

LIMITS TO REPUTATIONS

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Much of the interest in the adverse selection approach to reputations in repeated games arises from the fact that quite small departures from the complete information model seems to have large effects on the set of equilibrium payoffs. We show that this is not the case in reputation games where a long-run player plays a fixed stage-game against an infinite sequence of short-run players under imperfect public-monitoring. We conclude that in such games, introducing arbitrarily small incomplete information cannot open the possibility of new equilibrium payoffs that are far away from the complete information equilibrium payoff set, even when the long-run player's discount factor is very high but fixed. Our main result implies that the standard reputation results hold true due to a specific order of limits.

KEYWORDS: Reputations, Repeated Games, Short and Long Run Players, Continuity

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1. INTRODUCTION

One of the most prominent results in the reputation literature is due to Fudenberg and Levine (1989, 1992) who studied infinitely repeated “reputation” games where a long-run player faces an infinite sequence of short-run players. They showed that a *patient* strategic long-run player can guarantee herself a payoff close to her Stackelberg payoff when there is a small ex-ante probability that the long-run player is a commitment type who always plays the Stackelberg action.¹ This result seems to suggest that quite small departures from the complete information model have large effects on the set of equilibrium payoffs.²

The main result of this paper is in stark contrast with this interpretation. We study repeated games with incomplete information as in Fudenberg and Levine (1992), and we show that, if the discount factor of the long-run player is *fixed*, then arbitrarily small perturbations cannot open the possibility of equilibrium payoffs that are far away from the complete information equilibrium payoff set.

Therefore, Fudenberg and Levine’s (1992) result holds true due to a specific order of limits. That is, if the discount factor of the long-run player tends to 1 while holding the commitment type’s ex-ante probability fixed, then the reputation result a la Fudenberg and Levine (1992) holds true; however, if the commitment type’s ex-ante probability tends to 0 while holding the discount factor of the long-run player fixed, then the incomplete information equilibrium payoffs cannot be far away from the complete information equilibrium payoff set.

The next section describes our model. Section 3 presents the main result, and Section 4 concludes the paper.

2. THE MODEL

Our model is a standard model of an infinitely repeated game with incomplete information under imperfect public monitoring. The notation we employ is similar to that of Cripps, Mailath, and Samuelson (2004).³

¹Even though results of Fudenberg and Levine (1989, 1992) hold for both perfect and imperfect monitoring, Cripps, Mailath, and Samuelson (2004) showed that in the long-run the reputation effects disappear under imperfect public monitoring.

²The first papers which introduced the adverse selection approach to study reputations are Kreps, Milgrom, Roberts, and Wilson (1982), Kreps and Wilson (1982) and Milgrom and Roberts (1982)

³Interested readers are referred to Cripps, Mailath, and Samuelson(2004) for further technical details about the model.

A long-run player (player 1) plays an infinitely repeated stage-game with a sequence of different short-run players (player 2). The stage-game is a finite simultaneous-move game of imperfect public monitoring. With a slight abuse of notation, the action set of player 1 and player 2 in the stage-game are denoted by $I \equiv \{1, 2, \dots, I\}$ and $J \equiv \{1, 2, \dots, J\}$, respectively. The public signal, y , is drawn from a finite set, Y . The probability that y is realized under the action profile (i, j) is given by ρ_{ij}^y .

The ex ante stage-game payoffs are given by $u_1(i, j) = \sum_y f_1(i, j, y) \rho_{ij}^y$ and $u_2(i, j) = \sum_y f_2(j, y) \rho_{ij}^y$.

Player 1 (“she”) is a long-run player with the discount factor $\delta < 1$. Her payoff in the infinitely repeated game is the average discounted sum of stage-game payoffs, $(1 - \delta) \sum_{t=0}^{\infty} \delta^t u_1(i_t, j_t)$. Player 2 (“he”), on the other hand, denotes a sequence of short-run players, each of whom plays the stage-game only once.

Player 1’s actions are private. Hence, player 1 in period t has a private history, consisting of public signals and her past actions, denoted by $h_{1t} \equiv ((i_0, y_0), (i_1, y_1), \dots, (i_{t-1}, y_{t-1})) \in H_{1t} \equiv (I \times Y)^t$. Player 2 “short-lived players”, on the other hand, only observes the public history, i.e., $(y_0, y_1, \dots, y_{t-1}) \in Y^t$.

We assume that the monitoring structure has full support. That is, every signal y is possible after any action profile.

ASSUMPTION 1 (*Full Support*): $\rho_{ij}^y > 0$ for all $(i, j) \in I \times J$ and $y \in Y$.

REMARK 1 *Full support monitoring assumption ensures that all finite sequences of public signals occur with positive probability, hence must be followed by optimal behavior in any Nash equilibrium. Therefore, any Nash equilibrium outcome is also a perfect Bayesian equilibrium outcome. Furthermore, since there is only one long-run and one short-run player, Nash equilibrium outcomes coincide with perfect public equilibrium outcomes.*⁴

At time $t = -1$, a type of player 1 is selected. With probability $1 - p_0 > 0$, she is the “normal” type, with the preferences described above. With probability $p_0 > 0$, she is a “commitment” type who always, independent of history, plays the same (possibly mixed) action $s_1 \in \Delta(I)$ in each period.

⁴For technical details we refer the interested reader to Kandori and Matsushima (1998, Appendix) or Sekiguchi (1997, Proposition 3)

Player 2's posterior belief in period t that player 1 is the commitment type is given by the random variable p_t . By Assumption 1, Bayes' rule determines this posterior after all sequences of signals.

3. MAIN RESULT

Let $V(p_0)$ denote the equilibrium payoff set of the long-run player in the incomplete information repeated game with ex-ante commitment prior p_0 . In particular, $V(0)$ denotes the equilibrium payoff set of the long-run player in the repeated game with complete information.

Our main result is the following:

THEOREM 1 *Suppose the monitoring distribution ρ satisfies Assumption 1. For any fixed $\delta < 1$, and any $\zeta > 0$, there exists an $\eta > 0$ such that for any prior $p_0 \in (0, \eta)$, any equilibrium payoff of the long-run player in the incomplete-information repeated game with commitment prior p_0 is in the ζ neighborhood of $V(0)$.*

In words, our main result says that introducing arbitrarily small incomplete information cannot open the possibility of new equilibrium payoffs that are far away from the complete information equilibrium payoff set, even when the long-run player's discount factor is very high but fixed.

3.1. Outline of the Proof We proceed as follows: first, we introduce the standard set operator a la Abreu, Pearce and Stacchetti particular to our setting which gives us decomposable payoffs for the long-run player in a given set. When applied repeatedly to a compact set which includes all stage-game payoffs of the long-run player, this operator converges to the complete information equilibrium payoff set of the long-run player. Then, we slightly modify this operator to introduce a new set operator. The modification is that player 2, any of the short-run players, is not restricted to best respond to the enforcing action of player 1, but is allowed to best respond to some (possibly mixed) action which is close (in the Euclidean metric) to the enforcing action.

In our first lemma we show that there exists a distance, $\bar{\varepsilon} > 0$, such that all best responses to this particular action are also best responses to *some* (possibly mixed) action whose support is within the support of the action of the normal type of the long-run player. The essence of the argument is that the $\bar{\varepsilon}$ in Lemma 1 is

uniform over all possible supports. In Lemma 2, we show that the two operators coincide for any distance smaller than $\bar{\varepsilon}$. The essence of Lemma 2 is that the operators coincide uniformly over all possible subsets of the real line. Lemma 3 extends the result of Lemma 2 to all finite iterations using these uniformities.

The rest of the proof makes use of the fact that the discount factor is fixed and hence a finite number of iterations, t^* , suffices to approximate the set of all complete information equilibrium payoffs of the long-run player. When the commitment prior in the incomplete information game is small enough, the posterior after the first t^* periods stays below a certain threshold due to full support monitoring. Employing Lemma 3 allows us to show that one can identify a bound over the commitment prior so that the equilibrium payoffs of the long-run player in the incomplete information game with a prior less than this bound cannot be far away from her complete information equilibrium payoffs.

3.2. The Set Operator We start by introducing the standard set operator, T , of Abreu Pearce and Stacchetti (1990):

Consider the operator $T : 2^{\mathbb{R}} \setminus \{\emptyset\} \rightarrow 2^{\mathbb{R}} \setminus \{\emptyset\}$ defined as follows:

$v \in T(W)$ if and only if there exists a nonempty $I_v \subseteq I$, $\alpha_2 \in \Delta(J)$ and $w = (w_1, w_2, \dots, w_Y) \in W^{|Y|}$ such that:

- (i) $v = (1 - \delta)u_1(i, \alpha_2) + \delta(\sum_{y,j} w_y \rho_{ij}^y \alpha_2(j))$ for each $i \in I_v$
- (ii) $(1 - \delta)u_1(i, \alpha_2) + \delta(\sum_{y,j} w_y \rho_{ij}^y \alpha_2(j)) \leq v$ for each $i \in I$
- (iii) $\alpha_2 \in B_2(\alpha_1)$ for some $\alpha_1 \in \Delta(I_v)$

This operator identifies decomposable payoffs of the long-run player for a given set, $W \in \mathbb{R}$. Here, (i) corresponds to feasibility, (ii) corresponds to incentive compatibility conditions and, (iii) simply says that player 2 (a short run player) is best-responding to the enforcing action of the long-run player.

Let $M = \max_{i,j} |u_1(i, j)|$, and let $W_0 = [-M, M]$. It follows from the techniques introduced by Abreu, Pearce, and Stacchetti (1990) that $T^\infty(W_0) = V(0)$.

Next, consider the incomplete information game: Recall that, in period t , player 2 is best replying to α'_1 where $\alpha'_1 = p_t s_1 + (1 - p_t) \alpha_1$ and $\sigma_1(h_{1t}) = \alpha_1$, and p_t is the posterior at time t that Player 1 is a commitment type. Observe that, when

p_t happens to be arbitrarily small so is the distance $\|\alpha'_1 - \alpha_1\|$.

Utilizing this observation, we next define our set operator by relaxing condition (iii) of the operator T as follows:

For any $\varepsilon > 0$, let $T_\varepsilon : 2^{\mathbb{R}} \setminus \{\emptyset\} \rightarrow 2^{\mathbb{R}} \setminus \{\emptyset\}$ be such that:

$v \in T_\varepsilon(W)$ if and only if there exists a nonempty $I_v \subseteq I$, $\alpha_2 \in \Delta(J)$ and $w = (w_1, w_2, \dots, w_Y) \in W^{|Y|}$ such that:

- (i) $v = (1 - \delta)u_1(i, \alpha_2) + \delta(\sum_{y,j} w_y \rho_{ij}^y \alpha_2(j))$ for each $i \in I_v$
- (ii) $(1 - \delta)u_1(i, \alpha_2) + \delta(\sum_{y,j} w_y \rho_{ij}^y \alpha_2(j)) \leq v$ for each $i \in I$
- (iii $_\varepsilon$) $\alpha_2 \in B_2(\alpha'_1)$ for some $\alpha'_1 : \|\alpha'_1 - \tilde{\alpha}_1\| \leq \varepsilon$ for some $\tilde{\alpha}_1 \in \Delta(I_v)$

T_ε is slightly more permissive than T of Abreu, Pearce and Stacchetti (1990), inasmuch as it only requires the short-run player to best-respond to an action that is close, in Euclidean metric, to the one played by the long-run player.

The motivation for our operator is as follows: in the game of incomplete information, the mixed action to which the short-run player best-responds is a weighted average of the action taken by the normal type of the long-run player and the commitment type's action. Provided that the latter type is very unlikely, this means that the short-run player is taking a best-response to an action that is nearly the normal type's action. The key to our main result will be then to show that, if the short-run player best-responds to an action that is close to (the lotteries over) a set of actions, then he is actually also playing a best-response to an action within that set if the distance between the action and the set is sufficiently small.

It is clear that both operators T and T_ε are monotone. That is, $W_1 \subseteq W_2$ then $T(W_1) \subseteq T(W_2)$ and $T_\varepsilon(W_1) \subseteq T_\varepsilon(W_2)$. Moreover, for any $\varepsilon_1 > \varepsilon_2$, $T_{\varepsilon_2}(W) \subseteq T_{\varepsilon_1}(W)$.

3.3. Lemmata We start with a technical lemma which is key to our main result. In words, Lemma 1 says that if a short-run player best-responds to an action that is close to (the lotteries over) a set of actions, then he is actually also playing a best-response to an action within that set if the distance between the action and the set is sufficiently small.

Let $B(\Delta(X)) := \{\alpha_2 \in \Delta(J) : \alpha_2 \in B_2(\alpha_1) \text{ for some } \alpha_1 \in \Delta(X)\}$ for any $X \subseteq I$.

LEMMA 1 *There exists an $\bar{\varepsilon} > 0$ for all non-empty $X \subsetneq I$ such that if $\alpha_1 \in (\Delta(I) \setminus \Delta(X))$ with $\min_{\sigma_X \in \Delta(X)} \|\alpha_1, \sigma_X\| < \bar{\varepsilon}$, then $B_2(\alpha_1) \subseteq B(\Delta(X))$.*

PROOF: Suppose this is not the case. Since $\Delta(I)$ is compact, it is possible to construct convergent sequences $\alpha_{1,n} \in (\Delta(I) \setminus \Delta(X))$ and $\alpha_{2,n} \in \Delta(J)$ such that $\lim_{n \rightarrow \infty} \min_{\sigma_X \in \Delta(X)} \|\alpha_{1,n}, \sigma_X\| = 0$ for some $X \subset I$, with $\alpha_{2,n} \in B_2(\alpha_{1,n})$ and $\alpha_{2,n} \notin B(\Delta(X))$ for all n . Let $\alpha_{1,n} \rightarrow \bar{\alpha}_1$. Observe that we must have $\bar{\alpha}_1 \in \Delta(X)$.

Let $2^J = \{Y_1, Y_2, \dots, Y_m\}$. We know that $B(\Delta(X)) = \bigcup_{j \in H} \Delta(Y_j)$ for some $H \subset \{1, \dots, m\}$. We must have for all n , $\alpha_{2,n} \notin \Delta(Y_j)$ for all $j \in H$. Therefore, $\text{supp}(\alpha_{2,n}) = Y_n$ where $Y_n \neq Y_j$ for any $j \in H$. But since the number of subsets of J is finite; there exists a subsequence Y_k of Y_n such that $Y_k = \bar{Y}$ for some $\bar{Y} \neq Y_j$ for any $j \in H$. This implies $\text{supp}(\alpha_{2,k}) = \bar{Y}$ for all k .

Consider the stage-game action α_2^* of player 2 which gives equal probability to all the actions in $\bar{Y}$. It follows that α_2^* is a best response to $\alpha_{1,k}$ for all k . But, since the best response correspondence has a closed graph this implies that α_2^* is a best response to $\bar{\alpha}_1 \in \Delta(X)$. This contradicts the fact that $\bar{Y} \neq Y_j$ for any $j \in H$. $\square$

What is essential to note about Lemma 1 is that the $\bar{\varepsilon}$ is uniform over $X \subsetneq I$, i.e., a fixed $\bar{\varepsilon}$ works for all $X \subsetneq I$.

Next, we use Lemma 1 to show that for any $\varepsilon < \bar{\varepsilon}$ operators T and T_ε coincide. Note that this means $\bar{\varepsilon}$ is uniform over all $W \subset \mathbb{R}$.

LEMMA 2 *For any $0 < \varepsilon < \bar{\varepsilon}$ and $W \subset \mathbb{R}$, $T_\varepsilon(W) = T(W)$.*

PROOF: $T(W) \subseteq T(W)_\varepsilon$ is true for any $\varepsilon > 0$ and since, $\varepsilon_1 > \varepsilon_2$, implies $T_{\varepsilon_2}(W) \subseteq T_{\varepsilon_1}(W)$, it is enough to show that $T_{\bar{\varepsilon}}(W) \subseteq T(W)$.

Suppose $v \in T_{\bar{\varepsilon}}(W)$. Therefore, there exists $I_v \subseteq I$, $\alpha_2 \in \Delta(J)$ and $w = (w_1, w_2, \dots, w_Y) \in W^{|Y|}$ such that (i), (ii) and (iii) $_{\bar{\varepsilon}}$ holds.

We need to identify an $\hat{I}_v \subseteq I$, $\hat{\alpha}_2 \in \Delta(J)$ and $\hat{w} = (\hat{w}_1, \hat{w}_2, \dots, \hat{w}_Y) \in W^{|Y|}$ such that (i), (ii) and (iii) holds.

If $\alpha'_1 \in \Delta(I_v)$, we are trivially done. (simply take $\hat{I}_v = I_v$, $\hat{\alpha}_2 = \alpha_2$, $\hat{w} = w$ and $\alpha_1 = \alpha'_1$ in (iii).)

If not, $\alpha'_1 \in \Delta(I) \setminus \Delta(I_v)$ with $\min_{\sigma \in \Delta(I_v)} \|\alpha'_1, \sigma\| < \|\alpha'_1 - \tilde{\alpha}_1\| < \bar{\varepsilon}$ since $\tilde{\alpha}_1 \in \Delta(I_v)$. Hence, by Lemma 1, we must have $B_2(\alpha'_1) \subseteq B(\Delta(I_v))$. So there exists α''_1 in $\Delta(I_v)$ with $\alpha_2 \in B(\alpha''_1)$. Let $\bar{X}$ be the support of α''_1 . Clearly, $\bar{X} \subseteq I_v$.

Now take $\hat{I}_v = \bar{X}$, $\hat{\alpha}_2 = \alpha_2$, $\hat{w} = w$. Observe that, (i) and (ii) holds since $\bar{X} \subseteq I_v$. The fact that (iii) holds is clear, simply take $\alpha_1 = \alpha''_1$. $\square$

Since $\bar{\varepsilon}$ is uniform over $W \subset \mathbb{R}$ finite iterations of $T_{\bar{\varepsilon}}$ and T will coincide. Our next lemma formalizes this fact for W_0 .⁵

LEMMA 3 For any $0 < \varepsilon < \bar{\varepsilon}$, $T_\varepsilon^t(W_0) = T^t(W_0)$.

PROOF: By Lemma 2, for any $0 < \varepsilon < \bar{\varepsilon}$, $T_\varepsilon(W_0) = T(W_0)$. Since the ε identified in Lemma 2 is independent of W , with the same ε , $T_\varepsilon^2(W_0) = T^2(W_0)$. Iterating gives $T_\varepsilon^t(W_0) \subseteq T^t(W_0)$. $\square$

3.4. Proof of the Main Result Now, we are ready to give the proof of our main result:

PROOF OF THEOREM 1: Define $d_x = \min_{y \in V(0)} \|x, y\|$. Given any $\zeta > 0$, the fact $T^\infty(W_0) = V(0)$ implies that there must exist a t^* such that for $t > t^*$, $\max_{x \in T^t(W_0)} d_x < \zeta$.

Let $\eta := \frac{\varepsilon}{\sqrt{2}\kappa^{t^*}}$ for some $\varepsilon < \bar{\varepsilon}$ of Lemma 1, and let

$$\kappa := \sup_{\alpha_1 \in \Delta(I), j \in J} \max_y \left\{ \frac{\sum_i \rho_{i,j}^y s_1(i)}{\sum_i \rho_{i,j}^y \alpha_1(i)} \right\}.$$

Note first that by the full support assumption (Assumption 1), $\kappa < \infty$.⁶

In the incomplete information game with commitment prior $p_0 \in (0, \eta)$, at any period $t \leq t^*$, the probability with which player 1 is a commitment type is not more than $\frac{\varepsilon}{\sqrt{2}}$. To see why, observe that the posterior belief of Player 2 about the commitment type in any period can be at most κ times her prior from the preceding period and hence $p_t < p_0 \kappa^t < \frac{\varepsilon}{\sqrt{2}}$ for all $t \leq t^*$.

This implies that the set of continuation payoffs at period t^* are a subset of $T_\varepsilon(W_0)$. This is because at t^* , if $\tilde{\sigma} = (\tilde{\sigma}_1, \sigma_2)$ is an equilibrium in the incomplete

⁵Recall that $W_0 = [-M, M]$ where $M = \max_{i,j} |u_1(i, j)|$.

⁶Note that this is the only place in our proof where Assumption 1 (full-support monitoring) has a bite.

information game for normal types with $\tilde{\sigma}_1(h_{1t^*}) = \alpha_1$, player 2 is best replying to $\alpha'_1 = p_{t^*}s_1 + (1 - p_{t^*})\alpha_1$ and since $p_{t^*} < \frac{\varepsilon}{\sqrt{2}}$ we have $\|\alpha'_1 - \alpha_1\| < \varepsilon$.⁷ Similarly, at $t^* - 1$ the set of continuation payoffs are a subset of $T_\varepsilon^2(W_0)$. Thus, iterating backwards, the set of equilibrium payoffs at period 0 is a subset of $T_\varepsilon^{t^*+1}(W_0)$.

By Lemma 3, $T_\varepsilon^{t^*+1}(W_0) = T^{t^*+1}(W_0)$. Therefore $\max_{x \in T_\varepsilon^{t^*+1}(W_0)} d_x < \zeta$. Hence, any equilibrium payoff of player 1 (the long-run player) at time 0, as being in $T_\varepsilon^{t^*+1}(W_0)$, is in the ζ neighborhood of $V(0)$. $\square$

4. CONCLUSION

The main result of this paper is essentially an upper-hemi continuity result concerning the equilibrium payoffs in reputation games where a long-run player faces an infinite sequence of short-run players. Technically, we show that in these games the Nash equilibrium correspondence is, for a fixed discount factor, upper-hemicontinuous in the prior probability that the long-run player is a commitment type at zero when there is full support imperfect public monitoring.

Our result points out that the standard reputation results arise due to a particular order of limits. That is, if the discount factor of the long-run player tends to 1 sufficiently faster than the commitment type's ex-ante probability tends to 0, then the standard reputation result a la Fudenberg and Levine (1992) holds true; however, if the commitment type's ex-ante probability tends to 0 faster than the discount factor of the long-run player tends to 1, then the incomplete information equilibrium payoffs cannot be far away from the complete information equilibrium payoff set.

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⁷ $\|\alpha'_1 - \alpha_1\| \leq \varepsilon$ since $\|s_1 - \alpha_1\| \leq \sqrt{2}$.

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