2012), Devreux and Sutherland (2011), Tille and van Wincoop (2010), and Pavlova and Rigobon (2012), for example, all have the goal of introducing methodology that allows for sophisticated portfolio choice in two region, incomplete markets DSGE models. Pavlova and Rigobon (2012) use continuous time methods to derive a closed-form solution for a continuous time model with trade in equity and short-term bonds. The other three papers show how to introduce portfolio choice by approximating equilibrium at orders not usually considered in the DSGE literature. My analysis is distinct from this literature in three main ways. First, my framework allows for the computation of an exact solution (up to machine precision) of equilibrium spreads, haircuts, and portfolios. Aside from Pavlova and Rigobon (2012), these papers approximate equilibrium. Second, my model is one of a small open emerging economy. Indeed, while in my analysis bond spreads and haircuts are endogenous, international state prices are effectively exogenous.²¹ These papers, in contrast, model two regions that are each large enough to impact international state prices. Third, I allow for endogenous partial default, haircuts, and risk spreads. These papers, in contrast, do not.

3 Model

Consider an economy with an infinite number of time periods, $t = 0, 1, 2, \dots$, and an exogenous underlying shock process s_t . Throughout, I assume a period is one year. At time t , s_t is randomly equal to one of S possible values: $s_t \in S = \{1, \dots, S\}$, where S refers to both the set of possible values and its number of elements. s_t is a Markov process with transition matrix Π , where $\pi_{ss'}$ denotes element (s, s') of Π . In the recursive formulation below, I often drop references to t and let s and s' denote, respectively, the shock realization today and tomorrow. As I will explain, the process s contains information about both emerging market output growth and external financial market developments.

Two sets of agents populate the economy. First, there is a mass-one continuum of agents representing the citizens of an emerging market country. In the quantita-

²¹I do, however, discuss potential microfoundations for these state prices.

tive analysis below, I compare the model's predictions with data from Brazil, Mexico, Argentina, Colombia, Venezuela, Peru, and Chile, Latin America's seven biggest economies (by GDP). While the agents have identical utility functions and face the same budget constraints, they are subject to idiosyncratic wealth shocks, which generate inequality. Moreover, the agents are atomistic and anonymous. This implies that an agent will not internalize the impact his portfolio and default decisions have on risk spreads. However, in equilibrium, the collective actions of the emerging market agents will be consistent with the aggregate laws of motion investors take as given. In particular, individuals' debt and default policies generate the aggregate delivery rates on anonymous bond pools, which in turn determine spreads. This assumption of atomistic agents is in the tradition of Jeske (2006) and Wright (2006) in international finance and the Kehoe and Levine (1993) literature in general. In these papers, agents do not internalize the impact their portfolio decisions have on prices (through default risk).²²

At time t , the output or gross domestic product (GDP) of the emerging market economy is y_t , in units of the single consumption good, which is the numeraire. Between t and $t + 1$, GDP grows at rate $g(s_{t+1}) \in \{g(1), \dots, g(S)\}$. Therefore, GDP grows at an exogenous rate that follows a Markov process and takes on at most S values:

$$y' = g(s') y.$$

Note, however, that while GDP is exogenous, net liquid wealth ω (defined below) and gross national *income* (GNI) will be endogenous to the model. Let c_t denote the time t consumption of an emerging market agent. An agent's period utility from consumption takes the constant relative risk aversion (CRRA) form,

$$u(c_t) = \frac{(c_t)^{1-\sigma}}{1-\sigma},$$

and emerging market agents discount future utility flows at rate β , $0 < \beta < 1$. I assume throughout that $\sigma > 1$.

Given the Markov structure, it is clear that s_t is a state variable for the economy. Because markets are incomplete (as we will see below), one might suspect that the liquid net wealth distribution is also a state variable. Define Ω_t to be the cross-

²²In comparing my model with the data, I consider World Bank and Lane and Milesi-Ferretti (2007) datasets that lump together public and private quantities. That is, I interpret the emerging market agents as representing both the private and public sectors of the small open economy. While I blur the line between government and private decisions, this assumption is standard in representative agent macroeconomics.

sectional distribution of ω_t at time t . In my analysis below, while ω_t will still be a state variable for an individual agent, Ω_t will not be a state variable for the overall economy. In other words, in forecasting prices, agents will not need to forecast Ω_t . I will prove this, and to simplify notation before then I will suppress reference to Ω_t .

The second group of agents consists of rich, international investors. I assume that these foreign agents are extremely wealthy relative to the total value of the emerging market, which is thus a “small open economy.” Rather than explicitly modeling the utility maximization of these foreign agents, I simply represent them with a pricing kernel. This pricing kernel, which I will define and describe below, yields the economy’s prices and spreads, given default rates and the underlying shock and dividend processes. This is a reduced-form for a model, like that of Borri and Verdelhan (2012), in which the international investors maximize utility from consumption but where final consumption is independent of emerging market choices and assets. In either case, the point is that emerging market excess demand for assets does not exert pressure on prices. Rather, prices are uniquely determined such that the international investors are indifferent to all emerging market-related portfolios. In particular, they are willing to take positions opposite to those of the emerging market agents. At different prices, the international investors would effectively perceive arbitrage opportunities and asset markets would never clear. This does not mean that my analysis is entirely partial equilibrium: given bond default rates, the pricing kernel yields risk spreads. These risk spreads in turn imply emerging market portfolio and default policies. However, in the aggregate, these policies may not coincide with the original bond default rates. Therefore, solving the model entails finding a default rate fixed-point. In other words, spreads are a non-trivial equilibrium object.

3.1 Asset Markets

In my analysis, I allow the emerging market agents to trade five different assets. The first asset, denoted by a ’s, is emerging market equity. Its price is P . A share of this asset is a claim to future GDP. That is, y is the dividend. If emerging market agents sell shares of this asset to the foreign investors, it is like an American bank buying a stake in a Brazilian firm. Note that while GDP is exogenous in the model, selling equity entails dividend payments to foreigners and thus declines in future GNI. Suppressing state variables for now, define the return on equity to be $R'_a = (P' + y') / P$.

Next, the emerging market agents may issue short- and long-term bonds, shares of

which I denote $b_1 \leq 0$ and $b_L \leq 0$, respectively. Selling a share of the short-term bond, at price q_1 , raises q_1 units of the consumption good today and promises a payment of 1 in the next period. I model the long-term bond as a decaying perpetuity.²³ Selling a share of the long-term bond at time t , at price q_L , raises q_L and promises δ at $t+1$, δ^2 at $t+2$, δ^3 at $t+3$, etc., where $0 < \delta \leq 1$. For an arbitrary bond with price Q_t and coupons $C_{t+1}, C_{t+2}, C_{t+3}, \dots$, the yield (r_t) and duration (dur_t) are given by

$$Q_t = \frac{C_{t+1}}{(1+r_t)} + \frac{C_{t+2}}{(1+r_t)^2} + \dots$$

$$dur_t = \frac{1 \frac{C_{t+1}}{(1+r_t)} + 2 \frac{C_{t+2}}{(1+r_t)^2} + \dots}{Q_t}.$$

Intuitively, the yield is the internal rate of return, or effective interest rate, and the duration is the average repayment date, weighted by the contribution of future payments to net present value. For a one-period bond, the yield is $1/q_1$ and the duration is 1. Also, a t -year, zero coupon bond (not in the model) would have a duration of t . In short, duration generalizes the concept of maturity length to bonds with more than one payment date. For the perpetuity, the yield and duration admit simple expressions:

Lemma 1 *Suppose a decaying perpetuity promises to pay $\delta, \delta^2, \delta^3, \dots$ at subsequent future dates, where $0 < \delta \leq 1$, and currently sells at price q_L . Then the current yield and duration are, respectively,*

$$r = \frac{\delta - q_L(1 - \delta)}{q_L}$$

$$dur = \frac{1+r}{1+r-\delta} = \frac{1}{1 - \frac{\delta}{1+r}}.$$

Proof. These results follow quickly from the definitions of yield and duration and from: (i) $\sum_{t=1}^{\infty} \gamma^t = 1/(1-\gamma)$ and (ii) $\sum_{t=1}^{\infty} \gamma^t t = \gamma/((1-\gamma)^2)$, where $0 \leq \gamma < 1$. ■

From this lemma, it is apparent that, holding the yield fixed, the duration is increasing in δ , the slowness of decay. Also, holding δ fixed, the duration is decreasing in the yield. That is, bond maturity shortens mechanically as yields rise.²⁴ Lastly, note that 1 share of the long-term bond today becomes δ shares of an identical bond tomorrow. So, the one-period returns, or interest rates, are, respectively, $R'_{b_1} = 1/q_1$

²³Bianchi, Hatchondo, and Martinez (2013), Chatterjee and Eyigungor (2012), Arellano and Ramanarayanan (2012), and others similarly model long-term bonds.

²⁴This relationship is consistent with the finding of Arellano and Ramanarayanan (2012) that bond duration falls when spreads rise.

and $R'_{b_L} = \delta (1 + q'_L) / q_L$. Thus, while the long-term bond payments are deterministic, the short-term returns on this asset are stochastic, fluctuating with the price of bonds.

The final assets are the two international, risk-free bonds. Ignoring default, they are identical in structure to the emerging markets bonds. I denote shares of these short- and long-term bonds $B_1 \geq 0$ and $B_L \geq 0$, respectively. They trade at prices Q_1 and Q_L . These bonds differ from the emerging market ones in two ways. First, emerging market agents may hold only positive positions in the risk-free bonds (and only negative positions in domestically issued bonds). Second, in equilibrium we will have $q_1 \leq Q_1$ and $q_L \leq Q_L$: as I will explain below, emerging market agents may deliver less than promised, leading their bonds to trade at discounts.

3.2 Partial Default

3.2.1 The Emerging Market Default Choice

By selling domestic bonds, an emerging market agent promises to make future debt service payments, or “deliveries.” However, I assume there is no explicit international mechanism for inducing emerging market borrowers to meet their external debt obligations. Instead, as in Dubey, Geanakoplos, and Shubik (2005),²⁵ a borrower may deliver less than promised and thus partially default.²⁶ The cost of default is a utility penalty, which is proportional to the level of default. In particular, the utility cost of defaulting an amount $D_t \geq 0$ at t is

$$\lambda (\omega_t)^{-\sigma} D_t,$$

where $\lambda > 0$ is a constant, and ω_t (defined below) is the liquid net wealth of the defaulter. In short, there is a proportional, linear cost of default, but the marginal cost of default declines as the wealth of the borrower grows. This cost specification is a reduced-form for the myriad of losses that may accompany breaking a deal, including embarrassment, moral injury, legal fees, reputation decline, or material penalties (output loss, trade loss, or jail, for example). In the quantitative section

²⁵See Goodhart, Sunirand, and Tsomocos (2006) for another recent application of Dubey, Geanakoplos, and Shubik (2005).

²⁶As I mentioned above, the agents represent both private and public sectors. Therefore, I am implicitly assuming that public and private external defaults coincide. This assumption has empirical support: for corporate and sub-sovereign emerging market debt, Moody’s (2009) reports that for 1995-2008 71% of defaults coincided with sovereign debt crises. Moreover, Moody’s (2008) argues that during many sovereign default episodes governments *forced* default on private external liabilities.

Note also that the Eaton-Gersovitz models effectively have this assumption: they assume that private sector borrowing is done by the government for its citizens.

below, I set λ to generate a plausible frequency of default. The ω term in the cost of default is included to facilitate analytic solutions below, but it also has a natural interpretation. What the $\omega^{-\sigma}$ term implies is that the marginal cost of default is declining as the agent grows in wealth. Just as jails and punishments have become less draconian as society has progressed, I assume that punishment in the model “fits the crime:” the cost of default is proportional to the marginal utility of consumption, which declines as wealth grows. Without the ω term, agents would quickly “grow out” of default. In particular, as I will show below, this specification ensures that the default-wealth ratio is stationary. It might seem that this specification would generate lots of default in good economic times. I will show below that this is not the case.

Mechanically, this default cost introduces a minimum level of consumption, as a fraction of liquid net wealth ω : default occurs if and only if $u'(c) = \lambda\omega^{-\sigma}$. If $u'(c) > \lambda\omega^{-\sigma}$, the agent will default further and consume more at a net utility gain. If $u'(c) < \lambda\omega^{-\sigma}$, the agent should default less and cut consumption until either $D = 0$ or the wedge closes. Because $u'(c) = c^{-\sigma}$, the minimum consumption level is $\underline{c} = \lambda^{-1/\sigma}\omega$. All else equal, because default is unpleasant, emerging market agents want to fulfill their promises. However, if keeping promises entails low consumption, the agents will renege until $\underline{c}$ is possible.

I adopt this specification for two reasons. First, this is a simple and tractable way to introduce endogenous partial default and haircuts into an equilibrium model. As I explained above, external defaults are typically partial. In most default episodes, not all public and private external debt is involved, and haircuts are usually less than one. Moreover, the degree of default (the intensive margin) and not just whether or not it occurs (the extensive margin) is of central economic significance. Suppose a lender or guarantor knows there is a 10% probability of default. If default is all-or-nothing, then expected losses are 10%. If, instead, the expected haircut is 30%, then the expected losses are just 3%. This 7% gain may be a large difference for lenders or guarantors, and it may lead to significantly lower borrowing rates. Consequently, for a model to accurately predict borrowing rates, investor losses, and default frequencies, it should allow for the intensive margin. My specification is a simple and computationally convenient method of introducing partial default into an equilibrium model of international portfolio choice. This is in contrast with some recent sovereign default papers²⁷ that introduce a post-default, debt renegotiation game into the Eaton-Gersovitz framework and thus effectively allow for partial default. While these models

²⁷See, for example, D’Erasmus (2011) and Benjamin and Wright (2009).

have success in improving the empirical fit of the Eaton-Gersovitz model, their results are sensitive to the bargaining process parameters, of which there are many. Furthermore, these models still impose a post-default period of financial autarky, which is counterfactual, especially when considering private and public flows. My model is perhaps less “structural,” but it is tractable and provides a single-parameter (λ) reduced-form for a variety of potential haircut-generating mechanisms.

The second reason I choose my specification is that I find its implications broadly consistent with empirical evidence. With regard to sovereign default, Tomz (2007) argues that there is little evidence for the claim that lenders conspire to exclude defaulters from capital markets. Furthermore, considering post-1970 total gross external debt from the World Bank’s World Development Indicators, there is not evidence of substantial post-default capital market exclusion. See figure 1. Instead, Tomz (2007) argues, most borrowing countries have a *preference* for repayment of loans. When they default, it is because of bad economic times, when the pain of default is relatively low. Lenders understand this, so while they may demand a risk premium as economic conditions deteriorate, they do not charge “bad faith.” If risk premiums are high enough, the borrowers will effectively be excluded from markets by the price mechanism. Tomz (2007) writes, on page 102, “investors did not allege bad faith by most countries that defaulted during the [Great Depression], but they did refrain from extending new credit until the economic crisis passed and the debtors offered acceptable settlements.” Tomz (2007) emphasizes reputational mechanisms, but one could interpret my default model as a reduced-form for his narrative model. With the minimum consumption interpretation, my model captures these observations: default is painful but sometimes necessary when harsh consumption cuts are the only other option.

3.2.2 The Determination of Haircuts

The citizens and government of the small open economy borrow from the international investors via two anonymous bond pools, one for long-term bonds and one for short-term ones. To borrow via a pool, an emerging market agent simply sells a share of the pool at the market rate (q_1 or q_L) and promises to make payments in the future (either 1 in one period or the stream $\delta, \delta^2, \delta^3, \dots$). Because there are many, anonymous emerging market agents, each borrower takes the market price as given. However, the agents may deliver less than promised and thus default. Indeed, from the perspective of the lenders, buying shares of the pools is risky, as they may deliver less than 100% in some states. For simplicity, I assume that default takes the form of

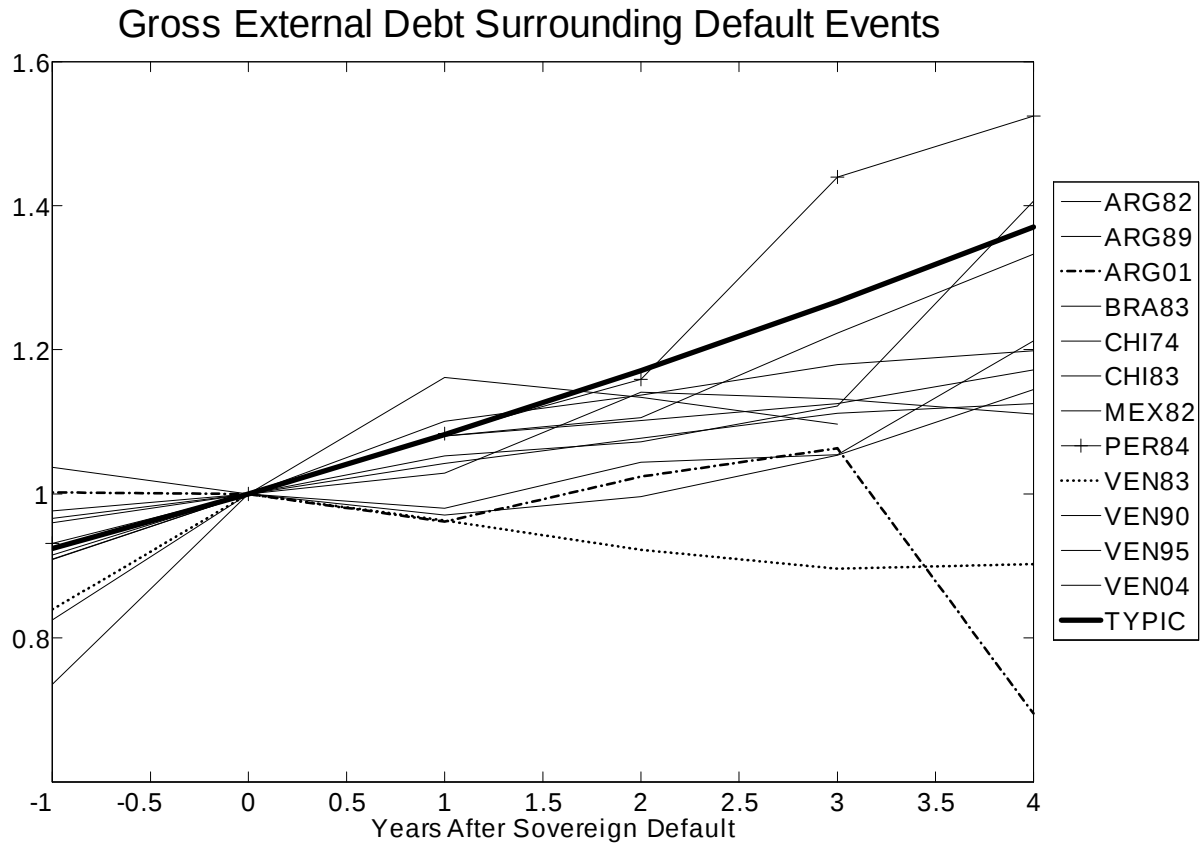


Figure 1: The figure illustrates gross external public and private debt surrounding sovereign default episodes in Latin America's biggest economies. Levels are normalized by debt in the year of default. The debt data are from the World Development Indicators (World Bank). The slope of TYPIC is the pooled, post-1970 average growth rate of debt. The dates are from Reinhart and Rogoff (2009). The figure includes all post-1970 sovereign defaults for the listed countries, except CHI72, PER75, PER78, and PER80, which immediately preceded further default.

missed debt service spread equally across outstanding bonds. Specifically, the pools have the same one-period ahead delivery rate d_{t+1} , which the lenders all take as given. The implication, as we will see below, is that haircuts will be higher on short-term bonds. This is because for long-term bonds payments currently due reflect only a portion of the their total principal and interest. Recall that, as I argued above, this relationship between maturity and haircuts is empirically accurate.

Given delivery rates, there are unique pool or bond prices that make the lenders willing to meet the demand for credit (in particular, the pricing kernel will yield unique prices, given delivery rates). All else equal, the more an agent borrows, the bigger the gap in some states of the world between his minimum consumption level and what he would consume after meeting his promises. That is, the current portfolio choices of agents will affect future delivery rates and thus current prices. Each agent is small relative to the pool and therefore does not internalize the impact of his actions on the prices he faces. However, in my definition of equilibrium, I impose a rationality or consistency requirement on lenders: they must correctly forecast the pool delivery rates. In particular, I assume the equilibrium delivery rates must satisfy

$$d(t+1) = 1 - \frac{\int D_i(t+1) di}{\int (b'_{1,i}(t) + \delta b'_{L,i}(t)) di},$$

where $D_i(t+1)$ is the realized default level of agent i at $t+1$. $b'_{1,i}(t)$ and $\delta b'_{L,i}(t)$ are the promises that agent i makes at time t . In other word, the actions of the emerging market agents must be consistent with the delivery rates assumed by the international investors. My justification for this equilibrium equation is a “no arbitrage” argument. Without it, more accurate international investors would effectively perceive arbitrage opportunities and aggressively invest, pushing prices and thus actual deliveries back to this equilibrium. Note, however, that my framework could easily accommodate irrational international investors. Later, I will show that d_{t+1} depends just on s_t and s_{t+1} (that d' depends just on s and s'). Conjecturing this fact for now, define $d_{ss'}$ to be the bond delivery rate going from state s to s' .

Given these delivery rates, the long- and short-term bond haircuts are

$$h_1 = 1 - d$$

$$h_L = \frac{1 - d}{1 + q_L}.$$

That is, the haircut is the present value loss in bond value, evaluated at market prices. Immediately, we see that because $q_L > 0$, the haircut is always higher on short-term

bonds.

3.3 Emerging Market Agent Optimization Problem

Written recursively, the problem of an arbitrary emerging market agent is (suppressing i subscripts denoting specific agents)

$$v(\omega; s) = \max_{\substack{c, a', B'_1, B'_L, \\ b'_1, b'_L, D}} \left\{ \frac{c^{1-\sigma}}{1-\sigma} - \lambda \omega^{-\sigma} D + \beta E[v(\omega'; s') | s] \right\} \text{ subject to}$$

$$(i) : c + Pa' + Q_1 B'_1 + Q_L B'_L + q_1 b'_1 + q_L b'_L = \omega + D$$

$$(ii) : \omega' = [(P' + y') a' + B'_1 + (1 + Q'_L) B'_L \delta + b'_1 + (1 + q'_L) b'_L \delta] \varepsilon'$$

$$(iii) : D \geq 0, B'_1, B'_L \geq 0, b'_1, b'_L \leq 0$$

$$(iv) : (c, D, a', B'_1, B'_L, b'_1, b'_L) \in \Phi(\omega; s)$$

where ε' is an agent-specific, i.i.d. shock to an agent's liquid net worth evolution. I assume that $\varepsilon' > 0$, $E[\varepsilon'] = 1$, and $\int \varepsilon'_i di = 1$. Assume also that $\tilde{\beta} = \beta E[(\varepsilon')^{1-\sigma}] < 1$. I have now explicitly defined ω as the liquid net worth or effective cash-on-hand with which an agent may invest and consume. It is the GDP dividend to which he is entitled, plus net debt service, plus the net market value of his assets (ex dividend). I will use the set Φ only in the quantitative exercises below.

A question that immediately arises is, why is ω the only agent-specific state variable? This is the case for two reasons. First, and most importantly, emerging market agents do not internalize their impact on prices (via delivery rates). If they were large or not anonymous, then they would perceive their choices changing the prices of emerging market bonds. As ω depends on current prices, ω could not be a state variable in this case. Furthermore, the extent to which changes in q_L affect ω depends on the maturity structure of outstanding debt. Indeed, without the price-taking assumption, there would be three agent-specific state variables: the debt-equity ratio, maturity structure, and ω . Second, and this is a technical point, I do not explicitly require that D be less than what is current promised. Also, I do not explicitly prevent debtless agents from defaulting. In other words, I am not ruling out negative deliveries. You could interpret negative deliveries as bailouts, but this is perhaps not necessary: in my quantitative analysis, in equilibrium agents neither default without debt nor default more than their debt. Also, one can rule out debtless default with a lower bound on λ coupled with a minimum equity level. Finding a primal assumption

that excludes negative deliveries in equilibrium would be trickier because of feedbacks between prices and deliveries. Note, finally, that the possibility for negative deliveries does not play a major role in the mechanics of the model. This possibility, which has no bite in my quantitative analysis, just serves to simplify the state space.

Because emerging market GDP is marketable and because ε' enters multiplicatively, one can show that the above optimization problem has a theoretically useful alternative formulation. For arbitrary asset $x \in \{a, B_1, B_L, b_1, b_L\}$ with price P_x , define the portfolio weight θ_x :

$$\theta_x = \frac{P_x x'}{\omega - c + D}.$$

For example, θ_a is the share of post-consumption, post-default wealth invested in equity. Let $\theta = (\theta_a, \theta_{B_1}, \dots)'$ be the vector of portfolio weights. The alternate but equivalent formulation of the optimization problem is

$$\begin{aligned} v(\omega; s) = \max_{c, D, \theta} & \left\{ \frac{c^{1-\sigma}}{1-\sigma} - \lambda \omega^{-\sigma} D + \beta E(v(\omega'; s') | s) \right\} \text{ subject to} \\ (i) : & \omega' = R(\theta; s, s')(\omega + D - c)\varepsilon' \\ (ii) : & \theta \in \Theta(s) \\ (iii) : & D \geq 0, \end{aligned}$$

where

$$R(\theta; s, s') = R'_a(s, s')\theta_a + R'_{B_1}(s, s')\theta_{B_1} + \dots,$$

and $\Theta(s)$ contains the constraint $\theta_a + \theta_{B_1} + \dots = 1$. Here and below I assume that the constraint set Θ is compact and convex and does not depend on ω . Note that this means $\Phi(\omega; s)$ may depend on ω . For example, if a constraint in Φ is $B'_1 > \alpha(\omega - c)$, the corresponding Θ constraint is

$$\theta_{B_1} > Q_1(s)\alpha.$$

Writing the emerging market problem this way, I have effectively cast it as a classic portfolio choice problem. Furthermore, introducing default in this fashion allows me to solve the problem using portfolio choice techniques.²⁸ The following proposition characterizes the solution:

²⁸I extend the method of Toda (2013), who was building on work by Samuelson (1969) and other authors.

Proposition 1 Assume $\lambda > 1$ and $\sigma > 1$. Define

$$\mathcal{U}_s(a_1, \dots, a_S) = \max_{\theta \in \Theta(s)} E \left[a_{s'} \frac{(R(\theta; s, s'))^{1-\sigma}}{1-\sigma} \mid s \right],$$

and assume further that $\tilde{\beta} \mathcal{U}_s(\mathbf{1})(1-\sigma) < 1$. Then there are S constants $a_1, \dots, a_S$ such that the Emerging Market Problem solution satisfies:

1. $v(\omega; s) = a_s \frac{\omega^{1-\sigma}}{1-\sigma}$
2. Consumption and default are proportional to ω and depend just on ω and market prospects $\mathcal{V}_s$:

$$c(\omega; s) = \omega \max(\lambda^{-1/\sigma}, \mathcal{V}_s)$$

$$D(\omega; s) = \omega \max\left(\frac{\lambda^{-1/\sigma}}{\mathcal{V}_s} - 1, 0\right)$$

3. Market prospects $\mathcal{V}_s$ are a monotonic transform of $\mathcal{U}_s$, utility from the optimal portfolio.

Proof. See Appendix. ■

What this says is that there is a separation of portfolio choice from the consumption/saving/default decision. Furthermore, relative to individual wealth ω_i , agents make the same decisions. In particular, they choose identical portfolio weights θ^* . Relative to ω_i , decisions just depend on λ , the cost of default, and $\mathcal{V}_s$, which is a measure of economic conditions going forward. $\mathcal{V}_s$ could be low, for example, when equity returns are expected to stay low or when borrowing rates are high. Note that $\mathcal{V}_s$ depends on β and the distribution of ε . See the appendix for details. Also, I will elaborate on $\mathcal{V}_s$ below. The key assumption in deriving this result is the $\omega^{-\sigma}$ term in the cost of default, which allows me to both include partial default and solve the problem using portfolio choice techniques. A consequence of this proposition is that given returns it is computationally easy to solve the emerging market problem: I know the shape of the value function, and, given the a 's, finding the optimal portfolio at each node s (solving for the $\mathcal{U}_s$'s) is computationally straightforward. The recursion that determines the a 's (see the proof in the appendix) converges quickly in practice, and, unlike with standard value function iteration, does not include a guess of the shape of value function.²⁹ This means I do not need to use interpolation methods.

²⁹Interestingly, the regularity condition that ensures existence, $\tilde{\beta} \mathcal{U}_s(\mathbf{1})(1-\sigma) < 1$, is the same as

Proposition 1 also immediately gives us a corollary, which will be useful below.

Corollary 1 *Realized delivery rates $d_{ss'}$ depend just on the exogenous shock process, not on the wealth distribution.*

Proof. See appendix. ■

In the representative agent version of the model in which $\varepsilon'_i = \varepsilon'_j = 1$, this corollary says that delivery rates do not depend on wealth ω . Basically, because both default and promises are proportional to wealth, their ratio (the default rate) does not depend on ω .

3.4 International Pricing Kernel

I represent the international investors with a pricing kernel μ , which I define to be S^2 strictly positive constants:

$$\begin{aligned} \mu_{ss'} &> 0 \\ \forall s, s' &\in S. \end{aligned}$$

Let M be the $S \times S$ matrix of μ 's. Note that I assume μ depends only on the transition of the exogenous underlying shock process s . In particular, it does not depend on the wealth distribution Ω . This is the small open economy assumption mentioned above.

Consider an arbitrary asset with dividends $\mathcal{D}(s)$ that depend just on the s process. I say that corresponding prices $\mathcal{P}(s)$ are *consistent with the pricing kernel* if for all s

$$\mathcal{P}(s) = \sum_{s'=1}^S \pi_{ss'} \mu_{ss'} (\mathcal{P}(s') + \mathcal{D}(s')),$$

which implies

$$\vec{\mathcal{P}} = [\mathbf{I} - (\Pi \circ M)]^{-1} ((\Pi \circ M)) \vec{\mathcal{D}},$$

where $\circ$ is element-by-element multiplication (Hadamard product), $\mathbf{I}$ is the identity matrix, and $\vec{\mathcal{X}} = (\mathcal{X}(1), \dots, \mathcal{X}(S))'$. Thus, given payoffs and probabilities, μ pins down prices and thus returns. A classic theorem is that asset prices admit no arbitrage opportunities as long as the elements of M are all strictly positive (see Cochrane (2005)).

in Toda (2013). Technically, this is because partial default necessarily makes the recursion operator for the a 's *more* tightly bounded. See the proof in the appendix.

Returning to my model, because international long-term bond payoffs are independent of Ω , it is clear that their prices depend just on s and satisfy

$$\overrightarrow{Q}_L = [\mathbf{I} - (\Pi \circ M) \delta]^{-1} (\Pi \circ M) \delta \mathbf{1},$$

where $\mathbf{1}$ is a vector of ones. Similarly, equity returns will depend just on s . A subtlety arises here, however, because GDP is growing. In this case, some algebra shows that

$$P_D(s) = \sum_{s'=1}^S \pi_{ss'} \mu_{ss'} (P_D(s') + 1) g(s'),$$

where P_D is the price-dividend ratio (P/y). That is, with a growing asset, it is the price-dividend ratio that is stationary. We have matrix form

$$\overrightarrow{P}_D = [\mathbf{I} - (\Pi \circ M \circ G)]^{-1} (\Pi \circ M \circ G) \mathbf{1},$$

where $G = (\overrightarrow{g}, \dots, \overrightarrow{g})'$ and is $S \times S$ (recall that $g' = y'/y$). Therefore, the respective returns on long-term risk-free bonds and equity are

$$R_{B_L}(s, s') = \frac{\delta (1 + Q_L(s'))}{Q_L(s)}$$

$$R_a(s, s') = \frac{P' + y'}{P} = \frac{P_D(s') + 1}{P_D(s)} g(s').$$

Also, the return on short-term risk-free bonds is $R_{B_1}(s, s') = 1/Q_1(s)$, which does not depend on s' .

In general, delivery rates and thus emerging market bond payoffs depend on Ω . Here however, as I proved in Corollary 1, delivery rates do not depend on Ω . Letting $\mathbf{d}$ be the matrix of $d_{ss'}$ elements, we have

$$q_1(s) = \sum_{s'=1}^S \pi_{ss'} \mu_{ss'} d_{ss'}$$

$$q_L(s) = \sum_{s'=1}^S \pi_{ss'} \mu_{ss'} \delta (q_L(s') + d_{ss'}),$$

implying, in matrix form,

$$\overrightarrow{q}_1 = (\Pi \circ M \circ \mathbf{d}) \mathbf{1}.$$

We also have

$$\vec{q}_L = [\mathbf{I} - \delta (\Pi \circ M)]^{-1} (\Pi \circ M \circ \mathbf{d}) \mathbf{1}\delta.$$

As I mentioned above, one can easily further “microfound” this theory of prices. For example, let $C(s)$ and $Y(s)$ denote the consumption and income of a representative international investor, who is very rich relative to the emerging market. While there may be correlation between emerging market and investor income, international investor consumption is exogenous. Let U be his CRRA period utility from consumption. Defining μ as

$$\mu_{ss'} = \xi_{ss'} \frac{U'(Y(s'))}{U'(Y(s))},$$

we can interpret the above pricing scheme as stemming from general equilibrium. In particular, the pricing conditions can be viewed as the international investor Euler equations coupled with the market clearing condition $C(s) = Y(s)$. The one deviation from standard consumption-based asset pricing à la Lucas (1978) is the ξ term. One may interpret it either as a subjective probability weight or as a taste shock. Alternatively, one could include behavioral elements, like habit, into U . This is the approach of, for example, Borri and Verdelhan (2012). I work with the more general M specification because, unlike previous studies, I allow for trade in equity and bonds of different maturities. With just one or two assets, pricing kernels with few parameters are sufficient in generating plausible prices³⁰. To match both equity premium and yield curve facts, I need more degrees of freedom³¹.

I now establish that Ω is not a state variable, which greatly simplifies the computation of equilibrium.

Proposition 2 *s is the only aggregate state variable.*

Proof. Prices depend just on s and the delivery rates. Delivery rates just depend on s (Corollary 1). Therefore, prices do not depend on the wealth distribution Ω . ■

3.5 Equilibrium

In light of Proposition 2, we can define and analyze equilibrium in which delivery rates and spreads are endogenous variables but independent of agents’ wealth.

³⁰See Arellano and Ramanarayanan (2012) and Borri and Verdelhan (2012).

³¹See Wachter (2006) for a microfoundation for a pricing kernel that matches moments similar to the ones I consider below.

Definition 1 *Equilibrium is*

1. *Delivery rates:* $d_{ss'}$
2. *State Prices:* $\mu_{ss'}$
3. *Prices:* $P_D(s), q_1(s), q_L(s), Q_1(s), Q_L(s)$
4. *Policy functions:* $c(\omega_i; s), \theta(\omega_i; s), D(\omega_i, s), \omega' = w(\omega_i; s)$

such that

1. *Given prices, policy functions are optimal for emerging market agents*
2. *Given delivery rates, prices are consistent with pricing kernel*
3. *Emerging market choices generate delivery rates d :*

$$d_{ss'} = 1 - \frac{\int D(w(\omega_i; s), s') di}{-\int (b_1(\omega_i, s) + \delta b_L(\omega_i, s)) di}$$

Given the pricing kernel μ , the prices and returns of the non-defaultable assets are immediately determined. In contrast, q_1 and q_L , the risky bond prices, depend on the delivery rates $d_{ss'}$, which are equilibrium objects. This means that solving for equilibrium entails solving a fixed point problem: delivery rates affect prices, which affect emerging market actions, which then determine realized delivery rates. Equilibrium obtains when delivery rates expected by the international investors yield prices and emerging market actions that generate those same delivery rates. This is not saying that international investors correctly predict what will happen in the random future. It is just saying that they correctly forecast default rates, conditional on the state of the world. They do not know if emerging market growth will be low. They just know how much default will occur if growth is indeed low.

Why is this a sensible equilibrium concept? If many international investors systematically misunderstood default rates across states of the world, they would effectively under or over charge on interest. This would leave money on the table for other world investors with better default rate forecasts. Their “smart money” would aggressively bet against the others, pushing prices towards the rational equilibrium.

I will now give a brief outline of my solution algorithm:

- Make an initial guess for delivery rates: $d_{ss'}^0$

- Given current delivery rate guess $d_{ss'}^t$, use pricing kernel to calculate all prices.
- Given prices, solve the emerging market problem via Proposition 1.
- Update delivery rates to $d_{ss'}^{t+1}$, the realized ones from the emerging market solution.
- Continue until $\max_{s,s' \in S} |d_{ss'}^t - d_{ss'}^{t+1}| \approx 0$.

Before turning to the quantitative exercise, I will establish some additional theoretical results.

3.6 Default in Bad Times and Additional Theoretical Results

The following corollary of Proposition 1 characterizes the relationship between the state of the economy, prices, and default.

Corollary 2 (Default in Bad Times) $\mathcal{V}_s \leq \mathcal{V}_{s'}$ implies

$$\begin{aligned} c(\omega; s) &\leq c(\omega; s') \\ D(\omega; s) &\geq D(\omega; s') \end{aligned}$$

Proof. This follows immediately from Proposition 1. ■

To the extent that $\mathcal{V}_s$ measures economic conditions, this corollary tells us that as market prospects deteriorate, the emerging market agents consume less, save more, and default more. When $\mathcal{V}_s$ is low, the marginal utility of consumption is high. When the marginal utility of consumption is high, it is worth defaulting and paying the marginal penalty, which is relatively low. In other words, emerging market agents default when they are desperate, in marginal utility terms. Unlike in the Eaton-Gersovitz models, potential defaulters are not implicitly playing a game with lenders who are threatening coordinated exclusion from capital markets. In short, my model predicts that agents will default when investment opportunities look bleak. *This is true for any degree of shock persistence and holds in spite of the fact that the marginal cost of default is declining in wealth.* Note that the $\omega^{-\sigma}$ term effectively neutralizes low wealth as a cause of default. When agents are poor, both the marginal utility of consumption and the marginal cost of default rise. We see from Proposition 1 that my specification ensures that these effects exactly cancel. Without the $\omega^{-\sigma}$ term, the solution to the model would not be so simple, but the relationship between default and bad times would be even stronger.

With respect to s , when exactly is $\mathcal{V}_s$ low? When there are many assets and a stochastic pricing kernel, the relationship between emerging market growth and market prospects are not necessarily intuitive. In particular, you might be able to find a pricing kernel and a set of assets such that $\mathcal{V}_s$ is *high* when emerging market growth is low. That is, you could invent an example in which bad and good times are not determined by the underlying exogenous GDP process. This is perhaps an attractive property for the model to have in a world with relatively open capital markets. Consider Mexican billionaire Carlos Slim. Given his stake in Mexican telecom, his market prospects and consumption are to some degree exposed to Mexican GDP fluctuations. However, with a large international portfolio, shocks to Wall Street alone could significantly affect him, perhaps more so than could Mexican GDP. On the other hand, these properties of $\mathcal{V}_s$ could be a downside: at least for post-1970s Latin American, major defaults have coincided with low GDP growth or below trend GDP. The recent Greek crisis was certainly triggered by recession. In the version of my model with multiple assets, however, default need not coincide with low GDP growth. While default occurs in bad times in my model, bad times need not correspond to low growth. That said, all else equal, poor growth must lead to declines in $\mathcal{V}_s$, assuming persistence. In particular, as long as emerging market agents have some positive exposure to equity, holding prices constant $\mathcal{V}_s$ must decline when GDP growth is relatively low and thus likely to stay so. Furthermore, in my calibration below, I find that default coincides with low emerging market growth.

Consider the following example that highlights the difference between the Eaton-Gersovitz default model and my model. Suppose there are just two assets, risky equity and a riskless bond with a constant return. Suppose the equity return is either bad (B) or good (G), and suppose ρ_s is the probability in state s that the next period will yield a good equity return. Assume also that equity holdings must be positive. As GDP growth is persistent, we may think of low ρ_s states as bad economic times: returns are low and likely to stay so. In my model, market prospects $\mathcal{V}_s$ are unequivocally low when ρ_s is low. By Corollary 2, there is more default as ρ_s declines and economic conditions deteriorate: low ρ_s means the marginal utility of consumption is high, perhaps triggering default. In Eaton-Gersovitz models, however, an additional effect is at play: when ρ_s is low, the exclusion punishment is particularly harsh. When times are bad and likely to stay so for a while, agents may place high value on the option to use capital markets for consumption smoothing. Therefore, with persistent GDP and the exclusion punishment, a low cost of default may trigger repudiation in good times, when the benefit from default is relatively low. As I explained above, some Eaton-

Gersovitz-based calibrations do indeed generate lots of boom time default. These defaults, which may coincide with relatively low marginal utility of consumption, are effectively “bad faith” defaults for which there are few empirical examples (see Tomz (2007)). In my model, in contrast, default necessarily coincides with a relatively high marginal utility of consumption.

Consider now the more general setting with j arbitrary assets indexed by $1, \dots, J$. Let $R_j(s, s')$ be the s to s' return of asset j . Assume $\varepsilon = 1$. Proposition 1 yields additional theoretical results in some special cases.

Corollary 3 (Closed-Form Solutions) *Suppose there are as many assets as states ($J = S$). Suppose also that $\Theta_s = \left\{ \theta \left| \sum_j \theta_j = 1 \text{ and } |\theta_j| \leq \bar{\theta} \right. \right\}$. If $|\theta_j| \leq \bar{\theta}$ is not binding for any j , then*

$$\begin{aligned} \theta^*(s) &= \mathbf{R}_s^{-1} \gamma_s^{-1/\sigma} (\mathbf{K}_s)_{-1/\sigma} \\ \gamma_s^{1/\sigma} &= \mathbf{1}' \mathbf{R}_s^{-1} (\mathbf{K}_s)_{-1/\sigma} \\ \mathbf{K}_s &= \begin{pmatrix} \beta \pi_{s1} a_1 R_1(s, 1) & \dots & \beta \pi_{sS} a_S R_1(s, S) \\ \dots & \dots & \dots \\ \beta \pi_{s1} a_1 R_J(s, 1) & \dots & \beta \pi_{sS} a_S R_J(s, S) \end{pmatrix}^{-1} \mathbf{1} \\ \mathbf{R}_s &= \begin{pmatrix} R_1(s, 1) & \dots & R_J(s, 1) \\ \dots & \dots & \dots \\ R_1(s, S) & \dots & R_J(s, S) \end{pmatrix}, \end{aligned}$$

where $()_x$ raises each element to the power of x .

Proof. See Appendix. ■

What this says is that when there are sufficiently many assets, you can solve the portfolio problem in closed-form, up to the a 's. If I assume log utility ($\sigma = 1$), then (following Toda (2013)), the a 's are known and constant:

$$a_s = 1 / (1 - \beta).$$

Consequently, I can solve the entire portfolio/consumption/default problem in closed-form. One caveat is that with log utility, consumption and default are constant, even

with Markov uncertainty:

$$c(\omega; s) = \omega \max(\lambda^{-1}, 1 - \beta)$$

$$D(\omega; s) = \omega \max\left(\frac{\lambda^{-1}}{1 - \beta} - 1, 0\right).$$

This allows me to establish a proposition relating debt and the maturity length of bonds.

Proposition 3 (Debt and Maturity) *Assume*

1. *Log utility*
2. *2 assets: equity (θ_a) and long-term bonds (θ_{b_δ})*
3. *2 states: bad (B) and good (G)*
4. *$R_a(s, G) > R_{b_\delta}(s, G)$ and $R_a(s, B) < R_{b_\delta}(s, B)$*
5. *bond yields are counter-cyclical.*

If returns are such that the agent is borrowing ($\theta_{b_\delta}(G) < 0$) in good times and lending in bad times, then $\theta_{b_\delta}(G)$ is declining (leverage is increasing) in the maturity of the bond, holding constant yields.

While these preference and state-space assumptions are perhaps strong, bond yield counter-cyclicalities are a reasonable assumption for emerging markets, and I am allowing for Markov uncertainty and infinite horizon. The pro-cyclicalities of debt are also realistic for emerging markets and are the natural case when interest rates are counter-cyclical. What this proposition tells us is that, all else equal, models with longer duration bonds should sustain higher levels of debt. This supports the conjecture of Tomz and Wright (2012) and others that many international finance models generate low debt because they include only one-period bonds.

The intuition for this result is the following. Consider the definition of the yield ($\mathcal{Y}$) of an arbitrary zero-bond with price $\mathcal{Q}$, face value F , and maturity length t :

$$\mathcal{Q} = \frac{F}{(1 + \mathcal{Y})^t}.$$

The one-period gross return or interest rate of this bond is $\mathcal{Q}'/\mathcal{Q}$, where $\mathcal{Q}' = F / ((1 + \mathcal{Y}')^{t-1})$. If yields are countercyclical, then by definition, prices and thus one-period interest rates must be pro-cyclical. Furthermore, it is easy to show that holding $\mathcal{Y}$ constant

the return becomes more procyclical as t increases. Note also that realized equity returns are generally pro-cyclical, and income is by definition pro-cyclical. Consider an agent who wants to borrow for one-period to finance an equity investment (if he issues a long-term bond, assume he buys it back after one period). If he uses a one-period bond, the one-period interest rate and thus debt payment is fixed. If he uses a longer maturity bond, the one-period effective interest rate, Q'/Q , becomes random. Because Q'/Q is pro-cyclical, the one-period debt payment is lowest exactly when income and equity returns are low. In other words, a negative position in the long-term bond serves to hedge equity and income risk. This hedging benefit, which increases with maturity, leads a risk averse agent to take a larger debt and thus equity position, all else equal.

It is somewhat unnatural to hold yields constant. In the following numerical example, I hold the pricing kernel constant, and let the yield change with maturity. Suppose equity is a claim on GDP, which grows either at gross rate .99 (B) or 1.04 (G). The probability of staying in the current state is $\pi_{ss} = .8$. If the pricing kernel is

$$M = \begin{pmatrix} 1.022 & .630 \\ 1.272 & .920 \end{pmatrix},$$

then one-period yields in B and G are, respectively, 6% and 1%. The equity premium is 1.6%, and the price-dividend ratio is 30 in B and 36.5 in G . For any maturity level δ , Corollary 3 allows us to solve the agent problem in closed-form. Note that in this case, unlike in the proposition, the agent is borrowing in both states. Holding M constant, we see in figure 2 that debt increases as maturity lengthens.

4 Quantitative Analysis

4.1 GDP Processes

I assume the emerging market agents and international investors have exogenous GDP growth following a joint 9 state Markov process. Specifically, growth for either agent is either bad (B), medium (M), or good (G). The nine possible combinations are the 9 states (BB, BM, BG, MB,...). In what follows, I will refer to the international investors as the US, to simplify exposition. Rows represent the US, and columns correspond to emerging market states. The following table gives the possible combinations of GDP growth:

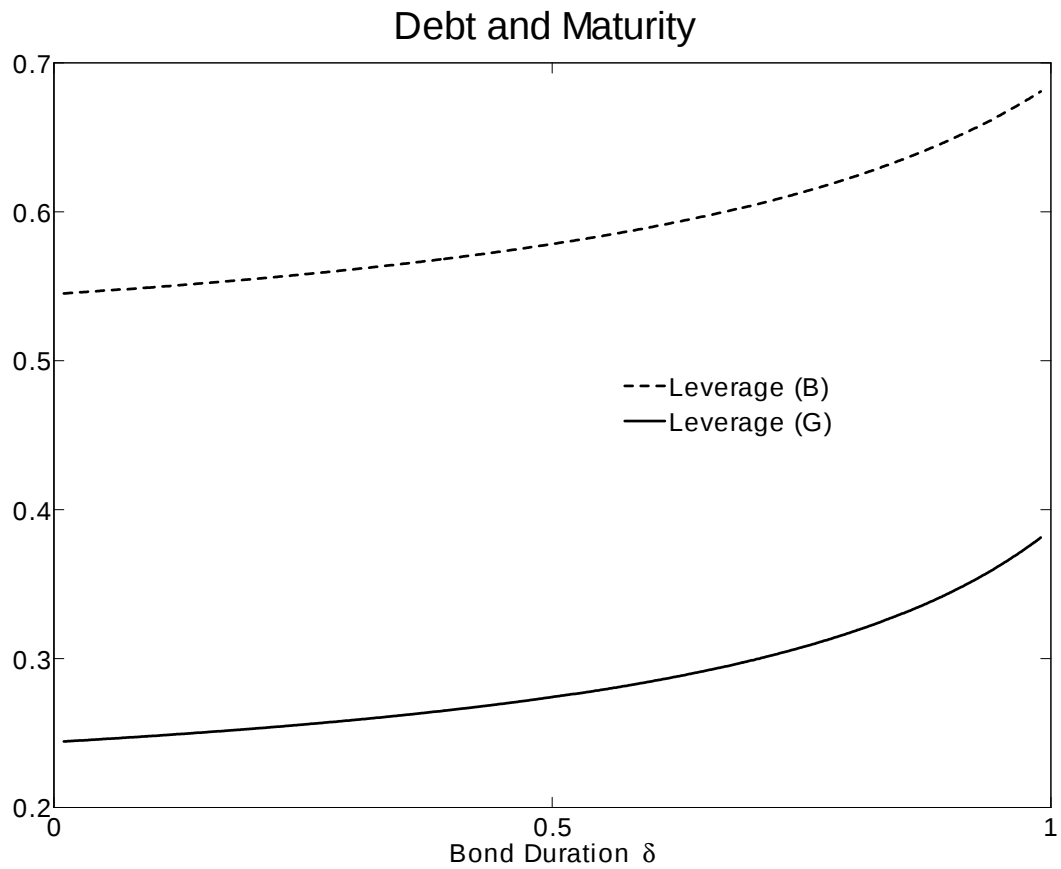


Figure 2: The figure displays the ratio of debt to equity, $-\theta_{b_\delta}(s)/\theta_a(s)$, as a function of the maturity of the bond, where $s \in \{B, G\}$ is the exogenous state.

Table 1: Exogenous GDP Growth Grid

		EM GDP Growth		
		G	M	B
US GDP Growth	G	(.05, .06)	(.05, .02)	(.05, −.02)
	M	(.02, .06)	(.02, .02)	(.02, −.02)
	B	(−.01, .06)	(−.01, .02)	(−.01, −.02)

I estimate a Markov transition matrix by maximum likelihood using annual real GDP growth (1960-2011) from the World Bank databank. Emerging market growth is the Latin America & Caribbean (developing only) aggregate, and international investor growth is the world aggregate.³² The estimated unconditional probabilities are

Table 2: Unconditional Probabilities (%)

	EM-G	EM-M	EM-B
US-G	32.4	16.9	.8
US-M	13.5	27.4	4.7
US-B	.3	2.4	1.6

The transition probabilities are

Table 3: Estimated Transition Probabilities (%)

	GG	GM	GB	MG	MM	MB	BG	BM	BB
GG	48	17	<1	13	18	2	<1	1	<1
GM	32	27	2	7	25	6	<1	1	1
GB	17	36	5	3	25	13	<1	1	1
MG	31	9	<1	24	28	2	1	4	1
MM	22	15	1	14	36	6	<1	4	2
MB	12	20	2	6	37	15	<1	3	5
BG	16	3	<1	35	29	1	3	11	2
BM	11	6	<1	20	40	5	1	11	5
BB	7	9	1	9	42	13	<1	9	10

Notice first that the most likely states are the good-medium combinations. Also, the probability of negative growth for either is only about 10%. The least likely state is GB. This will also happen to be the worst state (lowest $\mathcal{V}_s$) for emerging markets.

³²Since 1960, the correlation between U.S. real growth and world real growth is .82.

My gridpoints are perhaps not optimal from a statistical perspective. But, they reflect, roughly, what one usually thinks of as bad, medium, and good growth. Using such a small state space limits degrees of freedom and thus makes it harder to match moments below. The advantage is that one can easily inspect 9 graphs or numbers at once. This makes the results and mechanics of the model transparent. Standard moments are close to empirical counterparts:

Table 4: GDP Growth Moments (1960-2011)

	Latin America (EM)		World	
	Model	Data	Model	Data
Mean	.036	.038	.034	.036
Standard Deviation	.025	.025	.017	.017
Correlation with World	.42	.55	1	1

4.2 International State Prices or Pricing Kernel

Recall that the pricing kernel μ is meant to represent the marginal rate of substitution of the U.S. or international investors. For this reason, I try to find μ 's that generate plausible asset prices for the U.S. In particular, in the tradition of Lucas (1978), I assume that U.S. GDP is the dividend of the U.S. stock market. Let Y and $\mathcal{P}_{US}$ be the U.S. dividend and price-dividend ratio. Let $\mathcal{G}(s')$ be the growth rate of U.S. GDP so that $Y' = \mathcal{G}(s')Y$. As with emerging market equity above, we have

$$\mathcal{P}_{US}(s) = \sum_{s'=1}^S \pi_{ss'} \mu_{ss'} (\mathcal{P}_{US}(s') + 1) \mathcal{G}(s')$$

$$\overrightarrow{\mathcal{P}_{US}} = [\mathbf{I} - (\Pi \circ M \circ \mathcal{G})]^{-1} (\Pi \circ M \circ \mathcal{G}) \mathbf{1},$$

where $\mathcal{G}$ is the matrix of growth rates. Note this asset is not traded in the model. I introduce it simply to discipline the state prices.

As $S = 9$, there are 81 coefficients in μ . Based on asset price regularities and introspection, I restrict μ in the following ways. First, I impose that relative state prices do not depend on emerging market growth. I call this EMI (emerging market independence). To formalize this assumption, we need more notation. Let $\mathcal{W}(s)$ be the U.S. part of the state. For example, $\mathcal{W}(BG) = B$ and $\mathcal{W}(MB) = M$. What EMI says is if $\mathcal{W}(s_1) = \mathcal{W}(s_2)$ and $\mathcal{W}(s'_1) = \mathcal{W}(s'_2)$, then

$$\frac{\mu_{s_1 s'_1}}{\sum_s \mu_{s_1 s}} = \frac{\mu_{s_2 s'_2}}{\sum_s \mu_{s_2 s}}.$$

There are three immediate implications that illuminate the meaning of this assumption. First, EMI implies that $\mu_{ss_1} = \mu_{ss_2}$, provided the U.S. state is the same in s_1 and s_2 . In other words, given the state today, the state price for state s' tomorrow will vary only with the U.S. state $\mathcal{W}(s')$. Second, across emerging markets states today, the relative state prices are constant. In other words, for any s_1 and s_2 , μ_{ss_1}/μ_{ss_2} does not depend on the emerging market part of s . Third, the *level* of state prices may depend on the emerging market state. In particular, $\sum_{s'} \mu_{ss'}$ may depend on the emerging market part of s .

I restrict the pricing kernel in this way for two reasons. First, the pricing kernel represents how U.S. investors value consumption in one state versus another. Given the small open economy assumption, it does not make sense for emerging market outcomes to cause variation in the U.S. marginal rate of substitution. Second, risk-free rates *should* depend on the emerging market state. This is because the GDP growth processes are correlated. The emerging market state contains information about the expected average growth rate for the U.S., which should impact risk-free rates.

Consider the following microfoundation for μ of this kind:

$$\mu_{ss'} = \beta_s \left[\xi_{\mathcal{W}(s)\mathcal{W}(s')} (\mathcal{G}(s'))^{-\sigma} \right],$$

where $(\mathcal{G}(s'))^{-\sigma}$ is the CRRA marginal rate of substitution, ξ is a taste shock (prox-ying for, say, habit), and β_s is a discount rate shock. While β_s may depend on the emerging market, the terms inside the square brackets depend just on U.S. states.

The second set of constraints on μ are sign and monotonicity restrictions on U.S. prices and returns. I impose these so that prices satisfy intuitive qualitative properties. The constraints are

$$\begin{aligned} C1) : & \text{corr}(\mathcal{P}_{US}, \mathcal{G}) > 0 \\ C2) : & \text{corr}(P_D, g) > 0 \\ C3) : & R_{B_1}(BB) + .01 < R_{B_1}(MM) \\ C4) : & R_{B_1}(MM) + .01 < R_{B_1}(GG) \\ C5) : & R_{B_1}(jB) \leq R_{B_1}(jM) \leq R_{B_1}(jG), \forall j. \end{aligned}$$

$C1) - C4)$ say that price-dividend ratios and 1-year risk-free interest rates are procyclical. Indeed, in the U.S., price-dividend ratios and short-term risk-free rates generally move positively with the state of the economy. $C5)$ is related to the EMI

assumption. When the $C5$) constraints are equalities, 1-year risk-free rates do not depend on the emerging market state. In this case, most variation in prices comes from movement across U.S. states. So that emerging market states do not play a large role in matching asset pricing facts, I penalize deviations from equality in $C5$). That is, I restrict the extent to which $R_{B_1}(MM)$ differs from $R_{B_1}(MG)$, for example.

Subject to $C1) - C5$), EMI, and the penalties, I look for μ 's to minimize the distance between pricing moments from the model and from the data. The following table (table 5) displays the results:

Table 5: Calibration of State Prices

	Model	US, 1970-present
US Equity Premium	6.11%	6%
Average US P/D ($\mathcal{P}_{US}$)	52.37	50
St. Dev. of US P/D ($\mathcal{P}_{US}$)	13.42	19
EM Equity Premium	1.15%	
Correlation with US Growth		
10-1 risk-free yield curve	−.48	−.27 (−.45, 1984-present)
Expected US Eq. Premium	−.88	
Expected EM Eq. Premium	−.89	
Correlation with EM Growth		
Expected EM Eq. Premium	−.31	

Notes: Targeted moments in bold

Data Sources: Website of Ken French, FRED, Website of Amit Goyal

Figure 3 displays the zero risk-free yield curve. These zeroes are not traded in the model (except for the 1-year), but these plots are a convenient way to illustrate the prices of the model. As in the U.S. data, the yield curve is steepest in the lowest growth states and flattens as growth increases. Given the mean reversion of growth rates, this is related to the famous fact that flat or inverted yield curves predict recessions. See, for example, Estrella and Hardouvelis (1991).

Figure 4 displays the state prices $\mu_{ss'}$. Under the interpretation that the U.S. investors are risk neutral with subjective probabilities, one may think of this plot as subjective probabilities divided by true probabilities. Effectively, this figure shows how the representative investor must weight different states of the world in order to produce realistic asset prices. Roughly, I find that he must overweight his current state and underweight movement to different states.

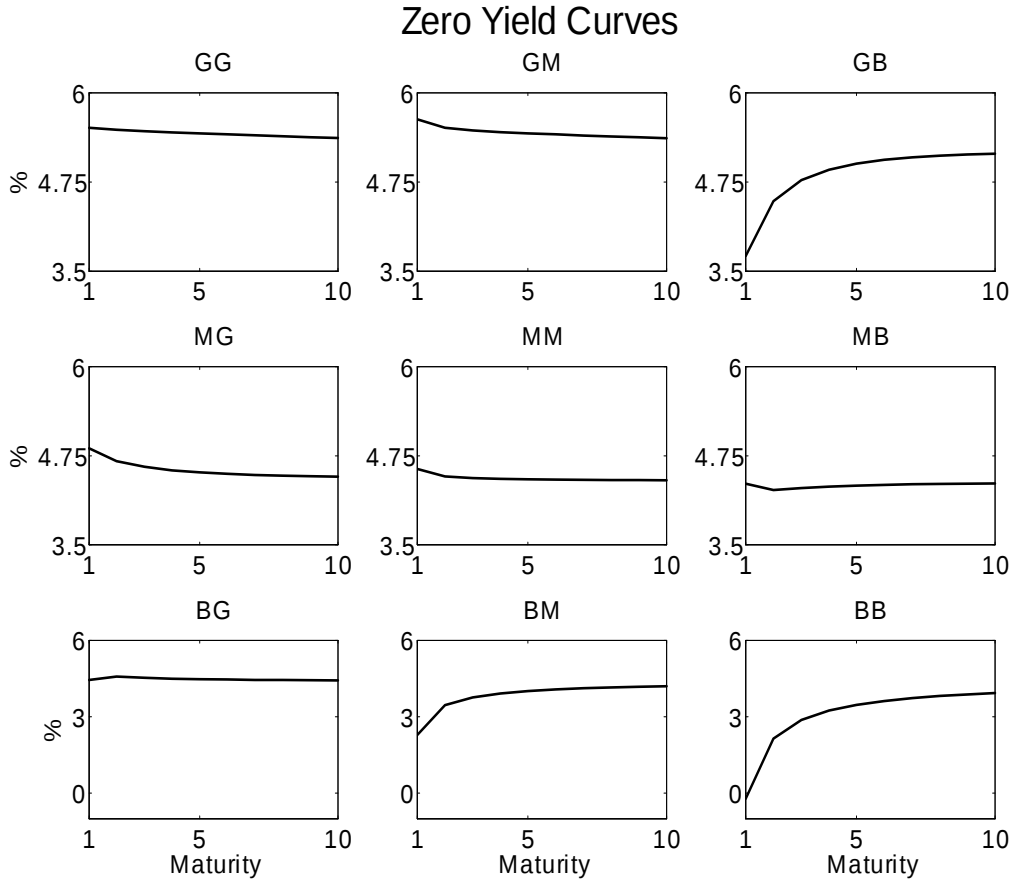


Figure 3: The figure displays, across states, the model's zero yield curves for maturities of 1 to 10 years. A t -year zero is a zero coupon bond with a maturity of t years.



Figure 4: The Estimated State Prices. For all $s, s' \in \{GG, GM, GB, MG, \dots\}$, the shade of box (s, s') represents the state price $\mu_{ss'}$. Light boxes indicate high weighting relative to the underlying probabilities. Darker boxes correspond to low weighting.

4.3 Calibration

Quantitatively, I find that emerging market agents have, in some states of the world, a strong desire to sell off equity. While it is true that some nations have substantial external equity liabilities, in no large country do foreigners own the majority of claims to output. Consider the distinction between GNI and GDP. While GDP measures output within a border, GNI reflects net income earned by residents. Therefore, with very high equity liabilities, GNI would be much less than GDP. Such a phenomenon is not a feature of the data. In Latin America, GNI tends to be slightly lower, around 2-5% less, than GDP.³³ For this reason, I restrict the extent to which agents are able to sell equity liabilities. In particular, I include the following home bias constraint:

$$\theta_a(s) \geq \alpha P_D(s),$$

where $\alpha > 0$. Using the definition of θ_a , the constraint is equivalent to

$$a' E[y' | s, y] \geq \alpha E[g(s') | s] (\omega - c(\omega; s) + D(\omega; s)).$$

I include the P_D term in the first constraint so that it does not appear in the second. The second constraint puts a lower bound on GNI tomorrow. These constraints are a reduced-form for part of GDP being nonmarketable.

The following table describes my calibration:

Table 6: Calibration of Emerging Market Parameters

Parameter	Value	Target
σ	20	
β	.86	$E[\omega'/\omega] = E[y'/y]$
α	.015	$\theta_a(GB) \geq 1.05$
$\lambda^{-1/\sigma} (= \frac{\varepsilon}{\omega})$	.0391	Default in GB and GM

This value for σ is arguably high. I choose a high value to limit portfolio volatility. Note, however, that lesser values for σ also generate portfolio pro-cyclicality. I will discuss this further below. The discount rate β governs the savings rate out of wealth (via the average level of $\mathcal{V}_s$) and thus the average growth rate of wealth. However, as the model's quantities of interest are proportional to ω , other implications of the model do not depend strongly on β . $\beta = .86$ means that wealth grows, on average, at the rate of GDP growth. A value of .96 would cause wealth to grow faster than

³³Sources: World Development Indicators (World Bank), and my calculations.

GDP but would not greatly impact other results. Given σ , choosing λ is equivalent to choosing the minimum consumption level (normalized by wealth). I set λ so that consumption is at least 3.91% of wealth. Recall that $D(\omega; s) = \omega \max\left(\frac{\lambda^{-1/\sigma}}{\mathcal{V}_s} - 1, 0\right)$. This value for λ implies that default occurs in just the two worst states, which are GB and GM in equilibrium. That is, $\lambda^{-1/\sigma} > \mathcal{V}_s$ only when $s \in \{GB, GM\}$. Note that I did not simply assume GB and GM are the worst states: $\mathcal{V}_s$ is an equilibrium quantity that depends on the prices of emerging market bonds. Finally, $\alpha = .015$ implies the following portfolio constraints

Table 7: $\theta_a(s)$ lower bounds

	EM-G	EM-M	EM-B
US-G	1.065	1.0513	1.050
US-M	.863	.852	.845
US-B	.870	.874	.888

For this calibration, I find that these constraints only bind in the US-G states. In US-M and US-B, equity prices fall, the emerging market agents buy back equity, and θ_a constraints do not bind. In the US-G states, when prices are high and the expected equity premium is low, the agents try to sell off almost all of their GDP equity. As US-G coincides with EM-G on average, this sell off creates pro-cyclicality in equity liabilities. I choose α so that $\theta_a(GB) \geq 1.05$.

Finally, to simplify exposition in what follows, I impose $\theta_{B_L} = 0$. That is, I exclude the long-term U.S. bond (but keep the short-term U.S. bond). As I explained above, introducing it would create no additional computational burden. I am currently pursuing extensions with additional states and additional assets including U.S. equity.

4.4 Quantitative Results

As default occurs in states GB and GM, the probability of default is 17.7% ($\Pr(GB) = 16.9\%$ and $\Pr(GM) = .8\%$). This is in the 1.8-19% range from Tomz and Wright (2012). Many sovereign default papers target a default probability of around 2%. This number reflects the unconditional probability of entering default, which may be the appropriate target when the model's default punishment is market exclusion. In my model, however, default simply means missing payments. In reality (and in my model), default episodes may last many years and entail a series of missed payments. The 19% number from Tomz and Wright (2012) is based on years spent in default and is thus more appropriate for my study. Consider post-1970 Argentina. They

defaulted 3 times but spent almost 15 years in default.³⁴

Table 8: Equilibrium Haircuts

		Previous State (s)								
		BB	BM	BG	MB	MM	MG	GB	GM	GG
s'=GB	h_1	.14	.18	.16	.74	.48	.30	.34	.34	.27
	h_L	.02	.02	.02	.08	.05	.03	.04	.04	.03
s'=GM	h_1	.02	.03	.02	.11	.07	.05	.05	.05	.04
	h_L	.00	.00	.00	.01	.01	.01	.01	.01	.00

Table 8 displays the long- and short-term haircuts (h_L and h_1) in GB and GM. As explained above, haircuts depend on both the current state and predecessor state (which determines the level and maturity structure of debt). Realized haircuts depend on equilibrium maturity structure. The average realized haircuts in GB and GM are, respectively, 20% and 3%. So, while payments are missed in about 17% of years, large defaults (with haircuts $>7\%$) occur only about 1% of the time. As in the data, haircuts are harsher on short-term debt.³⁵ While my calibration is not targeting the level of haircuts, these numbers are reasonable relative to empirical studies. In the Benjamin and Wright (2009) sample for the 7 largest Latin American economies, the haircut interquartile range is 2-33%.³⁶

Table 9 displays the equilibrium spreads, which are not targets of my calibration.

Table 9: Equilibrium Spreads (%)

	GB	GM	GG	other states
10-year Spread	.26	.21	.18	~ 0
1-year Spread	6.93	3.50	1.28	~ 0
Spread Curve	-6.67	-3.29	-1.10	~ 0

Long-term spreads are (counterfactually) less than short-term spreads. The spread curve, the difference between the 10-year spread and 1-year spread, is strongly procyclical as in the data.³⁷

Figure 5 plots the optimal portfolio weights θ in each state. Inspection of the figure yields four main observations. First, within U.S. states (rows), portfolios are very similar. Qualitatively, the emerging market states (columns) do not cause much

³⁴See Benjamin and Wright (2009).

³⁵Note that this is a feature of my default model and not a result per se.

³⁶This range is for the following episodes: ARG89, ARG01, BRA83, CHI83, MEX82, VEN90, VEN95, VEN04, COL85, PER80, PER83.

³⁷See Arellano and Ramanarayanan (2012) or Broner, Lorenzoni, and Schmukler (2013).

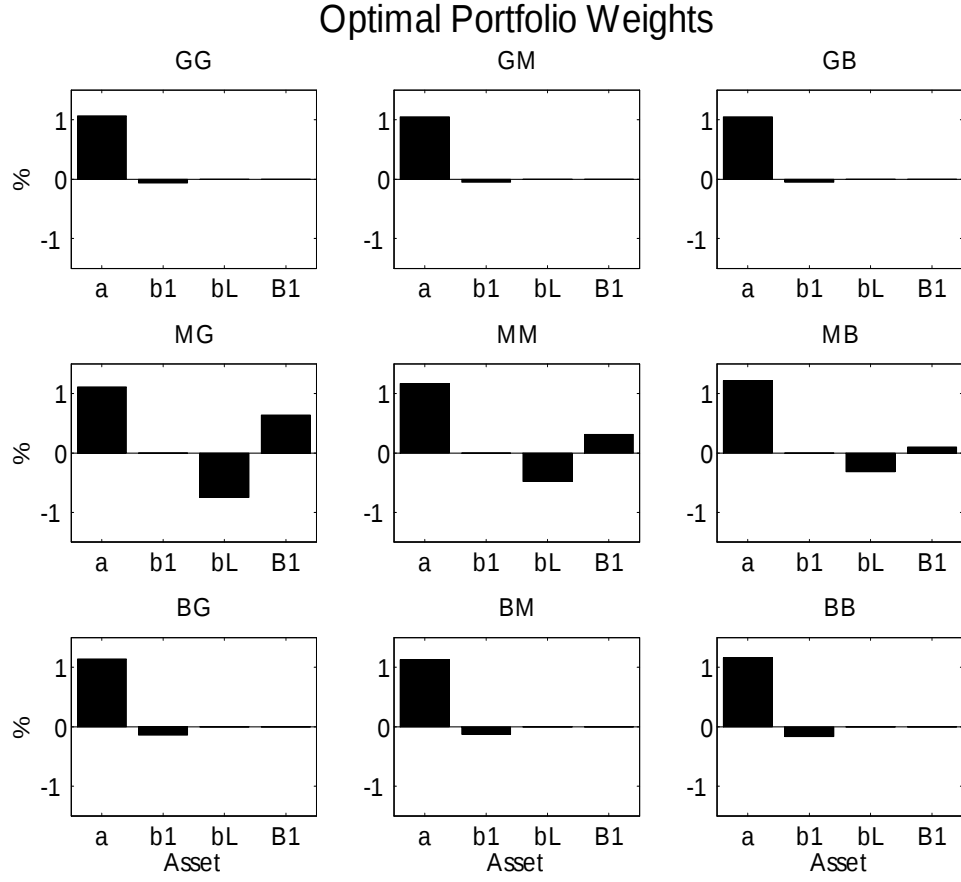


Figure 5: The portfolio weights across states. The columns a , $b1$, bL , and $B1$, which sum to 1 in each subplot, are the optimal portfolios θ .

variation in portfolios. In other words, international price fluctuations are key in understand emerging market portfolio flows and composition. This is true also for default in my model: bad investment prospects, which trigger default, are caused by the combination of bad emerging market growth, high risk-free rates, and low expected equity returns (high equity prices). Second, the emerging market agents use long-term debt in medium U.S. states and only short-term debt otherwise. That is, U.S. fluctuations alone determine the maturity structure of gross emerging market debt. As US-B is relatively unlikely and because the EM state is positively correlated with the U.S. state, this effect induces pro-cyclicality in the short-term debt share (which is either 0 or 1 in this calibration). Third, across US-M states, there is substantial movement in gross bond positions. In particular, as the EM states goes from good to bad, both gross long-term borrowing and gross lending decline. In other words, total bond capital flows are pro-cyclical in US-M. Net debt, the difference between bL and $B1$, does not appear to be moving much. Note that in the US-M states, the emerging market agents exploit the risk-free yield, borrowing long and lending short. More on this below. Fourth, the equity shares are lowest in the US-G states. As mentioned above, this is when the home bias constraints are binding. The U.S. boom pushes up asset prices and causes an inflow of equity into the emerging market. As the EM and U.S. states are positively correlated, this generates pro-cyclicality in equity inflow. Combining the third and fourth points, it seems that gross inflow and outflow are pro-cyclical. Indeed they are, as we see in Table 10.

Table 10 describes the cyclical properties of gross flows and portfolio composition in the model and their empirical counterparts across the largest Latin American economies.

Table 10: Cyclical Properties of Gross Flows and Portfolios

	Model	Argentina	Brazil	Colombia	Mexico	Average‡
Correlation with EM GDP Growth						
Equity Inflow*	.20	.42	.11	.31	.01	.21
Short-term Debt Share†	.29	.22	-.09	.01	.19	.08
Gross Capital Inflow*	.12	.22	.25	.26	.04	.19
Gross Capital Outflow*	.13	.24	.10	.38	-.24	.12

Data Sources: Model, Lane and Milesi-Ferretti (2007), WDI (World Bank)

* Model: normalized by wealth. Data: normalized by nominal GDP, 1970-2007.

† Fraction of debt maturing in 1 year. In data, detrended by HP filter, 1970-2011.

‡ Average of numbers for Argentina, Brazil, Colombia, Mexico

Note: See appendix for panel regressions.

While I have not targeted these moments in my calibration, the correlations with GDP growth from the model are quantitatively reasonable. While the correlation of the short-term debt share with GDP growth is high, the other moments fall within the cross-country ranges. Moreover, the numbers of .2, .12, and .13 for equity inflow, capital inflow, and capital outflow are close to the cross-country averages (considering just Argentina, Brazil, Colombia, and Mexico). A caveat in comparing these numbers is that while I normalize inflow and outflow by wealth in the model, in the data I normalize by GDP. Note that to make quantities stationary, one must normalize by *something*. Here, in using GDP I follow Broner, Didier, Erce, and Schmukler (2013). Alternatively, one could look at growth rates in the data. In the appendix, I conduct panel regressions showing, for a larger set of Latin American countries, that equity inflow, the short-term debt share, gross inflow, and gross outflow are significantly pro-cyclical. This holds with respect to both growth rates and normalization by GDP. With respect to capital inflow and outflow, this is consistent with the findings of Broner, Didier, Erce, and Schmukler (2013).

The pro-cyclicity of short-term debt is not inconsistent with the recent findings of Arellano and Ramanarayanan (2012) or Broner, Lorenzoni, and Schmukler (2013). These studies show that short-term debt issuance rises when spreads are high. This actually arises in my model also: spreads only spike in US-G, which is when emerging markets switch to short-term debt. Effectively, my model exhibits a hump-shaped relationship between maturity shortness and emerging market growth. Neither are my empirical findings inconsistent with these papers. While I look at the level or change in level of the short-term share of total public and private external debt, these

studies look at post-1990 sovereign bond issuance.³⁸

I explained above that the pro-cyclicality in equity inflow is likely the result of pro-cyclicality in prices and the resulting counter-cyclicality in the expected equity premium. But, what explains the other relationships? First, consider the maturity structure of debt. When risk-free rates are high (US-G states), long-term bond prices are at their lowest. Given mean reversion in the model, in these states there is a decent chance long-term prices will subsequently spike. When borrowing long-term, this corresponds to a large negative 1-year holding period return. In other words, the effective short-term interest rate (R'_{b_L}) on long-term bonds is very high in US-G states. For this reason, the emerging market uses only short-term debt. The agents use short-term debt in US-B states for a different reason. Here, the yield curve is steep, which induces short-term borrowing only. In these states, mean reversion implies that long-term borrowing actually provides hedging (bond prices will fall in the future). However, quantitatively the yield curve effect dominates. In US-M states, the yield curve is not too steep. Also, as US-B is relatively unlikely, there is a decent chance long-term bond prices will fall (in the event of a transition to US-G). In short, in medium times for the U.S., long-term borrowing is not too expensive *and* provides beneficial hedging (recall that US-G is often bad for the emerging market). Therefore, the emerging market switches to long-term debt in US-M states.

Next, consider gross inflow and outflow. The changes in equity liabilities described above contribute to pro-cyclicality in gross inflow. However, the emerging market yield curve investments also play a role. When the emerging market is borrowing long-term (in US-M), it is mostly to finance equity holdings. However, when the agents are borrowing long, they also purchase foreign risk-free bonds, which provide a substantial return. Furthermore, in this calibration, 1-year expected returns on long-term bonds are relatively low in US-M states. Now, within US-M, consider what happens as the EM state goes from G to B (as we move to the right in figure 5). One effect is that the risk-free return on foreign bonds declines. A second effect is that, by the positive correlation of EM and U.S. states, a transition to US-G (in which bond prices collapse) becomes less likely. In total, the yield curve investment of the US-M states has a lower return and more risk as the EM states moves from G to B. It is for this reason that total bond capital flows are pro-cyclical within the US-M states. *Simply put, the pro-cyclicality of quantities in my model is a natural consequence of pro-cyclicality in stock prices and risk-free rates.*

³⁸Rodrik and Velasco (1999) also present evidence of a positive correlation between GDP and maturity shortness.

Finally, in my baseline calibration gross debt/GNI is too *high* on average and too volatile. While about 20% of the time the ratio is below 50%, Debt/GNI often spikes to very high levels. The basic reason for this is the marketability of GDP, which is not in other models. Intuitively, the issue is that agents in my model can sell stock to repay debt. This means that liquid wealth is on the order of the *price* of the GDP asset, not its dividend flow. Note that future liquid wealth is what limits the debt of a borrower. As prices are, say, 50 times greater than dividends, this means that agents may borrow very large amounts relative to the dividend flow (GDP). There are two main implications of this: (i) The quantitative results of previous models may not be robust to the inclusion of equity liabilities, which are empirically substantial.³⁹ (ii) By including financial frictions like borrowing or leverage constraints that limit gross debt, my model *can* produce reasonable gross debt/GNI levels.

Another contributor to high gross debt is the yield curve investment, which I described above, that the agents adopt in US-M states. Many asset pricing puzzles say, basically, that given empirical quantities (for example, consumption) and CRRA utility, equilibrium asset prices are unrealistic. The flip side of this is that CRRA agents adopt unrealistic positions in response to realistic asset prices. In US-M states, the pricing kernel is such that the CRRA emerging market agents perceive highly attractive investment opportunities. These investments involve large gross long-term debt positions. By setting $\sigma = 20$, equilibrium portfolios are not excessively volatile. High risk aversion makes the agents very afraid of the potential but relatively unlikely losses from the yield curve investments. As I mentioned above, the cyclicity, spread, and haircut results are not especially sensitive to σ . Another way to limit volatility in gross positions is to adjust risk-free rates, in particular how they depend on emerging market states. I am currently pursuing an extension in which I take portfolio volatility into consideration when estimating state prices. I expect this will yield reduced debt levels.

5 Conclusion

What are the determinants and economic consequences of cross-border asset positions? This question is as old as the economics discipline itself. The pre-18th century mercantilists, whom Adam Smith later opposed, argued that power and prosperity stem from positive net exports and the resulting accumulation of gold and silver. Re-

³⁹In 2011, 71.34% of Brazil's external liabilities were in the form of equity. Sources: Lane and Milesi-Ferretti (2007) and my calculations.

lated discussions have continued to present times. Recently, Caballero (2009) argued that global demand for safe U.S. assets precipitated the late 2000s financial crisis. Shin (2012) argued that gross capital flows between American and European banks created the easy credit conditions that may have helped cause the crisis. Reinhart and Rogoff (2009) and others have suggested that high levels of gross external debt are associated with low economic growth. In short, the level and composition of gross capital flows are at the center of current policy debates and empirical macroeconomics and international finance.

My goal in this paper has been to further our ability to model cross-border portfolios. I developed a new quantitative portfolio choice model and applied it to emerging market international finance. Allowing for a rich set of assets, partial default, and world financial market fluctuations, the model is quantitatively consistent with many empirical regularities that are either beyond the scope of or difficult to explain in the standard models of external debt and default. While my focus has been positive, throughout the paper I have alluded to policy questions and counterfactuals. The next step is to explore the normative implications of the model.

6 Appendix

6.1 Panel Regressions

Table A1: Portfolio Composition: Fixed Effects Panel Regressions (Detrending)

	Dependent Variable	
	Short-term Debt Share*	Equity Liabilities†
Constant	.11	−1.65
(p-val)	(.00)	(.00)
Real GDP‡	.35	2.56
(p-val)	(.00)	(.05)
R ²	.13	.19
obs	282	265
sample	annual, 1970-2011	annual, 1970-2007

Countries: Argentina, Brazil, Chile, Colombia, Mexico, Peru, Venezuela

* HP filtered ($\lambda = 100$), public and private external debt, WDI (World Bank)

† HP filtered ($\lambda = 100$), Lane and Milesi-Ferretti (2007)

‡ HP filtered ($\lambda = 100$), WDI (World Bank)

p-val from robust standard errors

Table A2: Portfolio Composition: Fixed Effects Panel Regressions (Differencing)

	Dependent Variable		
	Δ Short-term Debt Share*	% Δ Equity Liabilities [†]	Δ Equity Liabilities [†] /GDP
Constant	-.80	9.50	1.11
(p-val)	(.00)	(.00)	(.00)
% Δ Real GDP [‡]	.23	1.36	.25
(p-val)	(.00)	(.00)	(.01)
R ²	.06	.08	.08
obs	275	258	258
sample	annual, 1970-2011	annual, 1970-2007	annual, 1970-2007

Countries: Argentina, Brazil, Chile, Colombia, Mexico, Peru, Venezuela

* Public and private external debt, WDI (World Bank)

† Lane and Milesi-Ferretti (2007)

‡ WDI (World Bank)

p-val from robust standard errors

Table A3: Gross Capital Flows: Fixed Effects Panel Regressions

	Dependent Variable			
	Capital Inflow* (% Δ)	Capital Inflow*/GDP	Capital Outflow [†] (% Δ)	Capital Outflow [†] /GDP
Constant	8.76	3.77	13.33	2.76
(p-val)	(.00)	(.00)	(.00)	(.00)
% Δ Real GDP	.57	.23	.62	.10
(p-val)	(.01)	(.00)	(.04)	(.02)
R ²	.06	.04	.02	.01
obs	258	258	258	258
sample (annual)	1970-2011	1970-2007	1970-2007	1970-2007

Countries: Argentina, Brazil, Chile, Colombia, Mexico, Peru, Venezuela

Sources: Lane and Milesi-Ferretti (2007) and WDI (World Bank)

* Capital Inflow: change in external liabilities

† Capital Outflow: change in external assets

p-val from robust standard errors

6.2 Proof of Corollary 1

By definition, the delivery rate from s to s' is

$$d_{ss'} = 1 - \frac{\int D(\omega_i; s), s' di}{-\int (b_1(\omega_i, s) + \delta b_L(\omega_i, s)) di}.$$

Let $D_r(s)$ and $c_r(s)$ denote, respectively, $D(\omega_i; s)/\omega_i$ and $c(\omega_i; s)/\omega_i$ for an arbitrary agent. Proposition 1 implies that $D_r(s)$, $c_r(s)$, and the optimal portfolio weights $\theta(s)$ are the same for all agents. From $\theta_x = P_x x' / (\omega - c + D)$ we then have

$$d_{ss'} = 1 - \frac{D_r(s')}{-(1 - c_r(s) + D_r(s)) \left(\frac{\theta_{b_1}(s)}{q_1(s)} + \delta \frac{\theta_{b_L}(s)}{q_L(s)} \right)} \left(\frac{\int \omega'_i di}{\int \omega_i di} \right). \quad (1)$$

From $\omega'_i = R(\theta^*(s); s, s')(\omega_i - c_i + D_i)\varepsilon'_i$, we have $\omega'_i = R(\theta^*(s); s, s')(1 - c_r(s) + D_r(s))\omega_i\varepsilon'_i$ and thus

$$\begin{aligned} \frac{\int \omega'_i di}{\int \omega_i di} &= \frac{\int (R(\theta^*(s); s, s')(1 - c_r(s) + D_r(s))\omega_i\varepsilon'_i) di}{\int \omega_i di} \\ &= R(\theta^*(s); s, s')(1 - c_r(s) + D_r(s)) \left(\frac{\int \omega_i \varepsilon'_i di}{\int \omega_i di} \right) \\ &= R(\theta^*(s); s, s')(1 - c_r(s) + D_r(s)), \end{aligned}$$

where the last line follows from i.i.d. ε 's: $\int \varepsilon'_i \omega_i di = \int \omega_i di$. Combining this with equation 1, we have

$$d_{ss'} = 1 - \frac{D_r(s') R(\theta^*(s); s, s')}{-\left(\frac{\theta_{b_1}(s)}{q_1(s)} + \delta \frac{\theta_{b_L}(s)}{q_L(s)} \right)},$$

which depends just on s and s' . ■

6.3 Proof of Proposition 1

The problem is

$$\begin{aligned} v(\omega; s) &= \max_{c, D, \theta} \left\{ \frac{c^{1-\sigma}}{1-\sigma} - \lambda \omega^{-\sigma} D + \beta E(v(\omega'; s') | s) \right\} \text{ subject to} \\ (i) : \omega' &= R(\theta; s, s')(\omega + D - c)\varepsilon' \\ (ii) : \theta &\in \Theta(s) \\ (iii) : D &\geq 0. \end{aligned}$$

The proof has three steps:

1. Guess that we can find a_s such that $v(\omega; s) = a_s \frac{\omega^{1-\sigma}}{1-\sigma}$. Plugging in this guess, the optimization problem becomes

$$v(\omega; s) = \max_{c, D} \left\{ \frac{c^{1-\sigma}}{1-\sigma} - \lambda \omega^{-\sigma} D + (\omega + D - c)^{1-\sigma} \tilde{\beta} \mathcal{U}_s \right\} \text{ subject to}$$

$$(iii) : -D \leq 0 \quad (L_s),$$

where L_s is the multiplier on constraint (iii) and

$$\mathcal{U}_s = \max_{\theta \in \Theta(s)} E \left[a_{s'} \frac{(R(\theta; s, s'))^{1-\sigma}}{1-\sigma} | s \right] < 0$$

$$\tilde{\beta} = \beta E \left((\varepsilon')^{1-\sigma} \right).$$

2. The remaining optimization problem is concave in c and D . The c and D FOCs are

$$c : c = \frac{\omega + D}{1 + \left(\tilde{\beta} \mathcal{U}_s (1 - \sigma) \right)^{1/\sigma}}$$

$$D : -\lambda \omega^{-\sigma} + (1 - \sigma) (\omega + D - c)^{-\sigma} \tilde{\beta} \mathcal{U}_s + L_s = 0. \quad (2)$$

When $D \geq 0$ is binding,

$$c = \frac{\omega}{1 + \left(\tilde{\beta} \mathcal{U}_s (1 - \sigma) \right)^{1/\sigma}},$$

and L_s is given by equation 2. When $D > 0$, $L_s = 0$, and we have

$$c : c = \frac{\omega + D}{1 + \left(\tilde{\beta} \mathcal{U}_s (1 - \sigma) \right)^{1/\sigma}}$$

$$D : D = \left(\left(\tilde{\beta} \mathcal{U}_s (1 - \sigma) \right)^{1/\sigma} - \frac{1}{\lambda^{-1/\sigma}} \right) \lambda^{-1/\sigma} \omega + c.$$

Let market prospects $\mathcal{V}_s$ be

$$\mathcal{V}_s = \frac{1}{1 + \left(\tilde{\beta} \mathcal{U}_s (1 - \sigma) \right)^{1/\sigma}}.$$

Because, $\mathcal{U}_s < 0$ and $\mathcal{U}_s(1 - \sigma) > 0$, $\mathcal{V}_s$ is a monotonic transformation of $\mathcal{U}_s$, and $0 < \mathcal{V}_s < 1$. We can now write the FOCs in the $L_s = 0$ case as

$$\begin{aligned} c &= \mathcal{V}_s(\omega + D) \\ D &= \left(\frac{\lambda^{-1/\sigma}}{\mathcal{V}_s} - \lambda^{-1/\sigma} - 1 \right) \omega + c. \end{aligned}$$

Algebra then yields, concluding the binding case, that

$$\begin{aligned} c &= \omega \lambda^{-1/\sigma} \\ D &= \left(\frac{\lambda^{-1/\sigma}}{\mathcal{V}_s} - 1 \right) \omega. \end{aligned}$$

It immediately follows that

$$\begin{aligned} c(\omega; s) &= \omega \max(\lambda^{-1/\sigma}, \mathcal{V}_s) \\ D(\omega; s) &= \omega \max\left(\frac{\lambda^{-1/\sigma}}{\mathcal{V}_s} - 1, 0\right). \end{aligned}$$

Also, define

$$\begin{aligned} c_r(s) &\equiv \frac{c(\omega; s)}{\omega} = \max(\lambda^{-1/\sigma}, \mathcal{V}_s) \\ D_r(s) &\equiv \frac{D(\omega; s)}{\omega} = \max\left(\frac{\lambda^{-1/\sigma}}{\mathcal{V}_s} - 1, 0\right). \end{aligned}$$

3. It remains just to confirm that these policy functions are consistent with the original value function guess. Plugging the solutions back into the original problem, we have

$$\begin{aligned} a_s \frac{\omega^{1-\sigma}}{1-\sigma} &= \frac{(\omega c_r(s))^{1-\sigma}}{1-\sigma} - \lambda \omega^{-\sigma} \omega D_r(s) + (\omega + \omega D_r(s) - \omega c_r(s))^{1-\sigma} \tilde{\beta} \mathcal{U}_s \\ &\implies \\ a_s \left(\frac{\omega^{1-\sigma}}{1-\sigma} \right) &= \left(\frac{\omega^{1-\sigma}}{1-\sigma} \right) (c_r(s))^{1-\sigma} - \left(\frac{\omega^{1-\sigma}}{1-\sigma} \right) \lambda (1-\sigma) D_r(s) \\ &\quad + \left(\frac{\omega^{1-\sigma}}{1-\sigma} \right) (1 + D_r(s) - c_r(s))^{1-\sigma} \tilde{\beta} \mathcal{U}_s (1-\sigma). \end{aligned}$$

So, the ω terms cancel, confirming the original conjecture. Finally, the a_s 's

follow the recursion:

$$a_s = (c_r(s))^{1-\sigma} - \lambda(1-\sigma)D_r(s) + (1+D_r(s)-c_r(s))^{1-\sigma}\tilde{\beta}\mathcal{U}_s(1-\sigma). \quad (3)$$

Specifically, given values $\{a_1^j, \dots, a_S^j\}$, we can calculate $\mathcal{U}_s$, $\mathcal{V}_s$, D_r , and c_r . These quantities then imply a new set of a's $\{a_1^{j+1}, \dots, a_S^{j+1}\}$ via equation 3. Under the regularity condition $\tilde{\beta}\mathcal{U}_s(\mathbf{1})(1-\sigma) < 1$, the recursion has a solution by the boundedness of $\mathcal{U}_s$ and by the continuity and monotonicity of the recursion operator. To see this, let $\mathbf{a} = (a_1, \dots, a_S)$ and define the operator $T\mathbf{a} : \mathbb{R}_{++}^S \rightarrow \mathbb{R}_{++}^S$ by

$$(T\mathbf{a})_s = (c_r(s; \mathbf{a}))^{1-\sigma} - \lambda(1-\sigma)D_r(s, \mathbf{a}) + (1+D_r(s, \mathbf{a}) - c_r(s, \mathbf{a}))^{1-\sigma}\tilde{\beta}\mathcal{U}_s(\mathbf{a})(1-\sigma).$$

Note I am now explicitly writing c_r and D_r as functions of $\mathbf{a}$ (recall that they depend on $\mathcal{U}_s$, which is a function of $\mathbf{a}$, by definition). Define also the post-default savings rate

$$\begin{aligned} \psi(s, \mathbf{a}) &= 1 + D_r(s, \mathbf{a}) - c_r(s, \mathbf{a}) \\ &= 1 + \max\left(\frac{\lambda^{-1/\sigma}}{\mathcal{V}_s(\mathbf{a})} - 1, 0\right) - \max(\lambda^{-1/\sigma}, \mathcal{V}_s(\mathbf{a})) \\ &= \max\left(\frac{\lambda^{-1/\sigma}}{\mathcal{V}_s(\mathbf{a})}, 1\right) - \max(\lambda^{-1/\sigma}, \mathcal{V}_s(\mathbf{a})) \\ &= \max\left(\frac{\lambda^{-1/\sigma}}{\mathcal{V}_s(\mathbf{a})} - \lambda^{-1/\sigma}, 1 - \mathcal{V}_s(\mathbf{a})\right) \\ &= (1 - \mathcal{V}_s(\mathbf{a})) \max\left(\frac{\lambda^{-1/\sigma}}{\mathcal{V}_s(\mathbf{a})}, 1\right). \end{aligned}$$

The second to last equation holds because when $\lambda^{-1/\sigma} \leq \mathcal{V}_s$, we have

$$\begin{aligned} \frac{\lambda^{-1/\sigma}}{\mathcal{V}_s} - \lambda^{-1/\sigma} &= \frac{\lambda^{-1/\sigma}}{\mathcal{V}_s} (1 - \mathcal{V}_s) \\ &\leq (1 - \mathcal{V}_s). \end{aligned}$$

Therefore, we can write the operator as

$$\begin{aligned} (T\mathbf{a})_s &= \max(\lambda^{-1/\sigma}, \mathcal{V}_s(\mathbf{a}))^{1-\sigma} - \lambda(1-\sigma) \max\left(\frac{\lambda^{-1/\sigma}}{\mathcal{V}_s(\mathbf{a})} - 1, 0\right) \\ &\quad + (1 - \mathcal{V}_s(\mathbf{a}))^{1-\sigma} \tilde{\beta}\mathcal{U}_s(\mathbf{a}) (1-\sigma) \min\left(\left(\frac{\lambda^{-1/\sigma}}{\mathcal{V}_s(\mathbf{a})}\right)^{1-\sigma}, 1\right). \end{aligned}$$

Then, using

$$\begin{aligned} \mathcal{V}_s(\mathbf{a}) &= \frac{1}{1 + \left(\tilde{\beta}\mathcal{U}_s(\mathbf{a}) (1-\sigma)\right)^{1/\sigma}} \\ \implies \\ \tilde{\beta}\mathcal{U}_s(\mathbf{a}) (1-\sigma) &= \frac{1}{\mathcal{V}_s(\mathbf{a})^\sigma} (1 - \mathcal{V}_s(\mathbf{a}))^\sigma, \end{aligned}$$

we have

$$\begin{aligned} (T\mathbf{a})_s &= \max(\lambda^{-1/\sigma}, \mathcal{V}_s(\mathbf{a}))^{1-\sigma} - \lambda(1-\sigma) \max\left(\frac{\lambda^{-1/\sigma}}{\mathcal{V}_s(\mathbf{a})} - 1, 0\right) \\ &\quad + \frac{1 - \mathcal{V}_s(\mathbf{a})}{\mathcal{V}_s(\mathbf{a})^\sigma} \min\left(\left(\frac{\lambda^{-1/\sigma}}{\mathcal{V}_s(\mathbf{a})}\right)^{1-\sigma}, 1\right). \end{aligned}$$

Now, one can show continuity of T by the maximum theorem. T is also monotonic: in the case with no default, T is monotonic as in Toda (2013).

When $D > 0$, one can show the operator has the simple form

$$(T\mathbf{a})_s = \lambda \left[1 + \sigma \left(\frac{\lambda^{-1/\sigma}}{\mathcal{V}_s(\mathbf{a})} - 1 \right) \right] \quad (4)$$

$$= \sigma \frac{\lambda^{-(1-\sigma)/\sigma}}{\mathcal{V}_s(\mathbf{a})} - \lambda(\sigma - 1) \quad (5)$$

which is monotonic. Finally, we need boundedness. By $\mathcal{V}_s(\mathbf{a}) < 1$ and $\lambda > 1$, it is clear that $T\mathbf{1} \geq \mathbf{1}$. Lastly, we know by the regularity condition $\tilde{\beta}\mathcal{U}_s(\mathbf{1}) (1-\sigma) < 1$ that we can find $M > 1$ such that if $\mathbf{a} \leq \mathbf{1}M$ then $T\mathbf{a} \leq \mathbf{1}M$. This is because if $T\mathbf{1}$ is bounded above in the case with no default (as in Toda (2013)), it must also be bounded above when there is default: with fewer constraints binding, utility must rise (a 's must fall). In either case, the implication is that T is a monotonic, continuous self map on $[\mathbf{1}, \mathbf{1}M]$, which completes the proof. ■

6.4 Proof of Corollary 3

Define, as above,

$$\mathbf{R}_s = \begin{pmatrix} R_1(s, 1) & \dots & R_J(s, 1) \\ \dots & \dots & \dots \\ R_1(s, S) & \dots & R_J(s, S) \end{pmatrix}$$

$$\mathbf{K}_s = \begin{pmatrix} \beta\pi_{s1}a_1R_1(s, 1) & \dots & \beta\pi_{sS}a_S R_1(s, S) \\ \dots & \dots & \dots \\ \beta\pi_{s1}a_1R_J(s, 1) & \dots & \beta\pi_{sS}a_S R_J(s, S) \end{pmatrix}^{-1} \mathbf{1},$$

where $R_j(s, s')$ is the return of asset j going from state s to state s' . Let $R(s, s')$ be the vector of returns across the J assets, that is, $R(s, s') = (R_1(s, s'), \dots, R_J(s, s'))$. Therefore, we have

$$\mathbf{R}_s = \begin{pmatrix} R(s, 1) \\ \dots \\ R(s, S) \end{pmatrix}.$$

At any state s , the portfolio problem is thus

$$\max_{\theta} \beta \sum_{s'=1}^S \pi_{ss'} a_{s'} \frac{(R(s, s') \theta)^{1-\sigma}}{1-\sigma} \quad s.t.$$

$$\theta_1 + \dots + \theta_J = 1 \quad (\gamma_s),$$

where γ_s is the multiplier. The J FOCs are thus

$$\beta \sum_{s'=1}^S \pi_{ss'} a_{s'} (R(s, s') \theta)^{-\sigma} R_1(s, s') = \lambda_s$$

$$\dots$$

$$\beta \sum_{s'=1}^S \pi_{ss'} a_{s'} (R(s, s') \theta)^{-\sigma} R_J(s, s') = \lambda_s.$$

We can write this as

$$\begin{pmatrix} \beta\pi_{s1}a_1R_1(s, 1) & \dots & \beta\pi_{sS}a_S R_1(s, S) \\ \dots & \dots & \dots \\ \beta\pi_{s1}a_1R_J(s, 1) & \dots & \beta\pi_{sS}a_S R_J(s, S) \end{pmatrix} \begin{pmatrix} (R(s, 1) \theta)^{-\sigma} \\ \dots \\ (R(s, S) \theta)^{-\sigma} \end{pmatrix} = \gamma \mathbf{1}.$$

By $S = J$ and the linear independence of asset returns, the left-most term is invertible, meaning we have:

$$\begin{aligned} \begin{pmatrix} (R(s, 1) \theta)^{-\sigma} \\ \vdots \\ (R(s, S) \theta)^{-\sigma} \end{pmatrix} &= \gamma \mathbf{K}_s \\ \implies \\ \mathbf{R}_s \theta &= \gamma^{-1/\sigma} (\mathbf{K}_s)_{-1/\sigma}, \end{aligned}$$

where the subscript is element by element inversion. Inverting $\mathbf{R}_s$, we then have

$$\theta = \mathbf{R}_s^{-1} \gamma^{-1/\sigma} (\mathbf{K}_s)_{-1/\sigma}.$$

Using the budget constraint

$$1 = \mathbf{1}' \theta = \mathbf{1}' \mathbf{R}_s^{-1} \gamma^{-1/\sigma} (\mathbf{K}_s)_{-1/\sigma},$$

it follows that

$$\begin{aligned} \theta^*(s) &= \mathbf{R}_s^{-1} \gamma_s^{-1/\sigma} (\mathbf{K}_s)_{-1/\sigma} \\ \gamma_s^{1/\sigma} &= \mathbf{1}' \mathbf{R}_s^{-1} (\mathbf{K}_s)_{-1/\sigma}, \end{aligned}$$

finishing the proof. ■

6.5 Proof of Proposition 3

Recall from lemma 1 that in any state,

$$r = \frac{\delta - q(1 - \delta)}{q},$$

where q is the bond price and r is the yield. Therefore,

$$\begin{aligned}
q &= \frac{\delta}{1+r-\delta} \\
&\implies \\
\frac{\partial q}{\partial r} &= \frac{-\delta}{(1+r-\delta)^2} < 0 \\
&\implies \\
\frac{\partial}{\partial \delta} \left(\frac{\partial q}{\partial r} \right) &= -\frac{1+r+\delta}{(1+r-\delta)^3} < 0.
\end{aligned}$$

These derivatives imply: (1) counter-cyclical r implies pro-cyclical q , and (2) the pro-cyclicity becomes stronger as δ increases. With respect to returns, we have

$$\begin{aligned}
R'_{b_\delta} &= \frac{\delta(1+q')}{q} \\
&= \frac{\delta \left(1 + \frac{\delta}{1+r'-\delta} \right)}{\frac{\delta}{1+r-\delta}} \\
&= (1+r') \frac{1+r-\delta}{1+r'-\delta}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\frac{\partial}{\partial \delta} R'_{b_\delta} &= (1+r') \frac{-(1+r'-\delta) + (1+r-\delta)}{(1+r'-\delta)^2} \\
&= (1+r') \left[\frac{r-r'}{(1+r'-\delta)^2} \right].
\end{aligned}$$

The implication, which will be important below, is that

$$\frac{\partial}{\partial \delta} R_{b_\delta}(G, G) = 0 \tag{6}$$

$$\frac{\partial}{\partial \delta} R_{b_\delta}(G, B) < 0 \tag{7}$$

because yields are counter-cyclical by assumption: $r_B > r_G$.

The portfolio problem in state $s \in \{B, G\}$ is

$$\max_{\theta_{b_\delta}} \left[\begin{array}{l} \pi \log(\theta R_{b_\delta}(B) + (1-\theta) R_a(B)) \\ + (1-\pi) \log(\theta R_{b_\delta}(G) + (1-\theta) R_a(G)) \end{array} \right],$$

where I have suppressed reference to the current state and let θ be the bond share.

The FOC is

$$\begin{aligned}
& \pi_B \frac{R_{b_\delta}(B) - R_a(B)}{R_a(B) + (R_{b_\delta}(B) - R_a(B))\theta} + \pi_G \frac{R_{b_\delta}(G) - R_a(G)}{R_a(G) + (R_{b_\delta}(G) - R_a(G))\theta} = 0 \\
& \implies \\
& \frac{\pi_B (R_{b_\delta}(B) - R_a(B))}{R_a(B) + (R_{b_\delta}(B) - R_a(B))\theta} = \frac{\pi_G (R_a(G) - R_{b_\delta}(G))}{R_a(G) + (R_{b_\delta}(G) - R_a(G))\theta} \\
& \implies \\
& \pi_B (R_{b_\delta}(B) - R_a(B)) R_a(G) + \pi_B (R_{b_\delta}(B) - R_a(B)) (R_{b_\delta}(G) - R_a(G)) \theta \\
& = \pi_G (R_a(G) - R_{b_\delta}(G)) R_a(B) + \pi_G (R_a(G) - R_{b_\delta}(G)) (R_{b_\delta}(B) - R_a(B)) \theta,
\end{aligned}$$

implying

$$\begin{aligned}
& \pi_G (R_a(G) - R_{b_\delta}(G)) (R_{b_\delta}(B) - R_a(B)) \theta + \pi_B (R_{b_\delta}(B) - R_a(B)) (R_a(G) - R_{b_\delta}(G)) \theta \\
& = \pi_B (R_{b_\delta}(B) - R_a(B)) R_a(G) + \pi_G (R_{b_\delta}(G) - R_a(G)) R_a(B) \\
& \implies \\
& \theta (R_{b_\delta}(B) - R_a(B)) (R_a(G) - R_{b_\delta}(G)) \\
& = \pi_B (R_{b_\delta}(B) - R_a(B)) R_a(G) + \pi_G (R_{b_\delta}(G) - R_a(G)) R_a(B),
\end{aligned}$$

which yields

$$\theta_{b_\delta}^* = \pi_B \frac{R_a(G)}{R_a(G) - R_{b_\delta}(G)} + \pi_G \frac{R_a(B)}{R_a(B) - R_{b_\delta}(B)}.$$

Suppose we are in state G . Then by 6 and 7, we have

$$\frac{\partial}{\partial \delta} \theta_{b_\delta}(G) = \pi_G \frac{\left(\frac{\partial}{\partial \delta} R_{b_\delta}(G, B) \right) R_a(G, B)}{(R_a(G, B) - R_{b_\delta}(G, B))^2} < 0.$$

By assumption, borrowing occurs in state G . Therefore, leverage is increasing in the maturity δ of the bond.

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