

The Race Between Technology and Human Capital

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Abstract

This paper develops a model in which heterogeneous firms invest in R&D to improve technology, and heterogeneous workers invest in human capital to increase their earnings. Both investment technologies have stochastic components, and the balanced growth path has stationary, nondegenerate distributions of technology and human capital.

Technology and human capital are complements in production, so the labor market produces assortative matching between firms and workers: firms with higher productivity employ higher quality workers and pay higher wages. Thus, wage differentials across firms have two sources: differences in firm productivity and differences in labor quality.

1. OVERVIEW

This paper develops a model in which heterogeneous firms invest in R&D to improve technology, and heterogeneous workers invest in human capital to increase their earnings. Both investment technologies have stochastic components, and the balanced growth path has stationary, nondegenerate distributions of technology and human capital.

Technology and human capital are complements in production, so the labor market produces assortative matching between firms and workers: firms with higher productivity employ higher quality workers and pay higher wages. Thus, wage differentials across firms have two sources: differences in firm productivity and differences in labor quality.

The heterogeneous firms produce intermediate goods, which are combined with a Dixit-Stiglitz aggregator to produce the single final good. Final goods are used for consumption and three kinds of investment. Incumbent firms invest to improve their productivity, and they die stochastically. Entering firms pay a fixed cost to obtain an initial technology. Workers invest to increase their human capital.

One goal is to assess the sources of wage inequality. Empirically, it is not easy to distinguish the importance of technology and human capital differences in generating wage differentials across firms. A theoretical framework that incorporates complementarity between the two may be useful in assessing the importance of each.

A second goal is to examine the interplay between human accumulation and technological change as contributors to long-run growth. From an empirical point of view, the chicken-and-egg issue makes it difficult to distinguish a single “engine” of growth. A theoretical framework that builds in the symbiotic nature of improvements in the two factors may provide insights for assessing, in particular contexts, the role of each.

On the technology side, the model builds on Luttmer (*QJE*, 2007), incorporating

the investment technology in Atkeson and Burstein (*JPE*, 2010). On the human capital side, it develops a similar investment model.

To start, we will assume that both the average productivity of entering firms and the average initial human capital of new workers grow at a common rate g . We will look for a BGP where the cross-sectional distribution of productivities across firms and of human capital across workers are both lognormal, with constant variances. The entering productivities for new firms and workers can then be made endogenous. Specifically, at each date they will be draws from a distribution that depends on the current cross-sectional distribution.

A. Variables

Exogenous:

population L is exogenous and constant, with birth and death at rate δ_L ;

exit rate for firms, $\delta_X > 0$;

productivity of entering firms at date t is lognormally distributed, with a fixed variance and a mean that grows at the fixed rate g ,

$$\ln X_{it0} \sim N\left(\mu_{EX} + gt, \sigma_{EX}^2\right);$$

human capital of entering workers at date t is lognormally distributed, with fixed variance and a mean that grows at the fixed rate g ,

$$\ln H_{jt0} \sim N\left(\mu_{EH} + gt, \sigma_{EH}^2\right).$$

Individual decisions:

productivity X_{ita} at age a of incumbent firm i that entered at date t

is a geometric Brownian motion, with fixed variance σ_X^2 .

The firm's investment decision is the choice of the drift g_X .

human capital H_{jta} at age a of worker j who entered at date t

is a geometric Brownian motion, with fixed variance σ_H^2 .

The worker's investment decision is the choice of the drift g_H .

Endogenous:

N , the (constant) number of incumbent firms, is determined by free entry.

Z_t is an aggregate variable, Luttmer's "total productivity." It grows at rate g .

Its level depends on the number of firms, N , and the

cross-sectional distribution of productivities X_{it} .

$x_{ita} = X_{ita}/Z_{t+a}$ is firm i 's relative productivity at age a . On a BGP, x_i has

a stationary cross-sectional distribution that depends on $\mu_{EX}, g_X, \sigma_{EX}^2, \sigma_X^2$.

Firm i 's labor demand and profits at date $t + a$ depend on (x_{ita}, Z_{t+a}) .

$h_{jta} = H_{jta}/Z_{t+a}$ is j 's relative human capital. Along a BGP, h_{jta} has a stationary

cross-sectional distribution that depends on $\mu_{EH}, g_H, \sigma_{EH}^2$ and σ_H^2 .

$W(h, Z)$ is the wage of a worker with relative human capital h in an economy with total productivity Z .

2. THE MODEL: STATIC BEHAVIOR OF FIRMS

A. Final good technology

The final good is produced by competitive firms using intermediate goods as inputs. All intermediates enter symmetrically into final good production, but demands for them differ because their prices differ. Specifically, intermediate producers are indexed by their productivity $X > 0$, which determines the price $p(X)$ for their good. Let $\lambda(X)$ denote the density for X across intermediate producers, and let N be the number (mass) of firms. Each final good producer has the CRS technology

$$Y = \left[N \int y(X)^{(\rho-1)/\rho} \lambda(X) dX \right]^{\rho/(\rho-1)},$$

where $\rho > 1$ is the substitution elasticity. The final goods sector takes the prices $p(X)$ as given. As usual, the price of the final good is

$$P = \left[N \int p(X)^{1-\rho} \lambda(X) dX \right]^{1/(1-\rho)}, \quad (1)$$

and input demands are

$$y^d(X) = \left(\frac{p(X)}{P} \right)^{-\rho} Y, \quad \text{all } X. \quad (2)$$

B. Intermediate producers: choice of labor quality

Intermediate producers use heterogeneous labor, differentiated by its human capital level H , as the only input. The output of a firm depends on the size and quality of its workforce, as well as its technology. In particular, if a firm with technology X employs ℓ workers with human capital H , then its output is

$$y = \ell \phi(H, X),$$

where $\phi(H, X)$ is the CES function

$$\phi(H, X) \equiv \left[\omega H^{(\eta-1)/\eta} + (1-\omega) X^{(\eta-1)/\eta} \right]^{\eta/(\eta-1)}, \quad \eta, \omega \in (0, 1). \quad (3)$$

The elasticity of substitution η between technology and human capital is assumed to be less than unity, and ω is the relative weight on human capital. Firms could employ workers with different human capital levels, and in this case their outputs would simply be summed. In equilibrium firms never choose to do so, however, and for simplicity the notation is not introduced.

Let $W(H)$ denote the wage function. For a firm with technology X , the cost of producing one unit of output with labor of quality H is $W(H)/\phi(H, X)$. Optimal labor quality $H^*(X)$ minimizes this expression, and unit cost is

$$\min_H \frac{W(H)}{\phi(H, X)}. \quad (4)$$

Hence H^* satisfies

$$\frac{W'(H^*)}{W(H^*)} = \frac{\phi_H(H^*, X)}{\phi(H^*, X)}.$$

If the wage function has the constant elasticity form

$$W(H) = W_0 H^{1-\varepsilon}, \quad \varepsilon \in (0, 1), \quad (5)$$

then labor quality $H^*(X)$ is proportional to X , with a constant of proportionality that depends on ε ,

$$H^*(X) = a_H X, \quad a_H \equiv \left(\frac{1-\varepsilon}{\varepsilon} \frac{1-\omega}{\omega} \right)^{\eta/(\eta-1)}. \quad (6)$$

Unit cost is then

$$\frac{W(a_H X)}{\phi(a_H X, X)} = \frac{a_H^{1-\varepsilon}}{\phi_0} W_0 X^{-\varepsilon}, \quad (7)$$

where

$$\phi_0 \equiv \phi(a_H, 1). \quad (8)$$

For $W(H)$ as in (5), the problem in (4) is concave if (and only if) $\eta \in (0, 1)$, as assumed here. The constants W_0 and ε in the wage function are determined by clearing in the labor market.

The quantity of labor hired is proportional to the target level of output,

$$\ell^*(y; X) = \frac{y}{\phi_0} X^{-1}. \quad (9)$$

Figure 1 displays isoquants for output and expenditure (total wage bill) in quality-quantity space, for the model parameters

$$\eta = 0.5, \quad \omega = 0.5, \quad W_0 = 1, \quad \varepsilon = 0.5,$$

and the technology, output, and expenditure levels

$$\begin{aligned} X_1 &= 0.5, & X_2 &= 1, & y_1 &= 0.5, & y_2 &= 1, \\ E_{11} &= 0.7071, & E_{12} &= 1.4142, & E_{21} &= 0.5, & E_{22} &= 1. \end{aligned}$$

The dashed curves are output isoquants for the technology level X_1 , and the broken curves are isoquants for $X_2 > X_1$. The four solid curves are expenditure isoquants, and the small circles indicate the four cost-minimizing input mixes. With X fixed, a higher output level y increases only the quantity ℓ of labor input. With y fixed, a higher technology level X increases labor quality H and reduces quantity ℓ . Note that cost minimization by firms implies positively assortative matching: firms with better technologies hire workers with more human capital.

C. Intermediate goods: pricing problem

Assume the wage function has the form in (5), so unit cost is as in (7). Given the aggregate price level P and total demand Y for final goods, each intermediate firm chooses price $p(X)$ to maximize profits. As usual, the optimal price is a markup $\rho/(\rho - 1)$ over unit cost. Let $w_0 \equiv W_0/P$ denote the scale for the real wage. Then price, quantity, labor input, and (real) profits for the intermediate firm involve various powers of X ,

$$p(X) = \frac{\rho}{\rho - 1} \frac{a_H^{1-\varepsilon}}{\phi_0} W_0 X^{-\varepsilon} \equiv P p_0 w_0 X^{-\varepsilon}, \quad (10)$$

$$y(X) = \left(\frac{p(X)}{P} \right)^{-\rho} Y \equiv (p_0 w_0)^{-\rho} Y X^{\rho\varepsilon}, \quad (11)$$

$$\ell(X) = \frac{y(X)}{\phi_0} X^{-1} \equiv \frac{1}{\phi_0} (p_0 w_0)^{-\rho} Y X^{\rho\varepsilon-1}, \quad (12)$$

$$\pi(X) = \frac{1}{\rho} \frac{1}{P} p(X) y(X) \equiv \frac{1}{\rho} (p_0 w_0)^{1-\rho} Y X^{(\rho-1)\varepsilon}, \quad (13)$$

where the constant

$$p_0 \equiv \frac{\rho}{\rho - 1} \frac{a_H^{1-\varepsilon}}{\phi_0},$$

depends on ε . The nominal price $p(X)$ in (10) is proportional to the aggregate price level P , while the real values $y(X)$, $\ell(X)$, and $\pi(X)$ in (11) - (13) depend only on the scale for real wages w_0 . Real wages $W(H)/P = w_0 H^{1-\varepsilon}$ can grow because w_0 grows

or H grows or both.

D. Total productivity and aggregate output

Recall that $\lambda(X)$ is the density for productivity, and define “total productivity”

$$Z \equiv \left[N \int X^{(\rho-1)\varepsilon} \lambda(X) dX \right]^{1/(\rho-1)\varepsilon}, \quad (14)$$

which involves the number of firms N , as well as the density λ . On the BGP Z grows at the rate g . Use (10) and (14) in the price index (1) to find that

$$\begin{aligned} P^{1-\rho} &= N \int [P p_0 w_0 X^{-\varepsilon}]^{1-\rho} \lambda(X) dX \\ &= (P p_0 w_0)^{1-\rho} Z^{(\rho-1)\varepsilon}, \end{aligned}$$

so the scale for the real wage is

$$w_0 = p_0^{-1} Z^\varepsilon. \quad (15)$$

Thus, on a BGP w_0 grows at the rate εg .

In Luttmer’s model X is lognormally distributed on the BGP. Conjecture that the same is true here, since the stochastic process for technology is the same. From (10), (11), and (13), we see that firms with higher X have lower prices, higher sales, and higher profits. In Luttmer’s model they also demand more labor.

Suppose human capital also has a lognormal distribution. Then a linear mapping from X to H can clear the market for labor at every human capital level only if the wage function is steep enough so that labor demand is constant across firms. From (12), it is immediate that this holds if and only if $\varepsilon = 1/\rho$.

Recall that L is total population. With $\varepsilon = 1/\rho$, use (15) in (12) to find that labor market clearing in aggregate requires

$$Y = \frac{L}{N} \phi_0 Z. \quad (16)$$

Thus, on a BGP Y grows like Z , at the rate g .

E. State variables for the firm

Let $x \equiv X/Z$ denote the firm's relative productivity. The state variables (x, Z) are convenient for analyzing the firm. Use (15) and (16) in (10)-(13) to write the firm's price, output, labor input, and profits as

$$\begin{aligned} p(x) &= Px^{-1/\rho}, \\ y(x, Z) &= \frac{\phi_0 L}{N} Zx, \\ \ell &= \frac{L}{N}, \\ \pi(x, Z) &= \frac{\phi_0 L}{\rho N} Zx^\beta, \end{aligned} \tag{17}$$

where

$$\beta \equiv \frac{\rho - 1}{\rho}.$$

Thus, x^β measures revenue, the wage bill, and profits. More productive firms set lower prices, and they have higher sales and profits. They also employ higher quality workers, and hence pay higher wages. By construction, employment is the same at every technology level.

Since $X_{it} = x_i Z_t$, from (14), N is determined by

$$Z_0^\beta = NE \left[(x_i Z_0)^\beta \right],$$

or

$$N = \left[E \left(x_i^\beta \right) \right]^{-1}. \tag{18}$$

The next tasks are to describe the stochastic process for x for an individual firm and the stationary distribution for x across firms.

3. INTERMEDIATE PRODUCERS: DYNAMICS

A. The value function

Let X_{ita} denote the stochastic process for the productivity of incumbent firm i that enters at date t , as function of its age a . In log form, it is a diffusion with parameters (g_X, σ_X^2) , where the firm chooses the drift g_X , which is its investment decision, and the variance σ_X^2 is fixed. Since Z_t grows at the constant rate g , the firm's relative productivity $x_{ita} \equiv X_{ita}/Z_t$ is a similar diffusion, with parameters $(g_X - g, \sigma_X^2)$.

Consider firm's choice of g_X . The cost of investment is paid in goods and, as in Atkeson and Burstein, the cost is scaled by current profitability. In particular, the cost of investment for a firm with state (x, Z) that chooses drift g_X is

$$\Gamma(g_X, x, Z) = \gamma(g_X) Z x^\beta, \quad (19)$$

where the function γ is strictly increasing and strictly convex.

The payoff of a firm is its profit flow in (17), net of its investment cost in (19). Since there is no fixed cost of operating, there is no voluntary exit, and the firm operates until the exogenous exit shock hits. Hence the HJB equation for the firm is

$$\begin{aligned} (r + \delta_X) V(x, Z) = & \max_{g_X} \left\{ \left[\frac{\phi_0 L}{\rho N} - \gamma(g_X) \right] Z x^\beta \right. \\ & \left. + (g_X - g) x V_x + \frac{1}{2} \sigma_X^2 x^2 V_{xx} + g Z V_Z \right\}. \end{aligned} \quad (20)$$

It is straightforward to show that V has the homogeneous form $V(x, Z) = v_0 x^\beta Z$, so the HJB equation can be written as

$$\begin{aligned} (r + \delta_X) v_0 = & \max_{g_X} \left[\frac{\phi_0 L}{\rho N} - \gamma(g_X) \right. \\ & \left. + (g_X - g) \beta v_0 + \frac{1}{2} \sigma_X^2 \beta (\beta - 1) v_0 + g v_0 \right]. \end{aligned} \quad (21)$$

B. Investment by incumbents

The first-order condition for investment is

$$\gamma'(g_X) = \beta v_0, \quad (22)$$

and

$$v_0 = \frac{\phi_0 L / \rho N - \gamma(g_X)}{r + \delta_X - g - \bar{\nu}(g_X)}, \quad (23)$$

where

$$\bar{\nu}(g_X) \equiv \beta (g_X - g) + \frac{1}{2} \sigma_X^2 \beta (\beta - 1). \quad (24)$$

The value of the firm is finite if and only if

$$\bar{\nu}(g_X) < r + \delta_X - g,$$

which puts an upper bound on g_X . In particular, we require

$$g_X < \bar{g}_X \equiv g + \frac{1}{\beta} (r + \delta_X - g) - \frac{1}{2} \sigma_X^2 (\beta - 1). \quad (25)$$

Use (23) in (22) to write the FOC for g_X as

$$\begin{aligned} \frac{\phi_0 L}{\rho N} &= \gamma(g_X) + \frac{1}{\beta} [r + \delta_X - g - \nu(g_X)] \gamma'(g_X) \\ &\equiv \Psi(g_X). \end{aligned} \quad (26)$$

Assume $\gamma(-\chi) = 0$ for some $\chi > 0$, representing the fact that the technology depreciates at the rate χ if there is no investment. Then we require $g_X \in [-\chi, \bar{g}_X)$, and on this range

$$\Psi'(g_X) = \frac{1}{\beta} [r + \delta_X - g - \nu(g_X)] \gamma''(g_X) > 0.$$

Assume in addition that $\gamma'(-\chi) = 0$, and that γ' is bounded for $g_X \in [-\chi, \bar{g}_X)$. Then

$$\Psi(-\chi) = 0 \quad \text{and} \quad \lim_{g_X \rightarrow \bar{g}_X} \Psi(g_X) = \gamma(\bar{g}_X).$$

In this case (26) has a solution if and only if

$$\gamma(\bar{g}_X) > \frac{\phi_0 L}{\rho N}, \quad (27)$$

which says that the cost of investment for a drift of $\bar{g}_X$ exceeds the firm's profit flow. Assume that (27) holds.

C. Firm size by age

Since g_X is the same for all firms, X_{ita} is a geometric Brownian motion with parameters (g_X, σ_X^2) . The initial values for each cohort of entrants are lognormally distributed, with a fixed variance σ_{EX}^2 and a mean that grows at the constant rate g over time. Thus,

$$\ln X_{ita} \sim N(\mu_{EX} + gt + g_X a, \sigma_{EX}^2 + a^2 \sigma_X^2), \quad \text{all } t, a, \quad (28)$$

where μ_{EX} is the mean for entrants at $t = 0$.

Since total productivity Z_t grows at the constant rate g , the distribution of relative size $x_{ita} = X_{ita}/Z_{t+a}$ for the cohort also has a lognormal distribution,

$$\ln x_{ita} \sim N(\mu_x(a), \Sigma_x^2(a)), \quad \text{all } t, a, \quad (29)$$

where

$$\begin{aligned} \mu_x(a) &= \mu_{EX} - \ln Z_0 + (g_X - g) a, \\ \Sigma_x^2(a) &= \sigma_{EX}^2 + a^2 \sigma_X^2. \end{aligned} \quad (30)$$

Note that the distribution of relative size is stationary.

D. Entry

Entry costs are also paid in goods. At any date t , a potential entrant can invest $I_E Z_t$ units of goods and obtain, for sure, a new product. Hence the entry condition

is

$$I_E Z_t \geq \mathbb{E}[V(X_{it0}/Z_t, Z_t)] = v_0 Z_t \mathbb{E}[x_{it0}^\beta],$$

with equality if firms enter. From (29), the mean for entrant relative productivity is constant. Therefore, since the entry rate is strictly positive on the BGP, the required condition is

$$I_E = v_0 \mathbb{E}[x_{i0}^\beta]. \quad (31)$$

Use (29) in this condition to get a relationship between Z_0 and g_X ,

$$\ln\left(\frac{I_E}{v_0}\right) = \ln \mathbb{E}[x_{i0}^\beta] = \beta \left[\mu_{EX} - \ln Z_0 + \frac{1}{2} \sigma_{EX}^2 \right], \quad (32)$$

where v_0 involves the number of firms N .

E. Stationary size distribution

Next consider the distribution of relative firm size, which is stationary on the BGP. For each age cohort, $\ln x_{ia}$ has a Normal distribution, described in (29) and (30). Hence for the entire population of incumbents, $\ln x_i$ has a distribution that is a mixture of normals, and hence itself normal. That is, $\ln x_i \sim N(\mu_x, \Sigma_x^2)$, where μ_x, Σ_x^2 are weighted averages of the means and variances for various age cohorts.

The exit rate $\delta_X > 0$ is fixed, so an entry rate of δ_X is required to maintain a unit mass of firms. Hence the cohort of age a gets weight $\delta_X e^{-a\delta_X}$, all $a \geq 0$, so

$$\begin{aligned} \bar{\mu}_x &= \int_0^\infty \delta_X e^{-a\delta_X} \mu(a) da \\ &= \int_0^\infty \delta_X e^{-a\delta_X} [\mu_{EX} - \ln Z_0 + (g_X - g)a] da \\ &= \mu_{EX} - \ln Z_0 - (g_X - g) \left[e^{-a\delta_X} \left(a + \frac{1}{\delta_X} \right) \right]_0^\infty \\ &= \mu_{EX} - \ln Z_0 + \frac{1}{\delta_X} (g_X - g), \end{aligned} \quad (33)$$

$$\bar{\Sigma}_x^2 = \int_0^\infty \delta_X e^{-a\delta_X} (\sigma_{EX}^2 + a^2 \sigma_X^2) da$$

$$\begin{aligned}
&= \sigma_{EX}^2 - \sigma_X^2 \left[e^{-a\delta_X} \left(a^2 + \frac{2}{\delta_X} a + \frac{2}{\delta_X^2} \right) \right]_0^\infty \\
&= \sigma_{EX}^2 + \frac{2}{\delta_X^2} \sigma_X^2.
\end{aligned} \tag{34}$$

Note that the coefficient of variation $\bar{\Sigma}_x^2/\bar{\mu}_x$ is a decreasing function of g_X .

Use $\bar{\mu}_x$ and $\bar{\Sigma}_x^2$ in (18) to determine the number of firms N , noting that $\bar{\mu}_x$ depends on g_X .

4. HOUSEHOLDS

Individuals, who are finite-lived, are organized into infinitely-lived dynastic household, with each household comprising a representative cross-section of the population. Individual members of a dynasty pool their earnings, and the dynasty allocates family income to investment in human capital and consumption.

A. Consumption

There is a continuum of identical households of total mass one, each composed of heterogeneous members. Individual household members die at a constant rate δ_L , and are replaced by an equal inflow of new members. Thus, the total size of each household, L , is constant. Each household member supplies one unit of labor inelastically, so aggregate labor supply is L .

All household members share equally in consumption, and the household has the usual constant-elasticity preferences

$$U = \int_0^\infty e^{-\hat{r}t} \frac{1}{1-\theta} c(t)^{1-\theta} dt,$$

where $\hat{r} > 0$ is the pure rate of time preference and $1/\theta > 0$ is the elasticity of intertemporal substitution. On the BGP consumption grows at the rate g , so the real interest rate is

$$r = \hat{r} + \theta g. \tag{35}$$

Income also grows at the rate g , so its PDV is finite if and only if $r > g$, or

$$\hat{r} > (1 - \theta) g. \quad (36)$$

Assume $(\hat{r}, \theta, g)$ satisfy (36).

B. Investment in human capital

New entrant i into the workforce at date t has initial human capital H_{i0t} , where $\ln H_{i0t} \sim N(\mu_{EH} + gt, \sigma_{EH}^2)$. Investments are then made continuously over the individual's lifetime to maximize the expected discounted value of lifetime earnings, net of investment costs. The investment process is like the one for firms. Specifically, the individual chooses the drift g_H for his human capital, and pays the associated cost. The variance σ_H^2 for the process is fixed.

Define $h \equiv H/Z$. The pair of state variables (h, Z) is convenient for analyzing the individual's investment problem. Recall from (5) and (15) that the individual's (real) wage rate is $p_0^{-1} Z h^\beta$. Assume the cost of investing is scaled like earnings, so the cost is $\psi(g_H) Z h^\beta$, where the function ψ is strictly increasing and strictly convex. Thus, the individual's earnings net of the investment cost is $[p_0^{-1} - \psi(g_H)] Z h^\beta$.

Let $B(h, Z)$ denote the expected discounted earnings of this individual over the rest of his life, if he follows an optimal investment plan. Since Z grows at the rate g and h is a geometric Brownian motion with parameters $(g_H - g, \sigma_H^2)$, the HJB equation for the investment decision is

$$\begin{aligned} (r + \delta_L) B(h, Z) = & \max_{g_H} \left\{ [p_0^{-1} - \psi(g_H)] Z h^\beta \right. \\ & \left. + (g_H - g) h B_h + \frac{1}{2} \sigma_H^2 h^2 B_{hh} + g Z B_Z \right\}. \end{aligned} \quad (37)$$

Again, it is easy to show B has the homogeneous form $B(h, Z) = b_0 Z h^\beta$. Hence the HJB equation can be written as

$$(r + \delta_L) b_0 = \max_{g_H} \left[p_0^{-1} - \psi(g_H) + (g_H - g) \beta b_0 + \frac{1}{2} \sigma_H^2 \beta (\beta - 1) b_0 + g b_0 \right]. \quad (38)$$

The FOC for optimal investment is

$$\psi'(g_H) = \beta b_0, \quad (39)$$

so g_H is independent of (h, Z) .

The solution consists of (b_0, g_H) satisfying (38) and (39). Use g_H in (38) to find that

$$b_0 = \frac{p_0^{-1} - \psi(g_H)}{r + \delta_L - g - \bar{v}(g_H)}, \quad (40)$$

where

$$\bar{v}(g_H) \equiv \beta (g_H - g) + \frac{1}{2} \beta (\beta - 1) \sigma_H^2.$$

The PDV of the household's income is finite, $b_0 < +\infty$, if and only if

$$\bar{v}(g_H) < r + \delta_L - g,$$

which puts an upper bound on g_H , call it $\bar{g}_H$. In particular,

$$\bar{g}_H = g - \frac{1}{2} \beta \sigma_H^2 + \frac{1}{\beta} (\delta_H + r - g).$$

C. The distribution of human capital

Let H_{jta} denote the stochastic process for the human capital of an individual j that is born at date t , as function of his age a . It is a geometric Brownian motion with parameters (g_H, σ_H^2) . The individual chooses the drift g_H , as described above, and the variance σ_H^2 is fixed. The initial values for each cohort of newborns are lognormally distributed, with a fixed variance σ_{EH}^2 and a mean that grows at the constant rate g over time. Thus,

$$\ln H_{jta} \sim N(\mu_{EH} + gt + g_H a, \sigma_{EH}^2 + a^2 \sigma_H^2),$$

where μ_{EH} is the mean for entrants at $t = 0$.

Since total productivity Z_t grows at the constant rate g , the distribution for relative human capital for the cohort also has a lognormal distribution:

$$\ln h_{jta} = \ln (H_{jta}/Z_{t+a}) \sim N \left(\mu_h(a), \Sigma_h^2(a) \right), \quad (41)$$

where

$$\begin{aligned} \mu_h(a) &= \mu_{EH} - \ln Z_0 + (g_H - g) a, \\ \Sigma_h^2(a) &= \sigma_{EH}^2 + a^2 \sigma_H^2. \end{aligned} \quad (42)$$

D. Stationary distribution of relative human capital

As with firms, the distribution of relative human capital in the total population is a weighted average of the $\mu_h(a)$'s and $\Sigma_h^2(a)$'s, with weights that reflect the size of each cohort. Since individuals die at rate δ_L , a birth rate of δ_L is needed to maintain the population, so the cohort of individuals of age a gets weight $\delta_L e^{-\delta_L a}$, all $a \geq 0$, so

$$\begin{aligned} \bar{\mu}_h &= \int_0^\infty \delta_H e^{-\delta_H a} \mu_h(a) da \\ &= \int_0^\infty \delta_H e^{-\delta_H a} [\mu_{EH} - \ln Z_0 + (g_H - g) a] da \\ &= \mu_{EH} - \ln Z_0 - (g_H - g) \left[e^{-\delta_H a} \left(a + \frac{1}{\delta_H} \right) \right]_0^\infty \\ &= \mu_{EH} - \ln Z_0 + \frac{1}{\delta_H} (g_H - g), \end{aligned} \quad (43)$$

$$\begin{aligned} \bar{\Sigma}_x^2 &= \int_0^\infty \delta e^{-\delta_H a} (\sigma_{EH}^2 + a^2 \sigma_H^2) da \\ &= \sigma_{EH}^2 - \sigma_H^2 \left[e^{-\delta_H a} \left(a^2 + \frac{2}{\delta_H} a + \frac{2}{\delta_H^2} \right) \right]_0^\infty \\ &= \sigma_{EH}^2 + \frac{2}{\delta_H^2} \sigma_H^2. \end{aligned} \quad (44)$$

5. MARKET CLEARING

A. Labor market

Let Λ and M denote the cdfs for x and h . Recall that aggregate clearing in the labor market requires (16). With $\varepsilon = 1/\rho$, use (15) in (12) to find that labor market clearing for every level of human capital requires

$$N \frac{1}{\phi_0} Z^{-\rho} Y \Lambda(x) = LM(a_H x), \quad \text{all } x, \quad (45)$$

or, using (16),

$$\Lambda(x) = M(a_H x), \quad \text{all } x.$$

Under the conjecture that Λ and M are lognormal distributions, this condition holds if and only if their parameters, $(\bar{\mu}_x, \bar{\Sigma}_x)$ and $(\bar{\mu}_h, \bar{\Sigma}_h)$, respectively, satisfy

$$a_H = \frac{\bar{\mu}_h}{\bar{\mu}_x}, \quad \frac{\bar{\Sigma}_h}{\bar{\mu}_h} = \frac{\bar{\Sigma}_x}{\bar{\mu}_x}.$$

The first condition says that a_H must offsets the difference between the two means. The second requires the two distributions to have the same coefficient of variation. Taken together, they imply

$$a_H = \frac{\bar{\Sigma}_h}{\bar{\Sigma}_x}, \quad (46)$$

so a_H is determined by the standard deviations. Equilibrium requires the two means to have this ratio as well, a restriction involving g_X and g_H . The race is on!

B. Goods market

Clearing in the final goods market determines the level of consumption, which is output minus investment by workers, incumbents, and entrants,

$$C = c_0 Z = Y - L\psi(g_H)ZE \left[h^\beta \right] - N\gamma(g_X)ZE \left[x^\beta \right] - \delta_X N Z I_E.$$

[To be completed.]

REFERENCES

Atkeson, Andrew, and Ariel Burstein. 2010. Innovation, firm dynamics, and international trade, *Journal of Political Economy*, 118(June): 433-484.

Luttmer, Erzo J.G. 2007. Selection, growth and the size distribution of firms, *Quarterly Journal of Economics* 122(3) 1103–44.

[To be completed.]