Uncertainty and Investment Options

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Abstract

This paper develops a simple model in which uncertainty about a future tax change leads to a temporary reduction in investment. When the uncertainty is resolved, investment recovers, generating a temporary boom.

Legislative bodies rarely act quickly, and during the period when a piece of new legislation is being formulated, there can be substantial uncertainty about its final form. If the legislation involves tax rates or other matters (trade policy, financial regulation) that affect the profitability of investments, this uncertainty increases the option value of delaying decisions.

This paper develops a simple model in which uncertainty about future tax policy leads to a temporary reduction in investment. The basic idea is that policy uncertainty creates uncertainty about the profitability of investment. If the uncertainty is likely to be resolved in the not-too-distant future, firms rationally delay committing resources to irreversible projects, reducing current investment. When the uncertainty is resolved, investment recovers, generating a temporary boom. The size of the boom depends on the realization of the fiscal uncertainty, with lower realizations of the tax rate producing larger booms.

The model here is formulated in terms of tax policy and business investment, but the idea could as well be applied to business hiring decisions and household decisions about purchases of housing and other durables, and to uncertainty about financial regulation, trade policy, energy policy, and other matters that affect the profitability/desirability of various types of investment.

The mechanism studied here may be most important in prolonging and amplifying the consequences of other shocks. If a financial crisis produces a severe downturn, the private sector may wait for legislative decisions about whether to turn to fiscal stimulus, what form it will take, and how it will be financed. If the fiscal stimulus is ineffective, investors may wait again for decisions about a second round. If central bankers and political leaders stall on decisions about how to deal with a currency crisis or a potential default on sovereign debt, investors may choose to delay until the main outlines of a policy have been agreed upon.

In the model studied here investment has two inputs, projects and cash. Projects can be thought of as specific investment opportunities. For a retail chain or a service provider, projects might be cities or locations where new outlets could be built. For a manufacturing firm, a project might be the construction of a new plant. For a real estate developer, a project might be a parcel of land that could be built on. The key feature of a project is that it is an investment opportunity not available to others: it is exclusive to one particular investor. This feature is important in generating delay: the investor is willing to delay because he does not have to worry that someone else will exploit the opportunity if he waits.

Both projects and liquid assets can be stored, and delay is here defined as a situation where the stocks of both inputs are positive. When is delay an optimal strategy? In the model here, delay occurs if and only if there is uncertainty about future policy.

That is, uncertainty necessarily produces a period of delay, although that period may not occur immediately if the uncertainty is in the distant future. Conversely, anticipated policy changes can lead investors to store one input or the other, but unless there is uncertainty they will not store both.

Although there is a vast literature on investment under uncertainty, most of it focuses on uncertainty about idiosyncratic shocks to the demand for a firm's product or to its cost of production. Recent work extends this literature to look at the aggregate effects of increased variance in these idiosyncratic shocks.[1]

Most closely related to the model here are papers by Cukierman (1980) and Bernanke (1983). Cukierman looks at the decision problem of an individual firm with a single investment opportunity. The project is characterized by an unknown scale parameter, drawn from a known distribution. Each period the firm receives a signal about the parameter and updates its beliefs. The firm must decide when to invest—how long to wait and receive more information—and how much to invest. The paper shows that an increase in the variance of the distribution from which the parameter is drawn either increases or leaves unchanged the number of periods that investment is delayed.

Bernanke (1983) looks at a dynamic inference model, in which investment opportunities arrive every period and the underlying distribution from which these are drawn is, at random dates, replaced with a new one. When this happens, investors learn about it slowly, by observing the outcomes of previous investment decisions. There-

[1] See Dixit and Pindyck (1994) and Stokey (2008) for more detailed discussions. More recently, Bloom (2009) and Arellano, Bai, and Kehoe (2011) develop aggregate models with idiosyncratic shocks to firm-level productivity. In these models more uncertainty means a higher variance for the distribution of shocks, and changes in the variance affects aggregate investment. In Bloom's model the effects of more uncertainty come through fixed costs of investment, while in ABK they come through financing constraints. Lee (2012) looks at a setting in which investment opportunities must be created, and higher idiosyncrataic volatility has a beneficial effect by inducing investors to create more opportunities and then set a high threshold for selection.

fore, after a switch occurs there is likely to be at least one period when investors are very uncertain which distribution is in place. The paper provides an example in which the switch from one distribution to another necessarily produces at least one period in which investors adopt a "wait and see" strategy and no investment takes place.

The empirical analysis in Baker, Bloom and Davis (2011) provides some confirmation of the idea that a temporary increase in policy uncertainty may depress investment. They construct an index of policy uncertainty that averages information from news media, the number of federal tax code provisions set to expire, and the extent of forecaster disagreement over future inflation and federal government purchases. Their VAR estimates show that an increase in policy uncertainty equal to the actual increase between 2006 and 2011 leads to a 13% decline in private investment. Fernández-Villaverde, et. al. (2011) look at a model with time-varying volatility and also find support for the adverse effect of an increase in fiscal volatility on economic activity.

The rest of the paper is organized as follows. Section 1 provides an overview of the model. In section 2 a benchmark model with no storage is studied, and two tax experiments are discussed. Section 3 sets out the full model in more detail, and studies the transitional dynamics after the announcement of a policy reform. The main result is Proposition 3, which shows that uncertainty about the new tax policy necessarily leads to delay. Section 4 looks at a numerical example. Section 5 extends the model to allow a Poisson arrival date, and Proposition 4 shows that the main result carries over, provided the arrival rate is not too small. Section 6 concludes. The proofs of all propositions are in Appendix A.

1. OVERVIEW

The model uses an investment technology designed to produce an option value.[2] Briefly, the key features are that each investment opportunity is specific to one firm, there is an intensity decision that is irreversible, and there are storage possibilities that permit delay. This rest of this section describes these features in more detail.

First, investment requires a project, as well as an input of cash. As noted above, projects should be thought of as specific investment opportunities. For retailers they might be new locations for outlets, for manufacturers they might be new plants, and so on. The key feature of a project is that it is available only to one particular investor. Hence that investor can wait to make a decision about how best to exploit it. This exclusivity assumption could be relaxed to some extent. For example, similar conclusions would hold if projects had a positive hazard rate of becoming available to other investors. The assumption cannot be dropped altogether, however. If a specific project were immediately available to multiple investors, there would be Bertrand-like competition to be the first to exploit it, precluding the possibility of delay.

Second, the intensity of investment in a project is an irreversible decision. Specifically, all investment in a particular project must take place at a single date: the capital cannot be increased or decreased later on. Thus, investment intensity has a putty-clay character: it is flexible ex ante but fixed ex post. This feature could also be relaxed. A model with costly reversibility would deliver similar conclusions, at the cost of added complexity.

Third, the firm can store projects. Here it is assumed that stored projects do not depreciate. A positive depreciation rate could be incorporated, but storage is key for creating an option value.

Fourth, the total cost of investment is linear in the number of projects for a fixed

[2]It is similar to the "seeds" model in Jovanovic (2009).

intensity and strictly convex in the intensity for a fixed number of projects. Strict convexity in the intensity is critical for making projects a valuable commodity. Without it, investment could be concentrated on a small set of projects, at no additional cost.

Finally, the firm cannot borrow, although it can hold stocks of liquid assets. The interest rate on liquid assets is less than the discount rate for dividends, however. Thus, in the absence of uncertainty holding liquid assets is unattractive, and the firm pays out dividends as quickly as possible. But in the presence of uncertainty, liquid assets can be attractive as a temporary investment while waiting for the uncertainty to be resolved. The assumption of a low interest rate on liquid assets makes the firm's dividend policy determinate, and the no-borrowing assumption is primarily for convenience. Allowing the firm to borrow at an interest rate higher than the discount rate would make the firm's financing decision more complex, and reduce or eliminate the incentive to hold liquid assets. Investment decisions would be qualitatively the same however, and the firm would borrow only to accelerate investment.

Formally, time is continuous, and new projects arrive at a constant rate μ. At each date a firm chooses n, the number of projects (a flow), and i, the intensity of investment in each project. Total investment is the product $I = ni$ (a flow). The cost of implementing a project with intensity $i > 0$ is $g(i)$, where the function g is strictly increasing and strictly convex. Hence cost minimization implies that the firm chooses the same intensity for all projects implemented at the same date, and investing at the total rate $I = ni$ has total cost $ng(i) = ng(I/n)$ (a flow) if it is allocated across n projects. If projects and liquid assets have been accumulated, there may also be a discrete investment with intensity $\hat{\imath}$, where the number of projects $\hat{n}$, the increment to capital $\hat{n}\hat{\imath}$, and the total cost $\hat{n}g(\hat{\imath})$ are masses. This type of investment is discussed in more detail later.

The role played by projects can be seen in a two-period example, $t = 1, 2$, with a project inflow of $\mu = 1$ in periods $t = 1, 2$, and k_0 given. The firm chooses the investment scale in each period, $0 \leq n_1 \leq 1$ and $0 \leq n_2 \leq 2 - n_1$, as well as the intensities, $i_1, i_2 \geq 0$. The capital stock and total investment costs are then

$$\begin{aligned} k_t &= (1-\delta)\, k_{t-1} + n_t i_t, \qquad t = 1, 2, \\ TC &= n_1 g(i_1) + \frac{1}{1+r} n_2 g(i_2). \end{aligned}$$

The option model coincides with the usual model with convex investment costs if the firm uses the projects evenly over the two periods, if $n_1 = n_2 = 1$. But if the firm chooses to concentrate investment in the second period, they are different. For example, if total investments are $I_1 = 0$ and $I_2 > 0$, the option model reduces total cost from $g(I_2)$ to $2g(I_2/2)$. Because it allows (forward) smoothing over time, the option model reduces the cost of delay.

Installed capital k produces the revenue flow $\pi(k)$ (net of variable costs) and depreciates at the constant rate $\delta > 0$. The revenue from installed capital is taxed at a flat rate τ, and uncertainty about τ is the only risk the firm faces. We will study the effect of uncertainty about a one-time tax reform that is expected in the future. Two slightly different versions are considered. In the first version the date T of the tax reform is known, and only the new tax rate is uncertain. In the second version the date of the reform is also random, with a Poisson arrival time. In both cases the new tax rate $\hat{\tau}$ is drawn from a known distribution $F(\cdot)$.

The following assumptions are maintained throughout:

—dividends are discounted at the constant rate $\rho > 0$;

—liquid assets held by the firm earn interest at the constant rate $0 \leq r < \rho$;

—the firm cannot borrow: all investment is from retained earnings, and the dividend must be nonnegative;

—capital cannot be sold: gross investment must be nonnegative;

—the firm receives a constant flow $\mu > 0$ of new projects;

—the revenue function π is strictly increasing, strictly concave, and twice differentiable, with $\pi(0) = 0$, $\pi'(0) = \infty$, and $\lim_{k\to\infty} \pi'(k) = 0$;

—the cost function g is strictly increasing, strictly convex, and twice differentiable, with $g(0) = 0$, $g'(0) \geq 0$, and $\lim_{i\to\infty} g'(i) = \infty$;

—the time horizon is infinite.

2. THE MODEL AND THE TRANSITION AFTER T

Consider a one-time tax reform, announced at date $t = 0$, that will take effect at date $T > 0$. There are no changes in the tax rate during the interval $[0, T)$, and after the single reform there are no further changes. The new tax rate is not announced at $t = 0$, however. Instead, the firm knows only that it will, at T, be drawn from a known distribution F. Here we will assume that T is fixed. Section 5 looks at an extension where T is also unknown.

In the option model the state variable for the firm is $s = (k, a, m)$, where $k > 0$ and $a, m \geq 0$ are its stocks of capital, liquid assets, and projects. We will compare the firm's optimal strategy in the option model with its behavior in a benchmark model where projects cannot be stockpiled. In the benchmark model the state variable is just (k, a).

The goal is to characterize the firm's optimal investment over $[0, T)$, in anticipation of the change, and after T, when the new rate is in effect. As usual, it is convenient to start by looking at the continuation value after T.

a. The firm's problem at T

Consider the firm's problem at date T, after the resolution of the uncertainty. The post-reform tax rate $\hat{\tau}$ is known, and there will be no further changes. Consider the optimal investment path for a firm with initial state s_T, where $k_T > 0$ and $a_T, m_T \geq 0$.

If $a_T, m_T > 0$, the firm can make a one-time discrete adjustment (DA), using some or all of its stocks of cash and projects to produce an increment to its capital stock. In particular, it can invest in a mass of projects $\hat{n} \geq 0$, with intensity $\hat{\imath} \geq 0$, producing a mass of new capital goods $\hat{I} = \hat{n}\hat{\imath}$. The cost of this investment, $\hat{n}g(\hat{\imath})$, must be financed out of its stock of liquid assets. In addition, the firm can use any remaining liquid assets to pay a discrete dividend $\hat{D}$. After these one-time adjustments, if any, the firm faces a standard control problem.

Thus, the firm's problem is to choose $\left(\hat{D}, \hat{\imath}, \hat{n}\right)$ and the subsequent dividend flow and investment intensity $\{D(t), i(t), n(t)\}_{t=T}^{\infty}$ to maximize the PDV of total dividends, discounted at the rate $\rho > 0$. Let $v(s_T; \hat{\tau})$ denote the maximized value of the firm,

$$v\left(s_T; \hat{\tau}\right) \equiv \max \left[\hat{D} + \int_T^{\infty} e^{-\rho(t-T)} D(t) dt\right] \tag{1}$$

$$\begin{aligned} \text{s.t.} \qquad \hat{k}_T &= k_T + \hat{n}\hat{\imath}, \\ \hat{a}_T &= a_T - \hat{D} - \hat{n}g(\hat{\imath}), \\ \hat{m}_T &= m_T - \hat{n}, \\ 0 &\leq \hat{D}, \hat{\imath}, \hat{n}, \hat{a}_T, \hat{m}_T, \end{aligned} \tag{2}$$

$$\begin{aligned} \dot{k} &= ni - \delta k, \\ \dot{a} &= ra + (1 - \hat{\tau})\,\pi(k) - D - ng(i), \\ \dot{m} &= \mu - n, \\ 0 &\leq D, i, n, a, m, \qquad\qquad \text{all } t > T, \end{aligned} \tag{3}$$

where $\hat{s}_T = \left(\hat{k}_T, \hat{a}_T, \hat{m}_T\right)$ denotes the firm's state after the DA. Letting q_k, q_a, q_m, denote the costates for the three state variables, the discrete adjustment $\left(\hat{D}, \hat{\imath}, \hat{n}\right)$

satisfies[3]

$$\begin{aligned}
1 &\leq q_{aT}, && \text{w/ eq. if } \hat{D} > 0, \\
q_{kT} &\leq q_{aT} g'(\hat{\imath}), && \text{w/ eq. if } \hat{n}\hat{\imath} > 0, \\
q_{kT}\hat{\imath} &\leq q_{mT} + q_{aT} g(\hat{\imath}), && \text{w/ eq. if } \hat{n}\hat{\imath} > 0, \qquad \text{all } \hat{\tau},
\end{aligned} \tag{4}$$

where $q_{kT}(\hat{\tau})$, $q_{aT}(\hat{\tau})$, and $q_{mT}(\hat{\tau})$ are the costate values at date T, after $\hat{\tau}$ is realized. Thereafter the solution satisfies

$$\begin{aligned}
1 &\leq q_a, && \text{w/ eq. if } D > 0, \\
q_k &\leq q_a g'(i), && \text{w/ eq. if } ni > 0, \\
q_k i &\leq q_m + q_a g(i), && \text{w/ eq. if } ni > 0, \qquad \text{all } t > T,
\end{aligned} \tag{5}$$

$$\begin{aligned}
\dot{q}_k &= (\rho + \delta)\, q_k - q_a\,(1 - \hat{\tau})\,\pi'(k), \\
\dot{q}_a &\leq (\rho - r)\, q_a, && \text{w/ eq. if } a > 0, \\
\dot{q}_m &\leq \rho q_m, && \text{w/ eq. if } m > 0, \qquad \text{all } t > T,
\end{aligned} \tag{6}$$

and the transversality conditions $\lim_{t\to\infty} e^{-\rho t} q_x(t) x(t) = 0$, $x = k, a, m$.

The benchmark firm faces a similar problem except that it cannot accumulate projects. Hence it cannot make a DA to its capital stock at T, and after T its scale of investment is equal to the inflow of projects at every date. Thus, its problem is as in (1) - (3), but with $m_T = 0$, $\hat{n} = 0$, $[n(t) \equiv \mu,\ m(t) \equiv 0,\ \ t > T]$. The solution is as in (4) - (6), except that q_m does not appear, and we drop the second and third lines in (4) and the third lines in (5) - (6). We will call this the benchmark system and let $w\,(k_T, a_T; \hat{\tau})$ denote the maximized value of the benchmark firm.

[3]See Kamien and Schwartz (1991) or Seierstad and Sydsaeter (1977) for a detailed discussion of these conditions.

The option and benchmark models have them same steady state (SS), which is unique and has $a^{ss} = 0$, $m^{ss} = 0$, and

$$
\begin{aligned}
(1-\tau)\,\pi'\left(k^{ss}\right) &= (\rho+\delta)\,g'(i^{ss}), \\
k^{ss} &= \frac{\mu}{\delta} i^{ss}, \\
D^{ss} &= (1-\tau)\,\pi(k^{ss}) - \mu g(i^{ss}),
\end{aligned}
\tag{7}
$$

where k^{ss}, i^{ss}, and D^{ss} are strictly decreasing in τ. As will be shown below, for any $k_T > 0$ and $a_T, m_T \geq 0$, the solution to (1) - (3) converges asymptotically to the SS.

Before proceeding, it is useful to bound the state space and the range for tax rates. Let $\overline{k} = k^{ss}(0)$ be the SS capital stock when the tax is $\tau = 0$. Only nonnegative tax rates are of interest, so $\overline{k}$ is a natural upper bound on the set of capital stocks to consider. Then define $\overline{\tau} > 0$ as the tax rate for which investment to maintain the capital stock at $\overline{k}$ just exhausts after-tax profits,

$$(1-\overline{\tau})\,\pi(\overline{k}) - \mu g(\delta\overline{k}/\mu) = 0.$$

For any lower tax rate and smaller capital stock, $\tau \in [0, \overline{\tau}]$ and $k \in (0, \overline{k}]$, after-tax revenue is sufficient to finance investment to maintain the capital stock.

b. The transition after T in the benchmark model

Proposition 1 describes the solution to the benchmark version of (1) - (3), which has a partial 'bang-bang' form with two critical values for capital, $0 < k_1^{\text{crit}} < k^{ss} < k_2^{\text{crit}} \leq \infty$. While the capital stock $k(t)$ is below the first critical value, all earnings are invested and the dividend is zero. While $k(t)$ is above the second critical value, all earnings are paid out as dividends and there is no investment. While $k(t)$ is between the two critical values, both investment and the dividend are positive.

If the initial asset stock is zero, liquid assets are never acquired. If initial assets are positive and $k_T \geq k_1^{\text{crit}}$, the entire asset stock is paid as an initial dividend, $\hat{D} = a_T$, and rest of the solution is unchanged. If $k_T < k_1^{\text{crit}}$, the initial dividend is less than initial assets, $0 \leq \hat{D} < a_T$, and the remaining assets are used to finance investment.

Proposition 1: For any $\hat{\tau} \in [0, \overline{\tau}]$, $k_T \in (0, \overline{k}]$, and $a_T \geq 0$, the solution to the benchmark version of (1)-(3) has the following properties:

(a) If $k_T < k^{ss}$, then $i(t) > 0$ and $k(t)$ is strictly increasing along the transition path. There is a critical value $0 < k_1^{\text{crit}} < k^{ss}$ with the property that $D(t) = 0$ and $i(t)$ is strictly increasing while $k(t) < k_1^{\text{crit}}$, and $D(t) > 0$ is strictly increasing and $i(t)$ is strictly decreasing while $k(t) > k_1^{\text{crit}}$.

If $k_T > k^{ss}$, then $k(t)$ and $D(t) > 0$ are strictly decreasing along the transition path. If $g'(0) > 0$, there is a critical value $k_2^{\text{crit}} > k^{ss}$ with the property that $i(t) = 0$ while $k(t) \geq k_2^{\text{crit}}$, and $i(t) > 0$ is strictly increasing while $k(t) < k_2^{\text{crit}}$. If $g'(0) = 0$, then $k_2^{\text{crit}} = +\infty$.

(b) If $a_T = 0$, the stock of assets remains at zero throughout the transition, $a(t) \equiv 0$.

If $a_T > 0$ and $k_T \geq k_1^{\text{crit}}$, the initial dividend is $\hat{D} = a_T$, and the rest of the transition is as above.

If $a_T > 0$ and $k_T < k_1^{\text{crit}}$, there is a critical value $\alpha(k_T) > 0$ for liquid assets with the following property. For $0 < a_T < \alpha(k_T)$, there is no initial dividend: $\hat{D} = 0$ and $\hat{a}_T = a_T$. For $a_T \geq \alpha(k_T)$, the initial dividend is $\hat{D} = a_T - \alpha(k_T)$, and $\hat{a}_T = \alpha(k_T)$. In either case the stock of assets is exhausted while $k(t) < k_1^{\text{crit}}$ and remains at zero thereafter.

Figure 1 shows the phase diagram in (k, q_k) −space, where $a(t) \equiv 0$ and q_a is not displayed. The value $\overline{k} \equiv k^{ss}(0)$ is indicated on the horizontal axis. Above the threshold $\chi(k)$, the broken curve, the firm is cash constrained, with $D = 0$ and $q_a > 1$. The $\dot{k} = 0$ and $\dot{q}_k = 0$ loci are the dotted curves, and the stable manifold SM_0 is the solid curve. The critical value k_1^{crit} is defined by the intersection of SM_0 and $\chi(k)$. In this example $k_2^{\text{crit}} >> \overline{k}$, so the region where $i^* = 0$ does not appear in the figure.

c. The transition after T in the option model

Next consider the post-reform behavior of the firm with the storage option. The following proposition describes one key feature of the dynamics.

Proposition 2: For any $\hat{\tau} \in [0, \overline{\tau}]$, $k_T \in (0, \overline{k}]$, and $a_T, m_T \geq 0$, the solution to (1) - (3) has the property that the choice $\left(\hat{D}, \hat{\imath}, \hat{n}\right)$ exhausts at least one of the input stocks: $\hat{n} = m_T$ or $\hat{n}g(\hat{\imath}) + \hat{D} = a_T$ or both.

Thus, the transition after the DA has $\hat{a}_T = 0$ or $\hat{m}_T = 0$ or both. Before analyzing the DA, it is useful to characterize the post-DA transition. The next results extends Proposition 1 to the option model.

Proposition 3: For any $\hat{\tau} \in [0, \overline{\tau}]$, $k_T \in (0, \overline{k}]$, $a_T \geq 0$, and $m_T \geq 0$, with $a_T m_T = 0$, the solution to (1)-(3) is as follows. The capital stock converges monotonically to the steady state, and in addition, the solution has the following properties:

(a) If $\hat{a}_T = 0$ and $\hat{m}_T > 0$, liquid assets are never accumulated, and the initial stock of projects is exhausted in finite time.

(b) If $\hat{a}_T = 0$ and $\hat{m}_T = 0$, there exist thresholds k_A, k_B, with $k_A < k^{\text{crit}} < k^{ss}$ and $k^{ss} < k_B < k_2^{\text{crit}}$ such that the solution is as in Proposition 1 for $\hat{k}_T \in [k_A, k_B]$. For $\hat{k}_T < k_A$ and $\hat{k}_T > k_B$, the solution involves accumulating and then decumulating projects. Liquid assets are never held, $a(t) = 0$, all t.

(c) If $\hat{a}_T > 0$ and $\hat{m}_T = 0$, the solution is as in Proposition 1 if $k(t) \geq k_A$ at the date when the liquid assets are exhausted. Otherwise, the solution has four phases:

(i) with $n = \mu$, $\dot{a} < 0$, and $D = 0$;

(ii) with $n < \mu$, $a = 0$, and $D = 0$;

(iii) with $n > \mu$, $a = 0$, and $D = 0$; and

(iv) on the SM, with $n = \mu$, $a = m = 0$, and $D > 0$.

The qualitative properties of the DA depend on the realization of $\hat{\tau}$. For a fixed initial condition with positive stocks of all assets, $k_T, a_T, m_T > 0$ Figure 2 shows, qualitatively, how $\hat{n}, \hat{\imath}, \hat{n}g(\hat{\imath}), \hat{D}$, and the initial costate value $\hat{q}_{aT}$ vary with $\hat{\tau}$. As shown there, the support of F can be divided into three regions, with different qualitative behavior for the transition. Note that the description of $\hat{\tau}$ as low, moderate or high means relative to other values in the support of F. The initial tax rate τ may be higher or lower than all these values.

For low realizations of $\hat{\tau}$, Region A in Figure 2, the firm is cash constrained. The DA exhausts the firm's stock of liquid assets, $\hat{n}g(\hat{\imath}) = a_T$, but some projects remain, $\hat{n} < m_T$. No initial dividend is paid, $\hat{D} = 0$, and cash is at a premium, $\hat{q}_{aT} > 1$. Let $\Delta_T > 0$ denote the length of time required to exhaust the remaining stock of projects. During this period no dividend is paid, and all earnings are used for investment. At $T + \Delta_T$ the solution lies on $SM_0(\hat{\tau})$, and the remaining transition is as in the benchmark model. In this region $\hat{n}$ is strictly increasing in $\hat{\tau}$, while $\hat{\imath}$, $\hat{q}_{aT}$ and Δ_T are strictly decreasing. At the threshold value for $\hat{\tau}$ separating this region from the next, $\hat{n} = m_T$, $\hat{q}_{aT} = 1$, and $\Delta_T = 0$.

For moderate realizations of $\hat{\tau}$, Region B in Figure 2, the DA uses the entire stock of projects, $\hat{n} = m_T$, and the firm has more than enough assets to finance investment at the desired intensity. Excess cash remains and is paid as a dividend, $\hat{D} = a_T - \hat{n}g(\hat{\imath}) > 0$, which implies $\hat{a}_T = 0$ and $\hat{q}_{aT} = 1$. The DA puts the state on $SM_0(\hat{\tau})$, and the rest of the transition is as in the benchmark model. Since $q_{aT} = 1$, the post-DA capital stock must satisfy $\hat{k}_T \geq k_1^{\text{crit}}(\hat{\tau})$.

For high realizations of $\hat{\tau}$, Region C in Figure 2, neither the stock of projects nor the stock of liquid assets is exhausted by the initial investment, and the excess assets are paid as a dividend. That is, $\hat{n} < m_T$, $\hat{D} = a_T - \hat{n}g(\hat{\imath}) > 0$, and $\hat{q}_{aT} = 1$. Indeed, for $\hat{\tau}$ sufficiently large, $\hat{n} = 0$. The post-DA adjustment starts with a stock of projects $\hat{m}_T > 0$, which is used up over some period of time Δ_T, at which point (k, q_k) lies on $SM_0(\hat{\tau})$.

One important feature of the solution is clear from Figure 2: the firm's optimal strategy before T necessarily produces a positive probability of being cash constrained when $\hat{\tau}$ is realized. To see this, note that since liquid assets are acquired before T, it follows that q_a is increasing on $(0, T)$, so $q_{aT} > 1$. At date T, the post-realization value satisfies $\hat{q}_{aT} > 1$ only if the new tax rate lies in Region A. Hence (9) implies that the solution lies in Region A—where the firm is cash constrained—with strictly positive probability.

In addition, notice that while the intensity $\hat{\imath}$ of the DA is decreasing in $\hat{\tau}$ over the entire range, the scale $\hat{n}$ is increasing in Region A, constant in Region B, and decreasing in Region C. Thus, a stock of projects remains after the DA in Regions A and C. The economic motivation for holding investment options after T is different in those two regions, however. In Region A the firm is accumulating capital, but it is cash constrained. Thus, it holds some projects back in order to finance them later, out of retained earnings, at higher intensities. In Region C the firm is decumulating capital, and it is profitable to hoard some projects to reduce the cost of replacement investment later on.

3. THE FIRM'S DECISIONS ON $[0, T)$

Next consider the firm's problem at $t = 0$, when the reform is announced. Assume that the firm has no initial stocks of liquid assets or projects, $a_0 = m_0 = 0$, and that its initial capital stock is at or below the steady state for the old tax rate, $k_0 \leq k^{ss}(\tau)$. Values for k_0 near $k^{ss}(\tau)$ represent mature firms, while smaller values represent younger firms. The new tax rate $\hat{\tau}$, which takes effect at T, is drawn from the known distribution $F(\hat{\tau})$, and there are no further changes thereafter. If F puts unit mass on a single point, the change is deterministic.

Since there are no initial stocks of liquid assets or projects, there can be no DA or discrete dividend at $t = 0$. Hence the firm chooses $\{(D, i, n)\}_{t=0}^{T}$ to solve

$$\max \int_0^T e^{-\rho t} D(t)dt + e^{-\rho T} \mathrm{E}_{\hat{\tau}} \left[v(s(T); \hat{\tau}) \right] \qquad \text{s.t. (3)}, \tag{8}$$

where the expectation uses F. The conditions in (5) - (6) are again necessary for an optimum, but the terminal conditions are now

$$\begin{aligned} \lim_{t \uparrow T} q_k(t) &= \mathrm{E}_{\hat{\tau}} \left[q_{kT} \left(\hat{\tau} \right) \right], \\ \lim_{t \uparrow T} q_x(t) &\geq \mathrm{E}_{\hat{\tau}} \left[q_{xT} \left(\hat{\tau} \right) \right], \qquad \text{w/ eq. if } x_T > 0, \qquad x = a, m, \end{aligned} \tag{9}$$

where $q_{xT}(\hat{\tau})$, $x = k, a, m$, are the initial costate values for the problem in (1) - (3), given the initial state $(k_T, a_T, m_T) = (k(T), a(T), m(T))$ and the realized tax rate $\hat{\tau}$. Thus, the costate for capital approaching date T, before the uncertainty is resolved, must equal its expected value ex post. The costates for liquid assets and projects may exceed their expected ex post values if the stock is zero.

The qualitative behavior of the transitional dynamics is sensitive to T. Specifically, if T is large the anticipation effect is small until t is close to T. In the discussion below, we will focus on the case where T is small enough so the anticipation effect is important at $t = 0$.

a. Deterministic changes in the benchmark model

Consider a deterministic tax change in the benchmark model. The phase diagram in (k, q_k)-space is useful for studying the transitional dynamics. At $t = 0$ the firm is on the stable manifold for the old tax rate, call it $SM(\tau)$, with initial capital $k_0 \in (0, k^{ss}(\tau)]$. The announcement at $t = 0$ induces a jump in q_k, to a transition that path pulls away from $SM(\tau)$. At T the pair $(k(T), q_k(T))$ must lie on the stable manifold for the new tax rate, $SM(\hat{\tau})$. Working backward, this requirement determines the direction and size of the initial jump in q_k. Figure 3a shows two such transitions, a tax increase and a decrease, both starting at tax rate τ^M and capital stock $k^{ss}(\tau^M)$.

The phase diagram assumes that liquid assets are not accumulated, $a(t) \equiv 0$, a conjecture that is correct if and only if $\dot{q}_a/q_a \leq \rho - r$ throughout the transition. There are three cases.

In the region below the threshold $\chi(\cdot;\tau)$, the firm is not cash constrained, so $q_a \equiv 1$. Hence $\dot{q}_a/q_a = 0$, and the required condition holds.

In the region above the threshold $\chi(\cdot;\tau)$, the firm is cash constrained, so $q_a = q_k/g'(i) > 1$, where the intensity i is determined by the cash constraint. For $\tau \in [0, \overline{\tau}]$ and $k \in (0, \overline{k}]$, this region lies entirely above the $\dot{k} = 0$ locus, so k is increasing. Hence revenue is increasing over time, so i and $g'(i)$ are also increasing. In the portion of the region to the left of the $\dot{q}_k = 0$ locus, q_k is falling. Hence in that region $\dot{q}_a/q_a < 0 < \rho - r$, and the required condition again holds.

The transition path for a tax increase (of any size) or a modest tax decrease lies entirely in these two regions, so in these cases liquid assets are not accumulated.

In the region above $\chi(\cdot;\tau)$ and to the right of $\dot{q}_k = 0$, both q_k and $g'(i)$ are increasing, and the condition on $\dot{q}_a/q_a$ may fail. The transition path lies in this region if the anticipated tax change is a large decrease and k_0 is small relative to the new steady state. In these cases the firm may accumulate liquid assets. Nevertheless, if it does so, it spends them before date T. If the firm accumulates liquid assets before T, then $\dot{q}_a/q_a > 0$ before T. Thus $a(T) > 0$ implies $q_a(T) > 1$. In this case q_a jumps down at T, when the tax cut increases the profit flow and relaxes the cash constraint. But (9) implies $a(T) = 0$ if q_a jumps.

Thus, for a large tax decrease and small k_0, the solution in general has two phases, with no dividend paid in either phase. During the first phase, call it $[0, T_0]$, no assets are held. Intensity $i(t)$ is determined by the cash constraint, the costate for assets is determined by $q_a = q_k/g'(i)$, and the latter satisfies $\dot{q}_a/q_a \leq \rho - r$. During the second phase, $(T_0, T]$, the firm first acquires and then spends liquid assets, using them to boost the investment intensity as date T draws near. During the second phase the costate grows, $\dot{q}_a/q_a = \rho - r > 0$, the intensity satisfies $g'(i) = q_k/q_a$, and assets evolve according to the law of motion in (3). Liquid assets first rise and then fall over this phase, with $a(T_0) = a(T) = 0$, while the costate grows throughout, $q_a(T) = e^{(\rho-r)(T-T_0)} > 1$.

As noted above, for small tax cuts no assets are acquired: $T_0 = T$ and the second phase disappears. For large tax cuts the firm starts acquiring liquid assets immediately after the announcement: $T_0 = 0$ and the first phase disappears.

b. Deterministic changes in the option model

The solution for the option model coincides with the benchmark solution for deterministic changes that are small or moderate in magnitude, but differs for sufficiently large changes in either direction.

For a sufficiently large tax increase, the benchmark solution for a firm with initial capital $k_0 = k^{ss}(\tau)$ sets $i = 0$ on $[0, T]$, and unused projects are discarded. In the option model those projects are accumulated and the subsequent transition is altered.

In all other cases the benchmark solution has $n = \mu$ and $i > 0$ on $(0, T)$. Hence $q_k = q_a g'(i)$ and

$$\begin{aligned} q_m &= q_k i - q_a g(i), \\ \dot{q}_m &= \dot{q}_k i - \dot{q}_a g(i), \qquad t \in (0, T). \end{aligned}$$

Thus, the solution for the benchmark model is also a solution for the option model if

$$\dot{q}_m/q_m \leq \rho \qquad \text{and} \qquad \lim_{t \uparrow T} q_m(t) \geq q_{mT}(\hat{\tau}). \tag{10}$$

In the region below $\chi(\cdot;\tau)$, $\dot{q}_a = 0$ and $\dot{q}_m = \dot{q}_k i$. Hence in the region below $\chi(\cdot;\tau)$ and to the left (below) the $\dot{q}_k = 0$ locus, the first condition in (10) holds. In addition, q_a and i are continuous at T in the benchmark transition, so q_m is also continuous, and the second condition in (10) holds as well. Hence in this region the benchmark transition is also a solution for the option model. This is true for large k_0 and a tax increase of any size.

The same is true for a tax decrease that is not too large. Two conditions limit the size of the tax cut. First, it must be modest enough so that the benchmark transition stays in the region below the threshold χ. In addition, q_k may rise over $(0, T)$ for a tax decrease. In this case q_m also rises, and the cut must be small enough so that first condition in (10) holds.

In the region below $\chi(\cdot;\tau)$ and to the right (above) the $\dot{q}_k = 0$ locus, the first condition in (10) can fail. In this region q_k is growing, so q_m is also growing.

If the benchmark solution has a terminal condition at T that lies above $\chi(\cdot;k)$, the option and benchmark solutions necessarily differ. In the benchmark solution $q_a(t) > 1$ jumps down at T, and the investment intensity jumps upward, since the tax cut increases the profit flow and relaxes the cash constraint. In the option model the firm holds some projects in inventory before T and uses them after T, smoothing the intensity.

[For a very large tax cut and $(r+\delta)/(\rho+\delta)$ close to unity, it seems possible that both inputs are held before T.]

c. Stochastic changes in the benchmark model

If there is uncertainty about the new tax rate, the transition for the benchmark model is similar to the one described above, except there is a second jump in the costates and controls at date T, when the uncertainty is resolved. At date T, given k_T and the realized value $\hat{\tau}$ for the tax rate, the costate jumps so the pair $(k_T, q_{kT}(\hat{\tau}))$ lies on $SM(\hat{\tau})$. The values $q_{kT}(\hat{\tau})$ must jointly satisfy (9), and the direction and size of the jump in q_k at $t = 0$ is determined by this requirement.

As before, if the reform involves tax increases (of any size) or decreases that are not too large, no liquid assets are accumulated. If F puts sufficiently high probability on a set of large tax decreases, however, liquid assets may be acquired and then spent over $(0, T)$, as in the deterministic case.

Figure 3b shows an example where the initial tax rate is τ^M and the support of F consists of the two points τ^L and τ^H. The example, F puts a high probability on τ^H, so q_k jumps down at $t = 0$. Both k and q_k decline over $(0, T)$, so the transition lies below $\chi(\cdot; \tau^M)$, in the region where $q_a = 1$. At date T the costate jumps again, so the pair (k_T, q_{kT}) lies on $SM(\hat{\tau})$, and the rest of the adjustment follows that trajectory.

Figure 4 shows the short-run dynamics for a tax cut anticipated one year in advance. The dashed lines show the adjustment when the new rate is known, and the solid lines the adjustment if it uncertain. Uncertainty slows down the adjustment, but the qualitative behavior of the transition is not much changed.

d. Stochastic changes in the option model: the period of delay

In the option model, uncertainty about the new tax rate always leads to a period of delay: there is an interval of time before T during which investment ceases.

PROPOSITION 4: Suppose a tax change at $T > 0$, drawn from the distribution F, is announced at $t = 0$. Unless F puts unit mass at a single point, there exists $\Delta > 0$ such that $n(t)i(t) = 0$, for $t \in (T - \Delta, T)$.

The proof is by contradiction. Suppose the contrary. Because g is convex, smoothing the intensity of investment across projects reduces the total cost. Delaying some projects from before T until just after T permits this type of smoothing. Of course, if the uncertainty is small in magnitude, the period of delay is short. Thus, if T is large, the period of delay may not begin at $t = 0$.

Proposition 3 implies that $a_T, m_T > 0$, so all three conditions in (9) must hold with equality. The size of the stocks that the firm accumulates can be determined as follows.

Suppose T is not too large, and conjecture that $n(t) = 0$ on $(0, T)$. Then

$$\begin{aligned} k_T &= k^{ss}(\tau)e^{-\delta T}, \\ m_T &= \mu T. \end{aligned}$$

Since $r < \rho$, the firm does not pay a dividend while it holds liquid assets. Thus, the strategy for acquiring liquid assets is to choose a date $S \in (0, T)$. Before date S, all earnings are paid as dividends, and after S all earnings are retained. Given S, the stock of liquid assets and the value of the costate q_a at T are

$$\begin{aligned} a_T &= \int_S^T (1 - \tau)\, \pi \left[k^{ss}(\tau)e^{-\delta t}\right] e^{r(T-t)} dt > 0, \\ q_{aT} &= e^{(\rho - r)(T-S)} > 1. \end{aligned}$$

Thus, increasing S reduces both a_T and q_{aT}.

When the new tax rate is realized, a larger initial stock of liquid assets a_T reduces $\hat{q}_{aT}(\hat{\tau})$ in region A, and shrinks the size of that region. Hence $\mathrm{E}_{\hat{\tau}}\left[\hat{q}_{aT}\right]$ is a decreasing function of a_T, and there is at most one value S for which $\mathrm{E}_{\hat{\tau}}\left[\hat{q}_{aT}\right] = q_{aT}$. If no solution exists, the conjecture that $n(t) = 0$ on $(0, T)$ is incorrect, and a smaller value for T is needed.

For the other two costates, use (9) to determine q_{kT} and q_{mT}, and then use the laws of motion in (6) to construct solutions over $(0, T)$.

4. AN EXAMPLE

The example uses the revenue and cost functions

$$\pi(k) = Ak^{\alpha}, \qquad g(i) = i + \frac{1}{2}g_2 i^2,$$

and the parameter values

$$\begin{aligned} &A = 1, \quad \alpha = 0.70, \quad g_2 = 1.5, \quad \delta = 0.10, \\ &\mu = 1, \quad \rho = 0.04, \quad r = 0.03. \end{aligned}$$

The initial tax rate is $\tau = \tau^L = 0.20$, and the tax cut is anticipated $T = 2$ years in advance. The post-reform rate is

$$\hat{\tau} = \begin{cases} \tau^M = 0.22, & \text{with probability } 0.566, \\ \tau^H = 0.40, & \text{with probability } 0.434. \end{cases}$$

Thus, the tax reform could raise the tax rate by 2 or 20 percentage points.

Figure 5 shows the short run consequences of the policy. At $t = 0$ the firm has the steady state capital stock for the initial tax rate, and the initial level for each displayed variable is indicated by a black diamond on the vertical axis. The period of delay begins immediately, so $ni = 0$ over $(0, T)$, as shown in panel (a) and the capital stock declines through depreciation, as shown in panel (b). The firm continues paying a dividend for about 18 months, as shown in panel (c). Indeed the dividend is higher than before date 0, since no funds are needed for investment. After about 18 months the dividend drops to zero, and the firm starts accumulating liquid assets. Panels (e) and (f) show the stocks of projects and liquid assets, and panels (d), (g), and (h) show the marginal values of the three stocks.

At date T, after the realization of the new tax rate, the firm begins adjusting toward its new steady state. For the high realization, $\hat{\tau} = \tau^H$, the firm's long-run goal involves decumulating even more capital. As shown in panels (e) and (f) of Figure 5, it uses the entire stock of projects in its initial DA, $\hat{n} = m_T$ and chooses an intensity $\hat{\imath}_T$ that exactly exhausts the liquid assets, so $\hat{n}g(\hat{\imath}_T) = a_T$. Hence no discrete dividend is paid, $\hat{D} = 0$, and $\hat{a}_T = \hat{m}_T = 0$. Thereafter the firm adjusts smoothly towards the new (much lower) steady state.

For the low realization, $\hat{\tau} = \tau^H$, the firm's long-run goal involves rebuilding its capital stock to almost the old steady state level. In this case the DA uses only part of the stock of projects $\hat{n} < m_T$, but a higher intensity $\hat{\imath}_T$. Again, liquid assets are exhausted, $\hat{n}g(\hat{\imath}_T) = a_T$, so no discrete dividend is paid, $\hat{D} = 0$, and $\hat{a}_T = 0$. The remaining stock of projects $\hat{m}_T > 0$ is used over the next 6 months, with investment at an intensity very similar to $\hat{\imath}$, and during this period no dividend is paid. Thereafter the firm adjusts smoothly towards the new steady state.

Figure 6 shows the capital stock and expenditure on investment over a longer period, for each realization of the tax rate.

5. STOCHASTIC ARRIVAL DATE T

Suppose the date of the tax change is also stochastic. In particular, assume the arrival is Poisson, with arrival rate θ. The firm's problem after the arrival of the tax change is exactly as before, in (1)-(3). Before the arrival its problem is to choose $\{(D,i,n)\}_{t=0}^{\infty}$ to solve

$$\max \int_0^{\infty} e^{-(\rho+\theta)t} \left\{ D(t) + \theta \mathrm{E}_{\hat{\tau}} \left[v(s(t);\hat{\tau}) \right] \right\} dt, \qquad \text{s.t. } (3),$$

where the extra term in the objective function is the post-reform continuation value, and the extra term in the discount rate represents the probability that the tax change has not yet occurred. The Hamiltonian now includes the extra term $\theta \mathrm{E}_{\hat{\tau}} \left[v(s;\hat{\tau}) \right]$.

The first order conditions in (5) are unchanged, but the law of motion for each costate now includes a term that picks up the expected capital gain (or loss) on the asset when the tax change occurs. Let T denote the random date—a stopping time—when the tax change arrives. Then, (6) is replaced by

$$\begin{aligned} \dot{q}_k &= (\rho+\delta)\, q_k - q_a \,(1-\tau)\, \pi'(k) + \theta \left\{ q_k - \mathrm{E}_{\hat{\tau}} \left[q_{kT}(s;\hat{\tau}) \right] \right\}, && (11) \\ \dot{q}_a &\leq (\rho - r)\, q_a + \theta \left\{ q_a - \mathrm{E}_{\hat{\tau}} \left[q_{aT}(s;\hat{\tau}) \right] \right\}, && \text{w/ eq. if } a > 0, \\ \dot{q}_m &\leq \rho q_m + \theta \left\{ q_m - \mathrm{E}_{\hat{\tau}} \left[q_{mT}(s;\hat{\tau}) \right] \right\}, && \text{w/ eq. if } m > 0, \end{aligned}$$

where $q_{xT}(s;\hat{\tau})$ denotes the initial value of the costate at T, conditional on the new tax rate $\hat{\tau}$, and we have used the fact that $v_x(s;\hat{\tau}) \equiv q_{xT}(s;\hat{\tau})$.

After T the transition is as before, although the initial condition now depends on the realization of T. Over the (random) interval $(0,T)$ the system asymptotically approaches a SS, call it $s^* = (k^*, a^*, m^*)$. If the stopping time happens to arrive quickly, the initial condition is close to $s(0) = (k^{ss}(\tau), 0, 0)$, so the transition is essentially from one SS to another, with negligible initial stocks of assets or projects. If the stopping time happens to arrive slowly, the initial condition for the post-arrival transition is close to s^*.

a. The pre-arrival SS

In the pre-arrival SS the firm may hold stocks of cash and projects, but it is no longer adding to those stocks. It is investing to maintain the current capital stock, and paying the rest of its profit flow as a dividend. Hence (5) implies the SS values for the costates are $q_a^* = 1$ and $q_k^* = g'(i^*)$, where $n^* = \mu$ and $i^* = \delta k^*/\mu$. Then (11) requires

$$\begin{aligned}
\theta \mathrm{E}_{\hat{\tau}}\left[q_{kT}(s^*;\hat{\tau})\right] &= (\theta+\rho+\delta)\, g'(i^*) - (1-\tau)\,\pi'(k^*), && \\
\theta \mathrm{E}_{\hat{\tau}}\left[q_{aT}(s^*;\hat{\tau})\right] &\leq (\theta+\rho-r)\,, && \text{w/ eq. if } a^* > 0, \\
\theta \mathrm{E}_{\hat{\tau}}\left[q_{mT}(s^*;\hat{\tau})\right] &\leq (\theta+\rho)\, q_m^*, && \text{w/ eq. if } m^* > 0.
\end{aligned} \tag{12}$$

As a check, note that for $\theta = 0$, these conditions hold for $s^* = (k^{ss}(\tau), 0, 0)$, the benchmark SS for the initial tax rate τ. If F puts unit mass on $\hat{\tau} = \tau$, so the arrival entails no change, then these conditions hold at the benchmark SS for any θ. If F puts unit mass on $\hat{\tau} \neq \tau$, so the change is deterministic, then as $\theta \to \infty$, they hold for $s^* = (k^{ss}(\hat{\tau}), 0, 0)$, the benchmark SS for the new tax rate $\hat{\tau}$.

b. A tax change with known $\hat{\tau}$

Consider first the case where F puts unit mass on $\hat{\tau} \neq \tau$, so the new tax rate is known. To see whether liquid assets or projects are accumulated, conjecture that $s^* = (k^*, 0, 0)$, and ask whether (12) holds for some k^*. Consider the first line in (12).

For $\theta > 0$ the LHS is positive and is strictly decreasing in k^*, asymptotically approaching zero as $k^* \to \infty$. The RHS (with $i^* = \delta k^*/\mu$) is strictly increasing in k^*. It is negative for k^* sufficiently small and increases without bound as $k^* \to \infty$. Hence there is a unique solution $k^* > 0$. It is easy to verify that k^* lies between $k^{ss}(\tau)$ and $k^{ss}(\hat{\tau})$. For fixed θ, an increase in τ or $\hat{\tau}$ reduces k^*. In addition, for $\hat{\tau} < \tau$ an increase in θ increases k^*, and for $\hat{\tau} > \tau$ an increase in θ reduces k^*.

For $a^* = 0$ and $m^* = 0$, the second and third lines in (12) are inequalities. The second line holds if k^* is in or sufficiently close to the region where $D > 0$ for $\hat{\tau}$, so $q_{aT} = 1$. Hence it holds if $\hat{\tau} > \tau$. It also holds if $\hat{\tau} < \tau$, provided the change is not too large. The third line also holds if $\hat{\tau} > \tau$, since in this case the marginal value of a project falls when the tax change arrives. If $\hat{\tau} < \tau$, the third line holds if the tax change is not too large, but it may fail for a large tax cut. Thus, for large tax cuts, either condition or both may fail. In these cases (presumably) inputs are held in the pre-arrival SS.

Whether inputs are accumulated also depends on θ. For θ large, k^* is close to $k^{ss}(\hat{\tau})$, so $q_{aT}((k^*, 0, 0)\,; \hat{\tau}) = 1$ and $q_{mT}((k^*, 0, 0)\,; \hat{\tau})$ is close to q_m^*. Hence the second and third lines in (12) hold. For θ small, k^* is close to $k^{ss}(\tau)$, which determines $q_{aT}(s^*; \hat{\tau})$ and $q_{mT}(s^*; \hat{\tau})$. Thus, the second and third lines in (12) also hold as $\theta \to 0$. Hence it seems most likely that inputs are acquired in the case of a large tax decrease and an intermediate value for θ.

c. A tax change with uncertainty about $\hat{\tau}$

Next consider the case where the new tax rate is also unknown. As an analog of Proposition 3 we have the following.

Proposition 5: Suppose a tax change, drawn from the distribution F, arrives with hazard rate $\theta > 0$. Unless F puts unit mass at a single point, for all θ sufficiently large, the pre-reform SS $s^* = (k^*, a^*, m^*)$ has $k^*, a^*, m^* > 0$.

6. CONCLUSIONS

The positive predictions of the option model are very stark: policy uncertainty leads to a sharp swings in both investment and dividends. In this sense it may provide a useful framework for looking at firm-level investment in periods of high uncertainty.

To incorporate this model of firm-level investment into a macro model, it would be useful to let stored projects depreciate. There are two interpretations: that the market changes, making the investment less profitable, or that a rival firm gets access to the project and exploits it. If depreciation rates vary across projects, those with high depreciation rates are less storable. Thus, for a policy reform with a given level of uncertainty, projects with depreciation rates below a certain threshold would be stored, while those above the threshold would be exploited immediately.

In highly competitive sectors, presumably the 'depreciation' due to rivals is greater, making delay less feasible. Thus, highly competitive sectors should behave more like the benchmark model, and sectors with more products that are more strongly differentiated should behave more like the options model.

The welfare implications in a macroeconomic setting are less clear. In the model here, the decline in investment during the period of delay is largely offset by a boom after the uncertainty is resolved. But the same is true in many models of investment over the business cycle, so the welfare costs here might be similar to the costs of cyclical fluctuations.

[To be added: The value of the firm is a weighted sum of the values of the installed capital, liquid assets, and stored projects that it holds. The model describes the marginal values of these three stocks, and the changes in firm valuation that accompany delay might provide a useful signal about when delay is occurring.]

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APPENDIX A: PROOFS OF PROPOSITIONS

PROOF OF PROPOSITION 1: Let q_k, q_a denote the costate variables for capital and liquid assets. The solution $\hat{D}$, $\{D, i, k, a, q_k, q_a,\ t \geq T\}$ satisfies the benchmark version of (2) - (6). Clearly (7) is the steady state, where the assumptions on π and g ensure it is unique.

Suppose $a_T = 0$, and conjecture the solution has $a(t) \equiv 0$. Then D, i satisfy

$$q_k \leq q_a g'(i), \qquad \text{w/ eq. if } i > 0, \tag{13}$$

$$D = (1 - \hat{\tau})\, \pi(k) - \mu g(i), \qquad \text{all } t \geq T. \tag{14}$$

Use (13) and (14) to define $\chi(k)$ by

$$\mu g \left[g'^{-1}(\chi(k)) \right] \equiv (1 - \hat{\tau})\, \pi(k).$$

For any k, the intensity i satisfying $g'(i) = \chi(k)$ is just sufficient to absorb all of after-tax earnings. The function χ is strictly increasing, with $\chi(0) = g'(0)$, and the threshold $\chi(k)$ divides (k, q_k) −space into two regions.

Above the threshold the firm is cash constrained. In this region the dividend is zero and cash is at a premium, $D = 0$ and $q_a > 1$. The investment intensity, call it $i^*(k, q_k)$, is determined by (14), so it is strictly increasing in k and independent of q_k, and (13) determines $q_{a.}$.

Below the threshold the firm is not cash constrained, so $D > 0$ and $q_a = 1$. In this region the investment intensity is determined by (13), so it is strictly increasing in q_k and independent of k, and the dividend is determined by (14).

Hence the intensity isoquants in (k, q_k) space are L-shaped, with kinks on the $(k, \chi(k))$ threshold. If $g'(0) > 0$, there is a second threshold, the horizontal line where $q_k = g'(0)$. Below this threshold $i^* = 0$, and above it $i^* > 0$.

The locus where $\dot{k} = 0$ satisfies $\mu i^*(k, q_k) = \delta k$, so it is upward sloping, hitting the vertical axis at $q_k = g'(0)$. For $\hat{\tau} \in [0, \overline{\tau}]$ and $k_0 \in (0, \overline{k}]$, the $\dot{k} = 0$ locus lies entirely in the region where $D > 0$, below $\chi(k)$. The locus where $\dot{q}_k = 0$ satisfies

$$(\rho + \delta)\, g'\left[i^*(k, q_k)\right] = (1 - \hat{\tau})\, \pi'(k),$$

so it is downward sloping in the region where $D, i^* > 0$, and vertical in the regions where $D = 0$ and $i^* = 0$.

The stable manifold, call it SM_0, slopes downward, and for any $k_T \in (0, \overline{k}]$ there is unique value $q_{kT} > 0$ for which the system converges to the steady state. The critical values $k_1^{\text{crit}} < k^{ss} < k_2^{\text{crit}}$ are defined by the points where SM_0 cuts the $\chi(k)$ and $g'(0)$ thresholds. Along SM_0, intensity i^* increases with k (as q_k falls) for $k < k_1^{\text{crit}}$, it decreases with k for $k \in \left(k_1^{\text{crit}}, k_2^{\text{crit}}\right)$, and it is constant at $i^* = 0$ for $k \geq k_2^{\text{crit}}$. The dividend is $D = 0$ for $k \leq k_1^{\text{crit}}$ and increases with k for $k > k_1^{\text{crit}}$.

To verify that the conjecture $a(t) \equiv 0$ is correct, it suffices to show that $\dot{q}_a/q_a \leq \rho - r$. If $D > 0$, then $q_a = 1$ and $\dot{q}_a = 0$. If $D = 0$, then $k < k_1^{\text{crit}} < k^{ss}$, so k and i^* are increasing, and q_k is decreasing. Since $q_a = q_k/g'(i^*)$, it follows that $\dot{q}_a < 0$.

If $a_T > 0$ and $k_T \geq k_1^{\text{crit}}$, then $\hat{D} = a_T$, so $\hat{a}_T = 0$, and the rest of the solution is as above.

For $a_T > 0$ and $k_T < k_1^{\text{crit}}$, solutions can be constructed as follows. Choose any point (k, q_k) on SM_0 with $k < k_1^{\text{crit}}$, use (14) with $D = 0$ to determine i^*, and calculate $q_a = q_k/g'(i^*) > 1$. Construct trajectories for (k, a, q_k, q_a) by running the relevant ODEs in (3) and (6) backward in time, with $\dot{q}_a/q_a = \rho - r > 0$ and $g'(i^*) = q_k/q_a$. The restriction $q_a \geq 1$ limits the length of the extension. The terminal pairs $(k_T, \alpha(k_T))$ for the longest extensions define the function α.

Varying the length of the backward extension traces out a one-dimensional family of initial conditions (k_T, a_T), and varying the initial point on SM_0 gives a two-dimensional family. Lower initial values for k on SM_0 have higher initial values for q_a, allowing longer extensions. Hence α is a continuous, decreasing function, with $\alpha(k) \to 0$ as $k \uparrow k_1^{\text{crit}}$.

For $0 < a_T \leq \alpha(k_T)$, the initial dividend is $\hat{D} = 0$, and the transition begins with $\hat{a}_T = a_T$ and $q_{aT} > 1$. The solution follows a constructed trajectory until assets are exhausted. This occurs while $k < k_1^{\text{crit}}$, and thereafter the solution follows SM_0. For $a_T > \alpha(k_T)$, the initial dividend is $\hat{D} = a_T - \alpha(k_T)$, and the transition begins with $\hat{a}_T = \alpha(k_T)$ and $q_{aT} = 1$. ■

Proof of Proposition 2: It suffices to show that $\hat{m}_T > 0$ implies $\hat{a}_T = 0$.

Fix $\hat{\tau}$ and suppose $\hat{m}_T > 0$. If $n(T+z)i(T+z) = 0$ for $z \in (0, \Delta)$, then $\dot{k}(T+z) < 0$, which implies $k > k^{ss}(\hat{\tau})$. In this region SM_0 lies below $\chi(\cdot; \hat{\tau})$, so $q_a = 1$ and $a = 0$. Hence the solution requires $\hat{a}_T = 0$.

If $n(T+z)i(T+z) > 0$ for $z \in (0, \Delta)$, then the second and third lines in (5) hold with equality over $(T, T+\Delta)$. Differentiate them to get two equations involving $\dot{i}$,

$$\begin{aligned} \frac{\dot{q}_k}{q_k} &= \frac{\dot{q}_a}{q_a} + \frac{g'' \dot{i}}{g'}, \\ \frac{\dot{q}_m}{q_m} &= \frac{\dot{q}_a}{q_a} + \frac{g'' \dot{i}}{g' - g/i}. \end{aligned}$$

By hypothesis $m > 0$, so the third line in (6) holds with equality, and if $a > 0$ the second line also holds with equality, so

$$\begin{aligned} (r+\delta)\, g' - (1-\hat{\tau})\, \pi' &= g'' \dot{i}, \\ r\left[g' - g/i\right] &= g'' \dot{i}, \end{aligned} \tag{15}$$

and hence

$$(1-\hat{\tau})\, \pi' = \delta g' + r\frac{g}{i}.$$

Suppose this condition holds at T. It continues to hold on $(T, T+\Delta)$ if and only if

$$(1-\hat{\tau})\, \pi'' \dot{k} = \left[\delta g'' + \frac{r}{i}\left(g' - \frac{g}{i}\right)\right] \dot{i}. \tag{16}$$

The term in brackets on the right is positive, and the second line in (15) implies $\dot{i} \geq 0$. Since $\pi'' < 0$, (16) holds only if $\dot{k} \leq 0$, which implies $k > k^{ss}(\hat{\tau})$. The rest of the argument is as before. ■

Proof of Proposition 3: (a) Solutions with $\hat{m}_T > 0$ can be constructed as follows. Choose $k_S > 0$, let $(k, a, m) = (k_S, 0, 0)$, and let (q_k, q_a, q_m) be the associated costate values on the SM. Construct the solution for (k, m, q_k, q_m) by running the ODEs

$$\begin{aligned} \dot{k} &= ni - \delta k, \qquad (17) \\ \dot{m} &= \mu - n, \\ \dot{q}_k &= q_k \left[\rho + \delta - \frac{(1-\hat{\tau})\,\pi'}{g'} \right], \\ \dot{q}_m &= \rho q_m, \end{aligned}$$

backward in time. No liquid assets are accumulated, $a = 0$, all t.

If $k_S \leq k^{ss}$, the controls satisfy $D = 0$ and

$$\begin{aligned} \phi(i) &= \frac{q_m}{q_k}, \\ n &= \frac{(1-\hat{\tau})\,\pi(k)}{g(i)}, \\ q_a &= \frac{q_k}{g'(i)}, \end{aligned}$$

where $\phi(i) \equiv i - g(i)/g'(i)$ is strictly increasing. To verify that $q_a \geq 1$ and $\dot{q}_a/q_a \leq \rho - r$, note that with time running forward, q_m is increasing and q_k is decreasing. Hence $\phi(i)$ and i are increasing, so q_a is falling. Since $q_a \geq 1$ on the SM, both of the required condition holds.

To verify that $\dot{m} = \mu - n < 0$ along the constructed trajectory, recall that $n = \mu$ on the SM. For any fixed k,

$$\mu g(i^{SM}) \leq (1-\tau)\,\pi(k) = n g(i),$$

where i^{SM} denotes the intensity on the SM. Hence it suffices to show that $i^{SM} > i$. On the SM ...

If $k > k^{ss}$, then $q_a = 1$, $g'(i) = q_k$, and

$$q_m = ig'(i) - g(i).$$

To determine n, note that the LOMs require

$$\begin{aligned}
\frac{\dot{q}_m}{q_m} &= \frac{g''\dot{i}}{g' - g/i} = \rho, \\
\frac{\dot{q}_k}{q_k} &= g''\dot{i} = \rho + \delta - \frac{(1-\hat{\tau})\,\pi'}{g'}.
\end{aligned}$$

Combine these conditions to get

$$(1-\hat{\tau})\,\pi'(k) = (\rho+\delta)\,g' - \rho g'\left(g' - \frac{g}{i}\right),$$

and differentiate w.r.t. t to get

$$(1-\hat{\tau})\,\pi''\dot{k} = \left[(\rho+\delta)\,g'' - \rho g''\left(g' - \frac{g}{i}\right) + \rho g'\frac{g}{i^2}\right]\dot{i}.$$

Since $\dot{k} < 0$ and $\pi'' < 0$, the term on the right is positive. Substitute for $\dot{k}$ and $\dot{i}$ to find that n satisfies

$$\begin{aligned}
ni &= \delta k + \frac{\rho\,(g' - g/i)}{(1-\hat{\tau})\,\pi''}\left[(\rho+\delta) - \rho\left(g' - \frac{g}{i}\right) + \rho\frac{g'}{ig''}\frac{g}{i}\right] \\
&= \delta k + \frac{\rho q_m/i}{(1-\hat{\tau})\,\pi''}\left[\rho + \delta - \rho\frac{q_m}{i} + \rho\frac{g'}{ig''}\frac{g}{i}\right],
\end{aligned}$$

as long as the term on the right is positive. The dividend $D > 0$ is the residual from the cash constraint.

To verify that $\dot{m} = \mu - n < 0$ along the constructed trajectory, recall that $n = \mu$ on the SM.

(b) The solution for the option model coincides with the solution to the benchmark model if and only if $\dot{q}_m/q_m \leq \rho$, all t. In the region where $ni > 0$ and $q_a = 1$,

$$q_m = ig'(i) - g(i), \qquad \text{and} \qquad g'(i) = q_k.$$

Hence

$$\dot{q}_m = ig''\dot{i} = i\dot{q}_k,$$

so the required condition holds if $k < k^{ss}$. It also holds in some region above k^{ss}, but fails for k sufficiently large. Define k_B as the threshold where $\dot{q}_m/q_m = \rho$.

In the region where $ni > 0$ and $q_a > 1$,

$$q_m = q_k \left[i - g/g'\right],$$

so

$$\frac{\dot{q}_m}{q_m} = \frac{\dot{q}_k}{q_k} + \frac{gg''}{g'}\frac{\dot{i}}{ig' - g}.$$

In this region $\dot{q}_k < 0$ and $\dot{k} > 0$. Since the investment intensity i is determined by the cash flow constraint, $\dot{i} > 0$, and the required condition fails if the growth in intensity is rapid. Define k_A as the threshold where $\dot{q}_m/q_m = \rho$.

(c)

Proof of Proposition 4: Suppose to the contrary that for any $\Delta > 0$, the optimal policy has $n(t)i(t) > 0$, all $t \in [T-\Delta, T)$. Consider the following perturbation to the conjectured solution over $(T-\Delta, T+\Delta)$, where $\Delta > 0$ is small. Reduce the flow of projects by $\varepsilon > 0$ over $(T-\Delta, T)$ and accumulate the projects and cash. For $\varepsilon > 0$ sufficiently small, this is feasible. Then increase the flow of projects by ε over $(T, T+\Delta)$, and adjust the intensity on an additional group of projects of size ε. For each $\hat{\tau}$, choose the intensity for this group of 2ε projects as follows.

Let $i(T)$ denote the intensity on $(T-\Delta, T)$ and let $i_T(\hat{\tau})$ denote the intensity on $(T, T+\Delta)$, conditional on $\hat{\tau}$. Since Δ is small, these intensities are approximately constant before and after T, although the latter varies with $\hat{\tau}$. For the 2ε projects use the intensity

$$i_P(\hat{\tau}) = \frac{1}{2}\left[i(T) + i_T(\hat{\tau})\right],$$

so the capital stock at $k(T+\Delta)$ is unaltered.

The perturbation changes the investment cost by

$$\Delta_C(\hat{\tau}) = \varepsilon\Delta\left\{2g\left[i_P(\hat{\tau})\right] - g\left[i(T)\right] - g\left[i_T(\hat{\tau})\right]\right\}, \qquad \text{all } \hat{\tau}.$$

Since g is strictly convex, $\Delta_C(\hat{\tau}) \leq 0$, with equality if and only if $i_T(\hat{\tau}) = i(T)$. Unless F puts unit mass at a single point, this condition must fail on a set of $\hat{\tau}$'s with positive probability. Therefore, unless F puts unit mass on a single point,

$$X \equiv \mathrm{E}_{\hat{\tau}}\left\{2g\left[i_P(\hat{\tau})\right] - g\left[i(T)\right] - g\left[i_T(\hat{\tau})\right]\right\} < 0.$$

Since the perturbation reduces the cost of investment, at least weakly, for every $\hat{\tau}$, and delays the timing of expenditures, it is also feasible in the sense that it can be financed without any additional liquid assets.

The cost of the delay is the foregone revenue. The perturbation changes the capital stock by

$$\begin{aligned}
\Delta_k(T-z) &\approx -\varepsilon\left(\Delta - z\right)i(T), \qquad z \in (0,\Delta), \\
\Delta_k(T+z;\hat{\tau}) &\approx -\varepsilon\Delta i(T) + \varepsilon z\left[2i_P(\hat{\tau}) - i_T(\hat{\tau})\right] \\
&= -\varepsilon\left(\Delta - z\right)i(T), \qquad z \in (0,\Delta),
\end{aligned}$$

where the changes after T are conditional on $\hat{\tau}$. Hence the change in revenue is

$$\begin{aligned}
\Delta_\Pi(\hat{\tau}) &\approx -\left(1-\hat{\tau}\right)\pi'\left[\int_0^\Delta \Delta_k(T-z)dz + \int_0^\Delta \Delta_k(T+z)dz\right] \\
&\approx -2\varepsilon\left(1-\hat{\tau}\right)\pi' i(T)\int_0^\Delta (\Delta - z)dz \\
&= -\varepsilon\Delta^2\left(1-\hat{\tau}\right)\pi' i(T).
\end{aligned}$$

The reduction in revenue is of order $\varepsilon\Delta^2$, while the reduction in investment costs is of order $\varepsilon\Delta$. As noted above, $X < 0$. Hence for $\Delta > 0$ sufficiently small,

$$\mathrm{E}_{\hat{\tau}}\left[\Delta_\Pi(\hat{\tau}) - \Delta_C(\hat{\tau})\right] \approx \varepsilon\Delta\left[-\Delta\pi' i(T) - X\right] > 0,$$

and the perturbation raises expected profits. ■

Proof of Proposition 5: Suppose to the contrary that $a^* = 0$ or $m^* = 0$ or both. Let T_0 be a date before the arrival of the tax change, with $s(T_0) = s^*$, and let T denote the (random) arrival date. Consider the following perturbation to the strategy of keeping $s(t) = s^*$ on (T_0, T).

Let i^* be the SS intensity, and to simplify notation, let $T_0 = 0$. Choose $\varepsilon, \Delta > 0$ small. Over $(0, \Delta)$, reduce the flow of projects by ε, keeping the intensity unchanged. At $t = \Delta$, the capital stock is reduced by $\varepsilon\Delta i^*$, and the firm has a stock of $m = \varepsilon\Delta$ untapped projects and a stock of $a = \varepsilon\Delta g(i^*)$ liquid assets. Over (Δ, T) reduce the intensity of replacement investment by $\varepsilon\Delta i^*\delta/\mu$, so the capital stock remains constant. Pay the interest on the accumulated liquid assets and the savings in replacement cost as dividends. The EDV of the additional dividends is

$$\begin{aligned}\Delta_D &= \varepsilon\Delta\left[rg(i^*) + \frac{\delta}{\mu}i^*g'(i^*)\right]\int_0^\infty e^{-(\rho+\theta)t}dt \\ &= \left[rg(i^*) + \frac{\delta}{\mu}i^*g'(i^*)\right]\frac{\varepsilon\Delta}{\rho+\theta}. \end{aligned} \tag{18}$$

These terms are positive and have order $\varepsilon\Delta$.

After the tax change arrives, over $(T, T+\Delta)$ increase the scale of investment by ε and alter the intensity for an additional ε projects as in the proof of Proposition 3, so the capital stock at $T+\Delta$ is as it would have been under the original plan. Define $i_T(\hat{\tau})$ and $i_P(\hat{\tau})$ as in the proof of Prop. 3. Conditional on the new tax rate $\hat{\tau}$, the perturbation to the investment cost is

$$\Delta_C(\hat{\tau}) = \varepsilon\Delta\left\{2g\left[i_P(\hat{\tau})\right] - g(i^*) - g\left[i_T(\hat{\tau})\right]\right\}, \qquad \text{all } \hat{\tau}.$$

As shown in the proof of Prop. 3, $\Delta_C(\hat{\tau}) \leq 0$, with equality if and only if $i_T(\hat{\tau}) = i^*$. Therefore, unless F puts unit mass on a single point,

$$X(i^*) \equiv \mathrm{E}_{\hat{\tau}}\left\{2g\left[i_P(\hat{\tau})\right] - g(i^*) - g\left[i_T(\hat{\tau})\right]\right\} < 0.$$

Hence this contribution of the perturbation to the EDV of profits is

$$\begin{aligned} -\Delta_C &= -\varepsilon \Delta X(i^*) \int_0^\infty \theta e^{-(\rho+\theta)t} dt \\ &= -\theta X(i^*) \frac{\varepsilon \Delta}{\rho+\theta}. \end{aligned} \tag{19}$$

This term is positive and has order $\varepsilon\Delta$.

The perturbation changes the capital stock by

$$\Delta_k(t) = \begin{cases} -\varepsilon i^* t, & t \in (0, \Delta), \\ -\varepsilon i^* \Delta, & t \in (\Delta, T), \\ \varepsilon i^* \left[t - (T+\Delta)\right], & t \in (T, T+\Delta). \end{cases}$$

By assumption, the original solution can have no DA at T. Hence $k(t;\hat{\tau}) \approx k^*$, $t \in (T, T+\Delta)$, all $\hat{\tau}$, and marginal revenue $\pi'(k^*)$ is approximately constant. The EDV of the change in revenues after T, evaluated at T, is

$$\begin{aligned} \Delta_{Ra} &= \mathrm{E}_{\hat{\tau}} \left[(1-\hat{\tau})\right] \pi' \varepsilon i^* \int_T^{T+\Delta} \left[t - (T+\Delta)\right] e^{-\rho(t-T)} dt \\ &\approx -\mathrm{E}_{\hat{\tau}} \left[(1-\hat{\tau})\right] \pi' i^* \varepsilon \frac{\Delta^2}{2}, \end{aligned} \tag{20}$$

which has order $\varepsilon\Delta^2$. The PDV of the change in revenues over $(0, \Delta)$ also has order Δ^2, so both terms can be dropped.

The EDV of the change in revenue over (Δ, T) is

$$\begin{aligned} \Delta_{Rb} &= -\varepsilon \Delta i^* (1-\tau) \pi' \int_\Delta^\infty e^{-(\rho+\theta)} \theta dt \\ &\approx -(1-\tau) \pi' i^* \frac{\varepsilon \Delta}{\rho+\theta}. \end{aligned} \tag{21}$$

This term is negative and has order $\varepsilon\Delta$.

Summing the components in (18)-(21) and dropping those of order higher than $\varepsilon\Delta$, we find that the perturbation is profitable for Δ sufficiently small if and only if

$$0 < rg(i^*) + \frac{\delta}{\mu} i^* g'(i^*) - \theta X(i^*) - (1-\tau) \pi' i^*.$$

The first three terms are positive, but the last is negative. But as θ grows without bound, with F fixed, k^* and $i^* = k^*\delta/\mu$ converge to some limiting values. Hence for θ sufficiently large, the third term, which is positive, dominates. ■

APPENDIX B: COMPUTATION METHOD

This Appendix describes the computational methods used for the simulations.

Region A, low $\hat{\tau}$: The solution can be constructed by backward shooting. Choose a candidate terminal value (k, q_k) on $SM_0(\hat{\tau})$ near $k^{ss}(\tau)$ and a candidate value $\Delta_T > 0$. The stock of projects is $m = 0$ at (k, q_k), and the costate for projects is $q_m = q_k \left[i - g(i)/g'(i)\right]$. Construct trajectories for (i, n) and (k, m, q_k, q_m) by running backward for Δ_T units of time the system of ODEs

$$\begin{pmatrix} \dot{k} \\ \dot{m} \\ \dot{q}_k/q_k \\ \dot{q}_m/q_m \end{pmatrix} = -\begin{pmatrix} ni - \delta k \\ \mu - n \\ \rho + \delta - (1-\hat{\tau})\,\pi'(k)/g'(i) \\ \rho \end{pmatrix}, \tag{22}$$

with investment satisfying

$$\begin{aligned} i - \frac{g(i)}{g'(i)} &= \frac{q_m}{q_k}, \\ n &= \frac{(1-\hat{\tau})\,\pi(k)}{g(i)}. \end{aligned} \tag{23}$$

The endpoint from this exercise is a candidate for $\left(\hat{k}_T, \hat{m}_T\right)$. Let $\hat{\imath}$ be the investment intensity at this point, and let $\hat{n} = m_T - \hat{m}_T$. Check whether

$$\hat{k}_T - k_T = \hat{n}\hat{\imath}, \qquad \text{and} \qquad a_T = \hat{n}g(\hat{\imath}).$$

Adjust the candidate value (k, q_k) on $SM_0(\hat{\tau})$ and time interval Δ_T until both conditions are satisfied.

With time running forward, q_m is rising and q_k is falling, so i is rising. Hence $q_a = q_k/g'(i)$ is falling, so $\dot{q}_a/q_a < \rho - r$. In addition, since $q_a \geq 1$ on $SM_0(\hat{\tau})$, it follows that $q_a \geq 1$ on all of $(T, T + \Delta_T)$.

Region B, moderate $\hat{\tau}$: Let $i^{SM}(k; \hat{\tau})$ denote the investment intensity on $SM_0(\hat{\tau})$, given the capital stock k. For the DA choose the intensity $\hat{\imath}$ satisfying

$$\hat{\imath} = i^{SM}(k_T + m_T \hat{\imath}; \hat{\tau}).$$

Since i^{SM} is decreasing in its first argument, there is a unique solution. And since i^{SM} is also decreasing in $\hat{\tau}$, the solution $\hat{\imath}$ is decreasing in $\hat{\tau}$. At the threshold value for $\hat{\tau}$ separating region B from region A, the entire stock of liquid assets is used by the initial investment, $\hat{n}g(\hat{\imath}) = m_T g(\hat{\imath}) = a_T$. For higher values of $\hat{\tau}$ the excess cash is paid out in the initial dividend $\hat{D}$.

After the initial investment, the rest of the transition to the new SS follows SM_0, with $n(t) \equiv \mu$, $m(t) = a(t) \equiv 0$, $q_a(t) \equiv 1$, $g'(i) = q_k$, and

$$q_m = g'(i)i - g(i).$$

The solution must also satisfy $\dot{q}_m/q_m \leq \rho$, where

$$\frac{\dot{q}_m}{q_m} = \frac{i g'' \dot{i}}{i g'(i) - g(i)}.$$

If $\hat{k}_T \leq k^{ss}$, then $\dot{q}_k \leq 0$. Hence $\dot{i} \leq 0$, and the required condition holds. If $\hat{k}_T > k^{ss}$, then $\dot{q}_k > 0$, and the required condition holds only if $\hat{k}_T(\hat{\tau})$ is not too far above $k^{ss}(\hat{\tau})$. The boundary separating regions B and C is determined by the restriction on $\dot{q}_m/q_m$.

Region C, high $\hat{\tau}$: As in Region A, the transition over $(T, T + \Delta_T)$ can be constructed using backward shooting. Choose a candidate terminal point (k, q_k) on $SM_0(\hat{\tau})$, with $k \geq k^{ss}(\hat{\tau})$, and candidates for $\Delta_T > 0$ and $\{n(t),\ t \in (T, T + \Delta_T)\}$. Calculate q_m as before, and construct solutions for i and (k, m, q_k, q_m) over $(T, T + \Delta_T)$, with the intensity i satisfying

$$q_k = g'(i),$$

and the ODEs in (22). The endpoint from this construction is a candidate for $\hat{k}_T, \hat{m}_T$, with

$$\hat{m}_T = \int_T^{T+\Delta_T} [n(t) - \mu]\, dt \leq m_T.$$

Let $\hat{\imath}$ be the intensity at this point and let $\hat{n} = m_T - \hat{m}_T$. Check whether

$$\hat{k}_T - k_T = \hat{n}\hat{\imath}, \qquad \text{and} \qquad a_T \leq \hat{n} g(\hat{\imath}).$$

In addition, the first line in (23) must hold on $(T, T + \Delta_T)$. Adjust candidate terminal point (k, q_k) on $SM_0(\hat{\tau})$, time interval Δ_T, and $\{n(t)\}$ until all conditions are satisfied.

Figure 1: phase diagram for the benchmark model, a = 0

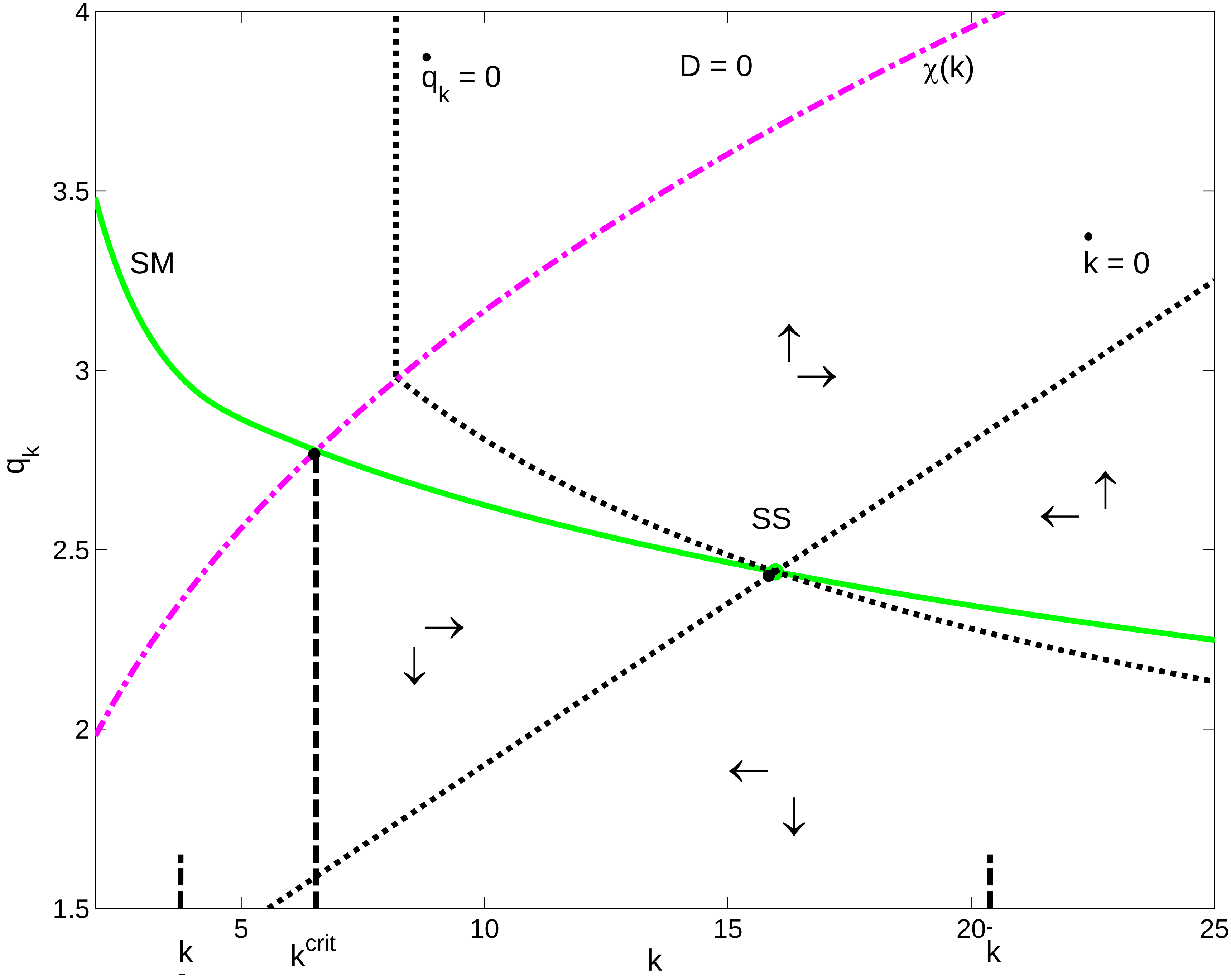


Figure 2: the discrete adjustment at T

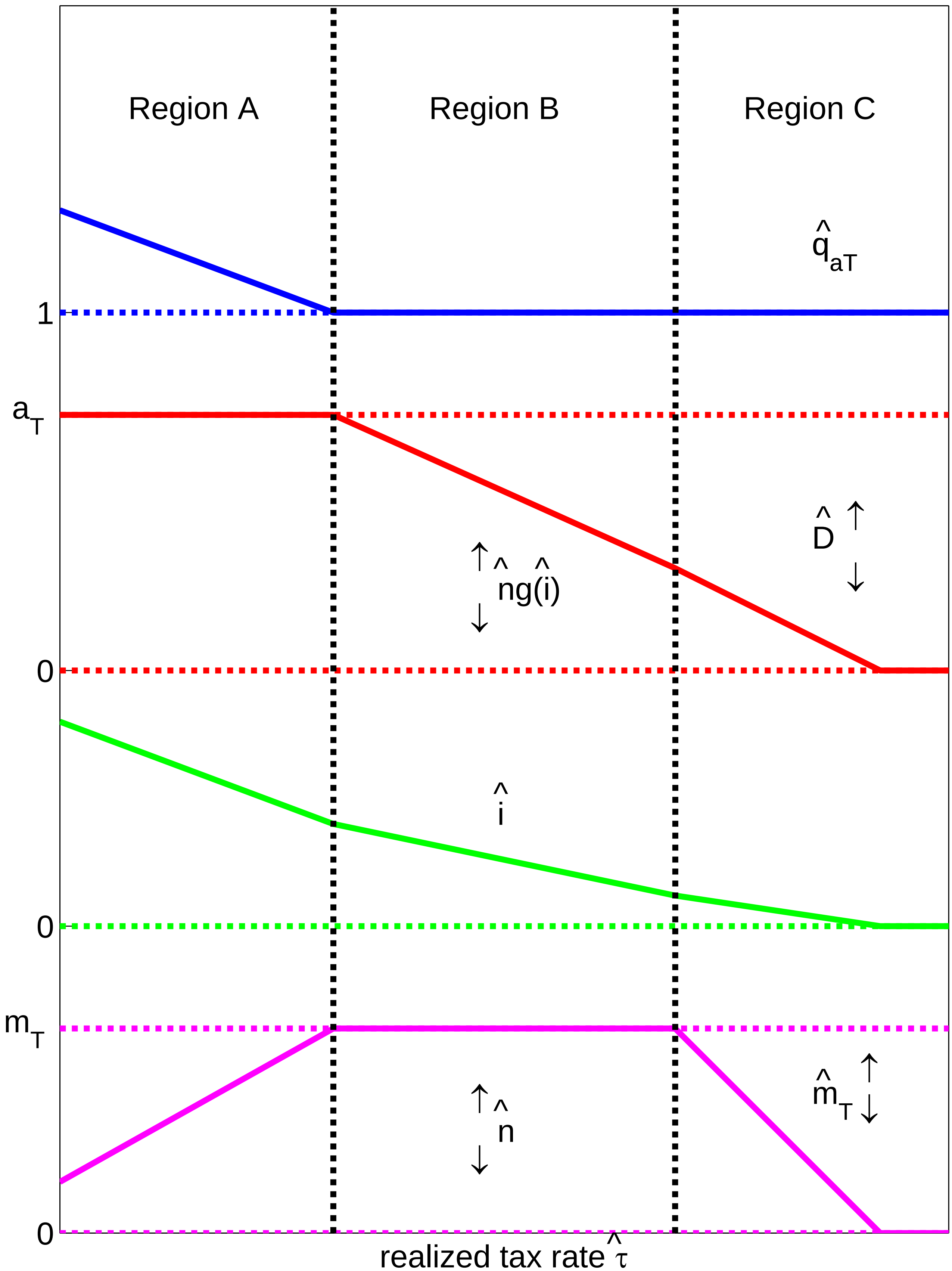


Figure 3a: deterministic tax change at T > 0 in the benchmark model

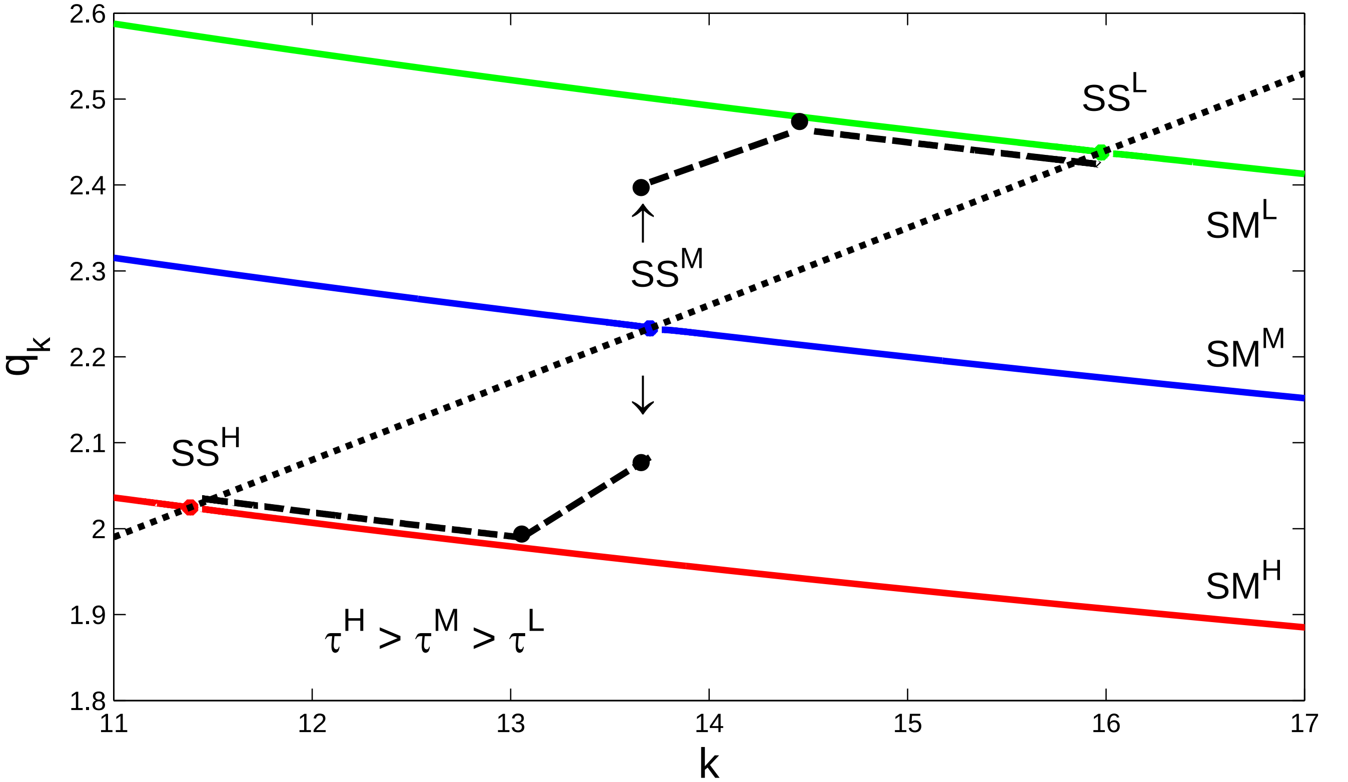


Figure 3b: stochastic tax change at T > 0 in the benchmark model

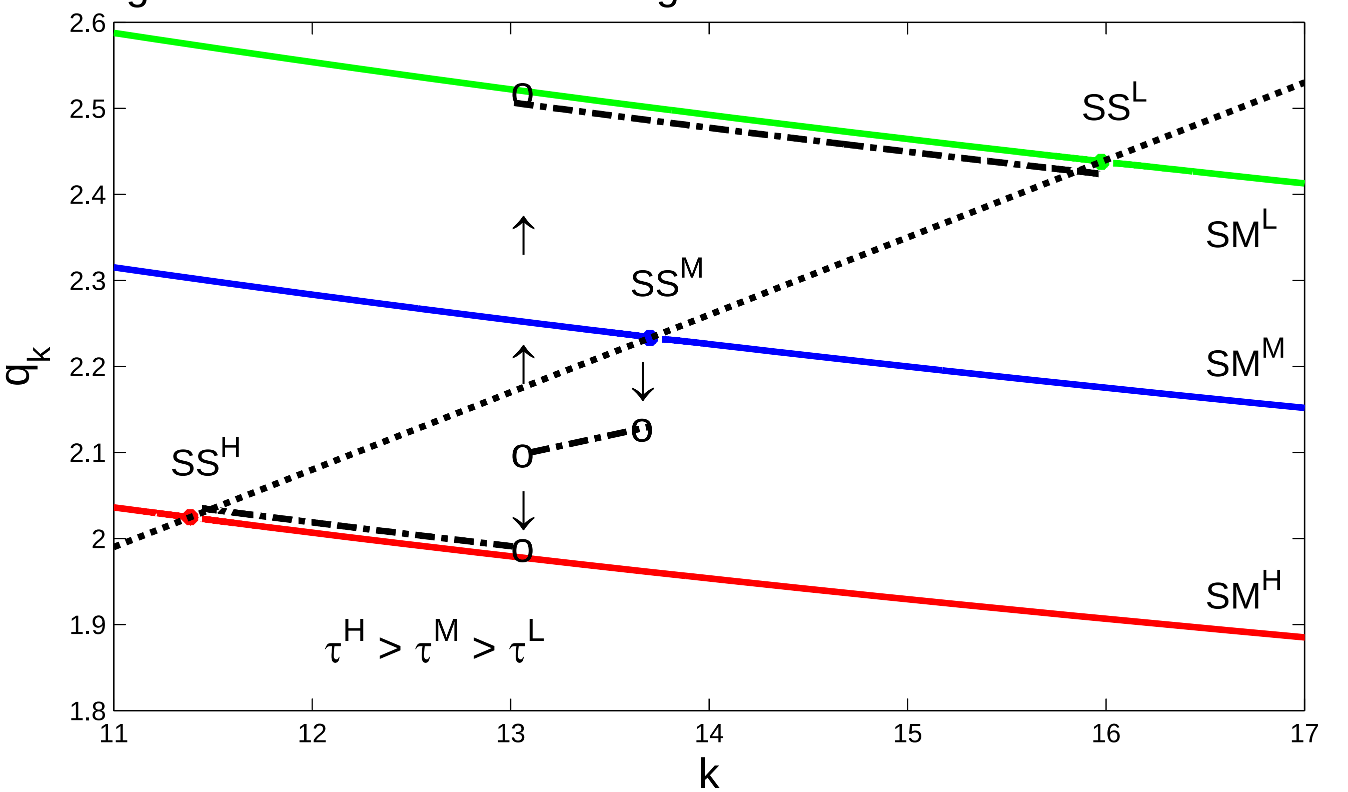


Figure 4a: Benchmark transition: capital stock

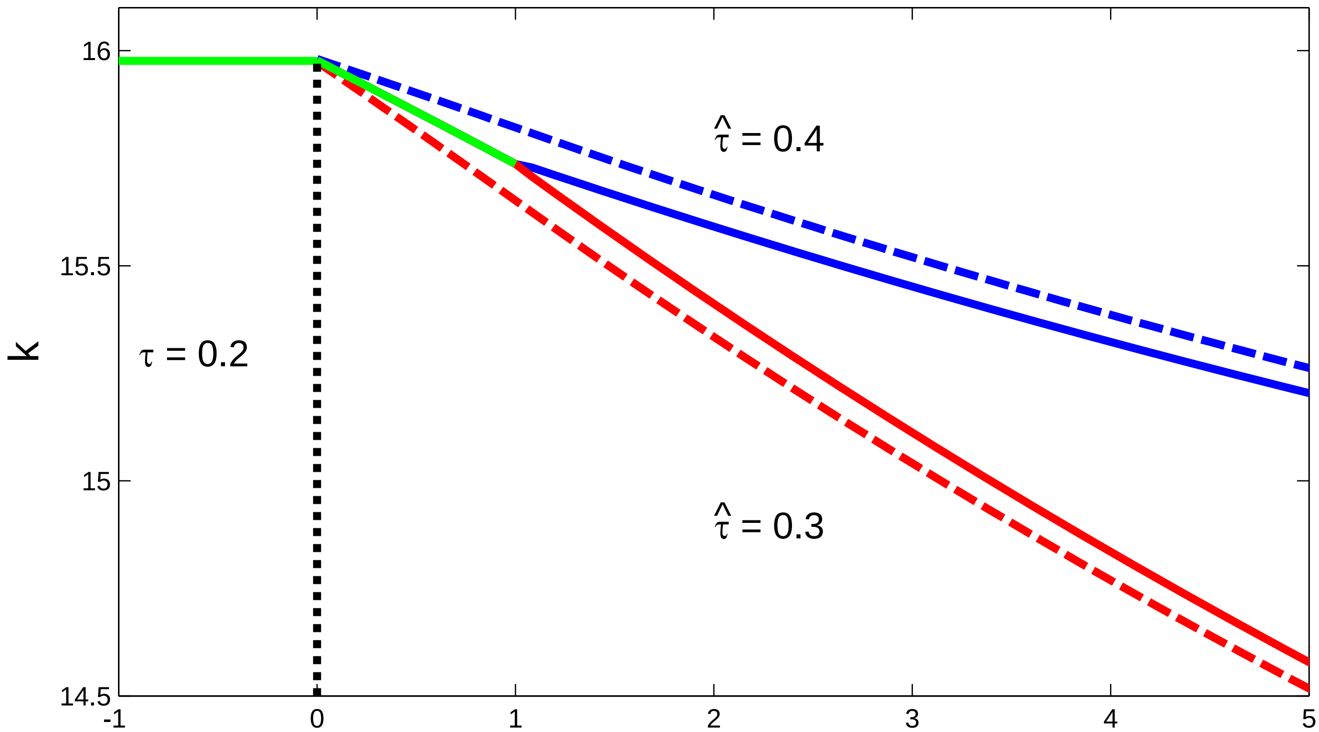
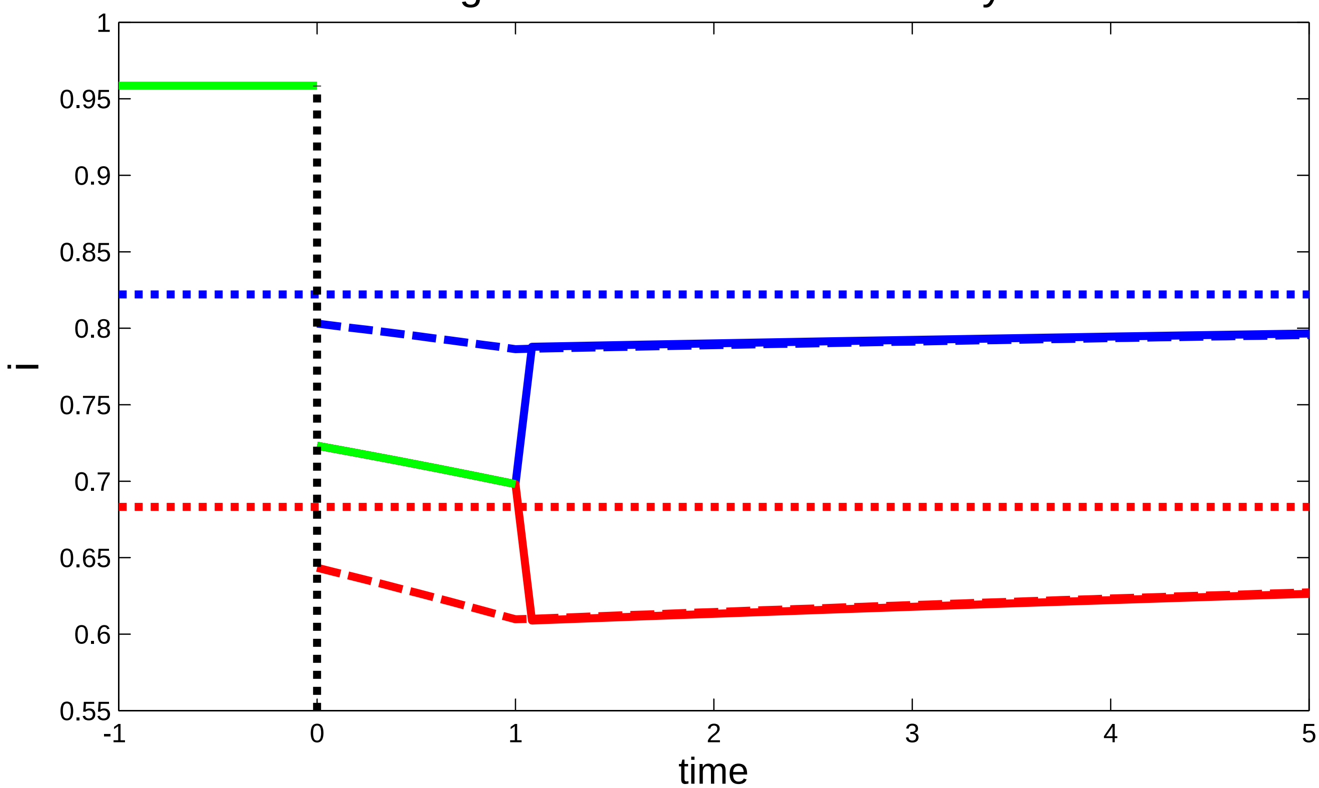


Figure 4b: Investment intensity



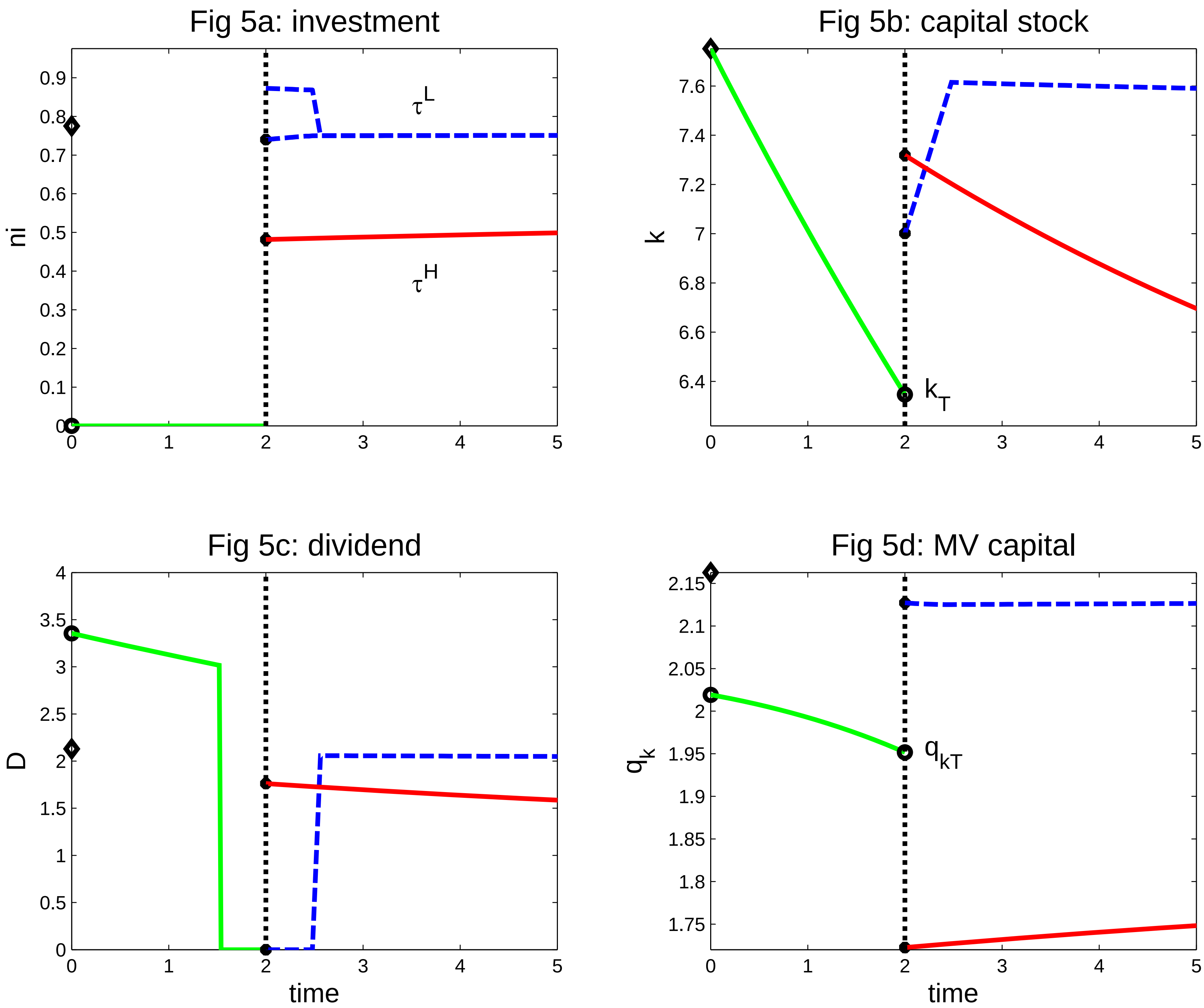


Fig 5a: investment
ni
τ^L
τ^H
Fig 5b: capital stock
k
k_T
Fig 5c: dividend
D
time
Fig 5d: MV capital
q_k
q_{kT}
time

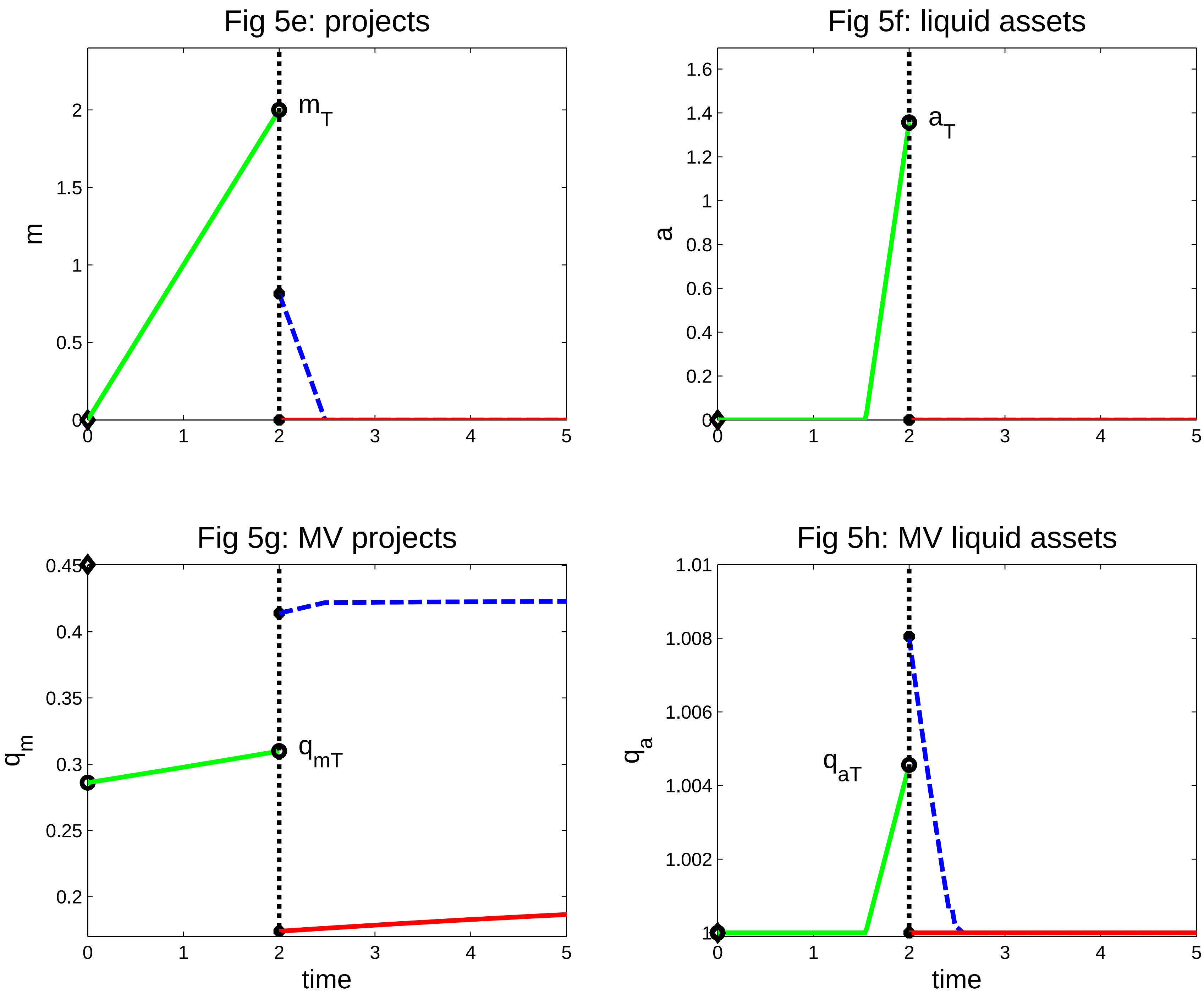


Fig 5e: projects
m
m_T
time
Fig 5f: liquid assets
a
a_T
Fig 5g: MV projects
q_m
q_mT
time
Fig 5h: MV liquid assets
q_a
q_aT
time

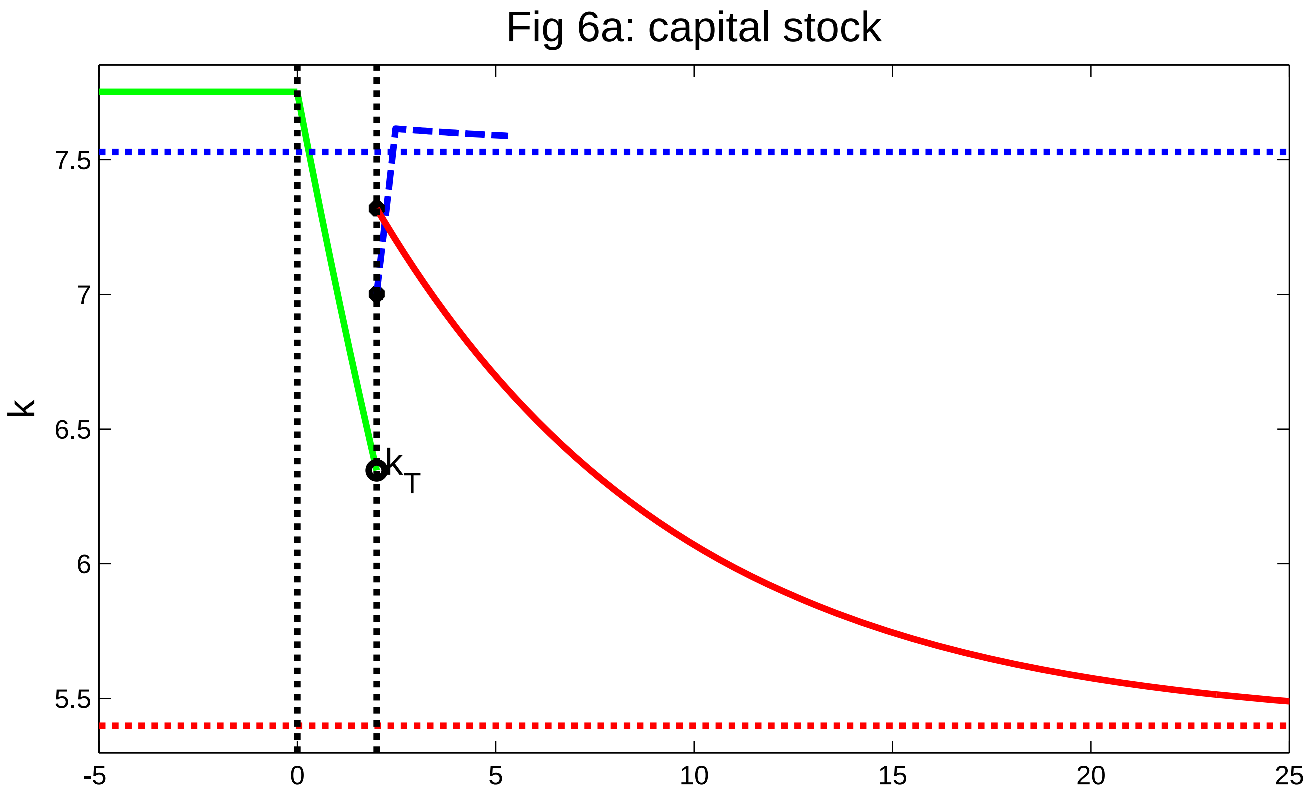


Fig 6a: capital stock
k
7.5
7
6.5
6
5.5
-5
0
5
10
15
20
25
k_T

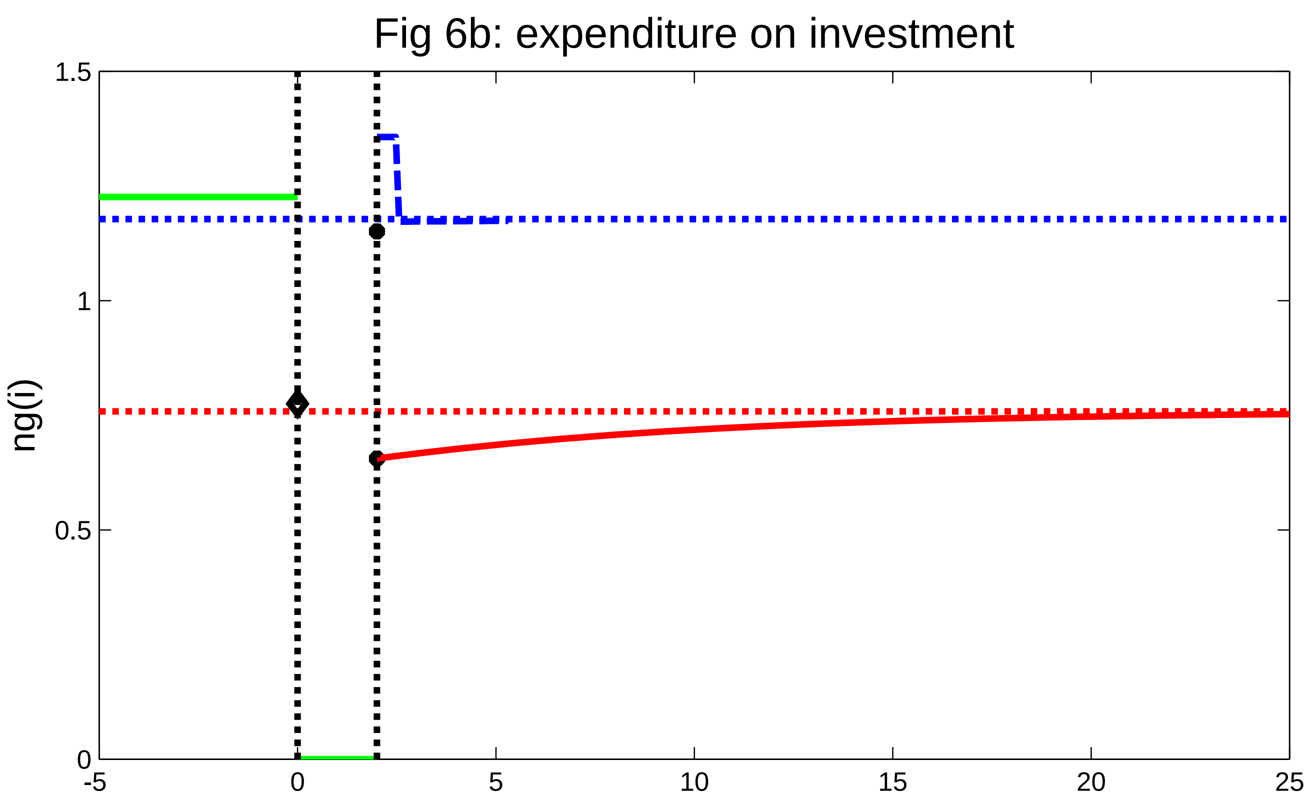


Fig 6b: expenditure on investment
ng(i)
1.5
1
0.5
0
-5
0
5
10
15
20
25