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Estimation of non-linear DSGE models made easy: taking second-order model approximations to the data (with an application to a DSGE model with a banking sector)

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February 15, 2013

A vast literature during the past two decades has provided econometric estimates of *linearized* DSGE models. Recently researchers have begun to estimate *non-linear* DSGE models using Sequential Monte Carlo ('particle filter') methods (see Fernández-Villaverde and Rubio-Ramírez (2007) for an early application). SMC methods are computationally intensive, which (given current computing technology) limits their use to small models with few shocks and endogenous state variables.

This paper presents a novel approach for the estimation of second-order accurate model approximations. I show how to construct a quadratic Kalman filter, and how to derive the (quasi-)likelihood function of the second-order accurate model. Importantly, the method is very simple and computationally very fast. No Monte Carlo simulations are needed for the algorithm. The method can thus be applied to large models. The approach can readily be extended to third-order (and higher-order) model approximations.

I provide an application of the method by estimating a non-linear DSGE model of an economy with a banking sector that faces a collateral constraint. In this setting, bank capital is a key state variable. The economy exhibits an important non-linearity, as negative shocks to bank capital have a much more detrimental effect on real activity when the health of the banking sector is poor, than when banks are well-capitalized. Estimates of the non-linear model, based on the novel method here, suggest that this non-linearity is quantitatively powerful in US and Euro Area data.

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1. Introduction

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Section 2 presents the new econometric approach. Section 3 studies the properties of the estimator, using a set of Monte Carlo experiments; that Section also compares the performance of the new method to existing methods based on the particle filter. Section 4 applies the method by estimating a DSGE model with a banking sector (Kollmann et al. (2011, 2012, 2013), Kollmann (2013)).

2. A new method for taking second-order model approximations to the data

Many widely studied DSGE models can be expressed as:

$$E_t \underset{n\times 1}{G}(\underset{n\times 1}{\Omega_{t+1}}, \Omega_t, \underset{1\times 1}{\xi}\cdot\underset{m\times 1}{\varepsilon_{t+1}}) = \underset{n\times 1}{0}, \qquad t \geq 0 \tag{1}$$

where E_t denotes the mathematical expectation conditional upon complete information about periods t and earlier; $G: R^{2n+m} \rightarrow R^n$ is a function, and Ω_t is an $n\times 1$ vector of variables known at date t; $\xi \geq 0$ is a scalar, and ε_{t+1} is an $m\times 1$ vector of date t+1 exogenous independent white noise innovations to exogenous forcing variables. The following discussion assumes that ε_t is multivariate normal with covariance matrix Σ_ε: $\varepsilon_t \sim N(0, \Sigma_\varepsilon)$.

The solution of (1) is a "policy function"

$$\Omega_{t+1} = F(\Omega_t, \xi), \tag{2}$$

that satisfies the condition $E_t G(F(\Omega_t, \xi\varepsilon_{t+1}, \xi), \Omega_t, \xi\varepsilon_{t+1}) = 0 \quad \forall \Omega_t, \forall \xi \geq 0$.

Schmitt-Grohé and Uribe (2004) show how to construct a second-order Taylor series expansion of the policy function around a deterministic steady state, i.e. around a point Ω such that $\Omega = F(\Omega, 0)$. Let $\omega_t \equiv \Omega_t - \Omega$ denote the deviation of the vector Ω_t from the steady state vector. The second-order accurate model solution has the following form:

$$\omega_{t+1}^i = f_0^i \xi^2 + f_1^i \omega_t + f_2^i \varepsilon_{t+1} + \tfrac{1}{2}\omega_t' f_{11}^i \omega_t + \omega_t' f_{12}^i \varepsilon_{t+1} + \varepsilon_{t+1}' f_{22}^i \varepsilon_{t+1} \quad \text{for } i=1,..,n, \tag{3}$$

where ω_{t+1}^i is the i-th element of the vector ω_t. $f_0^i, f_1^i, f_2^i, f_{11}^i, f_{12}^i, f_{22}^i$ are vectors/matrices of dimensions 1x1, 1x*n*, 1xm, *n*x*n*, nx*m* and *m*x*m*, respectively. (Schmitt-Grohé and Uribe (2004) show that these matrices are independent of the volatility of the exogenous shocks, i.e. independent of ξ).

For estimation is proves useful to write the quadratic forms $\omega_t' f_{11}^i \omega_t$ as a linear function of the squares and cross-products of the elements of the vector ω_t: $\frac{1}{2}\omega_t' f_{11}^i \omega_t = F_{11}^i \cdot P_2(\omega_t)$, where F_{11}^i and $P_2(\omega_t)$ are a row and a column vector consisting of the $\frac{1}{2}n\cdot(n+1)$ squares and cross-products of the elements of the vector ω_t. Similarly, we can write $\frac{1}{2}\varepsilon_{t+1}' f_{22}^i \varepsilon_{t+1} = F_{22}^i P_2(\varepsilon_{t+1})$. Also, note that $\omega_t' f_{12}^i \varepsilon_{t+1} = F_{12}^i P(\omega_t, \varepsilon_{t+1})$, where F_{12}^i is a row vector, while $P(\omega_t, \varepsilon_{t+1})$ is a column vector consisting of all cross products of elements of ω_t and ε_{t+1}. In the computer code that implements the estimation method, I define $P_2(x)$ and $P(x, y)$ as follows, for vectors x and y with n_x and n_y elements, respectively:

$$P_2(x) = [(x_1)^2, x_1x_2, .., x_1x_{n_x}, (x_2)^2, x_2x_3, .., x_2x_{n_x}, ..., (x_{n_x-1})^2, x_{n_x-1}x_{n_x}, (x_{n_x})^2],$$

$$P(x, y) = x \otimes y. \tag{4}$$

Using these definition, we can write the *n* equations shown in (3) in matrix form:

$$\omega_{t+1} = F_0\xi + F_1\omega_{t+1} + F_2\varepsilon_{t+1} + F_{11}P_2(\omega_t) + F_{12}P(\omega_t, \varepsilon_{t+1}) + F_{22}P_2(\varepsilon_{t+1}), \tag{5}$$

where $F_0, F_1, F_2, F_{11}, F_{12}, F_{22}$ are vectors/matrices obtained by stacking $f_0^i, f_1^i, f_2^i, F_{11}^i, F_{12}^i, F_{22}^i$ for i=1,..,n.

Note that the first-order accurate model solution is given by:

$$\omega_{t+1}^{(1)} = F_1\omega_t^{(1)} + F_2\varepsilon_{t+1}. \tag{6}$$

Pruning

When simulating second-order solutions it is common to replace the term $\omega_t' f_{11}^i \omega_t$ in (3) by $\omega_t^{(1)\prime} f_{11}^i \omega_t^{(1)}$, and to replace $\omega_t' f_{12}^i \varepsilon_{t+1}$ by $\omega_t^{(1)\prime} f_{11}^i \varepsilon_{t+1}$; in other terms, (3) is replaced by

$$\omega_{t+1}^i = f_0^i \xi^2 + f_1^i \omega_t + f_2^i \varepsilon_{t+1} + \tfrac{1}{2}\omega_t^{(1)\prime} f_{11}^i \omega_t^{(1)} + \omega_t^{(1)\prime} f_{11}^i \varepsilon_{t+1} + \varepsilon_{t+1}' f_{22}^i \varepsilon_{t+1}, \tag{7}$$

and thus (5) is replaced by:

$$\omega_{t+1} = F_0\xi + F_1\omega_{t+1} + F_2\varepsilon_{t+1} + F_{11}P_2(\omega_t^{(1)}) + F_{12}P(\omega_t^{(1)}, \varepsilon_{t+1}) + F_{22}P_2(\varepsilon_{t+1}). \tag{8}$$

This simulation procedure is known as ‘pruning’ (see Kollmann, Kim and Kim (2011)). The motivation for pruning is that equation (3) has extraneous steady states (that are not present in the original model)--some of these steady states mark transitions to unstable behavior. Large shocks thus can move the model into an unstable region. The pruning procedure overcomes this problem. It is motivated by the observations that in repeated applications of the second-order perturbation solution (3), third or higher-order terms of state variables appear. For example, when ω_{t+1} is quadratic in ω_t, then ω_{t+2} is quartic in ω_t. The pruning procedure removes these higher-order terms by computing the second-order terms using the squares of the linearized solution. Provided the first-order expansion is stable, the pruned version of a second-order approximation too is stable.

The estimation method presented below assumes that the data generating process is given by the combined system (8),(6).

Application of the Kalman filter to the pruned second-order system requires knowledge of the conditional expectation and covariance of the vector of squares and cross-products of first-order accurate states, $P_2(\omega_{t+1}^{(1)})$, as well as the conditional covariance $P_2(\omega_{t+1}^{(1)})$ with ω_{t+1} and $\omega_{t+1}^{(1)}$. To compute those conditional moments, I use the fact that

$$P_2(\omega_{t+1}^{(1)})=K_{11}P_2(\omega_t^{(1)})+K_{12}P(\omega_t^{(1)},\varepsilon_{t+1})+K_{22}P_2(\varepsilon_{t+1}), \qquad (10)$$

where K_{11}, K_{12}, K_{22} are matrices of dimensions n_z *x* n_z, n_z *x nm, and* n_z *x m*, respectively.

Stacking (5), (10)and (6) gives:

$$\begin{bmatrix} \omega_{t+1} \\ P_1(\omega_{t+1}^{(1)}) \\ \omega_{t+1}^{(1)} \end{bmatrix} = \begin{bmatrix} F_0\xi^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} F_1 & F_{11} & 0 \\ 0 & K_{11} & 0 \\ 0 & 0 & F_1 \end{bmatrix} \begin{bmatrix} \omega_t \\ P_1(\omega_t^{(1)}) \\ \omega_t^{(1)} \end{bmatrix} + \begin{bmatrix} F_2 \\ 0 \\ F_2 \end{bmatrix} \varepsilon_{t+1} + \begin{bmatrix} F_{12} \\ K_{12} \\ 0 \end{bmatrix} P(\omega_t^{(1)},\varepsilon_{t+1}) + \begin{bmatrix} F_{22} \\ K_{22} \\ 0 \end{bmatrix} P_2(\varepsilon_{t+1}). \qquad (11)$$

Denoting $Z_{t+1} \equiv (\omega_{t+1}', P_2(\omega_{t+1}^{(1)})', \omega_{t+1}^{(1)}{}')'$, the recursion (11) can be written as:

$$Z_{t+1}=G_0+G_1Z_t+u_{t+1}, \qquad (12)$$

where $u_{t+1} \equiv G_2\varepsilon_{t+1}+G_{12}P(\omega_t^{(1)},\varepsilon_{t+1})+G_{22}[P_2(\varepsilon_{t+1})-E(P_2(\varepsilon_{t+1}))]$ is a serially uncorrelated disturbance.

The econometrician is assumed to observe a subset, or a linear combination of the state vector ω_t for periods *t=1,..,T,* plus i.i.d. measurement error. The date *t* observables are, thus, given by:

$$y_{t+1} = \Gamma\omega_{t+1} + \nu_{t+1} = [\Gamma\ 0\ 0]Z_{t+1} + \nu_{t+1}, \qquad (9)$$

where Γ is a deterministic n_y *x n* matrix, with $n_y \le n$; while $\nu_{t+1} : N(0,\Sigma_\nu)$ is i.i.d. normally distributed measurement error that is independent of the innovations to exogenous forcing variable ε_s $\forall s$.

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