

Financial Frictions in Production Networks^{*}

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Abstract

We show that the organization of production among firms in an economy has important implications for the impact of financial frictions. We set up a model in which firms use output of other firms as inputs for their own production. We allow for arbitrary network structures such that aggregate production functions are constant. Therefore, in the absence of frictions these structures are allocatively equivalent. We then provide several examples which illustrate that when firms face liquidity constraints, different input-output structures require vastly different amounts of aggregate liquidity in order to implement identical allocations. This implies that the input-output structure of the economy is an important determinant of its response to a financial shock. Our main result is that financial constraints have a stronger impact on aggregate output when firms are engaged in a larger amount of transactions among themselves. Finally, we calibrate the model to match the input-output matrix of the U.S. economy and use this to explore the extent to which these interrelationships can explain the drop in output during the latest recession.

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1 Introduction

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In this paper we look at how financial frictions distort the aggregate economy under different network structures.

Layout. The rest of the paper is organized as follows. Section 2 introduces the model and characterizes the general equilibrium. Section 3 then explores the implications of networks and financial frictions for business cycles. Section 4 examines the general N-by-N case and generalizes the results from the simple example. Using the general network structure, we then calibrate this model to the input-output structure of the U.S. economy. Section 6 concludes.

2 Two Simple Economies

In this section we consider a very simple example which illustrates the main idea of how the production network structure interacts with the financial frictions. We consider the implications of collateral constraints in two economies which differ in their organizational structure of production. The vertical economy is an economy in which firms are arranged in a vertical supply chain. The horizontal economy is then engineered to be allocatively equivalent. We then characterize the general equilibrium in both economies

Vertical Economy. There are three firms that use labor and intermediate inputs to produce output. We assume that these firms are perfectly competitive, in that they take prices as given.¹ The firms are organized in a vertical supply chain. Specifically, their production functions are given by

$$\begin{aligned}y_{v1} &= A_1 n_{v1}^{\alpha_1} \\y_{v2} &= A_2 n_{v2}^{\alpha_2} y_{v1}^{\beta_2} \\y_{v3} &= A_3 n_{v3}^{\alpha_3} y_{v2}^{\beta_3}\end{aligned}$$

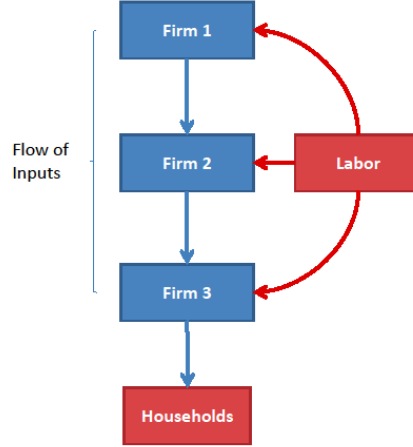
where y_{vi} is the amount produced by firm i and n_{vi} is the amount of labor employed by firm i . Thus, firm 1 uses labor as its sole input, however for $i = 2, 3$, firm i also uses as input the output of firm $i - 1$. Finally, the final consumption good is the output of firm three, that is $Y_v = y_{v3}$. We can therefore write the aggregate production function of the economy in terms

¹One can think of this simply as three sectors, each composed of a continuum of perfectly competitive firms.

of labor as follows

$$Y_v = y_{v3} = A_3 n_{v3}^{\alpha_3} (A_2 n_{v2}^{\alpha_2})^{\beta_3} (A_1 n_{v1}^{\alpha_1})^{\beta_2 \beta_3}$$

For simplicity, assume CRS: $\alpha_3 + \alpha_2 \beta_3 + \alpha_1 \beta_2 \beta_3 = 1$. The following figure illustrates the flow of inputs and output in the vertical economy.



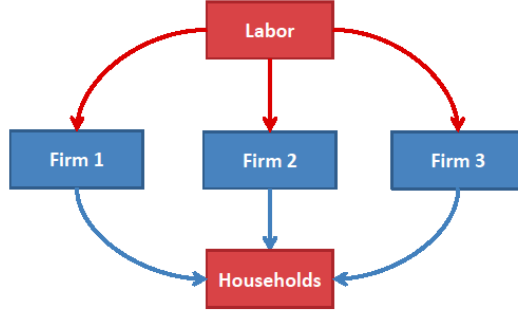
Horizontal Economy. Now consider an equivalent, but completely horizontal economy. There are three representative firms that use labor to produce a good. The production functions of these representative firms are as follows

$$\begin{aligned} y_{h1} &= A_1 n_{h1}^{\alpha_1} \\ y_{h2} &= A_2 n_{h2}^{\alpha_2} \\ y_{h3} &= A_3 n_{h3}^{\alpha_3} \end{aligned}$$

where y_{hi} is the amount produced by firm i and n_{hi} is the amount of labor employed by firm i . These three goods aggregated into a final consumption good, Y_h . We normalize this consumption basket so as to make it equivalent to the final good in the vertical economy: $Y_h = Y_v$

$$Y_h = y_{h1}^{\beta_2 \beta_3} y_{h2}^{\beta_3} y_{h3}$$

Therefore, both economies have the same aggregate production function. However, unlike the vertical economy, the firms in the horizontal economy operate in isolation from one another, only combining at the end in terms of consumption, as in Dixit-Stiglitz. The following figure illustrates the flow of inputs and output in the horizontal economy.



Households and Market Clearing. To close the economy we introduce households. In either economy, there is a representative household with preferences given by

$$U(C) - V(N)$$

where $U : \mathbb{R} \rightarrow \mathbb{R}$ is increasing and concave, $V : \mathbb{R} \rightarrow \mathbb{R}$ is increasing and convex, C is the final good consumption, and N is labor supplied competitively to the market. The household's budget constraint is given by

$$C = wN$$

where w is the competitive real wage rate and where we have normalized the price of the final good to 1. Finally, for markets to clear, we have that consumption is equal to output $Y = C$, and labor supply equals labor demand $N = n_1 + n_2 + n_3$.

Remarks and Notation. We will use $\varepsilon \in \{v, h\}$ to denote the economy of interest, where $\varepsilon = v$ denotes the vertical economy and $\varepsilon = h$ the horizontal.

Our first remark is that in this paper we abstract from investment. The model is static, so that firms only have static inputs. As one will see later, the financial constraint will be on working capital.

Second, note that we are taking the network structure of these economies as exogenous. As will be seen later, there will be incentives for firms to merge or vertically integrate when there are frictions. This then begs the question of why firms are structured in the economy as they are. The theory of the firm and its boundaries is an interesting one, however, here we abstract from these considerations and take the firm boundaries as given. See Antras

(2011) for a sequential production chain regarding the optimal allocation of ownership rights along a supply chain.

Equilibrium Definition. We define the competitive equilibrium in either economy as follows

Definition 1. A competitive equilibrium in economy $\varepsilon \in \{v, h\}$ is a collection of quantities $\{n_{\varepsilon 1}, n_{\varepsilon 2}, n_{\varepsilon 3}, y_{\varepsilon 1}, y_{\varepsilon 2}, y_{\varepsilon 3}, N_{\varepsilon}, Y_{\varepsilon}\}$ and prices $\{p_{\varepsilon 1}, p_{\varepsilon 2}, p_{\varepsilon 3}, w_{\varepsilon}\}$ such that

- (i) each representative firm maximizes profits,
- (ii) the representative household maximizes utility,
- (iii) markets clear.

This is a standard definition for a Walrasian equilibrium in a production economy.

2.1 Frictionless Benchmark

As a benchmark, we first consider the equilibrium in either economy in the absence of frictions. By construction, the vertical and horizontal economies are allocationally equivalent.

Proposition 1. In either economy without frictions, there exists a unique equilibrium allocation. In either economy $\varepsilon \in \{v, h\}$, the unique equilibrium allocation is given by

$$\tilde{\alpha}_3 \frac{Y_{\varepsilon}}{n_{\varepsilon 3}} = \tilde{\alpha}_2 \frac{Y_{\varepsilon}}{n_{\varepsilon 2}} = \tilde{\alpha}_1 \frac{Y_{\varepsilon}}{n_{\varepsilon 1}} = \frac{V'(N)}{U'(Y)} \quad (1)$$

and

$$N = n_1 + n_2 + n_3$$

where

$$\tilde{\alpha}_3 = \alpha_3, \quad \tilde{\alpha}_2 = \alpha_2 \beta_3, \quad \tilde{\alpha}_1 = \alpha_1 \beta_2 \beta_3$$

denote the each firm's labor share in the aggregate production function.

The aggregate production function simply transforms each type of labor into aggregate output. In equilibrium, the marginal rate of transformation of each type of labor is equal to the marginal rate of substitution. Thus, absent any frictions, the way production of an economy is broken down into different firms or network structures is irrelevant. This is a simple extension of the Modigliani-Miller () result. Here, instead of considering how an individual firm is sliced up in terms of financing, we consider how a macroeconomic production function is sliced up into different units. Aside from any frictions, for a given aggregate production function, the way it is cut into different production units does not matter for allocations.

Proposition 1 further implies that we may write

$$\bar{\alpha} \frac{Y}{N} = \frac{V'(N)}{U'(Y)}$$

where $\bar{\alpha} = \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3$ is the total labor share of output. Thus, these economies admit a representative firm, with production function $Y = \bar{A}N^{\bar{\alpha}}$, where $\bar{A} = A_1^{\beta_2\beta_3}A_2^{\beta_3}A_3$ is aggregate productivity.

3 Implications of Financial Frictions

After establishing the allocational equivalence of the two economies in the absence of frictions, we now consider the implications of adding financial frictions. Financial frictions introduce distortions into the economy, relative to the frictionless benchmark; however, depending on the network structure, these frictions distort the two economies in different ways.

We introduce financial frictions by adding collateral constraints on input purchases. We assume that each firm faces a constraint in which their expenditure on inputs is constrained to be less than or equal to a fraction χ of their revenue. One can think of this as follows: firms can credibly commit to pay only a fraction χ of their revenue to laborers or suppliers, and can abscond with the rest after production and sales are realized. Hence, expenditure on inputs cannot exceed the pledgeable portion of their revenue. In the horizontal economy this pledgeability constraint is given by the following.

$$w_h n_{hi} \leq \chi_{hi} p_{hi} y_{hi} \quad (2)$$

On the other hand, in the vertical economy, only firm 1 uses labor as a sole input so faces a constraint as in (2), while firms 2 and 3 face the following constraint

$$w_v n_{vi} + p_{v,i-1} y_{v,i-1} \leq \chi_{vi} p_{vi} y_{vi} \quad (3)$$

Thus, firms face working capital constraints.²

The financial frictions introduce distortions into the two economies. However, we note that these constraints are not directly comparable across the two economies, as the vertical economy firms must finance both labor and intermediate goods whereas firms in the horizontal economy need only finance their wage bill. We take this into consideration in our analysis.

²This working-capital constraint is similar to the static-input financing example in Chari, Kehoe, and McGrattan ().

That is, our results will not depend on how χ_{hi} directly compares to χ_{vi} , but instead will rest on how these constraints manifest themselves differently in terms of distorting the two economies.

Equilibrium Characterization. We first examine these distortions at the individual firm level. The pledgeability constraint faced by any firm introduces a wedge between the firm's marginal benefit and marginal cost of production. In the horizontal economy, each firm's production is pinned down by

$$w_h = \phi_{hi} p_{hi} \alpha_i \frac{y_{hi}}{n_{hi}}$$

where

$$\phi_{h1} = \min \left\{ 1, \frac{\chi_{h1}}{\alpha_1} \right\}, \quad \phi_{h2} = \min \left\{ 1, \frac{\chi_{h2}}{\alpha_2} \right\}, \quad \phi_{h3} = \min \left\{ 1, \frac{\chi_{h3}}{\alpha_3} \right\} \quad (4)$$

This simply states that the marginal cost is equal to the marginal benefit, times some wedge. This individual wedge represents the distortion for that firm away from its optimal labor usage due to the collateral constraint. That is, for any firm i , the wedge $\phi_{hi} \in [0, 1]$. When $\chi_{hi} < \alpha_i$, the firm's pledgeability constraint is binding, and the wedge is given by $\phi_{hi} = \chi_{hi}/\alpha_i$. The constraint is binding whenever χ_{hi} is less than the labor share of output of firm i . On the other hand, if $\chi_{hi} \geq \alpha_i$, then the firm's pledgeability constraint is not binding—firms operate at their optimal level and have enough funds to cover their expenses. In this case, there is no wedge between the firm's marginal benefit and marginal cost of production, i.e. $\phi_{hi} = 1$.

In the vertical economy, firms solve a cost minimization problem in terms of its expenditure on each of its inputs: labor and the intermediate good. This cost minimization implies that their expenditure on each good is equal to the ratio of the relative shares of each input in production.

$$\frac{w_v n_{vi}}{p_{v,i-1} y_{v,i-1}} =$$

Given this condition, each firm's production is then pinned down by the following condition

$$w_v = \phi_{vi} p_{vi} \alpha_i \frac{y_{vi}}{n_{vi}}$$

where

$$\phi_{v1} = \min \left\{ 1, \frac{\chi_{v1}}{\alpha_1} \right\}, \quad \phi_{v2} = \min \left\{ 1, \frac{\chi_{v2}}{\alpha_2 + \beta_2} \right\}, \quad \phi_{v3} = \min \left\{ 1, \frac{\chi_{v3}}{\alpha_3 + \beta_3} \right\} \quad (5)$$

Again, for any firm i , the wedge $\phi_{hi} \in [0, 1]$ represents the distortion in optimal production level due to the collateral constraint. This the same condition as in the horizontal economy;

however, the only difference here is that for firms 2 and 3, the constraint is binding whenever $\chi_{vi} < \alpha_i + \beta_i$, that is, when the pledgeability ratio χ is less than the total output share of inputs (the labor share plus the share of intermediate goods).

Combining the individual firm conditions with market clearing and household optimality conditions, we reach the following proposition which fully characterizes the equilibrium allocation.

Proposition 2. *Suppose firms face pledgeability constraints*

(i) *In the horizontal economy, the unique equilibrium allocation is given by*

$$\begin{aligned} (\phi_{h3}) \tilde{\alpha}_3 \frac{Y_h}{n_{h3}} &= V'(N_h) / U'(Y_h) \\ (\phi_{h2}) \tilde{\alpha}_2 \frac{Y_h}{n_{h2}} &= V'(N_h) / U'(Y_h) \\ (\phi_{h1}) \tilde{\alpha}_1 \frac{Y_h}{n_{h1}} &= V'(N_h) / U'(Y_h) \\ N_h &= n_{h1} + n_{h2} + n_{h3} \end{aligned}$$

(ii) *In the vertical economy, the unique equilibrium allocation is given by*

$$\begin{aligned} (\phi_{v3}) \tilde{\alpha}_3 \frac{Y_v}{n_{v3}} &= V'(N_v) / U'(Y_v) \\ (\phi_{v2} \phi_{v3}) \tilde{\alpha}_2 \frac{Y_v}{n_{v2}} &= V'(N_v) / U'(Y_v) \\ (\phi_{v1} \phi_{v2} \phi_{v3}) \tilde{\alpha}_1 \frac{Y_v}{n_{v1}} &= V'(N_v) / U'(Y_v) \\ N_v &= n_{v1} + n_{v2} + n_{v3} \end{aligned}$$

In either economy, for each type of labor there is now a wedge between its marginal rate of transformation into aggregate output and its marginal rate of substitution. In the horizontal economy this wedge between the MRT and the real wage is simply the same wedge that arises in the individual firm decisions. This is due to the fact that each firm in the horizontal economy operates in isolation. Whatever distortion shows up at the firm level only effects the marginal rate of transformation for that type of labor, but not for others. On the other hand, in the vertical economy, this is not the case. The individual wedge of firm 2 affects the wedge for both firm 2 and firm 1. Similarly, the wedges of firm 3 affect the wedges found in firms 1 and 2. Thus, the downstream financial frictions distort upstream input use. As in Jones (2011,), it has a multiplier effect, which we will evaluate in the next subsection.

Finally, note that these economies with financial frictions are isomorphic to an economy without frictions, but with taxes given by $(1 - \tau_i) = \phi_i$ for $\tau \in [0, 1]$. That is, it is isomorphic

to an economy where firms face taxes but not subsidies. This relates to the literature on taxation and supply chains, for example, the input-output model of Jones (2011).

3.1 The Aggregate Labor Wedge

We now look at how these distortions at the individual firm level affect the economy at the aggregate level. One variable of interest is the aggregate labor wedge. A large literature has documented large labor wedges as well as countercyclicality of this wedge (e.g., Hall, 1997; Rotemberg and Woodford, 1999; Chari, Kehoe, and McGrattan, 2007; Shimer, 2009). Following the literature, we define the aggregate labor wedge $(1 - \tau)$ implicitly by

$$(1 - \tau) \bar{\alpha} \frac{Y}{N} = \frac{V'(N)}{U'(C)}$$

That is, the aggregate labor wedge is simply the wedge between the aggregate marginal rate of transformation of aggregate labor, and the marginal rate of substitution. Using our results for the equilibrium, we may also back out the aggregate labor wedge in these economies. Note that in the frictionless economy, $\tau = 0$ so that there is no wedge or aggregate distortion. The following proposition then characterizes the aggregate labor wedge when firms face pledgeability constraints.

Proposition 3. *Suppose firms face pledgeability constraints. (i) In the horizontal economy, the aggregate labor wedge is given by*

$$(1 - \tau_h) = \tilde{\alpha}_3 (\phi_{h3}) + \tilde{\alpha}_2 (\phi_{h2}) + \tilde{\alpha}_1 (\phi_{h1})$$

(ii) In the vertical economy, the aggregate labor wedge is given by

$$(1 - \tau_v) = \tilde{\alpha}_3 (\phi_{v3}) + \tilde{\alpha}_2 (\phi_{v2}\phi_{v3}) + \tilde{\alpha}_1 (\phi_{v1}\phi_{v2}\phi_{v3})$$

Thus, the pledgeability constraints introduce aggregate labor wedges between the aggregate marginal product of labor and the real wage. Proposition 3 makes clear that the aggregate labor wedge is simply a weighted sum of the individual labor wedges—the weighting is simply given by each labor share of aggregate output. However, depending on how the constraints ϕ 's affect the individual labor wedges, the aggregate labor wedges differ across the two economies. In the horizontal economy all constraints are weighted according to their respective labor share. On the other hand, in the vertical economy, the downstream constraint, ϕ_3 , has the greatest impact on the aggregate wedge. This is because downstream constraints distort upstream labor choices.

Thus, we see that the way the economy is sliced into different sectors and components, and where each constraint binds, matters for the aggregate labor wedge. That is, when there are financial frictions, different network structures lead to different aggregate distortions.

A number of other papers have shown that financial frictions lead to static wedges. Chari, Kehoe, McGrattan (2007) consider a static-input financing friction and show that this leads to an efficiency wedge.³ Buera and Moll (2012) also find a labor wedge due to financial frictions in a search model of labor. Here, we show that if individual firms face constraints, the network structure is important in determining how each constraint contributes to the aggregate labor wedge. This result shows that financial frictions can lead to labor wedges; in particular, this labor wedge will depend on the network structure.

3.2 Aggregate Liquidity

Another way these two economies differ is in the aggregate amount of liquidity needed for any allocation. Let us first define the aggregate amount of pledgeable funds, i.e. liquidity, as follows.

Definition 2. For economy $\varepsilon \in \{v, h\}$, let M_ε denote the aggregate amount of liquidity

$$M_\varepsilon \equiv \chi_{\varepsilon 1} p_{\varepsilon 1} y_{\varepsilon 1} + \chi_{\varepsilon 2} p_{\varepsilon 2} y_{\varepsilon 2} + \chi_{\varepsilon 3} p_{\varepsilon 3} y_{\varepsilon 3}$$

That is, we define liquidity M_ε to be the aggregate amount of pledgeable funds. As we've mentioned previously, we cannot directly compare the constraints across the two economies. Hence, fixing the vectors $\{\}$ and $\{\}$ and then comparing the liquidity across the two economies would be uninformative. However, we can instead ask what is the liquidity needed in either economy to implement a given allocation. First, we define a feasible allocation in the economy as follows.

Definition 3. An allocation $\{n_1, n_2, n_3, N, Y\}$ is feasible if and only if $n_1 + n_2 + n_3 = N$ and $Y = A_3 n_{v3}^{\alpha_3} (A_2 n_{v2}^{\alpha_2})^{\beta_3} (A_1 n_{v1}^{\alpha_1})^{\beta_2 \beta_3}$.

Thus, an allocation is feasible if and only if it satisfies the economy's resource constraints. Now, suppose we fix a feasible allocation. In order to implement this allocation as an equilibrium outcome, what is the minimum amount of liquidity needed in order to do so?

Proposition 4. Fix some feasible allocation $\{n_1, n_2, n_3, N, Y\}$. Then, the minimum liquidity needed to implement this allocation is given by

³See Section ?? of their paper.

(i) the liquidity needed in the horizontal economy to achieve this allocation is given by

$$M_h = \frac{V'(N)}{U'(Y)}N \quad (6)$$

(ii) the liquidity needed in the vertical economy to achieve this allocation is given by

$$M_v = \frac{V'(N)}{U'(Y)} \left(N + \frac{\beta_2}{\alpha_2}n_2 + \frac{\beta_3}{\alpha_3}n_3 \right) \quad (7)$$

Therefore

$$M_v > M_h$$

Thus, we find that the amount of liquidity needed to implement any feasible allocation is strictly greater in the vertical economy than in the horizontal economy. Note that this proposition is stated in terms of the allocation alone, not in terms of the constraints ϕ , thereby making the two measures directly comparable.

The intuition for this result is quite simple. In the horizontal economy, firms need only to finance their own cost of labor, that is, their own added value. Thus, the aggregate amount of liquidity needed to implement a feasible allocation is simply just the sum of the equilibrium wage bills

$$M_h = wn_1 + wn_2 + wn_3$$

where the real wage w is equal to the marginal rate of substitution $V'(N)/U'(C)$ in equilibrium. In the vertical economy, on the other hand, it is as if there is double counting. In addition to the value of their own labor, firms in the vertical economy must also finance their expenditure on intermediate goods, thereby also pledging collateral for the labor purchased upstream.

$$M_v = wn_1 + \left(wn_2 + \frac{1}{\chi_1}wn_1 \right) + \left(wn_3 + \frac{1}{\chi_2}wn_2 + \frac{1}{\chi_2\chi_1}wn_1 \right)$$

Furthermore, from comparing (6) to (7), we see that the greater the output shares β_2 and β_3 of the intermediate goods, the greater the amount of liquidity needed in the vertical economy relative to the horizontal in order to implement the same allocation.

Moreover, note that in the vertical economy the aggregate amount of liquidity is greater than aggregate expenditure on labor; yet, despite this difference in liquidity and labor expenditure, collateral constraints are still binding. Chari, Christiano, Kehoe (2009) find that in the aggregate, among public companies, retained earnings plus dividends are greater than capital expenditures. One potential conclusion that they draw from this is that financial frictions do not matter, as firms could clearly finance their own capital expenditures with

their own liquidity. There are a number of caveats to this finding—first, that these are only public companies; second, that this is only looking at the aggregate rather than individual firms and hence not taking into account distributional effects. Our findings here challenge this conclusion in another way—which firms are constrained and where they are in the production network matter. Here, in the vertical supply chain, the aggregate amount of funds in equilibrium is greater than the aggregate expenditure on labor (not including intermediate inputs), yet firms are still constrained by their collateral. If instead this were a representative firm economy or a horizontal economy, this would not be the case. Thus, the conclusion we obtain from this simple exercise is that the aggregate amount of available funds may not indicate the bite of financial frictions.

Finally, we see a close relationship here to the quantity theory of money. Effectively, one can think of the pledgeability constraints on the firms as analogous to cash-in-advance constraints. Money can be thought of as anything used to make transactions. Hence, the aggregate amount of pledgeable funds is similar to the amount of money in the economy. With this interpretation in mind, our results are similar to the following representation of the quantity theory of money,

$$MV = PT$$

where M is money, V is the velocity of money, P is the aggregate price level, and T is the aggregate number of transactions. Suppose that the aggregate price level P is normalized to 1 and that the velocity V is a constant. This implies that the level of transactions in the economy is proportional to amount of money. This general concept holds true in our model: more money is needed in the vertical economy than in the horizontal economy because there is a greater level of transactions occurring between firms. Therefore, given any equilibrium allocation, the more transactions made in the economy, that is, the more times goods change hands between firms, the more money is necessary to complete these transactions and implement the allocation.

3.3 Aggregate effects of Tightening Constraints

Next, we consider how the tightening of these collateral constraints affects aggregate output. For simplicity, we specify a particular utility function in order to solve for aggregate output in closed form. Suppose that utility over consumption and labor is given by

$$U(C) - V(N) = \log C - N$$

The assumption of log-linear utility over consumption and linear disutility of labor is not crucial for any of our results, but simplifies the expressions considerably. Given this specification, aggregate output is given in the following Lemma.

Lemma 1. (i) *Aggregate output in the frictionless economy is given by*

$$\bar{Y} = \bar{A} (\tilde{\alpha}_3)^{\tilde{\alpha}_3} (\tilde{\alpha}_2)^{\tilde{\alpha}_2} (\tilde{\alpha}_1)^{\tilde{\alpha}_1}$$

(ii) *Suppose firms face pledgeability constraints and fix vectors $\{\phi_{v1}, \phi_{v2}, \phi_{v3}\}$ and $\{\phi_{h1}, \phi_{h2}, \phi_{h3}\}$. Given these constraints, aggregate output in the horizontal economy is given by*

$$Y_h = \bar{Y} (\phi_{h3})^{\tilde{\alpha}_3} (\phi_{h2})^{\tilde{\alpha}_2} (\phi_{h1})^{\tilde{\alpha}_1} \quad (8)$$

and aggregate output in the vertical economy is given by

$$Y_v = \bar{Y} (\phi_{v3})^{\tilde{\alpha}_3} (\phi_{v2}\phi_{v3})^{\tilde{\alpha}_2} (\phi_{v1}\phi_{v2}\phi_{v3})^{\tilde{\alpha}_1} \quad (9)$$

Lemma 1, thus provides expressions for aggregate output in the frictionless economy, as well as in the vertical and horizontal economies with financial frictions. Note that because $\phi \in [0, 1]$ and strictly less than 1 whenever collateral constraints are binding, this implies that output in the horizontal economy is lower when constraints are binding than when they are not, as expected. Similarly output in the vertical economy is lower when constraints are binding than when they are not. Using these expressions for aggregate output, we now consider the aggregate effect of tightening individual collateral constraints. We obtain the following result.

Proposition 5. *In the horizontal economy*

$$\begin{aligned} \frac{d \log Y_h}{d \log \phi_{h1}} &= \tilde{\alpha}_1 > 0 \\ \frac{d \log Y_h}{d \log \phi_{h2}} &= \tilde{\alpha}_2 > 0 \\ \frac{d \log Y_h}{d \log \phi_{h3}} &= \tilde{\alpha}_3 > 0 \end{aligned}$$

In the vertical economy

$$\begin{aligned}\frac{d \log Y_v}{d \log \phi_{v1}} &= \tilde{\alpha}_1 \\ \frac{d \log Y_v}{d \log \phi_{v2}} &= \tilde{\alpha}_2 + \tilde{\alpha}_1 \\ \frac{d \log Y_v}{d \log \phi_{v3}} &= \tilde{\alpha}_3 + \tilde{\alpha}_2 + \tilde{\alpha}_1\end{aligned}$$

Consider first the horizontal economy. The effect on aggregate output of tightening any of the collateral constraints is simply equal to its labor share of total output. That is, tightening a collateral constraint leads to a 1-for-1 decrease in the labor employed at firm i . This is due to the fact that with log-utility over consumption and linear disutility of labor, income and substitution effects cancel, and thus labor simply falls 1-for-1 with the collateral constraint. The effect of a fall in labor also affects aggregate output according to its labor share.

On the other hand, in the vertical economy the constraints downstream have a greater impact on aggregate output than those upstream. For example, a percentage change in the collateral constraint of firm 1 leads to the a fall in aggregate output equal to its labor share—the same as in the horizontal case. In contrast, a percentage change in the collateral constraint on firm 3 has a greater effect than its own labor share, instead it is the sum of the labor shares of all firms 1, 2, and 3. The reason for this is that not only is there a direct effect on firm 3 employment, but it also directly affects the labor chosen by firms 1 and 2, due to reduced demand for their products. We study these spill-over effects more closely in the following subsection. For now, we see that in terms of aggregate output, downstream liquidity has the largest effect.

Finally, suppose all constraints were to tighten at the same time—how much would aggregate output fall in response to this aggregate tightening?

Proposition 6. *Suppose we scaled down all collateral constraints by x percent so that each firm faces a collateral constraint given by $\phi(1 - x)$. Then aggregate output falls more in the vertical economy than in the horizontal economy*

$$\frac{d \log Y_v}{d \log x} < \frac{d \log Y_h}{d \log x} < 0$$

That is, the liquidity multiplier is greater in the vertical economy than in the horizontal.

Formally, suppose we scaled down all constraints by x percent. The drop in aggregate output due to this fall in aggregate liquidity is given by $d \log Y / d \log x < 0$. We call this

object $|d \log Y_v / d \log x|$ the liquidity multiplier—it tells us how much aggregate output falls due to a 1 percent decrease in collateral constraints across the board. First, note that in the horizontal economy, this multiplier is equal to the aggregate labor share, and therefore Note that if the share of labor is 1, as in CRS production function, then the liquidity multiplier must be equal to 1. On the other hand, the multiplier in the vertical economy must necessarily be greater than 1. In fact, in our calibration results in Section ??, we find a liquidity multiplier of 3.5 in the U.S. economy. In conclusion, aggregate output falls more in the vertical economy than in the horizontal economy.

3.4 Spill-over effects of Tightening Constraints

In the previous subsection we explored the aggregate effects of tightening credit constraints. However, underlying these results is the behavior of individual firms and prices in response to any liquidity shock. To understand this, we now look at the spill-over effects from tightening the collateral constraints of an individual firm. How does the tightening of a constraint of one firm affect the production of other firms in the economy? First, for the horizontal economy, clearly there are no direct spill over effects since there are no linkages among firms. The only effects that could be coming from the centralized labor market—income effects on the wage. Our specification for utility kills these indirect effects (see appendix for an explanation), so there are only direct spill-over effects.

On the other hand, in the vertical economy there are direct spill-over effects to other firms from a tightening of the constraint of one firm. To understand this more generally, we extend the economy from 3 firms to K firms and shock the constraint of sector i . The effects are described in the following proposition.

Proposition 7. *(i) In the horizontal economy, there are no direct spill-over effects of tightening constraints. A drop in the constraint of firm i leads to a fall in firm i employment and output and an increase in its price. However, there are no direct spill-over effects for the output, employment, and prices of firms $j \neq i$.*

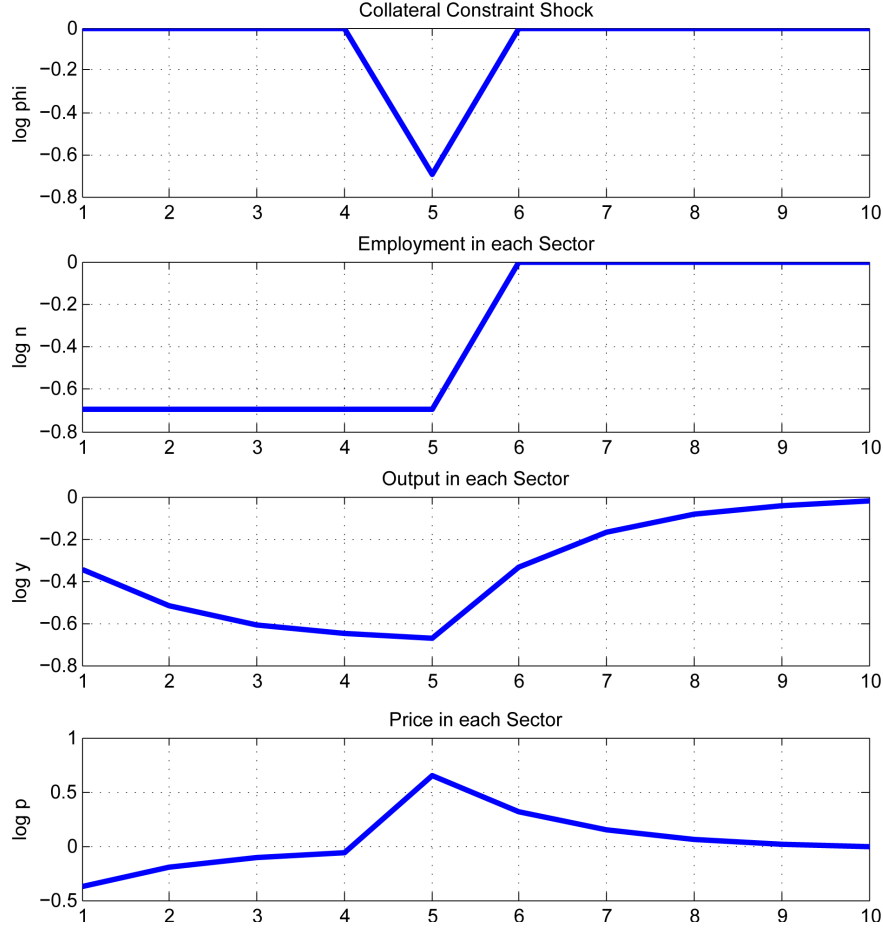
(ii) In a vertical economy with K firms, there are direct spill-over effects following a tightening of a collateral constraint of firm i

for firm i , employment falls, intermediate input use falls, and output falls. Its equilibrium price rises.

for firms $i - k$: employment falls, intermediate input use falls, output falls. Its equilibrium price falls.

for firms $i + k$: employment remains unchanged but intermediate input use falls; thus, production falls. Its equilibrium price rises.

We can further summarize these effects in the following graphs. In these graphs, the vertical supply chain so that there are 10 firms. We shock the collateral constraint of firm 5 and study what happens in equilibrium to all firms.



Consider the effects of tightening the collateral constraint of firm 5. This implies that this firm purchases both less labor and less intermediate inputs. Thus, its labor and inputs decrease and hence its output falls. Given that its supply decreases, its price thereby increases.

For firm 4, there is now less demand for its good from firm 5. This implies that in equilibrium its output and price falls. In order to produce less output, it both hires less labor and buys less intermediate inputs. Furthermore, this implies that the demand for the good of firm 3 falls. Firm 3 therefore undergoes the same qualitative effects as firm 4: its output, employment, and intermediate input use all falls. But this implies that the demand for the good of firm 2 falls, and so on.

For firm 6, the price of its input (the output of firm 5) is now higher. Thus, it demands less of its intermediate inputs, however, its employment remains unchanged. This firm produces less and because its supply decreases, its price thereby increases.

Thus, there are numerous spill-over effects coming from the tightening of one firm's collateral constraint. In summary, for firms $i - 1$ there is less demand for their good, so this is like a demand effect. For firms $i + 1$, there is an increase in input prices, so this acts like a supply effect. For all firms, these are adverse effects, so that production decreases across the board. One would not see this in a horizontal model nor a representative firm model.

Again, if we had allowed for a different specification of utility such as power utility, we would also have had indirect effects coming from a change in the real wage. However, we've shut these down by using log utility in consumption and linear disutility in labor. If we had allowed for effects coming from the real wage (income and substitution effects) these would be aggregate effects—thereby affecting all firms equally. Therefore, these indirect aggregate effects would have not affected the qualitative results in Proposition 7, and hence without loss of generality we can take them out.

3.5 Misallocation

There is a large and growing literature on misallocation in growth and development. Restuccia and Rogerson (2008) show that misallocation of resources across firms can have important effects on aggregate TFP. Banerjee and Duflo (2005) emphasize the importance of resource misallocation in understanding aggregate TFP differences across countries. Furthermore, Hsieh and Klenow (2009) provide quantitative evidence on the potential impact of resource misallocation on aggregate TFP. They present empirical evidence that misallocation across plants may reduce TFP in manufacturing by a factor of two to three in China and India. Jones (2010).. We now consider the question of measuring misallocation in our economy. Here we make the case that standard measures of misallocation are not appropriate for network economies and we propose a new measure which takes into account the location of each firm within the network.

There are a number of ways to look at misallocation. In general the misallocation literature has looked at the dispersion of marginal revenue products across plants for industries. Hsieh and Klenow (2009) define marginal revenue product of labor as follows

$$MRP_i \equiv \alpha_i \frac{p_i y_i}{n_i} \quad (10)$$

where α_i is the labor-share of output in firm i , n_i is the amount of labor that firm hires, and $p_i y_i$ is its revenue.⁴

⁴Hsieh and Klenow (2009) define both Marginal revenue product of labor and Marginal revenue product of capital. Here, since we have only labor, we provide only the definition for marginal revenue product of labor.

Hsieh and Klenow (2009) consider something similar to a horizontal economy—firms are heterogeneous and monopolistically competitive. Firm products are aggregated... We now look at the composition of dispersion in marginal products in our economy. Note that in either economy, the following relation holds: $w_v = \phi_{vi}\alpha_i p_{vi} y_{vi} / n_{vi}$. That implies that

$$MRP_i = \alpha_i \frac{p_{vi} y_{vi}}{n_{vi}} = \frac{w_v}{\phi_{vi}}$$

If $\phi_i = 1$ for all firms, then there is no misallocation—marginal revenue products are equalized across all sectors. Now suppose ϕ_{vi} is strictly less than 1 but constant across all firms. We thus reach the following lemma. Thus, MRP is constant across all firms by construction. However, we show that this hides the aggregate distortions that are in fact present.

Lemma 2. *Suppose all firms are equally constrained: $\phi_i = \phi < 1$. Then, there is no dispersion in MRP_i in the horizontal economy nor in the vertical economy. However, the horizontal economy is less distorted: it has greater TFP and a greater labor wedge than the vertical economy.*

However, note that in the horizontal economy TFP is greater than in the vertical economy. Thus, at the micro level, if all firms are constrained equally, one would not see any dispersion in marginal products in either economy. However, the vertical economy would have much lower "efficiency" as measured by a greater labor wedge. We also show the same is true if one were to measure this with aggregate TFP. This implies that the misallocation found in Hsieh and Klenow (2009) may in fact be *greater* once one takes into account the network structure of these firms. That is, the horizontal economy provides a *lower bound* for the amount of misallocation generated in a network economy with frictions.

Thus, here we argue that MRP , as defined in (10) is not the right measure to use when thinking about misallocation in network economies with vertical supply chains. We argue for a new measure of misallocation in which we look at the dispersion of the following measure which we call the "Marginal Aggregate Revenue Product"

$$MARP \equiv \tilde{\alpha}_i \frac{PY}{n_i} \tag{11}$$

In our economy, this should be equal to

$$MARP = \tilde{\alpha}_i \frac{PY}{n_i} = \frac{w}{\phi_i}$$

Again, in the frictionless economy, there is no misallocation—the Marginal Aggregate Revenue Product is equated across all firms according to (1): $\tilde{\alpha}_i \frac{PY}{n_i} = w$. Thus, there is no dispersion

in marginal aggregate products. In the horizontal economy, there again is no misallocation in the sense that there is no dispersion in marginal products. However, in the vertical economy, there is misallocation in the sense that there is dispersion in this measure in marginal aggregate product.

Finally, we can also calculate the marginal products across each of the vertical supply chain firms. An important empirical implication emerges here. In terms of the vertical supply chain, wedges are unambiguously larger for upstream labor, than for downstream labor. Thus, if one has data on the amount of labor used in each sector, and its labor share in aggregate output, one could easily back out the wedges. For example, in the vertical supply chain.

Proposition 8. *Suppose all firms are equally constrained: $\phi_i = \phi < 1$. Then*

- (i) in the horizontal economy with frictions, there is no dispersion in MARP.*
- (ii) in the vertical economy, there is dispersion in MARP.*

Thus, *MARP* is a better indicator of misallocation in a vertical economy. Furthermore, the marginal aggregate product of labor of firm i is less than the marginal aggregate product of firm $i - 1$.

$$(\phi_{v1}\phi_{v2}\phi_{v3})\tilde{\alpha}_1\frac{Y_v}{n_{v1}} = (\phi_{v3})\tilde{\alpha}_3\frac{Y_v}{n_{v3}}$$

This is one empirical implication that one may be able to look for in the data. Not sure whether we should put this in a proposition or not.

3.6 Summary

In this section we studied two economies, a horizontal economy in which firms did not transact with each other and a vertical economy in which firms were arranged in a supply chain. These economies were allocationally equivalent under no frictions. We found that financial frictions then drove wedges between each firm's marginal benefit and marginal cost of production. However, due to the different economy structures this lead to different wedges between marginal aggregate product... We summarize our main results from this simple model as follows.

Result 1. Pledgeability constraints introduce wedges between the marginal product of labor and marginal cost, however these wedges differ across the two economies. Aggregate labor wedges are simply a weighted sum of these individual wedges.

Result 2. For any allocation, more liquidity is needed in the vertical economy than in the horizontal economy to implement this allocation.

Result 3a. In vertical economies, the most downstream firm has the greatest impact on aggregate output.

Result 3b. The liquidity multiplier in the vertical economy is greater than the liquidity multiplier in the horizontal economy. Tightening constraints in the vertical economy leads to a greater drop in output than in the horizontal economy.

Result 4. There are no direct spill-over effects in the horizontal economy. However, there are direct spill-over effects in the vertical economy. The collateral constraint acts like an adverse demand shock on firms upstream $(i - k)$, while it works as an adverse supply shock on firms downstream $(i + k)$.

Result 5. When firms are arranged in production networks with vertical chains, the right measure of misallocation should be the dispersion in marginal aggregate products across firms, not the dispersion in marginal individual products. If firms were equally constrained, the former measure would pick up misallocation in the vertical economy, whereas the latter measure would not.

In the next section look at more general input-output structures, and use this to calibrate the U.S. economy.

4 The General Production Network

We now consider a more general environment. We follow Acemoglu, Carvalho Ozdaglarz, and Tahbaz-Salehi (2012) and Jones (2012) in working with a general input-output structure. This allows us then to go to compare this to the input-output economy of the U.S. From this we try to back out

4.1 The Model

The general sectoral model is as follows. The economy is populated by a representative household and N production sectors. Each production sector consists of a continuum of firms. Let $I = \{0, 1, \dots, N\}$, be the collection of all productive sectors; we adopt the convention that households compose the 0-th sector.

Households. The preferences of the household is given by⁵

$$U(x_o, y_0) \equiv \frac{x_0^{1-\gamma}}{1-\gamma} - \frac{y_0^{1+\varepsilon}}{1+\varepsilon}.$$

⁵In many ways the household is treated as a sector, hence the use of x to denote consumption and y to denote labor (output).

where x_0 is the final consumption good and y_0 is the household's labor supply. The parameters $\gamma > 0$ and $\varepsilon > 0$ correspond to the inverse elasticity of intertemporal substitution and the inverse Frish elasticity. The composite consumption good x_0 is given by

$$x_0 = \prod_{j \in I_0} x_{0j}^{\alpha_{0j}}$$

where x_{0j} is the consumption of goods produced by sector j and $I_0 \subseteq I$ is the set of sectors from which these goods are purchased.⁶ The parameter $\alpha_{0j} \in (0, 1]$ corresponds to the j th sector's share in the composite consumption good. Without loss of generality, we set $\sum_{j \in I_0} \alpha_{0j} = 1$.

The budget constraint of the household is given by [need to fix?]

$$\sum p_j c_j = \sum \pi_j + p_0 x_0$$

Production. Each good in the economy is produced by firms within competitive sector. Each sector consists of a unit-mass continuum of identical firms. Goods are differentiated across sectors, but not across firms within a sector. The production of any given sector is purchased either by consumers or by other sectors to be used as inputs (intermediate goods) for their own production. We may drop the terminology of firms and just think of sectors as the production unit, with the understanding that it is perfectly competitive.

Sectors produce using labor and intermediate goods purchased from other sectors. In particular, the output of sector i , denoted by y_i , is given by Cobb-Douglas technology

$$y_i = z_i x_i^{\alpha_i}.$$

where z_i is sector specific productivity and x_i is a composite of inputs. Here, $\alpha_i \leq 1$ is the parameter that governs the returns to scale to the firms output. The composite of inputs is given by the following Cobb-Douglas production function:

$$x_i = \prod_{j \in I_i} x_{ij}^{\alpha_{ij}}.$$

where x_{ij} is the amount of commodity j used in the production of good i and $\alpha_{ij} \in (0, 1]$ is the j -share in the production of i 's composite input, and $I_i \subseteq I$ is the set of sectors that supplies sector i with inputs. In general, α_{ii} need not be equal to 0; sectors may use their

⁶ In particular, x_{00} is the labor input used to directly produce the consumption composite (home production).

own product as an input.⁷ Again, because the 0th sector is the household, we adopt the convention that the 0 input, x_{0i} , is labor. Finally, we assume that productivity z_i is drawn from known distribution F_i . By assumption $\sum_{j \in N_i} \alpha_{ij} = 1, \forall i$.

Financial Frictions. We impose that firms face trade credit contracts are subject to a limited enforcement problem. These depend on an amount of liquidity that characterize the firm's problem. We begin describing the firm's problem. Firm i maximizes profits

$$\Pi_i = \max_{\sigma_i, x_i} p_i y_i - c_i x_i$$

subject to

$$y_i = z_i x_i^{\alpha_i}$$

$$(1 - \sigma_i) c_i x_i \leq w_i \tag{12}$$

$$(1 - \theta_i) p_i y_i \leq p_i y_i - \sigma_i c_i x_i. \tag{13}$$

In the expression above c_i is the marginal cost of x_i which by the assumption that $\sum_{j \in N_i} \alpha_{ij} = 1$, is constant and independent of x_i . The first constraint is the technological constraint of the firm. The second constraint states that a fraction $(1 - \sigma_i)$, (chosen by the firm) has to be paid up-front with liquid funds w_i . Liquid funds are given by an exogenously. Changes in the availability of liquid funds are the focus of interest in this paper. In addition we assume, following the contracting literature, that firm's can pledge at most θ_i of their output to pay their suppliers. Thus, $\sigma_i c_i x_i$ is the amount of trade credit obtained from suppliers. If they choose to default on suppliers, they loose the fraction θ_i of the firm's income. Hence, upon default, the firm keeps the fraction $(1 - \theta_i) p_i y_i$. Thus, the third constraint is an incentive constraint that states that the fraction of output they get to keep should they choose to default on suppliers must exceed the revenue firm's expect to make minus the amount it owes after it pays for part of its inputs with liquid funds. By rearranging this constraint, we obtain an equivalent constraint, $\sigma_i c_i x_i \leq \theta_i p_i y_i$. This one reads that the amount that the firm can owe to its suppliers after it payed from a certain fraction in advance must not exceed the pledgeable amount of output, $\theta_i p_i y_i$. In the specific examples that follow, we let the liquid funds to be some proportion of the firms output $w_i = \omega_i p_i y_i$.

Market Clearing. We assume perfectly competitive markets. Sectors take the price of each sector $\{p_i\}_{i \in I}$ as given (where p_0 is the wage) and maximize profits.

Notation. The following table summarizes notation. I is a collection of sectors. $\alpha_{ij} \in$

⁷This is in fact shown to be true in the data.

$(0, 1]$ is the j -share in the production of i 's composite input. x_{ij} is the amount of commodity j used in production of good i .

Variable	Definition
I	collection of sectors
I_i	subset of i 's input suppliers
α_{ij}	j -share of i 's input composite
α_i	i decreasing returns
x_{ij}	j input of sector i
x_i	input composite of sector i
y_i	output of sector i
θ_i	enforceability of sector i
σ_i	??
ϕ_i	enforceability of sector i
γ	$CRRA$ households
ε	Inverse Frisch Elasticity

4.2 Equilibrium Characterization

Equilibrium Definition. We now describe an equilibrium for this economy.

Definition 4. An equilibrium is defined as a vector of prices, $\{p_i\}_{i \in I}$ for sectoral outputs, and an allocation $(N + 1) \times (N + 1)$ matrix of inputs x_{ij} , an $(N + 1 \times 1)$ vector of sectoral outputs $\{y_i\}$ and an $(N + 1) \times 1$ vector of intermediate good composites such that, (a) Given prices, $(\{x_{ij}\}_{j \in I_i}, \sigma_i, x_i)$ solve sector i firm's problem for every $i \in I \setminus \{0\}$. (b) Given prices, $(\{x_{0j}\}_{j \in I_0}, y_0, x_0)$ solves the household's problem. (c) The allocations are consistent with technology: $x_i = \prod_{j \in I_N} x_{ji}^{\alpha_{ij}}, i \in I$ and $y_i = z_i x_i^{\alpha_i}$ and $i \in I \setminus \{0\}$. (d) The resource constraint: $y_i \geq \sum_{j \in N} x_{ji}$ is satisfied, $i \in I$.

We now move to characterizing the equilibrium.

Lemma 3. The Liquidity and Enforcement constraints ((12) and 13) bind jointly if and only if

$$\theta_i + \omega_i < \alpha_i$$

The firm's problem is characterized by the following first order condition:

$$c_i x_i = \phi_i p_i y_i$$

where $\phi_i = \min \{\alpha_i, (\theta_i + \omega_i)\}$.

[NEED TO REDO PROOF]

A first important observation is that whether firm's production is constrained or not, is only a function of θ_i and ω_i . But the constraint is independent of prices. The Lemma also implies that the firm's problem is also characterized by the same first order condition to the problem of firm that faces a sales tax, or equivalently to a firms facing input taxes. It is convenient then to characterize the firm's problem in terms of a firm facing sales taxes. To be more precise, a firm faces a sales taxes maximizes profits:

$$\max_{x_i} (1 - \tau_i) p_i z_i x_i^{\alpha_i} - c_i x_i.$$

The solution to this problem is $(1 - \tau_i) \alpha_i p_i y_i = c_i x_i$. Thus, the firm's problem is equivalent to a the problem of a firm facing a sales tax of $(1 - \tau_i) \alpha_i = \phi_i$. Thus, the corresponding tax for firm i is:

$$\tau_i \equiv 1 - \frac{\phi_i}{\alpha_i} = \frac{\alpha_i - \min \{\alpha_i, \theta_i + \omega_i\}}{\alpha_i}.$$

This tax has an immediate interpretation. At the unconstrained optimal, α_i is the fraction of the firm's revenue that is needed to pay for inputs, or 1 minus the markup. $\theta_i + \omega_i$ represents the fraction of revenues that can be obtained as the firm's credit. This credit is the sum of the firm's liquid funds ω_i and the fraction that can be obtain directly as trade credit θ_i . Thus, the tax is the fraction between the gap between the firm's liquidity needs and its actual liquidity over its entire liquidity needs. With this, we establish the following,

Proposition 9. *An equilibrium allocation is equivalent to the equilibrium allocation of an economy with sales taxes, and lump-sum transfers where the sales tax in sector i, is given by:*

$$\tau_i = 1 - \frac{\min \{\alpha_i, \theta_i + \omega_i\}}{\alpha_i}.$$

Thus, the environment is isomorphic to the input-output model of Jones (2011). Various setups of production economies with sales taxes and lump-sum transfers have been widely studied in general equilibrium theory. Existence and uniqueness in know for this environments so we omit such proofs in this paper. The only distinction is that here we will affect τ_i . We are ready to provide a characterization. We need to introduce some notation first.

Input-Output Matrix and Notation. The resource constraint says that the output of every i-th sector must be larger than the sum of the usage of i sector inputs in every sector j. At this point, it is convenient to define an input output matrix of an economy.

As in Acemoglu, Carvalho Ozdaglarz, and Tahbaz–Salehi or Jones we define a frictionless input-output matrix A . For any given pair of sectors i and j with $j \in I_i$, $\alpha_{ij} > 0$ is the share of good j in the total intermediate input use of firms in sector i . The fact that firms in a given sector use the output of firms in other sectors as inputs for production is the source of interconnectivity in the economy. α_{ij} is the ij generic entry of the input-output matrix A . We also adopt the convention that $\alpha_{ij} = 0$ if sector j is not an input supplier to sector i . By definition, A is a non-negative $n \times n$ stochastic matrix (its rows sum up to one (and thus has an eigenvalue equal to one with the corresponding right eigenvector consisting of all ones)).

We denote by $\tilde{A}$ the input-output matrix whose rows are scaled by ϕ_i . Thus, this matrix has a typical ij entry $\phi_i \alpha_{ij}$. Vector valued quantities of a variable are given expressed in bold. We also denote by α_j be the j -th column of A .

Characterization. The first order condition for the firm's optimal input choice is given by (15):

$$\frac{p_j x_{ij}}{p_k x_{ik}} = \frac{\alpha_{ij}}{\alpha_{ik}}$$

which holds for every sector and every input. One can substitute this relationship into the the cost function to obtain the relation between the sector i 's composite input and the j sector input,

$$\alpha_{ij} c_i x_i = p_j x_{ij}. \quad (14)$$

Together with these conditions, an equilibrium is therefore fully characterized by the following system of equations:

$$\begin{aligned} z_i x_i^{\alpha_i} &= \sum_{j \in I} \frac{\alpha_{ji} c_j x_j}{p_i}, i \in \{1, 2, \dots, N\} \\ y_o &= \sum_{j \in I} \frac{\alpha_{j0} c_j x_j}{p_0} \\ c_i x_i &= \phi_i p_i z_i x_i^{\alpha_i}, i \in \{1, 2, \dots, N\} \\ x_o^{-\sigma} &= p_0^{-1} y_o^\varepsilon \end{aligned}$$

This is an autonomous system of $2 * (N + 1)$ equations and same number of unknowns. The following Proposition describes the computation of equilibria how to compute equilibria:

Proposition 10. *In equilibrium, log hours are given by:*

$$\tilde{y}_o = C_1 / (1 - A_{11}) + \frac{A_{12}}{(1 - A_{11})} \left[I - \left(\frac{A_{21}}{(1 - A_{11})} A_{12} \mathbf{e}' + A_{22} \right) \right]^{-1} \left[C_2 + \frac{C_1}{(1 - A_{11})} A_{21} \right]$$

and log GDP is given by:

$$\tilde{x}_0 = \frac{(\varepsilon - 1)}{(1 - \sigma)} \tilde{y}_o - \frac{1}{(1 - \sigma)} \ln \left(1 - \sum_{i=1}^N (1 - \phi_i) \gamma_i \right).$$

where,

$$A_{11} = \left[\frac{1 - \sigma}{1 + \varepsilon} \alpha_{00} \right]$$

$$A_{12} = \left[\begin{array}{cccc} \frac{1 - \sigma}{1 + \varepsilon} \alpha_{01} & \frac{1 - \sigma}{1 + \varepsilon} \alpha_{02} & \cdots & \frac{1 - \sigma}{1 + \varepsilon} \alpha_{0N} \end{array} \right]$$

$$A_{21} = \left[\begin{array}{c} \alpha_1 \alpha_{10} \\ \alpha_2 \alpha_{20} \\ \vdots \\ \alpha_N \alpha_{N0} \end{array} \right]$$

$$A_{22} = \left[\begin{array}{cccc} \alpha_1 \alpha_{11} & \alpha_1 \alpha_{12} & \cdots & \alpha_1 \alpha_{1N} \\ \alpha_2 \alpha_{21} & \alpha_2 \alpha_{22} & \cdots & \alpha_2 \alpha_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_N \alpha_{N1} & \alpha_N \alpha_{N2} & \cdots & \alpha_N \alpha_{NN} \end{array} \right]$$

$$C_1 = \frac{1}{1 + \varepsilon} \ln \left(1 - \sum_{i=1}^N (1 - \phi_i) \gamma_i \right) + \frac{1 - \sigma}{1 + \varepsilon} \sum_{j \in I} \alpha_{0j} \ln \left(\frac{\alpha_{0j}}{\gamma_j} \right)$$

and

$$C_2 = \left[\begin{array}{c} \tilde{z}_1 + \alpha_1 \ln \phi_1 + \alpha_1 \sum_{j \in I_1} \alpha_{1j} \ln \left(\alpha_{1j} \frac{\gamma_1}{\gamma_j} \right) \\ \tilde{z}_1 + \alpha_1 \ln \phi_1 + \alpha_1 \sum_{j \in I_1} \alpha_{1j} \ln \left(\alpha_{1j} \frac{\gamma_1}{\gamma_j} \right) \\ \vdots \\ \tilde{z}_N + \alpha_N \ln \phi_N + \alpha_N \sum_{j \in I_N} \alpha_{Nj} \ln \left(\alpha_{Nj} \frac{\gamma_N}{\gamma_j} \right) \end{array} \right].$$

Here, the terms γ_i are the elements of the vector that solves:

$$\boldsymbol{\gamma} = (\mathbf{I} - \tilde{A})^{-1} \tilde{A}.$$

Sectoral prices are given by:

$$\mathbf{p} = [\mathbf{e}'\alpha_0 - \mathbf{I}] (\tilde{\mathbf{y}} - \tilde{\boldsymbol{\gamma}}) .$$

We begin with a theorem that establishes an observational equivalence between observed sectoral TFP and sector specific financial frictions. Theorem xxx then uses this to establish which economies are more sensible to the financial shocks.

In this section we use Proposition 10 to analyze how more general production network structures react differently to liquidity shocks. Both, at an aggregate and cross-sectional levels. In the first of our exercises, we compare the response of GDP to a tailored sequence of liquidity shocks that are common to every sector. The sequence of liquidity shocks is chosen so that when we calibrate the model to match the I-O matrix of the U.S. economy, we obtain a pattern that replicates aggregate output during a chosen sample period. The exercise allows us to compute a liquidity multiplier for the 5 sample economies.

Figure ?? shows the behavior of GDP in the model for various network structures to the same liquidity shock sequence. The figure is divided into 5 rows, each corresponding to a different production structure. The boxes on the left of each row are the graphical representations of the I-O matrix in each model. The top row for example, corresponds to the case of a horizontal network. No sector uses any other input but labor which is why the box has a solid color. The liquidity multiplier is approximately one and the sequence of GDP (solid line) tracks almost perfectly the sequence of the liquidity shock. The intuition has been discussed earlier in the paper. With a horizontal structure, firms operate in isolation and the liquidity shock directly constrains the amount of workers each firm hires. The percentage drop in liquidity, to a first order, is equivalent to the percent reduction in hours and roughly the labor share of GDP. However, there is an indirect effect that operates through the reduction in wages that reduces total hours worked. The multiplier of 1 turns out as a result of the labor share and the Frisch elasticity. The following cases are a triangular production chain, a vertical chain, a production cycle, and a star network. These sample economies are organized in increasing order of their impact. Note that in all cases, the multiplier is about twice as large. It shouldn't be a surprise that the star network yields the largest impact. However, it is a surprise that its multiplier is roughly close to the multiplier in all of the other models with a sectoral linkage. The reason is that we are studying an aggregate shock. It is worth discussing what is going on at the cross-section level of each model. We do this for selected network structures of size $N=10$. In all examples, households demand goods from every sector.

Horizontal. Figure ?? shows the cross sectional impact of liquidity shocks in a horizontal

economy. The top left panel once again the I-O matrix. The three panels to the right describe the cross-section effects on the GDP of different sectors to an increase in the liquidity of a single sector. For example, the second panel shows (red stem) a reduction of liquidity in sector 1 of 1%. This shock induces a reduction in that sector's GDP of 1.8%. Since the horizontal economy has no spill over effects beyond an indirect effect via wages in the labor market, the impact on the GDP of other sectors is mute. The next panel shows a symmetric response to a shock in sector 6 and the right panel to sector 10. The bottom panel presents the response of GDP as we shock each sector by 1%. Since this economy is symmetric, what sector is hit doesn't matter for the aggregate response.

Cycle. Figure ?? repeats the information of Figure ?? for a cycle production network economy. As can be noted from the top left panel, in this economy, sector i requires inputs from sector $i-1$ for all $i > 1$ and $i=1$ requires inputs from sector 10. A first thing to note is that this network has similar cross-sectional properties as the horizontal economy because its symmetric (bottom panel). However, the top panels show that here spill-over effects are present. Sector's that supply inputs to the affected sector are affected with a negative reduction in their GDP. The sectors affected will experience an increase in their price. Sectors that require inputs from those sectors will substitute that input with labor, something that can be done here because labor is substitutable with the production input. The supplied sector's output can actually expand because they benefit from lower wages and take over part.

Vertical. A vertical economy's response is quite different. In the vertical economy is no longer symmetric. In fact, it matters a lot what sector is being affected for aggregate GDP and the cross-sectional impact. Here sector's of lower i supply inputs to sector's of higher i with sector 1 not hiring inputs other than labor. The bottom panel shows that hitting the most downstream sector is more important. This was discussed earlier in the paper. Downstream firms have a larger impact because... [Jen-Jen fill in]. Also, spill-over effects matter a lot for the sector that is being affected. It is interesting to observe that when the last sector is hit, the firms upstream experience an expansion in their output because they supply goods to consumer who substitute consumption from the last sector to lower i sectors. When the first sector is affected by liquidity, it will affect the supply of inputs to the sector above it and that sector to the sector above it. This causes a chain reaction with reductions in GDP in these sectors. Output of downstream firms can actually expand because of the reduction in labor.

Star. Finally, we examine a star network where every sector requires the input of a key sector, a sector like utilities or transportation in the real world. In Figure ??, that key sector is sector 1. This sector's shock has a much larger impact on aggregate out. All other sectors

have a minor impact because they don't affect the supply of intermediate inputs for other sectors. In fact, shocking the star in the network has the effect of an aggregate shock in the vertical economy, something that is known in the networks literature.

5 Calibration

We calibrate the general production network described in Section () to the input-output matrix of 3-digit level sectors of the U.S. economy. [NEED TO ADD]

Idea for calibration: can back out wedges. Similar to Chari-Kehoe-McGrattan business cycle accounting. But instead, sectoral accounting with networks.

That is, the elasticity of the response of aggregate GDP to a reduction in liquidity of any or all sector's of the economy. The theoretical analysis presented earlier shows that when production is organized horizontally and labor is inelastically supplied, that multiplier is 0. If instead, labor is supplied elastically, the response depends on the labor supply elasticity and the elasticity of output to labor. However, production in the United States is organized via a highly interconnected network. This section shows that calibrating the input output matrix of our model to that of the U.S. economy, the multiplier turns out to be 3.8. We will explain how we estimate this number and what additional empirical implications our model has. We organize this section into ten questions. The first set of questions are about aggregates and the rest correspond to sectoral analysis. We begin discussing the calibration.

For our empirical analysis, we set household's parameters to $\gamma = 2$ and $\varepsilon = 1/2$ which is standard in macro calibrations. We use the Input-Output accounting tables reported by the Beareau of Economic Analysis (BEA) to calibrate the rest of the model.

The BEA organizes the summary level of Input Output matrixes at a three-digit level according to the North American Industry Classification System (NAICS). There are 65 NAICS sectors at three-digit level. We collect yearly data from 1998 until 2011. The Use of Commodities by Industries after Redefinitions (USES) table is used to calibrate the α_{oj} shares of consumption. This table reports the expenditures in US\$ billions of all industries and households for every commodity produced in every industry. Thus, we the following property of our model to calibrate α_{oj} :

$$\alpha_{oj} = \frac{p_j x_{0j}}{\sum_{n=1:65} p_n x_{0n}}.$$

Let $u_{ij,t}$ be the US\$ Billion expenditures by sector i in sector j's commodities. This data is the corresponding analog for the product $p_j x_{ij}$ in our model. The USES table reports personal consumption expenditures (Column F010 of that table). Thus, $p_j x_{0j}$ is reported so

we compute the consumption share of sector j 's commodity at time t , $\alpha_{oj,t}$ by

$$\alpha_{oj,t} = \frac{u_{oj,t}}{\sum_{n=1:65} u_{on,t}}$$

Figure ?? reports the evolution of these shares for the 1998-2011 showing a very stable behavior for most sectors. The exceptions are mining and petroleum industries which happen to have the lowest values to begin with.

The technology parameters are directly calibrated from the Input-Output Matrix. Each columns of the Matrix reports the fraction of expenditures of each sector create a dollar of sales in that sector. Let, $\alpha_{ij,t}^{Data}$ represent the share of sector's i expenditures of on j sector commodities over sector i 's sales. Since the matrix columns includes added value and taxes, we calibrate the coefficients by normalizing by expenditures on inputs and labor. Thus we leave outside 3 entries: (1) scrap, used and secondhand goods, (2) non-comparable imports and rest-of-the-world adjustment and (3) taxes on production and imports, less subsidies without including taxes. The coefficients of the production function are obtained by setting:

$$\alpha_{ij,t} = \frac{\alpha_{ij,t}^{Data}}{\sum_{n=1:65} \alpha_{ij,t}^{Data} + \alpha_{0j,t}^{Data}}.$$

The input-output matrix presents a highly stable pattern. Figure ?? shows the contour plot for the log of the entries of the matrix during the years of the sample 1998:2011.

We cannot compute the coefficients of decreasing returns to scale of sector i at period t , $\alpha_{i,t}$ directly because our model yields predictions only about the fraction of costs. Thus, we obtain an estimate of $\chi_{i,t}$

$$\chi_{i,t} = 1 - \left(\sum_{n=1:65} \alpha_{ij,t}^{Data} + \alpha_{0j,t}^{Data} \right).$$

These shares are very unstable compared to the expenditure shares of consumption in the U.S.. This pattern can be seen from Figure ?. Through the lens of the model, the instability of costs/sales of each sector is purely the outcome of liquidity shocks. We will treat $\chi_{i,t}$ as our primitives. We cannot identify the term $\alpha_{i,t}$ from our observation of $\chi_{i,t}$. This is the main drawback of the paper. Because we are aware of this inconvenience, for the rest of the section we adopt a cautious approach. That is, we will make assumptions that minimize the impact of liquidity by assuming that $\alpha_{i,t}$ is as low as possible. We are now ready to answer some questions.

5.1 Aggregate Analysis

Question 1. Suppose liquidity is drawn down by in 1% in every sector. What is the multiplier? By 3.80%. The answer to this question depends on the parameters that we input to the model that in turn, depend on a given year's input output matrix. To compute the number, we numerically approximate the derivative of output to a reduction in liquidity in every sector. The corresponding directional derivative is $\langle \frac{\partial y(\Theta_t)}{\chi_{i,t}}, I \rangle$ computed along the unit vector. Since the derivative depends on the calibration for a given year, we average across all years of our sample.

$$\mu = \sum_{t=1998:2011} \frac{\langle \frac{\partial y(\Theta_t)}{\chi_{i,t}}, I \rangle}{y_{i,t}(\Theta_t)} = 3.8$$

The range of answers for given years goes from 3.5 to 4.

Question 2. What is the aggregate model implied liquidity shock that could have caused the recession? Figure ?? reports the response of GDP together with several equilibrium objects to a particular sequence of liquidity shocks. The sequence corresponds to the quarters ranging from the third quarter of 2007 until the second quarter of 2011. The sequence is chosen so that GDP in the model follows the same path as the deviations from its the average growth rate of the U.S. economy over that last 30 years. The top panel reports the reduction in output, both for the model and the data which by construction coincide. The same panel also reports the aggregate liquidity shock that hits every sector symmetrically that would generates the path for GDP. At the trough of the cycle, the required reduction in liquidity would have been of 1.28% of sales for every sector. A model with a representative firm would require at least 7.6% reduction in liquidity.⁸ Now, it is important to note that in our model, this is coming mostly from a production missallocation effect. The data shows a much higher reduction in labor and much lower adjustment in prices. Possibly, nominal rigidities would magnify the multiplier.

Question 3. How big are the spill-over effects? A natural question at this stage is to ask, how much of the reduction in GDP has to do with the spill-over effects described in the previous answer. Spill-over effects will show up in the price scheduled faced by the firm. In the analysis of a vertical economy we argued that the sales price of a firm will faces a force that pushes it downwards when other sectors downstream are constrained. Similarly, the purchase cost of commodities produced by sectors upstream that face a constraint should

⁸Consider the case where wages are fixed. if Labor is the only factor for production and GDP falls by $\Delta y\% = 5\%$ away from its linear trend at the trough of the cycle, the model requires $\frac{1}{\alpha}\Delta y\%$ which calibrated at $\alpha = 2/3$ yields the quoted number

increase in this environment. Indeed, such mechanisms are operating in the hypothetical exercises we are studying here. One statistic that captures this missallocation is obtained by the following:

$$\tau_{i,t} = \left[\frac{(c_{i,t}/p_{i,t})}{c_{i,1998}/p_{i,1998}} \right].$$

The lower panel of Figure ?? presents the average $\tau_{i,t}$ across sectors for every period. One can see that at the peak of the crisis, the average firm in our model would have experienced a change in prices that would have made it act as if facing a 2% increase in intermediate input taxes. The standard deviation of this tax across sectors is about 1%. Thus, the amount of sectoral missallocation had an effect on the individual firm which would have been equivalent to the direct effect. Spill-over effects are roughly as big as the direct effects. Putting it differently, a firm in a given sector should be on average indifferent between removing its liquidity shocks or removing the liquidity shocks in the other sector of the economy. Now, such small taxes can have a large impact on output because, as argued in the public finance literature, the effects on intermediate taxes get compounded.

Question 4. What sectors are the most affected by the aggregate implied liquidity shock? Figure ?? answers this question. The sectors most affected are manufacturing sectors that provide intermediate inputs for final goods industries. These include metal products, chemical products, fabricated metal products, hydrocarbons and other industries related to the extraction and transformation of raw materials. Amongst the most sensible sectors we also find some service industries such as miscellaneous professional services and administrative, brokerage and management firms. Sectors that are least affected are retail, water transportation, and health related industries and entertainment. We don't have an answer for why this pattern. However, this pattern has to do more with the sectoral linkages than with the liquidity need of each sector. Figure ?? shows that sectors that are most sensible to the liquidity shock are in turn sectors that are suffering more from the misapplication. For example, the metal products is the most sensible sector because an aggregate 1% in liquidity shocks has the effects of an equivalent 7% increase in intermediate input taxes. We now turn the attention to answering questions about the relative importance of liquidity in the 3-digit sectors of the U.S. economy.

5.2 The Key Sectors

Question 5. Which sector is the most sensitive to its own liquidity shock? We now ask which sector responds most to a decrease in 1% in the liquidity of the same sector.

Figure ?? shows the estimates of $\frac{\partial y_i(\Theta_t)}{\chi_{i,t}}$ for every sector. The sectors with the highest internal sensitivity are roughly the same as the sector's that react the most to the aggregate liquidity shock but in this case, the ranking is lead by the motors (automotive) sector. Figure ?? hints towards the explanation behind this ranking. It reports a scatter plot of the sectors impact on itself against the share of intermediate inputs and the share of the sector itself. Here the pattern is clear, the higher the share of intermediate inputs, α_i , the higher the response to liquidity. Sectors with high α_{ii} also respond more to liquidity. This last aspect is important because firms operating within the same sector should receive very similar liquidity shocks and if they are important supplying inputs for the sector itself, we should expect a large multiplier.

Question 6. To what sector is the U.S. most sensitive? Figure ?? reports the estimates of the response of aggregate GDP to the shock in one sector, i.e., $\langle \frac{\partial y(\Theta_t)}{\chi_{i,t}}, I \rangle$ for all i . The bars suggest that most important sector is *retail*. Other important sectors are industries such as motors and computer electronics, the most sensitive sector to themselves. This patterns points out to two sources of aggregate impact. The first force behind this ranking is the extent of sectoral linkages. For example, manufacturing industries are more interrelated than primary resource extraction industries, and this leads to larger aggregate effects. The second factor of influence is that sector's such as retail have important consumption shares. Hence, their aggregate impact is large simply because the sector is large in the computation of GDP although the sector is not dramatically affected as industrial sectors. Figure ?? presents the consumption shares in each sector, and the labor share of each sector against the sector's aggregate impact. Figure ?? shows that sector's with larger consumption shares lead to larger aggregate effects.

Question 7. What sector causes the biggest spill overs?

?? shows the average intermediate input tax rate induced on the rest of the sectors by every sector against the sector's aggregate impact. In this case, we find a low correlation between the aggregate impact rank and this spill-over rank. This suggests that the first order effect of a high intermediate shares or consumer shares are important in determining the sector's impact more than a possible misallocation. It also shows that shocks to sector's such as retail will have a very large impact on the rest of the economy by affecting the demand for goods. Finally, Figure ?? describes the sector's that are affected most by a shock in a given other sector. The outcomes are natural.

5.3 Firm Level Analysis

Question 8. What sector had the biggest impact? We now try to go in more depth towards the sector's that could have been key during the recession. We limit the analysis to 9 manufacturing industries and construction.⁹ We gather data from COMPUSTAT for all the industries and compute the ratio of costs over sales for each sector. We corroborated a negative relationship with this measure and sales growth (in agreement with the countercyclical markups). This feature is in line with our model. We also corroborated a negative relationship with our measure of illiquidity and an increasing working capital over costs share. The implied reduction in GDP is shown in Figure ???. The figure shows that these sector's implied liquidity drop could explain roughly 1/3 of the drop in GDP purely from missallocation effects. That is, keeping hours constant, prices would have to fall in this hypothetical world by roughly 0.5%. Figure ??? presents a decomposition of the drop in GDP by the contribution of each sector. Roughly all sector's play a similar role except for wood products and construction that have a lower impact. This result is surprising given that construction fell dramatically during the crises. Our result could be affected because of our imperfect measure of liquidity or a size bias on the COMPUSTAT sample.

Question 9. Which one caused the biggest spill overs?

Figure ??? presents a measurement of the sector's induced taxes during the recession as a measure of the spill-over effects induced by each sector. At the top of the rank are leather and apparel as well as printing.

5.4 Failure of the Model

It is important to discuss what features is this model missing. We can note several things. [1] The model lacks any form of price rigidities. In particular, wage rigidities could possibly explain the much sharper reduction in hours as suggested by Figure ???. We see in that hours fall more in the data compared to the model. If anything, a rigidity of the sort would increase the multiplier that we calculated in the first part. [2] A second issue is related to the fact that we are neglecting any form of durability in final goods. Consumer durables would potentially amplify the output of industries in the durables sectors by causing changes in relative demand. In our model, consumption shares are constant fractions of labor income. [3] A third concern is that we are allowing a high degree of production elasticity substitution. In the model, sectors whose suppliers are affected by reductions in liquidity easily substitute production inputs with other inputs or labor. The model may be missing a

⁹Construction, Wood Products, Non-metallic Mineral Products, Metallic Products, Electrical and Computer Products, Furniture, Food and Beverage, Apparel and Leather and Printing.

larger amount of rigidities in the production process. [4] A final concern is that we are treating sectors in a symmetric way. This is a problem because sectors such as retail, wholesale and warehousing and transportation are sectors that don't transform products in a the same way manufacturing industries. Often these are simple intermediaries charging a constant markup so the decreasing returns to scale assumption may not be appropriate.

6 Conclusion

This paper argues that the network of production links in an economy can matter substantially for the transmission of financial shocks. To make this point, we formulated an economy in the most simple way possible. We introduced financial shocks that are orthogonal across sectors. That is, we formulated a model such that the liquidity shock in one sector has no effects on the intermediate inputs wedge of other sectors, although there are spill-over effects via the effects on prices that cause inefficient reallocation.

We provided several analytic examples of liquidity shocks to analyze their propagation in particular network structures. We then took the structure of the U.S. I-O and calibrated our model. We asked what is the liquidity multiplier in the U.S. Our experiment showed a multiplier of 3.5. compared with a horizontal economy whose multiplier is 1.

We believe the exercise illustrates the that the effects of liquidity shocks can be quite dramatic if production is organized with industrial linkages. We speculate that if one were to introduce the possibility of demand changes via durable consumption preferences, nominal rigidities, or low short-run input substitution, the effects could be even more dramatic. This extensions can be studied bringing this framework into richer environments. [Literature here]

There are other questions that are relevant for the theoretical study of financial frictions in networks. Early work of Kiyotaki and Moore [Credit Chains, Moore] noted that disruptions in the payments chain have important welfare implications. We abstracted from any stratetigic/time dimension form of trade credit because in our model, trade credit moves jointly with liquidity. However, we should study a model which can explain disruptions in the supply chain. In our model, the net-work structure is exogenous. There is a growing literature on endogenous network formation [Oberfeldt]. Part of this network formation has to do with technological processes but a lot can be do to financial innovation in a way that can give us a better understanding of the theory of the firm.

7 Appendix

Proof of Proposition 1 *Vertical Economy.* Let us first consider the Vertical economy. Firm 1 maximizes the following objective function.

$$\max p_1 A_1 n_{v1}^{\alpha_1} - w n_1$$

This yields FOC

$$p_1 \alpha_1 \frac{y_1}{n_1} = w$$

Firm 2 maximizes the following objective function.

$$\max p_2 A_2 n_{v2}^{\alpha_2} y_{v1}^{\beta_2} - w n_2 - p_1 y_1$$

This yields FOCs

$$\alpha_2 p_2 \frac{y_2}{n_2} = w \quad \text{and} \quad \beta_2 p_2 \frac{y_2}{y_1} = p_1$$

Firm 2's expenditure on goods is given by

$$w n_2 + p_1 y_1 = \alpha_2 p_2 y_2 + \beta_2 p_2 y_2 = (\alpha_2 + \beta_2) p_2 y_2$$

Firm 3 solves a similar problem to that of firm 2; it's FOC's are given by

$$\alpha_3 P \frac{y_3}{n_3} = w \quad \text{and} \quad \beta_3 P \frac{y_3}{y_2} = p_2$$

Thus, combining () with () and (), we get that.

$$\begin{aligned} \alpha_3 \frac{Y}{n_3} &= \frac{w}{P} \\ \alpha_2 \beta_3 \frac{Y}{n_2} &= \frac{w}{P} \\ \alpha_1 \beta_2 \beta_3 \frac{Y}{n_1} &= \frac{w}{P} \end{aligned}$$

Finally, the optimality condition for the household is given by

$$\frac{v'(N)}{u'(C)} = \frac{w}{P}$$

Combining this with ()-() yields the conditions for th

Horizontal Economy. In the horizontal economy, all firms are identical and maximize the

following objective.

$$\max p_i A_i n_i^{\alpha_i} - w n_i$$

the FOC's for each firm is given by

$$p_i \alpha_i \frac{y_i}{n_i} = w$$

Finally the household maximizes the following objective

$$\max P y_1^{\beta_2 \beta_3} y_2^{\beta_3} y_3 - p_1 y_1 - p_2 y_2 - p_3 y_3$$

It's FOC is given by

$$\begin{aligned} \beta_2 \beta_3 P \frac{Y}{y_1} &= p_1 \\ \beta_3 P \frac{Y}{y_2} &= p_2 \\ P \frac{Y}{y_3} &= p_3 \end{aligned}$$

Combining, we get the following conditions, which are exactly the same as in the vertical economy.

$$\begin{aligned} \alpha_1 \beta_2 \beta_3 \frac{Y}{n_1} &= \frac{w}{P} \\ \alpha_2 \beta_3 \frac{Y}{n_2} &= \frac{w}{P} \\ \alpha_3 \frac{Y}{n_3} &= \frac{w}{P} \end{aligned}$$

The household's optimality condition remains the same as (). Thus, combining this with ()-() yields the same conditions

Proof of Proposition 2 *Vertical Economy.* Let's first consider the vertical economy with collateral constraints. Firm 1 maximizes the following objective

$$\max p_1 y_1 - w n_1$$

subject to

$$\begin{aligned} y_1 &= A_1 n_{v1}^{\alpha_1} \\ w n_1 &\leq \chi_1 p_1 y_1 \end{aligned}$$

If collateral constraint is binding then

$$wn_1 = \chi_1 p_1 y_1$$

otherwise the firm chooses labor according to it's unconstrained FOC given by

$$p_1 \alpha_1 \frac{y_1}{n_1} = w$$

We may summarize this in the following way.

$$wn_1 = \phi_1 \alpha_1 p_1 y_1$$

where $\phi_i = \min \left\{ 1, \frac{\chi_{hi}}{\alpha_i} \right\}$.

Now consider the problem of firm 2:

$$\max p_2 y_2 - wn_2 - p_1 y_1$$

subject to

$$\begin{aligned} y_2 &= A_2 n_{v2}^{\alpha_2} y_{v1}^{\beta_2} \\ wn_2 + p_1 y_1 &\leq \chi_2 p_2 y_2 \end{aligned}$$

Firm 2's cost minimization is given by

$$\min wn_2 + p_1 y_1$$

subject to

$$y_2 = A_2 n_2^{\alpha_2} y_1^{\beta_2}$$

This implies that the firm's optimal choices for inputs must satisfy

$$\frac{1}{\alpha_2} wn_2 = \frac{1}{\beta_2} p_1 y_1$$

Firm 2's unconstrained expenditure on goods would be given by

$$wn_2 + p_1 y_1 = (\alpha_2 + \beta_2) p_2 y_2$$

Thus firm 2 is constrained iff

$$\alpha_2 + \beta_2 > \chi_2$$

If constrained, then

$$\begin{aligned} wn_2 + wn_2 \frac{\beta_2}{\alpha_2} &= \chi_2 p_2 y_2 \\ wn_2 \left(\frac{\alpha_2 + \beta_2}{\alpha_2} \right) &= \chi_2 p_2 y_2 \end{aligned}$$

If unconstrained, then

$$\begin{aligned} wn_2 \left(\frac{\alpha_2 + \beta_2}{\alpha_2} \right) &= (\alpha_2 + \beta_2) p_2 y_2 \\ wn_2 &= \alpha_2 p_2 y_2 \end{aligned}$$

Thus,

$$wn_2 = \phi_2 \alpha_2 p_2 y_2$$

where $\phi_2 = \min \left\{ 1, \frac{\chi_2}{\alpha_2 + \beta_2} \right\}$.

Similarly for firm 3 we have that

$$\frac{1}{\alpha_3} wn_3 = \frac{1}{\beta_3} p_2 y_2$$

Thus

$$wn_3 = \phi_3 \alpha_3 p_2 y_2$$

where $\phi_3 = \min \left\{ 1, \frac{\chi_3}{\alpha_3 + \beta_3} \right\}$.

Plugging this into firm 2's problem

$$\begin{aligned} p_2 y_2 &= wn_3 \frac{\beta_3}{\alpha_3} \\ &= \frac{\beta_3}{\alpha_3} \alpha_3 \frac{\chi_3}{\alpha_3 + \beta_3} PY \\ &= \chi_3 \frac{\beta_3}{\alpha_3 + \beta_3} PY \end{aligned}$$

Therefore, for firm 3 we have

$$wn_3 = \phi_3 \alpha_3 PY$$

which corresponds with equation (). For firm 2 we have

$$wn_2 = \phi_2 \alpha_2 p_2 y_2$$

where

$$p_2 y_2 = \frac{\beta_3}{\alpha_3} wn_3 = \beta_3 \phi_3 PY$$

This implies

$$wn_2 = \phi_2 \phi_3 \alpha_2 \beta_3 PY$$

which corresponds with equation (). And finally for firm 1 we have

$$wn_1 = \phi_1 \alpha_1 p_1 y_1$$

where

$$p_1 y_1 = \frac{\beta_2}{\alpha_2} wn_2 = \frac{\beta_2}{\alpha_2} \phi_2 \phi_3 \alpha_2 \beta_3 PY$$

This implies

$$wn_1 = \phi_1 \phi_2 \phi_3 \alpha_1 \beta_2 \beta_3 PY$$

which corresponds with equation ().

Horizontal Economy. Again, each firm solves an identical problem. Firm i chooses n_i to maximize

$$\max p_i A_i n_i^{\alpha_i} - wn_i$$

subject to

$$wn_i \leq \chi_i p_i y_i$$

thus, if it is binding then

$$\chi_i p_i y_i = wn_i$$

otherwise

$$p_i \alpha_i \frac{y_i}{n_i} = w$$

Therefore,

$$\phi_i \alpha_i p_i y_i = wn_i$$

where $\phi_i = \min \left\{ 1, \frac{\chi_{hi}}{\alpha_i} \right\}, \forall i$.

Finally the household maximizes the following objective

$$\max PY_1^{\beta_2 \beta_3} Y_2^{\beta_3} Y_3 - p_1 y_1 - p_2 y_2 - p_3 y_3$$

It's FOC is given by

$$\begin{aligned}\beta_2\beta_3P\frac{Y}{y_1} &= p_1 \\ \beta_3P\frac{Y}{y_2} &= p_2 \\ P\frac{Y}{y_3} &= p_3\end{aligned}$$

Combining these with () yields

$$\begin{aligned}\phi_1\alpha_1\beta_2\beta_3PY &= wn_1 \\ \phi_2\alpha_2\beta_3PY &= wn_2 \\ \phi_3\alpha_3PY &= wn_3\end{aligned}$$

which corresponds with equations ()-().

Proof of Proposition 3 In the horizontal economy, we have that

$$\begin{aligned}n_{h3} &= (\phi_{h3})\alpha_3\frac{Y}{V'(N)/U'(C)} \\ n_{h2} &= (\phi_{h2})\alpha_2\beta_3\frac{Y}{V'(N)/U'(C)} \\ n_{h1} &= (\phi_{h1})\alpha_1\beta_2\beta_3\frac{Y}{V'(N)/U'(C)}\end{aligned}$$

thus

$$\frac{Y}{N} = \frac{1}{\alpha_3(\phi_{h3}) + \alpha_2\beta_3(\phi_{h2}) + \alpha_1\beta_2\beta_3(\phi_{h1})} \frac{V'(N)}{U'(C)}$$

From this, we can again back out the aggregate labor wedge $1 - \tau$ defined by ().

In the vertical economy, we have that

$$\begin{aligned}n_{h3} &= (\phi_{h3})\alpha_3\frac{Y}{W/P} \\ n_{h2} &= (\phi_{v2}\phi_{v3})\alpha_2\beta_3\frac{Y}{W/P} \\ n_{h1} &= (\phi_{v1}\phi_{v2}\phi_{v3})\alpha_1\beta_2\beta_3\frac{Y}{W/P}\end{aligned}$$

thus

$$\frac{Y}{N} = \frac{1}{\alpha_3(\phi_{v3}) + \alpha_2\beta_3(\phi_{v2}\phi_{v3}) + \alpha_1\beta_2\beta_3(\phi_{v1}\phi_{v2}\phi_{v3})} \frac{W}{P}$$

From this, we can again back out the aggregate labor wedge $1 - \tau$ defined by (). QED.

Proof of Proposition 4 [NEED TO FIX ALPHAS AND BETAS]

In the horizontal economy

$$M_h = w(n_1 + n_2 + n_3)$$

In the vertical economy

$$M_v = w \left[n_{v1} + \left(1 + \frac{\alpha_2}{\beta_2} \right) n_{v2} + \left(1 + \frac{\alpha_3}{\beta_3} \right) n_{v3} \right]$$

We can translate that into real terms

(i) the liquidity needed in the vertical economy to achieve $Y_v = Y$ is given by

$$\Phi_v = Y \left[\phi_{v3} + \phi_{v2} \alpha_3 \left(\frac{\phi_{v3}}{\beta_3 + \alpha_3} \right) + \phi_{v1} \alpha_2 \alpha_3 \left(\frac{\phi_{v2}}{\beta_2 + \alpha_2} \right) \left(\frac{\phi_{v3}}{\beta_3 + \alpha_3} \right) \right]$$

(i) the liquidity needed in the horizontal economy to achieve $Y_h = Y$ is given by

$$\Phi_h = Y [\phi_{h3} + \phi_{h2} \alpha_3 + \phi_{h1} \alpha_2 \alpha_3]$$

Proof of Proposition ?? Given our specification for utility in (), we have that

$$\frac{V'(N)}{U'(Y)} = Y$$

Frictionless Economy. Thus, in the economy without frictions, we have that

$$\alpha_3 \frac{Y_\varepsilon}{n_{\varepsilon 3}} = \alpha_2 \beta_3 \frac{Y_\varepsilon}{n_{\varepsilon 2}} = \alpha_1 \beta_2 \beta_3 \frac{Y_\varepsilon}{n_{\varepsilon 1}} = \frac{V'(N)}{U'(Y)}$$

Thus,

$$\begin{aligned} n_3 &= \alpha_3 \\ n_2 &= \alpha_2 \beta_3 \\ n_1 &= \alpha_1 \beta_2 \beta_3 \end{aligned}$$

Substituting these values for n into () gives us our expression for aggregate output ().

Horizontal Economy. In the horizontal economy, the unique equilibrium allocation is

given by

$$(\phi_{h3}) \alpha_3 \frac{Y_h}{n_{h3}} = (\phi_{h2}) \alpha_2 \beta_3 \frac{Y_h}{n_{h2}} = (\phi_{h1}) \alpha_1 \beta_2 \beta_3 \frac{Y_h}{n_{h1}} = Y_h$$

This implies

$$\begin{aligned} n_{h3} &= \phi_{h3} \alpha_3 \\ n_{h2} &= \phi_{h2} \alpha_2 \beta_3 \\ n_{h1} &= \phi_{h1} \alpha_1 \beta_2 \beta_3 \end{aligned}$$

Thus, there are no spill over effects other than income-substitution effects coming from the real wage. Therefore, output is given by

$$Y_h =$$

which corresponds with equation ().

Vertical Economy. In the vertical economy, the unique equilibrium allocation is given by

$$(\phi_{v3}) \alpha_3 \frac{Y_v}{n_{v3}} = (\phi_{v2} \phi_{v3}) \alpha_2 \beta_3 \frac{Y_v}{n_{v2}} = (\phi_{v1} \phi_{v2} \phi_{v3}) \alpha_1 \beta_2 \beta_3 \frac{Y_v}{n_{v1}} = Y_v$$

This implies

$$\begin{aligned} n_{v3} &= \phi_{v3} \alpha_3 \\ n_{v2} &= \phi_{v2} \phi_{v3} \alpha_2 \beta_3 \\ n_{v1} &= \phi_{v1} \phi_{v2} \phi_{v3} \alpha_1 \beta_2 \beta_3 \end{aligned}$$

Aggregate output is given by

$$Y_v =$$

which corresponds with equation (). QED

Proof of Proposition 5 These expressions follow from taking the derivative of ().

Proof of Proposition 6 Now, let's think of a fall in aggregate liquidity. We drop ϕ down by x -percent.

In the horizontal economy.

$$\begin{aligned}\frac{d \log Y_h}{d \log \phi} &= \frac{d \log Y_h}{d \log \phi_{h1}} + \frac{d \log Y_h}{d \log \phi_{h2}} + \frac{d \log Y_h}{d \log \phi_{h3}} \\ &= \alpha_1 \beta_2 \beta_3 + \alpha_2 \beta_3 + \alpha_3\end{aligned}$$

In the vertical economy

$$\begin{aligned}\frac{d \log Y_v}{d \log \phi} &= 3 \frac{d \log Y_h}{d \log \phi_{h1}} + 2 \frac{d \log Y_h}{d \log \phi_{h2}} + \frac{d \log Y_h}{d \log \phi_{h3}} \\ &= 3\alpha_1 \beta_2 \beta_3 + 2\alpha_2 \beta_3 + \alpha_3\end{aligned}$$

It follows that $\frac{d \log Y_v}{d \log \phi} > \frac{d \log Y_h}{d \log \phi}$.

Proof of Proposition 7 Follows directly from equations ()-(). QED.

Proof of Proposition 7 In order to answer this question, we need only consider a tightening of the collateral constraint of firm 2. First, consider the effects of employment from a tightening of the collateral constraint of firm 2 implies that it will buy both less labor and less intermediate inputs

$$n_{v2} = \phi_{v2} \phi_{v3} \alpha_2 \beta_3$$

thus,

$$\frac{d \log n_{v2}}{d \log \phi_{v2}} = 1$$

labor of firm 1 is given by $n_{v1} = \phi_{v1} \phi_{v2} \phi_{v3} \alpha_1 \beta_2 \beta_3$, which implies that

$$\frac{d \log n_{v1}}{d \log \phi_{v2}} = 1$$

and labor of firm 3 is unchanged.

$$\frac{d \log n_{v3}}{d \log \phi_{v2}} = 0$$

Now, let's look at output. Output of firm 1 $y_1 = A_1 n_{v1}^{\alpha_1}$ decreases

$$\log y_{v1} = \log A_1 + \alpha_1 \log n_{v1}$$

thus

$$\frac{d \log y_{v1}}{d \log \phi_{v2}} = \alpha_1 \frac{d \log n_{v1}}{d \log \phi_{v2}} = \alpha_1$$

output of firm 2 $y_{v2} = A_2 n_{v2}^{\alpha_2} y_{v1}^{\beta_2}$

$$\frac{d \log y_{v2}}{d \log \phi_{v2}} = \alpha_2 \frac{d \log n_{v2}}{d \log \phi_{v2}} + \beta_2 \frac{d \log y_{v1}}{d \log \phi_{v2}} = \alpha_2 + \beta_2 \alpha_1$$

output of firm 3 $y_{v3} = A_3 n_{v3}^{\alpha_3} y_{v2}^{\beta_3}$

$$\begin{aligned} \frac{d \log y_{v3}}{d \log \phi_{v2}} &= \alpha_3 \frac{d \log n_{v3}}{d \log \phi_{v2}} + \beta_3 \frac{d \log y_{v2}}{d \log \phi_{v2}} \\ &= \alpha_2 \beta_3 + \alpha_1 \beta_2 \beta_3 \end{aligned}$$

which corresponds with the fall in aggregate output in ().

Finally, let's look at prices. What happens to the real wage?

$$\frac{w}{p} = \frac{V'(N)}{U'(Y)} = Y$$

Thus the effect on the real wage is given by

$$\frac{d \log w}{d \log \phi_{v2}} = \frac{d \log Y}{d \log \phi_{v2}} = \alpha_2 \beta_3 + \alpha_1 \beta_2 \beta_3$$

Thus, for firm 2 we have that

$$w n_2 = \phi_2 \alpha_2 p_2 y_2$$

therefore

$$p_2 = \frac{w n_2}{\alpha_2 \phi_2 y_2}$$

thus

$$\begin{aligned} \frac{d \log p_2}{d \log \phi_{v2}} &= \frac{d \log w}{d \log \phi_{v2}} + \frac{d \log n_2}{d \log \phi_{v2}} - \frac{d \log \phi_{v2}}{d \log \phi_{v2}} - \frac{d \log y_2}{d \log \phi_{v2}} \\ &= \beta_3 (\alpha_2 + \alpha_1 \beta_2) + 1 - 1 - (\alpha_2 + \beta_2 \alpha_1) \\ &= -(1 - \beta_3) (\alpha_2 + \alpha_1 \beta_2) < 0 \end{aligned}$$

therefore, the price of firm 2 goes up. Also expenditure of firm 3 on intermediate goods $p_2 y_2$ goes down

$$\frac{d \log p_2}{d \log \phi_{v2}} + \frac{d \log y_2}{d \log \phi_{v2}} = \frac{d \log w}{d \log \phi_{v2}} = \beta_3 (\alpha_2 + \alpha_1 \beta_2) < 0$$

thus price of firm 2 goes up.

price of firm 1. Can pin it down either from this

$$wn_1 = \phi_1 \alpha_1 p_1 y_1$$

or from this.

$$\frac{1}{\alpha_2} wn_2 = \frac{1}{\beta_2} p_1 y_1$$

thus

$$\begin{aligned} \frac{d \log p_1}{d \log \phi_{v2}} &= \frac{d \log w}{d \log \phi_{v2}} + \frac{d \log n_1}{d \log \phi_{v2}} - \frac{d \log y_1}{d \log \phi_{v2}} \\ &= \beta_3 (\alpha_2 + \alpha_1 \beta_2) + 1 - \alpha_1 > 0 \end{aligned}$$

or

$$\begin{aligned} \frac{d \log p_1}{d \log \phi_{v2}} &= \frac{d \log w}{d \log \phi_{v2}} + \frac{d \log n_2}{d \log \phi_{v2}} - \frac{d \log y_1}{d \log \phi_{v2}} \\ &= \beta_3 (\alpha_2 + \alpha_1 \beta_2) + 1 - \alpha_1 > 0 \end{aligned}$$

price of firm 1 goes down after collateral constraint tightens, since there is less demand for this good.

suppose there is a collateral shock to firm 1. just to understand,

$$wn_2 = \phi_2 \alpha_2 p_2 y_2$$

thus where

$$y_2 = A_2 n_{v2}^{\alpha_2} y_{v1}^{\beta_2}$$

so that

$$\begin{aligned} \frac{d \log y_2}{d \log \phi_{v1}} &= \beta_2 \frac{d \log y_1}{d \log \phi_{v1}} \\ &= \beta_2 \alpha_1 \end{aligned}$$

$$\begin{aligned} p_2 &= \frac{d \log w}{d \log \phi_{v1}} + \frac{d \log n_2}{d \log \phi_{v1}} - \frac{d \log y_2}{d \log \phi_{v1}} \\ &= \frac{d \log Y}{d \log \phi_{v1}} + 1 - \frac{d \log y_2}{d \log \phi_{v1}} \\ &= \alpha_3 + 1 - \beta_2 \alpha_1 > 0 \end{aligned}$$

SOMETHING IS WRONG HERE—SEE MATLAB CODE

price of firm 2 goes down after collateral constraint 1 tightens, since inputs are now more expensive.

Proof of Proposition 8 In the horizontal economy

$$\begin{aligned}(\phi_{h3}) \alpha_3 \frac{Y_h}{n_{h3}} &= V'(N_h) / U'(Y_h) \\(\phi_{h2}) \alpha_2 \beta_3 \frac{Y_h}{n_{h2}} &= V'(N_h) / U'(Y_h) \\(\phi_{h1}) \alpha_1 \beta_2 \beta_3 \frac{Y_h}{n_{h1}} &= V'(N_h) / U'(Y_h)\end{aligned}$$

with power utility, linear disutility of labor, we have

$$\begin{aligned}n_{h3} &= \phi_{h3} \alpha_3 \\n_{h2} &= \phi_{h2} \alpha_2 \beta_3 \\n_{h1} &= \phi_{h1} \alpha_1 \beta_2 \beta_3\end{aligned}$$

Thus,

$$N = n_{h1} + n_{h2} + n_{h3}$$

In the vertical economy

$$\begin{aligned}(\phi_{v3}) \alpha_3 \frac{Y_v}{n_{v3}} &= V'(N_v) / U'(Y_v) \\(\phi_{v2} \phi_{v3}) \alpha_2 \beta_3 \frac{Y_v}{n_{v2}} &= V'(N_v) / U'(Y_v) \\(\phi_{v1} \phi_{v2} \phi_{v3}) \alpha_1 \beta_2 \beta_3 \frac{Y_v}{n_{v1}} &= V'(N_v) / U'(Y_v) \\N_v &= n_{v1} + n_{v2} + n_{v3}\end{aligned}$$

with power utility, linear

$$\begin{aligned}n_{v3} &= \phi_{v3} \alpha_3 \\n_{v2} &= \phi_{v2} \phi_{v3} \alpha_2 \beta_3 \\n_{v1} &= \phi_{v1} \phi_{v2} \phi_{v3} \alpha_1 \beta_2 \beta_3\end{aligned}$$

Can we get TFP?

Proof of Lemma Composite cost The marginal cost of the input composite is the solution to a standard cost minimization problem subject to a capacity constraint:

PROBLEM. The optimal input use problem is given by,

$$\begin{aligned} c_i x_i &= \min_{x_{ij} \geq 0} \sum_{j \in I} p_j x_{ij} \\ &\text{subject to} \\ x_i &= \prod_{j \in I_i} x_{ji}^{\alpha_{ij}} \end{aligned}$$

It is convenient to give an exact expression for the marginal cost.

Lemma 4 (Composite Cost). *The marginal cost for the firm is given by:*

$$c_i = \prod_{j \in N_i} \left(\frac{p_j}{\alpha_{ij}} \right)^{\alpha_{ij}}.$$

Proof. The fact that $\sum_{j \in I_i} \alpha_{ij} = 1$ implies that problem is homogeneous and hence c_i is independent of x_i . The solution to this cost minimization problem is obtained by forming a Lagrangean. Let λ_i be the Lagrangean associated with the capacity constraint. Then we obtain the following first order condition:

$$p_j x_{ij} = \lambda_i \alpha_{ij} \prod_{j \in I_i} x_{ji}^{\alpha_{ij}} = \lambda_i \alpha_{ij} x_i, \quad j \in N$$

With this, we know that the expense share firm i's use of j's input is proportional α_{ij}

$$\frac{p_j x_{ij}}{p_k x_{ik}} = \frac{\alpha_{ij}}{\alpha_{ik}} \quad (15)$$

or alternatively:

$$p_j x_{ij} = \alpha_{ij} c_i x_i$$

This result implies hence,

$$\begin{aligned} x_i &= c_i^{\sum_{j \in N_i} \alpha_{ij}} \cdot \prod_{j \in N_i} \left(\frac{\alpha_{ij}}{p_j} \right)^{\alpha_{ij}} \cdot \prod_{j \in N_i} x_{ji}^{\alpha_{ij}} \rightarrow \\ c_i &= \prod_{j \in N_i} \left(\frac{p_j}{\alpha_{ij}} \right)^{\alpha_{ij}} \end{aligned}$$

We are in time to describe the household's problem. □

Proof of Lemma Household's problem *Household's Problem.* Household's solve,

PROBLEM. Household's maximize utility,

$$\max_{x_o, y_0} U(x_o, y_0)$$

subject to the household's budget constraint,

$$c_0 x_0 \leq p_0 y_0 + \sum_{j \in I \setminus \{0\}} p_j y_j - c_j x_j.$$

The budget constraint implies that consumption expenditures are financed with labor income $p_0 y_0$ and profits. The cost of the consumption composite is again the solution to a similar intermediat input problem:

PROBLEM. The final good minimiation problem is given by:

$$\begin{aligned} c_0 x_0 &= \min_{x_{0j} \geq 0} \sum_{j \in I} p_j x_{0j} \\ &\text{subject to} \\ x_0 &= \prod_{j \in I_0} x_{0j}^{\alpha_{0j}} \end{aligned}$$

Lemma 4 also applies to this problem so $c_0 = \prod_{j \in I_0} \left(\frac{p_j}{\alpha_{0j}} \right)^{\alpha_{0j}}$. We adopt the convention of normalizing the price index to 1 so $c_0 = 1$.

Proof of Proposition 10 Proof. The idea of the proof is to find a linear self-map for the vector of sectoral outputs (including labor). This requires the computation of an intermediate step.

The i -th sector's resource constraint can be written as:

$$\begin{aligned} y_i &= x_{0i} + \sum_{j=1}^N x_{ji} \\ &= x_{0i} + \sum_{j=1}^N \frac{\alpha_{ji} c_j x_j}{p_i} \\ &= x_{0i} + \sum_{j=1}^N \frac{\alpha_{ji} p_j \phi_i y_i}{p_i} \end{aligned}$$

where the second line uses (14) and the third uses the first order condition of firm i . Multiplying both sides by p_i yields:

$$p_i y_i = p_i x_{0i} + \sum_{j=1}^N \alpha_{ji} p_j \phi_j y_j.$$

One can use the optimality condition from the household, $\frac{\alpha_{0j} x_0}{x_{0j}} = p_j$ to obtain the following equation without prices:

$$\frac{\alpha_{0i} x_0}{x_{0i}} y_i = \frac{\alpha_{0i} x_0}{x_{0i}} x_{0i} + \sum_{j=1}^N \phi_j \alpha_{ji} \frac{\alpha_{0j} x_0}{x_{0j}} y_j.$$

One can clear consumption expenditures x_0 from both sides. Define $\gamma_i \equiv \frac{\alpha_{0i}}{x_{0i}/y_i}$, which represents the ratio of the consumption share of input i , relative to the fraction of the output of i that is destined to consumption. Then we can express the firm's output this equation in vector form as:

$$\boldsymbol{\gamma} = \boldsymbol{\alpha}_0 + \tilde{A} \boldsymbol{\gamma}$$

where $\boldsymbol{\gamma}$ is the $(N+1) \times 1$ vector of entries γ_i . Here, the matrix $\tilde{A}$ is defined as:

$$\tilde{A} \equiv [\boldsymbol{\phi} \mathbf{e}'] \cdot \mathbf{A}' \quad (16)$$

The vector solves:

$$\boldsymbol{\gamma}_{[1:N]} = (\mathbf{I} - \tilde{A})^{-1} \tilde{A}. \quad (17)$$

This vector yields values for the fraction of the output of i that is destined to consumption, y_i/x_{0i} . Solving for this ratios is an intermediate step in the solution to the model. Now, by definition, the first element of $\boldsymbol{\gamma}$, is:

$$\gamma_0 = \alpha_0 + \sum_{j=1}^N \phi_j \alpha_{j0}. \quad (18)$$

The idea now is to find a self map for sectoral outputs. Using firm i 's first order conditions we have:

$$x_{ij} = \alpha_{ij} \phi_i \frac{p_i}{p_j} y_i.$$

From the household's optimal input ratio, $\frac{p_i}{p_j} = \frac{\alpha_{0i} x_{0j}}{\alpha_{0j} x_{0i}}$, we can clear prices from this expression:

$$x_{ij} = \alpha_{ij} \phi_i \frac{\alpha_{0i} x_{0j}}{\alpha_{0j} x_{0i}} \frac{y_i}{y_j} = \alpha_{ij} \phi_i \frac{\gamma_i}{\gamma_j} y_j.$$

Back in firm i 's output we have:

$$\begin{aligned}
y_i &= z_i \prod_{j \in I_i} (x_{ij})^{\alpha_i \alpha_{ij}} \\
&= z_i (\phi_i)^{\alpha_i} \prod_{j \in I_i} \left(\alpha_{ij} \frac{\gamma_i}{\gamma_j} y_j \right)^{(\alpha_i \alpha_{ij})} \\
&= z_i (1 - \tau_i)^{\alpha_i} \prod_{j \in I_i} \left(\alpha_i \alpha_{ij} \frac{\gamma_i}{\gamma_j} y_j \right)^{(\alpha_i \alpha_{ij})}
\end{aligned} \tag{19}$$

This is a map from vector of sectoral outputs $\mathbf{y}$ which includes labor to firm outputs. To make this a self map in $\mathbf{y}$, we need an additional equation that maps $\mathbf{y}$ into y_0 . For this we use the household's budget constraint. We express x_0 as a function of $\mathbf{y}$ and substitute this expression into the household's budget constraint. Notice also from this expression that term $(1 - \tau_i)^{\alpha_i}$ multiplies z_i so the liquidity shocks act like an adjustment to TFP. The τ_i terms thus capture direct effect of ϕ_i on the i -th sector's scale. The terms $\frac{\gamma_i}{\gamma_j}$ capture the effects on the allocations across sectors. We will use this distinction to decompose losses stemming from direct effects and missallocation of resources across the network.

The household's first order condition is given by:

$$x_0^{-\sigma} = p_o y_o^\varepsilon.$$

Manipulating this expression we obtain:

$$x_0^{-\sigma} = y_o^{\varepsilon-1} p_o y_o$$

and from his budget constraint, notice that

$$p_0 y_0 = x_0 - \sum_{i=1}^N \Pi_i.$$

Now, from the first order conditon of the firm, profits satisfy

$$\begin{aligned}
\Pi_i &= p_i y_i - \phi_i p_i y_i \\
&= x_0 \frac{\alpha_{0i} y_i}{x_{0i}} (1 - \phi_i) \\
&= x_0 (1 - \phi_i) \gamma_i
\end{aligned}$$

where the second line follows from $\frac{\alpha_{0i} x_0}{x_{0i}} = p_i$ and the third uses the definition of γ_i . Hence,

the firm's budget constraint may be written as:

$$x_0 \left(1 - \sum_{i=1}^N (1 - \phi_i) \gamma_i \right) = p_0 y_0.$$

We can combine this expression with the household's first order condition to get rid of p_0 . The outcome of this substitution is:

$$y_0^{\varepsilon-1} = x_0^{1-\sigma} \left(1 - \sum_{i=1}^N (1 - \phi_i) \gamma_i \right).$$

Now, using the definition of γ_j we obtain,

$$x_0 = \prod_{j \in I} (x_{0j})^{\alpha_{0j}} = \prod_{j \in I} \left(\frac{\alpha_{0j} y_j}{\gamma_j} \right)^{\alpha_{0j}}.$$

Thus, the desired self-map is:

$$y_0 = \left(\prod_{j \in I} \left(\frac{\alpha_{0j} y_j}{\gamma_j} \right)^{\alpha_{0j}} \right)^{\frac{1-\sigma}{1+\varepsilon}} \left(1 - \sum_{i=1}^N (1 - \phi_i) \gamma_i \right)^{\frac{1}{1+\varepsilon}}. \quad (\text{SelfMap2})$$

Taking logs, (19) and (??) define a linear map:

$$\begin{aligned} \ln y_o &= \ln \left[\left(\prod_{j=0}^N \left(\frac{\alpha_{0j} y_j}{\gamma_j} \right)^{\alpha_{0j}} \right)^{\frac{1-\sigma}{1+\varepsilon}} \left(1 - \sum_{i=1}^N (1 - \phi_i) \gamma_i \right)^{\frac{1}{1+\varepsilon}} \right] \\ \ln y_i &= \ln \left[z_i (\phi_i)^{\alpha_i} \prod_{j \in I_i} \left(\alpha_{ij} \frac{\gamma_i}{\gamma_j} y_j \right)^{(\alpha_i \alpha_{ij})} \right]. \end{aligned}$$

We now use $\tilde{x}$ to denote the log of a variable or parameter x . Using this notation, these equations read:

$$\begin{aligned} \tilde{y}_o &= \frac{1}{1+\varepsilon} \ln \left(1 - \sum_{i=1}^N (1 - \phi_i) \gamma_i \right) + \frac{1-\sigma}{1+\varepsilon} \sum_{j \in I} \alpha_{0j} \ln \left(\frac{\alpha_{0j}}{\gamma_j} \right) + \frac{1-\sigma}{1+\varepsilon} \sum_{j \in I} \alpha_{0j} \tilde{y}_j \\ \tilde{y}_i &= \tilde{z}_i + \alpha_i \ln \phi_i + \alpha_i \sum_{j \in I_i} \alpha_{ij} \ln \left(\alpha_{ij} \frac{\gamma_i}{\gamma_j} \right) + \alpha_i \sum_{j \in I_i} \alpha_{ij} y_j. \end{aligned}$$

We can write the expressions in matrix form.

$$[\tilde{\mathbf{y}}] = \begin{bmatrix} C_1 \\ C \end{bmatrix} + \begin{bmatrix} \frac{1-\sigma}{1+\varepsilon} \alpha_{00} & \frac{1-\sigma}{1+\varepsilon} \alpha_{01} & \frac{1-\sigma}{1+\varepsilon} \alpha_{02} & \cdots & \frac{1-\sigma}{1+\varepsilon} \alpha_{0N} \\ \alpha_1 \alpha_{10} & \alpha_1 \alpha_{11} & \alpha_1 \alpha_{12} & \cdots & \alpha_2 \alpha_{2N} \\ \alpha_2 \alpha_{20} & \alpha_2 \alpha_{21} & \alpha_2 \alpha_{22} & \cdots & \alpha_2 \alpha_{2N} \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ \alpha_N \alpha_{N0} & \alpha_N \alpha_{N1} & \alpha_N \alpha_{N2} & \cdots & \alpha_N \alpha_{NN} \end{bmatrix} [\tilde{\mathbf{y}}]$$

where

$$C_1 = \frac{1}{1+\varepsilon} \ln \left(1 - \sum_{i=1}^N (1 - \phi_i) \gamma_i \right) + \frac{1-\sigma}{1+\varepsilon} \sum_{j \in I} \alpha_{0j} \ln \left(\frac{\alpha_{0j}}{\gamma_j} \right) \quad (20)$$

and,

$$C_2 = \begin{bmatrix} \tilde{z}_1 + \alpha_1 \ln \phi_1 + \alpha_1 \sum_{j \in I_1} \alpha_{1j} \ln \left(\alpha_{1j} \frac{\gamma_1}{\gamma_j} \right) \\ \tilde{z}_2 + \alpha_2 \ln \phi_2 + \alpha_2 \sum_{j \in I_2} \alpha_{2j} \ln \left(\alpha_{2j} \frac{\gamma_2}{\gamma_j} \right) \\ \vdots \\ \tilde{z}_N + \alpha_N \ln \phi_N + \alpha_N \sum_{j \in I_N} \alpha_{Nj} \ln \left(\alpha_{Nj} \frac{\gamma_N}{\gamma_j} \right) \end{bmatrix}.$$

In matrix form, C_1 and C_2 may be written as,

$$C_1 = \frac{1}{1+\varepsilon} \ln (1 - (\mathbf{e} - \boldsymbol{\phi})' \boldsymbol{\gamma}) + \frac{1-\sigma}{1+\varepsilon} \boldsymbol{\alpha}'_0 (\tilde{\alpha}_0 - \tilde{\gamma})$$

and

$$C_2 = \left[\tilde{\mathbf{Z}} + \boldsymbol{\alpha} \cdot \tilde{\boldsymbol{\phi}} + \boldsymbol{\alpha} \cdot \text{diag}(\mathbf{A} [\log(\mathbf{A})]') + (\boldsymbol{\alpha} \cdot \tilde{\boldsymbol{\gamma}}) \cdot (\mathbf{A} \mathbf{1}_{N+1}) - \boldsymbol{\alpha}' \cdot (\mathbf{A} \tilde{\boldsymbol{\gamma}}) \right]$$

Define the following matrixes:

$$\begin{aligned}
A_{11} &= \left[\frac{1-\sigma}{1+\varepsilon} \alpha_{00} \right] \\
A_{12} &= \left[\begin{array}{cccc} \frac{1-\sigma}{1+\varepsilon} \alpha_{01} & \frac{1-\sigma}{1+\varepsilon} \alpha_{02} & \cdots & \frac{1-\sigma}{1+\varepsilon} \alpha_{0N} \end{array} \right] \\
A_{21} &= \left[\begin{array}{c} \alpha_1 \alpha_{10} \\ \alpha_2 \alpha_{20} \\ \vdots \\ \alpha_N \alpha_{N0} \end{array} \right] \\
A_{22} &= \left[\begin{array}{cccc} \alpha_1 \alpha_{11} & \alpha_1 \alpha_{12} & \cdots & \alpha_2 \alpha_{2N} \\ \alpha_2 \alpha_{21} & \alpha_2 \alpha_{22} & \cdots & \alpha_2 \alpha_{2N} \\ \vdots & \vdots & \ddots & \dots \\ \alpha_N \alpha_{N1} & \alpha_N \alpha_{N2} & \cdots & \alpha_N \alpha_{NN} \end{array} \right]
\end{aligned}$$

The solutio to sector outputs can be expressed in terms of these matrices and C_1 and C_2 .

$$[\tilde{\mathbf{y}}] = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} [\tilde{\mathbf{y}}] \quad (21)$$

We can use the first row of this equation to obtain:

$$\tilde{y}_o = C_1 / (1 - A_{11}) + A_{12} / (1 - A_{11}) \tilde{\mathbf{y}}_{[1:N]}. \quad (22)$$

Using this expression and replacing it into the final N rows of (21) we have an expression for sectoral outputs that is homogeneous. This expression is given by:

$$\begin{aligned}
\tilde{\mathbf{y}}_I &= C_2 + A_{21} (C_1 / (1 - A_{11}) + A_{12} / (1 - A_{11}) \tilde{\mathbf{y}}_I) + A_{22} \tilde{\mathbf{y}}_I \\
&= C_2 + \frac{A_{21} C_1}{(1 - A_{11})} + \left(\frac{A_{21}}{(1 - A_{11})} A_{12} + A_{22} \right) \tilde{\mathbf{y}}_I
\end{aligned}$$

which yields:

$$\tilde{\mathbf{y}}_I = \left[I - \left(\frac{A_{21}}{(1 - A_{11})} A_{12} + A_{22} \right) \right]^{-1} \left[C_2 + \frac{C_1}{(1 - A_{11})} A_{21} \right]. \quad (23)$$

Hours can be solved from equation (22):

$$\tilde{y}_o = C_1 / (1 - A_{11}) + A_{12} / (1 - A_{11}) \left[I - \left(\frac{A_{21}}{(1 - A_{11})} A_{12} \mathbf{e}' + A_{22} \right) \right]^{-1} \left[C_2 + \frac{C_1}{(1 - A_{11})} A_{21} \right].$$

The expression for log GDP is obtained through the first order condition of the household:

$$\tilde{x}_0 = \frac{(\varepsilon - 1)}{(1 - \sigma)} \tilde{y}_o - \frac{1}{(1 - \sigma)} \ln \left(1 - \sum_{i=1}^N (1 - \phi_i) \gamma_i \right).$$

Sectoral Prices. We can use the vector of sectoral prices.

$$p_i = \frac{\alpha_{0i}}{\alpha_{00}} \frac{x_{00}}{x_{0i}} p_0$$

By definition of γ_i we have that:

$$p_i = \left[\frac{\gamma_i}{y_i} \frac{y_0}{\gamma_0} \right] p_0$$

and using the definition of c_0 , we obtain:

$$\begin{aligned} c_0 &= \prod_{j \in I_0} \left(\frac{p_j}{\alpha_{0j}} \right)^{\alpha_{0j}} \\ &= p_0 \frac{y_0}{\gamma_0} \prod_{j \in I_0} \left(\frac{\gamma_j}{y_j} \right)^{\alpha_{0j}} \end{aligned}$$

and using the normalization constant, we obtain the real wage rate of this economy:

$$p_0 = \frac{\gamma_0}{y_0} \prod_{j \in I_0} \left(\frac{y_j}{\gamma_j} \right)^{\alpha_{0j}}$$

so in logs we obtain:

$$\tilde{p}_0 = \tilde{\gamma}_0 - \tilde{y}_0 + \sum_{j \in I_0} \alpha_{0j} (\tilde{y}_j - \tilde{\gamma}_j)$$

and for the rest of prices we have:

$$\tilde{p}_i = \tilde{\gamma}_i - \tilde{y}_i + \sum_{j \in I_0} \alpha_{0j} (\tilde{y}_j - \tilde{\gamma}_j).$$

Thus, in matrix form, prices are given by:

$$\tilde{p} = [\mathbf{e}' \boldsymbol{\alpha}_0 - \mathbf{I}] (\tilde{\mathbf{y}} - \tilde{\boldsymbol{\gamma}})$$