

International Portfolios: A Comparison of Solution Methods

Katrin Rabitsch ^{*} Serhiy Stepanchuk [†] Viktor Tsyrennikov [‡]

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Abstract

Substantial progress has been made in recent years in integrating optimal portfolios into (open macro) general equilibrium models using standard local approximation (perturbation) methods. We compare these perturbation-based portfolio solution methods with a global portfolio solution method to evaluate their relative performance. We find that the local method performs very well when the model features and parameters are aimed at capturing macro stylized facts only, and in symmetric country setups. It performs much less well when the model includes features from the finance literature that have been shown to be important in jointly explaining macro- and asset pricing stylized facts, such as Epstein-Zin preferences, or in a model with positive expected excess returns (equity premia). Also, in asymmetric country setups a 'wrong' approximation of the net foreign asset (NFA) position can strongly compromise the performance of the local portfolio solution method whenever the true policy functions are nonlinear in the NFA.

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^{*}Vienna University of Economics and Business. E-mail: katrin.rabitsch@wu.ac.at.

[†]Magyar Nemzeti Bank (The Central Bank of Hungary). E-mail: stepanchuks@mnbb.hu

[‡]Cornell University. E-mail:

1 Introduction

This paper presents a comparison and an evaluation of local (perturbation-based) and global solution methods for solving (country) portfolios. The recent advances of Devereux and Sutherland (2011, 2010), Tille and van Wincoop (2007), and Evans and Hnatkovska (2005) in solving portfolio choice models and in integrating them into standard dynamic stochastic general equilibrium (international) macro models are typically based on local approximation techniques around a (near-) non-stochastic steady state. We show that these methods are well suited to address setups in which countries are symmetric and in which the model does not display strong nonlinearities. However, many of the questions that concern international asset trade require setups in which countries are asymmetric, are subject to strong nonlinearities and/or large shocks, setups local approximation methods may have difficulties in dealing with. We use a global, nonlinear solution method to solve for country portfolios and compare the results to the perturbation-based solutions. In this way, we are able to shed light on the question under which circumstances local methods can only be used with caution and may lead to imprecise results. Furthermore, we also provide indications of how best to incorporate asymmetric setups into the DS method.

The contributions by the above cited authors have been path-breaking in many aspects: for most of their existence, general equilibrium international macro models have ignored portfolios –that is, the existence of and choice between several available asset types– altogether.^{1,2} Typically, models adopted either a setup of complete international financial markets, or the case where only a single asset (typically a risk-free bond) is traded and focused their attention on *net* external flows only. Until recently, portfolio issues were only considered in the special case of complete financial markets, in which case the behavior of macro variables and portfolios are independent, and the latter can be backed out after having obtained a solution to macro dynamics.

Instead, the solution approach developed by Devereux and Sutherland (2011, 2010) applies explicitly to models of incomplete markets, and makes use of perturbation methods (similar

¹An earlier literature, of the late 1970s and early 1980s, has looked at portfolio balance models (see e.g., the review of Branson and Henderson (1985)). Those were, however, typically cast in a partial equilibrium setup.

²There is also a more recent literature on (also partial equilibrium) country portfolios in continuous time (see Kraay and Ventura (2000, 2003)), and (Kraay et al. (2005)).

to Judd and Guu (2001)). This methodological progress is of interest not only from theoretical considerations but also allows macroeconomic models to (begin to) do justice to a number of important questions in open economy macroeconomics: the existence of substantial gross external asset and liability positions and their rapid growth in recent decades, the increasing empirical importance of (two-way) asset trade, the role of portfolio allocation in determining (net) capital flows, the potential influences of size and composition of gross portfolios on macroeconomic outcomes themselves (e.g., through exchange rate and asset-price driven 'valuation effects').

Nevertheless, several questions concerning the validity of the proposed local approximation method arise naturally. Local approximation methods may be inaccurate for evaluating the affects of large shocks (which may particularly apply to e.g. emerging market economies or the case of the Great Recession), or when, along simulated paths, the economy departs significantly from the point of approximation. Also, in asymmetric setups local approximation methods have the additional difficulty that –when the true policy functions are very nonlinear– results rely heavily on the choice of the approximation point of net foreign asset, the 'right' value of which is, however, itself determined endogenously in a true solution. Furthermore, the (closed economy) finance literature has shown that nonlinearities may be of particular importance in jointly explaining macro- and asset pricing stylized facts. Such nonlinearities may come from the presence of explicit borrowing or short-selling constraints, or from consideration of richer preference specifications such as, e.g. Epstein-Zin preferences.

Our paper provides a comprehensive evaluation of the performance of perturbation-based solution methods to portfolio allocation, by comparing policy functions, simulated short time-paths, moments from simulated model data, stationary distributions, and Euler equation errors of the local and global solution approaches. It is important to note that the aim of the comparison is not to dismiss the perturbation-based method. While it is clear that, by definition, a global solution method will always perform better in some dimensions, the local approximation approach offers also substantial advantages over the global method, e.g. it can handle a large number of state variables easily. It therefore is not subject to a 'curse of dimensionality' and can be applied in large, relatively complex models. We believe, however, that it is important to know when the perturbation-based solution approach works well, what potential fallacies open when using this solution approach and how best to resolve them.

Section 2 briefly lays out the baseline model economy used in the comparison, which closely corresponds to the model in Devereux and Sutherland (2009): a two-country model with capital and labor endowments, in which the economies can engage in financial trade in equities and where agents have CRRA preferences. Section 3 discusses some of the specifics of the solution techniques of the local and global approaches. We then proceed in the following manner: in section 4.1, we first focus on a *symmetric* setup of the countries. We start by considering the special case where an analytical solution is known and investigate how well the found portfolio in the perturbation-based solution and the global solution fares against the true portfolio. We then move to more general (but still symmetric) specifications. In these cases, the fact that applying the DS solution method requires knowledge about the right approximation point for the NFA, $\bar{W}$, is not problematic, as it is natural to take $\bar{W} = 0$ as the stipulated value around which to approximate. We find that the DS method performs very well in such cases.

We then, in section 4.3, turn to an evaluation of the found portfolio solution in a *non-symmetric* setup and show how the DS method is best adopted in asymmetric setups. Arguably, asymmetric setups are more relevant if one wants to study realistically calibrated open economy models. Even more so, as the DS method can be applied not only to two-country settings, but in principle to any heterogeneous agent setting with portfolio choice, in which case the assumption of symmetry (and a resulting zero net position between agents in steady state) may make even less sense.³ In this case, solving for the portfolio for a given value of $\bar{W}$ that is exogenously chosen is no longer innocuous, but will have a strong influence on the solution found. In a true global solution $\bar{W}$ should itself be determined (together with the portfolio, α) by countries' (relative) risk characteristics of the two countries, together with other factors that influence the (relative) motives for precautionary asset holdings. To solve for $\bar{W}$ endogenously, we propose to follow an iterative method in which wealth is updated from the mean stochastic steady state based on a 2nd order approximation of the model (including steady state and portfolio dynamics). In this we follow the idea of Devereux and Sutherland (2009), yet with a slightly different execution of the algorithm that appears to lead

³A recent example from the literature on macro models with financial frictions is Gertler et al. (2011). In their paper banks' liability side consists of either debt or risky outside equity ('preferred stocks'). The applicability of the methods discussed in this paper is thus wider – they can be applied to determine the composition of bank balance sheets, though it should be clear that the setup (consumers are creditor, banks debtors) is naturally asymmetric.

to a superior performance. In particular, we propose a modified formulation of the endogenous discount factor in which we use the *stochastic* steady state value for the NFA together with the *deterministic* steady state values implied by such NFA for all other variables. This way the 'updating algorithm' is used model-consistently and also performs much better, i.e. closer to the its global counterpart.

Finally, we turn to a number of cases in which the differences between local and global portfolio solution approaches become more aggravated, which typically corresponds to including model elements that enable the model to perform better also in matching stylized facts in finance: Section 4.1.4 considers the presence of tighter inequality or short-selling constraints, and section 4.1.5 performs sensitivity analysis with respect to the choice of preferences (drawing comparisons for the case of Epstein-Zin preferences and the case of risk-sensitive preferences). Section 6 concludes.

2 Example model of Devereux and Sutherland (JEEA, 2011)

Our baseline model follows closely the model of Devereux and Sutherland (2009), which is an economy of two countries, each of which obtain 'capital' and 'labor income' endowments ('coconuts from trees') and are able to purchase and hold their own and the other country's equity. The representative agent in the home economy has a utility function of the form

$$U_t \equiv E_t \sum_{\tau=t}^{\infty} \vartheta_{\tau} u(C_{\tau}), \quad (1)$$

where C_t is consumption and the utility function is assumed to be of the constant relative risk aversion type, that is, $u(C_t) = (C_t^{1-\rho}) / (1-\rho)$. ϑ_{τ} is the (endogenous) discount factor, which follows the following law of motion:

$$\vartheta_{\tau+1} = \vartheta_{\tau} \beta(C_{A\tau}), \vartheta_0 = 1,$$

where C_A is aggregate home consumption, $0 < \beta(C_A) < 1$, and $\beta'(C_A) \leq 0$.

We will look at two versions for the functional form of the endogenous discount factor

(EDF). In its first specification, we follow Devereux and Sutherland (2011, 2010) and assume

$$\beta(C_{A,t}) = \beta C_{A,t}^{-\eta}, \quad (2)$$

where $0 \leq \eta < \rho$ and $0 < \beta \bar{C}_A^{-\eta} < 1$, where $\bar{C}_A$ is the steady-state level of (aggregate) consumption. We also consider an alternative specification for the functional form of $\beta(C_A)$, which we call the 'modified' endogenous discount factor:

$$\beta(C_{A,t}) = \beta \left(\frac{C_{A,t}}{\bar{C}_A} \right)^{-\eta}. \quad (3)$$

In both specifications, if $\eta = 0$, then $\beta'(C_A) = 0$, and the discount factor $\beta(C_A)$ is exogenous and constant, and equal to β . As is well known, in this case the incompleteness of financial markets leads to two problems when the model is solved based on a Taylor-series approximation of the model's equilibrium conditions (i.e. with a local approximation method). One, the non-stochastic steady state is not uniquely determined, and two, there is a unit root in wealth dynamics in the first-order approximated model.⁴ In both of the above specifications of the discount factor, making it endogenous, i.e. $\beta'(C_A) < 0$, eliminates the unit root in the first order dynamics. However, only in the specification of equation (12) the deterministic steady state is pinned down uniquely, whereas the modified EDF in equation (3) continues to fail to determine the non-stochastic steady state uniquely. It should be noted, that this is intentional, as we will propose to find the 'right' approximation point for wealth endogenously from the mean of the stochastic steady state distribution, which will be of particular importance in setups in which the two countries are asymmetrically parameterized.⁵

The home agent maximizes equation (1) subject to the budget constraint:

$$\theta_{H,t}q_t + \theta_{F,t}q_t^* = \theta_{H,t-1}(q_t + Y_{K,t}) + \theta_{F,t-1}(q_t^* + Y_{K,t}^*) + Y_{L,t} - C_t, \quad (4)$$

where θ_H and θ_F denote the domestic country's holdings of domestic and foreign equity, q and q^* are the prices of domestic and foreign equity, and Y_K and Y_L is the domestic capital

⁴Schmitt-Grohé and Uribe (2003) discuss a number of other stationary introducing devices, other than the endogenous discount factor.

⁵Section 4.3 discusses this method in detail. Appendix B further discusses the choice of the precise functional form and the influence of the endogenous discount factor on the model solution.

and labor income respectively. The home agent also faces a borrowing constraint, namely, that the value of her future wealth cannot be smaller than $\bar{w}$:

$$\theta_{H,t}(q_{t+1} + Y_{K,t+1}) + \theta_{F,t}(q_{t+1}^* + Y_{K,t+1}^*) \geq \bar{w} \quad (5)$$

The aggregate output Y_t is the sum of capital and labor income endowments, $Y_t = Y_{K,t} + Y_{L,t}$, which are given by exogenous autoregressive processes:

$$\begin{aligned} \log(Y_{K,t}/\bar{Y}_K) &= \rho_K \log(Y_{K,t-1}/\bar{Y}_K) + \varepsilon_{K,t}, & \log(Y_{K,t}^*/\bar{Y}_K^*) &= \rho_K^* \log(Y_{K,t-1}^*/\bar{Y}_K^*) + \varepsilon_{K,t}^*, \\ \log(Y_{L,t}/\bar{Y}_L) &= \rho_L \log(Y_{L,t-1}/\bar{Y}_L) + \varepsilon_{L,t}, & \log(Y_{L,t}^*/\bar{Y}_L^*) &= \rho_L^* \log(Y_{L,t-1}^*/\bar{Y}_L^*) + \varepsilon_{L,t}^*, \end{aligned} \quad (6)$$

where $\varepsilon_{K,t}$, $\varepsilon_{L,t}$, $\varepsilon_{K,t}^*$ and $\varepsilon_{L,t}^*$ are zero-mean i.i.d. shocks symmetrically distributed over the interval $[-\epsilon, \epsilon]$ with $Var(\varepsilon_K) = \sigma_K^2$, $Var(\varepsilon_L) = \sigma_L^2$, $Var(\varepsilon_K^*) = \sigma_{K^*}^2$, $Var(\varepsilon_L^*) = \sigma_{L^*}^2$, $Cov(\varepsilon_K, \varepsilon_L) = Cov(\varepsilon_K^*, \varepsilon_L^*) = \sigma_{KL}$. We assume that there is no international correlation of endowment, such that $Cov(\varepsilon_K, \varepsilon_K^*) = Cov(\varepsilon_L, \varepsilon_L^*) = 0$.

Goods and asset market clearing require

$$\begin{aligned} C_t + C_t^* &= Y_t + Y_t^*, \\ \theta_{H,t} + \theta_{H,t}^* &= 1, \theta_{F,t} + \theta_{F,t}^* = 1. \end{aligned} \quad (7)$$

Table 1 summarizes the model's equilibrium conditions.

3 Solution techniques of global versus local approximation

In the following we provide a brief description of the numerical solution methods for global and local approximations to obtaining the portfolio solutions. For the more interested reader we refer to a more complete description of these methods in the references listed within this section.

Table 1: Model A: Two-Equity Endowment Model of Devereux and Sutherland (JEEA, 2009)

(A1):	$q_t u_{C,t} = E_t \beta(C_t) u_{C,t+1} (q_{t+1} + Y_{K,t+1}) + \mu_t (q_{t+1} + Y_{K,t+1})$
(A2):	$q_t^* u_{C,t} = E_t \beta(C_t) u_{C,t+1} (q_{t+1}^* + Y_{K,t+1}^*) + \mu_t (q_{t+1}^* + Y_{K,t+1}^*)$
(A3):	$q_t u_{C,t}^* = E_t \beta(C_t) u_{C,t+1}^* (q_{t+1} + Y_{K,t+1}) + \mu_t^* (q_{t+1} + Y_{K,t+1})$
(A4):	$q_t^* u_{C,t}^* = E_t \beta(C_t) u_{C,t+1}^* (q_{t+1}^* + Y_{K,t+1}^*) + \mu_t^* (q_{t+1}^* + Y_{K,t+1}^*)$
(A5):	$C_t + C_t^* = Y_t + Y_t^*$
(A6):	$q_t \theta_{H,t} + q_t^* \theta_{F,t} = (q_t + Y_{K,t}) \theta_{H,t-1} + (q_t^* + Y_{K,t}^*) \theta_{F,t-1} + Y_{L,t} - C_t$
(A7):	$\theta_{H,t} + \theta_{H,t}^* = 1$
(A8):	$\theta_{H,t} + \theta_{H,t}^* = 1$

3.1 Global solution method

To obtain a globally valid approximation we make guesses and iterate on the policy functions of equity prices and marginal utilities as functions of a discretized grid of the five state variables of our model. The state variables in the model economy consist of exogenous states Y_K , Y_L , Y_K^* , and Y_L^* , over which a grid is constructed by discretizing the continuous VAR given in equation (6) using the Rouwenhorst method.⁶ Denote the vector of exogenous states by $\mathbf{Y} = (Y_K, Y_K^*, Y_L, Y_L^*)$. We choose 3 discretization point in direction of each variable, which results in an exogenous grid and probability transition matrix, Π , of dimension 3^4 . Following Stepanchuk and Tsyrennikov (2011), Judd et al. (2011), Kubler and Schmedders (2003), we recast the above equilibrium conditions in terms of a wealth-recursive equilibrium, such that the dimensionality of the problem is reduced, and the model's only endogenous state variable is the wealth share, ω_t (for which we specify 501 discretization points). More precisely, this transformed state variable expresses the domestic country's financial wealth in total (world) financial wealth, which can be written as

$$\omega_t = \frac{\theta_{H,t-1} (q_t + Y_{K,t}) + \theta_{F,t-1} (q_t^* + Y_{K,t}^*) - \bar{w}}{(q_t + Y_{K,t} + q_t^* + Y_{K,t}^* - 2\bar{w})}, \quad (8)$$

⁶A number of recent papers have shown that traditional discretization approaches such as, e.g. the method of Tauchen and Hussey (1991), can perform rather poorly when the number of discretization nodes is low or when shock persistence of the underlying process becomes large (Flodén (2006), Kopecky and Suen (2010)). We follow Lkhagvasuren and Galindez (2010) and Lkhagvasuren and Gospodinov (2011), in their implementation of the Rouwenhorst discretization for two-dimensional VARs (this is sufficient for our case, as we assume that international shock correlations are zero).

where parameter $\bar{w}$ governs the degree to which countries can short-sell their equity, which determine the tightness of the countries' borrowing limits. The assumption of short-selling constraints is common in the literature of global portfolio solution methods and is required to insure that the wealth share can take on only values in the interval $[0, 1]$. In principle, a constraint on the maximum amount of short-selling allowed can be placed on either individual holdings of equity positions (i.e. on $\theta_{H,t}$ and $\theta_{F,t}$ individually, and on $\theta_{H,t}^*$ and $\theta_{F,t}^*$ individually) or could be placed on the value of the joint equity position (i.e., that $\theta_{H,t}(q_{t+1} + Y_{K,t+1}) + \theta_{F,t}(q_{t+1}^* + Y_{K,t+1}^*) \geq \bar{w}$, and similarly for Foreign).⁷ Since we are to compare the portfolio solution from the global approach to the local portfolio solution (DS) method in which such short-selling constraints are absent, we are interested in a version of short-selling constraints that are as loose as possible (but still avoid the need to extrapolate). For this reason, we specify a 'joint constraint' and choose parameter $\bar{w} = -3.5$, such that we are very close to the 'natural borrowing constraint'.

Using the definition in (8), we rewrite the budget constraint, equation (4), as:

$$\theta_{H,t}q_t + \theta_{F,t}q_t^* = \omega_t(q_t + Y_{K,t} + q_t^* + Y_{K,t}^*) + Y_{L,t} - C_t. \quad (9)$$

We start the iterative algorithm with guesses $q(\omega; \mathbf{Y})$, $q^*(\omega; \mathbf{Y})$, $u_C(\omega; \mathbf{Y})$, and $u_C^*(\omega; \mathbf{Y})$. Using these guesses one period ahead as functions of $(\omega'; \mathbf{Y}')$ we can compute the expectation terms in the Euler equations in Table 1 by summing over all probability-weighted future term the on right hand side. For this one needs to know ω' , which can be found by linearly interpolating the equity price functions q' and q'^* .⁸ Once the expectation terms on the RHS of the Euler equations are found the system of model equations in Table 1 can be solved, which gives updated functional guesses for $q(\omega; \mathbf{Y})$, $q^*(\omega; \mathbf{Y})$, $u_C(\omega; \mathbf{Y})$, and $u_C^*(\omega; \mathbf{Y})$. For some states of nature the borrowing constraint may bind, with we account for by using the Garcia-Zangwill trick (alternatively, imposing a 'barrier methods', following the functional form of ??). We continue this procedure until convergence is achieved (with stopping criterium $1e-9$).

⁷For example, an assumption often made is a no-short-selling constraint on (individual) equity positions, such that $\theta_{H,t}, \theta_{F,t}, \theta_{H,t}^*, \theta_{F,t}^* \geq 0$. In this case, ω_t simplifies to $\omega_t = \frac{\theta_{H,t-1}(q_t + Y_{K,t}) + \theta_{F,t-1}(q_t^* + Y_{K,t}^*)}{(q_t + Y_{K,t} + q_t^* + Y_{K,t}^*)}$. The numerator is domestic financial wealth, the denominator is world financial wealth $\theta_{H,t-1}(q_t + Y_{K,t}) + \theta_{F,t-1}(q_t^* + Y_{K,t}^*) + \theta_{H,t-1}^*(q_t + Y_{K,t}) + \theta_{F,t-1}^*(q_t^* + Y_{K,t}^*) = (q_t + Y_{K,t} + q_t^* + Y_{K,t}^*)$.

⁸That is, we can find ω' from equation $\omega' = \frac{\theta_H(q'(\omega'; \mathbf{Y}') + Y_{K'}') + \theta_F(q'^*(\omega'; \mathbf{Y}') + Y_{K'}'^*) - \bar{w}}{(q'(\omega'; \mathbf{Y}') + Y_{K'}') + q'^*(\omega'; \mathbf{Y}') + Y_{K'}'^* - 2\bar{w}}$.

3.2 Local solution method

In obtaining the local, perturbation-based portfolio solution we follow the method of Devereux and Sutherland (2011, 2010). To apply their method, we define asset shares in terms of a country's net foreign asset position, transforming the budget constraint, equation (4), such that the tradable assets are expressed in zero-net supply terms:⁹

$$\begin{aligned} (\theta_{H,t} - 1) q_t + \theta_{F,t} q_t^* &= \left[\begin{aligned} &(\theta_{H,t-1} - 1) q_{t-1} \frac{(q_t + Y_{K,t})}{q_{t-1}} + \\ &\theta_{F,t-1} q_{t-1}^* \frac{(q_t^* + Y_{K,t}^*)}{q_{t-1}^*} + Y_{K,t} + Y_{L,t} - C_t \end{aligned} \right], \\ \alpha_{H,t} + \alpha_{F,t} &= \alpha_{H,t-1} r_t + \alpha_{F,t-1} r_t^* + Y_{K,t} + Y_{L,t} - C_t, \\ W_t &= r_t^* W_{t-1} + Y_t - C_t + \alpha_{H,t-1} r x_t, \end{aligned} \quad (10)$$

where W_t is net foreign assets defined as

$$W_t = \alpha_{H,t} + \alpha_{F,t},$$

and where $\alpha_{H,t}$ and $\alpha_{F,t}$ are the domestic agent's portfolio share, r_t and r_t^* are the returns on domestic and foreign equity, and $r x_t$ is excess returns, all defined as:

$$\begin{aligned} \alpha_{H,t} &= (\theta_{H,t} - 1) q_t, \alpha_{F,t} = \theta_{F,t} q_t^*, \\ r_t &= \frac{(q_t + Y_{K,t})}{q_{t-1}}, r_t^* = \frac{(q_t^* + Y_{K,t}^*)}{q_{t-1}^*}, r x_t = (r_t - r_t^*). \end{aligned}$$

The model's state variables when solving the model with the local approximation method are the NFA, W_t , lagged equity prices, and the four exogenous output endowments. While the local solution can be re-written in terms of functions of the wealthshare, ω_t , it should be

⁹It should be noted that, in the way DS write the budget constraint, the home (foreign) country typically is the default owner of home (foreign) equity, as a result of which the tradable assets are expressed in zero net supply terms. In this case, the asset market clearing conditions equations (7) are replaced by:

$$\begin{aligned} \alpha_{H,t} + \alpha_{H,t}^* &= 0, \alpha_{F,t} + \alpha_{F,t}^* = 0, \\ \alpha_{H,t} + \alpha_{F,t} &= W_t = -W_t^* = -(\alpha_{H,t}^* + \alpha_{F,t}^*). \end{aligned}$$

noted that in the local solution there is nothing that constrains the wealthshare to fall into the $[0, 1]$ interval.¹⁰

The approach of Devereux and Sutherland towards characterizing the portfolio solution is based on Taylor-series approximations of the model's equilibrium conditions. The common procedure in macroeconomics is to take the non-stochastic steady state as the point of approximation. When solving for portfolios this is, however, not a trivial exercise to do for two reasons. One, portfolios are not defined in a first-order approximation to a DSGE model, since such approximation satisfies certainty equivalence, so that all assets are perfect substitutes. Moreover, two, optimal portfolios are not uniquely defined in a non-stochastic steady state, so there is no natural point around which to approximate. DS show that the first problem can be overcome by considering higher-order approximations to the portfolio problem. That is, in order to solve for the steady state (or 'zero-order') portfolio positions, one requires that the solution to the first-order approximated macro model is combined with a second-order approximation to the portfolio equations.¹¹ In a similar fashion, solving for the (first-order) portfolio dynamics requires that a second-order Taylor approximation to the macro dynamics be combined with a third-order approximation to the portfolio equations.

Devereux and Sutherland (2011) show that the latter problem, of not having a uniquely determined portfolio at the non-stochastic steady state, can be overcome by treating the value of portfolio holdings at the approximation point as unknown that is determined endogenously as part of the solution. In particular, they solve for steady state portfolio holdings by looking at the first order optimality conditions of the portfolio problem in the (stochastic) neighborhood of the non-stochastic steady state. DS show that the second-order approximations to portfolio equations and the 1st order solution to the macro model will be interdependent, but that this simultaneous system can be solved to give a simple closed-form analytical solution

¹⁰To make the two solution approaches more directly comparable, we initially intended to include nonlinear borrowing constraints also in the model version solved with the local solution method through barrier constraints (see e.g. Kim et al. (2010) for an application of barrier methods in a model solved with perturbation methods). Unfortunately though, such barriers would have led to portfolio terms appearing in a non-multiplicative form with rx in the model's equilibrium conditions, in which case the DS method cannot be applied.

¹¹The steady state portfolio is defined as $\alpha(\bar{W})$, the constant (or zero-order) term in the below Taylor series approximation of the true equilibrium portfolio function. Similarly, first order portfolio dynamics are described by $\alpha'(\bar{W})$ in such approximation:

$$\alpha(W_t) \simeq \alpha(\bar{W}) + \alpha'(\bar{W})(W_t - \bar{W})$$

for the equilibrium steady state portfolio.¹²

We now want to draw attention to a problem that is not explicitly addressed in the description of the DS portfolio solution method. The problem comes from the fact that at the non-stochastic steady state not only the portfolio positions are not uniquely identified (to which the DS method offers a solution), but, in general (without additional assumptions of a stationarity introducing-device as the EDF), *also the net foreign asset position itself* is not uniquely pinned down at the non-stochastic steady state. Because all assets are identical at the non-stochastic steady state and their return equal to the $1/\beta$ the system is 'short the equations' for the wealth position, the NFA, to be uniquely determined. Instead there exists a continuum of deterministic steady states, one for each assumed value of $\bar{W}$. Unfortunately, since the DS method relies on a (1st and 2nd order) approximation to the budget constraint around steady state value $\bar{W}$, the solution for steady state portfolios and portfolio dynamics hinges on the assumed value of $\bar{W}$. As a result the DS portfolio solution method is directly affected by this complication. While it is common practise in models of the open economy to make the non-stochastic steady state uniquely determined by, e.g., an endogenous discount factor (EDF), it can be argued that pinning down $\bar{W}$ by 'exogenous' assumptions of stationary introducing devices is not in any way less exogenous or more satisfactory than simply postulating it. This problem may be relatively inconsequential for symmetric country setups, where a value of $\bar{W}$ equal zero seems natural to assume, but may become problematic in asymmetric country setups where countries on average are either net debtors or net creditors in the 'true' solution.¹³ Instead, in a true solution $\bar{W}$ should itself be determined (together with the portfolio, α) by the (relative) risk characteristics of the two countries. Devereux and Sutherland (2009) propose a workaround around the problem and propose to solve for $\bar{W}$ in an iterative method in which wealth is updated from the mean stochastic steady state based

¹²The endogenous portfolio weights depend on the variance-covariance matrix of excess returns produced by the general equilibrium model, but those in turn depend on the portfolio positions themselves. The solution approach then makes use of the insight that excess returns, $\overline{\alpha_H} \widehat{r} \widehat{x}_t$, can be considered an i.i.d. random variable...

log-linearized budget constraint:

$$\widehat{W}_t = \frac{1}{\beta} \widehat{W}_{t-1} + \widehat{Y}_t - \widehat{C}_t + \overline{\alpha_H} \widehat{r} \widehat{x}_t$$

second-order approximation to the Home and Foreign Euler Equations gives:

$$E_t \left(\widehat{r} \widehat{x}_t + \frac{1}{2} \widehat{r} \widehat{x}_{t+1}^2 - \rho \widehat{C}_{t+1} \right) = 0, \quad E_t \left(\widehat{r} \widehat{x}_{t+1} + \frac{1}{2} \widehat{r} \widehat{x}_{t+1}^2 - \rho \widehat{C}_{t+1}^* \right) = 0, \quad \text{which, combining gives}$$

$$E_t \left[-\rho \left(\widehat{C}_{t+1} - \widehat{C}_{t+1}^* \right) \widehat{r} \widehat{x}_{t+1} \right] = 0.$$

¹³In a true global solution approach the portfolio composition and net foreign asset positions are jointly simultaneously determined.

on a 2nd order approximation of the model (including steady state and portfolio dynamics). In section 4.3 we evaluate their method to find $\overline{W}$ in an asymmetric setting and show that it performs relatively poorly compared to the global solution. We also show, however, how their idea can be improved substantially, in which case the performance of local and global solution method becomes much better, even in asymmetric settings.

A final issue that deserves mention before turning to the comparison of results of the two solution approaches concerns a technical difficulty that arises when generating simulated time paths of 2nd (or higher) order perturbation methods. Making direct use of the found second order solutions to generate simulated time paths is second-order accurate, but introduces some higher-order terms into the expansion. As first emphasized by Kim et al. (2003) these extra higher order-terms do not in general increase accuracy of the approximation as they do not correspond to higher-order coefficients in a Taylor expansion. In practise they often lead to explosive time paths. Therefore, one should choose among the second-order accurate expansions the one that implies stability. A stable solution can be obtained by ‘pruning’ out the the extraneous higher order terms in each iteration by computing the projections of the second order terms based on a first-order expansion. In deriving our results we therefore follow the state-of-the-art approach and present all results from simulated data from the local approximation method as computed by applying the ‘pruning’ method. It should be noted, however, that ‘pruning’ is essentially a trick and is not based on indisputable foundations (see Kim et al. (2003), Den Haan and de Wind (2009) and Lombardo (2010) for a discussion of advantages and disadvantages of the “pruning” procedure). We refer to Appendix C for the alternative results without the pruning approach.

4 Results from the Devereux Sutherland (JEEA, 2011) model

Subsection 4.1 starts our comparison of the performance of global and local portfolio solution methods for setups in which countries are symmetric. As mentioned, this facilitates the comparison in an important dimension, since, in this case, it is ‘natural’ to assume that the net foreign asset position in steady state, $\overline{W}$, is equal to 0 (or equivalently, the wealth share

$\bar{\omega} = 0.5$), which we can use as the approximation point in the DS solution method.¹⁴ Section (4.1.1) focuses on a special case in which the analytical solution is known, section (4.1.2) and (4.1.3) looks at more general – but always symmetric – parameterizations. In all cases, as long as we stay in a ‘symmetric world’ the DS method performs extremely well, delivering solutions that are (close to) indistinguishable from the global solution.

4.1 Symmetric country setup

The baseline parameterization of our example model, when parameterized symmetrically, is summarized in table 2.

Table 2: Baseline Parameter Specification

Discount Factor	β	0.95
Endog. Discount Factor Parameter	η	$0, 1e - 4$
Coeff. of Risk Aversion	ρ	2
Capital Income Share	$\frac{Y_K}{Y} = \frac{Y_K^*}{Y^*}$	0.3
Persistences	$\rho_Y, \rho_{Y^*}, \rho_{Y_K}, \rho_{Y_K^*}, \rho_{Y_L}, \rho_{Y_L^*}$	0.8
Shock volatilities	$\sigma_Y, \sigma_{Y^*}, \sigma_{Y_K}, \sigma_{Y_K^*}, \sigma_{Y_L}, \sigma_{Y_L^*}$	0.02
Shock correlation, Labor and Capital	$\rho_{Y_K, Y_L} = \rho_{Y_K^*, Y_L^*}$	0.2

4.1.1 The case with known analytic solution – constant factor shares

Suppose that “capital” and “labor” incomes are constant fractions of total output in the two countries¹⁵:

$$Y_{K,t} = \omega_K Y_t, \quad Y_{L,t} = \omega_L Y_t, \quad Y_{K,t}^* = \omega_K Y_t^*, \quad Y_{L,t}^* = \omega_L Y_t^*,$$

$$\omega_K + \omega_L = 1.$$

It is easy to show that in our model, Pareto optimality is achieved via the so-called “linear sharing rule”, where the worldwide output is shared between the two countries in fixed

¹⁴When moving to asymmetric specifications of countries in section 4.3 we discuss the best method to find the ‘right’ $\bar{W}$ to approximate around. It should be noted that applying this method precisely leads to $\bar{W} = 0$ for symmetric country setups.

¹⁵This would be the case in the production economy with Cobb-Douglas production function.

proportion:

$$C_t = \lambda (Y_t + Y_t^*), \quad C_t^* = (1 - \lambda) (Y_t + Y_t^*).$$

λ in the above formula can be interpreted as the weight the planner attaches to the welfare of the domestic consumer. In the decentralized equilibrium, it will depend on the initial wealth distribution, and can be computed using the Negishi algorithm if the 2 countries start with equal wealth¹⁶:

$$(q_0 + Y_{K,0}) \theta_{H,-1} + (q_0^* + Y_{K,0}^*) \theta_{F,-1} + Y_{L,0} = (q_0 + Y_{K,0}) \theta_{H,-1}^* + (q_0^* + Y_{K,0}^*) \theta_{F,-1}^* + Y_{L,0}^*,$$

In this case, we get $\lambda = 1/2$, and the Pareto optimal allocation is $C_t = C_t^* = (Y_t + Y_t^*)/2$ for all t . Given our assumption of constant factor shares, this allocation can be supported in a decentralized equilibrium by the following time-invariant portfolio¹⁷:

$$\theta_{H,t} = \frac{1/2 - \omega_L}{\omega_K}, \quad \theta_{F,t} = \frac{1/2}{\omega_K}.$$

Since the consumption allocation depends only on the realization of the exogenous shock, so do the asset prices. If we replace the continuous output processes with their discrete-state Markov chain approximation, we can compute the implied equilibrium equity prices by solving the following systems of linear equations:

$$\begin{aligned} q(s) \left(\frac{Y(s) + Y^*(s)}{2} \right)^{-\rho} &= \beta \sum_{s' \in \sigma} \pi(s'|s) \left(\frac{Y(s'^*(s'))}{2} \right)^{-\rho} Y_k(s'), \quad \forall s \in \sigma, \\ q^*(s) \left(\frac{Y(s) + Y^*(s)}{2} \right)^{-\rho} &= \beta \sum_{s' \in \sigma} \pi(s'|s) \left(\frac{Y(s'^*(s'))}{2} \right)^{-\rho} Y_k^*(s'), \quad \forall s \in \sigma, \end{aligned}$$

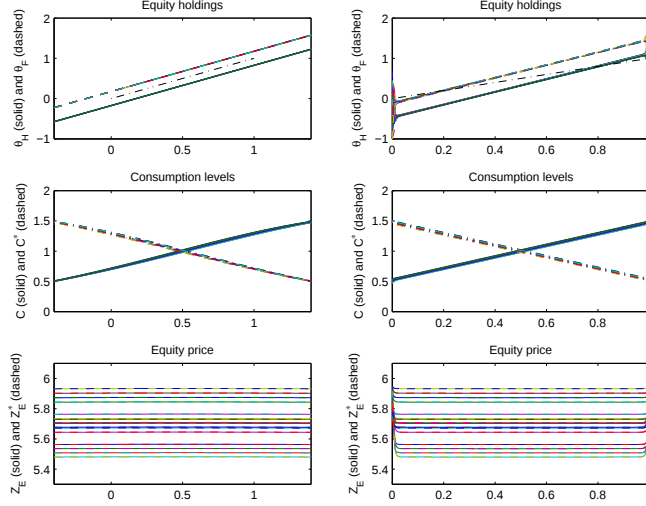
where σ is the set of all possible realizations of exogenous state, and s and s' are some particular realizations of this state. Given the solution for portfolio positions and equity prices, we can compute the implied NFA (or “wealth”) positions.

We compare one simulated path of the economy generated by the “global” solution, the Devereux-Sutherland “local perturbation” solution and the analytic solution and the same

¹⁶This corresponds to $\omega_0(q_0 + Y_{K,0} + q_0^* + Y_{K,0}^*) + Y_{L,0} = \omega_0^*(q_0 + Y_{K,0} + q_0^* + Y_{K,0}^*) + Y_{L,0}^*$ in the “global” solution formulation, and to ... in the DS formulation.

¹⁷Note that this does not depend on any stochastic properties of the output in the two countries.

Figure 1: Figure Policy Functions



sequence of exogenous shocks. For this economy, we assume that $Y_{K,t} = 0.3Y_t$, $Y_{K,t}^* = 0.3Y_t^*$, $Y_{L,t} = 0.7Y_t$, $Y_{L,t}^* = 0.7Y_t^*$, $\sigma(y_t) = \sigma(y_t^*) = 0.02$, $\text{corr}(y_t, y_{t-1}) = \text{corr}(y_t^*, y_{t-1}^*) = 0.8$ ¹⁸. These comparisons show that both numerical approaches produce solutions that are quite precise and stay close to the true analytic solution during all 1000 periods of the simulated path, although, as one would expect, global solution is an order of magnitude better than the local perturbation solution. The maximum (absolute) differences of the global approach from the analytical solutions are of order $1e-9$, those of the local approximation method are of order $1e-3$.¹⁹

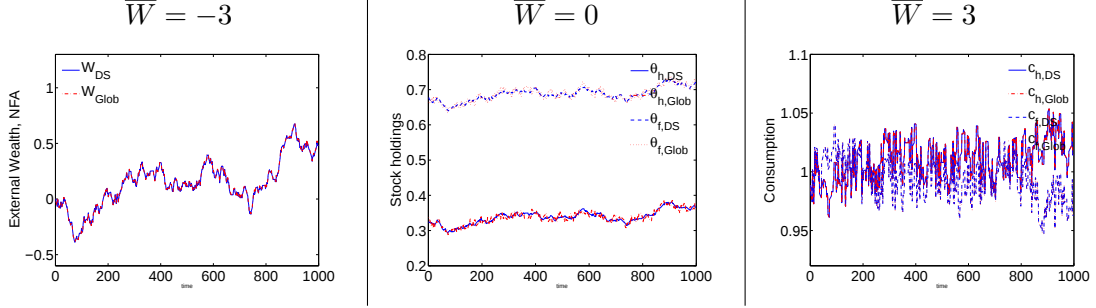
4.1.2 General symmetric country cases

We now turn to a general (symmetric) case in which the exogenous variables follow the exogenous process described in equation (6). Figure 1 presents policy functions for the domestic

¹⁸Small letters denote logarithms, e.g. $y_t = \log(Y_t)$. The stochastic properties of the output shocks (i.e. standard deviations and autocorrelations) are the “targets” that we use when approximating the continuous VAR process with a discrete Markov chain process. Since the approximation is not exact, we use the stochastic properties of the resulting Markov chain process when computing the solutions. Stochastic properties of the output shocks do not change the computation of the analytic solution for the equity portfolio positions and consumption allocations (as a function of the output shocks), but do play a role in computing the stock prices, and hence the NFA position.

¹⁹The figures can be found in the Appendix A as Figure 9 and 10.

Figure 2: Simulated Short Time Paths



country's equity holdings, consumptions, and equity prices from the local and global solution methods, Figure 2 compares the behavior of simulated time paths of the same variables. The results are generated for the case of an exogenous discount factor ($\beta(C_A) = \beta$) and the time paths in DS are derived from the 2nd order solution with pruned simulations. As can be seen the time paths of all variables are extremely close, the only noticeable difference being that the portfolio holdings in the global solution appear to be somewhat more volatile. In particular, the maximum (absolute) difference of the time paths of equity holdings (consumption, equity prices) generated from local and global solution methods over a 1000 period simulation is $1.2e-2$ ($1.6e-3$, $6e-4$).

We are also interested in a comparison of simulated moments (means, standard deviations and correlations) of key model variables, and in the stationary distributions generated by the local and global solution approach. In doing this we face a technical difficulty that is well known in the literature. While the stationary distribution of wealth is well defined when the model is solved by the global approximation method, and is a result of the (relative) motives for precautionary asset holdings (such as the risk properties) of the two countries, in the local approximation the unit root in wealth in the first order part of the solution translates into a stationary distribution that is not well defined/ cannot be pinned down. As a consequence, in order to derive simulated model moments from the model solved with the DS method first requires rendering stationary by making the discount factor endogenous, $\beta'(C_A) < 0$. We –for now– follow Devereux and Sutherland (2011, 2010) in the functional form of the endogenous discount factor that follows equation (2). We are strict in treating the endogenous discount factor only as a technical device to introduce stationarity and set η in the DS solution method

to a 'small' value. While this is to some degree arbitrary, our guideline in determining what 'small enough' means, is such that the dynamics of short simulated time paths are hardly affected. For our choice of set $\eta = 1\text{e-}4$, the maximum absolute differences in the time paths over 1000 periods with or without EDF of equity holdings (consumption, equity prices) are $1.4\text{e-}3$ ($8\text{e-}4$, $1\text{e-}4$). Also, we continue to leave the discount factor in the global portfolio solution method exogenous (i.e., $\eta = 0$), as the problem is a stationary one and no technical device to introduce stationarity is needed. In addition, we find that the stationary distribution is hugely affected by the size of the EDF parameter in the global solution. This concerns on the one hand the shape of the distribution, but, even worse, in asymmetric settings also the mean of stationary distributions is very strongly affected in the global, with $\eta \geq 1\text{e-}5$ very strongly determining not only the shape of the distribution but also affecting the mean of its distribution (always tilting it towards 0.5). Appendix B turns to these issues in detail. For this reason we abstract from set $\eta > 0$ in the global solution method and set η to a small number in DS ($=1\text{e-}4$).

Tables 4 and 5 report the model's key moments from 'panel estimations', that is, from 1000 simulations of length 1000 periods each.²⁰ The moments of these panel estimations from local and global portfolio solution method are virtually identical. This is true not only for the (portfolio) means but also its standard deviations and correlations with relative outputs. From this exercise, we therefore conclude that in a symmetric country setting with standard CRRA preferences and parameter values in line with matching macroeconomic fluctuations the perturbation-based portfolio solution method of Devereux and Sutherland performs extremely well. One might argue though that the local portfolio solution method performs that well because the policy functions actually are close to linear. To ascertain whether this could be the case we next turn to more extreme parameterizations and consider a model version that is subject to more nonlinearities (from limits to short selling constraints or) from the specification of recursive preferences.

²⁰Because of high persistence of the net foreign asset positions and the fact that it may travel a large range generally leads to quite different stationary moments, with generally way too high standard deviations.

Figure 3: Stationary Distributions

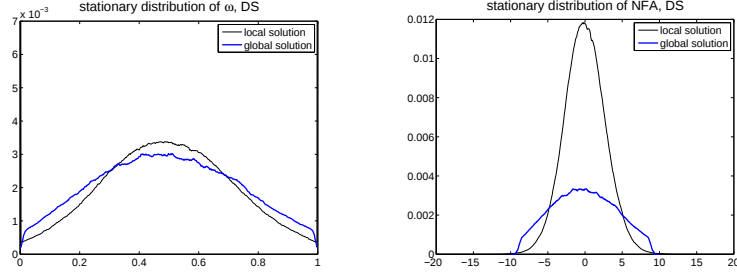


Table 3: Model Means from Simulated Stationary Distributions

	ω	W	θ_H	θ_F	θ_H^*	θ_F^*	q	q^*	C	C^*
DS Sol.	0.497	-0.039	0.322	0.671	0.678	0.329	5.706	5.706	0.998	1.003
Global Sol.	0.492	-0.150	0.312	0.662	0.688	0.338	5.705	5.705	0.992	1.008

Table 4: Model Means and Standard Deviations from Panel Simulations

Variable X	DS Solution		Global Solution	
	$\bar{X}$	$\sigma(X)$	$\bar{X}$	$\sigma(X)$
NFA	0.000	0.320	-0.001	0.323
θ_h	0.325	0.030	0.325	0.031
θ_f	0.674	0.036	0.674	0.037
θ_h^*	0.674	0.036	0.674	0.037
θ_f^*	0.325	0.030	0.325	0.031
C	0.999	0.039	0.999	0.039
C^*	0.999	0.039	0.999	0.039
Z_E	5.700	0.224	5.698	0.224
Z_E^*	5.700	0.224	5.698	0.224

Table 5: Model Correlations from Panel Simulations

Variable X	DS solution			Global solution		
	$\frac{Y}{Y^*}$	$\frac{Y_L}{Y_L^*}$	$\frac{Y_K}{Y_K^*}$	$\frac{Y}{Y^*}$	$\frac{Y_L}{Y_L^*}$	$\frac{Y_K}{Y_K^*}$
NFA	-0.014	0.005	-0.055	-0.015	0.004	-0.054
θ_h	0.289	0.259	0.267	0.292	0.268	0.256
θ_f	0.482	0.434	0.439	0.482	0.438	0.429
θ_h^*	0.492	0.447	0.440	0.469	0.421	0.431
θ_f^*	0.303	0.276	0.267	0.275	0.246	0.256
C	0.667	0.615	0.575	0.665	0.613	0.574
C^*	0.606	0.536	0.576	0.605	0.535	0.575
Z_E	0.643	0.569	0.611	0.643	0.569	0.611
Z_E^*	0.599	0.554	0.512	0.599	0.554	0.512

4.1.3 Varying risk aversion, shock volatility and shock persistence

We perform some sensitivity analysis w.r.t. a number of key parameters, that generally tend to increase to increase the amount of uncertainty or aversion to it, and therefore intensifies the nonlinearities of the model. Table ? shows that in the baseline model with CRRA preferences and a symmetric country scenario, the DS methods continues to perform well. Nevertheless, it can be noticed that while the mean of the portfolio is matched well, the standard deviations of the optimal θ_H and θ_F are less well matched. add table with key moments of portfolio and other variables from simulation.. for baseline case, $\sigma = 10$, $\sigma_{YK} = \sigma_{YL} = \sigma_{YK}^* = \sigma_{YL}^* = 0.1$, $\rho_{YK} = \rho_{YL} = \rho_{YK}^* = \rho_{YL}^* = 0.99, \dots$

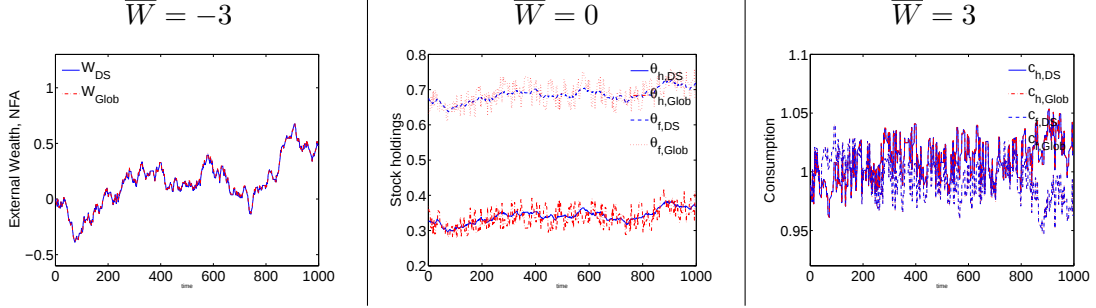
4.1.4 Adding inequality constraints: explicit short-selling constraints

explicit borrowing/ short-selling constraints

4.1.5 Epstein-Zin preferences

The baseline model presented so far is a standard parameterization in macroeconomics, that matches macro (business cycle) properties reasonably well. However, the setup (preferences) and parameterization would have a hard time in matching finance stylized facts, such as volatile pricing kernels and high risk premia . Since portfolio choice and country portfolios

Figure 4: Simulated Short Time Paths, Epstein-Zin Preferences



naturally relate to international finance aspects it seems all the more important to incorporate these into a setup that does not miserably fail on the finance side. A feature that has become very popular in finance and in macroeconomics as modeling devices to account for business cycle fluctuations and asset pricing is the use of time-nonseparable (recursive) preferences of the form of Epstein-Zin. In this case, preferences are described by:

$$U_t = \max_{C_t} \left[(1 - \beta) C_t^{\frac{1-\rho}{\theta}} + \beta \left(E_t U_{t+1}^{1-\rho} \right)^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\rho}},$$

where ρ is the coefficient of risk aversion, $\theta = \frac{1-\rho}{1-\frac{1}{\psi}}$, and where ψ is the elasticity of intertemporal substitution. In a recent paper ? compare the performance perturbation methods applied to a neoclassical growth model with these type of recursive preferences with the solution from (several) global approximation methods, and find that the perturbation methods perform very well even under this type of preferences. We therefore expect that the solution for macro variables should be well captured by a perturbation method, which is indeed what we find (Figure 4 and Tables 6 and 7). The differences that emerge between local and global portfolio solution methods therefor refer to the portfolio positions themselves. Here we find that while the mean portfolios are still well, yet the local solution method fails to account for the increased variance in the portfolio positions that result from the increased risk aversion.

Table 6: Model Means and Standard Deviations from Panel Simulations, Epstein-Zin Preferences

Variable X	DS Solution		Global Solution	
	$\bar{X}$	$\sigma(X)$	$\bar{X}$	$\sigma(X)$
NFA	0.000	0.320	-0.001	0.318
θ_h	0.328	0.030	0.326	0.038
θ_f	0.671	0.036	0.673	0.042
θ_h^*	0.671	0.036	0.673	0.042
θ_f^*	0.328	0.030	0.326	0.038
C	0.999	0.039	0.999	0.039
C^*	0.999	0.039	0.999	0.039
Z_E	5.722	0.225	5.706	0.225
Z_E^*	5.722	0.225	5.706	0.225

Table 7: Model Correlations from Panel Simulations, Epstein-Zin Preferences

Variable X	DS solution			Global solution		
	$\frac{Y}{Y^*}$	$\frac{Y_L}{Y_L^*}$	$\frac{Y_K}{Y_K^*}$	$\frac{Y}{Y^*}$	$\frac{Y_L}{Y_L^*}$	$\frac{Y_K}{Y_K^*}$
NFA	-0.014	0.005	-0.055	-0.014	0.006	-0.055
θ_h	0.289	0.259	0.267	0.228	0.210	0.198
θ_f	0.482	0.434	0.439	0.406	0.369	0.361
θ_h^*	0.492	0.447	0.440	0.394	0.353	0.363
θ_f^*	0.303	0.276	0.267	0.213	0.190	0.198
C	0.667	0.615	0.575	0.668	0.616	0.575
C^*	0.606	0.536	0.576	0.607	0.536	0.577
Z_E	0.643	0.569	0.611	0.643	0.568	0.611
Z_E^*	0.599	0.554	0.512	0.599	0.554	0.512

4.2 The role of the approximation point ($\bar{W}$) in the local portfolio solution method

Until now, we considered only symmetric country setups in which it was natural to assume that the mean of the underlying stochastic steady state distribution of net foreign assets is centered around zero. In this section we show in a still symmetric setup that the point of approximation, $\bar{W}$, that is chosen in the DS solution method can greatly affect the performance of the method, when the true policy functions are nonlinear (in the net foreign asset position). In particular, the experiment we perform is to use the model with Epstein-Zin preferences and compare the time paths of global solution with three versions of the model solved with the DS method: a case when the model is solved with the DS method and taking approximation point $\bar{W} = 0$, a case where the model is solved taking approximation point $\bar{W} = 3$ under the DS method, and a case where the model is solved with approximation point $\bar{W} = -3$. To save space we only report the results of these comparisons in terms of the figures of the short time paths and some core moments from the *panel* simulations. Figure 5 shows the short time paths of these three cases. While the DS method performs quite well (as documented in the previous sections) even with Epstein-Zin preferences as long as the model is approximated around the "true" mean of the stationary distribution, of $\bar{W} = 0$, the time paths of the DS method differ to a great extent when $\bar{W} = -3$ or 3 is used as the approximation point. The error introduced through the wrong approximation point has a substantial influence on the panel simulations, as can be seen from Table 5 (the consequences for the stationary distribution are enormous).

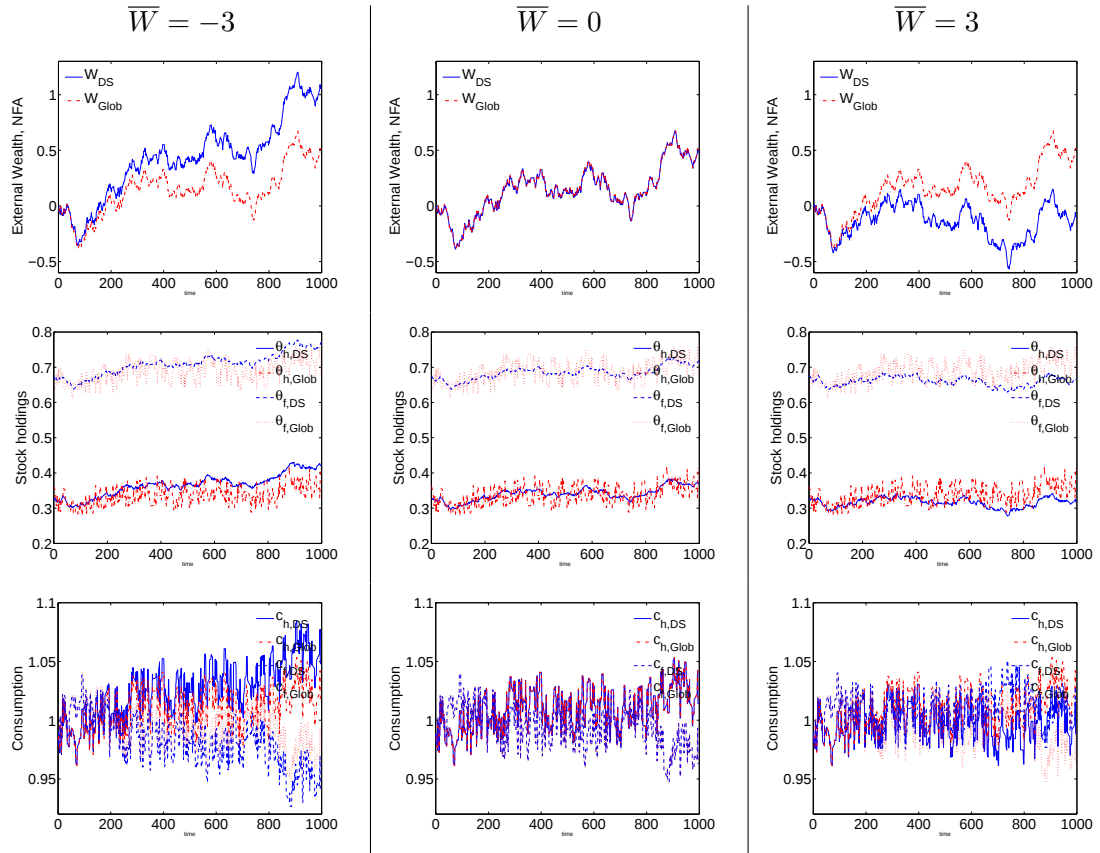
It should be noted that the point of approximation should not have an affect on the performance of the DS solution if the true policy functions were in fact linear in the net foreign asset position. For the case of CRRA preferences with coefficient of risk aversion equal to 2, for example, we find that the approximation point indeed has very little consequence on the difference of the DS time paths from the global time paths.²¹

4.3 Asymmetric specifications and endogenous external wealth

We now turn to an evaluation of portfolio solution methods in explicitly non-symmetric setups. Arguably, this is the more relevant case empirically (e.g. Lane and Milesi-Ferretti

²¹Figure ?? provides the equivalent to Figure 5 for the case of CRRA preferences.

Figure 5: Varying the point of approximation of $\bar{W}$



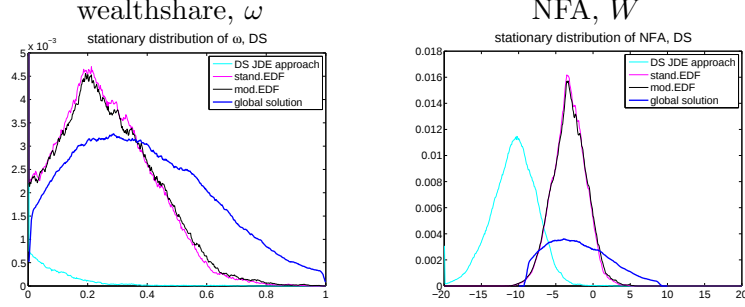
(2001) document that countries' NFA positions are non-zero over very long time horizons for most countries). Unlike for the analysis of symmetric country setups, the application of local portfolio solution methods when countries are asymmetric is far less straightforward, and its performance much less perfect. As explained in section 3, the DS method involves the difficulty of not knowing the correct value of the NFA to take the approximation around, as $\bar{W}$ is, in general, not well defined in the non-stochastic steady state. It is perfectly possible to introduce an EDF that uniquely pins down $\bar{W}$ in the deterministic steady state. Furthermore, it is equally possible to calibrate models to cases where $\bar{W}$ is positive or negative in the deterministic steady state, for example, by assuming different degrees of discount factors for the two economies, reflecting different degrees of patience or impatience, that as a result give rise to countries being net debtors or net creditors. Yet, in an exact solution the stationary distribution of the NFA will be determined not only by differences in time preferences, but also by the assumed menu of assets available for risk sharing and the relative characteristics of the exogenous shocks that give rise to varying degrees of precautionary saving demands.

Since, as section 4.2 has shown, the correctness of the found portfolio solution in the DS method can be crucially affected by the value of $\bar{W}$, this suggests that the ideal approximation point of $\bar{W}$ should be the one that explicitly accounts for these factors of a stochastic setting. In fact, Devereux and Sutherland (2009) propose to solve for $\bar{W}$ in an iterative method in which wealth is updated from the mean stochastic steady state based on a 2nd order approximation of the model (including steady state and portfolio dynamics).

Devereux and Sutherland (2009) follow an updating scheme in which only net foreign assets are updated from the non-stochastic steady state, and all other variables remain at their (initially assumed) steady state.

Instead, we propose to update the implied mean NFA position from the stochastic steady state, but then also update the steady state variables of all other variables by recomputing the implied deterministic steady state given the new value of $\bar{W}$. Ideally one might have considered such updating from the stochastic steady state for *all* variables, but the DS method requires that the returns of all assets at steady state ($\bar{r}$ and $\bar{r}^*$) are identical (see Appendix ?). It is here where we need to use the modified specification for the EDF. In this modified specification of the EDF the (first order) macro *dynamics* are still rendered stationary, but the steady state value of $\bar{W}$ in the *deterministic* steady state is still left undetermined. Instead, we

Figure 6: Stationary Distributions, Baseline model with CRRA preferences



find the stochastic steady state value of $\bar{W}$ from the stochastic steady state (by computing the 'expected path'²²), and use this value of $\bar{W}$ to update all other variables from the deterministic steady state using that particular $\bar{W}$.

We use the simulate and update algorithm on our baseline model that is slightly modified to an asymmetric setup where country 1's volatility of labor and capital income endowments is half as volatile as country 2's volatility (i.e., $\sigma_{Y_K} = \sigma_{Y_L} = 0.01$, $\sigma_{Y_K}^* = \sigma_{Y_L}^* = 0.02$).

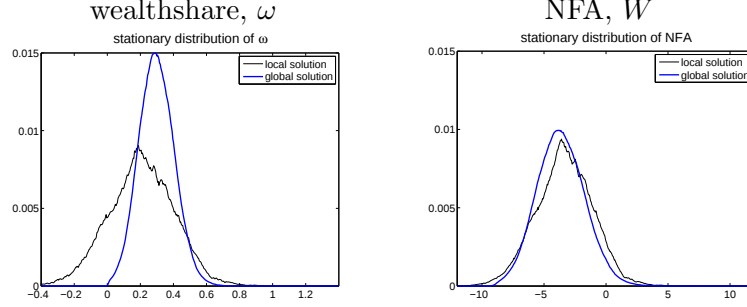
Figure 6 and Table 8 shows the stationary distribution and moments of interest in the DS in each of these cases and the comparison with the global solution method.

²²To facilitate computation we compute the approximate stochastic steady state not from a long stochastic simulation, but instead use the 'expected path': assuming that all exogenous variables are at their mean values forever, what value does the NFA position converge to as implied by the underlying policy functions.

Table 8: Model Moments of local (DS JDE approach, standard EDF, modified EDF) and global solution, Baseline model with CRRA preferences

Variable, X	DS solution, DS JDE approach			DS solution, stand.EDF			DS solution, mod.EDF			Global solution		
	$\bar{X}$	$\sigma(X)$	$\rho(X, Y)$	$\bar{X}$	$\sigma(X)$	$\rho(X, Y)$	$\bar{X}$	$\sigma(X)$	$\rho(X, Y)$	$\bar{X}$	$\sigma(X)$	$\rho(X, Y)$
ω	-0.430	0.265		0.217	0.191		0.222	0.199		0.390	0.226	
W	-10.613	3.033		-3.230	2.182		-3.173	2.275		-1.994	4.085	
θ_H	-0.945	0.323	0.004	0.003	0.233	0.003	0.009	0.243	0.003	0.135	0.437	-0.000
θ_F	0.084	0.207	0.002	0.431	0.149	0.014	0.435	0.156	0.014	0.516	0.280	0.013
θ_H^*	1.945	0.323	0.009	0.997	0.233	0.006	0.991	0.243	0.006	0.865	0.437	0.011
θ_F^*	0.916	0.207	0.009	0.569	0.149	0.015	0.565	0.156	0.014	0.484	0.280	-0.002
C	0.995	0.152	0.044	0.830	0.113	0.060	0.833	0.118	0.058	0.895	0.215	0.034
C^*	1.005	0.159	0.033	1.170	0.115	0.045	1.167	0.119	0.043	1.105	0.215	0.022
Z_E	5.705	0.109	0.582	5.704	0.109	0.582	5.704	0.109	0.582	5.703	0.109	0.581
Z_E^*	5.705	0.116	0.492	5.704	0.116	0.492	5.704	0.116	0.492	5.703	0.116	0.491
$\rho(nfa_t, nfa_{t-1})$	0.999040			0.999762			0.999780			0.999904		
$\rho(\omega_t, \omega_{t-1})$	0.999987			0.999990			0.999991			0.999987		

Figure 7: Stationary Distributions, Baseline model with Epstein-Zin preferences

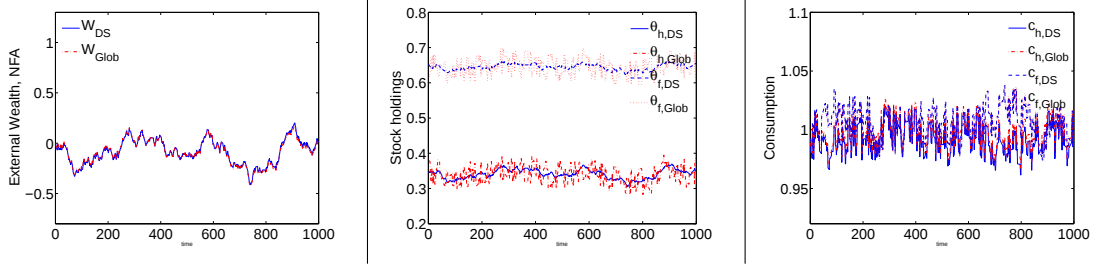


Using the simulate and update algorithm and applying it to the asymmetric country case also in the model with Epstein-Zin preferences shows that the DS method achieves a good performance (the issue of the too low volatility of the portfolio remains though), once the right approximation point of the NFA is found. Figure 7 shows the resulting stationary distribution from the global method and from the local method approximated around the point that the simulate and update algorithm converged to. Figure 8 shows that also the model moments are well matched in this asymmetric setup when the 'right' approximation point has been found and used.²³

Unfortunately, there is nothing that guarantees that the simulate and update algorithm will be successful and will always converge to a mean of a stationary distribution. For extreme parameterizations the simulate and update algorithm typically will lead to very large values of the net foreign asset position, such that the solution breaks down. Also, there are model examples where, even with moderate parameterizations, the simulate and update algorithm discussed here is always bound to fail. The model in Section 8 is an example.

²³Figure ?? in Appendix A reports the short time paths for the asymmetric country setting with Epstein-Zin preferences, where the DS method is wrongly approximated around $\bar{W} = 0$.

Figure 8: Simulated Short Time Paths, Asymmetric Country Setup with Epstein-Zin Preferences



5 A model with positive expected excess return (equity premium)

To further highlight a factor for a potential shortcoming of the DS method we want to construct a slightly modified model. Instead of the labor and capital income endowments, assume that both countries receive their income from a joint global endowment (tree) that fluctuates stochastically over time. Let the set of tradable assets be assumed to consist of equity on that world endowment, and a risk-free bond. Suppose every period t , there is a new tree born that lives and gives fruit for only 1 period, $t + 1$. Suppose that for each such new tree, half of it is initially owned by the domestic investor, and the other half is owned by the foreign investor. Let $\theta_t^0 = 1/2$ and $(\theta^*)^0_t = 1/2$ denote these initial ownership rights.

Also, let θ_t and θ_t^* denote the domestic and foreign investor's holdings of the tree born in period t that they take to period $t + 1$ with them.

Then we can write domestic investor's budget constraint as:

$$\theta_t q_t^e + b_t q_t^b = \theta_{t-1} Y_t^{\text{glob}} + \theta_t^0 q_t^e + b_{t-1} + Y_{\min} - C_t$$

or:

$$(\theta_t - \theta_t^0) q_t^e + b_t q_t^b = (\theta_{t-1} - \theta_{t-1}^0) q_{t-1}^e \left(\frac{Y_t^{\text{glob}}}{q_{t-1}^e} \right) + b_{t-1} q_{t-1}^b \left(\frac{1}{q_{t-1}^b} \right) + \theta_{t-1}^0 Y_t^{\text{glob}} + Y_{\min} - C_t$$

or:

$$\alpha_t^e + \alpha_t^b = \alpha_{t-1}^e R_t^e + \alpha_{t-1}^b R_t^b + \theta_{t-1}^0 Y_t^{\text{glob}} + Y_{\min} - C_t$$

and finally:

$$W_t = \alpha_{t-1}^e (R_t^e - R_t^b) + W_{t-1} R_t^b + \theta_{t-1}^0 Y_t^{\text{glob}} + Y_{\min} - C_t$$

For the global solution, we need to specify a “borrowing constraint”:

$$\theta_t Y_{t+1}^{\text{glob}} + b_t + \theta_t^0 q_{t+1}^e \geq \bar{\omega} Y_{\min}.$$

We can then define $\omega_{t+1} = \frac{\theta_t Y_{t+1}^{\text{glob}} + b_t + \theta_t^0 q_{t+1}^e - \bar{\omega} Y_{\min}}{Y_{t+1}^{\text{glob}} + q_{t+1}^e - 2\bar{\omega} Y_{\min}}$, so that $\omega \in [0, 1]$, and rewrite the budget constraint as:

$$\theta_t q_t^e + b_t q_t^b = \omega (Y_t^{\text{glob}} + q_t^e - 2\bar{\omega} Y_{\min}) + (1 + \bar{\omega}) Y_{\min} - C_t$$

outline of why DS method differs from global solution under such model

$$\begin{aligned} u_{C,t} &= \beta E_t u_{C,t+1} R_{t+1}^{\text{risky}} \\ u_{C,t} &= \beta E_t u_{C,t+1} R_{t+1}^{\text{safe}} \\ u_{C,t}^* &= \beta E_t u_{C,t+1}^* R_{t+1}^{\text{risky}} \\ u_{C,t}^* &= \beta E_t u_{C,t+1}^* R_{t+1}^{\text{safe}} \end{aligned}$$

$$\begin{aligned} E_t [u_{C,t+1} R_{t+1}^{\text{risky}}] &= E_t [u_{C,t+1}] R_{t+1}^{\text{safe}} \\ E_t [u_{C,t+1}^* R_{t+1}^{\text{risky}}] &= E_t [u_{C,t+1}^*] R_{t+1}^{\text{safe}} \end{aligned}$$

$$\begin{aligned} E_t [u_{C,t+1}] (E_t [R_{t+1}^{\text{risky}}] - R_{t+1}^{\text{safe}}) + \text{cov}_t (u_{C,t+1}, R_{t+1}^{\text{risky}}) &= 0 \\ E_t [u_{C,t+1}^*] (E_t [R_{t+1}^{\text{risky}}] - R_{t+1}^{\text{safe}}) + \text{cov}_t (u_{C,t+1}^*, R_{t+1}^{\text{risky}}) &= 0 \end{aligned}$$

$$E_t [u_{C,t+1} - u_{C,t+1}^*] (E_t [R_{t+1}^{\text{risky}}] - R_{t+1}^{\text{safe}}) + \text{cov}_t (u_{C,t+1} - u_{C,t+1}^*, R_{t+1}^{\text{risky}}) = 0 \quad (11)$$

Because DS assume that have $E(R_x) = 0$, the first term disappears.

6 Conclusions

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A Additional figures and tables

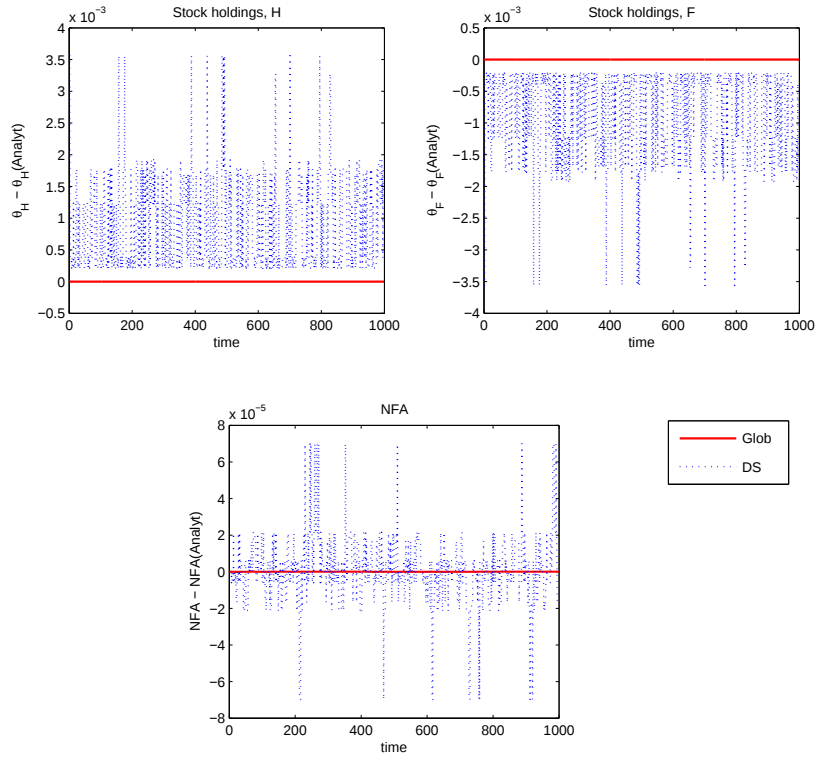


Figure 9: Global, local (with pruning) and analytic solutions

B Sensitivity analysis w.r.t. the endogenous discount factor

$$\beta(C_A) = \beta C_A^{-\eta}, \quad (12)$$

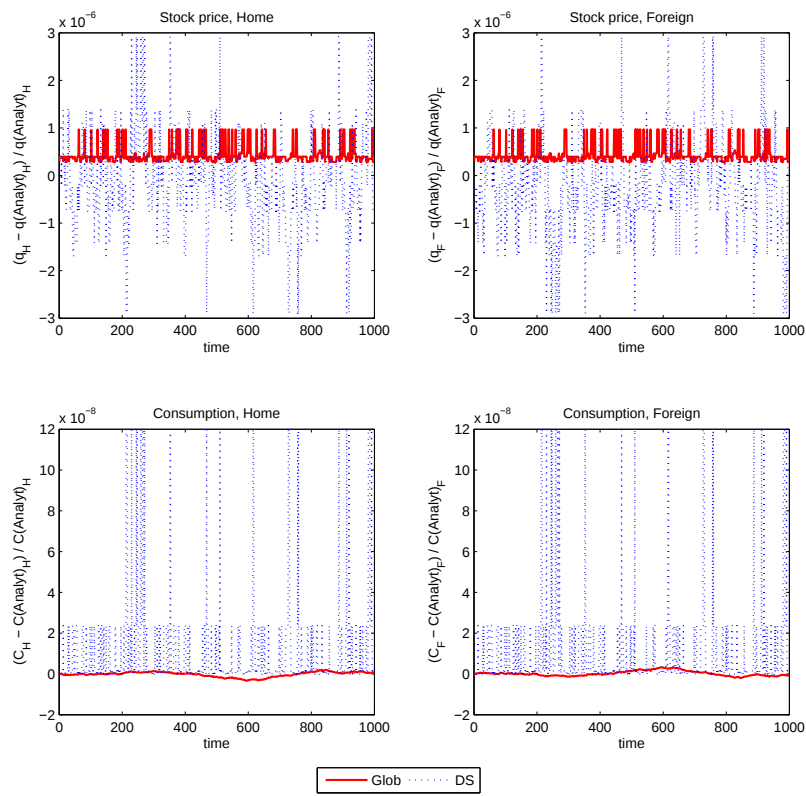


Figure 10: Global, local (with pruning) and analytic solutions

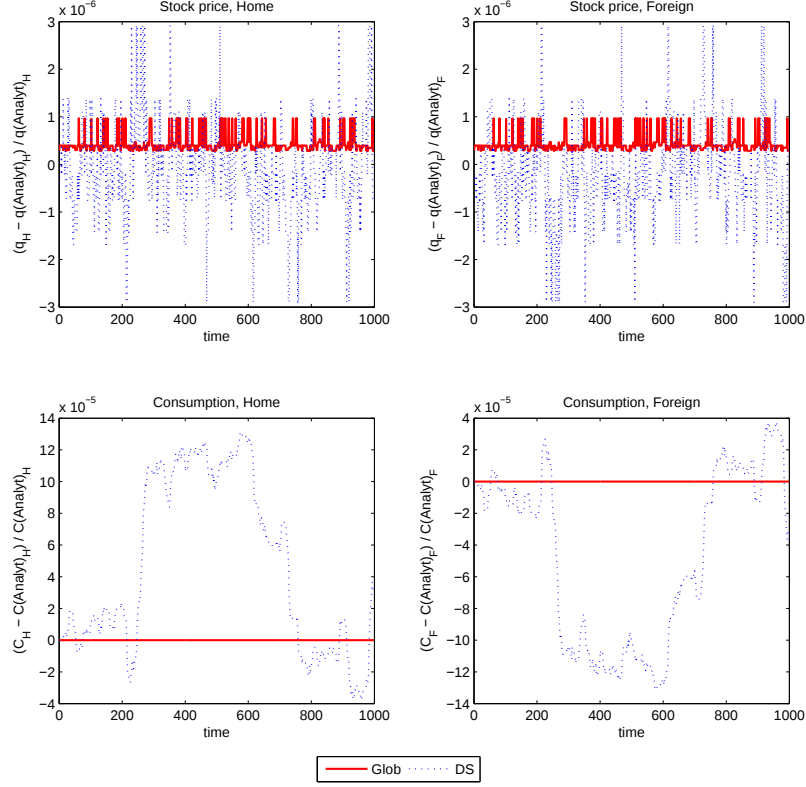


Figure 11: Global, local (no pruning) and analytic solutions

Figure 12: Simulated Short Time Paths, Asymmetric Country Setup with Epstein-Zin Preferences approx. around $W=0$

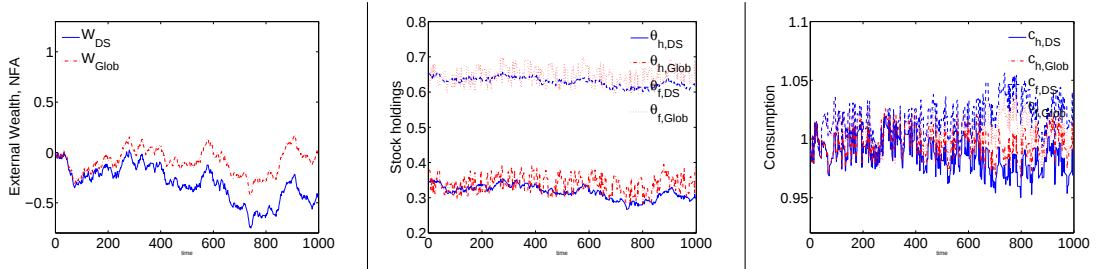
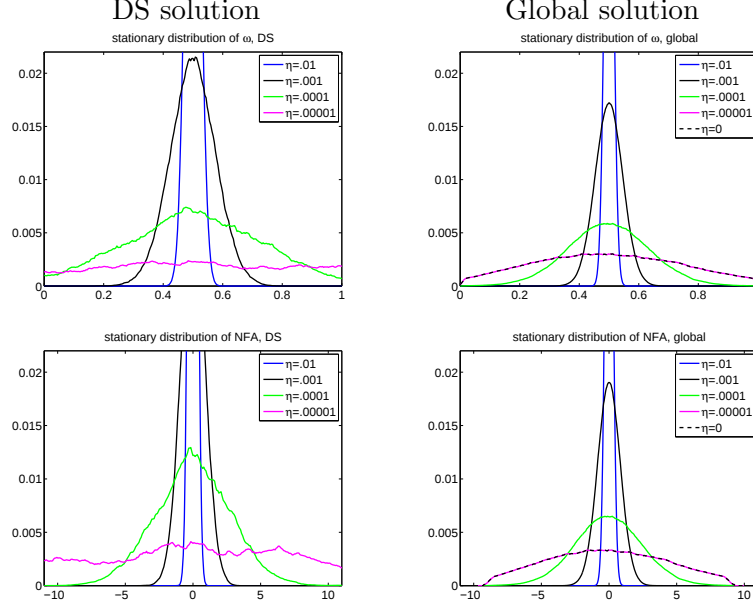


Figure 13: Stationary Distributions



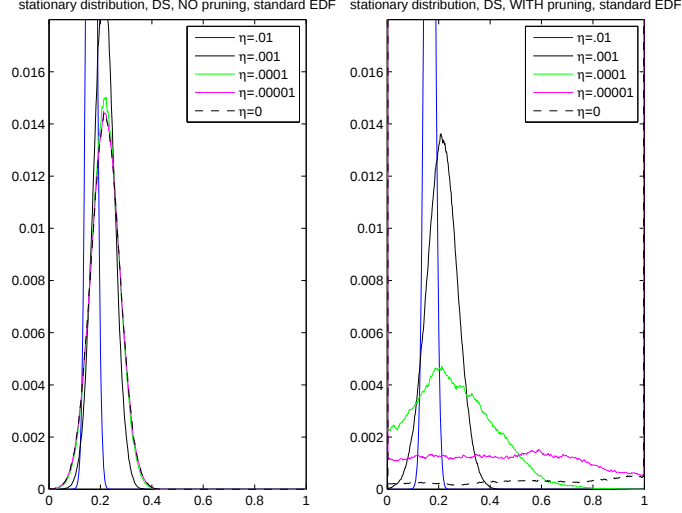
where $0 \leq \eta < \rho$ and $0 < \beta \bar{C}_A^{-\eta} < 1$, where $\bar{C}_A$ is the steady-state level of consumption. If $\eta = 0$ the discount factor $\beta(C_A)$ is constant and equal to β .

Figure ?? displays variations in η when using the traditional functional form assumed by Devereux and Sutherland, and their effect on the stationary distribution of wealth under the local and under the global portfolio solution method, when the model economies are symmetric in all aspects (and the mean wealth share is 0.5).

We then turn to asymmetric settings, to the parameterization of section 4.3, where the labor and capital income shocks of the domestic economy are half as volatile as foreign's. In this case, when sticking to functional form in equation (12) the mean of the stationary distribution is very strongly influenced when $\eta > 0$, tilted towards 0.5. Figure 14 shows that for the DS method, the size of η determines the shape of the stationary distribution but appears to have little effect on the mean of the stationary distribution.

Figure 15 repeats the variations of η in the asymmetric case and for DS, but in the case when the functional form of the endogenous discount factor follows its modified version in equation (13).

Figure 14: Role of etta in asymmetric case, pruning vs no pruning, standard EDF



$$\beta(C_A) = \beta \left(\frac{C_A}{\overline{C_A}} \right)^{-\eta}, \quad (13)$$

As outlined in section 4.3 , we show that the stationary distribution (both mean and shape) are hardly affected whether one uses the traditional functional form or the modified one. (As explained, only in case of the modified version of the EDF the iterative scheme to find the (mean) of the stochastic steady state is consistent with the DS method.

C Results without pruning

Section with analytical solution:

but not in figure ?? (and figure 11 in the appendix) (see Kim et al. (2003), Den Haan and de Wind (2009) and Lombardo (2010) for a discussion of advantages and disadvantages of the “pruning” procedure).

These comparisons show that both numerical approaches produce solutions that are quite precise and stay close to the true analytic solution during all 1000 periods of the simulated path, although, as one would expect, global solution is an order of magnitude better than the

Figure 15: Role of etta in asymmetric case, pruning vs no pruning, modified EDF

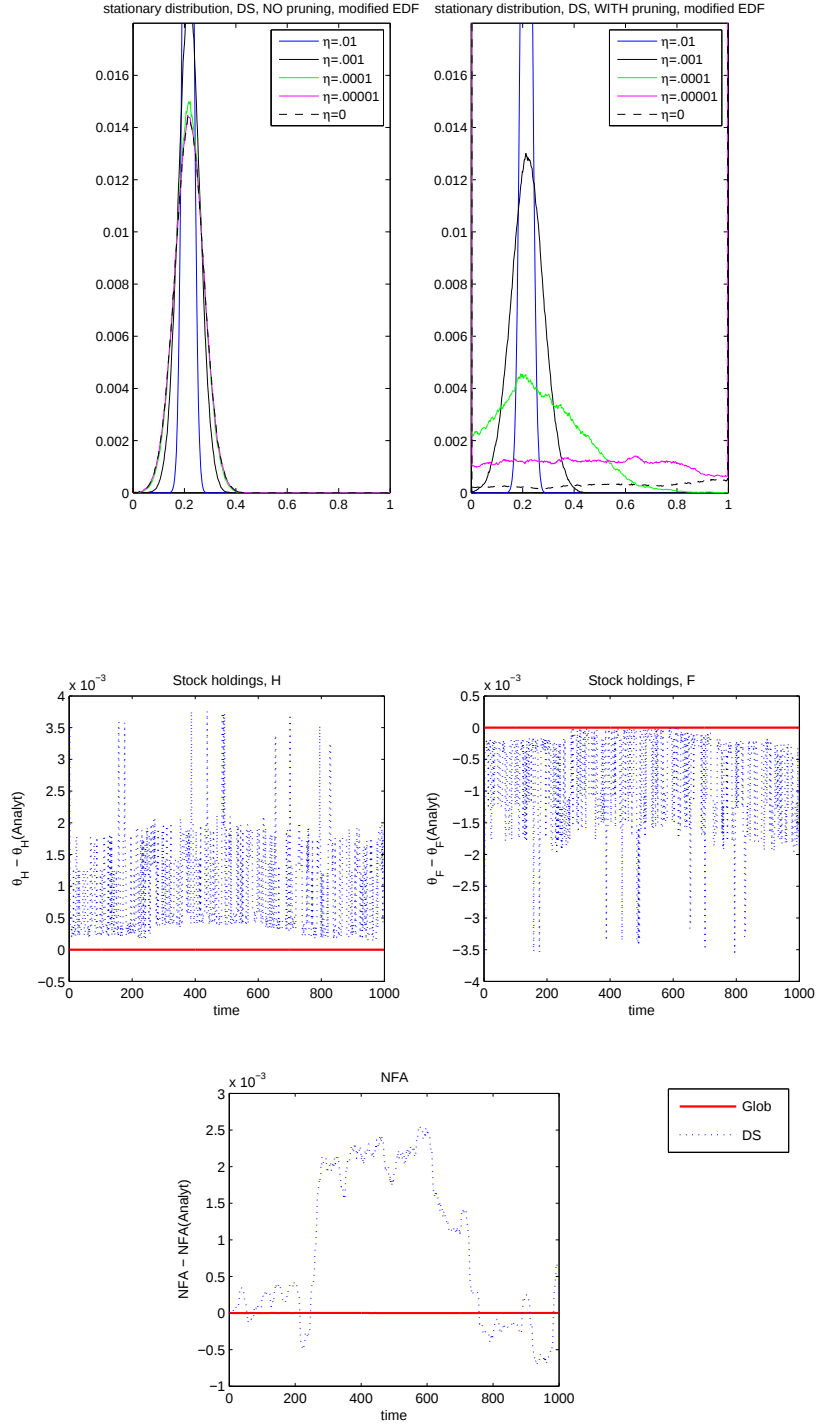


Figure 16: Global, local (no pruning) and analytic solutions

local perturbation solution. Interestingly, the simulation that uses local perturbation solution and pruning performs noticeably better in terms of the NFA position and consumption allocations than the simulation that does not use pruning.

Another potentially interesting finding is that stationary distribution in the DSSGU, without pruning, is found to be stationary. after having fed back the 0-order (stst.) portfolio (before not stationary)

It should be noted that the unit root and therefore the non-stationarity applies only the first order approximated model. If the model is solved by a second order perturbation method the stationary distribution is typically stationary (as explained in more detail in section ??).

D Short-selling constraints

Model version with 'joint short-selling constraints' (baseline):

$$\sum_{t=0}^{\infty} E_t \vartheta_t \left\{ \begin{aligned} & u(C_t) + \lambda_t \left[(q_t + Y_{K,t}) \theta_{H,t-1} + (q_t^* + Y_{K,t}^*) \theta_{F,t-1} + Y_{L,t} - C_t - q_t \theta_{H,t} + q_t^* \theta_{F,t} \right] \\ & + \mu_t \left[\theta_{H,t} (q_{t+1} + Y_{K,t+1}) + \theta_{F,t} (q_{t+1}^* + Y_{K,t+1}^*) - \bar{w} \right] \end{aligned} \right\}$$

Home's FOC:

$$\begin{aligned} \vartheta_t \lambda_t &= \vartheta_t u_{C,t} \\ q_t \vartheta_t \lambda_t &= \vartheta_t \mu_t (q_{t+1} + Y_{K,t+1}) + E_t \vartheta_{t+1} \lambda_{t+1} (q_{t+1} + Y_{K,t+1}) \\ q_t^* \vartheta_t \lambda_t &= \vartheta_t \mu_t (q_{t+1}^* + Y_{K,t+1}^*) + E_t \vartheta_{t+1} \lambda_{t+1} (q_{t+1}^* + Y_{K,t+1}^*) \\ q_t \theta_{H,t} + q_t^* \theta_{F,t} &= (q_t + Y_{K,t}) \theta_{H,t-1} + (q_t^* + Y_{K,t}^*) \theta_{F,t-1} + Y_{L,t} - C_t \\ 0 &= \mu_t \left[\theta_{H,t} (q_{t+1} + Y_{K,t+1}) + \theta_{F,t} (q_{t+1}^* + Y_{K,t+1}^*) - \bar{w} \right] \end{aligned}$$

Model's equilibrium conditions:

$$\begin{aligned}
q_t u_{C,t} &= \beta(C_t) E_t u_{C,t+1} (q_{t+1} + Y_{K,t+1}) + E_t \mu_t (q_{t+1} + Y_{K,t+1}) \\
q_t^* u_{C,t} &= \beta(C_t) E_t u_{C,t+1} (q_{t+1}^* + Y_{K,t+1}^*) + E_t \mu_t (q_{t+1}^* + Y_{K,t+1}^*) \\
q_t u_{C,t}^* &= \beta(C_t) E_t u_{C,t+1}^* (q_{t+1} + Y_{K,t+1}) + E_t \mu_t^* (q_{t+1} + Y_{K,t+1}) \\
q_t^* u_{C,t}^* &= \beta(C_t) E_t u_{C,t+1}^* (q_{t+1}^* + Y_{K,t+1}^*) + E_t \mu_t^* (q_{t+1}^* + Y_{K,t+1}^*) \\
C_t + C_t^* &= Y_t + Y_t^* \\
q_t \theta_{H,t} + q_t^* \theta_{F,t} &= (q_t + Y_{K,t}) \theta_{H,t-1} + (q_t^* + Y_{K,t}^*) \theta_{F,t-1} + Y_{L,t} - C_t \\
\theta_{H,t} + \theta_{H,t}^* &= 1 \\
\theta_{F,t} + \theta_{F,t}^* &= 1
\end{aligned}$$

approximate the expressions $+E_t \mu_t (q_{t+1} + Y_{K,t+1})$, $+E_t \mu_t (q_{t+1}^* + Y_{K,t+1}^*)$, $+E_t \mu_t^* (q_{t+1} + Y_{K,t+1})$ and $+E_t \mu_t^* (q_{t+1}^* + Y_{K,t+1}^*)$ by barrier constraints with the following functional form:

$$\begin{aligned}
E_t \mu_t (q_{t+1} + Y_{K,t+1}) &= E_t \left\{ (q_{t+1} + Y_{K,t+1}) \varkappa \kappa \max(0, -[\theta_{H,t} (q_{t+1} + Y_{K,t+1}) + \theta_{F,t} (q_{t+1}^* + Y_{K,t+1}^*) - \bar{w}]) \right\} \\
E_t \mu_t (q_{t+1}^* + Y_{K,t+1}^*) &= E_t \left\{ (q_{t+1}^* + Y_{K,t+1}^*) \varkappa \kappa \max(0, -[\theta_{H,t} (q_{t+1} + Y_{K,t+1}) + \theta_{F,t} (q_{t+1}^* + Y_{K,t+1}^*) - \bar{w}]) \right\} \\
E_t \mu_t^* (q_{t+1} + Y_{K,t+1}) &= E_t \left\{ (q_{t+1} + Y_{K,t+1}) \varkappa \kappa \max(0, -[\theta_{H,t}^* (q_{t+1} + Y_{K,t+1}) + \theta_{F,t}^* (q_{t+1}^* + Y_{K,t+1}^*) - \bar{w}]) \right\} \\
E_t \mu_t^* (q_{t+1}^* + Y_{K,t+1}^*) &= E_t \left\{ (q_{t+1}^* + Y_{K,t+1}^*) \varkappa \kappa \max(0, -[\theta_{H,t}^* (q_{t+1} + Y_{K,t+1}) + \theta_{F,t}^* (q_{t+1}^* + Y_{K,t+1}^*) - \bar{w}]) \right\}
\end{aligned}$$

where we can write $[\theta_{H,t} (q_{t+1} + Y_{K,t+1}) + \theta_{F,t} (q_{t+1}^* + Y_{K,t+1}^*) - \bar{w}]$ as

$$\begin{aligned}
&= \left[q_t \theta_{H,t} \frac{(q_{t+1} + Y_{K,t+1})}{q_t} + q_t^* \theta_{F,t} \frac{(q_{t+1}^* + Y_{K,t+1}^*)}{q_t^*} - \bar{w} \right] \\
&= \left[q_t (\theta_{H,t} - 1) \frac{(q_{t+1} + Y_{K,t+1})}{q_t} + q_t^* \theta_{F,t} \frac{(q_{t+1}^* + Y_{K,t+1}^*)}{q_t^*} - \bar{w} + (q_{t+1} + Y_{K,t+1}) \right] \\
&= \left[\alpha_{H,t} \frac{(q_{t+1} + Y_{K,t+1})}{q_t} + \alpha_{F,t} \frac{(q_{t+1}^* + Y_{K,t+1}^*)}{q_t^*} - \bar{w} + (q_{t+1} + Y_{K,t+1}) \right] \\
&= [\alpha_{H,t} (r_{t+1} - r_{t+1}^*) + (\alpha_{H,t} + \alpha_{F,t}) r_{t+1}^* - \bar{w} + (q_{t+1} + Y_{K,t+1})] \\
&= [W_t r_{t+1}^* + \alpha_{H,t} r_{t+1} - \bar{w} + (q_{t+1} + Y_{K,t+1})]
\end{aligned}$$

and $\left[\theta_{H,t}^* (q_{t+1} + Y_{K,t+1}) + \theta_{F,t}^* (q_{t+1}^* + Y_{K,t+1}^*) - \bar{w} \right]$ as

$$\begin{aligned}
&= \left[q_t \theta_{H,t}^* \frac{(q_{t+1} + Y_{K,t+1})}{q_t} + q_t^* (\theta_{F,t}^* - 1) \frac{(q_{t+1}^* + Y_{K,t+1}^*)}{q_t^*} - \bar{w} + (q_{t+1}^* + Y_{K,t+1}^*) \right] \\
&= \left[\alpha_{H,t}^* r_{t+1} + \alpha_{F,t}^* r_{t+1}^* - \bar{w} + (q_{t+1}^* + Y_{K,t+1}^*) + \alpha_{H,t}^* r_{t+1}^* - \alpha_{H,t}^* r_{t+1}^* \right] \\
&= \left[-W_t r_{t+1}^* - \alpha_{H,t} r_{t+1} - \bar{w} + (q_{t+1}^* + Y_{K,t+1}^*) \right]
\end{aligned}$$

therefore:

$$\begin{aligned}
E_t \mu_t (q_{t+1} + Y_{K,t+1}) &= E_t \left\{ (q_{t+1} + Y_{K,t+1}) \nearrow \kappa \max(0, -[W_t r_{t+1}^* + \alpha_{H,t} r_{t+1} - \bar{w} + (q_{t+1} + Y_{K,t+1})])^{\kappa-1} \right\} \\
E_t \mu_t (q_{t+1}^* + Y_{K,t+1}^*) &= E_t \left\{ (q_{t+1}^* + Y_{K,t+1}^*) \nearrow \kappa \max(0, -[W_t r_{t+1}^* + \alpha_{H,t} r_{t+1} - \bar{w} + (q_{t+1} + Y_{K,t+1})])^{\kappa-1} \right\} \\
E_t \mu_t^* (q_{t+1} + Y_{K,t+1}) &= E_t \left\{ (q_{t+1} + Y_{K,t+1}) \nearrow \kappa \max(0, -[-W_t r_{t+1}^* - \alpha_{H,t} r_{t+1} - \bar{w} + (q_{t+1}^* + Y_{K,t+1}^*)])^{\kappa-1} \right\} \\
E_t \mu_t^* (q_{t+1}^* + Y_{K,t+1}^*) &= E_t \left\{ (q_{t+1}^* + Y_{K,t+1}^*) \nearrow \kappa \max(0, -[-W_t r_{t+1}^* - \alpha_{H,t} r_{t+1} - \bar{w} + (q_{t+1}^* + Y_{K,t+1}^*)])^{\kappa-1} \right\}
\end{aligned}$$

Model version with no-short selling constraints on individual equity holdings:

$$\sum_{t=0}^{\infty} E_t \vartheta_t \left\{ u(C_t) + \lambda_t \left[(q_t + Y_{K,t}) \theta_{H,t-1} + (q_t^* + Y_{K,t}^*) \theta_{F,t-1} + Y_{L,t} - C_t - q_t \theta_{H,t} + q_t^* \theta_{F,t} \right] \right. \\
\left. + \mu_{H,t} [\theta_{H,t} - 0] + \mu_{F,t} [\theta_{F,t} - 0] \right\}$$

Home's FOC:

$$\begin{aligned}
\vartheta_t \lambda_t &= \vartheta_t u_{C,t} \\
q_t \vartheta_t \lambda_t &= \vartheta_t \mu_{H,t} + E_t \vartheta_{t+1} \lambda_{t+1} (q_{t+1} + Y_{K,t+1}) \\
q_t^* \vartheta_t \lambda_t &= \vartheta_t \mu_{F,t} + E_t \vartheta_{t+1} \lambda_{t+1} (q_{t+1}^* + Y_{K,t+1}^*) \\
q_t \theta_{H,t} + q_t^* \theta_{F,t} &= (q_t + Y_{K,t}) \theta_{H,t-1} + (q_t^* + Y_{K,t}^*) \theta_{F,t-1} + Y_{L,t} - C_t \\
0 &= \mu_t [\theta_{H,t} (q_{t+1} + Y_{K,t+1}) + \theta_{F,t} (q_{t+1}^* + Y_{K,t+1}^*) - \bar{w}]
\end{aligned}$$

Model's equilibrium conditions:

$$\begin{aligned}
q_t u_{C,t} &= \beta(C_t) E_t u_{C,t+1} (q_{t+1} + Y_{K,t+1}) + \mu_{H,t} \\
q_t^* u_{C,t} &= \beta(C_t) E_t u_{C,t+1} (q_{t+1}^* + Y_{K,t+1}^*) + \mu_{F,t} \\
q_t u_{C,t}^* &= \beta(C_t) E_t u_{C,t+1}^* (q_{t+1} + Y_{K,t+1}) + \mu_{H,t}^* \\
q_t^* u_{C,t}^* &= \beta(C_t) E_t u_{C,t+1}^* (q_{t+1}^* + Y_{K,t+1}^*) + \mu_{F,t}^* \\
C_t + C_t^* &= Y_t + Y_t^* \\
q_t \theta_{H,t} + q_t^* \theta_{F,t} &= (q_t + Y_{K,t}) \theta_{H,t-1} + (q_t^* + Y_{K,t}^*) \theta_{F,t-1} + Y_{L,t} - C_t \\
\theta_{H,t} + \theta_{H,t}^* &= 1 \\
\theta_{F,t} + \theta_{F,t}^* &= 1
\end{aligned}$$