

The Shadow Rate, Taylor Rules, and Monetary Policy Lift-off*

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Abstract

When the policy rate is constrained by the zero lower bound (ZLB), a new set of tools is needed to answer crucial questions about monetary policy, regarding the impact of the ZLB, expected lift-off, and the appropriateness of the policy stance. We document the shortcomings of affine dynamic term structure models (DTSMs) at the ZLB, and the benefits of shadow rate DTSMs. Using these we are able to appropriately answer the questions of interest: First, over recent years U.S. monetary policy has become increasingly constrained by the zero bound. Second, we estimate that in December 2012 the expected duration of the period of near-zero policy rates was 33 months, in line with survey-based and private-sector forecasts. Third, incorporating macroeconomic information in ZLB models is beneficial, improving inference about future policy, and allowing us to derive model-based Taylor rules and the resulting policy prescriptions. We find that in December 2012 the stance of monetary policy was in line with the desired stance based on simple policy rules.

Keywords: dynamic term structure models, monetary policy, zero lower bound, shadow rate, lift-off, Taylor rules

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1 Introduction

Dynamic term structure models (DTSMs) in the affine class are among the workhorse models of modern finance—see [Duffie and Kan \(1996\)](#), [Dai and Singleton \(2002\)](#), and [Ang and Piazzesi \(2003\)](#) among many others. However, these models do not prevent interest rates from becoming negative. This theoretical short-coming is inconsequential in practice during normal times, and the benefits of analytical bond pricing afforded by affine models outweigh this issue. However, when nominal interest rates are near the zero lower bound (ZLB), as it has been the case in the U.S. since 2008, the approximation of affine models deteriorates substantially. In such a context it becomes to explicitly incorporate the ZLB constraint on nominal interest rates into the term structure model. In this paper, we use DTSMs that account for this constraint, based on the shadow rate approach proposed by [Black \(1995\)](#), to answer some of the most pressing questions about monetary policy at the zero bound. In particular, we assess the stance of monetary policy, infer expected future policy, and estimate the future lift-off date. We also document the benefits of including macroeconomic information in ZLB models, which improves the inference about future monetary policy, and allows us to derive model-implied policy rules and desired policy rates.

Only few studies have used DTSMs that respect the ZLB constraint. The main reason is that this constraint introduces a nonlinearity that makes it difficult to solve for bond prices. In particular, one loses affine bond pricing, the major advantage of affine DTSMs. Since this usually gives rise to the need for numerical methods, the calculation of model-implied interest rates is computationally costly. Therefore, existing studies have only used models with a small number of risk factors, and full estimation of the model is the exception rather than the rule. However, due to advances in computing power and the efficiency of modern Monte Carlo methods, we can obtain accurate approximations of bond prices and filter latent risk factors with manageable computational cost. We implement a variety of shadow rate DTSMs, which replace the affine short rate specification of affine models by the maximum of zero and the shadow rate, which itself is affine in the risk factors. Shadow rate models have been shown by [Kim and Singleton \(2012\)](#) to perform well in comparison to other DTSMs that incorporate the ZLB. Using Treasury yield data for the U.S., we estimate both yields-only models as well as macro-finance models, and include models with different numbers of risk factors to ensure the robustness of our results. Our paper is the first to apply estimated shadow rate models to the recent U.S. experience, and the first to estimate macro-finance DTSMs that respect the ZLB constraint.

Our first contribution is to evaluate affine and shadow rate models in the presence of the

ZLB constraint. During the recent ZLB period, shadow rate models fit the cross section of yields substantially better than affine models, because the latent shadow rate constitutes an added degree of flexibility. Affine models frequently and starkly violate the ZLB constraint: Estimated forward curves and expected future short rates often drop below zero, and model-implied probabilities of negative future short rates are large. Shadow rate models avoid all of these violations by construction. Finally, affine models cannot capture the phenomenon of the policy rate remaining near zero. Consequently, such models produce inaccurate short rate forecasts at the ZLB. In contrast, shadow rate models accurately capture and forecast near-zero policy rates.

Our second contribution is to debunk the concept of the shadow rate, which has received much attention in academic and policy circles (see, for example, [Bullard, 2012](#)). We estimate shadow rates using a variety of models, and show that these estimates are highly model-dependent, in line with the findings of [Christensen and Rudebusch \(2013\)](#). It is hard to interpret the estimated level of the shadow rate at any given point in time, because it is simply a feature of the model that is used to fit the short end of the yield curve. More interesting are shadow yield curves and shadow forward rates. Such shadow term structures summarize the stance of monetary policy at any given point in time. Importantly, alternative estimated DTSMs agree much more closely on the shadow term structures than on the level of the shadow rate. A comparison of shadow yield curves with actual yield curves reveals that monetary policy has become more constrained by the ZLB between mid 2011 and late 2012, consistent with [Swanson and Williams \(2012\)](#).

Our modeling framework allows us to answer the question of when monetary policy is expected to lift off from the zero bound. A common approach is to take the horizon where the forward curve crosses a given threshold, say 25 basis points, as an estimate of the future lift-off date. This is misguided, because it ignores the asymmetry of the distribution of future short rates. Instead, one needs to consider the path of forward *shadow rates*, or expected future shadow rates, to appropriately estimate the expected duration of the ZLB period. We estimate this duration, based on the horizon at which forward shadow rates start to exceed 25 basis points, and find that our estimates accord closely with survey-based estimates and private-sector forecasts of policy lift-off. For December 2012, two of our four ZLB models and the two alternative outside estimates exactly coincide, predicting a lift-off 33 months in the future, i.e., in September 2015. Estimates of the ZLB duration based on forward rate curves are systematically too low, due to the inherent bias of this approach. Our paper is the first to estimate the date of policy lift-off using the appropriate state-of-the-art methodology.

When the nominal term structure is constrained by the ZLB, it is particularly beneficial

to add macroeconomic variables to the information set for inference about future monetary policy. We compare yields-only and macro-finance DTSMs and document the added benefit of the inclusion of macro factors. At the ZLB, macro-finance models forecast future policy rates much more accurately than yields-only models. Furthermore, estimates of the duration until lift-off are more closely in line with outside estimates when macroeconomics information is included. Intuitively, the ZLB constraint limits the information content of the yield curve, and it pays off to take into account the current macroeconomic situation when trying to predict how long the policy rate will remain near zero.

Finally, we evaluate the stance of monetary policy in relation to the prescriptions of simple policy rules. For this purpose, we first tie our model-based analysis in with the Taylor rule literature, by illuminating the relationship between Taylor rules and macro-finance DTSMs. In contrast to a related paper by [Ang et al. \(2007\)](#), we find that the policy rule implied by the model is practically identical to the one estimated using OLS. Importantly, we require orthogonality of the policy shock and calculate the correct policy rule coefficients. The desired policy rates from the rules implied by our macro models closely coincide with those from regression-based policy rules. A comparison of the shadow rate to the desired policy rate for our favored macro-finance model yields the result that both are at a similar level in December 2012.

We caution against evaluating the stance of monetary policy using current shadow rates for two reasons. First, shadow rate estimates are highly model-dependent. Second, in a ZLB situation, the short end of the term structure does not convey sufficient information about the stance of monetary policy. We propose instead to evaluate the appropriateness of the policy stance by considering the entire term structure. We compare the desired path of future policy rates, derived from model-implied macro forecasts and policy rule coefficients, to the expected path of future policy rates. We find that monetary policy was neither too tight nor too easy at the end of our sample period—the desired and actual forward policy paths closely coincide.

Some existing studies have used shadow rate term structure models to analyze yield curves in the proximity or presence of the ZLB. [Bomfim \(2003\)](#) employs a two-factor model to estimate the probability of the future policy rate hitting the ZLB during the 2002/2003 period. [Ueno et al. \(2006\)](#) analyze Japanese interest rates over the period 2001-2006 using a one-factor model, for which [Gorovoi and Linetsky \(2004\)](#) have derived an analytical solution, and [Ichiue and Ueno \(2007\)](#) apply a two-factor model to the same data. [Kim and Singleton \(2012\)](#) estimate two-factor models using Japanese yield data, and demonstrate the good performance of shadow rate models in comparison to alternative DTSM specifications that incorporate the ZLB. [Christensen and Rudebusch \(2013\)](#) estimate one-, two- and three-factor models on the

Japanese data and document the sensitivity of shadow rate estimates to the model specification. [Krippner \(2012\)](#) is the only study focusing on the recent U.S. experience. He uses a restricted two-factor model, and estimates a very negative shadow rate for recent years. No study has applied flexible and accurately fitting ZLB models to the U.S. data, estimated the expected duration until lift-off of the Federal funds rate, or applied macro-finance models that incorporate the ZLB restriction—our paper is the first to do so.

The paper is structured as follows: Section 2 lays out our econometric framework. In Section 3 we evaluate affine and shadow rate models in terms of cross-sectional fit, violations of the ZLB, and short rate forecasts. Section 4 presents estimates of shadow rates and shadow yield curves. Section 5 discusses estimation of the future lift-off date, and compares model-implied and alternative estimates of the duration of the ZLB period. In Section 6, we present the policy rules and desired policy rates implied by our macro-finance DTSMs, and evaluate the stance of monetary policy relative to rule-based policy.

2 Dynamic term structure models

Dynamic term structure models (DTSMs) impose absence of arbitrage in the sense that the cross section of bond yields is consistent with their time series behavior, allowing for a risk adjustment. This assumption implies the existence of a risk-neutral probability measure ($\mathbb{Q}$) in addition to the real-world probability measure ($\mathbb{P}$), and asset prices are discounted expected future payoffs under $\mathbb{Q}$. In addition, it is assumed that a low-dimensional vector of state variables, which we denote by X_t , contains all the information at time t that is relevant for investors. The specification of a DTSM includes (i) a function relating the short-term interest rate to X_t , (ii) a stochastic process describing the evolution of X_t under $\mathbb{Q}$, and (iii) a stochastic process describing the evolution of X_t under $\mathbb{P}$. In this section we discuss the role of the ZLB on interest rates in DTSMs, specify the models we use in this paper, and provide details about our empirical implementation. We use monthly data so t is measured in months, and our models are set in discrete time.

2.1 Affine models

Affine DTSMs have since [Duffie and Kan \(1996\)](#) become the work-horse models of modern finance. Here we focus on Gaussian models, in which the risk factors are homoskedastic. The canonical affine Gaussian DTSM is based on three assumptions:

Short rate The short-term interest rate—the one-month rate in our context—is affine in the

N risk factors X_t , i.e.,

$$r_t = \delta_0 + \delta_1' X_t, \quad (1)$$

for scalar δ_0 and N -vector δ_1 .

Risk-neutral distribution Under $\mathbb{Q}$, the risk factors are assumed to follow a Gaussian vector autoregression (VAR),

$$X_t = \mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} Z_{t-1} + \Sigma \varepsilon_t^{\mathbb{Q}}, \quad (2)$$

where Σ is lower triangular and $\varepsilon_t^{\mathbb{Q}}$ is an *iid* standard normal random vector under $\mathbb{Q}$.

Physical distribution Under the physical probability distribution $\mathbb{P}$, X_t also follows a Gaussian VAR,

$$X_t = \mu + \Phi Z_{t-1} + \Sigma \varepsilon_t, \quad (3)$$

where ε_t is an *iid* standard normal random vector under $\mathbb{P}$.

Note that these assumptions imply the existence of an essentially-affine stochastic discount factor as in [Duffee \(2002\)](#). The price of a bond with a maturity of m periods is determined by

$$P_t^m = E_t^{\mathbb{Q}} \left[\exp \left(- \sum_{i=0}^{m-1} r_{t+i} \right) \right]. \quad (4)$$

In an affine model this expectation can be found analytically, and it is exponentially affine in the risk factors. Model-implied yields therefore are affine functions of the factors, and the same holds for forward rates, expectations of future short-term interest rates, and risk-neutral yields. Term premia are defined as the difference between model-implied yields and risk-neutral yields. The details are well-known—for completeness we summarize them in [Appendix A](#).

Importantly, a Gaussian model implies that interest rates can turn negative with non-zero probability. During times when the short rate r_t is significantly above zero, this probability is negligibly small. Therefore the violation of the ZLB restriction on nominal interest is immaterial for all practical purposes, which explains the widespread use of the Gaussian model in academics, policy circles, and in practice. However, during times of near-zero interest rates, the ZLB restriction becomes practically relevant. First, the model-implied forward rates (short rate expectations under $\mathbb{Q}$) and the policy path (short rate expectations under $\mathbb{P}$) can turn negative. Second, the model-implied probability of a negative short rate in the future can

become substantial. We have

$$\alpha_{t,h} = P(r_{t+h} < 0 | X_t) = \Phi(-\tilde{\mu}_{t,h}/\tilde{\sigma}_{t,h}), \quad (5)$$

where $\tilde{\mu}_{t,h}$ and $\tilde{\sigma}_{t,h}$ are the conditional moments of the short rate:

$$\begin{aligned} \tilde{\mu}_{t,h} &= E_t r_{t+h} = \rho_0 + \rho_1' E_t X_{t+h} \\ \tilde{\sigma}_{t,h}^2 &= Var_t r_{t+h} = \rho_1' Var_t X_{t+h} \rho_1. \end{aligned}$$

The conditional variance $\tilde{\sigma}_{t,h}^2$ only depends on the horizon h . The conditional mean is typically positive, and large relative to $\tilde{\sigma}_{t,h}$, so that $\alpha_{t,h}$ is close to zero. When interest rates are near zero, however, $\tilde{\mu}_{t,h}$ will also be near zero and $\alpha_{t,h}$ becomes large. In Section 3.2 we will document violations of the ZLB and show the evolution of $\alpha_{t,h}$ over time for various affine models.

2.2 Shadow rate models

There are several alternative possibilities to model the term structure of interest rates under no-arbitrage and to observe the ZLB restriction. In this paper we use the shadow rate approach based on the work of Black (1995). Our shadow rate DTSMs are specified similarly to the affine models, the only difference being that the affine short rate equation (1) is replaced by a shadow rate specification as follows:

$$r_t = \max(s_t, 0), \quad s_t = \delta_0 + \delta_1' X_t. \quad (6)$$

The shadow rate s_t is unconstrained, and the short rate is equal to s_t or zero, whichever is larger. The interpretation of the shadow rate is that in the absence of currency, we would observe s_t as the (possibly negative) short-term interest rate. However, in the presence of currency, nominal interest rates cannot go below zero.¹ As Black (1995) pointed out, currency has the effect of introducing an option-like behavior of the short-term interest rate, which theoretically will always remain non-negative.

In addition to its intuitive appeal, the shadow rate DTSM has proved empirically more successful than alternative DTSM specifications that incorporate the ZLB—see Kim and Singleton (2012), who compare alternative model specifications using Japanese yield data. Shadow rate models have been used in several other studies, including Ueno et al. (2006), Ichiue and Ueno

¹This theoretical restriction has been violated in some historic instances, but even in those cases nominal interest rates remained very close to zero, albeit turning negative.

(2007), Krippner (2012) and Christensen and Rudebusch (2013). In the following we use the terms shadow rate model and ZLB model interchangeably.

Due to the nonlinearity of the short rate equation, we lose affine bond pricing. There is no simple closed-form solution for bond prices and yields, hence the need for numerical methods arises.² We use Monte Carlo simulations to evaluate the expectation in equation (4), which is a flexible and reliable method. Importantly, the computational cost of Monte Carlo simulation does not substantially increase with a higher number of risk factors. This stands in contrast to alternative numerical methods, such as approximating the solution to the fundamental partial differential equation (Kim and Singleton, 2012), which becomes prohibitively costly for more than two risk factors. Details about the implementation and evidence of the accuracy of our method are in Appendix B.

2.3 Risk factors

A key modeling choice is which risk factors to include in a DTSM. We will estimate both “yields-only” models, where the risk factors in X_t reflect only information in the yield curve, and “macro-finance” models, where X_t also includes macroeconomic variables.

Econometric identification of a DTSM requires normalizing parameter restrictions. For our yields-only models we use the canonical form of Joslin et al. (2011). We estimate affine and shadow rate models using both two-factor and three-factor specifications, and denote the affine models by $YA(2)$ and $YA(3)$, and the ZLB models by $YZ(2)$ and $YZ(3)$. The yield factors are taken as the first N principal components of the observed yields. They are assumed to be measured with error, so that the true yield factors are latent and need to be filtered.³ Our affine yields-only models correspond to the RKF model specification in Joslin et al. (2011).

Because the focus of this paper is on current and expected future monetary policy, it is crucial to incorporate macroeconomic information in the modeling framework. For this purpose we estimate macro-finance DTSMs that include measures of inflation and economic activity in addition to the yield factors. Our empirical results demonstrate the advantages of including macroeconomic factors into the information set during a ZLB period. To ensure identification we use the canonical form of JLS. We estimate affine and ZLB models with

²Krippner (2012) has proposed to express forward rates in a shadow rate context as the sum of forward shadow rates and an option effect. This can lead to quasi-analytical solutions if the value of the option effect can be derived using an option pricing framework. Christensen and Rudebusch (2013) perform the necessary derivations for their Affine-Nelson-Siegel model and are able to apply this approach empirically.

³In other words, all yields are measured with error, the yield factors are linear combinations of the true model-implied yields, and the weightings for these linear combinations correspond to the loadings of the first N principal components of the yield data.

one and two yield factors in addition to the two macro factors, and denote these by $MA(1)$, $MA(2)$, $MZ(1)$, and $MZ(2)$. We denote by L and M the number of yield and macro factors, respectively. The yield factors are the first L principal components of yields, again measured with error. Notably, the yield factors are latent and the macro factors are observable. The affine macro-finance models correspond to the TS^f specification in JLS.

In our macro-finance models, the macroeconomic variables are spanned by the yield curve.⁴ An alternative is to have unspanned macro risks as in [Joslin et al. \(2010\)](#) and [Wright \(2011\)](#). However, in our context it is important to have spanning, because otherwise there can be no Taylor rule. In a model where macro variables are not spanned, current yields and the current short rate are completely independent of macroeconomic information. This would be undesirable for our purpose, since one of our goals is to estimate model-implied policy rules and assess the stance of monetary policy relative to the desired policy stance.

2.4 Data, filtering, and estimation

Our data sample consists of monthly observations of interest rates and macroeconomic variables from January 1985 to December 2012. For the short end of the yield curve, we use three-month and six-month T-Bill rates⁵. The remaining rates are smoothed zero-coupon Treasury yields with maturities of one, two, three, five, seven, and ten years from [Gürkaynak et al. \(2007\)](#).⁶ We measure economic activity by the unemployment gap, using the estimate of the natural rate of unemployment from the Congressional Budget Office. Inflation is measured by percentage changes from the previous year in the consumer price index (CPI) for all items excluding food and energy, i.e., by core CPI inflation. We choose core inflation and activity gap measures because these are closely linked to the target Federal funds rate, the policy instrument of the Federal Reserve ([Judd and Rudebusch, 1998](#); [Rudebusch, 2009](#)). Figure 1 gives a view of our data. The top panel shows the first three principal components of yields, which due to the shape of the loadings can be interpreted as level, slope, and curvature. The bottom panel shows the macroeconomic data series.

All $J = 8$ yields are measured with error. Denote the model-implied vector of yields by Y_t and the actual, observed yields by Y_t^o . The observation equation for yields is

$$Y_t^o = Y_t + e_t, \quad e_t \stackrel{iid}{\sim} N(0, \sigma_e^2 I_J). \quad (7)$$

⁴A subtlety is that for the ZLB models, the macro factors are spanned by the unobservable “shadow yields,” and not by the model-implied yields.

⁵T-Bill rates are obtained from the Federal Reserve’s H.15 release, see <http://www.federalreserve.gov/releases/h15/data.htm>.

⁶These yields are available at <http://www.federalreserve.gov/pubs/feds/2006/200628/200628abs.html>.

This measurement error specification implies that the yield factors are latent. As mentioned above, for the macro-finance models the macroeconomic variables are assumed to be measured without error.

For the affine models, we have $Y_t = A + BX_t$, with J -vector A and $J \times N$ -matrix B containing the usual affine loadings. In this case the state space system is linear, and the Kalman filter can be used for inferring the latent factors and calculating the likelihood. We can carry out fast and reliable maximum likelihood estimation using the following approach: First, we obtain accurate starting values by estimating versions of the models in which yield factors are priced perfectly. In that setting, estimates of μ and Φ can be obtained using least squares, due to the particular form of the normalization that we use. The remaining parameters are easily obtained by maximizing the log-likelihood function for given VAR parameters. Second, we estimate the models with measurement error using the Kalman filter—the optimization quickly converges to the global maximum because of the good starting values. [Joslin et al. \(2011\)](#) make the point that this procedure delivers excellent starting values and works well for their yields-only model, and we find that this is also the case for the macro-finance models.

For the shadow rate models, we have $Y_t = g(X_t)$, where the function $g(\cdot)$ is non-linear and not known in closed form, but can be approximated using Monte Carlo simulation (see [Appendix B](#)). Since the observation equation becomes non-linear, a different filtering method is needed. We use the Extended Kalman Filter (EKF) in this case, which is computationally efficient since it only requires the calculation of the Jacobian of $g(\cdot)$. We approximate the Jacobian numerically. For a given set of parameters, it takes less than one minute to run the EKF and obtain risk factors and yields, for our chosen level of accuracy of the Monte Carlo simulations.

Estimation of the ZLB models using maximum likelihood and the EKF is computationally costly but feasible. In ongoing work, we are implementing a fast estimation method to obtain ML estimates of all models in this paper. At this stage, however, we use the following simple work-around, which dramatically reduces computational cost and is satisfactory in practice: We obtain the parameters by estimating the affine models over a subsample during which the ZLB was largely irrelevant, specifically, the sample period ending in December 2007. We use the parameters obtained in this way not only for the estimated affine model, but also for the corresponding shadow rate model. There are two assumptions underlying the validity of this approach. The first assumption is that the affine model and the ZLB model have close to identical implications on the estimation subsample, so that maximum likelihood estimates of the model parameters are interchangeable between the two models. [Appendix C](#) provides evidence that this is a reasonable assumption, by comparing affine and ZLB models over this

subsample. The second assumption is that cutting off the end of the sample before 2008 does not invalidate the parameter estimates. We are comfortable with this assumption as well because for those cases where we have included the full sample of data, we have found that extending the sample period does not materially affect the parameter estimates and economic implications.

3 Model evaluation

From a theoretical perspective, shadow rate models have a fundamental advantage over affine models in that they impose the non-negativity of nominal interest rates. But how relevant is this in practice? In this section we evaluate the affine and shadow rate models, and demonstrate what is being missed when affine models are used during a period of near-zero interest rates.

3.1 Cross-sectional fit

The first dimension of model evaluation for any DTSM is the accuracy of its cross-sectional fit, because a low-dimensional factor model cannot fit all yields perfectly. Table 1 shows the root-mean-square pricing errors (RMSEs) across models, for the whole cross section of yields and for each yield maturity separately. The top panel reports RMSEs for the whole sample, while the bottom panel reports the fit for the ZLB subsample, here taken to be the period from January 2008 to December 2012.

The ZLB models fit the cross section of yields more accurately than their affine counterparts. The differences are larger for those models that have worse fit. For example, the macro-finance model $MA(1)$, which was analyzed in detail by [Joslin et al. \(forthcoming\)](#), has trouble fitting yields, with a total RMSE for the full sample of 0.6 percentage points. The ZLB counterpart to this model, $MZ(1)$, cuts these RMSEs roughly in half. For affine models that fit the cross section well, and in particular for the canonical three-factor yields-only model $YA(3)$, the improvements from introducing a shadow rate specification are more modest.

The increased accuracy in cross-sectional of the ZLB models is explained by their better fit of yields during the ZLB period from 2008 to 2012. The bottom panel of Table 1 shows that the improvements in RMSEs are very substantial for this subsample, most dramatically for the $MA(1)/MZ(1)$ models, where the difference in fit is an order of magnitude. In contrast, for the pre-2008 period the affine and ZLB models have essentially identical cross-sectional fit, as shown in Appendix C. Intuitively, ZLB models have an additional degree of freedom

to fit yields, the shadow rate. This, however, only comes into play when yields are close to zero, and the policy rate is stuck at the ZLB, because only then does the shadow rate differ from the short rate. During ZLB periods, the added flexibility of shadow rate models helps to substantially improve cross-sectional fit.

3.2 Violations of the ZLB by affine models

To understand the relevance of the ZLB constraint for term structure modeling, we now show the extent to which affine models violate the this constraint in U.S. data.

One way for a DTSM to violate the constraint of the zero bound is when the path of model-implied forward rates or expected short rates drops below zero at some future horizon. Figure 2 shows an instance of this type of ZLB violation. The forward curve implied by model YA(3) is shown for June 30, 2011, together with the policy path, i.e., expected future short rates. The forward curve starts near zero, drops below zero for horizons from three to six months, and then turns positive and increases toward the unconditional mean under $\mathbb{Q}$. For the expected policy path, the violation is more severe, with negative values dipping lower and the path staying below zero for horizons from three to 15 months.

To systematically evaluate this type of violation, Table 2 shows for each affine model the number of observations in the sample where the forward curve and the policy path, respectively, drop below zero. Also shown are the dates of the first and last violation, and the average number of periods or which the path stays below zero. All affine models imply forward curves and policy paths that consistently violate the ZLB restriction during the period of near-zero policy rates from 2008 to 2012. For yields-only models, the paths typically stay below zero for several months. For the macro-finance models, forward rates often are negative for one year or more, and policy expectations often remain negative for more than two years. The more frequent and starker violations of the ZLB by macro-finance models is due to the added information from macroeconomic variables, which during and after the Great Recession drag down forward rates and expected policy rates.

Even when the expectation for the short rate at some future point in time is positive, its probability distribution implied by an affine model might put non-negligible probability mass on negative outcomes. Figure 3 plots $\alpha_{t,h}$, the conditional probability of negative future short rates as defined in equation (5). Each of the four panels shows, for a specific affine model, $\alpha_{t,h}$ for $h = 6, 12$, and 24 months. Even before the most recent episode, the probability of negative future short rates has at times been noticeably above zero, namely during the period of monetary easing after the 2001 recession. or the period from 2008 to 2012, all models imply that these probabilities are very sizable. The macro-finance affine models lead to even larger

probabilities over this period than the corresponding yields-only models. Because of the high unemployment rates and low levels of inflation toward the end of the sample, future short rates are estimated to be very depressed, and consequently the implied $\alpha_{t,h}$ is close to one for this period.

Affine models imply frequent and stark violations of the ZLB restriction during the data period that we consider. We now turn to a comparison of affine and shadow rate models which clearly demonstrates how important it is to explicitly incorporate the ZLB into the modeling framework.

3.3 Short rate forecasts at the ZLB

The Federal funds rate and other short-term interest rates have remained at the effective ZLB for the entire period from December 2008 to the time of this writing. This phenomenon of the short rate being “stuck” at zero is impossible to capture by an affine DTSM. In such a model, forecasts of the short rate will necessarily revert back to the unconditional mean, and the smooth forecast trajectory will remain at zero for longer than one period, due to the Gaussian character of the model. Shadow rate models, however, are able to capture such short rate behavior.

To empirically assess the importance of this issue, we evaluate the accuracy of model-based short rate forecasts at the ZLB. On December 16, 2008, the FOMC lowered the target for the Federal funds rate to a range from 0 to 25 basis points, hence we choose December 2008 as the first month of our forecast exercise. The forecast target is the three-month T-Bill rate, the shortest interest rate in our data set, which has remained stuck at zero, just like the Federal funds rate, over the remainder of the sample. For each month from December 2008 to June 2011 we calculate model-based forecasts of this rate for horizons of 3, 6, 9, 12, 15, and 18 months, and construct forecast errors as the difference between future observed values and model-based forecasts.

Table 3 shows the RMSEs in percentage points for each horizon across models. The ZLB models predict the short rate more accurately than the affine models, and the differences in forecast accuracy are large. This holds for yields-only models as well as for macro-finance models, but in the latter case the improvements are particularly striking. For the model pair $MA(2)/MZ(2)$ the improvements of the shadow rate model over the affine model are between 54% and 79%, depending on the forecast horizon. Clearly, the ability of shadow rate models to forecast near-zero short rates is empirically important, and leads to substantially more accurate short rate forecasts than for affine models.

This evidence, together with our results above on cross-sectional fit and the ZLB violations

of affine models, show the danger of seriously distorted inference when using affine DTSMs during a period of near-zero short-term interest rates. While an affine model with a sufficient number of risk factors would be able to accurately fit the cross section of interest rates, any type of economic inference is prone to be misleading. In particular, affine model estimates of near-term forward curves and policy paths, of risk-neutral rates and term premia estimates, as well as point and interval forecasts of future interest rates are likely to be seriously distorted when interest rates are near zero. Hence, a researcher will be well-advised to instead use models that incorporate the ZLB constraint, and shadow rate models are well suited for this purpose.

Another interesting result emerges from Table 3, regarding the effect of incorporating macroeconomic information into DTSMs. Comparing the models $MZ(2)$ and $YZ(2)$, both of which have two yield factors and exhibit similar cross-sectional fit, we see that for horizons longer than six months the macro-finance model substantially outperforms the yields-only model. The inclusion of macroeconomic factors leads to improvements in forecast accuracy that range from 29%, at the nine-month horizon, to 73%, at the 18-month horizon. More generally, both macro-finance ZLB models produce much more accurate forecasts than either of the yields-only models. This illustrates that for forecasts and economic inference at the ZLB, it is particularly useful to augment the information set by macroeconomic factors. The yield curve is constrained by the zero bound and therefore carries less information about the future path of monetary policy than during normal times. Therefore, at the zero bound there are substantial benefits if one takes into consideration the current economic situation.

4 Shadow rates and the stance of monetary policy

A key feature of the ZLB models we consider is the shadow rate, the short-term interest rate that would be observed in the absence of physical currency. Much interest in the ZLB term structure literature (Kim and Singleton, 2012; Krippner, 2012; Christensen and Rudebusch, 2013) and in policy circles (Bullard, 2012) has centered around the shadow rate. Intuitively, it appears that the shadow rate would signal the stance of monetary policy. This is appealing because during a period of near-zero policy rates, nothing about the stance of monetary policy can be learned from observing the short rate, which is stuck at zero.

Figure 4 shows the time series of the shadow rate implied by our four models, together with the three-month T-Bill rate. During times when the short rate is above zero, the shadow rates generally agree closely with the observed short rate and with each other. During the ZLB period, however, the models disagree substantially about the value of the shadow rate

at each given point in time. The $MA(1)$ model implies by far the most negative shadow rate, on average around minus four percent. The $MA(2)$ and YA_1 models lead to shadow rates that are slightly negative, whereas the $YA(3)$ model produces a shadow rate that is mostly positive and very close to zero.

Evidently the estimates of the shadow rate during times of a binding ZLB constraint are highly model dependent. This is consistent with the findings of [Christensen and Rudebusch \(2013\)](#) who show that shadow rate DTSMs estimated on Japanese bond yields differ substantially in their implications about the level of the shadow rate depending on the number of factors used. Notably, [Krippner \(2012\)](#) estimates a shadow rate for the recent U.S. period that drops far below zero, reaching around minus eight percent. His estimates are based on a restricted two-factor model which likely does not have very good cross-sectional fit.⁷ An explanation that is consistent with our evidence is that DTSMs with good cross-sectional fit produce a shadow rate that is closer to the actual short rate, while models that have larger yield fitting errors, such as $MA(1)$ and Krippner’s model, produce more negative shadow rates during ZLB periods. Intuitively, good-fitting models do not need the additional degree of freedom that the shadow rate provides in fitting the cross section of yields at any given time, while worse-fitting models use the shadow rate to tweak the current yield curve closer to actual yields.

A consequence of these observations is that the shadow rate does not easily lend itself to an interpretation regarding the current stance of monetary policy. [Bullard \(2012\)](#) has taken the Krippner’s estimates of a very negative shadow rate in the U.S. as evidence for a very easy stance of monetary policy. Our estimates and the discussion in [Christensen and Rudebusch \(2013\)](#) suggest that it is difficult to make such a conclusion, and that estimates of the current level of the shadow rate need to be taken with a large grain of salt. [Bullard \(2012\)](#) and others have also compared the shadow rate to the short-term interest rate prescribed by policy rules such as the Taylor rule. We will take up the issue of policy rules in [Section 6](#).

To answer the question how tightly the ZLB is constraining policy and yields, it turns out to be more useful to consider shadow yield curves. These represent the yields that would prevail in the absence of the ZLB, as estimated by a shadow rate DTSM. Shadow yields are calculated from bond prices with the short rate used for discounting being replaced by the shadow rate, i.e., in equation (4) r is replaced by s . Hence, calculation of shadow yields is carried out by using conventional affine loadings in combination with the risk factors of the ZLB model.⁸ Shadow yields approximately equal average expected future shadow rates under

⁷[Krippner \(2012\)](#) does not report RMSEs to assess the accuracy of the model fit.

⁸Note that the shadow yield curve for a ZLB model differs from the fitted yield curve for the affine model, even if the same parameters are used in both models, because the filtered factors differ. For the shadow

$\mathbb{Q}$, just as actual yields reflect $\mathbb{Q}$ -expected future short rates.

Figures 5 and 6 show actual yields together with fitted yield curves and shadow yield curves implied by each of the four ZLB models, on June 30, 2011, and December 31, 2012, respectively. The shadow yield curves naturally lie below the fitted yield curves. Importantly, there is much more agreement about the general level and shape of the shadow yield curve across models, than there is about the current level of the shadow rate. With the exception of $MA(1)$, the models agree that for June 2011, the shadow yield curves cross the zero bound at a maturity of about two years, and that from there on out shadow yields are about 20 to 50 basis points below zero. Thus, there was a noticeable effect of the ZLB on yields, which were somewhat constrained by it. But comparing this to December 2012, it becomes clear that at that time the ZLB was constraining yields more tightly. The maturity where shadow yields turn positive is longer, around three to five years, depending on the model (leaving model $MA(1)$ aside), and the differences between fitted and shadow yields are larger.

The result that the ZLB has become a tighter constraint later in the period of near-zero policy rates is consistent with the findings of [Swanson and Williams \(2012\)](#). Their analysis of the time-varying sensitivity of interest rates to economic news during this period reveals that some yields have become substantially constrained “only in late 2011” (p. 4). We show here and below that this tendency continued, with the ZLB constraining monetary policy and yields ever more tightly.

5 Policy expectations and lift-off

During times when the policy rate is at the zero bound, the question of the expected duration of this situation becomes front and center. The yield curve conveys information about the expectations of market participant for the timing of the lift-off of the policy rate from zero. It is common practice to gage the date of expected lift-off by considering the horizon when forward rates rise above a certain threshold near zero. Here we argue that this is the wrong way to interpret information in the yield curve.

The forward rate curve reflects expectations of the future short-term interest rate, under the pricing measure $\mathbb{Q}$, that is, $E_t^{\mathbb{Q}} r_{t+h}$ across horizons h . We leave aside the issue of risk adjustment, as is commonly done in practice in this context.⁹ It appears natural to take the

models, the EKF takes into account the ZLB and produces different risk factors than the Kalman filter does for the affine model.

⁹We do not consider P-measure expectations here and in the remainder of the paper because of the difficulties for inference resulting from the high persistence of interest rates—see, for example, [Kim and Orphanides \(2012\)](#) and [Bauer et al. \(2012\)](#).

horizon at which these expectations lift off from zero as an estimate of the duration of the ZLB period. This, however, ignores the asymmetry of the distribution of the future short term interest rate induced by the ZLB. Importantly, this distribution has a point mass at zero.¹⁰ In a Gaussian shadow rate model, the distribution of the future shadow rate is symmetric around $E_t^Q s_{t+h}$. In contrast, the distribution of the future short rate is identical for positive values, but piles up the probability $P^Q(s_{t+h} < 0 | X_t)$ at zero. Therefore it is always the case that $E_t^Q s_{t+h} < E_t^Q r_{t+h}$, and this difference becomes sizable near the zero bound. As a consequence, the horizon at which short rate expectations lift off from the zero bound will generally be shorter than the horizon at which shadow rate expectations, i.e., “forward shadow rates,” lift off. Since the latter is the relevant horizon for the duration of the ZLB period, considering forward rate curves will systematically underestimate this duration.¹¹

Figures 7 and 8 illustrate this issue. They show the forward rate curves implied by affine models, and the forward rate and forward shadow rate curves from the ZLB models, for June 30, 2011, and December 31, 2012, respectively. We include horizontal lines at 25 basis points as the relevant threshold. The forward rate curves from affine and ZLB models are typically close to each other (with the exception of the macro-finance models with $L = 1$), with the ZLB models implying slightly lower forward rates which do not violate the ZLB restriction. Forward shadow rates are always below the corresponding forward rates from the same model, in line with the argument made above. The lift-off implied by forward shadow rates is therefore later as the lift-off estimated from forward rates. The difference is larger in December 2012, because the ZLB was binding more tightly at that time.

Interestingly, the ZLB models agree much more closely on the expected horizon for lift-off than they agree on the current value of the shadow rate. Leaving aside model $MZ(1)$, lift-off is estimated to be 13 to 17 months in the future in June 2011, and 30 to 33 months in December 2012. Evidently, models with similar fit to the cross section of yields provide similar answers to the important question of policy lift-off, which is a comforting result.

The ZLB models allow us to appropriately estimate the time until lift-off for each date in the sample. In Figure 9 these estimated durations are shown for each of the four models for the time period from January 2008 to December 2012. We include both the conceptually correct estimates based on forward shadow rates, as well as the estimates based on forward rates. To evaluate the model-based estimates, we obtain two alternative estimates of future lift-off dates. The first is from the Primary Dealer (PD) Survey of the Federal Reserve Bank of New

¹⁰See also [Kim and Singleton \(2012\)](#) who provide an explicit formula for shadow rate expectations.

¹¹The distinction that we draw here is exactly the same as the one between the mean of the distribution of future short rates, and the mode of this distribution.

York, which is publicly available since January 2011.¹² In particular, we take as an estimate of the lift-off date the median of the dealers’ estimates for the most likely quarter and year of the first target rate increase. The second source we use are the forecasts of Macroeconomic Advisers (MA) for the future path of the target federal funds rate. We take as the lift-off estimate the first quarter in which this forecast exceeds 25 basis points. Table 4 shows the durations based on forward shadow rates and the two alternative estimates for the period starting in December 2010.

The results show that our approach accurately estimates the duration of the ZLB period. The model-based durations are generally close to the alternative duration estimates, with the exception of model $MA(1)$. In particular, the models capture the substantial increase in the expected duration between mid 2011 and late 2011. The Federal Open Market Committee (FOMC) announced on August 9, 2011, that it expected a near-zero policy rate until at least “mid-2013,” which caused the expected duration to jump up (see also Swanson and Williams, 2012). At the end of our sample, in December 2012, the model-based estimates are 30 months for $YZ(2)$ and 33 months for both $YZ(3)$ and $MZ(2)$, in accordance with the 33 months from both PD and MA.

Using forward rates instead of forward shadow rates to estimate the time of lift-off leads to durations that are too short, and the difference is particularly pronounced late in the ZLB period. In December 2012, estimates based on forward rates imply durations of 9 and 14 months for the yields-only models, and 21 months for the model $MZ(2)$, i.e., lift-off is estimated to occur between one and two years too early. Evidently, the asymmetry of the future short rate distribution introduces a very significant downward bias into these estimates.

The expected duration of the ZLB period increased quite substantially over the period from 2008 to 2012. One interpretation of this is that the constraint of the ZLB on monetary policy and interest rates has become tighter. Swanson and Williams (2012) measure the tightness of the ZLB constraint using the sensitivity of interest rates to macroeconomic news, and argue that two-year yields became constrained once the expected duration jumped in 2011. We document that this duration has in fact increased over the whole period until the end of our sample, and that in this sense the yield curve has become ever more constrained by the zero bound.

The analysis of these model-based estimates of the time until policy lift-off also shows the benefit of incorporating macroeconomic information. Visual inspection of Figure 9 appears to indicate that model $MZ(2)$ is more accurate than model $YZ(2)$, in the sense that the

¹²See http://www.newyorkfed.org/markets/primarydealer_survey_questions.html (accessed February 8, 2013) for the questions and answers of each survey.

estimated durations are closer, on average, to the outside estimates. To make this concrete we calculate the root-mean-square error of model-based vs. alternative estimates. For the Primary Dealer estimates, this RMSE is 4.6 months for model $MZ(2)$, compared to 5.7 months for $YZ(2)$, and 5.5 months for $YZ(3)$. For the estimates from Macro Advisers, the RMSEs are 3.5 months for $MZ(2)$, 5.3 months for $YZ(2)$, and 4.2 months for $YZ(3)$. Measured in this way, the macro-finance model $MZ(2)$ estimates the duration of the ZLB period more accurately than either of the yields-only models. The results here parallel our evidence in Section 3.3, demonstrating that at the ZLB the inference about future monetary policy can substantially benefit from the inclusion of macro factors in the information set. Another benefit of having macroeconomic variables in a DTSM is that this allows for an analysis of desired policy rates based on simple policy rules, to which we now turn.

6 Taylor rules

A vast literature, starting with Taylor’s seminal 1993 paper, has documented that actual policy rates have tracked quite closely the prescriptions from simple regression-based policy rules. At the zero bound, the difference between the actual policy rate (zero) and the prescribed policy rate has been interpreted as a measure of how much monetary policy falls short in providing the desired stimulus to the economy, i.e., the extent to which policy was *too tight* (Rudebusch, 2009). Recently, Bullard (2012) compared the shadow rate from a ZLB model to the rule-based policy prescription. Using Krippner’s very negative shadow rate, which is much lower than the predicted value from any reasonable policy rule, Bullard concluded that monetary policy was *too easy*. In this section, we first explain and present the policy rules implied by macro-finance DTSMs, discuss the relationship between desired policy rates and shadow rates, and then answer the question of whether the stance of monetary policy in December 2012 was too tight, too easy, or appropriate.

6.1 Regression-based and model-implied policy rules

We first estimate a simple policy rule by regressing the three-month T-Bill rate (TB_t) on core CPI inflation (π_t) and the unemployment gap (u_t). Using a sample period from January 1985 to December 2007, a period during which the T-Bill rate was unconstrained by the zero bound, we obtain the following estimates:

$$TB_t = 0.44 + 1.47\pi_t + 1.27u_t + \epsilon_t, \quad R^2 = 0.7732, \quad (8)$$

$\begin{matrix} (0.18) & (0.06) & (0.06) \end{matrix}$

where numbers in parentheses represent robust standard errors. Inflation and unemployment explain very well the time variation in this short-term interest rate, with an R^2 of about 77%. The coefficient estimates are roughly in line with typical Taylor-rule estimates. The coefficient on inflation exceeds one, indicating that the Fed responded more strongly than one-for-one to inflation, thus raising real rates in response to inflationary pressures (the Taylor principle).

What is the policy rule that is implied by a macro-finance DTSM and how does it correspond to regression-based estimates? One might be tempted to take the coefficients on the macro factors in the short rate equation (1) or in the shadow rate equation (6) as estimates of the Taylor rule coefficients. For our model $MZ(1)$ the element of δ_1 corresponding to π_t is estimated as 0.24, and the coefficient on u_t is 0.66, which is nowhere close to the values estimated in (8), despite identical macro variables and estimation sample. Ang et al. (2007) take the coefficients in the short rate equation of their macro-finance DTSM as estimates of Taylor rule parameters, and also find that these differ substantially from OLS estimates of the policy rule. Neither our nor their coefficients on inflation in the short rate equation are above one, so these estimates violate the Taylor principle. The discrepancy of parameters between model estimates and OLS estimates appears puzzling.

The explanation lies in the fact that the regression error term ϵ_t is orthogonal to π_t and u_t by construction, whereas the yield factors in the short rate equation are highly correlated with the macro factors. To make matters concrete, consider the $MF(1)$ model, denote the yield factor by l_t (level factor), and the coefficients in the shadow rate equation by δ_0 , δ_π , δ_u and δ_l . The model-implied policy rule is of the form

$$s_t = \beta_0 + \beta_\pi \pi_t + \beta_u u_t + e_t, \quad \text{cov}(\pi_t, e_t) = 0, \quad \text{cov}(u_t, e_t) = 0, \quad (9)$$

where we require orthogonality of the deviation of policy from the rule, e_t . The model-implied policy rule coefficients can easily be derived analytically. Let $m_t = (\pi_t, u_t)'$. We have

$$\begin{pmatrix} \gamma_\pi \\ \gamma_u \end{pmatrix} = \text{cov}^{-1}(m_t) \text{cov}(m_t, s_t) \quad (10)$$

$$= \begin{pmatrix} \delta_\pi \\ \delta_u \end{pmatrix} + \frac{\delta_l}{\sigma_\pi^2 \sigma_u^2 - (\sigma_{\pi u})^2} \begin{pmatrix} \sigma_{\pi l} \sigma_u^2 - \sigma_{ul} \sigma_{\pi l} \\ \sigma_{ul} \sigma_\pi^2 - \sigma_{\pi l} \sigma_{\pi u} \end{pmatrix} \quad (11)$$

The second term accounts for the covariance between the level factor and the macro factors. If we calculate the implied values for the estimated DTSM, we obtain $\beta_\pi = 1.41$ and $\beta_u = 1.25$.¹³

¹³For this calculation, we use the sample covariance matrix of the filtered risk factors, instead of the population covariance matrix implied by Φ and Σ .

These values correspond very closely to the OLS estimates. Note that they can alternatively be obtained for any macro-finance model without analytical work by regressing the model-implied short rate/shadow rate on the macro factors. Policy rules implied by macro-finance DTSMs and those estimated using OLS are practically identical. The minor differences in estimated coefficients are due to the difference between the model-implied short rate and the observed short rate used in policy rule regressions.

Notably, [Ang et al. \(2007\)](#) argue that one should interpret δ_π and δ_u directly as the Taylor rule coefficients, taking $\delta_l l_t$ as the “policy shock.” They do not require orthogonality, and view this as an advantage, stating that estimating policy rules using a DTSM “can free up the contemporaneous correlation between the macro and latent factors.” We take a different stance: In our view, the orthogonality of the deviation of the policy rate from the desired value is at the core of the idea of a policy rule.

6.2 The desired policy rate and shadow rates

We now compare the desired policy rate implied by macro-finance DTSMs to the shadow rate. Figure 10 shows the desired rate obtained from a Taylor rule regression, the model-implied desired rate, the shadow rate, and the three-month T-Bill rate. The top panel shows the model-based desired policy rate and the shadow rate for the model $MZ(1)$, and the bottom panel shows these for the model $MZ(2)$. All estimates are obtained on the subsample ending in December 2008, and the series are shown for the whole period until December 2012.

OLS and the macro-finance DTSMs imply desired policy rates that are indistinguishable, reflecting our finding above that the Taylor rule coefficients are very similar. Furthermore, both track the actual short rate very closely for the period before 2008, consistent with the high explanatory power of simple policy rules.

We might now compare the desired policy rate to the shadow rate, and, like [Bullard \(2012\)](#), draw conclusions from this about the stance of monetary policy was at a given point in time. However, the obvious problem here is that the shadow rate estimates are quite different between models. For the model $MF(1)$ it remains very negative until the end of the sample. The desired rate, in contrast, returns from negative territory to zero during 2011-2012. Based on this model, a possible conclusion would then be that monetary policy in fact was too easy over the last two years of the sample. However, model $MF(2)$ would lead to a different conclusion. Both the shadow rate and the desired rate are near zero in late 2012, so the stance of policy appears to have been appropriate. Based on our previous results in this paper one would have to prefer model $MF(2)$, due to its superior fit and more plausible implications regarding lift-off. Hence, if one takes the comparison between shadow

rate and desired policy rate seriously, the conclusion would have to be that monetary policy has recently been consistent with the prescription of simple policy rules.

While this is one possible interpretation, there are some reasons to take this with a grain of salt. First, the shadow rate is highly model-dependent, so the conclusion would be made on unstable grounds. Second, during times when the ZLB is binding, considering only the short end of the term structure is intuitively less appealing than viewing and judging the term structure as a whole.

6.3 The desired term structure and the stance of monetary policy

The stance of monetary policy is determined by the level and shape of the yield curve, not only by the short-term interest rate—after all, longer-term rates determine the cost of consumption and investment, which are central to the transmission of monetary policy. During normal times, yields at all maturities correlate strongly with movements at the short end of the term structure, so the policy stance can to some extent be summarized by the level of the short rate. But when the policy rate is at the zero bound, nothing can be learned about the stance of monetary policy by considering the level of the short rate.

To accurately gauge the policy stance at the ZLB, it is necessary to take into consideration the entire term structure. The question which naturally arises then is whether and how simple policy rules can still be used to measure the desired stance of policy. One possible answer is to construct a “desired term structure” from forecasts of the desired policy rate, using forecasts of macroeconomic variables in the policy rule. Macro-finance DTSMs imply macroeconomic forecasts, with the information set consisting of current values of yield factors and macro factors. Alternatively, one could obtain such forecasts from surveys or sophisticated macroeconomic models. We use our model $MZ(2)$, which produces both macro forecasts and a path for the desired future policy rate, based on the model-implied Taylor rule.

Figure 11 shows the forecasts and term structure implications. The top panel displays the forward curve, the forward shadow curve, and the forecasts of the desired short rate, e.g., the desired term structure. The bottom panel shows the underlying macro forecasts. Inflation is predicted to fall further and to remain subdued for an extended period of time, and the output gap to close only slowly.¹⁴ Consequently, the desired policy path falls below zero and does not rise above zero for 38 months. The appropriate object of comparison for this path is

¹⁴Based on this forecast and the CBO’s forecast of the natural rate of unemployment (5.5%), the unemployment rate is predicted to first drop below 6.5% at a horizon of 25 months. Since this is the FOMC’s stated numerical threshold that would have to be crossed before the first funds rate hike, the first increase in the policy rate is here predicted to occur no earlier than in January 2015.

the forward shadow rate path, since both paths can turn negative.¹⁵ Forward shadow rates, as discussed above, lift off at 33 months, and this path lies above the desired policy path for the most relevant range of maturities. Therefore actual monetary policy appears to have been ever so slightly tighter than desired monetary policy in December 2012. The difference, however, is small, and would be sensitive to slight changes in the macro forecasts. Overall, there is no strong evidence that policy was, at this date, substantially easier or tighter than prescribed by simple policy rules. The stance of monetary policy in December 2012 can therefore be viewed as appropriate with regard to the prescriptions of simple policy rules.

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¹⁵Another natural comparison is with expected shadow rates, since forecasts of both desired policy rates and shadow rates would in that case be constructed under the physical measure.

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A Affine bond pricing

Under the assumptions of Section 2.1, bond prices are exponentially affine functions of the pricing factors:

$$P_t^m = e^{\mathcal{A}_m + \mathcal{B}_m X_t},$$

and the loadings $\mathcal{A}_m = \mathcal{A}_m(\mu^Q, \Phi^Q, \delta_0, \delta_1, \Sigma)$ and $\mathcal{B}_m = \mathcal{B}_m(\Phi^Q, \delta_1)$ follow the recursions

$$\begin{aligned}\mathcal{A}_{m+1} &= \mathcal{A}_m + (\mu^Q)' \mathcal{B}_m + \frac{1}{2} \mathcal{B}_m' \Sigma \Sigma' \mathcal{B}_m - \delta_0 \\ \mathcal{B}_{m+1} &= (\Phi^Q)' \mathcal{B}_m - \delta_1\end{aligned}$$

with starting values $\mathcal{A}_0 = 0$ and $\mathcal{B}_0 = 0$. Model-implied yields are determined by $y_t^m = -m^{-1} \log P_t^m = A_m + B_m X_t$, with $A_m = -m^{-1} \mathcal{A}_m$ and $B_m = -m^{-1} \mathcal{B}_m$. Risk-neutral yields, the yields that would prevail if investors were risk-neutral, can be calculated using

$$\tilde{y}_t^m = \tilde{A}_m + \tilde{B}_m X_t, \quad \tilde{A}_m = -m^{-1} \mathcal{A}_m(\mu, \Phi, \delta_0, \delta_1, \Sigma), \quad \tilde{B}_m = -m^{-1} \mathcal{B}_m(\Phi, \delta_1).$$

Risk-neutral yields reflect policy expectations over the life of the bond, $m^{-1} \sum_{h=0}^{m-1} E_t r_{t+h}$, plus a convexity term. The yield term premium is defined as the difference between actual and risk-neutral yields, $ytp_t^m = y_t^m - \tilde{y}_t^m$.

B Monte Carlo bond pricing

To obtain bond prices and yields for the ZLB model, we resort to Monte Carlo simulation. For given values of the risk factors, the price of a bond with maturity m is

$$P_t^m = E_t^Q \left[\exp \left(- \sum_{i=0}^{m-1} r_{t+i} \right) \right],$$

where $\theta = (\gamma_0, \gamma_1, \mu^Q, \Phi^Q)$. Since this expectation cannot be found analytically, we approximate it by simulating $M = 500$ paths of the risk factors of length m , where each sample path is obtained using the Q-measure VAR in equation (2), starting from the given initial value X_t . Using the ZLB short rate equation (6) we obtain the sampled paths for the short rate. Denote the value of the short rate in simulation j at time $t+i$ by $r_{t+i}^{(j)}$. The approximate bond price is given by

$$\tilde{P}_t^m = M^{-1} \sum_{j=1}^M \exp \left(- \sum_{i=0}^{m-1} r_{t+i}^{(j)} \right).$$

We use antithetic sampling to obtain the shock sequences, by taking the shock sequence for replication j as the negative of the shock sequence for replication $j-1$. This improves the accuracy of the approximation for any given M , because it introduces negative dependence between pairs of replications.

To assess the accuracy of our Monte Carlo simulation for bond prices, we compare Monte

Table B.1: Accuracy of Monte Carlo yields

Model	Total	3m	6m	1y	2y	3y	5y	7y	10y
YA(2)	0.14	0.00	0.00	0.01	0.04	0.08	0.06	0.11	0.35
YA(3)	0.85	0.00	0.00	0.01	0.04	0.20	0.72	1.17	1.96
MA(1)	0.47	0.00	0.00	0.01	0.04	0.11	0.42	0.75	1.01
MA(2)	0.38	0.00	0.00	0.01	0.03	0.07	0.48	0.78	0.56

Notes: Root-mean-square difference, in basis points, between analytical and Monte Carlo yields for affine models. Sample period: January 1985 to December 2012.

Carlo bond prices and analytical bond prices for the affine model. The Monte Carlo bond prices are obtained as described above, with the only difference being that we use the affine short rate equation (1) instead of the ZLB short rate equation (6). We use our estimated parameters, together with the risk factors filtered using the Kalman filter in the affine model, and consider the accuracy of Monte Carlo yields over the full sample from 1985 to 2012.

Table B.1 shows the approximation error, measured as the root-mean-square difference between exact affine yields and approximate Monte Carlo yields, in basis points. The approximation error is miniscule at short and medium maturities, and rarely exceeds one basis point. Evidently, the approximation of model-implied yields using our Monte Carlo method, even for a moderate amount of replications ($M = 500$), is very accurate.

C Affine vs. shadow rate models over the 1985–2007 subsample

Here we provide evidence that the implications of the affine and ZLB models on the estimation subsample 1985 to 2007 are very similar. This is part of the justification for using in our shadow rate models those parameters that are estimated for the affine models.

We use the parameters estimated for the affine model and calculate model-implied yields using both the affine and ZLB models over the pre-2008 sample period. For each model we first filter the latent risk factors appropriately from the observed yields, using the Kalman filter for the affine models, and the Extended Kalman filter for the ZLB models.

Table C.1 compares the model fit of the affine and ZLB models over the estimation subsample. In the first panel, it shows the root-mean-square error (RMSE) in basis points, to assess whether affine and ZLB models differ in terms of cross-sectional fit over this subsample. The RMSEs are very similar for each pair of affine and ZLB models, which demonstrates that the cross-sectional fit is essentially identical. The second panel of the table shows the discrepancy between yields implied by the affine and ZLB models, again measured as a root-mean-square difference. We see that the discrepancy is very small—model-implied yields from affine and ZLB models typically differ by less than one basis point.

These results show that the affine model and the ZLB model have close to identical implications on the estimation subsample. Therefore, it appears that the assumption is reasonable

Table C.1: Comparison of affine and ZLB model from 1985 to 2007

Model	Total	3m	6m	1y	2y	3y	5y	7y	10y
<i>Cross-sectional fit</i>									
YA(2)	13.98	17.73	11.22	19.60	15.98	11.25	9.06	8.91	13.97
YZ(2)	13.96	17.77	11.27	19.59	16.03	11.23	8.85	8.88	13.86
YA(3)	6.07	6.36	9.25	9.29	2.89	4.44	4.17	2.76	5.46
YZ(3)	6.10	6.33	9.22	9.39	2.86	4.56	4.25	2.72	5.45
MA(1)	28.77	40.85	33.80	22.86	12.08	13.55	22.93	30.78	38.51
MZ(1)	28.75	40.85	33.71	22.90	12.06	13.56	22.95	30.76	38.52
MA(2)	9.56	13.74	10.26	10.31	9.98	8.45	4.15	5.15	10.80
MZ(2)	9.58	13.64	10.39	10.36	9.98	8.44	4.15	5.19	10.88
<i>Discrepancy between affine and ZLB fitted yields</i>									
YA(2)/YZ(2)	0.41	0.26	0.19	0.08	0.20	0.37	0.45	0.13	0.93
YA(3)/YZ(3)	0.28	0.14	0.13	0.25	0.24	0.37	0.42	0.24	0.31
MA(1)/MZ(1)	0.47	0.34	0.36	0.36	0.32	0.23	0.21	0.61	0.89
MA(2)/MZ(2)	0.58	0.58	0.32	0.12	0.57	0.70	0.34	0.34	1.10

Notes: Cross-sectional fit, measured by root-mean-square errors of model-implied yields, and discrepancy between affine and ZLB fitted yields, measured by root-mean-square differences, all measured in basis points. Sample period: January 1985 to December 2007.

that maximum likelihood estimates of the model parameters are interchangeable between each affine model and its ZLB counterpart, if estimation is constrained to the pre-2008 subsample.

Table 1: Cross-sectional fit

Model	Total	3m	6m	1y	2y	3y	5y	7y	10y
<i>Full sample</i>									
<i>YA</i> (2)	14.0	17.1	10.8	18.1	16.2	13.1	9.0	8.8	15.4
<i>YZ</i> (2)	13.5	16.8	11.0	18.3	15.3	12.0	8.7	8.5	14.2
<i>YA</i> (3)	6.8	7.2	8.7	10.2	3.0	5.9	5.7	2.8	7.1
<i>YZ</i> (3)	6.7	7.0	8.8	10.1	3.4	6.0	6.0	2.9	6.6
<i>MA</i> (1)	60.2	78.6	71.8	54.9	19.0	20.0	52.2	68.3	80.3
<i>MZ</i> (1)	29.1	37.7	31.7	25.0	22.0	20.4	22.2	29.5	38.3
<i>MA</i> (2)	12.0	17.1	9.9	14.9	12.8	9.8	8.0	7.4	12.3
<i>MZ</i> (2)	10.3	13.7	10.6	12.4	10.7	9.2	6.3	5.6	11.2
<i>Subsample 2008-2012</i>									
<i>YA</i> (2)	14.0	14.0	8.7	7.8	16.9	19.3	8.7	8.1	20.7
<i>YZ</i> (2)	11.4	10.9	10.0	10.1	11.6	14.9	8.1	6.5	15.7
<i>YA</i> (3)	9.3	10.3	5.8	13.6	3.3	10.2	10.1	3.0	11.9
<i>YZ</i> (3)	9.1	9.7	6.7	12.7	5.0	10.1	10.8	3.7	10.4
<i>MA</i> (1)	128.3	164.0	153.7	120.3	36.8	37.3	113.3	147.7	171.2
<i>MZ</i> (1)	30.7	16.5	20.3	33.0	45.2	38.4	18.1	22.5	37.3
<i>MA</i> (2)	19.5	27.6	8.0	27.4	21.4	14.5	16.7	13.7	17.8
<i>MZ</i> (2)	13.2	14.1	11.3	19.3	13.6	12.2	12.1	7.2	12.6

Notes: Root-mean-square errors of model-implied yields. Full sample: January 1985 to December 2012. ZLB subsample: January 2008 to December 2012.

Table 2: Violations of ZLB by forward curves and policy paths

Model	Forward curve				Short rate expectations			
	# viol.	from	to	avg. dur.	# viol.	from	to	avg. dur.
<i>YA</i> (2)	25	8/2009	12/2011	1.4	30	12/2008	11/2011	3.3
<i>YA</i> (3)	11	10/2010	1/2012	4.4	16	11/2009	8/2011	9.2
<i>MA</i> (1)	50	11/2008	12/2012	10.5	50	11/2008	12/2012	27.7
<i>MA</i> (2)	34	12/2008	12/2012	10.5	49	12/2008	12/2012	22.3

Notes: Number of observations for which the forward curve (left panel) or the policy path, i.e., expected future short-term interest rates (right panel) drop below zero, first and last month for which this occurred, and average duration that the path stayed in negative territory.

Table 3: Forecast accuracy at the ZLB

Model	3m	6m	9m	12m	15m	18m
<i>YA</i> (2)	0.086	0.134	0.233	0.323	0.416	0.519
<i>YZ</i> (2)	0.102	0.119	0.150	0.178	0.229	0.307
<i>YA</i> (3)	0.110	0.149	0.169	0.223	0.336	0.493
<i>YZ</i> (3)	0.094	0.096	0.104	0.143	0.228	0.353
<i>MA</i> (1)	1.719	1.496	1.266	1.035	0.802	0.566
<i>MZ</i> (1)	0.131	0.120	0.110	0.109	0.110	0.101
<i>MA</i> (2)	0.252	0.444	0.514	0.496	0.414	0.292
<i>MZ</i> (2)	0.115	0.114	0.106	0.104	0.099	0.084

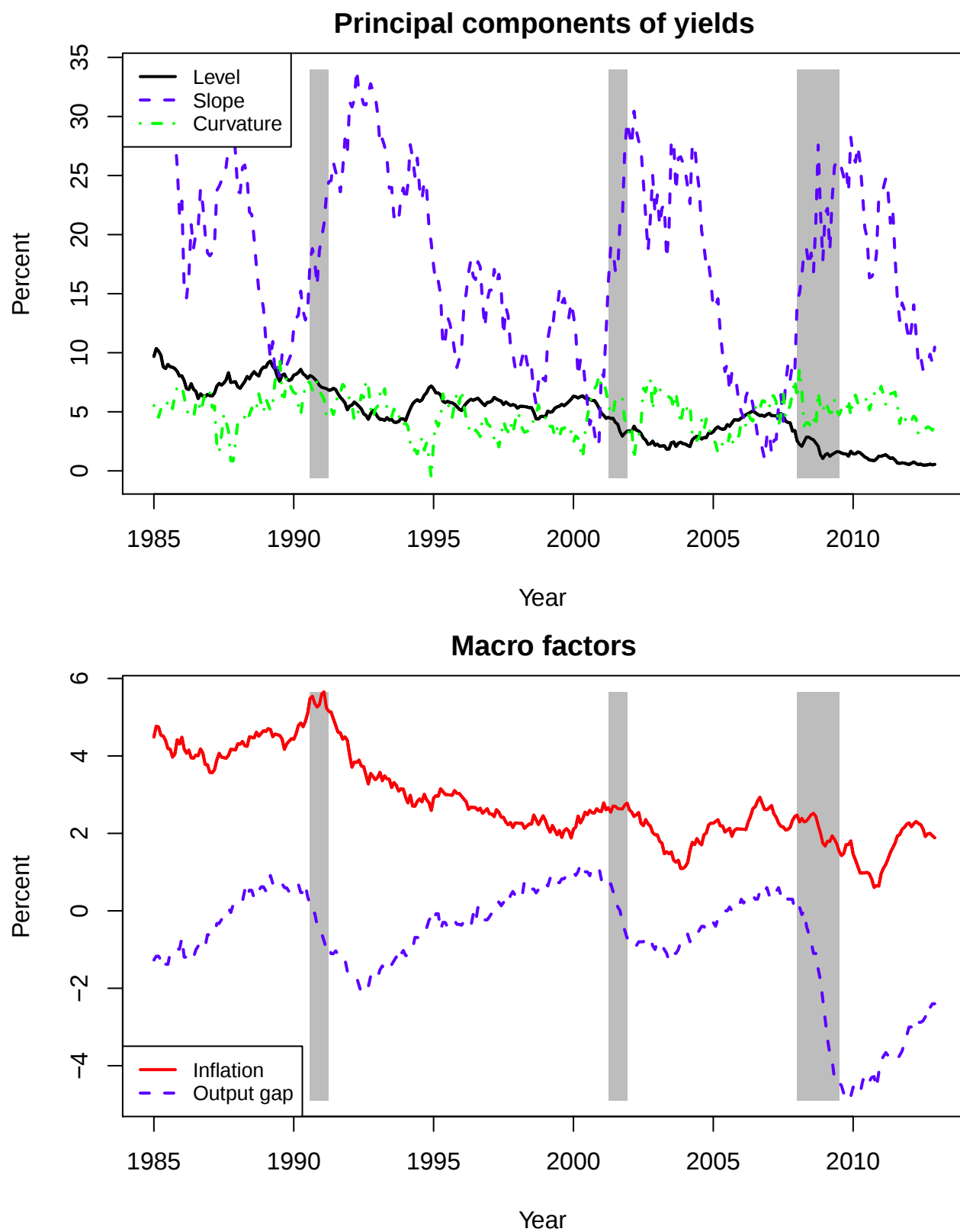
Notes: RMSEs in percentage points for model-based forecasts of the future three-month T-Bill rate at various forecast horizons. Forecast period: December 2008 to June 2011.

Table 4: Estimated duration of ZLB period

Month	PD	MA	YZ(2)	YZ(3)	MZ(1)	MZ(2)
2010-Jan			7	9	20	11
2010-Feb		16	8	10	21	11
2010-Mar		15	6	6	20	9
2010-Apr		14	7	7	20	9
2010-May		13	8	10	21	11
2010-Jun		12	10	13	23	13
2010-Jul		11	11	16	25	15
2010-Aug		16	14	19	28	19
2010-Sep		21	14	20	29	19
2010-Oct		20	16	22	30	21
2010-Nov		19	13	18	29	19
2010-Dec		15	9	12	25	14
2011-Jan	20		10	14	23	15
2011-Feb		13	10	12	21	14
2011-Mar	18	12	9	10	20	12
2011-Apr	14	11	11	14	21	15
2011-May			13	17	23	18
2011-Jun	15	15	13	17	23	18
2011-Jul		14	17	21	26	22
2011-Aug	16	25	23	28	30	28
2011-Sep	27	36	26	30	33	30
2011-Oct			24	27	34	29
2011-Nov	28		24	28	34	30
2011-Dec	30		27	30	36	32
2012-Jan	29	35	30	34	37	34
2012-Feb			25	28	36	31
2012-Mar	30		18	21	34	27
2012-Apr	29	32	23	29	35	30
2012-May			31	37	38	35
2012-Jun	30		30	34	39	35
2012-Jul		35	37	42	40	38
2012-Aug	31		38	41	41	38
2012-Sep	36		36	39	41	37
2012-Oct	35		31	34	40	34
2012-Nov			35	39	40	36
2012-Dec	33	33	30	33	39	33

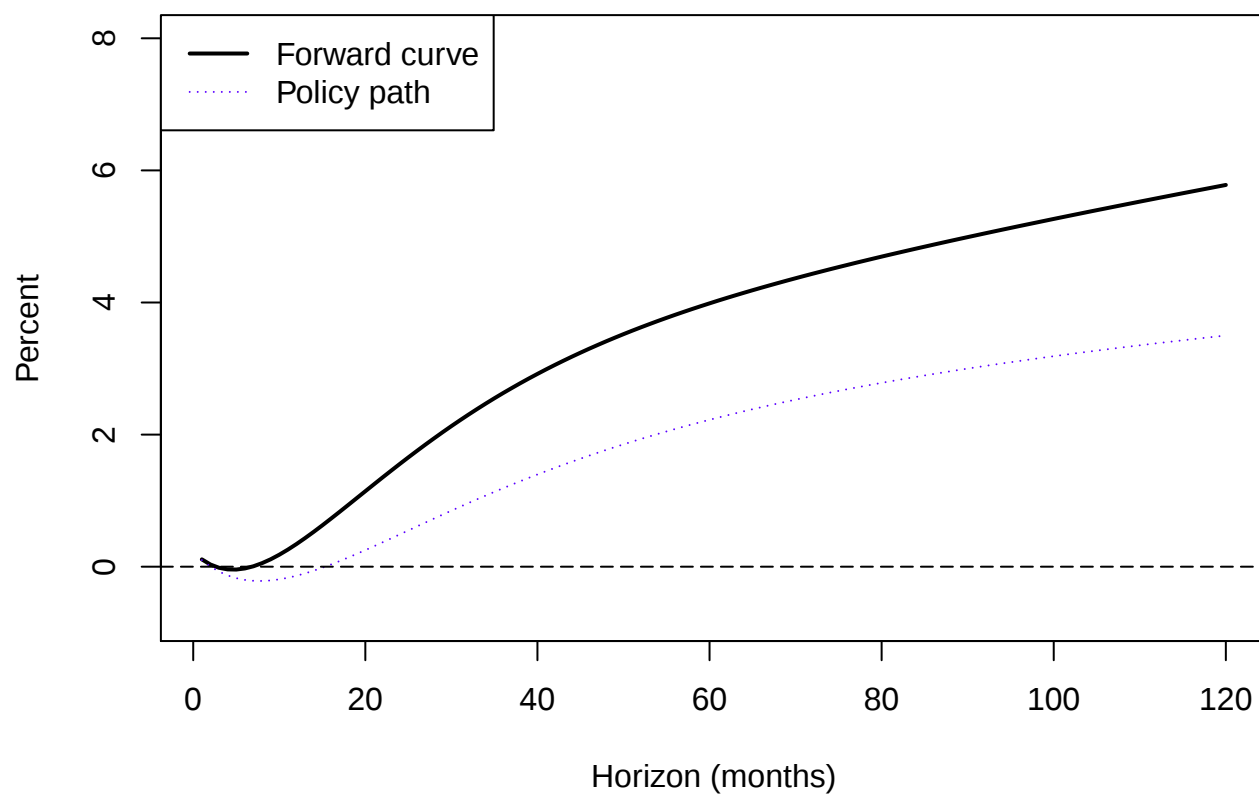
Notes: Estimated time (in months) until policy lift-off from the ZLB period, defined as the first period in which the short rate exceeds 25 basis points, based on forward rates and forward shadow rates. Also shown are survey-based expectations from the Primary Dealer (PD) survey and the predicted duration from Macroeconomic Advisers (MA).

Figure 1: Yield factors and macroeconomic data



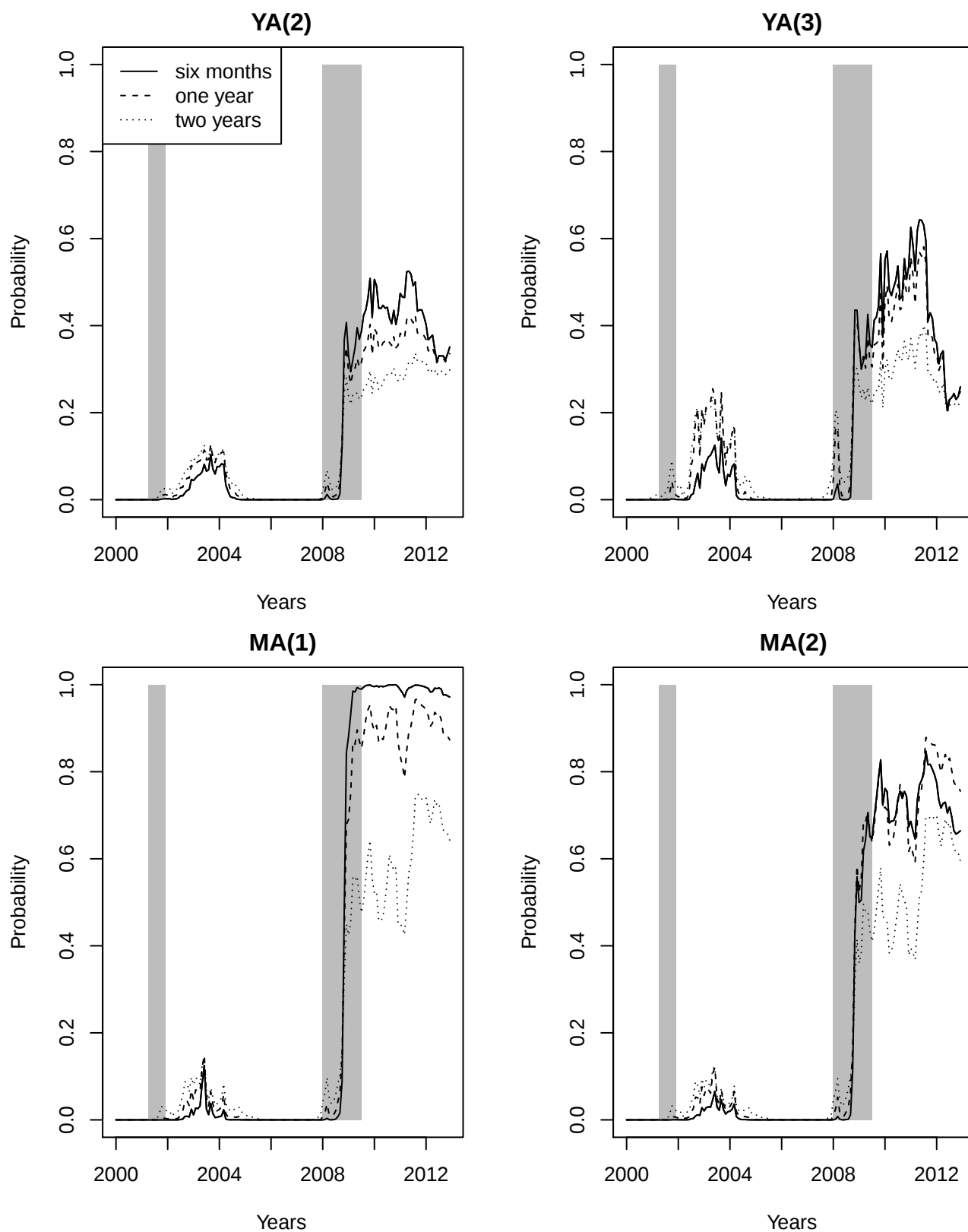
Notes: Top panel shows the yield factors, the first three principal components of yields. Bottom panel shows the macroeconomic data. Sample period: January 1985 to December 2012

Figure 2: Violation of the ZLB by affine model $YA(3)$ in June 2011



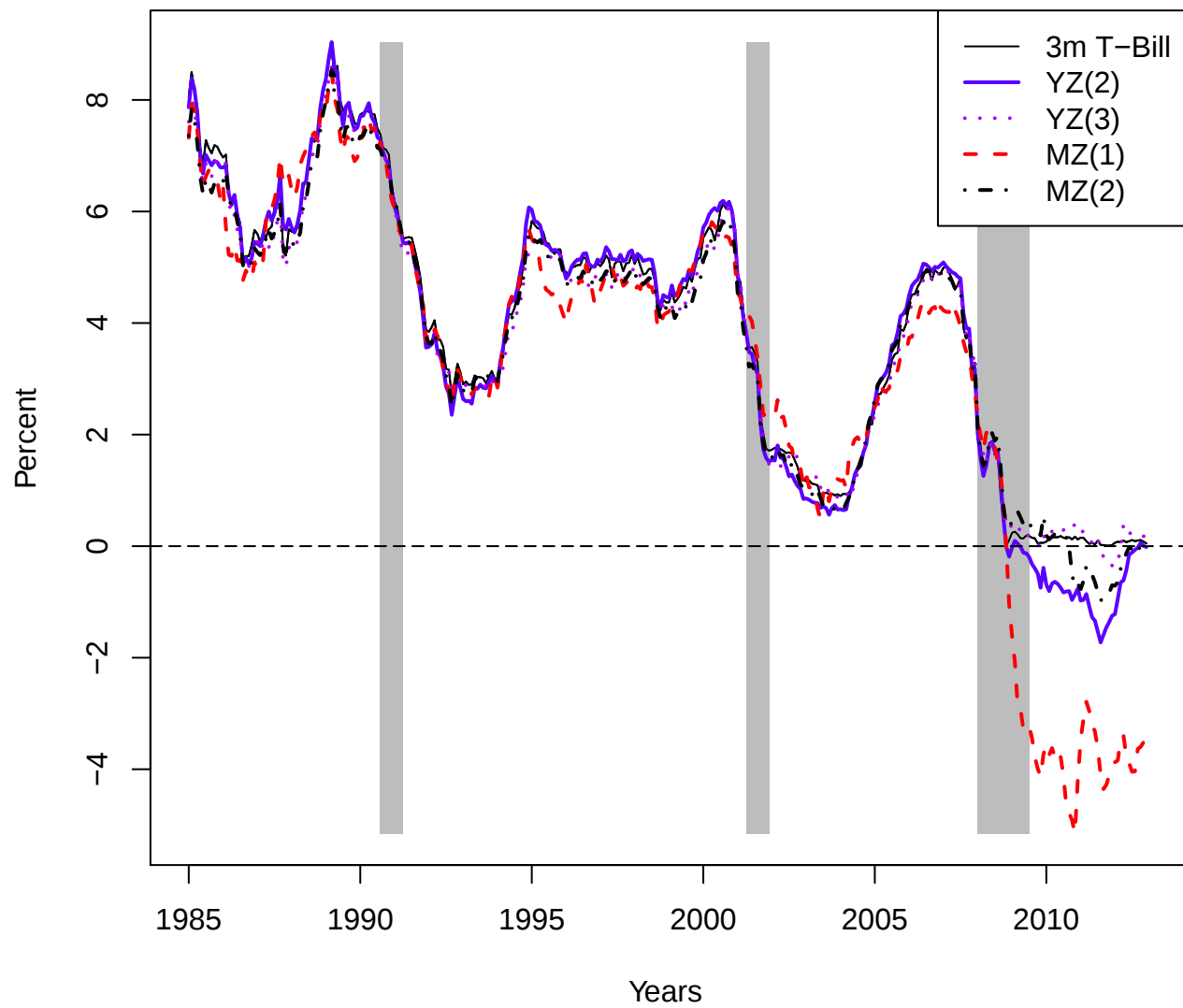
Notes: Solid black line shows the forward curve, i.e., expectations of future short rates under $\mathbb{Q}$. Dotted blue line shows the policy path, i.e., expectations under $\mathbb{P}$.

Figure 3: Affine model probabilities of negative future interest rates



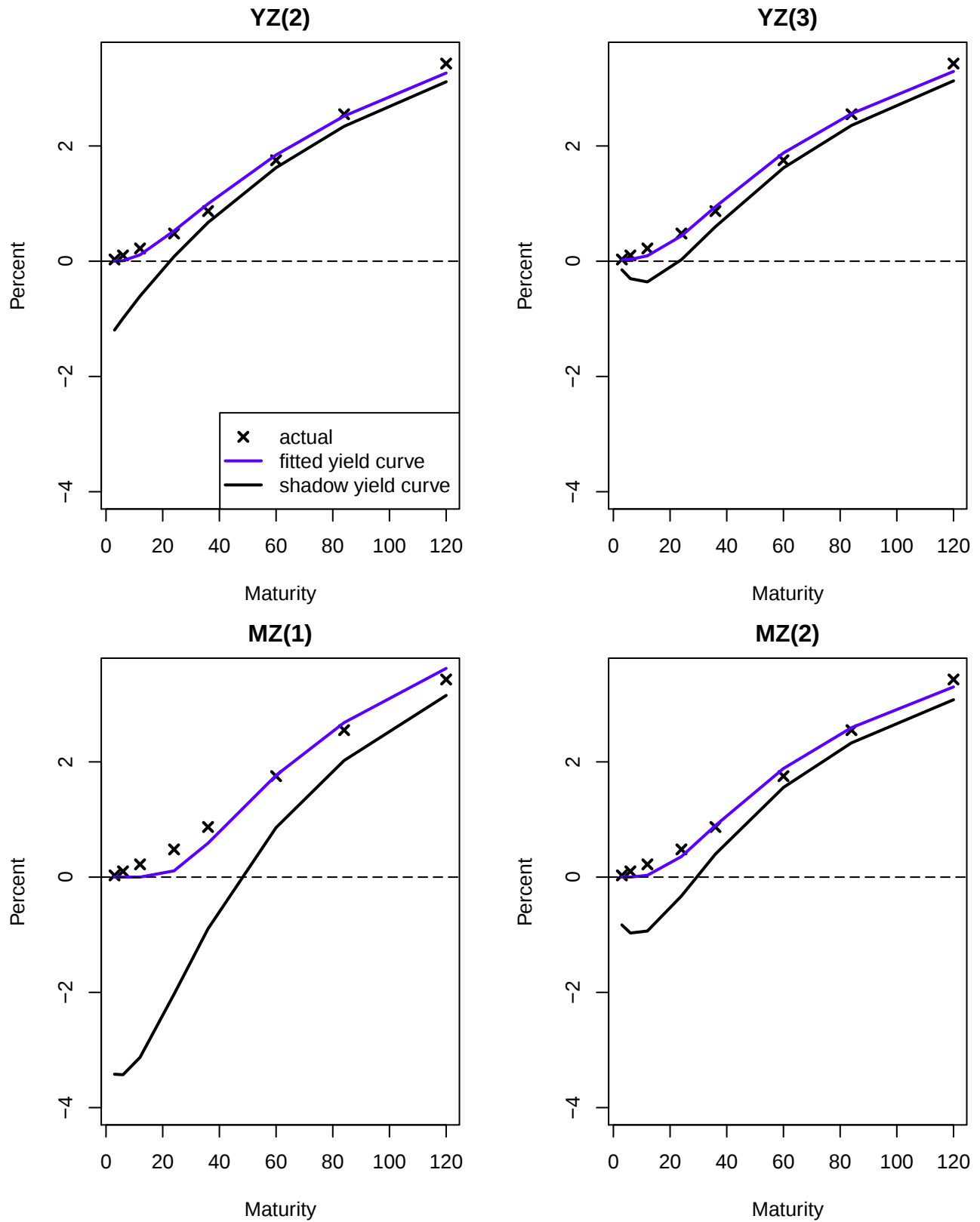
Notes: Model-implied probabilities of negative future short-term interest rates at horizons of six months, one year, and two years. Shaded areas correspond to NBER recessions. Sample period: January 2000 to December 2012

Figure 4: Shadow rates



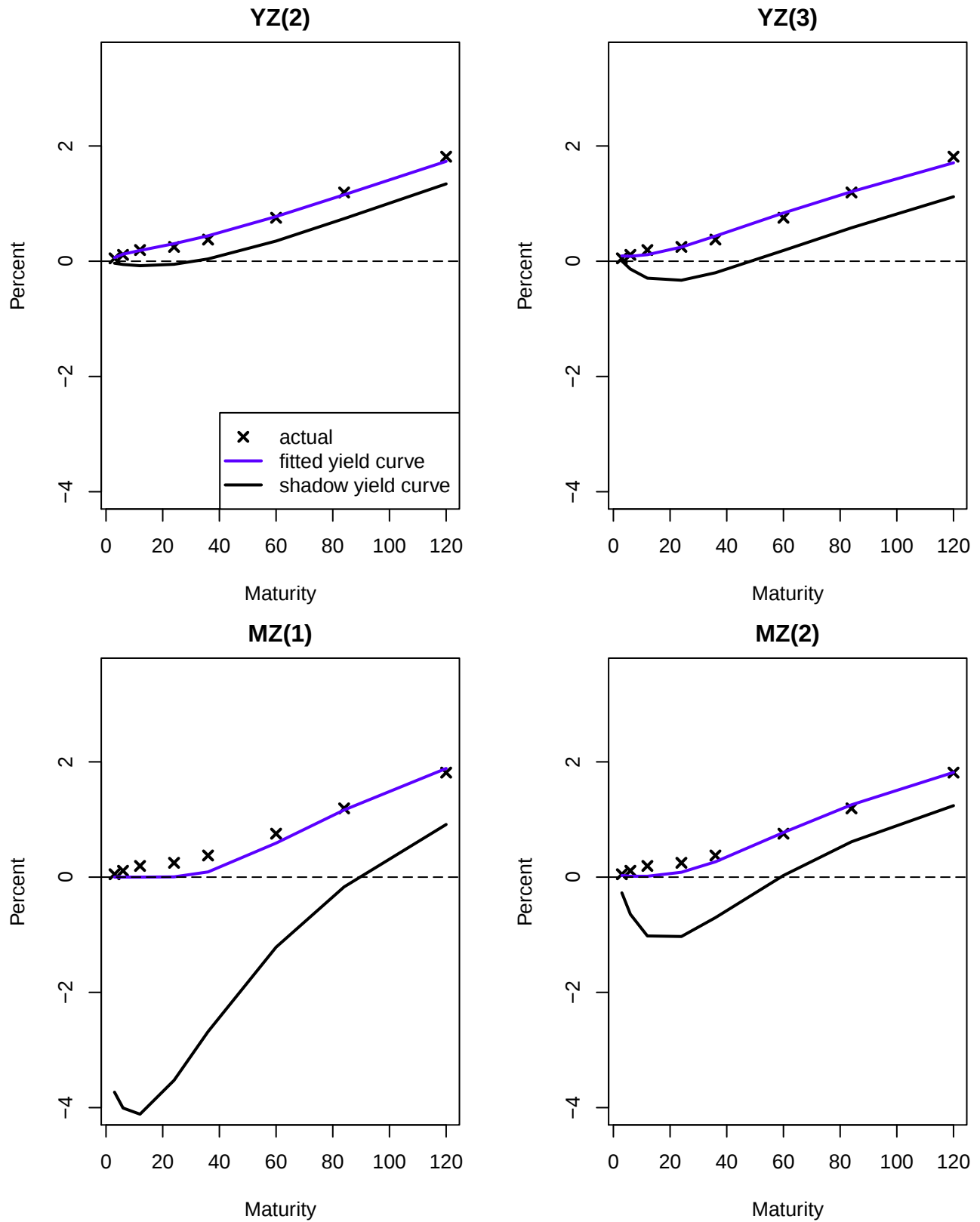
Notes: Shadow rates implied by ZLB models, and three-month T-Bill rate. Shaded areas correspond to NBER recessions. Sample period: January 1985 to December 2012.

Figure 5: Shadow yield curves in June 2011



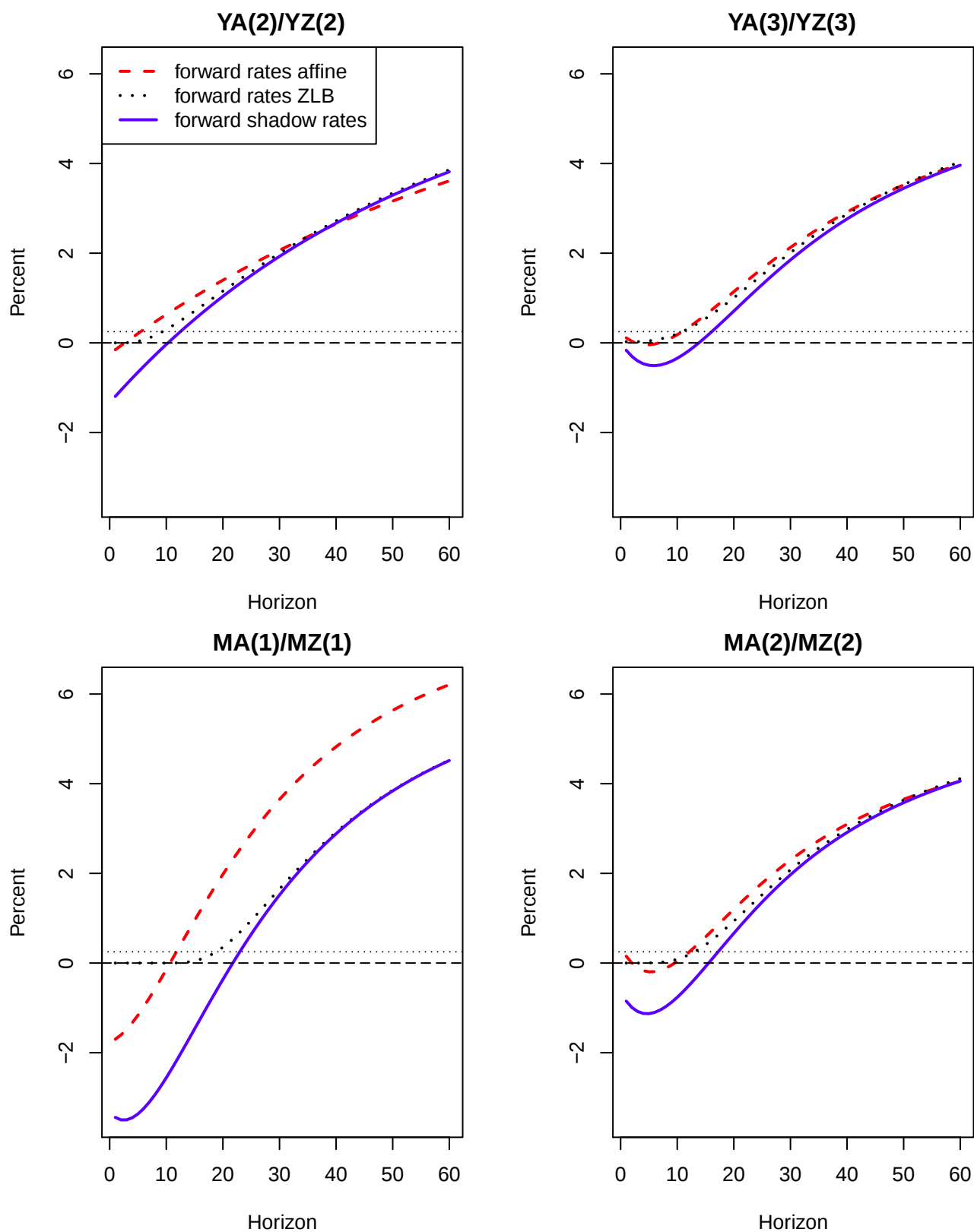
Notes: Actual, fitted, and shadow yield curves for ZLB models on June 30, 2011.

Figure 6: Shadow yield curves in December 2012



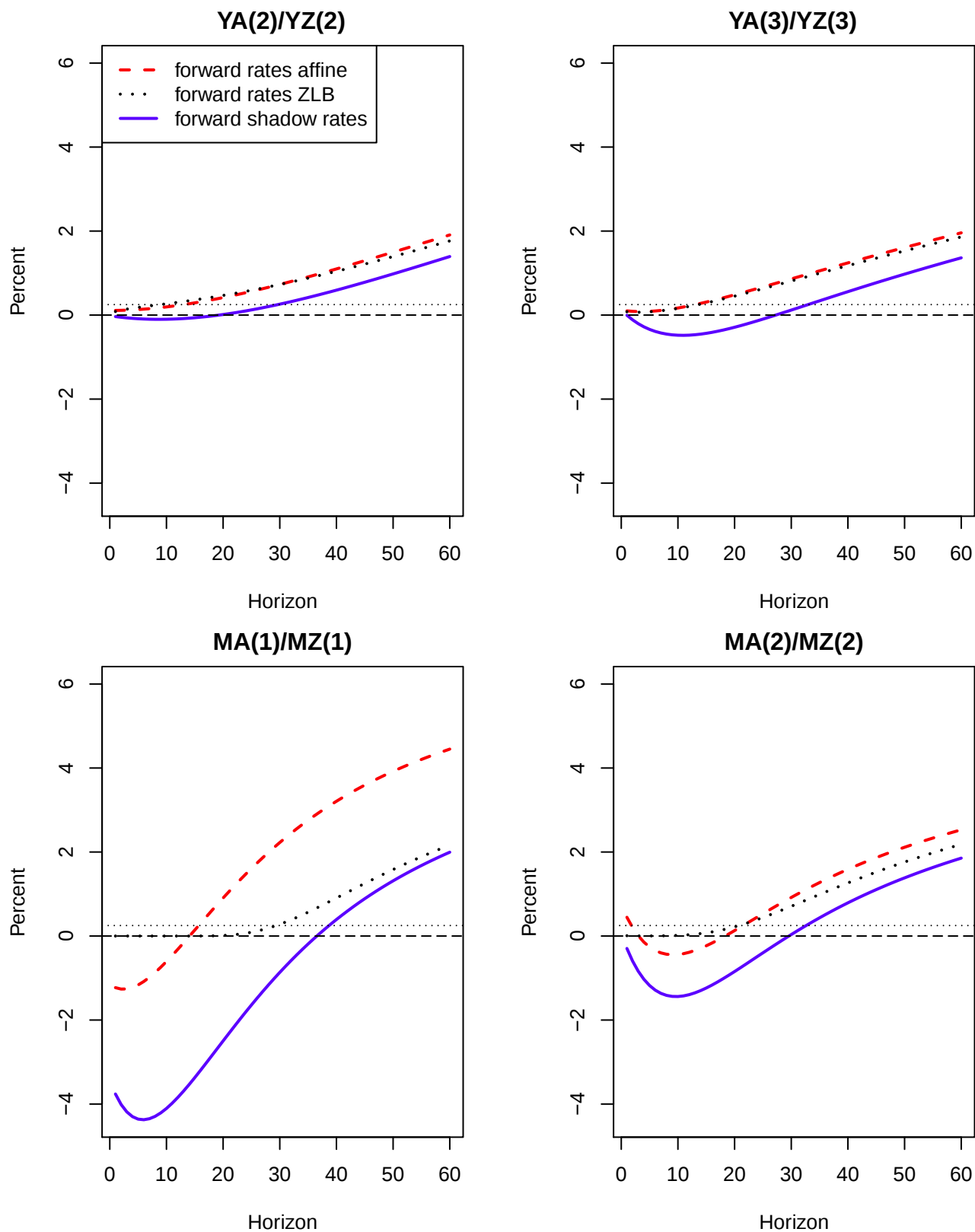
Notes: Actual, fitted, and shadow yield curves for ZLB models on December 31, 2012.

Figure 7: Forward rates and forward shadow rates in June 2011



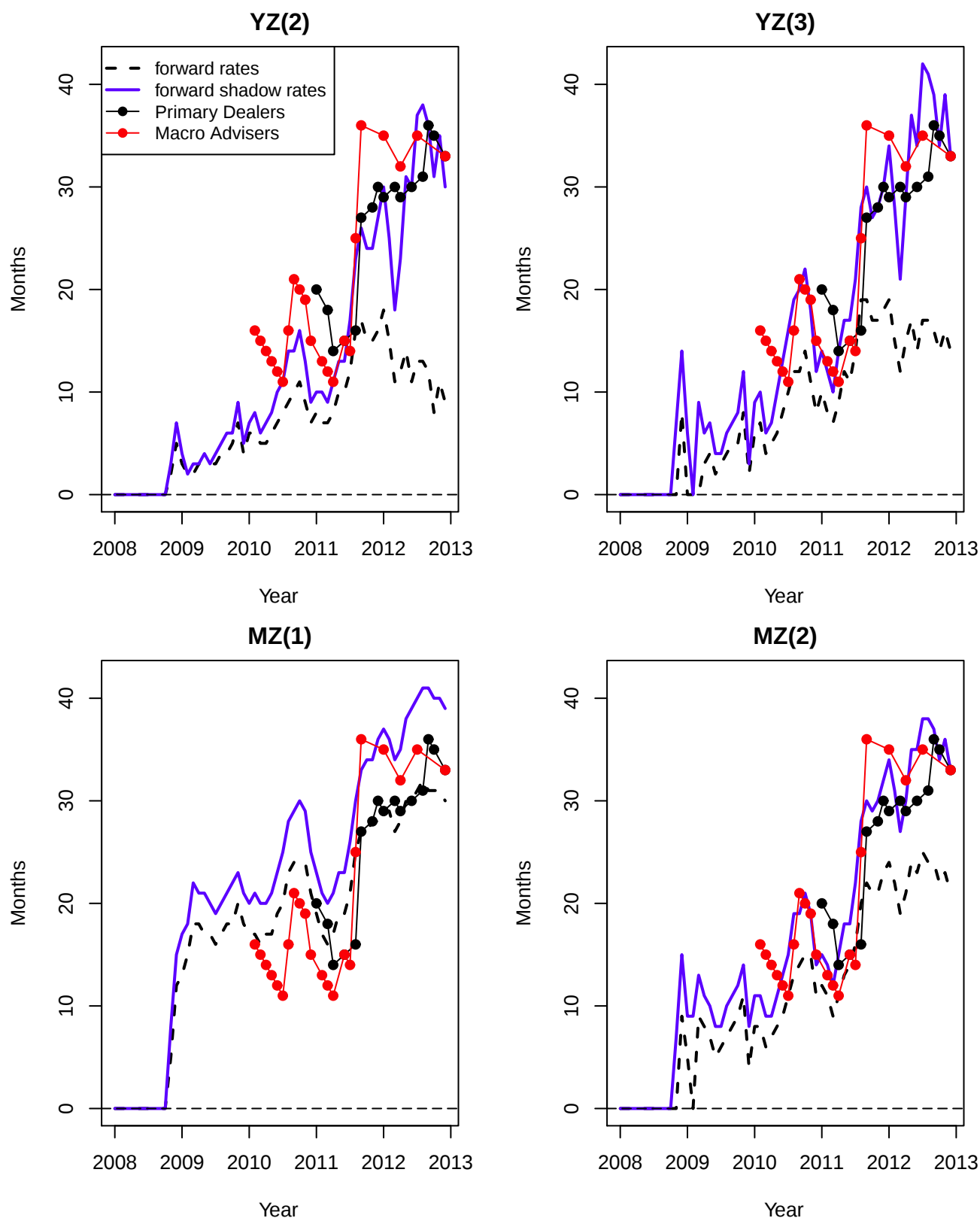
Notes: Forward rate path from affine and ZLB models, and forward (Q-expected) shadow rates across models on June 30, 2011.

Figure 8: Forward rates and forward shadow rates in December 2012



Notes: Forward rate path from affine and ZLB models, and forward (Q-expected) shadow rates across models on June 31, 2012.

Figure 9: Estimated duration of ZLB period



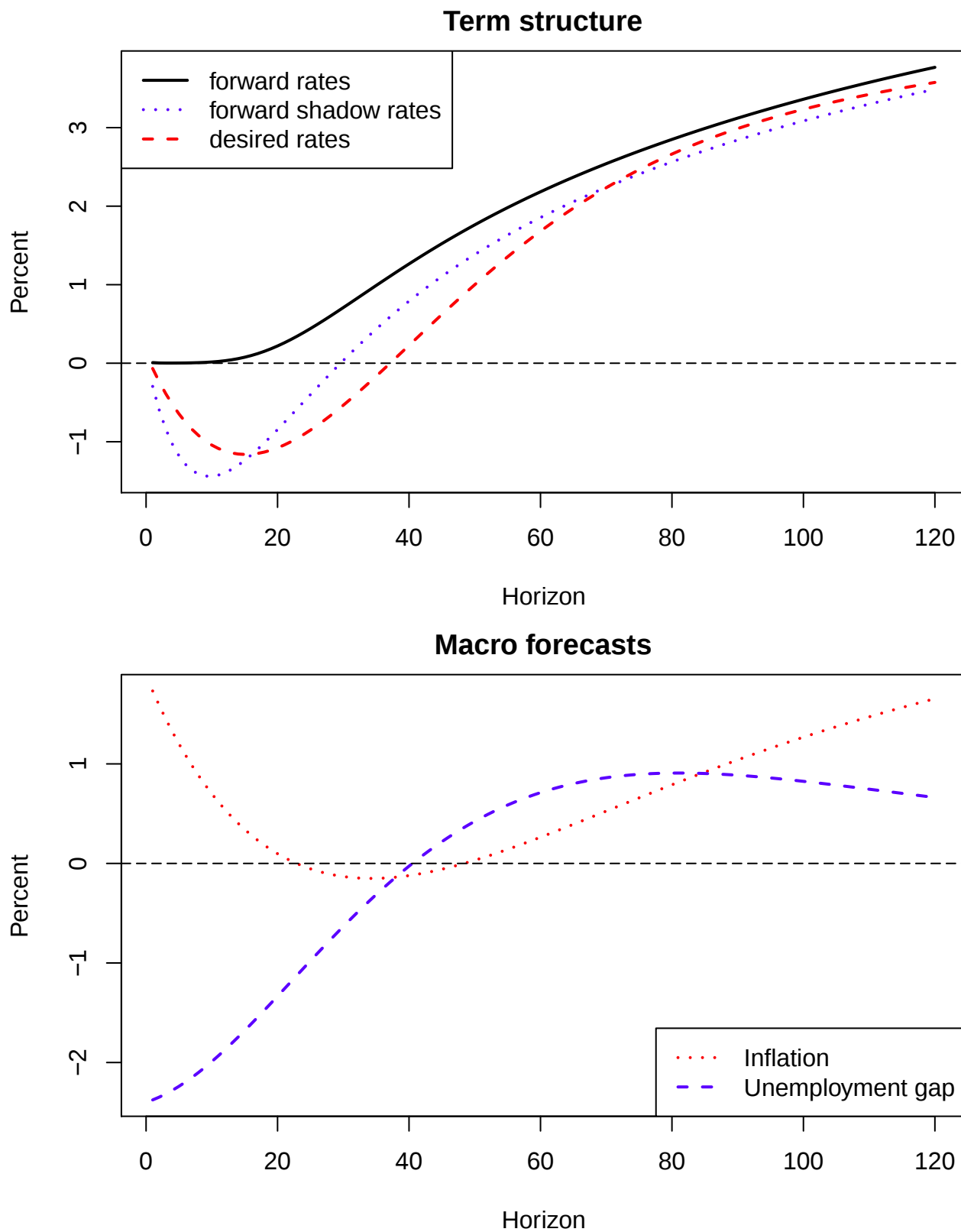
Notes: Estimated time (in months) until policy lift-off from the ZLB period, defined as the first period in which the short rate exceeds 25 basis points, based on forward rates and forward shadow rates. Also shown are survey-based expectations from the Primary Dealer survey and the predicted duration from Macroeconomic Advisers. Period: January 2008 to December 2012.

Figure 10: Desired policy rate vs. shadow rate



Notes: Desired policy rate from Taylor rule estimates, as obtained using OLS and implied by macro-finance DTSM, in comparison to estimated shadow rate. Also shown is the three-month T-Bill rate. Estimation sample: January 1985 to December 2008. Post-estimation sample extends until December 2012.

Figure 11: Actual vs. desired term structure in December 2012



Notes: Top panel shows forecasts of desired policy rate, forward curve, and forward shadow curve from model $MZ(2)$. Bottom panel shows forecasts of inflation and the unemployment. Date: December 31, 2012