

UNEMPLOYMENT RISK AND THE DISTRIBUTION OF ASSETS^{*}

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– PRELIMINARY –

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Abstract

How does the distribution of assets affect job search decisions? We analyze unemployed workers and how their asset holdings affect the allocation to jobs of different productivity. In the absence of insurance, workers with low asset holdings direct their search to low productivity jobs because they offer a low wage and low risk. We show that this occurs under a condition closely related to Decreasing Relative Risk Aversion. There is perfect segregation of asset holders into job productivities even when assets holdings are private. We also find that for a given worker, the productivity of jobs she applies for is decreasing in the duration of unemployment. As assets gradually deplete, she takes more secure, low wage jobs. When workers are heterogeneous in skills, there is a trade off between wages and insurance. The skilled but poor worker will necessarily go for the less ambitious, low wage job in order to hedge risk.

Keywords. Unemployment Risk. Precautionary Savings. Sorting. Mismatch. Optimal Unemployment Insurance.

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1 Introduction

In this paper we analyze how the distribution of assets affect job search decisions, and in particular how the workers' asset holdings affect the allocation to jobs of different productivity. In the absence of insurance, workers with low asset holdings direct their search to low productivity jobs because they offer a low wage and low risk. We show that this occurs under a condition closely related to Decreasing Relative Risk Aversion. There is perfect segregation of asset holders into job productivities even when assets holdings are private. We also find that for a given worker, the productivity of jobs she applies for is decreasing in the duration of unemployment. As assets gradually deplete, she takes more secure, low wage jobs. When workers are heterogeneous in skills, there is a trade off between wages and insurance. The skilled but poor worker will necessarily go for the less ambitious, low wage job in order to hedge risk.

Unemployment risk is, arguably, one of the biggest causes of income uncertainty. In this model, we analyze labor market equilibrium in the presence of three sources of market incompleteness: uninsurable unemployment risk, job search and private information about assets. The key aspect that we focus on is heterogeneity in asset holdings. How do workers with different asset holdings insure against unemployment risk in the absence of a formal insurance market? While our focus is on incomplete markets, we do allow for complete capital markets with consumption smoothing. However, there is no full insurance and therefore there is a role for precautionary savings.

Our main finding is that in equilibrium there is full separation in the job search decision under a condition closely related to Decreasing Absolute Risk Aversion. Workers with more asset holdings apply to more productive jobs even though they are not more productive. This is due to the fact that high productive firms set higher wages because their opportunity cost of not filling the vacancy is higher. On the other side of the market, high asset holders are more willing to take risk. Therefore they apply for high paying, high risk jobs.

condition: under a property on preferences, there is “single crossing”, but there is no technology, hence single crossing stems from the preferences only!!!

This paper is related to a large literature on unemployment risk and risk averse agents. Danforth (1979) is one of the first to analyze optimal search in a partial equilibrium setting. Hopenhayn and Nicolini (1997), Shimer and Werning (2007) and Shimer and Werning (2008) analyze optimal unemployment insurance in a similar setting. Our paper is a general equilibrium search model with risk averse agents and closely related to Acemoglu and Shimer (1999). They analyze workers with identical asset holdings and focus on the incentives for firms to create jobs. Golosov, Maziero, and Menzio (2012) consider the setup in Acemoglu and Shimer (1999) and analyze private job search decisions by identical agents, driven by the participation constraint. Here, we focus on private assets and where the distribution of those assets is non-degenerate.

Also Guerrieri, Shimer, and Wright (2010) and Eeckhout and Kircher (2010) have private infor-

mation, but there the source of complementarities and hence the separation is technological, namely the complementarity between worker skills and job productivity or the single crossing between worker skill and effort. Here instead, the complementarity derives entirely from preferences and how the risk attitude changes with assets. This is an entirely novel approach to matching since it basically involves two-sided matching with non-linear pairwise Pareto frontiers.

Unemployment insurance (UI) is an important policy tool to help families insure against the risk of job loss. In 2010, the US federal government spent \$162 billion on UI. Since the onset of the great recession, UI payments have gone up fivefold, up from \$33 billion in 2007. This is mainly due to the extension of the eligibility period from 26 weeks to 99 weeks. One of the central policy questions in economics is how those UI payments affect aggregate labor market variables such as the unemployment rate, job creation and productivity. The focus of attention of the literature is on the incentive effects of the presence or absence of UI.¹ How does UI affect workers' reservation wage? How much effort are they willing to put in to search for a job when generous benefits are available?

In this paper, we analyze the job search decision by workers who are heterogenous in their asset holdings. Since the needs to smooth consumption by the rich are very different from those of the poor, asset holdings will affect workers' job search behavior. In particular, workers with high asset holdings will apply for the more productive jobs that pay higher wages. For the workers with low asset holding those jobs have too much unemployment risk and they prefer to apply for low productivity jobs with lower wages yet higher employment chances. Because workers are homogeneous in productivity, this allocation of asset holdings to job productivities is inefficient, i.e., there is mismatch. When UI benefits are not conditioned on assets/income, there are two reasons why the high asset holders get too much benefits. First, because they do not need the smoothing since they can rely on their assets. Second, due to the fact that they choose jobs with higher unemployment risk they tend to receive unemployment benefits more often.

2 The Model

Timing. This is a two-period economy in which agents make a joint consumption-savings and job search decision. Endowed with assets, in there first period they choose their consumption-savings level and in the second period they choose the optimal job search. In section 5 we extend the model to an infinite horizon setting.

Agents. There is a measure one of workers, indexed by their heterogeneous asset holdings $a \in \mathcal{A} = [a, \bar{a}] \subset \mathbb{R}_+$. Let $G(a)$ denote the measure of workers with asset levels weakly below $a \in \mathcal{A}$. We assume a is private information. Each worker supplies her labor and can only apply to one job at a time. Firms are heterogeneous in their productivities y and each have one job. Let $y \in \mathcal{Y} = [\underline{y}, \bar{y}] \subset \mathbb{R}_+$ and assume

¹There may of course also be aggregate demand effects of transferring consumption to the unemployed. For a recent analysis of such a mechanism, see Kaplan and Menzio (2012).

the firm type is observable. $F(y)$ denotes the measure of firms with a type weakly below y . The total measure of firms is $F(\bar{y})$. F and G are C^2 with strictly positive derivatives f and g .

Preferences and Technology. Workers are risk averse and their preferences are represented by the Bernoulli utility function $u(c)$ over consumption level c , where $u : \mathbb{R}_+ \rightarrow \mathbb{R}_{++}$. We assume that u is increasing and concave: $u' > 0, u'' < 0$. Agents discount utility with factor $\beta < 1$. Savings can be invested in a risk free bond at a fixed rate $R > 1$. We assume that firms are owned by entrepreneurs who are risk neutral and who do not participate in the labor market.² Output produced at a firm of type y is $v(y)$.

Search Technology. Job search is directed and there is a search technology that governs those frictions. The frictions crucially depend on the degree of competition for jobs, as captured by ratio of workers to firms, denoted by $\lambda \in [0, \infty]$. This represents the relative supply and demand for jobs, as it determines the probability of a match $m(\lambda)$, where $m : [0, \infty] \rightarrow [0, 1]$: the higher the value of λ , the easier it is for a firm to fill its vacancy, so m is a strictly increasing function: $m' > 0$. In contrast, the higher the ratio of workers to firms, the harder it is for the worker to find a match. We denote the probability that a worker gets matched by $q(\lambda)$, where $q : [0, \infty] \rightarrow [0, 1]$ is a strictly decreasing function, $q' < 0$. Since matching is always in pairs, matching probability of workers must be consistent with those of firms, in particular, it must be the case that $q(\lambda) = m(\lambda)/\lambda$. We also require the standard assumptions hold: m is twice continuously differentiable, strictly concave and has a strictly decreasing elasticity. The fact that we express the matching probability in terms of the ratio of workers to firms λ and not the number of workers and firms effectively means that we assume a matching technology that is constant returns. As the number of workers and firms doubles, the number of matches doubles, yet the matching probabilities remain unchanged.

Actions. In the first period, agents choose their consumption-savings bundle for both periods. Given assets a , each worker chooses consumption $c_1 = a - a'$ where a' are the assets saved, and second period consumption is contingent on the labor market outcome, $c_{2,e} = Ra' + w$ when employed and $c_{2,u} = Ra'$ when unemployed. Within the second period, a two-stage labor search extensive form game determines the labor market outcome. Firms first simultaneously announce wages $w \in \mathcal{W} = [\underline{w}, \bar{w}] \subset \mathbb{R}_+$. After observing wage-firm type pairs (w, y) , the workers then choose which pair to apply to. A worker always has the option not to apply for any job, the choice of which is denoted by $\emptyset$. Due to the presence of market frictions, not all applications are successful. Denote by $P(y, w)$ and $Q(a, a', y, w)$ the distribution of actions by firms and workers: $P(y, w)$ is the measure of firms that offers a productivity-wage pair below (y, w) and $Q(a, a', y, w)$ is the measure of workers with assets below a who save less than a' and who apply for productivity-wage pairs below (y, w) . We impose that those distributions of actions are

²We could also have assumed that profits are distributed as the risk free dividend of a mutual fund owned by all workers and that has all firms in its portfolio as in Golosov, Maziero, and Menzio (2012). This approach does not affect any of the results since the dividend deterministically increases the workers' asset holdings and merely shifts the asset distribution.

consistent with the initial distributions of types $F(y)$ and $G(a)$, i.e., that there is market clearing. In particular, it must be the case that $P_Y(\cdot) = F(\cdot)$ and $Q_A = G(\cdot)$, where P_Y and Q_A are the marginal distributions. This ensures that the allocation is measure preserving.

Equilibrium. We adopt the equilibrium concept used by Acemoglu and Shimer (1999). To accommodate the two sided heterogeneity of firm productivity and worker assets, we will use the version of their equilibrium adjusted by Eeckhout and Kircher (2010) to allow for two-sided heterogeneity and a continuum of agents. They consider the Acemoglu and Shimer (1999) setup as a large game where each individual's payoff is determined only by her own action and the distribution of actions in the economy, which consists of the optimal choices of each of the individuals in the distribution. Denote by the function $\lambda_{PQ} : \mathcal{Y} \times \mathcal{W} \rightarrow [0, \infty]$ the expected queue length at each productivity-wage combination (y, w) .³ The expected payoff of the workers is given by:

$$\begin{aligned} U(a, a', y, w, P, Q) &= u(c_1) + \beta \mathbb{E}u(c_2) \\ &= u(c_1) + \beta q(\lambda_{PQ})u(c_{2,e}) + (1 - q(\lambda_{PQ}))u(c_{2,u}), \end{aligned} \quad (1)$$

Since $c_1 = a - a'$, $c_{2,e} = Ra' + w$, $c_{2,u} = Ra'$, the consumption is completely pinned down by the choice of a' . The expected profits of the firm are

$$\pi(y, w, P, Q) = m(\lambda_{PQ})(v(y) - w). \quad (2)$$

In line with the literature on directed search (see for example McAfee (1993), Acemoglu and Shimer (1999)), we impose restrictions on the beliefs about off equilibrium path behavior. In the current setup, beliefs about the queue length corresponding to firm or worker choices that do not occur in equilibrium are not defined. Therefore, we define those off equilibrium path beliefs corresponding to the notion of subgame perfection.⁴ Firms expect workers to queue up for jobs as long as it is profitable for them to do so given the options they have on the equilibrium path. Formally, this defines the queue length over the entire domain as: $\lambda_{PQ}(a, w) = \sup \{ \lambda \in \mathbb{R}_+ : \exists a, q(\lambda)[y - w \geq \max_{y, w \in \text{supp } P} U(a, y, w, P, Q)] \}$. In all other cases, the queue length is zero. This now permits us to formally define equilibrium:

Definition 1 *An equilibrium is a pair of distributions (P, Q) such that the following conditions hold:*

1. *Worker optimality: $(a, a', y, w) \in \text{supp } Q$ only if (y, p) maximizes (1) for a ;*
2. *Firm optimality: $(y, w) \in \text{supp } P$ only if w maximizes (2) for y ;*

³Then along the support of the firms' wage setting distribution, $\lambda_{PQ} = dQ_{Y\mathcal{W}}/dP$ is given by the Radon-Nikodym derivative, where $Q_{Y\mathcal{W}}$ is the marginal distribution of Q with respect to $\mathcal{Y}$ and $\mathcal{W}$.

⁴Peters (1997) and Peters (2000) provide micro foundations for a version of this model where this assumption is indeed justified as the limit of deviations in a finite game.

This is a matching problem with a non-linear pairwise Pareto frontier. Existence is established in Legros and Newman (2007) and Kaneko (1982). Jerez (2012) establishes the existence of an equilibrium in a directed search model with a continuum of agents and a general matching technology.

The (measure preserving) market clearing condition is particularly simple when matching is monotone. Then there is one-to-one matching of a to y , which we represent by a function $\mu : \mathcal{Y} \rightarrow \mathcal{A}$. Under positive assortative matching (PAM), $\mu'(y)$ is positive and it is negative under negative assortative matching (NAM). Under PAM high asset workers match with high productivity jobs, and the market clearing condition can be written as:

$$\int_a^{\bar{a}} f(a) da = \int_{\mu(a)}^{\bar{y}} \lambda(y) g(y) dy.$$

3 The Decentralized Equilibrium Allocation

The firm problem is to set wages to maximize expected profits $\pi(y) = \max_w m(v(y) - w)$. The consumer's problem is to maximize expected utility from consumption while simultaneously making an optimal search decision. The worker's consumption-labor choice problem is:

$$\max_{a', \lambda} u(a - a') + \beta (q' u(Ra' + w) + (1 - q) u(Ra')),$$

We can therefore write the equilibrium worker and firm optimization as:

$$\begin{aligned} \max_{a', \lambda} u(a - a') + \beta (q' u(Ra' + w) + (1 - q) u(Ra')) \\ \text{s.t. } \pi = \max_w m(v(y) - w). \end{aligned}$$

Given $w = v(y) - \frac{\pi}{m}$ and the optimal choice of wages follows from the optimal choice of queue length λ , we can write this problem as a matching problem with non-linear Pareto frontier denoted by $\Phi(a, y, \pi)$:

$$\Phi(a, y, \pi) = \max_{a', \lambda} u(a - a') + \beta [qu(c_e) + (1 - q) u(Ra')]$$

where $c_e = Ra' + v(y) - \frac{\pi}{m}$. Then the solution to the maximization problem is $a'^*(a, y, \pi)$, $\lambda^*(a, y, \pi)$ and satisfies:

$$-u'(a - a') + \beta R [qu'(c_e) + (1 - q) u'(Ra')] = 0 \quad (3)$$

$$\beta q' [u(c_e) - u(Ra')] + \beta u'(c_e) \frac{m' \pi}{\lambda m} = 0 \quad (4)$$

The optimal savings behavior and optimal job search simultaneously implies a matching decision. That is a worker a chooses a firm y . Her optimal decision given a schedule $\pi(y)$ is to choose the firm

type y that maximizes her expected utility. The optimal y therefore satisfies $\Phi_y + \Phi_\pi \frac{\partial \pi}{\partial y} = 0$. This implies:

$$\beta q u'(c_e) \left(v_y - \frac{\pi'}{m} \right) = 0. \quad (5)$$

where the effect of y and π on π through a' and λ is zero from the envelope theorem: $\Phi_{a'} = 0, \Phi_\lambda = 0$. The details of the derivation of the partial derivatives are in the Appendix.

We want to ascertain under which circumstances there is monotone matching of asset holdings a in job productivities y . This is now a matching problem $\Phi(a, y, \pi)$ where a type a chooses the optimal y , given optimizing behavior. The first order condition to this problem is $\Phi_y + \Phi_\pi \frac{\partial \pi}{\partial y} = 0$. An allocation $a = \mu(y)$. The total cross derivative is positive provided

$$\frac{d^2}{dady} \Phi = \Phi_{ay} + \Phi_{\pi y} \frac{\partial \pi}{\partial y} = \Phi_{ay} - \frac{\Phi_y}{\Phi_\pi} \Phi_{\pi a},$$

where we use the first order condition to substitute for $\frac{\partial \pi}{\partial y}$. Therefore, there will be Positive Assortative Matching in types a, y provided $\Phi_{ay} > \frac{\Phi_y}{\Phi_\pi} \Phi_{a\pi}$. Then $\Phi_{ay} > \frac{\Phi_y}{\Phi_\pi} \Phi_{a\pi}$ provided (where the partial derivatives of Φ are derived in the Appendix):

$$\begin{aligned} -u''(a - a') a'_y &> \frac{\beta q u'(c_e) v_y}{\beta q u'(c_e) \frac{-1}{m}} (-u''(a - a') a'_\pi) \\ a'_y &> -m v_y a'_\pi \end{aligned}$$

We obtain the expressions for a'_y and a'_π from the first order conditions (in Appendix). Denote the maximand by $\phi(a', \lambda) = u(a - a') + \beta [q u(c_e) + (1 - q) u(Ra')]$, i.e., the objective function that is maximized with respect to a', λ . Then the condition for positive sorting of a on y becomes:

$$\begin{aligned} a'_y &> -m v_y a'_\pi \\ (\phi_{a'y} + m v_y \phi_{a'\pi}) \phi_{\lambda\lambda} &< (\phi_{\lambda y} + m v_y \phi_{\lambda\pi}) \phi_{a'\lambda} \end{aligned}$$

Observe that from the first order conditions to the maximization problem, we obtain the cross partials on ϕ . First, note that $\phi_{a'y} = -v_y m \phi_{a'\pi} = v_y \beta R q u''(c_e)$ so that the LHS is zero. Then we derive the following:

$$\begin{aligned} \phi_{\lambda y} &= \beta q' u'(c_e) v_y + \beta u''(c_e) v_y \frac{m' \pi}{\lambda m} \\ \phi_{\lambda\pi} &= \beta q' u'(c_e) \frac{-1}{m} + \beta u'(c_e) \frac{m'}{\lambda m} + \beta u''(c_e) \frac{-1}{m} \frac{m' \pi}{\lambda m} \\ &= \frac{-1}{v_y m} \phi_{\lambda y} + \beta u'(c_e) \frac{m'}{\lambda m} \end{aligned}$$

Therefore, the inequality can be written as:

$$0 < \beta u'(c_e) \frac{m'}{\lambda} v_y \phi_{a'\lambda}$$

The condition for positive sorting of a on y is $\phi_{a'\lambda} > 0$ is thus,

$$\beta R \left(q'[u'(c_e) - u'(Ra')] + u''(c_e) \frac{\pi m'}{\lambda m} \right) > 0$$

This now allows us to derive the first result:

Proposition 1 *Workers with higher initial asset levels a will apply for higher wage jobs provided*

$$\frac{u'(c_e) - u'(Ra')}{u(c_e) - u(Ra')} < \frac{u''(c_e)}{u'(c_e)}. \quad (\text{U})$$

Proof. From the first order condition $\phi_\lambda = 0$ we obtain:

$$\frac{m'\pi}{\lambda m} = -q' \frac{u(c_e) - u(Ra')}{u'(c_e)}.$$

Substituting in the condition $\phi_{a'\lambda} > 0$:

$$q'[u'(c_e) - u'(Ra')] - u''(c_e) q' \frac{u(c_e) - u(Ra')}{u'(c_e)} > 0,$$

or, noting that $q' < 0$,

$$u'(c_e)[u'(c_e) - u'(Ra')] < u''(c_e) [u(c_e) - u(Ra')].$$

or alternatively

$$\frac{u'(c_e) - u'(Ra')}{u(c_e) - u(Ra')} < \frac{u''(c_e)}{u'(c_e)}.$$

■

This Proposition establishes under what conditions of the utility function agents with higher levels of assets will choose more risky jobs. The does not immediately allow for a straightforward interpretation, and in the next two results we characterize the properties. First, we show that within the class of Hyperbolic Absolute Risk Aversion (HARA) utility functions, the condition is satisfied whenever absolute risk aversion is decreasing (DARA).

Proposition 2 *Consider the class of utility functions with Hyperbolic Absolute Risk Aversion (HARA):*

$$u(c) = \frac{1-\gamma}{\gamma} \left(\frac{\alpha c}{1-\gamma} + \beta \right)^\gamma \quad \text{where } \alpha > 0, \beta > \frac{\alpha c}{1-\gamma}.$$

Then condition (U) holds whenever there is Decreasing Absolute Risk Aversion (DARA): $\gamma < 1$. It holds with opposite inequality when there is Increasing Absolute Risk Aversion (IARA): $\gamma > 1$.

Proof. In Appendix. ■

A number of results for special cases of the HARA preferences immediately follow, including CRRA, logarithmic, CARA, risk neutrality and the quadratic.

Proposition 3 Consider the class of HARA utility functions. Condition (U) holds:

1. under CRRA $u(c) = \frac{1-\gamma}{\gamma} c^\gamma$ ($\alpha = 1 - \gamma, \gamma < 1, \beta = 0$) and Log utility: $u(c) = \log c$ (CRRA, $\gamma \rightarrow 0$);
2. with equality under CARA $u(c) = 1 - e^{-\alpha c}$ ($\beta = 1, \gamma \rightarrow -\infty$) and Risk Neutral $u(c) = \alpha c$ ($\gamma = 1$);
3. with opposite inequality under Quadratic utility: $u(c) = -\frac{1}{2}(-\alpha c + \beta)^2$ ($\gamma = 2$).

Proof. In Appendix. ■

The results for HARA may indicate that condition (U) holds more generally. The answer is partially true. For small differences between the level of consumption when a job is obtained and the consumption of unemployment ($c_e - Ra' = w$ small), we can indeed completely generalize the characterization: when there is DARA, condition (U) is satisfied and high asset types choose high productivity jobs. This is proven in Proposition 4. However, for general utility functions beyond HARA and with wages w large, this characterization does not hold. In Example 1 in the Appendix, we show by counterexample that for w large, Decreasing Absolute Risk Aversion (DARA) is not sufficient for the condition to hold.

Proposition 4 When w is small, condition (U) is satisfied for any utility function that exhibits Decreasing Absolute Risk Aversion (DARA), $-\frac{u''}{u'} < 0$, and thus has positive risk prudence, $u''' > 0$. Likewise, it holds with opposite inequality under IARA.

Proof. In Appendix. ■

Characterization. Condition (U) establishes that there are complementarities in the match value between a firm type y and a worker with assets a . In other words, the match value $\Phi(a, y, \pi)$ between types a and y is supermodular, and therefore the equilibrium expected payoff for the worker is increasing in a , and so is the equilibrium expected payoff to the firm in y . While there are no technological complementarities (all workers are identically skilled), the preferences generate a complementarity between assets and job productivity.

The implication of this condition is that high asset workers:

1. apply for high productivity jobs ($y \uparrow$)

2. earn higher wages ($w \uparrow$)
3. have higher unemployment ($\lambda \uparrow \Rightarrow q(\lambda) \downarrow$)
4. have higher expected consumption ($c \uparrow$)
5. have higher expected utility ($U \uparrow$)

And likewise, high productivity firms:

1. post higher wages ($w \uparrow$)
2. attract higher asset workers ($a \uparrow$)
3. have higher expected profits ($\pi \uparrow$)
4. fill vacancies faster ($\lambda \uparrow \Rightarrow m(\lambda) \uparrow$)

4 Private Assets

Participation Constraint. So far, we have considered a fixed population. With private information, we first introduce a participation constraint. When there is a cost of applying for a job, and the mechanism does not have the commitment power to make the worker apply, then there is a constraint on the payoff. In particular, the expected benefit of application must be no lower than the application cost. Otherwise, the worker will not apply. Then mechanism's optimal solution needs to satisfy the following participation constraint, where c denotes the ex ante cost for a job application:

$$\beta q(\lambda) u \left(Ra' + v(\mu) - \frac{\pi(\mu)}{m(\lambda)} \right) + (1 - q(\lambda)) \beta u(Ra') \geq c + \beta u(Ra')$$

and hence

$$\beta q(\lambda) \left[u \left(Ra' + v(\mu) - \frac{\pi(\mu)}{m(\lambda)} \right) - u(Ra') \right] \geq c.$$

If the payoff is increasing in assets – this is the case for example in the decentralized equilibrium allocation, irrespective of condition (U) – then this constraint is binding only for the lowest type $\underline{a}$:

$$\beta q(\lambda(\underline{a})) \left[u \left(Ra'(\underline{a}) + v(\mu(\underline{a})) - \frac{\pi(\mu(\underline{a}))}{m(\lambda(\underline{a}))} \right) - u(Ra'(\underline{a})) \right] \geq c.$$

Truthtelling Constraint. Now we analyze what happens when asset holdings of the workers are private. Consider a mechanism (the planner's or the market) in which workers participate and that consists of an allocation $\mu(\hat{a})$ and a payoff when matched $t(\hat{a})$. Because the mechanism will use all

the pairwise surplus, that is equivalent to specifying a payoff to the firm. Moreover, since we take the matching technology as a constraint of the mechanism, we can express the mechanism in terms of the expected payoff to the firm, denoted by $\pi(\mu(\hat{a}))$.

Truth-telling satisfies:

$$\begin{aligned}
& \Phi(a, \mu(a), \pi(\mu(a))) \geq \Phi(a, \mu(\hat{a}), \pi(\mu(\hat{a}))) \\
\iff & \hat{a} \in \arg \max_{\hat{a}} \Phi(a, \mu(\hat{a}), \pi(\mu(\hat{a}))) \\
\iff & \Phi_y \mu'(a) + \Phi_\pi \pi' \mu'(a) = 0 \\
\iff & \Phi_y + \Phi_\pi \pi' = 0.
\end{aligned}$$

Observe that this implies:

$$\begin{aligned}
& (-u'(a - a') + \beta R [qu'(c_e) + (1 - q) u'(Ra')]) \frac{\partial a'}{\partial \hat{a}} \\
& + \left(\beta q' [u(c_e) - u(Ra')] + \beta u'(c_e) \frac{m' \pi}{\lambda m} \right) \frac{\partial \lambda}{\partial \hat{a}} \\
& + \left(\beta qu'(c_e) \left(v_y - \frac{\pi'}{m} \right) \right) \frac{\partial \mu}{\partial \hat{a}} = 0.
\end{aligned} \tag{6}$$

So far, in the decentralized economy, we have assumed observable assets. The following result establishes that the decentralized outcome will be achieved even if asset holdings are private.

Proposition 5 *Let condition (U) hold. Then the complete information decentralized equilibrium outcome is incentive compatible and can therefore be attained under private asset holdings.*

Proof. Inspection of the incentive constraint (6) immediately reveals that each of the terms in brackets coincides with the first order conditions of the solution to the decentralized equilibrium allocation (3), (4), (5). ■

This is a feature of the the matching model. A planner with a social welfare function will in general have a different objective than to implement the equilibrium allocation in the presence of private information. We can also establish the following:

Proposition 6 *Let condition (U) hold. Then the private information decentralized equilibrium outcome is unique.*

Proof. The condition for monotone matching – that Φ is supermodular in a and y – is equivalent to uniqueness of the equilibrium allocation (see Legros and Newman (2007)). Condition (U) is equivalent to supermodularity of Φ , thus establishing the result. ■

Efficiency. The planner's problem, with weights ω_1, ω_2 (possibly functions of a) on worker utility and firm profits, is:

$$\begin{aligned} \max_{\mu, \pi(\cdot), \lambda(\cdot)} \int_0^{\bar{a}} & \left\{ \omega_1 \Phi(a, \mu(a), \pi) + \omega_2 \frac{\pi(\mu(a))}{\lambda(a)} \right\} f(a) da \\ \text{s.t.} \quad \mu'(a) &= \frac{f(a)}{g(\mu(a))} \frac{1}{\lambda(a)} \\ \pi'(a) &= -\frac{\Phi_y}{\Phi_\pi} \end{aligned}$$

We can write the Hamiltonian as:

$$\mathcal{H}(\mu, \pi) = \left\{ \omega_1 \Phi(a, \mu(a), \pi) + \omega_2 \frac{\pi(\mu(a))}{\lambda(a)} \right\} f(a) + \delta_1 \frac{f(a)}{g(\mu(a))} \frac{1}{\lambda(a)} - \delta_2 \frac{\Phi_y}{\Phi_\pi}.$$

which implies the first order conditions

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial a'} &= \omega_1 \Phi_{a'} f(a) - \delta_2 \frac{\partial}{\partial a'} \frac{\Phi_y}{\Phi_\pi} = 0 \\ \frac{\partial \mathcal{H}}{\partial \lambda} &= \omega_1 \Phi_\lambda f(a) - \omega_2 \beta \frac{\pi(\mu)}{\lambda^2} f(a) - \frac{\delta_1}{g(\mu) \lambda^2} - \delta_2 \frac{\partial}{\partial \lambda} \frac{\Phi_y}{\Phi_\pi} = 0 \end{aligned}$$

as well as the costate equations:

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial \mu} &= \left(\omega_1 \Phi_y + \omega_2 \frac{\pi'}{\lambda} \right) f(a) - \delta_1 \frac{f(a) g'}{g^2} - \delta_2 \frac{\partial}{\partial \mu} \frac{\Phi_y}{\Phi_\pi} = -\delta'_1 \\ \frac{\partial \mathcal{H}}{\partial \pi} &= \left(\omega_1 \Phi_\pi + \omega_2 \frac{1}{\pi} \right) f(a) - \delta_2 \frac{\partial}{\partial \pi} \frac{\Phi_y}{\Phi_\pi} = -\delta'_2 \end{aligned}$$

Compared with the equilibrium allocation, where we have first-order conditions:

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial a'} &= \Phi_{a'} f(a) = 0 \\ \frac{\partial \mathcal{H}}{\partial \lambda} &= \Phi_\lambda f(a) - \frac{\delta_1}{g(\mu) \lambda^2} = 0, \end{aligned}$$

and costate equations:

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial \mu} &= \Phi_y f(a) - \delta_1 \frac{f(a) g'}{g^2} = -\delta'_1 \\ \frac{\partial \mathcal{H}}{\partial \pi} &= \Phi_\pi f(a) = -\delta'_2 \end{aligned}$$

Observe that even with a Utilitarian Planner, i.e. when $\omega_1 = \omega_2 = 1$, the equilibrium solution does not coincide with the planner's solution.

5 Extensions

5.1 Infinite Horizon

We consider a job market where the unemployed search until they find a job, and once employed they hold the job forever. Denote by $U(a)$ the value of being unemployed with asset level a and by $E(a)$ the value of being employed. We maintain the cloning assumption (see Burdett and Coles (1997)), i.e., in the distribution of unemployed workers, a matched worker is replaced by a new unemployed worker with the same type. Then the Bellman equations are:

$$\begin{aligned} U(a) &= \max_{a', \lambda} \left\{ u(a - a') + \beta [qE(Ra') + (1 - q)U(Ra')] \right\} \\ E(a) &= \max_{a'} \left\{ u(w + a - a') + \beta E(Ra') \right\} \end{aligned}$$

Since the employment-consumption problem $E(a)$ is stationary and independent of $U(a)$, we can solve it separately. The FOC is $u'(w + a - a') = \beta RE'(Ra')$. With $\beta R = 1$ the solution is $a' = a/R = \beta a$ and we can explicitly write the value for employment:

$$E(a) = \frac{1}{1 - \beta} u(w + (1 - \beta)a).$$

We can then write the problem of the unemployed as:

$$U(a) = \max_{a', \lambda} \left\{ u(a - a') + \beta \left[q \frac{1}{1 - \beta} u(w + (1 - \beta)Ra') + (1 - q)U(Ra') \right] \right\}$$

with the problem of the firm in infinite horizon and in a stationary equilibrium:

$$V(y) = \max_w \left\{ m(v(y) - w) + \beta(1 - m)V(y) \right\},$$

which is equivalent to:

$$V(y) = \max_w \left\{ \frac{m}{1 - \beta(1 - m)} [v(y) - w] \right\}.$$

We can then write the problem as:

$$\begin{aligned} U(a) &= \max_{a', \lambda} \left\{ u(a - a') + \beta \left[q \frac{1}{1 - \beta} u(w + (1 - \beta)Ra') + (1 - q)U(Ra') \right] \right\} \\ \text{s.t. } \pi &= \max_w \frac{m}{1 - \beta(1 - m)} [v(y) - w] \end{aligned}$$

But now we can substitute for wages from the firm's problem and write the matching problem as:

$$\Phi(a, y, \pi) = \max_{a', \lambda} \left\{ u(a - a') + \beta \left[q \frac{1}{1 - \beta} u \left((1 - \beta)Ra' + v(y) - \pi \frac{1 - \beta(1 - m)}{m} \right) + (1 - q)\Phi(Ra') \right] \right\}$$

Analogously to the two-period model, we can establish the following result.

Proposition 7 *Workers with higher initial asset levels a will apply for higher wage jobs provided*

$$\frac{u'(c_e) - \Phi'(Ra')}{\frac{1}{1-\beta}u(c_e) - \Phi(Ra')} < \frac{u''(c_e)}{u'(c_e)}. \quad (\mathbf{U}_\infty)$$

Proof. In Appendix. ■

This has of course a similar interpretation of the condition in the two period model ($\mathbf{U}$), but now it necessarily involves the continuation value of the unemployed worker rather than the static utility. Though it does not admit a immediate closed form interpretation – the value function Φ is non stationary – we can of course recursively solve for this continuation value.

Interestingly, the dynamic nature of the problem now implies a time variant job choice decision. A worker who fails to become employed sees her assets gradually deplete (since, trivially, $a' < a$). But the optimal search decision dictates application to less productive, lower wage jobs when assets are lower. As a result, over the duration of unemployment and as assets deplete, workers will apply for less productive, lower wage jobs. This then follows immediately from Proposition 7.

Proposition 8 *Under condition ($\mathbf{U}_\infty$) and for a given worker with assets a , the job productivity y she applies for decreases in the duration of unemployment.*

The decrease in wages with employment duration is thus observationally equivalent to skill depreciation where wages decrease with employment duration since skills of the unproductive worker erode.

5.2 Heterogeneity in Worker Productivity

Consider an optimal savings decision where all agents are identical in assets a but have different productivity levels x . Output produced at firm y by worker x is $v(x, y)$. The matching problem is now:

$$\Phi(x, y, \pi) = \max_{a', \lambda} u(a - a') + \beta \left(q(\lambda) u \left(Ra' + v(x, y) - \frac{\pi}{m(\lambda)} \right) + (1 - q(\lambda)) u(Ra') \right)$$

Then the solution to the (interior) maximization problem is $s^*(x, y, \pi), \lambda^*(x, y, \pi)$ and satisfies:

$$\begin{aligned} -u'(a - a') + \beta R [qu'(c_e) + (1 - q) u'(Ra')] &= 0 \\ \beta q' [u(c_e) - u(Ra')] + \beta u'(c_e) \frac{m'\pi}{\lambda m} &= 0 \end{aligned}$$

Now we have monotone matching of x in y provided: $\Phi_{xy} > \frac{\Phi_y}{\Phi_\pi} \Phi_{x\pi}$.

$$\begin{aligned}
\Phi_y &= \beta q u'(c_e) v_y \\
\Phi_x &= \beta q u'(c_e) v_x \\
\Phi_\pi &= \beta q u'(c_e) \frac{-1}{m} = -\beta u'(c_e) \frac{1}{\lambda} \\
\Phi_{x\pi} &= -\beta \left[\frac{u''(c_e)}{\lambda} v_x + \frac{1}{\lambda} \left(u''(c_e) \frac{m'\pi}{m^2} - \frac{u'(c_e)}{\lambda} \right) \lambda_x + u''(c_e) \frac{1}{\lambda} R a'_x \right] \\
&= \beta v_x \left[\left(u' q' + u'' \frac{m'\pi}{\lambda m} \right) \lambda_\pi + q u'' R a'_\pi \right] \\
\Phi_{xy} &= \beta \left[q u'(c_e) v_{xy} + q u'' v_x v_y + \left(u'' \frac{m'\pi}{\lambda m} + q' u' \right) v_y \lambda_x + q u'' v_y R a'_x \right]
\end{aligned}$$

Then:

$$\begin{aligned}
\Phi_{xy} &> \frac{\Phi_y}{\Phi_\pi} \Phi_{x\pi} \\
q u' v_{xy} + q u'' v_x v_y + \left(u'' \frac{m'\pi}{\lambda m} + q' u' \right) v_y \lambda_x + q u'' v_y R a'_x &> m v_y \left[\frac{u''}{\lambda} v_x + \frac{1}{\lambda} \left(u'' \frac{m'\pi}{m^2} - \frac{u'}{\lambda} \right) \lambda_x + u'' \frac{1}{\lambda} R a'_x \right] \\
q u' v_{xy} + \left(q' + \frac{m}{\lambda^2} \right) u' v_y \lambda_x &> 0 \\
q u' v_{xy} + \frac{m'}{\lambda} u' v_y \lambda_x &> 0 \\
m v_{xy} + m' v_y \lambda_x &> 0
\end{aligned}$$

The sorting pattern depends crucially on the λ_x , where

$$\lambda_x = - \frac{\begin{vmatrix} \phi_{a'a'} & \phi_{a'x} \\ \phi_{\lambda a'} & \phi_{\lambda x} \end{vmatrix}}{|\mathbf{H}|} < 0 \quad \text{where} \quad |\mathbf{H}| = \begin{vmatrix} \phi_{a'a'} & \phi_{a'\lambda} \\ \phi_{\lambda a'} & \phi_{\lambda\lambda} \end{vmatrix}.$$

That immediately reveals that the condition also depends on the curvature of u through $\phi_{a'a'}$ and $\phi_{\lambda\lambda}$. In particular, the condition for PAM is:

$$(\phi_{a'a'} \phi_{\lambda\lambda} - \phi_{\lambda a'}^2) m v_{xy} - (\phi_{a'a'} \phi_{\lambda x} - \phi_{\lambda a'} \phi_{a'x}) m' v_y > 0$$

or

$$\begin{aligned}
&\left([u''(a - a') + (1 - q)u''(Ra')] \phi_{\lambda\lambda} - \left[q'[u'(c_e) - u(Ra')] + u''(c_e) \frac{m'\pi}{m\lambda} \right]^2 \right) m v_{xy} \\
&- \left([u''(a - a') + (1 - q)u''(Ra')] \beta \left[q' u'(c_e) + u''(c_e) \frac{m'\pi}{m\lambda} \right] v_x - \left[q'[u'(c_e) - u(Ra')] + u''(c_e) \frac{m'\pi}{m\lambda} \right] q u''(c_e) v_x \right) m' v_y > 0
\end{aligned}$$

The expression is quite involved, so let's try to get some insight by shutting down the smoothing decision, i.e., a' is exogenously given. Then the only FOC is the second one, and λ_x is given by:

$$\lambda_x = - \frac{\left(q'u' + u'' \frac{m'\pi}{\lambda m}\right) v_x}{q''[u(c_e) - u(Rs)] + 2q'u' \frac{m'\pi}{m^2} + qu''(c_e) \left(\frac{m'\pi}{m^2}\right)^2 + u'(c_e)q\pi \frac{2mm'^2 + m^2m''}{m^4}}$$

Note that when the utility is linear, we can write

$$\begin{aligned} \lambda_x &= - \frac{q'v_x}{q''w + 2q' \frac{m'\pi}{m^2} + \pi \frac{\lambda mm'' - \lambda(m')^2 - mm'}{\lambda m^2}} \\ &= - \frac{q'v_x}{q''w + 2q' \frac{m'\pi}{m^2} + m(v-w) \frac{\lambda mm'' - \lambda(m')^2 - mm'}{\lambda m^2}} \\ &= - \frac{(m'\lambda - m)v_x}{\lambda m''v} \end{aligned}$$

Therefore, the condition for positive sorting of x on y is

$$mv_{xy} - m'v_y \frac{(m'\lambda - m)v_x}{\lambda m''v} > 0$$

or

$$v_{xy} > \frac{m'(m'\lambda - m)}{\lambda mm''} \frac{v_x v_y}{v}$$

which is the condition in Eeckhout and Kircher (2010).

5.3 Heterogeneity in Worker Productivity, Identical Firms

Assume all firms are identical and workers are heterogenous in both a and their job productivity x . The joint distribution of worker types is denoted by $F(a, x)$. Output produced by a worker of type x is equal to $v(x)$. Suppose for now that the entire type bundle (a, x) is observable. Then the firms will offer wages contingent on x, a . The firm profits are $\pi = m(\lambda)(v(x) - w(x))$, and the firm essentially offers the worker a wage that ensures the firm obtains a constant residual profit π . Therefore, the offered wage is equal to the worker's productivity minus the ex post profits π/m . Since firms are homogeneous, π is a positive scalar.⁵ The problem can thus be written as:

$$\Phi(a, x, \pi) = \max_{a', \lambda} u(a - a') + \beta \left(q(\lambda) u \left(Ra' + v(x) - \frac{\pi}{m(\lambda)} \right) + (1 - q(\lambda)) u(Ra') \right)$$

⁵When there is free entry, say at a cost K , total profits are driven down to zero – the measure of firms is such that $m(\lambda)(v(x) - w(x)) - K = \pi - K = 0$ – yet the profits conditional on entering π must be strictly positive to cover the entry cost. If there is no entry cost, profits can be zero, but then the measure of firms goes to infinity and frictions vanish completely since $\lim_{\lambda \rightarrow 0} m(\lambda) = 0$ and $\lim_{\lambda \rightarrow 0} q(\lambda) = 1$.

Then the solution to the (interior) maximization problem is $a'^*(x, y, \pi), \lambda^*(x, y, \pi)$ and satisfies:

$$\begin{aligned} -u'(a - a') + \beta R [qu'(c_e) + (1 - q) u'(Ra')] &= 0 \\ \beta q' [u(c_e) - u(Ra')] + \beta u'(c_e) \frac{m'\pi}{\lambda m} &= 0 \end{aligned}$$

where $c_e = Ra' + v(x) - \frac{\pi}{m(\lambda)}$.

We now establish how wages change with assets. Since $w = v(x) - \frac{\pi}{m(\lambda)}$, we can write

$$\begin{aligned} \frac{\partial w(a, x)}{\partial a} &= \frac{\pi}{m(\lambda)^2} \frac{\partial \lambda}{\partial a} \\ &= -\frac{\pi}{m(\lambda)^2} \frac{\begin{vmatrix} \phi_{a'a'} & \phi_{a'a} \\ \phi_{\lambda a'} & \phi_{\lambda a} \end{vmatrix}}{|\mathbf{H}|} \\ &= -\frac{\pi}{m(\lambda)^2} \frac{\phi_{a'a'} \phi_{\lambda a} - \phi_{\lambda a'} \phi_{a'a}}{|\mathbf{H}|} \end{aligned}$$

Now given $|\mathbf{H}| > 0, \pi > 0$ and $m^2 > 0$, the sign of the change in wages as a result of a change in assets is positive provided:

$$\phi_{a'a'} \phi_{\lambda a} < \phi_{\lambda a'} \phi_{a'a}.$$

But note that $\phi_{\lambda a} = 0$ and $\phi_{a'a} = -u''(a - a') > 0$, and therefore this condition is satisfied provided $\phi_{\lambda a'} > 0$. Now this is exactly the same condition that leads to condition (U). We can therefore immediately establish the following result.

Proposition 9 *Workers with higher initial asset levels a and identical skills x will apply for higher wage jobs provided condition (U) holds.*

Now the effect of the skill level on wages for given asset holdings is in general ambiguous, since:

$$\begin{aligned} \frac{\partial w(a, x)}{\partial x} &= v_x + \frac{\pi}{m(\lambda)^2} \frac{\partial \lambda}{\partial x} \\ &= v_x - \frac{\pi}{m(\lambda)^2} \frac{\begin{vmatrix} \phi_{a'a'} & \phi_{a'x} \\ \phi_{\lambda a'} & \phi_{\lambda x} \end{vmatrix}}{|\mathbf{H}|} \\ &= v_x - \frac{\pi}{m(\lambda)^2} \frac{\phi_{a'a'} \phi_{\lambda x} - \phi_{\lambda a'} \phi_{a'x}}{|\mathbf{H}|}. \end{aligned}$$

Now we can derive

$$\begin{aligned} \phi_{\lambda x} &= \beta q' u'(c_e) v_x + \beta u''(c_e) v_x \frac{m'\pi}{\lambda m} \\ \phi_{a'x} &= \beta R q u''(c_e) v_x \end{aligned}$$

so that

$$-(\phi_{a'a'}\phi_{\lambda x} - \phi_{\lambda a'}\phi_{a'x}) = -v_x \left(\phi_{a'a'} \left(\beta q' u'(c_e) + \beta u''(c_e) \frac{m'\pi}{\lambda m} \right) - \phi_{\lambda a'} \beta R q u''(c_e) \right) < 0$$

since $q' < 0, u'' < 0$ and under condition (U), $\phi_{\lambda a'} > 0$. In other words, $\frac{\partial \lambda}{\partial x} < 0$. Higher productive types look for shorter queues because their opportunity cost of not being matched is higher. As a result, the net effect is ambiguous. The ambiguous effect of skills on prices in the presence of directed search frictions has been pointed out by Eeckhout and Kircher (2010).

Since $\phi_{\lambda a'} = \beta R \left(q'[u'(c_e) - u'(Ra')] + u''(c_e) \frac{\pi m'}{\lambda m} \right)$ we can write $\beta q' u'(c_e) + \beta u''(c_e) \frac{m'\pi}{\lambda m} = \frac{\phi_{\lambda a'}}{R} + \beta u'(Ra')$, we can further rewrite as:

$$-(\phi_{a'a'}\phi_{\lambda x} - \phi_{\lambda a'}\phi_{a'x}) = -v_x \left(\phi_{a'a'} \left(\frac{\phi_{\lambda a'}}{R} + \beta u'(Ra') \right) - \phi_{\lambda a'} \beta R q u''(c_e) \right).$$

But there is no simple interpretation for when the price effect is positive or negative. However, a sufficient condition for the effect of skill on wages to be positive is that u not too concave is, since the magnitude of u'' determines the magnitude of $\frac{\partial \lambda}{\partial x}$, which is zero when preferences are linear.

The presence of firm heterogeneity and complementarities between skills and firm productivity will be force towards making the wage profile increasing since it reinforces the effect of skills on marginal productivity and therefore on wages.

Now what is the Marginal Rate of Substitution between skills and assets? We can basically now derive at which rate skilled workers trade off assets at constant wages. This is fully captured in the iso-wage curve in (a, x) space.

From the expression for wages $w = v(x) - \frac{\pi}{m}$ we obtain

$$\begin{aligned} \frac{\partial a}{\partial x} &= -\frac{w_x}{w_a} \\ &= -\frac{\beta q u'(c_e) v_x}{\frac{\pi}{m(\lambda)^2} \frac{\phi_{\lambda a'} \phi_{a'a}}{|\mathbf{H}|}} \end{aligned}$$

5.4 Portfolio choice

Suppose the worker has access to two assets, on safe asset with return R , and one risky asset that offers a higher return. The agent now has to optimally choose the portfolio composition α , the share of the risky asset. Then in the decision problem, an agent with CRRA preferences will choose the same α independent of assets. However, here we have background risk, and the optimal choice of α jointly with a' and λ will imply that the optimal choice of α will depend on a . In particular, the conjectures are: 1. high a will choose more risky assets than low a types. This is consistent with the facts. 2. the portfolio choice will interact with the job search decision, so this should reduce the need to insure via the job

market and the wage profile should become flatter (and the queue length profile steeper).

Appendix

Derivatives of Φ and a'

$$\begin{aligned}
\Phi_y &= \beta qu'(c_e)v_y + \Phi_{a'}a'_y + \Phi_\lambda\lambda_y = \beta qu'(c_e)v_y \\
\Phi_a &= u'(a - a') + \Phi_{a'}a'_a + \Phi_\lambda\lambda_a = u'(a - a') \\
\Phi_\pi &= \beta qu'(c_e)\frac{-1}{m} + \Phi_{a'}a'_\pi + \Phi_\lambda\lambda_\pi = \beta qu'(c_e)\frac{-1}{m} \\
\Phi_{ay} &= -u''(a - a')a'_y (= \partial/\partial a \Phi_y) \\
\Phi_{a\pi} &= -u''(a - a')a'_\pi
\end{aligned}$$

where $\Phi_{a'} = 0$ and $\Phi_\lambda = 0$ from the envelope theorem.

We calculate the derivative of a' using the implicit function theorem. For the problem to have a maximum, we require that the Hessian of the maximand is positive $|\mathbf{H}| > 0$ (recall that $\phi_{\lambda\lambda}$ is assumed negative), where:

$$|\mathbf{H}| = \begin{vmatrix} \phi_{a'a'} & \phi_{a'\lambda} \\ \phi_{\lambda a'} & \phi_{\lambda\lambda} \end{vmatrix}$$

Applying the implicit function theorem,

$$a'_y = \frac{\partial a'}{\partial y} = -\frac{\begin{vmatrix} \phi_{a'y} & \phi_{a'\lambda} \\ \phi_{\lambda y} & \phi_{\lambda\lambda} \end{vmatrix}}{|\mathbf{H}|} = \frac{\phi_{a'y}\phi_{\lambda\lambda} - \phi_{\lambda y}\phi_{a'\lambda}}{|\mathbf{H}|} \quad \text{and} \quad a'_\pi = \frac{\partial a'}{\partial \pi} = -\frac{\begin{vmatrix} \phi_{a'\pi} & \phi_{a'\lambda} \\ \phi_{\lambda\pi} & \phi_{\lambda\lambda} \end{vmatrix}}{|\mathbf{H}|} = \frac{\phi_{a'\pi}\phi_{\lambda\lambda} - \phi_{\lambda\pi}\phi_{a'\lambda}}{|\mathbf{H}|}$$

Proof of Proposition 3

Proof. We can calculate the derivatives:

$$\begin{aligned}
u'(c) &= \alpha \left(\frac{\alpha c}{1-\gamma} + \beta \right)^{\gamma-1} \\
u''(c) &= -\alpha^2 \left(\frac{\alpha c}{1-\gamma} + \beta \right)^{\gamma-2}
\end{aligned}$$

and condition (U) becomes (where $c = Ra'$):

$$\begin{aligned}
&\alpha \left(\frac{\alpha c_e}{1-\gamma} + \beta \right)^{\gamma-1} \left[\alpha \left(\frac{\alpha c_e}{1-\gamma} + \beta \right)^{\gamma-1} - \alpha \left(\frac{\alpha c}{1-\gamma} + \beta \right)^{\gamma-1} \right] < \\
&-\alpha^2 \left(\frac{\alpha c_e}{1-\gamma} + \beta \right)^{\gamma-2} \left[\frac{1-\gamma}{\gamma} \left(\frac{\alpha c_e}{1-\gamma} + \beta \right)^\gamma - \frac{1-\gamma}{\gamma} \left(\frac{\alpha c}{1-\gamma} + \beta \right)^\gamma \right]
\end{aligned}$$

and after dividing by α^2 and by $\left(\frac{\alpha c_e}{1-\gamma} + \beta\right)^{2\gamma-2}$, which under our assumptions are both positive, this implies:

$$1 - \left(\frac{\frac{\alpha c}{1-\gamma} + \beta}{\frac{\alpha c_e}{1-\gamma} + \beta}\right)^{\gamma-1} < -\frac{1-\gamma}{\gamma} \left[1 - \left(\frac{\frac{\alpha c}{1-\gamma} + \beta}{\frac{\alpha c_e}{1-\gamma} + \beta}\right)^{\gamma}\right],$$

or

$$1 - x^{\gamma-1} < -\frac{1-\gamma}{\gamma} [1 - x^{\gamma}] \quad \text{where } x = \frac{\frac{\alpha c}{1-\gamma} + \beta}{\frac{\alpha c_e}{1-\gamma} + \beta} \in (0, 1).$$

First consider $\gamma > 0$. After rearranging and multiplying by $\gamma x^{1-\gamma}$, which is positive for $\gamma > 0$:

$$\begin{aligned} x^{1-\gamma} - (\gamma + (1-\gamma)x) &< 0 \\ G(\gamma) - H(\gamma) &< 0. \end{aligned}$$

At $\gamma = 0$ and $\gamma = 1$ the expression is exactly zero, i.e., G and H cross at 0 and 1. Now, $G'(\gamma) = -x^{1-\gamma} \log x$, $H'(\gamma) = 1 - x$, and $G''(\gamma) = x^{1-\gamma}(\log x)^2 > 0$, $H''(\gamma) = 0$. Observe that $G(\gamma)$ is convex, $G''(\gamma) = x^{1-\gamma}(\log x)^2 > 0$, while $H(\gamma)$ is linear. As a result, for $\gamma \in (0, 1)$ condition (U) holds with strict inequality. For $\gamma = 1$, (U) holds with equality and for $\gamma > 1$ it holds with opposite inequality.

Now consider $\gamma < 0$. Since we multiplied by $\gamma < 0$, condition (U) now implies that $G(\gamma) - H(\gamma) > 0$. Using the same logic, we establish that condition (U) holds for $\gamma < 0$.

This establishes that for a risk averse worker with HARA utility function, condition (U) holds strictly if and only if $\gamma < 1$, i.e., there is DARA. The condition holds with opposite inequality when there is IARA and $\gamma > 1$. ■

Proof of Proposition 3

Proof. All the cases can immediately be verified from Proposition 3, except for the case of CARA. There, $u'(c) = \alpha e^{-\alpha c}$, $u''(c) = -\alpha^2 e^{-\alpha c}$, so that condition (U) becomes:

$$\begin{aligned} \alpha e^{-\alpha c_e} (\alpha e^{-\alpha c_e} - \alpha e^{-\alpha c}) &< -\alpha^2 e^{-\alpha c_e} (1 - e^{-\alpha c_e} - 1 + e^{-\alpha c}) \\ e^{-\alpha c_e} - e^{-\alpha c} &< -(-e^{-\alpha c_e} + e^{-\alpha c}) \end{aligned}$$

which holds with equality. ■

Proof of Proposition 4

Proof. It is immediate that this condition is not satisfied when $u''' = 0$. To see this, observe that then $u'(c_e) - u'(Ra') = wu''(c_e)$ and the condition (U) can be written as $u'(c_e)wu''(c_e) <$

$u''(c_e)[u(c_e) - u(Ra')]$, or $u'(c_e)w > u(c_e) - u(Ra')$. This condition only holds under convexity of u , and therefore is never satisfied for risk averse agents.

When $u''' < 0$, we have instead that $u'(c_e) - u'(Ra') > wu''(c_e)$, so the left hand side is even smaller, and again, condition (U) implies $u'(c_e)w > u(c_e) - u(Ra')$, which is not satisfied for risk averse agents.

Now consider $u''' > 0$. Then we can write the utility function and its derivative as

$$\begin{aligned} u(c) &= u(c_e) + u'(c_e)(c - c_e) + \frac{u''(c_e)}{2}(c - c_e)^2 + \dots \\ u'(c) &= u'(c_e) + u''(c_e)(c - c_e) + \frac{u'''(c_e)}{2}(c - c_e)^2 + \dots \end{aligned}$$

and therefore condition (U) becomes:

$$\begin{aligned} u'(c_e) \left[u''(c_e)(c_e - c) - \frac{u'''(c_e)}{2}(c_e - c)^2 + \frac{u^{(4)}(c_e)}{6}(c_e - c)^3 - \dots \right] < \\ u''(c_e) \left[u'(c_e)(c_e - c) - \frac{u''(c_e)}{2}(c_e - c)^2 + \frac{u'''(c_e)}{6}(c_e - c)^3 - \dots \right]. \end{aligned}$$

Canceling terms and dividing by $(c_e - c)^2$, this condition implies that at least for small $c_e - c = w$ implies

$$u'''(c_e) > \frac{u''(c_e)^2}{u'(c_e)}.$$

This is equivalent to requiring that the coefficient of risk aversion $A(c) = -\frac{u''}{u'}$ is decreasing, i.e., $A' = -\frac{u'''u' - (u'')^2}{u'^2}$ or $u''' > \frac{(u'')^2}{u'} = -u''A(c)$. ■

A Counter Example

Example 1 Let w be large enough and find a u -function with u''' suitably chosen such that the condition is not satisfied. Let $u(c)$ be defined as:

$$u(c) = u(c_e) + u'(c_e)(c - c_e) + \frac{u''(c_e)}{2}(c - c_e)^2 + \frac{u'''(c_e)}{6}(c - c_e)^3 + \frac{u^{(4)}(c_e)}{24}(c - c_e)^4.$$

Evaluating u at $c = Ra'$ and observing that $c_e - Ra' = w$ we can then write

$$\begin{aligned} u(c_e) - u(Ra') &= u'(c_e)w - \frac{1}{2}u''(c_e)w^2 + \frac{1}{6}u'''(c_e)w^3 - \frac{1}{24}u^{(4)}(c_e)w^4 \\ u'(c_e) - u'(Ra') &= u''(c_e)w - \frac{1}{2}u'''(c_e)w^2 + \frac{1}{6}u^{(4)}(c_e)w^3. \end{aligned}$$

Now we can write condition (U) as (where u denotes $u(c_e)$):

$$\begin{aligned} u' \left[u''w - \frac{1}{2}u'''w^2 + \frac{1}{6}u''''w^3 \right] &< u'' \left[u'w - \frac{1}{2}u''w^2 + \frac{1}{6}u'''w^3 - \frac{1}{24}u''''w^4 \right] \\ u'u''' &> u''^2 - \frac{1}{3}u''u'''w + \frac{1}{3}u''''w \left[u' + \frac{1}{4}u''w \right] \end{aligned}$$

Observe that $u'u''' > u''^2$ is the standard condition for Decreasing Absolute Risk Aversion. But for any $u''' > 0$, however large, we can find a utility function with $\frac{1}{3}u''''w \left[u' + \frac{1}{4}u''w \right]$ sufficiently large such that the inequality is not satisfied. For example, if $u' + \frac{1}{4}u''w > 0$ we can choose u'''' positive and large. Conversely, if $u' + \frac{1}{4}u''w < 0$ we can choose u'''' sufficiently negative such that the inequality does not hold.

Proof of Proposition 7

Proof. The solution to the (interior) maximization problem is $a'^*(a, y, \pi)$, $\lambda^*(a, y, \pi)$ and satisfies:

$$\begin{aligned} -u'(a - a') + \beta q R u'(c_e) + \beta(1 - q) R \Phi_{a'}(R a') &= 0 \\ \beta q' \left[\frac{1}{1 - \beta} u(c_e) - \Phi(R a') \right] + \beta u'(c_e) \frac{m' \pi}{\lambda m} &= 0 \end{aligned}$$

Now we have monotone matching of a in y provided: $\Phi_{ay} > \frac{\Phi_y}{\Phi_\pi} \Phi_{a\pi}$.

$$\begin{aligned} \Phi_y &= \frac{\beta}{1 - \beta} q u'(c_e) v_y + \Phi_{a'} a'_y + \Phi_\lambda \lambda_y = \beta q u'(c_e) v_y \\ \Phi_a &= u'(a - a') + \Phi_{a'} a'_a + \Phi_\lambda \lambda_a = u'(a - a') \\ \Phi_\pi &= -\frac{\beta}{1 - \beta} q u'(c_e) \frac{1 - \beta(1 - m)}{m} + \Phi_{a'} a'_\pi + \Phi_\lambda \lambda_\pi = -\frac{\beta}{1 - \beta} q u'(c_e) \frac{1 - \beta(1 - m)}{m} \\ \Phi_{ya} &= -u''(a - a') a'_y \\ \Phi_{a\pi} &= -u''(a - a') a'_\pi \end{aligned}$$

where $\Phi_{a'} = 0$ and $\Phi_\lambda = 0$ from the envelope theorem. Then:

$$\begin{aligned} \Phi_{ay} &> \frac{\Phi_y}{\Phi_\pi} \Phi_{a\pi} \\ -u''(a - a') a'_y &> \frac{\frac{\beta}{1 - \beta} q u'(c_e) v_y}{\frac{\beta}{1 - \beta} q u'(c_e) \frac{1 - \beta(1 - m)}{-m}} (-u''(a - a') a'_\pi) \\ a'_y &> -\frac{m}{1 - \beta(1 - m)} v_y a'_\pi \end{aligned}$$

Writing the Hessian $|\mathbf{H}| > 0$ as:

$$|\mathbf{H}| = \begin{vmatrix} \phi_{a'a'} & \phi_{a'\lambda} \\ \phi_{\lambda a'} & \phi_{\lambda\lambda} \end{vmatrix}$$

then

$$a'_y = \frac{\partial a'}{\partial y} = -\frac{\begin{vmatrix} \phi_{a'y} & \phi_{a'\lambda} \\ \phi_{\lambda y} & \phi_{\lambda\lambda} \end{vmatrix}}{|\mathbf{H}|} \quad \text{and} \quad a'_\pi = \frac{\partial a'}{\partial \pi} = -\frac{\begin{vmatrix} \phi_{a'\pi} & \phi_{a'\lambda} \\ \phi_{\lambda\pi} & \phi_{\lambda\lambda} \end{vmatrix}}{|\mathbf{H}|}$$

$$\begin{aligned} a'_y &> -\frac{m}{1-\beta(1-m)}v_y a'_\pi \\ \phi_{a'y}\phi_{\lambda\lambda} - \phi_{\lambda y}\phi_{a'\lambda} &< -\frac{m}{1-\beta(1-m)}v_y (\phi_{a'\pi}\phi_{\lambda\lambda} - \phi_{\lambda\pi}\phi_{a'\lambda}) \\ \left(\phi_{a'y} + \frac{m}{1-\beta(1-m)}v_y\phi_{a'\pi}\right)\phi_{\lambda\lambda} &< \left(\phi_{\lambda y} + \frac{m}{1-\beta(1-m)}v_y\phi_{\lambda\pi}\right)\phi_{a'\lambda} \end{aligned} \quad (7)$$

Observe that from the first order conditions to the (interior) maximization problem, we obtain the cross partials on ϕ . First, note that:

$$\phi_{a'y} = -\frac{v_y m}{1-\beta(1-m)}\phi_{a'\pi} = v_y q u''(c_e)$$

so that the LHS is zero. Then deriving the expression for $\phi_{\lambda y}$ and $\phi_{\lambda\pi}$

$$\begin{aligned} \phi_{\lambda y} &= \beta q' \frac{1}{1-\beta} u'(c_e) v_y + \beta u''(c_e) v_y \frac{m'\pi}{\lambda m} \\ \phi_{\lambda\pi} &= \beta q' \frac{1}{1-\beta} u'(c_e) \left(-\frac{1-\beta(1-m)}{m}\right) + \beta u'(c_e) \frac{m'}{\lambda m} + \beta u''(c_e) \left(-\frac{1-\beta(1-m)}{m}\right) \frac{m'\pi}{\lambda m}, \end{aligned}$$

the RHS reduces to $\beta u'(c_e) \frac{m'}{\lambda(1-\beta(1-m))} v_y \phi_{a'\lambda}$. Therefore, the inequality (7) is satisfied provided $\phi_{a'y} > 0$.

$$\begin{aligned} \phi_{a'\lambda} &= \beta R q' [u'(c_e) - \Phi_{a'}(Ra')] + \beta R u''(c_e) \frac{m'\pi}{\lambda m} \\ &= \beta R q' [u'(c_e) - \Phi_{a'}(Ra')] - \beta R u''(c_e) q' \frac{\frac{1}{1-\beta} u(c_e) - \Phi(Ra')}{u'(c_e)}, \end{aligned}$$

since from the FOC for λ

$$\frac{m'\pi}{\lambda m} = -q' \frac{\frac{1}{1-\beta} u(c_e) - \Phi(Ra')}{u'(c_e)}.$$

Therefore $\phi_{a'\lambda} > 0$ provided

$$\frac{u'(c_e) - \Phi_{a'}(Ra')}{\frac{1}{1-\beta}u(c_e) - \Phi(Ra')} < \frac{u''(c_e)}{u'(c_e)},$$

where Φ' denotes the total derivative with respect to the state variable, assets. Finally, observe that $\Phi_{a'}(Ra') = u'(Ra' - a'')$ and

$$\Phi(a, y, \pi) = \max_{a', \lambda} \left\{ u(a - a') + \beta \left[q \frac{1}{1-\beta} u \left((1-\beta)Ra' + v(y) - \pi \frac{1-\beta(1-m)}{m} \right) + (1-q)\Phi(Ra') \right] \right\}$$

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