On the Composition Effects on the Labor Market Outcomes *

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February 14, 2013

Abstract

It is well documented that hazard rates and entry wages decline with unemployment duration and are procyclical. These outcomes may be explained on the basis of workers' invariant characteristics and composition variation with unemployment duration and over the business cycle. To explore it theoretically and quantitatively, we build a labor market model in which firms lack information on the qualifications of the candidates and react to the asymmetry of information by designing self-selection schemes. As pointed out by the personnel economics literature, performance-contingent compensation plans are central in such sorting schemes. The other key ingredient of our self-selection mechanism is capital-skill complementarity, which allows firms to target one type of applicants by playing with the capital margin. As a result, workers of different types direct their search towards distinct submarkets and high-skilled workers experience higher job-finding rates and entry wages. We calibrate the model to replicate some salient facts of the US economy. In line with the empirical literature, the sorting mechanism explains almost one third of the 48 percent decline of the monthly hazard rate after three months of unemployment relative to the first week. Regarding the procyclical pattern,...

Keywords: Unemployment Duration, Business Cycle, Sorting, Asymmetric Information, Directed Search

JEL Codes: J01, J21, J42, J64

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*Very preliminary. Please do not cite or circulate without permission. We benefited from the feedback of Manuel Arellano, Edgar Preugschat... Fernández-Blanco gratefully acknowledge financial support from the Spanish Ministry of Science and Technology under Grant Nos. SEJ2007-63098 and 2011/0031/001.

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1 Introduction

It is well documented that, after controlling for observable characteristics, hazard rates from unemployment and entry wages decline with unemployment duration and are procyclical. These patterns may result from state-contingent mechanisms that deteriorate the job market prospects over duration and during a recession for all job-seekers identically. Alternatively, these outcomes may be explained on the basis of workers' invariant characteristics and composition variation with unemployment duration and over the business cycle. We explore theoretically and quantitatively the latter explanation by building a labor market model in which firms lack information on the qualifications of the candidates and, following Salop and Salop (1976), react to the asymmetry of information by designing self-selection schemes.[1] As pointed out by the personnel economics, e.g. Lazear and Shaw (2007), performance-contingent compensation plans are central in such sorting schemes.[2] The other key ingredient of our self-selection mechanism is capital-skill complementarity, which allows firms to target one type of applicants by playing with the capital margin. As a result, workers of different types direct their search towards distinct submarkets and high-skilled workers experience higher job-finding rates and entry wages. We calibrate the model to replicate some salient facts of the US economy. In line with the empirical literature,[3] the sorting mechanism explains almost one third of the 48 percent decline of the monthly hazard rate after three months of unemployment relative to the first week. Regarding the procyclicality pattern, [...]

To be more precise, we build a directed search model with asymmetric information on workers' skills. We think of such skills as abilities that recruiters cannot grasp from a CV or an interviewing process, but can be assessed on the job. Firms commit to a capital-renting plan. Output is the result of the combination of capital and labor skills, whereas the labor compensation is bargained over. The decentralized assignment of workers to firms is driven by the capital-skill complementarity. That is, a production technology with a positive cross partial derivative leads firms and high-ability

[1] As they say, firms could alternatively or complementarily design screening devices.

[2] Personnel economics has highlighted that performance pay also aims at selecting workers who can respond to the incentive schemes by exerting more effort. We abstract away from the effort-related agency problem asymmetric information poses, which has been investigated in a frictional labor market by Moen and Rosén (2011).

[3] The empirical literature has emphasized, since Lancaster (1979) and Heckman and Singer (1984), that ex-ante (unobversable) differences account for a large part of the decline in hazard rates over unemployment duration. For example, van den Berg and van Ours (1994, 1996, 1999) conclude that the decline of the hazard rates is mostly due to unobserved heterogeneity.

workers to anticipate that their match would produce a larger surplus and, hence, higher profits and wages. As a result, a relatively larger entry of firms takes place in equilibrium in submarkets targeting high-ability workers leading to higher job-finding rates. Job-seekers who differ in their market ability find it optimal to direct their search to distinct submarkets in equilibrium. This is the case because although low-ability workers would benefit from a higher job-finding rate if applying to submarkets targeting high-skilled workers, the generated surplus would be much lower offsetting the matching gains. The equilibrium is thus block-recursive, which ensures the tractability of the model when adding aggregate shocks to the economy.

To quantitatively analyze the composition effect over the business cycle, we hit the economy with two types of shocks: productivity and separation shocks...

We contribute to the literature dealing with asymmetric information in the labor market by exploring the dynamic consequences of performance-based pay schemes as well as exploring one margin (the capital plan) firms may use to upgrade the productivity of their applicants. In a static setting, Michelacci and Suarez (2006) allow firms to either design performance-contingent compensation schemes or commit to a non contingent wage. They find that for a parameter subset both types of wage schemes coexist in equilibrium with high-skilled workers leading their search to performance-pay jobs. One other approach is in Gale (1992) and Guerrieri, Shimer, and Wright (2010) in which workers self-select according to the disutility of work. The former finds that good workers work more hours and are better paid in the absence of frictions. The latter show that this result does not directly extent to a frictional labor market in the sense that good workers do have a higher employability rate, but wages and working time comparisons result from additional assumptions. More specifically, if good workers are not only more productive at any working time, but also increasingly more as hours rise, then they enjoy both higher wages and working time. However, a sorting mechanism cannot be based on working hours since it would imply that working time would fall with unemployment duration, a result which is at odds with CPS data.[4] Furthermore, it is worth noting that their set of assumptions and, in particular, their key sorting assumption are not satisfied in our model.

Other articles have also tackled a sorting mechanism. To the best of our knowledge, Lockwood's

[4] Maybe because of institutional reasons, working time is very clustered at 40 hours a week, with almost 47% of the sample -after removing the 10% of individuals who report 'hours vary'-, and at 20 and 30 hours with less than 10% each.

(1991) random search model is the first in investigating a pure composition effect in hazard rates, although entry wages stay constant in duration. Gonzalez and Shi (2009) and Fernández-Blanco and Preugschat (2012) base their mechanism on symmetric incomplete information on workers' skills and directed search, then state-contingent and composition mechanisms coexist in equilibrium.

The paper is also related to the literature on labor market segmentation. For example, Moen (2003) analyzes a labor market with individual- and match-specific productivity in which firms do not observe the latter whereas firms do observe the overall productivity. He shows that if firms cannot commit to productivity-contingent wages, there are too few skilled jobs posted in equilibrium and, although the wage premium for skilled workers is larger than optimal, their welfare reduces. In our setting, high-ability workers are also worse off in case their low-ability counterparts are willing to enter the submarket identified by the perfect information capital. In such a case, firms commit to larger capital levels to get rid of low-ability applicants leading down the equilibrium wage and job-finding probability.

The rest of the paper is as follows. Section 2 describes the stochastic economy. The equilibrium is characterized in Section 3. Section 4 analyzes the steady state calibrated economy and the implications of the model regarding the average hazard rate and entry wage distributions over unemployment duration. The stochastic results are determined in Section 5. Section 6 concludes and the proofs are postponed to the Appendix.

2 Set Up

In this section, we describe an economy with ex-ante heterogeneity on the supply side of the labor market. There is asymmetric information on the market productivity of workers and idiosyncratic and aggregate shocks.

2.1 Description of the Labor Market

The economy is populated by a measure one of workers and a large continuum of firms determined in equilibrium by free entry. All agents are risk neutral and discount future payoffs at common rate r. Workers differ ex-ante from one another in their market ability. They can be either low-skilled or high-skilled, $i \in \{\ell, h\}$. Their market ability is denoted by y, with $y_\ell < y_h$. There is a mass

$\mu_h = \mu$ of high-ability workers.[5] The worker's type is only observable to the worker herself. Time is continuous. The time indices are suppressed for notational simplicity.

Production and Separation. Unemployed workers produce b units of output at home. The market output produced by a job-worker pair at a given instant depends on three objects: the worker's ability y, the capital level of the firm $k \in \mathcal{R}_+$, and the aggregative productivity of the economy z. The productivity component which is common across all matched pairs belongs to the set $\mathcal{Z} \equiv \{z_1, ..., z_n\}$, with $z_1 < ... < z_n$. Type-i employed workers produce $f(z, y_i, k)$ in the market. The production technology $f(\cdot, \cdot, \cdot)$ is increasing and concave in all its arguments with a positive cross partial derivative of capital and skill. Although other functional forms are also valid, to fix ideas we may think of a multiplicative function $f(z, y, k) = zyk^a$, with $a \in (0, 1)$.

Job-worker pairs see their match destroyed at Poisson rate λ, where $\lambda \in \Lambda \equiv \{\lambda_1, .., \lambda_m\}$. Let $\psi \equiv (u_\ell, u_h, z, \lambda)$ define the aggregate state of the economy, where u_i stands for the unemployment rate of type-i workers. The stochastic process of ψ is Markovian, $\pi(\psi'|\psi)$. The law of motion of the unemployment rates will be determined later. Let $\Psi \equiv [0, 1]^2 \times \mathcal{Z} \times \Lambda$ denote the set of possible values of the aggregate state of the economy.

Capital Plans and Commitment. Search and Matching. Firms and workers direct their search to a given submarket. Search is directed in the sense that agents anticipate that a larger entry of workers and hence a higher job-filling rate takes place in submarkets promising higher employment prospects. Commitment is short-term in the sense that firms commit to a capital level k given the current state of the economy ψ instead to a state-contingent capital plan.[6] That is, a submarket is defined by one capital level. Upon the arrival of an aggregate shock, the firm-worker pair renegotiate and agree to choose the capital level that maximizes the returns. Then, workers locate themselves in submarkets that maximize their expected discounted utility. Firms incur the capital rental price r plus the depreciation cost at rate δ at any instant while producing. The returns from capital investment are shared by the firm and the worker as wages are bargained over. Let α denote the worker's bargaining power. However, because of the capital renting and the depreciation costs, larger capital levels do not monotonically translate into higher wages. Instead, the flow returns $p(z, y, k) \equiv f(z, y, k) - (r + \delta)k$ show an inverted U shape as a function of k. Let

[5] Hence, the mass of type-ℓ workers is $\mu_\ell = 1 - \mu$. Their market productivity is y_ℓ.

[6] See the comments in this regard later in Section ???.

$\overline{k}_i(\psi)$ denote the capital level that maximize the flow returns of type-i workers, respectively. That is, $p_k(z, y_i, \overline{k}_i(\psi)) = 0$. To guarantee existence of equilibrium at all possible states, we assume that $\lim_{k\to 0} f_k(z, y, k) = \infty$, $\lim_{k\to\infty} f_k(z, y, k) = 0$, and $b < \max_k p(z_1, y_\ell, k)$.

Let $\Theta(k, \psi)$ denote the market tightness defined as the ratio of vacancies to workers in submarket k when the state variable is ψ. This ratio must be consistent with the agents' optimizing behavior in equilibrium, which will be further explained below. A firm fills a vacancy in submarket k at Poisson rate $\eta(\Theta(k, \psi))$. Workers of either type find jobs at equal rate $\nu(\Theta(k, \psi))$ because of the asymmetric information assumption about the applicant's skills. Functions η and ν are decreasing and increasing in their capital argument, respectively. Again, to guarantee existence of equilibrium, we assume that $\lim_{\theta\to 0} \eta(\theta) = \lim_{\theta\to\infty} \nu(\theta) = \infty$ and $\lim_{\theta\to\infty} \eta(\theta) = \lim_{\theta\to 0} \nu(\theta) = 0$. Because the mass of newly employed workers must coincide with the mass of newly filled jobs, it can be shown that the job-finding rate becomes $\nu(\Theta(k, \psi)) = \Theta(k, \psi)\eta(\Theta(k, \psi))$.

Timing. The timing of the events is as follows. At the beginning of every instant, firms hold one single job and can be either vacant or actively producing, whereas workers are either employed or unemployed. There is potentially a continuum of submarkets indexed by a capital level k. Without loss of generality, we assume that a submarket is identified by an state-contingent capital level k. A submarket where there are firms and workers seeking a trading partner is said to be active. Vacant firms enter a submarket where to locate their vacancy at cost c. Then, unemployed workers choose a submarket to search for a job. Matching and production takes place. Wages are continuously bargained over upon matching. Job-worker pairs see their match destroyed at Poisson rate λ. When this occurs, the firm vanishes and the worker becomes newly unemployed. At the end of the period, a shock hits the economy at Poisson rate φ and the new aggregate productivity or separation rate is drawn by Nature. New capital levels are also renegotiated at ongoing productive pairs.

Value Functions. An unemployed worker of type i derives utility from home production b. She chooses the submarket that maximizes her utility and becomes employed at rate $\nu(\Theta(k, \psi))$ obtaining value $\alpha S_i(k, \psi)$. Her continuation value switches to the expected value $\mathbb{E}_\psi(U_i(\psi'))$ if an aggregate shock arrives, an event that occurs at rate φ. The expectations are always conditional on the current aggregate productivity.

$$rU_i(\psi) = b + \max_k \nu(\Theta(k, \psi))\alpha S_i(k, \psi) + \varphi(\mathbb{E}_\psi(U_i(\psi')) - U_i(\psi)) \tag{1}$$

The (net) surplus, or value of a job-worker pair net of their outside options, $S_i(k, \psi)$, consists of the output net of capital-renting and -depreciation costs as well as unemployment value plus the continuation value of the match. It satisfies the following functional equation[7]

$$(r + \lambda + \varphi)S_i(k, \psi) = p(z, y_i, k(\psi)) - (r + \varphi)U_i(\psi) + \varphi\mathbb{E}_\psi \left(U_i(\psi') + S_i(\overline{k}_i(\psi'), \psi')\right) \tag{2}$$

Firms choose a submarket to post their vacancies to maximize expected profits. A vacant firm incurs one time cost c when posting the vacancy in submarket k. The job can be filled by either type-h or -ℓ workers. Let $\rho_i(k, \psi)$ denote the proportion of workers of type-i in submarket k given state ψ. By definition, it must be the case that $\rho_h(k, \psi) + \rho_\ell(k, \psi) \leq 1$ and $\rho_i(k, \psi) \geq 0$. That is, the vector $\rho(k, \psi)$ belongs to the simplex Δ^1 unless no worker enters the submarket. The returns of a filled vacancy are $(1 - \alpha)S_i(k, \psi)$. Thus, the value of the vacant firm is defined by

$$V(\psi) = -c + \eta(\Theta(k, \psi)) \sum_i \rho_i(k, \psi)(1 - \alpha)S_i(k, \psi) \tag{3}$$

Furthermore, expected profits must be zero in any active submarket in equilibrium because of free entry, $V = 0$. Therefore, a capital plan k that maximizes the surplus of their relationship is mutually beneficial for workers and firms. Nevertheless, firms may face workers of different types in a given submarket x and then will have to average the given surpluses with their respective matching probabilities.

Given the state ψ, the law of motion of the unemployment rates of low- and high-ability workers is

$$\dot{u}_i = \lambda(1 - u_i) - \nu(\Theta(k_i(\psi), \psi))u_i, \text{ for } i \in \{\ell, h\} \tag{4}$$

3 Equilibrium

We turn to define the equilibrium allocation. Although a natural extension of the equilibrium concept defined in the competitive search literature, we benefited from Gale 1992 and Guerrieri et al 2010.

Definition 1 *A stochastic equilibrium consists of a state-contingent distribution G_ψ of active markets with support $\mathcal{X}_\psi \subset X$ and $\psi \in \Psi$, a pair of value functions $U_i : \Psi \to \mathcal{R}_+$, a pair of surplus*

[7]The derivation of the surplus expression is left to the Appendix Subsection 7.1. As an abuse of language, we often use the term surplus to refer to the net surplus.

values $S_i : X \times \Psi \to \mathcal{R}_+$, a tightness function $\Theta : X \times \Psi \to \mathcal{R}_+$, a function $\rho : X \times \Psi \to \Delta^1$ such that

1. *Firms' profit-maximizing program and free entry:*

   $\forall k \in X$ and $\psi \in \Psi$, $\eta(\Theta(k,\psi)) \sum_i \rho_i(k,\psi)(1-\alpha)S_i(k,\psi) \leq c$, with equality if $k \in \mathcal{X}_\psi$.

2. *Workers' optimal search: $\forall \psi \in \Psi$,*

   the type-i worker's value satisfies

$$rU_i(\psi) = b + \max_{k \in \mathcal{X}_\psi} \nu(\Theta(k,\psi))\alpha S_i(k,\psi) + \varphi(\mathbb{E}_\psi(U_i(\psi')) - U_i(\psi))$$

   Furthermore, $\forall k \in X_\psi$, $rU_i(\psi) \geq b + \nu(\Theta(k,\psi))\alpha S_i(k,\psi) + \varphi(\mathbb{E}_\psi(U_i(\psi')) - U_i(\psi))$, with equality if $\Theta(k,\psi) < \infty$, and $\rho_i(k,\psi) > 0$

   If $S_i(k,\psi) < 0$, then either $\Theta(k,\psi) = \infty$ or $\rho_i(k,\psi) = 0$.

3. *The surplus values S_i satisfy the functional equation (2).*

4. *Market-clearing condition: $\forall \psi \in \Psi$,*

$$\int_{\mathcal{X}_\psi} \frac{\rho_i(k,\psi)}{\Theta(k,\psi)} dG_\psi(k) = u_i(\psi)\mu_i, \forall i$$

The interpretation of the equilibrium definition regarding active submarkets is standard. The first equilibrium condition guarantees that, first, vacant firms maximize profits and, second, the null expected profits of a vacant job because of free entry. Whereas the second condition establishes the optimal search behavior of workers given the distribution of active markets. The third condition determines equilibrium wages and, finally, the sum of job-seekers across active submarkets must be equal to the mass of unemployed workers for each type.

Let us turn to describe the off-the-equilibrium beliefs that support the equilibrium allocation by considering a trembling hand kind of argument. Think of an arbitrarily small mass of firms that consider to deviate and locate their vacancy in a given submarket k'. Those firms form rational expectations about the search behavior of all workers in the continuation *subgame* determining the triple $(\Theta(k',\psi), \rho_\ell(k',\psi), \rho_h(k',\psi))$. The second equilibrium condition establishes that the expectations on $\Theta(k',\psi)$ are pinned down by the workers who are willing to accept the lower tightness, determining in turn the ρ proportions, provided that the employment value promised by k' is large enough. In other words, if a mass of firms deviated to k', then there would be a flow of workers from the equilibrium submarkets until their utility falls down to their market value. This subgame perfection condition explains why the system of rational beliefs is determined by the type who is indifferent between applying to submarket k' and the equilibrium one.

3.1 Characterization of Equilibrium.

The following proposition claims that there cannot exist an equilibrium in which both types of workers apply to the same job offers. The result is very intuitive. If a submarket k were active with $\rho_\ell(k,\psi), \rho_h(k,\psi) > 0$, there would be a profitable deviation of firms committing to a capital level k' arbitrarily smaller or bigger than k for the current aggregate state. By entering submarket k', firms would manage to get rid of the applicants with lower surplus leading to a discrete jumps in profits.

Proposition 3.1 *There is no equilibrium in which workers of different types search in the same active submarket. That is, for all active submarket $k \in \mathcal{X}_\psi$ in equilibrium, it is the case that either $\rho_\ell(k,\psi) = 1$ and $\rho_h(k,\psi) = 0$ or vice versa.*

As a result, conditional on existence, the decisions made by the agents in equilibrium depend on the state of the economy only through the aggregate productivity and the separation rate, but not on the unemployment rate of each type of worker. This feature has been referred to in the literature as block-recursivity and it simplifies much the characterization of the equilibrium as the equilibrium objects are jump variables that vary over a finite set, $\mathcal{Z}$.

The following lemma states that in any submarket where high-ability workers enter the surplus derived from them is larger than the one linked to their low-ability counterparts. It follows that the value of the firm from hiring a type-ℓ worker at these submarkets is lower than filling the vacancy with a type-h worker as $(1-\alpha)S_\ell(k,\psi) < (1-\alpha)S_h(k,\psi)$. As a result, this lemma partially resembles Assumption 1 in Guerrieri, Shimer, and Wright (2010).

Lemma 3.2 *Let k be a submarket such that $\rho_\ell(k,\psi) \geq 0$, $\rho_h(k,\psi) > 0$ and $\Theta(k,\psi) < \infty$. Then, $S_\ell(k,\psi) < S_h(k,\psi)$.*

To characterize the equilibrium we proceed in several steps. We first focus on the necessary conditions the equilibrium allocation must satisfy regarding type-ℓ workers and then with regard to high-ability workers. Second, we establish that such conditions are also sufficient and state the existence of equilibrium. The following result claims that all low-ability workers enter the same submarket, if any, which is the surplus-maximizing capital level, $\overline{k}_\ell(\psi)$.

Proposition 3.3 *Let $\{G_\psi, \mathcal{X}_\psi, \Theta(\cdot,\psi), \{U_i(\psi), S_i(\cdot,\psi), \rho_i(\cdot,\psi)\}_i\}_\psi$ be an equilibrium. Let ψ be the state of the economy. If there exists an active submarket $k(\psi) \in \mathcal{X}_\psi$ such that $\theta_\ell(\psi) \equiv \Theta(k(\psi),\psi) < \infty$ and $\rho_\ell(k(\psi),\psi) > 0$, then $\rho_\ell(k',\psi) = 0$ for all $k' \in \mathcal{X}_\psi \setminus \{k(\psi)\}$. Let $S_\ell(\psi) \equiv S_\ell(k(\psi),\psi)$ denote the surplus derived at submarket $k(\psi)$. The vector $(\theta_\ell(\psi), k(\psi), S_\ell(\psi))_\psi$ is a solution of the following system of equations (5)-(7)*

$$c = \eta(\theta(\psi))(1-\alpha)S_\ell(\psi), \qquad \forall\ \psi \in \Psi \tag{5}$$

$$\tfrac{r+\lambda+\varphi}{\eta(\theta(\psi))} = (p(z,y_\ell,k(\psi)) - b)\,\tfrac{1-\alpha}{c} - \alpha\theta(\psi) + \varphi\mathbb{E}_\psi\left(\tfrac{1}{\eta(\theta(\psi'))}\right), \quad \forall\ \psi \in \Psi \tag{6}$$

$$k(\psi) = \overline{k}_\ell(\psi), \qquad \forall\ \psi \in \Psi \tag{7}$$

This set of necessary conditions (5)-(7) stems from the well-known equivalence to the problem of maximizing the utility of low-ability workers subject to guaranteeing non-negative profits to firms (see e.g. Acemoglu and Shimer (1999)). That is, given the aggregate state of the economy ψ, the equilibrium objects of the low-ability market are determined by the following program:

$$\begin{aligned} \max_{\theta(\psi),S_\ell(\psi)} \quad & \nu(\theta(\psi))S_\ell(\psi) \\ \text{s. to} \quad & \eta(\theta(\psi))(1-\alpha)S_\ell(\psi) \geq c \end{aligned} \tag{8}$$

The equations (5) constitute the zero-profit conditions, whereas the set (7) stands for the first-order condition with respect to capital, $p_k(z,y_\ell,k) = 0$. They deliver the values of the surplus linked to low-ability workers, given the tightness, and the capital level, respectively. The equilibrium conditions (6) in turn come out from replacing the surplus values from the functional equations (2) by using the zero-profit condition (5).

We now turn to high-ability workers. Let $\psi \in \Psi$ be a state of the economy. Given the vector $(U_i(\psi'))_{\psi'}$, we define problem $(P(\psi))$ as

$$\begin{aligned} \max_{\theta,k} \quad & \nu(\theta)\alpha S_h(k,\psi) \\ \text{s. to} \quad & \eta(\theta)(1-\alpha)S_h(k,\psi) \geq c \\ & b + \nu(\theta)\alpha S_\ell(k,\psi) + \varphi\mathbb{E}_\psi\left(U_\ell(\psi)\right) \leq (r+\varphi)U_\ell(\psi) \end{aligned}$$

where the functions $S_i(\cdot,\psi)$ are defined by (2). Notice that the problem for high-ability workers has one extra restriction relative to problem (8). This is the case to guarantee that low-ability workers do not enter the type-h submarket. To help us with the characterization of equilibrium, we define a solution of a bigger problem.

Definition 2 *We say that a vector $(\theta(\psi), k(\psi), U_h(\psi))_\psi$ is a solution of problem (P) if*

1. *Given $(U_\ell(\psi))_\psi$ and $(U_h(\psi))_\psi$, $(\theta(\psi), k(\psi))$ is a solution of problem $(P(\psi))$, $\forall\ \psi \in \Psi$.*
2. *The following equations hold:*

$$b + \nu(\theta(\psi))\alpha S_h(k(\psi), \psi) + \varphi \mathbb{E}_\psi \left(U_h(\psi')\right) = (r+\varphi)U_h(\psi),\ \ \forall\ \psi \in \Psi$$

Now, we state that an equilibrium must be a solution of problem (P).

Proposition 3.4 *Let $\{G_\psi, \mathcal{X}_\psi, \Theta(\cdot,\psi), \{U_i(\psi), S_i(\cdot,\psi), \rho_i(\cdot,\psi)\}_i\}_\psi$ be an equilibrium. Consider a vector $(k(\psi))_\psi$ of active submarkets such that $\theta(\psi) \equiv \Theta(k(\psi),\psi) < \infty$ and $\rho_h(k(\psi),\psi) > 0$, then the equilibrium vector $(\theta(\psi), k(\psi), U_h(\psi))_\psi$ is a solution of problem (P), given $(U_\ell(\psi))_\psi$.*

The following result states necessary conditions for the equilibrium allocation to satisfy.

Proposition 3.5 *There exists a solution $(\theta(\psi), k(\psi), U_h(\psi))_\psi$ of problem (P). Such a solution satisfies the following conditions for all $\psi \in \Psi$:*

$$c = \eta(\theta(\psi))(1-\alpha)S_h(k(\psi),\psi) \tag{9}$$

$$\tfrac{r+\lambda+\varphi}{\eta(\theta(\psi))} = (p(z,y_h,k(\psi)) - b)\tfrac{1-\alpha}{c} - \alpha\theta(\psi) + \varphi\tfrac{1-\alpha}{c}\mathbb{E}_\psi\left(S_h(\overline{k}_h(\psi'),\psi')\right) \tag{10}$$

$$b + \nu(\theta(\psi))\alpha S_h(k(\psi),\psi) + \varphi\mathbb{E}_\psi\left(U_h(\psi')\right) = (r+\varphi)U_h(\psi), \tag{11}$$

where

$$k(\psi) = \overline{k}_h(\psi),\ \ \textit{if}\ b + \nu(\theta(\psi))\alpha S_\ell(k(\psi),\psi) + \varphi\mathbb{E}_\psi\left(U_\ell(\psi')\right) \le (r+\varphi)U_\ell(\psi) \tag{12}$$

Otherwise,

$$k(\psi) > \overline{k}_h(\psi)\ \ \textit{and}\ b + \nu(\theta(\psi))\alpha S_\ell(k(\psi),\psi) + \varphi\mathbb{E}_\psi\left(U_\ell(\psi')\right) = (r+\varphi)U_\ell(\psi). \tag{13}$$

Figure 3.1 shows the two possible equilibrium outcomes resulting from conditions (12) and (13). The left panel shows the case in which low-ability workers are not willing to enter the submarket $\overline{k}_h$. In this case, the indifference curve of each type of workers is tangent to the isoprofit curve of the firms. That is, the first case ends up being the perfect information scenario. Instead, the right subfigure depicts the scenario in which type-ℓ workers are better off at such submarket. As a result, firms are willing to incur an excess of capital level forgoing potential returns from matching type-h workers to prevent low-ability workers from applying. As Guerrieri, Shimer, and Wright (2010) put it, firms commit to the less expensive contract that guarantees a selected pool of applicants. The forgone gains represented by the capital difference $k_h - \overline{k}_h$ and its associated smaller entry of firms targeting high-ability workers amount to the social cost of asymmetric information. The forgone gains can also be interpreted as the burden the presence of low-ability workers creates on their high-ability counterparts. Because of the assumption that contracting terms are renegotiated whenever an aggregate shock hits the economy, the forgone gains are not spread over the lifecycle of the job, but limited to the current state of the economy. Now, we show two final results to claim existence of the equilibrium allocation.

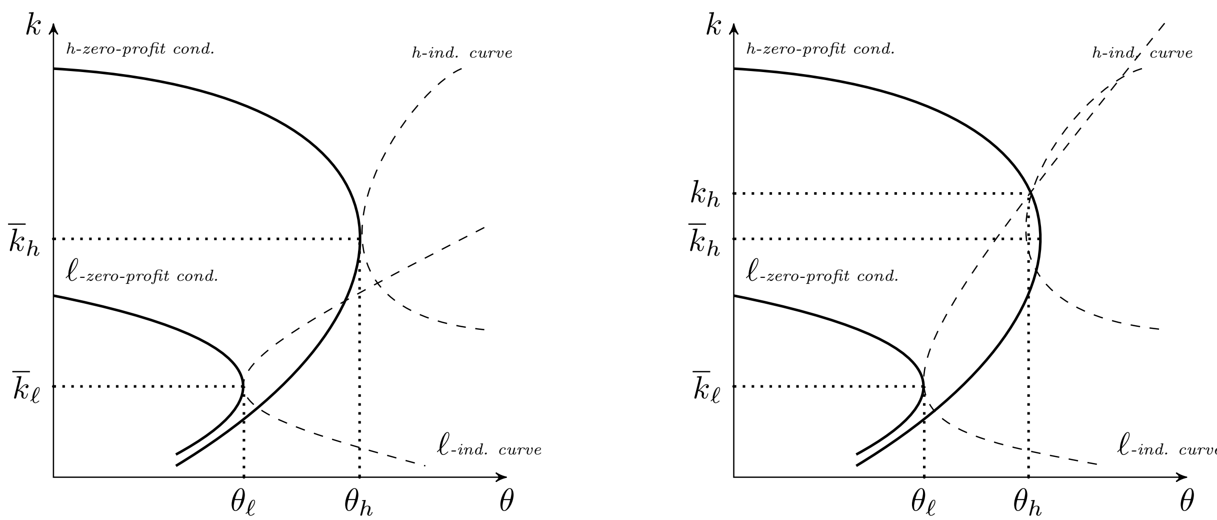


Figure 1: The two possible separating equilibrium allocations.

Proposition 3.6 *Let $(\theta_\ell(\psi), k_\ell(\psi), S_\ell(\psi))_\psi$ be the solution of the system (5)-(7), and $(U_\ell(\psi))_\psi$ be defined as the solution of the functional equation (1) evaluated at the given vector. Given $(U_\ell(\psi))_\psi$, let $(\theta_h(\psi), k_h(\psi), U_h(\psi))_\psi$ be a solution of problem (P), and $(S_h(\psi))_\psi$ denote the associated surplus. Then, there exists an equilibrium such that, given state $\psi \in \Psi$, the set of active submarkets $\mathcal{X}_\psi \equiv \{k_\ell(\psi), k_h(\psi)\}$, the unemployment value is $U_i(\psi)$ and the surplus $S_i(k_i(\psi), \psi) = S_i(\psi)$, the tightness of the active submarkets $\Theta(k_i(\psi), \psi) = \theta_i(\psi)$, and the beliefs $\rho_i(k_i(\psi), \psi) = 1$ for $i \in \{\ell, h\}$.*

Corollary 3.7 *There exists an equilibrium.*

Recall that the left panel of Figure 3.1 depicts the equilibrium allocation in which there is no economic cost from the asymmetric information assumption, whereas the right one stands for the case in which firms and high-skilled workers choose a higher capital level than the first best to screen out low-skilled candidates. As Guerrieri, Shimer, and Wright (2010) show, this second case is constrained inefficient. As they point out the key aspect of the inefficiency result is that the cost of the screening out of low-skilled applicants does not depend on the size of this group in the pool of unemployed. Therefore, for a sufficiently small share of low-skilled workers, a pooling allocation featured by $(\overline{k}_h, \theta_h)$ would Pareto-dominate the separating equilibrium because the social returns of posting vacancies in the former allocation are continuous in the size of the low-skilled population.

3.2 Equilibrium Wages and Comparison across skills.

Upon matching, a firm and a worker set the wage for the period, which will be renegotiated inasmuch as a shock hits the economy. Wages are Nash-bargained. Let α denote the bargaining power of the worker. Workers obtain a proportion α of the surplus of the match in addition to the unemployment value $rU_i(\psi)$.

That is, the equilibrium wages are determined as[8]

$$w_i(k_i(\psi), \psi) = p(z, y_i, k_i(\psi)) - (1-\alpha)\left((r+\lambda+\varphi)S_i(k_i(\psi), \psi) - \varphi\mathbb{E}_\psi\left(S_i(\overline{k}_i(\psi'), \psi')\right)\right) \tag{14}$$

The following proposition states that high-ability workers enjoy larger employment prospects along all dimensions relative to their low-ability counterparts.

Proposition 3.8 *The equilibrium capital levels, tightness, job-finding rates, surpluses, unemployment values and entry wages are larger for high-ability workers.*

Moreover, this result provides one simple intuition to understand why high-ability workers have no incentives in applying to low-type submarkets and, hence, the difference in the constraint set of problem $(P(\psi))$ and its low-ability counterpart. The returns from searching in the type-ℓ submarket, $\nu(\theta_\ell(\psi))\alpha S_h(x_\ell, \psi)$, are strictly lower than the returns of searching in the high-ability submarket, $\nu(\theta(\psi))\alpha S_h(x_\psi, \psi)$.

[8] The equilibrium wages are derived in detail in the Appendix Subsection 7.1.

4 Unemployment Duration Effects in the Steady State Economy

To analyze the effects of unemployment duration on hazard rates from unemployment and entry wages, we look at the steady state equilibrium. Therefore, in this section we impose that there are no aggregate shocks to the economy, $\varphi = 0$. We first state the uniqueness of the steady state equilibrium. Then, we show that both hazard rates and entry wages decline with unemployment duration in equilibrium. Finally, we calibrate the model to the US economy to quantitatively estimate the composition effect on the decline of hazard rates and entry wages from unemployment.

The following proposition states that the steady state equilibrium is unique.

Proposition 4.1 *There exists a unique steady state equilibrium.*

For notational concreteness, we refer in this section to the tuple $(\theta_i, k_i, w_i)_i$ as the equilibrium allocation. We can rewrite the unemployment dynamics (4) as

$$\dot{u_i} = -u_i\nu(\theta_i) + (1-u_i)\lambda, \text{ with } i \in \{\ell, h\} \tag{15}$$

As a result, the steady state unemployment rate of the economy amounts to

$$u = \mu\frac{\lambda}{\lambda+\nu(\theta_h)} + (1-\mu)\frac{\lambda}{\lambda+\nu(\theta_\ell)} \tag{16}$$

4.1 Duration Effects

We now move to the equilibrium properties related to unemployment duration. Let the length of an unemployment spell be denoted by $\tau \in (0, \infty)$ and assume a period is set one month. The average monthly job-finding rate and entry wages at any duration τ are determined by

$$\overline{\nu}_\tau = \frac{\mu(1-u_h)e^{-\tau\nu(\theta_h)}(1-e^{-\nu(\theta_h)}) + (1-\mu)(1-u_\ell)e^{-\tau\nu(\theta_\ell)}(1-e^{-\nu(\theta_\ell)})}{\mu(1-u_h)e^{-\tau\nu(\theta_h)} + (1-\mu)(1-u_\ell)e^{-\tau\nu(\theta_\ell)}} \tag{17}$$

$$\overline{w}_\tau = \frac{\mu(1-u_h)e^{-\tau\nu(\theta_h)}(1-e^{-\nu(\theta_h)})w_h + (1-\mu)(1-u_\ell)e^{-\tau\nu(\theta_\ell)}(1-e^{-\nu(\theta_\ell)})w_\ell}{\mu(1-u_h)e^{-\tau\nu(\theta_h)}(1-e^{-\nu(\theta_h)}) + (1-\mu)(1-u_\ell)e^{-\tau\nu(\theta_\ell)}(1-e^{-\nu(\theta_\ell)})} \tag{18}$$

The following proposition shows that the average hazard rate declines as duration increases in spite of constant individual rates. This is the case regardless of whether skilled workers leave unemployment relatively sooner or not. The key element underlying the composition effect is that one of the two types is systematically more likely to find a job than the other one. Nevertheless, since the wages newly employed skilled workers obtain are higher in equilibrium, it is central for the result of declining average wages over unemployment duration that type h job-finding rates are also higher.

Proposition 4.2 *The average monthly job-finding rates and entry wages decline with unemployment duration.*

4.2 Calibration and Dynamics over Unemployment Duration

In this section, we describe the calibration details and compare the simulated distribution of hazard rates and entry wages by unemployment duration with their actual counterparts.

4.2.1 CPS Data

We derive the actual distributions plotted in Figure 4.2.2 as well as some of the targets from the Current Population Survey (CPS) for the 1994-2011 period. Broadly speaking, we restrict the sample to individuals of the CPS outgoing rotation groups, aged 20 to 60 who were active in the labor market in the previous survey and stay active in the reporting month. We exclude workers on temporary layoff and those with farming-, army- and public sector-related jobs. The reason to exclude temporary layoffs is because their search behavior and prospects differ much from permanently laid-off workers. See e.g. Pries and Rogerson (2005) for a similar approach. Using SIPP data, Fujita and Moscarini (2012) document that the recall rates for temporarily laid-off unemployed stay above 85% since 1996. Katz and Meyer (1990) show that the search intensity and outcomes largely differ for workers with recall expectations relatively to those with no expectations. Unemployed workers on temporary layoff account for 14.65% of total unemployment in our sample. The targets we use in the calibration will be adjusted properly.

Regarding the job-separation and job-finding probabilities, we run a probit regression to control for observable characteristics such as experience, unemployment rate of the month and a number of dummies for industry, occupation, race, marital status, gender. We do a Hechman correction for sample selection before running a Mincerian regression of log wages. The distributions depicted in Figure 4.2.2 show the predicted hazard rates and entry wages at each duration computed by replacing the unemployment duration variable by the corresponding duration value and the remaining variables by their means. To compare the actual data with the data simulated from our model, we report the values normalized by the predicted value at unemployment duration of one week.

4.2.2 Calibration

A period is set to be a month. We assume the production function to be multiplicative in the worker's ability and capital, $f(z, y, k) = zyk^a$, with $a < 1$. To save on notation, we assume $z = 1$ and normalize the market productivity of the skilled to $y_h = 100$.[9]. Following the literature, the matching technology is Cobb-Douglas, $\eta(\theta) = \Gamma\theta^\gamma$ and $\nu(\theta) = \Gamma\theta^{\gamma-1}$. Table 1 depicts the twelve parameters to be calibrated. The discount factor r is set to match a 4% annual interest rate. For the elasticity of the matching technology, we use the standard value $\gamma = 0.5$, which lies within the range surveyed in Petrongolo and Pissarides (2001).

Backus, Henriksen, and Storesletten (2008) report that the depreciation rate has steadily increased, rising from 3% in 1970 to 4.15% in 2004 for government capital and from 5.25% to 8.50% for business capital, whereas the one for private residential capital has remained constant and equal to 1.5%. We take the intermediate annual value of 6%, which implies a monthly rate $\delta = 0.0051$.

Regarding the separation rate, the average monthly EU transition rate is 0.0086 in our dataset after excluding workers on temporary layoff. This number is much lower than those reported in the literature. For example, Shimer (2005), estimate a monthly separation rate at 0.035 using CPS data for the 1951-2003 period. Our numbers are instead very close to those reported by Fujita and Moscarini (2012) for the same time period conditioning on permanent separations and also by Pries and Rogerson (2005). For consistency reasons, we prefer the average of the probit predicted probabilities after controlling for observables characteristics, which amounts to the Poisson rate $\lambda = 0.009$.

Then, we are left with seven parameters to be jointly estimated: $a, y_\ell, \mu, \Gamma, c, b, \alpha$. We target the capital to annual output ratio and the labor income share. While the latter is mostly assumed to be 2/3 for the US economy, Backus, Henriksen, and Storesletten (2008) also document that the former ratio varies from 2.4 to 2.8 since 1970. We take the intermediate value 2.6 for this target. If the equilibrium submarket for type-h workers were $\overline{k}_h$, then this target would directly give a value of parameter a as $12 \cdot 2.6(r + \delta)$. We then have the following two equations

$$12 \cdot 2.6 = \frac{\mu(1-u_h)k_h + (1-\mu)(1-u_\ell)k_\ell}{\mu(1-u_h)y_h k_h^a + (1-\mu)(1-u_\ell)y_\ell k_\ell^a}$$

$$2/3 = \frac{\mu(1-u_h)w_h + (1-\mu)(1-u_\ell)w_\ell}{\mu(1-u_h)y_h k_h^a + (1-\mu)(1-u_\ell)y_\ell k_\ell^a}$$

[9]Indeed, the relevant object is the ratio y_h/y_ℓ. If both objects are factored by z, then multiplying b and c by $z^{1/(1-a)}$ delivers the same equilibrium queues q_h and q_ℓ and equal relative wages.

Table 1: Calibration

Parameter	Description	Value	Source/Target
Exogenously Set Parameters			
y_h	Productivity of skilled	100	normalization
Individually Calibrated Parameters			
r	Discount factor	0.0033	4% annual interest rate
δ	Depreciation rate	0.0051	Backus, Henriksen, and Storesletten (2008)
γ	Elasticity job-filling rate	0.5	Petrongolo and Pissarides (2001)
λ	Separation rate	0.0090	CPS data
Jointly Calibrated Parameters			
α	Workers' bargaining power	0.5986	tightness = 0.4142[1,2]
y_l	Productivity of unskilled	57.3926	vacancy duration = 1.25
μ	Share of high-skilled	0.2178	st dev of standardized log wages = 0.3285[2]
Γ	Scale job-filling rate	0.6547	average job-finding rate = 0.28[2]
a	Production elasticity	0.2716	capital-output ratio = 12·2.6[2]
c	Vacancy cost	322.8311	13% quarterly average wage[3]
b	Non-employment value	568.4515	70% average wage

Note: [1] Hall and Milgrom (2008); [2] CPS data; [3] Backus, Henriksen, and Storesletten (2008); [4] Abowd and Kramarz (2003) where $u_i \equiv \frac{\lambda}{\lambda+\nu(\theta_i)}$. The simulated capital output ratio and labor income share are 32.2646 and 0.7250, respectively.

We set the parameter b to correspond to 0.7 of the simulated average wage, which lies within the range from the 0.62 set by Ebell and Haefke (2009), 0.72 implied in Hall and Milgrom (2008) and 0.75 assumed in Costain and Reiter (2008). It encompasses both unemployment benefits and the value of home production and leisure. The calibrated b to average wage ratio is 0.6601. Hall and Milgrom (2008) also assume that the ratio of vacancy cost c to the simulated average quarterly wage per hire is 14%. Marginally different, we use the 13% ratio reported by Abowd and Kramarz (2003), who collect data from the 1992 French Wage Structure Survey for a representative sample of establishments of at least 20 employees in the manufacturing, construction and service industries.[10]

These two targets lead to the two following equations.

$$b = 0.7\frac{\mu(1-u_h)w_h + (1-\mu)(1-u_\ell)w_\ell}{\mu(1-u_h) + (1-\mu)(1-u_\ell)}$$

$$c\frac{\mu u_h\theta_h + (1-\mu)u_\ell\theta_\ell}{\mu u_h\nu(\theta_h) + (1-\mu)u_\ell\nu(\theta_\ell)} = 0.39\frac{\mu(1-u_h)w_h + (1-\mu)(1-u_\ell)w_\ell}{\mu(1-u_h) + (1-\mu)(1-u_\ell)}$$

We target the average monthly job-finding probability. In our dataset, the average predicted probit probability of becoming employed amounts to 0.28.[11] Its simulated counterpart is 0.3031. Hall and Milgrom (2008) report an unemployment to vacancy ratio of 2 for the 2000 to 2007 period. Adjusting for temporary layoffs both in the numerator and denominator, the ratio target amounts to 2.4144, whereas the simulated ratio is 2.6596.[12]

To have a sense of the dispersion in the economy and, hence, of parameters μ and y_ℓ, we target the standard deviation of the log wage residuals of a standard Mincerian regression. Such data value and its simulated counterpart amount to 0.3285 and 0.3157, respectively. Finally, we target the average vacancy duration. For the 1985-2001 period, Coles and Petrongolo (2008) estimate that about 40% of newly unemployed workers and of newly posted vacancies registered in UK Jobcenters

[10]Hagedorn and Manovskii 2008 add a cost of idle capital, which multiplies by a factor of four the management time cost. We do not include such potential idle capital costs in our analysis.

[11]This rate may look too low compared to the values reported in the literature. For example, Shimer (2005) estimates it at 0.45. The difference is explained by our exclusion of the job-seekers in temporary layoff. Using CPS data since 1994, Fujita and Moscarini (2012) also find comparable numbers for workers who have been permanently laid-off, and around a 0.5 job-finding probability for those in temporary layoff.

[12]Recall that the actual ratio $u/v = 2$ and $x = 0.17$ is the percentage of temporary layoffs. Assuming that all temporarily laid off workers are rehired and fill the vacancies of their former employers, the ratio of unemployed not in temporary layoff to the mass of vacancies not filled with a former worker equals $u(1-x)/(v-ux) = 2.515$.

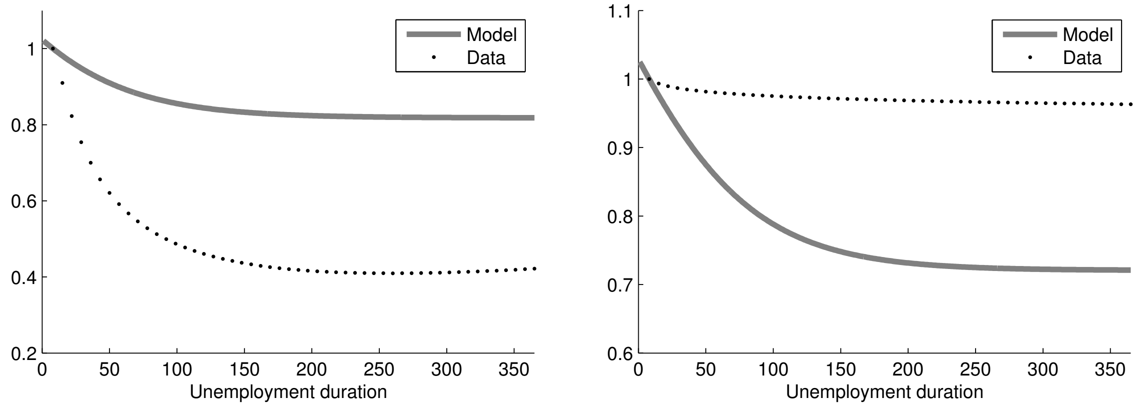
are on the short side of the market, i.e. they become re-employed and filled in days, respectively. Indeed, they estimate that 30% of the new vacancies are filled on the posting day. This large percentage of vacancies filled in a very short period of time drives the average duration down and is likely to be highly correlated with temporary layoffs. They report that the average duration of a vacancy was slightly over 3 weeks, whereas it jumps to 6 weeks for firms on the long side. van Ours and Ridder (1992) in turn report that vacancy duration ranges from 0.9 to 2.1 months in the Netherlands during the 1980s. Barron, Berger, and Black (1997) document that a US vacancy lasts 13.39, 17.21 and 30.35 days according to the Employment Opportunity Pilot Project 1980 (EOPP), the EOPP 1982 and the Upjohn 1993 surveys, respectively. DeVaro (2005) studies how the vacancy duration depends on the recruitment method and, using data from the Multi-City Study of Urban Inequality survey for 1992 to 1995, documents that the average duration is slightly below 4 weeks. As a compromise, our benchmark calibration targets a 5 weeks expected duration of a vacancy and the simulated duration is 5.3 weeks. We perform a robustness check with regard to this target and find no significant difference if duration is set at 4 weeks instead.

Note that our calibration does not explicitly target either the unemployment rate or duration-related variables. The latter has the obvious explanation of being the object of analysis. The unemployment rate of the calibrated economy is 2.36%, whereas it raises to 5.5% if temporary layoffs are not excluded from the sample. The reason for this low unemployment rate is because the separation rate falls from 0.035 to 0.009, whereas the job-finding rate only from 0.45 to 0.28 when discarding temporary laid-off workers

Figures 2(a) and 2(b) plot the weekly data of monthly hazard rates and entry wages as well as their simulated counterparts according to our benchmark calibration. The horizontal axis is at the daily frequency. The monthly job-finding rate declines by 48% after one quarter relative to the first week. The sorting mechanism accounts for 28% according to our calibration. The model also multiplies by a factor of 9 the 2.34% wage fall after the first quarter. This suggests that there is a state dependent component that restrains wages from falling that much, as shown by Fernández-Blanco and Preugschat (2012).

5 Cyclicality of Labor Market Outcomes

[To be added]



(a) Normalized hazard rates

(b) Normalized entry wages

Figure 2: Duration Distributions of Normalized Hazard Rates and Entry Wages

Note: The horizontal axis is measured in days.

6 Conclusions

[To be added]

References

Abowd, J., and F. Kramarz (2003): "The costs of hiring and separations," *Labour Economics*, 10(5), 499–530.

Acemoglu, D., and R. Shimer (1999): "Efficient Unemployment Insurance," *Journal of Political Economy*, 107(5), 893–928.

Backus, D., E. Henriksen, and K. Storesletten (2008): "Taxes and the global allocation of capital," *Journal of Monetary Economics*, 55(1), 48 – 61, ¡ce:title¿Carnegie-Rochester Conference Series on Public Policy: Advances in Dynamic Public Finance April 20-21, 2007¡/ce:title¿.

Barron, J., M. Berger, and D. Black (1997): "Employer search, training, and vacancy duration," *Economic Inquiry*, 35(1), 167–192.

Coles, M., and B. Petrongolo (2008): "A Test Between Stock-flow Matching and the Random Matching Function Approach," *International Economic Review*, 49(4), 1113–1141.

Costain, J., and M. Reiter (2008): "Business cycles, unemployment insurance, and the calibration of matching models," *Journal of Economic Dynamics and Control*, 32(4), 1120–1155.

DeVaro, J. (2005): "Employer recruitment strategies and the labor market outcomes of new hires," *Economic Inquiry*, 43(2), 263–282.

Ebell, M., and C. Haefke (2009): "Product market deregulation and the US employment miracle," *Review of Economic Dynamics*, 12(3), 479–504.

Fernández-Blanco, J., and E. Preugschat (2012): "On the effects of ranking by unemployment duration," *mimeo.*

Fujita, S., and G. Moscarini (2012): "Recall and Unemployment," *Unpublished Manuscript, Yale University.*

Gale, D. (1992): "A Walrasian theory of markets with adverse selection," *The Review of Economic Studies*, 59(2), 229–255.

Gonzalez, F., and S. Shi (2009): "An equilibrium theory of learning, search and wages," *Mimeo U. of Toronto.*

Guerrieri, V., R. Shimer, and R. Wright (2010): "Adverse selection in competitive search equilibrium," *Econometrica*, 78(6), 1823–1862.

Hall, R., and P. Milgrom (2008): "The Limited Influence of Unemployment on the Wage Bargain," *The American Economic Review*, 98(4), 1653–1674.

Heckman, J., and B. Singer (1984): "A Method for Minimizing the Impact of Distributional Assumptions in Econometric Models for Duration Data," *Econometrica*, 52(2), 271320.

Katz, L., and B. Meyer (1990): "Unemployment insurance, recall expectations, and unemployment outcomes," *The Quarterly Journal of Economics*, pp. 973–1002.

Lancaster, T. (1979): "Econometric methods for the duration of unemployment," *Econometrica: Journal of the Econometric Society*, pp. 939–956.

Lazear, E., and K. Shaw (2007): "Personnel economics: The economist's view of human resources," Discussion paper, National Bureau of Economic Research.

LOCKWOOD (1991): "Information Externalities in the Labour Market and the Duration of Unemployment," *The Review of Economic Studies*, 58(4), 733–753.

MICHELACCI, C., AND J. SUAREZ (2006): "Incomplete Wage Posting," *Journal of Political Economy*, 114(6), 1098–1123.

MOEN, E. (2003): "Do good workers hurt bad workersor is it the other way around?*," *International Economic Review*, 44(2), 779–800.

MOEN, E., AND Å. ROSÉN (2011): "Incentives in competitive search equilibrium," *The Review of Economic Studies*, 78(2), 733–761.

PETRONGOLO, B., AND C. PISSARIDES (2001): "Looking into the black box: A survey of the matching function," *Journal of Economic Literature*, 39(2), 390–431.

PRIES, M., AND R. ROGERSON (2005): "Hiring policies, labor market institutions, and labor market flows," *Journal of Political Economy*, 113(4), 811–839.

SALOP, J., AND S. SALOP (1976): "Self-selection and turnover in the labor market," *The Quarterly Journal of Economics*, pp. 619–627.

SHIMER, R. (2005): "The Cyclical Behavior of Equilibrium Unemployment and Vacancies," *American Economic Review*, 95(1), 25–49.

VAN DEN BERG, G., AND J. VAN OURS (1996): "Unemployment dynamics and duration dependence," *Journal of Labor Economics*, 14(1), 100–125.

——— (1999): "Duration dependence and heterogeneity in French youth unemployment durations," *Journal of Population Economics*, 12(2), 273–285.

VAN DEN BERG, G. J., AND J. C. VAN OURS (1994): "Unemployment Dynamics and Duration Dependence in France, the Netherlands and the United Kingdom," *The Economic Journal*, 104(423), pp. 432–443.

VAN OURS, J., AND G. RIDDER (1992): "Vacancies and the Recruitment of New Employees," *Journal of Labor Economics*, 10(2), 138–155.

7 Appendix

7.1 Surplus and Wage determination.

To determine wages as a Nash-bargaining solution, we better add some details to our setting. A type-i employed worker obtains the flow wage $w_i(k,\psi)$ and becomes displaced at exogenous rate λ. The aggregate state changes at rate φ. The employment value is defined as

$$rE_i(k,\psi) = w_i(k,\psi) + \lambda(U_i(\psi) - E_i(k,\psi)) + \varphi(\mathbb{E}_\psi(E_i(\overline{k}_i(\psi'),\psi')) - E_i(k,\psi)) \tag{19}$$

The value of a filled vacancy is determined by

$$rJ_i(k,\psi) = p(z,y_i,k) - w_i(k,\psi) - \lambda J_i(k,\psi) + \varphi(\mathbb{E}_\psi(J_i(\overline{k}_i(\psi'),\psi')) - J_i(k,\psi)) \tag{20}$$

The net surplus is defined as $S_i(k,\psi) \equiv J_i(k,\psi) + E_i(k,\psi) - U_i(\psi)$. Using above expressions for the value functions, we obtain expression (2).

Upon matching, a firm and a worker set the wage for the instant, and it will be renegotiated inasmuch as a shock hits the economy. Wages are Nash-bargained. Let α denote the bargaining power of the worker. That is, the equilibrium wage is the solution of the following problem.

$$\max_w \left(E_i(k,\psi) - U_i(\psi)\right)^\alpha \left(J_i(k,\psi) - V(\psi)\right)^{1-\alpha} \tag{21}$$

The solution is characterized by $J_i(k,\psi) = (1-\alpha)S_i(k,\psi)$. Using this last equation to replace the value J in the above Bellman equation (20), we conclude that equilibrium wages are determined as

$$w_i(k,\psi) = p(z,y_i,k) - (r+\lambda+\varphi)(1-\alpha)S_i(k,\psi) + \varphi(1-\alpha)\mathbb{E}_\psi\left(S_i(\overline{k}_i(\psi'),\psi')\right) \tag{22}$$

That is, workers obtain a proportion α, and firms the remaining $1-\alpha$, of the surplus of the match in addition to the unemployment value $(r+\varphi)U_i(\psi)$.

7.2 Proofs

Proof of Proposition 3.1.

The proof is by contradiction. Suppose there is an equilibrium in which a submarket k is active when aggregate conditions are ψ and $\rho_\ell(k,\psi), \rho_h(k,\psi) > 0$. Since both types of workers enter submarket k, we have

$$(r+\varphi)U_i(\psi) = b + \alpha\nu(\Theta(k,\psi))S_i(k,\psi) + \varphi\mathbb{E}_\psi\left(U_i(\psi')\right),\ \forall\ i \in \{\ell, h\} \tag{23}$$

Firms' profits are defined by expression (3) and average the ex-post profits across workers' types.

Consider now two alternative submarkets k' and k''. Instead of k, they specify capital $k' \equiv k+\epsilon$ and $k'' \equiv k-\epsilon$, respectively, with ϵ arbitrarily small, until the arrival of an aggregate shock. By differentiating expressions (23) with respect to k, we obtain

$$\frac{\partial \theta_i}{\partial k} = \frac{\nu(\Theta(k,\psi))}{\nu'(\Theta(k,\psi))} \frac{-p_k(z,y_i,k)}{S_i(k,\psi)(r+\lambda+\varphi)}, \text{ for } i \in \{\ell, k\} \tag{24}$$

where $\frac{\partial \theta_i}{\partial k}$ measures how much the tightness would change to keep the worker of type i indifferent between submarkets k and an alternative one. The above expression takes into account that

$$(r+\lambda+\varphi)\frac{\partial S_i(k,\psi)}{\partial k} = p_k(z,y_i,k) \tag{25}$$

The capital-skill complementarity assumption implies that $\frac{\partial S_\ell(k,\psi)}{\partial k} < \frac{\partial S_h(k,\psi)}{\partial k}$. The case study follows:

<u>Case 1:</u> Suppose that $k \in [\overline{k}_\ell(\psi), \overline{k}_h(\psi)]$. Then, $-p_k(z,y_h,k) \leq 0 \leq -p_k(z,y_\ell,k)$. Therefore, $\frac{\partial \theta_h}{\partial k} \leq 0 \leq \frac{\partial \theta_\ell}{\partial k}$, with at least one strict inequality. If $S_\ell(k,\psi) < S_h(k,\psi)$, then firms deviating to submarket k' see a discrete jump in profits as low-ability workers do not apply to k'. In contrast, if $S_h(k,\psi) < S_\ell(k,\psi)$, then type-$h$ workers do not enter submarket k'' and firms deviating to this submarket obtain a discrete jump in profits. Therefore, we reach a contradiction in either case. Therefore, submarket k cannot be active in equilibrium.

<u>Case 2:</u> Suppose now that $\overline{k}_h(\psi) < k$. If $S_\ell(k,\psi) < S_h(k,\psi)$, then $0 < \frac{\partial \theta_h}{\partial k} < \frac{\partial \theta_\ell}{\partial k}$. Therefore, firms deviating to submarket k' would obtain a discrete jump in profits. Instead, if $S_h(k,\psi) < S_\ell(k,\psi)$, expression (24) does not provide conclusions in the comparison of the partial derivative of the tightness across types, despite being positive. However, notice that if $\frac{\partial \theta_h}{\partial k} \leq \frac{\partial \theta_\ell}{\partial k}$, then a deviation to submarket k'' implies a discrete jump in profits by getting rid of high-ability workers. Otherwise, so does a deviation to submarket k' which only receives applications from type-ℓ workers. Once again, we reach a contradiction in either case and, hence, submarket k cannot be active in equilibrium.

It remains to analyze the subcase $S_\ell(k,\psi) = S_h(k,\psi)$ in the two cases above. In this case, firms' profits are

$$N(k,\psi) = \eta(\Theta(k,\psi))S_\ell(k,\psi),$$

and by differentiating with respect to k and using the above expressions for the partial derivatives of the tightness and the surplus, we obtain

$$\frac{\partial N(k,\psi)}{\partial k} = \eta'(\Theta(k,\psi))\frac{\partial \Theta(k,\psi))}{\partial k}S_y(k,\psi) + \eta(\Theta(k,\psi))S_y(k,\psi) = \frac{p_k(z,y,k)}{r+\lambda+\varphi}\frac{\nu(\Theta(k,\psi))}{\Theta(k,\psi)^2\nu'(\Theta(k,\psi))},$$

where S_y is the surplus linked to the type of workers that enter the submarket in question.

If the surpluses are equal to each other, we have $\frac{\partial \theta_h}{\partial k} < \frac{\partial \theta_\ell}{\partial k}$. Only do type-$\ell$ workers apply to k'' and $\frac{\partial N(k,\psi)}{\partial k} < 0$ as $p_k(z,y_\ell,k) < 0$. If $k = \overline{k}_\ell(\psi)$, then a deviation of firms to submarket k' obtains larger profits as $p_k(z,y_h,k) > 0$. Therefore, we reach a contradiction in either case and, hence, submarket k cannot be active in equilibrium.‖

Proof of Lemma 3.2. By the definition of the surplus value equation (2), we have that function $S_h - S_\ell$ satisfies the following equation at submarket k.

$$\begin{aligned}(r+\lambda+\varphi+\alpha\nu(\Theta(k,\psi)))\left(S_h - S_\ell\right)(k,\psi) \geq \quad & f(z,y_h,k) - f(z,y_\ell,k) + \\ & +\varphi\mathbb{E}_\psi\left(S_h(\overline{k}_h(\psi'),\psi') - S_\ell(\overline{k}_\ell(\psi'),\psi')\right)\end{aligned} \tag{26}$$

The first term of the right side of the inequality is positive because of the capital-skill complementarity. We now show that the second term is also positive. Let $k_i^*(\psi)$ denote the equilibrium submarket for type-i workers at the state ψ. Similarly, we can write the surplus gap of ongoing productive pairs as

$$\begin{aligned}S_h(\overline{k}_h(\psi),\psi) - S_\ell(\overline{k}_\ell(\psi),\psi) = \quad & \tfrac{1}{r+\lambda+\varphi}[p(z,y_h,\overline{k}_h(\psi)) - p(z,y_\ell,\overline{k}_\ell(\psi)) - \\ & -\alpha\left(\nu(\Theta(k_h^*(\psi),\psi))S_h(k_h^*(\psi),\psi) - \nu(\Theta(k_\ell^*(\psi),\psi))S_\ell(k_\ell^*(\psi),\psi)\right) + \\ & +\varphi\mathbb{E}_\psi\left(S_h(\overline{k}_h(\psi'),\psi') - S_\ell(\overline{k}_\ell(\psi'),\psi')\right)] \geq \\ \geq \quad & \tfrac{1}{r+\lambda+\varphi}[p(z,y_h,\overline{k}_h(\psi)) - p(z,y_\ell,\overline{k}_\ell(\psi)) - \\ & -\alpha\nabla(\psi)\left(S_h(k_h^*(\psi),\psi) - S_\ell(k_\ell^*(\psi)),\psi)\right) + \\ & +\varphi\mathbb{E}_\psi\left(S_h(\overline{k}_h(\psi'),\psi') - S_\ell(\overline{k}_\ell(\psi'),\psi')\right)],\end{aligned} \tag{27}$$

where the inequality comes from defining $\nabla(\psi)$ as $\min\{\nu(\Theta(k_h^*(\psi),\psi)),\nu(\Theta(k_\ell^*(\psi),\psi))\}$ if the first argument of the minimum operator is smaller or equal than the second one, and as the positive fraction $\frac{\nu(\Theta(k_h^*(\psi),\psi))S_h(k_h^*(\psi),\psi)-\nu(\Theta(k_\ell^*(\psi),\psi))S_\ell(k_\ell^*(\psi),\psi)}{S_h(k_h^*(\psi),\psi)-S_\ell(k_\ell^*(\psi),\psi)}$ otherwise. Notice that in the second case the denominator is positive because of the equilibrium zero-profit condition, which implies that $\eta(\Theta(k_h^*(\psi),\psi))S_h(k_h^*(\psi),\psi) = \eta(\Theta(k_\ell^*(\psi),\psi))S_\ell(k_\ell^*(\psi),\psi)$.

We now make a guess-and-verify type of argument. We guess the following inequality holds.

$$S_h(\overline{k}_h(\psi),\psi) - S_\ell(\overline{k}_\ell(\psi),\psi) \geq S_h(k_h^*(\psi),\psi) - S_\ell(k_\ell^*(\psi),\psi) \tag{28}$$

This is the case if and only if $p(z, y_h, \overline{k}_h(\psi)) - p(z, y_h, k_h^*(\psi)) \geq p(z, y_\ell, \overline{k}_\ell(\psi)) - p(z, y_\ell, k_\ell^*(\psi))$. Proposition 3.3 states that the right side of this last inequality is null and the capital-skill complementarity ensures that the left side is positive. Therefore, the guess is verified. Then, we can claim the surplus gap of inequality (27) satisfies

$$\begin{aligned} S_h(\overline{k}_h(\psi), \psi) - S_\ell(\overline{k}_\ell(\psi), \psi) \geq & \quad \tfrac{1}{r+\lambda+\varphi+\alpha\nabla(\psi)} [p(z, y_h, \overline{k}_h(\psi)) - p(z, y_\ell, \overline{k}_\ell(\psi)) + \\ & + \varphi \mathbb{E}_\psi \left(S_h(\overline{k}_h(\psi'), \psi') - S_\ell(\overline{k}_\ell(\psi'), \psi') \right)] \geq \\ \geq & \quad \textstyle\sum_{\psi'} \left(p(z', y_h, \overline{k}_h(\psi')) - p(z', y_\ell, \overline{k}_\ell(\psi')) \right) \omega(\psi' | \psi_1, ..., \psi_n), \end{aligned}$$

where the weights $\omega(\psi'|\psi_1, ..., \psi_n)$ are positive and so are the terms $p(z', y_h, \overline{k}_h(\psi')) - p(z', y_\ell, \overline{k}_\ell(\psi'))$ because of capital-skill complementarity. Thus, the surplus gap $S_h(\overline{k}_h(\psi), \psi) - S_\ell(\overline{k}_\ell(\psi), \psi)$ must be positive at any state ψ. Therefore, the expected surplus gap term in expression (26) is positive and $(S_h - S_\ell)(k, \psi) > 0.\|$

Proof of Proposition 3.3. We first state and prove the following result.

Claim 7.1 *Let k be a submarket such that either $\rho_\ell(k, \psi) > 0$ or $\rho_h(k, \psi) > 0$. Then*

$$\sum_i \rho_i(k, \psi)(1 - \alpha) S_i(k, \psi) \geq (1 - \alpha) S_\ell(k, \psi). \tag{29}$$

Proof of Claim 7.1.

Expression (29) is equivalent to

$$(1 - \rho_\ell(k, \psi)) S_h(k, \psi) \geq (1 - \rho_\ell(k, \psi)) S_\ell(k, \psi). \tag{30}$$

We analyze two cases. If $\rho_\ell(k, \psi) = 1$, then, expression (30) holds with equality. Otherwise, if $\rho_\ell(k, \psi) < 1$ and $\rho_h(k, \psi) > 0$, the inequality holds because of Lemma 3.2.$\|$

We first show that for any $k \in \mathcal{X}_\psi$ such that $\theta_\ell(\psi) \equiv \Theta(k, \psi) < \infty$ and $\rho_\ell(k, \psi) > 0$, the pair $(\theta_\ell(\psi), S_\ell(k, \psi))$ solves the following program (31). Then, we prove that the solution of the program is uniquely characterized by conditions (5)-(7).

$$\begin{aligned} \max_{\theta \in [0,\infty], S \in [0, \overline{S}_\psi]} & \qquad \nu(\theta) S \\ \text{s. to} & \quad \eta(\theta)(1 - \alpha) S \geq c \end{aligned} \tag{31}$$

Proposition 3.1 ensures that if k is an active submarket such that $\rho_\ell(k, \psi) > 0$, then $\rho_\ell(k, \psi) = 1$. As a result, the first equilibrium condition implies that the constraint of problem (31) evaluated at the pair $(\theta_\ell(\psi), S_\ell(\psi))$ holds with equality. We will prove that it is a maximizer by contradiction. Suppose there exists $(\theta', S') \in [0, \infty] \times [0, \overline{S}_\psi]$ such that $\nu(\theta')\alpha S' > \nu(\theta_\ell(\psi))\alpha S_\ell(k,\psi)$, and $\eta(\theta')(1-\alpha)S' \geq c$. The functional equation (2) is a contraction mapping on the set of continuous functions; hence, the solution function is continuous and there must exist k' such that $S_\ell(k', \psi) = S'$. Moreover, because ν is a continuous increasing function, there exists $\widetilde{\theta} < \theta'$ such that $\nu(\widetilde{\theta})\alpha S_\ell(k', \psi) = \nu(\theta_\ell)\alpha S_\ell(k, \psi)$. By definition of the equilibrium beliefs, $\Theta(k', \psi) \leq \widetilde{\theta}$. Therefore,

$$\eta(\Theta(k',\psi)) \sum_i \rho_i(k',\psi)(1-\alpha)S_i(k',\psi) \geq \eta(\Theta(k',\psi))(1-\alpha)S_\ell(k',\psi) > \eta(\theta')(1-\alpha)S_\ell(k',\psi) \geq c,$$

where the first inequality comes from Claim 7.1. This implies that a deviation to submarket k' would be profitable contradicting the assumption that k is an active submarket in equilibrium. Therefore, the pair $(\theta_\ell, S_\ell(k, \psi))$ is a solution of problem (31).

Now, we benefit from the fact that the constraint must hold with equality to rewrite program (31) as

$$\max_{\theta \in [0, b_\psi]} \theta,$$

where $\eta(b_\psi)$ equals $\frac{c}{(1-\alpha)\overline{S}_\psi}$, and $\overline{S}_\psi \equiv \max_k S_\ell(k, \psi)$ is the maximum surplus given the current aggregate state ψ. Thus, the maximizer θ is b_ψ and conditions (5) hold in equilibrium. The surplus $S_\ell(k, \psi)$ is a strictly concave function of k; hence, the maximizing capital level is uniquely determined by the first order condition (7) and equals $\overline{k}_\ell(\psi)$.

By using conditions (5) to replace the surplus value in the functional equation (2) and, then, dividing both sides of the resulting expression by $r + \lambda + \varphi$, we are left with

$$\frac{1}{\eta(\Theta(k(\psi),\psi))} = \frac{p(z, y_\ell, k(\psi)) - b}{r + \lambda + \varphi}\frac{1-\alpha}{c} - \alpha\frac{\Theta(k(\psi),\psi)}{r + \lambda + \varphi} + \frac{\varphi}{r + \lambda + \varphi}\mathbb{E}_\psi\left(\frac{1}{\eta(\Theta(k(\psi'),\psi'))}\right)$$

This last functional equation can be thought of as a contraction mapping; hence, it admits a unique solution, which must be the composite function $f(k, \psi) \equiv 1/\eta(\Theta(k, \psi))$. By evaluating it at the solutions of equations (7), we obtain the tightness vector $(\Theta(\overline{k}_\ell(\psi), \psi))_\psi$ as the inverse of function $1/\eta$ of the resulting values.$\|$

Proof of Proposition 3.4. We first show that condition 1 of problem (P) holds. Consider state $\psi \in \Psi$. We show arguing by contradiction that for any active submarket $k \in \mathcal{X}_\psi$ such that

$\theta \equiv \Theta(k,\psi) < \infty$ and $\rho_h(k,\psi) > 0$, the vector (θ, k) solves problem $(P(\psi))$ given the vector $(U_i(\psi'))_{\psi'}$.

Once again, Proposition 3.1 ensures that if k is such that $\rho_h(k,\psi) > 0$, then $\rho_h(k,\psi) = 1$. As a result, the first equilibrium condition implies that the first constraint of problem $(P(\psi))$ holds with equality. By definition of the equilibrium beliefs, the second constraint also holds given the equilibrium values $(U_\ell(\psi))_\psi$. Therefore, the pair (θ, k) belongs to the constraint set. Now, suppose that there exists (θ', k') such that $\nu(\theta')\alpha S_h(k',\psi) > \nu(\theta)\alpha S_h(k,\psi)$, and satisfies the two constraints of problem $(P(\psi))$. Because ν is an increasing continuous function, there must exist $\widetilde{\theta} < \theta'$ such that $\nu(\widetilde{\theta})\alpha S_h(k',\psi) = \nu(\theta)\alpha S_h(k,\psi)$. By definition of the equilibrium beliefs, $\Theta(k',\psi) \leq \widetilde{\theta}$. It must be the case that type-ℓ workers are strictly worse off in submarket k' because $b + \nu(\Theta(k',\psi))\alpha S_\ell(k',\psi) < b + \nu(\theta')\alpha S_\ell(k',\psi) \leq (r+\varphi)U_\ell(\psi) - \varphi \mathbb{E}_\psi\left(U_\ell(\psi)\right)$. It follows that $\rho_\ell(k',\psi) = 0$ and $\Theta(k',\psi) = \widetilde{\theta}$. Therefore,

$$\eta(\Theta(k',\psi))\sum_i \rho_i(k',\psi)(1-\alpha)S_i(k',\psi) = \eta(\Theta(k',\psi))(1-\alpha)S_h(k',\psi) > \eta(\theta')(1-\alpha)S_h(k',\psi) \geq c$$

This result contradicts the assumption that submarket k is active in equilibrium because a deviation to submarket k' would be profitable. Thus, the equilibrium pair (θ, k) is a solution of problem $(P(\psi))$. Finally, the second condition of problem (P) holds by definition of equilibrium.$\|$

Proof of Proposition 3.5. Consider the state $\psi \in \Psi$ and, given the vectors $(U_i(\psi'))_{\psi'}$, the associated problem $(P(\psi))$. Due to continuity of the objective function and compactness of the constraint set of the problem, the Weierstrass theorem applies to ensure the existence of a solution. Indeed, the constraint set is non-empty because e.g. the pair zero tightness and a capital level k being such that the surplus $S_h(k,\psi)$ is positive satisfies both constraints.

We will show later that the solution always satisfies the first constraint with equality. Then, using the constraint to replace the surplus from the objective function, the program can be rewritten as maximizing the identity function θ subject to the two constraints of program $(P(\psi))$. Therefore, the maximizer is the maximum feasible tightness value. Hence, for every state ψ, there exists a unique $\theta(\psi)$. Using the first constraint, we obtain a unique capital surplus value $S(\psi)$. Since the flow net returns from capital are strictly decreasing in capital in the interval of interest $[\overline{k}_h(\psi), \infty)$, the contraction mapping results for the functional equation (2) ensure that the surplus function is also strictly decreasing in capital within that interval; hence, there exists a capital level $k(\psi)$ whose associated surplus is $S_h(k(\psi),\psi) = S(\psi)$. Moreover, given the obtained vectors $(\theta(\psi))_\psi$ and

$(S_h(k(\psi),\psi))_\psi$, the unemployment values $(\tilde{U}_h(\psi))_\psi$ obtain as a solution of the functional equation (1).

Now, let H be a function defined on $\mathbb{R}^{|\Psi|}$ as $H((U_h(\psi))_\psi) = (\tilde{U}_h(\psi))_\psi$. To show the existence of solution of problem (P), we just need to show existence of a fixed point of H. Notice that function H is defined over the compact set $[0,\overline{U}]^{|\Psi|}$, with $\overline{U}$ sufficiently large. Furthermore, its continuity is inherited from the continuity of the functions involved in the maximization problem. Therefore, the Brower's fixed-point theorem applies.

Now, to characterize the solution of the program $(P(\psi))$ given state ψ, we define the Lagrangian as

$$\begin{aligned}\mathcal{L} = \quad & \nu(\theta)\alpha S_h(k,\psi) + \xi_1\left(\eta(\theta)(1-\alpha)S_h(k,\psi) - c\right) - \\ & -\xi_2\left(b + \nu(\theta)\alpha S_\ell(k,\psi) + \varphi\mathbb{E}_\psi\left(U_\ell(\psi)\right) - (r+\varphi)U_\ell(\psi)\right)\end{aligned}$$

where ξ_1 and ξ_2 stand for the non-negative multipliers associated to the first and second constraints, respectively.[13] The Kuhn Tucker (necessary) conditions are

$$\nu'(\theta)\alpha\left(S_h(k,\psi) - \xi_2 S_\ell(k,\psi)\right) + \xi_1\eta'(\theta)(1-\alpha)S_h(k,\psi) \leq 0,\ \theta \geq 0 \tag{32}$$

$$\text{and} \qquad \theta\nu'(\theta)\alpha\left(S_h(k,\psi) - \xi_2 S_\ell(k,\psi)\right) + \theta\xi_1\eta'(\theta)(1-\alpha)S_h(k,\psi) = 0.$$

$$\tfrac{\partial S_h(k,\psi)}{\partial k}\left(\nu(\theta)\alpha + \xi_1\eta(\theta)(1-\alpha)\right) - \xi_2\nu(\theta)\alpha\tfrac{\partial S_\ell(k,\psi)}{\partial k} \leq 0,\ k \geq 0 \tag{33}$$

$$\text{and} \qquad k\tfrac{\partial S_h(k,\psi)}{\partial k}\left(\nu(\theta)\alpha + \xi_1\eta(\theta)(1-\alpha)\right) - k\xi_2\nu(\theta)\alpha\tfrac{\partial S_\ell(k,\psi)}{\partial k} = 0.$$

$$\eta(\theta)(1-\alpha)S_h(k,\psi) \geq c,\ \xi_1 \geq 0 \text{ and } \xi_1\left[\eta(\theta)(1-\alpha)S_h(k,\psi) - c\right] = 0. \tag{34}$$

$$b + \nu(\theta)\alpha S_\ell(k,\psi) + \varphi\mathbb{E}_\psi\left(U_\ell(\psi)\right) \leq (r+\varphi)U_\ell(\psi),\ \xi_2 \geq 0 \tag{35}$$

$$\text{and} \qquad \xi_2\left[b + \nu(\theta)\alpha S_\ell(k,\psi) + \varphi\mathbb{E}_\psi\left(U_\ell(\psi)\right) - (r+\varphi)U_\ell(\psi)\right] = 0.$$

This set of complementary slackness conditions delivers the characterization of the solution of the problem. Let us analyze two different cases depending on whether the second constraint is binding or not.

<u>Case 1. $\xi_2 = 0$</u> Condition (33) implies that $k > 0$ as $\lim_{k\to 0}\frac{\partial S_h(k,\psi)}{\partial k} = \infty$. This same complementary slackness condition leads to the necessary condition (12), $p_k(z, y_h, k) = 0$, which has a unique solution due to the assumptions on the production technology. Notice that if the first constraint were slack, then $\xi_1 = 0$ because of condition (34). As a result, $\theta = \infty$ would follow

[13] To save on notation, we avoid to make the objects k, θ, ξ_j dependent on the state ψ.

from the complementary slackness condition (32), contradicting the same first constraint. Thus, the necessary condition (9) holds. However, the second constraint must hold for the pair (θ, x) obtained above to be the solution of the problem. The second case needs to be analyzed only if this second constraint fails to hold.

Case 2. $\xi_2 > 0$ Condition (13) must hold according to the complementary slackness condition (35). We now turn to show by contradiction that the first constraint cannot be slack either in this case. Suppose that $\xi_1 = 0$ and $\eta(\theta)(1-\alpha)S_h(k,\psi) > c$. If $\theta = 0$, then condition (35) fails to hold; hence, $\theta > 0$. According to condition (32), either $\nu'(\theta) = 0$ or $S_h(k,\psi) = \xi_2 S_\ell(k,\psi)$. The former implies $\theta = \infty$ and hence a zero job-filling rate for firms making the expected discounted profits null, which contradicts condition (34). As a result, it must be the case that $S_h(k,\psi) = \xi_2 S_\ell(k,\psi)$. In addition, if $\theta \in (0,\infty)$, then $\xi_2 = 1$ according to condition (33) since $\frac{\partial S_h(k,\psi)}{\partial k} > \frac{\partial S_\ell(k,\psi)}{\partial k}$ according to expression (25). Putting these two conditions together, we obtain that $S_h(k,\psi) = S_\ell(k,\psi)$. Proceeding along the same lines as in the proof of Lemma 3.2, we know that this fails to be true due to the complementarity between capital and ability. Moreover, condition (**??**) follows from the complementary slackness condition (33) as $\frac{\partial S_\ell(k,\psi)}{\partial k} < 0$. It implies that the solution is such that $k(\psi) > \overline{k}_h(\psi)$.

In both cases, we can replace the surplus value in the surplus equation (2) using the zero-profit condition to obtain equation (10).‖

Proof of Proposition 3.6. An equilibrium is a tuple $\{G_\psi, \mathcal{X}_\psi, \Theta(\cdot,\psi), \{U_i(\psi), S_i(\cdot,\psi), \rho_i(\cdot,\psi)\}_i\}_\psi$ that satisfies the equilibrium definition 1. Apart from the equilibrium objects already defined in the proposition statement, it remains to determine the surplus function, the off-the-equilibrium beliefs, and the continuum of distributions $(G_\psi)_\psi$. Given the vector $(U_i(\psi'))_{\psi'}$, the surplus function $S_i(\cdot,\psi)$ is uniquely defined as the solution of the contraction mapping (2) at any state ψ. The remaining equilibrium objects are set consistently with the equilibrium definition. In particular, if ψ is the aggregate state of the economy and $\mathcal{X}_\psi = \{k_\ell(\psi), x_h(\psi)\}$, then

$$dG_\psi(k) \equiv \begin{cases} (1-\mu)u_\ell\Theta(k,\psi), \text{ if } k = k_\ell(\psi) \\ \mu u_h\Theta(k,\psi), \text{ if } k = k_h(\psi) \\ 0, \text{ otherwise.} \end{cases}$$

The construction of distribution G trivially ensures the equilibrium market-clearing condition. We define the beliefs off-the-equilibrium for all submarket k as

$$\Theta(k,\psi) \equiv \min_i \hat{\theta}_i(k,\psi),$$

$$\rho_\ell(k,\psi) = \begin{cases} 1, \text{ if } \hat{\theta}_\ell(k,\psi) < \hat{\theta}_h(k,\psi) \\ 0, \text{ otherwise} \end{cases}, \text{ and } \rho_h(k,\psi) = \begin{cases} 1, \text{ if } \hat{\theta}_h(k,\psi) \leq \hat{\theta}_\ell(k,\psi) \\ 0, \text{ otherwise,} \end{cases}$$

where $\hat{\theta}_i(k,\psi) \equiv \begin{cases} \theta \text{ such that } b + \nu(\theta)\alpha S_i(k,\psi) + \varphi \mathbb{E}_\psi \left(U_i(\psi')\right) = (r+\varphi)U_i(\psi), \text{ if } S_i(k,\psi) > 0 \\ \infty, \text{ otherwise} \end{cases}$

Notice that the beliefs are well defined given the assumptions on the matching function ν and there is no submarket k such that both types of workers enter simultaneously. Now, we first show that firms maximize profits and those are zero in equilibrium for all active submarkets. Second, we prove that workers optimally search for jobs. Again, let ψ be the current state of the economy. The free entry condition holds in each active submarket because of Propositions 3.3 and 3.5. We turn to show by contradiction that firms maximize profits by entering either active submarket in $\mathcal{X}_\psi = \{k_\ell(\psi), k_h(\psi)\}$. We distinguish two cases.

Case 1. Suppose there exists a submarket k' such that $\Theta(k',\psi) < \infty$, $\rho_\ell(k',\psi) = 1$ and $\eta(\Theta(k',\psi))(1-\alpha)S_\ell(k',\psi) > c$. To attract low-ability workers, it must be the case that $b + \nu(\Theta(k',\psi))\alpha S_\ell(k',\psi) + \varphi \mathbb{E}_\psi \left(U_\ell(\psi)\right) = (r+\varphi)U_\ell(\psi)$. Because of continuity and monotonicity of function η, there exists $\theta' > \Theta(k',\psi)$ such that $\eta(\theta')(1-\alpha)S_\ell(k',\psi) = c$. Since ν is an increasing function of the tightness, we have

$$\nu(\Theta(k',\psi))\alpha S_\ell(k',\psi) = (r+\varphi)U_\ell(\psi) - b - \varphi \mathbb{E}_\psi \left(U_\ell(\psi)\right) < \nu(\theta')\alpha S_\ell(k',\psi),$$

which contradicts the assumption that $(\theta_\ell(\psi), k_\ell(\psi))$ is the maximizer of problem (8) and $(r+\varphi)U_\ell - b - \varphi \mathbb{E}_\psi \left(U_\ell(\psi)\right)$ is the maximum value. Therefore, there is no profitable deviation that attracts only low-ability workers.

Case 2. Suppose there exists a submarket k' such that $\Theta(k',\psi) < \infty$, $\rho_h(k',\psi) = 1$ and $\eta(\Theta(k',\psi))(1-\alpha)S_h(k',\psi) > c$. Again, high-ability workers enter the submarket if $b+\nu(\Theta(k',\psi))\alpha S_h(k',\psi) + \varphi \mathbb{E}_\psi \left(U_h(\psi)\right) = (r+\varphi)U_h(\psi)$. Further, the following inequality follows from the off-the-equilibrium beliefs, $b + \nu(\Theta(k',\psi))\alpha S_\ell(k',\psi) + \varphi \mathbb{E}_\psi \left(U_\ell(\psi)\right) \leq (r+\varphi)U_\ell(\psi)$. We distinguish two subcases. First, if this last inequality is strict, then there must exist $\theta' > \Theta(k',\psi)$ such that the two constraints of problem $(P(\psi))$ hold at (θ', k'). Because of the properties of the matching technology, we obtain that $\nu(\theta')(1-\alpha)S_h(k',\psi) > (r+\varphi)U_h(\psi) - b - \varphi \mathbb{E}_\psi \left(U_h(\psi)\right)$, which contradicts our assumption that $(\theta_h(\psi), k_h(\psi))$ is the maximizer. Second, if the inequality is indeed an equality, then $\frac{\partial \theta_\ell(k',\psi)}{\partial k'} > \frac{\partial \theta_h(k',\psi)}{\partial k'}$, where k' is the capital level at state ψ of k', as shown in the proof of Proposition 3.1. This means that there exists a submarket k'', which differs from k' by an arbitrarily small amount, such that the first constraint of problem $(P(\psi))$ holds, the second one holds with strict inequality and $b + \nu(\Theta(k'', \psi))(1-\alpha)S_h(k'', \psi) + \varphi\mathbb{E}_\psi(U_h(\psi)) = (r+\varphi)U_h(\psi)$. This leads us back to the first case.

It remains to show the equilibrium condition of workers' optimally search. This is the case by construction of the beliefs $\Theta(\cdot, \psi)$.‖

Proof of Proposition 3.8. Conditions (5)-(7) and (10)-(13) characterize the equilibrium outcome. Let ψ be the state of the economy. Let us define $\overline{k}_y(\psi)$ as the value of k that satisfies $p_k(z, y, k) = 0$, given parameter y. By differentiating with respect to y, we obtain that $\frac{\partial \overline{k}_y(\psi)}{\partial y} = -\frac{p_{ky}(z,y,\overline{k}_y(\psi))}{p_{kk}(z,y,\overline{k}_y(\psi))} > 0$. Because of the equilibrium conditions (7) and (12), we have $k_\ell(\psi) < k_h(\psi)$. If the capital level were determined by (13) instead, then we already stated that $k_\ell(\psi) < \overline{k}_h(\psi) \leq k_h(\psi)$.

From the zero-profit conditions (5) and (9), we have

$$\eta(\theta_\ell(\psi))S_\ell(k_\ell(\psi), \psi) = \eta(\theta(\psi))S_h(k_h(\psi), \psi). \tag{36}$$

Along similar lines as in the proof of Lemma 3.2, we can show that

$$\nu(\theta_\ell(\psi))S_\ell(k_\ell(\psi), \psi) < \nu(\theta(\psi))S_h(k_h(\psi), \psi).$$

Putting these two expressions together, it follows that $\theta_\ell(\psi) < \theta_h(\psi)$. The tightness gap together with expression (36) implies that $S_\ell(k_\ell(\psi), \psi) < S_h(k_\ell(\psi), \psi)$. As a result, by the definition of the unemployment value, we also obtain $U_\ell(\psi) < U_h(\psi)$. Finally, we can rewrite the entry wage expressions as

$$w_\ell(k_\ell(\psi), \psi) = \alpha p(z, y_\ell, k_\ell(\psi)) + (1-\alpha)((r+\varphi)U_\ell(\psi) - \varphi\mathbb{E}_\psi(U_\ell(\psi')))$$

$$w_h(k_h(\psi), \psi) = \alpha p(z, y_h, k_h(\psi)) + (1-\alpha)((r+\varphi)U_h(\psi) - \varphi\mathbb{E}_\psi(U_h(\psi')))$$

Because both terms on the right side are larger for high-ability workers, it follows that $w_\ell(k_\ell(\psi), \psi) < w_h(k_h(\psi), \psi)$.‖

Proof of Proposition 4.2 To show that hazard rates and entry wages decline with duration τ, it suffices to rewrite the expressions (17) and (18) as

$$\begin{aligned}
\overline{\nu}_\tau &= \frac{\mu\nu(q_h) + (1-\mu)e^{-\tau(\nu(q_\ell)-\nu(q_h))}\nu(q_\ell)}{\mu + (1-\mu)e^{-\tau(\nu(q_\ell)-\nu(q_h))}} = \nu(q_\ell) + \frac{\mu(\nu(q_h)-\nu(q_\ell))}{\mu + (1-\mu)e^{-\tau(\nu(q_\ell)-\nu(q_h))}} \\
\overline{w}_\tau &= \frac{\mu\nu(q_h)w_h + (1-\mu)e^{-\tau(\nu(q_\ell)-\nu(q_h))}\nu(q_\ell)w_\ell}{\mu\nu(q_h) + (1-\mu)e^{-\tau(\nu(q_\ell)-\nu(q_h))}\nu(q_\ell)} = w_\ell + \frac{\mu\nu(q_h)(w_h-w_\ell)}{\mu\nu(q_h) + (1-\mu)e^{-\tau(\nu(q_\ell)-\nu(q_h))}\nu(q_\ell)}
\end{aligned}$$

The first expression is decreasing in τ regardless of the sign of the term $\nu(q_h)-\nu(q_\ell)$. The derivative of the average wage with respect to τ is also negative as $\nu(q_h) > \nu(q_\ell)$ and $w_h > w_\ell$ according to Proposition 3.8.‖