

International Trade and the Firm Size Distribution^{*}

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PRELIMINARY AND INCOMPLETE

Abstract

This paper studies the differences in firm size distributions between European countries. We start by documenting large differences using the EFIGE database: in the countries under most severe distress in the sample (Italy and Spain) firms are relatively small compared with the remaining countries. To account for this, we develop a multi-country, continuous time version of Melitz (2003) with endogenous process innovation. Firms choose when to become exporters by paying a sunk export cost. This gives them access to a larger market, increases profits and provides incentives to grow faster. We analytically derive both an endogenous steady state Pareto size distribution of firms and the transitional distribution between steady states. The model allows us to identify the main source of the distribution's difference: export costs. This sharply contrasts the predictions of a closed economy model, where heterogeneous innovation costs account for the difference. Additionally we identify strong microeconomic similarities among countries that at first appear different, such as Spain and U.K., and strong microeconomic differences between countries that appear similar, such as Austria and Germany. International trade is essential to uncover these patterns.

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1 Introduction

What determines the observed differences in the size distribution of firms across countries? What are the implications of these differences for aggregate efficiency and welfare? The availability of firm-level information has led to a growing literature that endogenizes size distributions via innovation. However, this literature has an important drawback: it abstracts from international trade. In this paper we show quantitatively that trade frictions are a key determinant of the size distribution of firms. In particular, we find that sources of similarities in the size distributions change substantially once we add trade frictions. Apparently similar countries have strong microeconomic differences, and apparently different countries have strong microeconomic similarities.

We start by documenting large differences in employee-size distributions across seven European countries in the *European Firms In a Global Economy* (EFIGE) database: Austria, France, Germany, Hungary, Italy, Spain and U.K. While the distributions in Austria, France, Germany, Hungary and U.K. are similar to each other (and to the U.S.), the distributions in Italy and Spain are different, with a relatively large number of small firms. Figure 1 shows these distributions. The x-axis plots the log of employees, and the y-axis the log of the share of firms with more than x employees. The slope of this figure shows the "speed" at which the mass of given sizes decreases. That is, a steeper slope implies relatively higher number of small firms. We can see that exporters mainly drive the differences. Actually, after certain size level, all firms are exporters. In other words, the upper tail is entirely determined by exporters. This is consistent with Bernard and Jensen's (1999) findings for U.S. Still, figures 2 and 3 show that differences in slope remain for non-exporters.

Next, we develop a model to account for the observed differences. The model is a continuous time version of Melitz (2003) with process innovation and many heterogeneous countries. The use of continuous time allows us to describe the behavior of firms in great detail.

Any firm may innovate, i.e., increase its productivity, by incurring a convex cost, as in Rubini (2011) and Atkeson and Burstein (2010). Since more efficient firms sell to larger markets and hire more workers, **innovation endogenizes the size distribution** of firms. Additionally, firms can become exporters by incurring a sunk cost. This cost is heterogeneous across firms and independent of their size. In equilibrium, if a firm is born sufficiently large, it pays the sunk cost and exports immediately. Otherwise it grows by innovating and becomes an exporter after reaching a productivity threshold.

We fully characterize these decisions, including the instant in which they become exporters and the growth rates of both types of firms. Because exporters sell to a larger market, they make higher profits and grow faster, resulting in a flatter distribution, consistent with Figures 2 and 3. Bernard and Jensen (2001) observe these differential growth rates for the U.S. Finally, we derive the distribution of firms in closed form solution, both in steady state and in the transition between steady states. A key feature of the distributions is that they are close to Pareto, with curvature determined by the growth rate, independently of the exogenous distribution of entrants. Luttmer (2010) and Acemoglu and Cao (2010) derive similar conclusions for the closed economy case.

We introduce five dimensions of cross-country heterogeneity: (i) labor endowments; (ii) iceberg trade costs; (iii) corporate taxes; (iv) labor taxes; and (v) innovation costs. Labor endowments come from the EFIGE database. Trade costs match export volumes. Taxes come from independent sources. We calibrate the cost of innovation to match the observed slopes of the firm size distributions in the EFIGE database. We focus mainly on exporters, since these include relatively more large firms. Gabaix (2011) shows that large firms are key to understand aggregate effects.

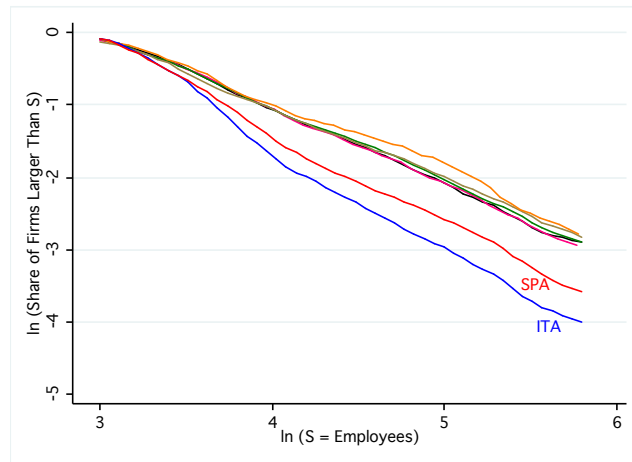


Figure 1: Firm Size Distributions

The importance of international trade can be understood by describing the shortcomings of a closed country model generating endogenous distributions. After controlling for observables, such as corporate and labor taxes, a closed economy model would attribute the differences in the distributions to innovation costs. That is, innovation being more expensive in Italy and Spain. Allowing for international trade adds

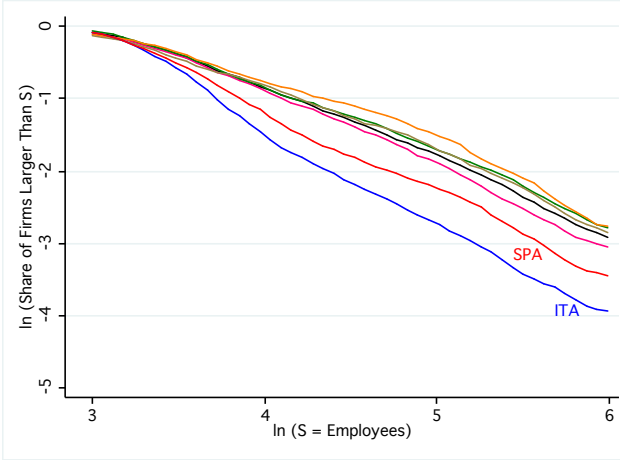


Figure 2: Exporters' Distributions

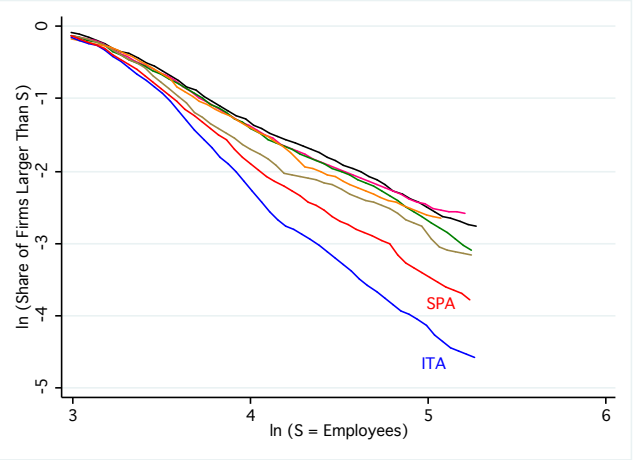


Figure 3: Non-exporters' Distributions

other mechanisms that make the above logic less clear, if not false, since innovation and trade costs have similar effects. Consider the effects of trade frictions first. Lower trade frictions drives exporters to increase innovation and growth rates, increasing aggregate export volumes. Because of current account balance, imports increase, tightening domestic competition and lowering the incentives for non-exporters to innovate. A lower cost of innovation has similar effects: it increases the incentives to innovate for all firms, so exporters become more productive, exports increase, then imports increase, and domestic competition tightens. Thus, a lower cost of innovation may lead to lower growth rates by non-exporters. Therefore, we can only identify the source of the difference in distributions by adding trade.

Contrary to the intuition under a closed economy, we do not find evidence of higher innovation costs in Italy and Spain. In fact, Austria and Hungary are the countries with the largest innovation cost, but they compensate with very low trade costs, generating a flat distribution. Spain and U.K. have the lowest innovation costs but face the largest trade costs. The difference between them is taxes. Spanish taxes are much higher, accounting for the steep distribution. Finally, controlling for trade costs, Italy, France and Germany have very similar fundamentals. Firms in these countries are taxed similarly and face similar innovations costs. However, Italy's trade costs are twice as large as those in France and Germany, generating the steep distribution.

The fact that we solve most of the model analytically makes the task of identifying the differences across countries much easier. We can directly pin down parameters to match the target moments. For instance, we identify the direct (unique) mapping between the tail of the distribution and the growth rates. Hence, each distribution tells

us the exact magnitude of the growth rates we need to replicate the observed slopes. Further, the full characterization of the firm's dynamics provides, again, a direct mapping between growth rates and each of the frictions. Thus, we can reverse-engineer the innovation cost that generate such growth rates, and hence, the slope of the distribution. The only stage in which we must rely on numerical solutions is when solving the equilibrium aggregate prices and wages. But this just involves finding the solution of a standard non-linear system of equations.

The model fails in that it cannot simultaneously match the slope of the distribution of exporters and non-exporters. In particular, the model generates size distribution of non-exporting firms that are too steep when compared with the data. A potential source of the failure is the different taxation policies applied to non-exporters. Most of the countries in the sample tax "small" firms (non-exporters in our model) with a reduced rate. Another potential source is the elasticity of substitution among goods. At this stage, and for tractability reasons, we use only an elasticity of substitution equal to two. While this is consistent with the international real business cycle theory (see Backus et al. (1994), Ruhl (2008), and Corsetti et al. (2008)), the estimations in Broda and Weinstein (2006) suggest this number is somewhat low. A larger number would reduce the difference between slopes.

Our work is related to di Giovanni et al. (2011), who account for the flatter distribution of exporters by assuming that more productive firms export to more destinations. Productivity is exogenous, so their mechanism amplifies these exogenous differences. We differ from them in that we are agnostic as to where the differences lie: we let the data tell us. It is interesting to point out that their model reaches the same conclusion as ours: Italian and Spanish firms have difficulties exporting.

2 The Basic Model

We start by presenting a simple version of the model, with two symmetric countries, no taxes, and a homogeneous sunk export cost. In the next section we extend this model to many asymmetric countries, labor and corporate taxes, and heterogeneous sunk export costs across firms.

The model builds on Melitz (2003). Time is continuous. There are two symmetric countries that produce a continuum of differentiated goods that can be traded. Each good can only be produced in one country. There is an infinitively lived representative consumer that derives utility from consuming as many goods as possible. There

are incumbent firms each period that make production, innovation, and exporting decisions. Firms die each period with an exogenous probability δ . There is a pool of potential entrants that can enter by paying an entry cost κ_E .

The preferences of the consumer in country i are given by the following utility function, for $i = 1, 2$:

$$U_i(q_i(\omega, t)) = \int_0^\infty e^{-\rho t} \ln Q_i(t) dt$$

$$Q_i(t) = \left[\int_{\Omega_i(t)} q_i(\omega, t)^{\frac{\sigma-1}{\sigma}} d\omega + \int_{\Omega_i^*(t)} q_i(\omega, t)^{\frac{\sigma-1}{\sigma}} d\omega \right]^{\frac{\sigma}{\sigma-1}}$$

where ω is the name of the good consumed, Ω_i is the set of goods produced in country i and Ω_i^* is the set of goods produced in country $j \neq i$ and imported into country i . $\sigma > 1$ is the elasticity of substitution between goods.

Each instant, there is a continuum of incumbent firms that produce the goods. Firms are owned by the domestic consumer. Each firm is a monopolist producing each good. Given a productivity level z and labor services n , the firm producing good ω has access to the following technology:

$$y(\omega; z, n) = zn$$

A firm can make innovation expenses to increase its productivity level z . We choose a functional form for the innovation cost that guarantees that in equilibrium Gibrat's law emerges. That is, in equilibrium, firm growth rate is independent of firm size. The innovation cost is in labor units. The cost of increasing productivity by an amount $\dot{z}$ depends on the current productivity level z , and is given by:

$$c(z, \dot{z}) = \frac{\kappa_I z^{\sigma-1}}{2} \left(\frac{\dot{z}}{z} \right)^2$$

To increase productivity by a certain proportion, a firm must incur a cost proportional to that proportion squared. Additionally, if a very productive firm wants to increase its productivity by 10%, it must incur a cost that is greater than what a low productivity firm would need to incur to increase its productivity by 10%. This is why the term $z^{\sigma-1}$ appears in the cost function. The term $\sigma - 1$ in the exponent is useful for the solution to be in closed form. κ_I determines how costly innovation is.

A firm can export by incurring a sunk export cost equal to κ_X units of labor. Once

a firm becomes an exporter, it remains an exporter until it dies, without the need of paying additional export costs.

There is a large pool of potential entrants that can enter anytime by incurring an entry cost equal to κ_E units of labor. After paying the entry cost, entrants draw their productivity z from an exogenous distribution $f(z)$. It is worth to notice that, in equilibrium, younger firms are relatively smaller firms. This distribution function $f(z)$ shapes the distribution of entrants, and therefore the lower tail of the size distribution of firms. We assume $z \in [1, \infty)$.

Exports are subject to iceberg trade costs. Transport depletes a proportion τ of the good. So if a consumer consumes an amount q of a good, the exporter must ship an amount $(1 + \tau)q$.

The labor market clearing condition closes the model. Let $M(t)$ be the measure of entrants at time t . The labor market clearing condition is

$$1 = \int_{\Omega_i(t)} (n(\omega, t) + \bar{c}(\omega, t)) d\omega + M(t)\kappa_E \quad (1)$$

where $\bar{c}(\omega, t)$ is the labor demand for innovation of firm ω at time t .

2.1 Symmetric Equilibrium

We identify a monopolistically competitive symmetric equilibrium for this economy. Symmetry allows us to drop the subindex i .

Let $w(t)$ be the wage rate at time t . We use this as numeraire, so set $w(t) = 1$ for all t . The price of good ω is $p(\omega)$. The equilibrium price before trade costs for an exported good is the same as the price of the same good sold domestically, so we do not introduce notation for the price of an exported good. This price is set by the monopolist to maximize profits subject to the demand for its product. This demand function comes from the consumer maximization problem. Each instant, consumers choose how much to consume of each good taking each price as given. In equilibrium, symmetry implies there is no borrowing and lending between countries, so the problem of the consumer

becomes a static problem:

$$\max \ln Q(t)$$

s.t.

$$Q(t) = \left[\int_{\Omega(t)} q(\omega, t)^{\frac{\sigma-1}{\sigma}} d\omega + \int_{\Omega^*(t)} q(\omega, t)^{\frac{\sigma-1}{\sigma}} d\omega \right]^{\frac{\sigma}{\sigma-1}}$$

$$\int_{\Omega(t)} p(\omega, t) q(\omega, t) d\omega + (1 + \tau(t)) \int_{\Omega^*(t)} p(\omega, t) q(\omega, t) d\omega = 1 + \int_{\Omega(t)} \pi(\omega) d\omega + R(t)$$

The last line is the budget constraint. $\pi(\omega, t)$ is profits of a firm ω . $R(t)$ is tax revenue at time t . When trade costs are iceberg costs, this term is equal to zero. Let the right hand side be equal to $I(t)$ (for income). The demand of a particular good is

$$q(\omega, t) = \begin{cases} p(\omega, t)^{-\sigma} P(t)^{\sigma-1} I(t) & \text{if } \omega \in \Omega(t) \\ ((1 + \tau(t))p(\omega, t))^{-\sigma} P(t)^{\sigma-1} I(t) & \text{if } \omega \in \Omega^*(t) \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

where $P(t)$ is the Dixit-Stiglitz aggregate price,

$$P(t) = \left[\int_{\Omega(t)} p(\omega, t)^{1-\sigma} d\omega + (1 + \tau)^{1-\sigma} \int_{\Omega^*(t)} p(\omega, t)^{1-\sigma} d\omega \right]^{\frac{1}{1-\sigma}} \quad (3)$$

The solution to the firms maximization problem is the mark-up rule

$$p(\omega, t) = \frac{\sigma}{\sigma - 1} \frac{1}{z(\omega, t)}$$

where $z(\omega, t)$ is the productivity z of the firm producing good ω at time t . Let $\pi_d(P, I, z)$ be the variable profits for a non exporter (profits before paying innovation or exporting costs). It is straightforward to show that profits for non exporters before paying for innovation expenses are

$$\pi_d(z, P, I) = \sigma^{-1} I P^{\sigma-1} z^{\sigma-1} = \pi_d(z, P, I) z^{\sigma-1} \quad (4)$$

Variable profits for exporters are

$$\pi_x(z, P, I, \tau) = (1 + (1 + \tau)^{1-\sigma}) \pi_d(z, P, I) z^{\sigma-1} = \pi_x(z, P, I, \tau) z^{\sigma-1} \quad (5)$$

From this point onwards, as is common in Dixit and Stiglitz (1977) models, it is convenient to change variables from the ω state to the z state, since firm decisions depend on the productivity of a firm and not on the name of the good. Let $\mu(z, t)$ be the measure of firms with productivity z at time t . Abusing our notation, the price of a good with productivity z is $p(z, t)$.

Firms decide how much to innovate each period, and non exporters choose whether to become exporters. We start by solving the problem of exporters. Their Hamilton-Jacobi-Bellman equation is

$$(\rho + \delta)V_x(z, \pi_x(t)) = \max_{\dot{z}} \pi_x(t) z^{\sigma-1} - c(z, \dot{z}) + V_{x1}(z, \pi_x(t)) \dot{z} + V_{x2}(z, \pi_x(t)) \dot{\pi}_x(t) \quad (6)$$

For non exporters, the dynamic problem consists on when to become exporters and how much to innovate¹. Their problem is a stopping time problem:

$$V_d(z, \pi_d(t), \pi_x(t)) = \max_{\dot{z}(t), T} \int_0^T e^{-(\rho+\delta)t} [\pi_d(t) z(t)^{\sigma-1} - c(z(t), \dot{z}(t))] dt + e^{-(\rho+\delta)T} [V_x(z(T), \pi_x(T)) - \kappa_X] \quad (7)$$

Let the decision to become an exporter be represented by

$$X(z, \pi_d(t), \pi_x(t)) = \begin{cases} 1 & \text{if become exporter} \\ 0 & \text{if not} \end{cases}$$

New firms enter the economy whenever their expected profits exceed the entry cost. That is, in equilibrium, the free entry condition is

$$\kappa_E = \int_1^\infty V_d(z, \pi_d(t), \pi_x(t)) (1 - X(z, \pi_d(t), \pi_x(t))) f(z) dz + \int_1^\infty (V_x(z, \pi_x(t)) X(z, \pi_d(t), \pi_x(t)) - \kappa_X) f(z) dz \quad (8)$$

¹It is straightforward to show that a non exporter will always choose to become an exporter if it survives long enough. Simply calculate its value given that it never exports and show that, for a sufficiently large z , the value of becoming an exporter exceeds the value of continuing as a non exporter.

2.2 Characterizing the Symmetric Steady State

In steady state, the aggregate state variables μ , P , and I do not change, so we omit the time index. The exporter value function is

$$(\rho + \delta)V_x(z) = \max_{\dot{z}} \pi_x z^{\sigma-1} - c(z, \dot{z}) + V_{x1}(z)\dot{z}$$

To solve this problem, we guess and verify that $V_x(z)$ is homogeneous of degree $\sigma - 1$. The solution is the productivity of exporters grows at a constant rate, and is therefore independent of firm size. Thus, Gibrat's law holds. This rate of growth is

$$\begin{aligned} g_x = \frac{\dot{z}}{z} &= \frac{\rho + \delta}{\sigma - 1} \left(1 - \sqrt{1 - h_x} \right) \\ h_x &= \frac{2\pi_x(\sigma - 1)^2}{(\rho + \delta)^2 \kappa_1} \end{aligned} \quad (9)$$

The rate of growth is increasing in exporter profits and decreasing in innovation costs.

The closed form solution for this value function is

$$V_x(z) = \kappa_1 \frac{\rho + \delta}{(\sigma - 1)^2} \left(1 - \sqrt{1 - h_x} \right) z^{\sigma-1} \quad (10)$$

The non exporter value function is

$$V_d(z) = \max_{\dot{z}(t), T} \int_0^T e^{-(\rho+\delta)t} [\pi_d z(t)^{\sigma-1} - c(z(t), \dot{z}(t))] dt + e^{-(\rho+\delta)T} [V_x(z(T)) - \kappa_X]$$

This is a stopping time problem. Divide this problem into two steps. Taking T as given, solve the problem

$$\begin{aligned} &\max_{\dot{z}(t)} \int_0^T e^{-(\rho+\delta)t} [\pi_d z(t)^{\sigma-1} - c(z(t), \dot{z}(t))] dt \\ &s.t. \\ &z(0) = \underline{z}, z(T) = z_x(\underline{z}) \end{aligned}$$

where $\underline{z}$ is the starting point and $z_x(\underline{z})$ is an ending point, taken as given for now. This is a problem of calculus of variations. We solve it in the Appendix. The solution is that

non exporter productivity grows at a constant rate

$$\begin{aligned} g_d = \frac{\dot{z}}{z} &= \frac{\rho + \delta}{\sigma - 1} \left(1 - \sqrt{1 - h_d} \right) \\ h_d &= \frac{2\pi_d(\sigma - 1)^2}{(\rho + \delta)^2 \kappa_1} \end{aligned} \quad (11)$$

Once again, Gibrat's law holds, and the rate of growth is increasing in non exporter profits, and decreasing in κ_I .

Step 2 requires us to plug in this solution into the value function, and take derivatives with respect to T to find the optimal stopping time. The solution, solved in the Appendix, is that all non exporters grow until they hit a productivity level z_x , point at which they become exporters. Thus, only large firms export.

The non exporter value function is

$$V_d(z) = \frac{\kappa_I g_d}{\sigma - 1} z^{\sigma-1} + \frac{\kappa_x g_d (\sigma - 1)}{(\rho + \delta - g_d(\sigma - 1))} \left(\frac{z}{z_x} \right)^{\frac{\rho + \delta}{g_d}} \quad (12)$$

And

$$z_x = \left[\frac{(\rho + \delta) \kappa_X (\sigma - 1)}{\kappa_I (\rho + \delta - g_d(\sigma - 1)) (g_x - g_d)} \right]^{\frac{1}{\sigma-1}} \quad (13)$$

Proposition 1 *The smooth pasting condition holds. That is, the non exporter value function smooth pastes into the exporter value function net of the export cost. Moreover, the smooth pasting is from above.*

Proof: See the Appendix.

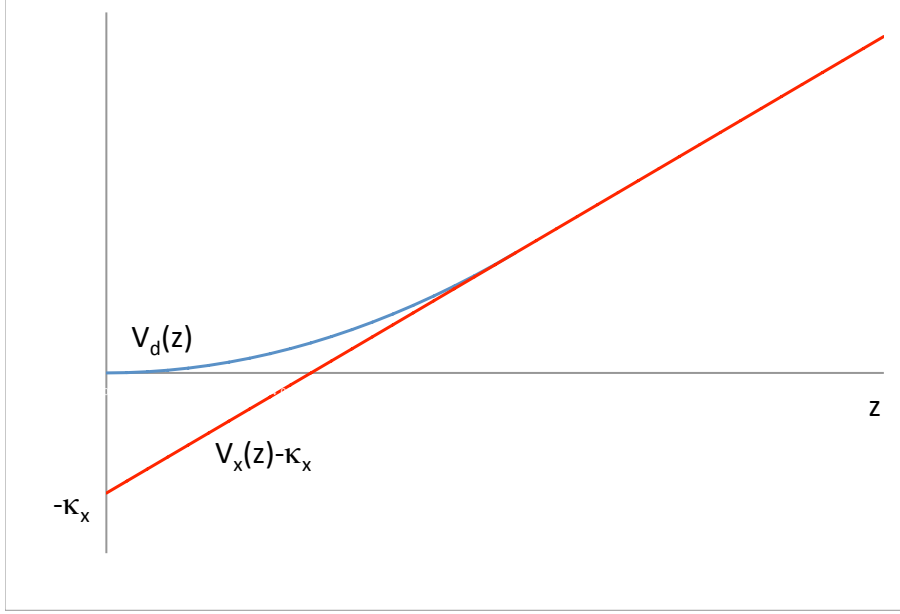
Figure 4 shows these value functions for $\sigma = 2$, in which case $V_x(z)$ is linear.

We next characterize the steady state distribution. Define $\hat{\mu}(t, z)$ as the measure of firms with productivity z in period t . Define $\mathcal{Z} = [z_1, z_2]$. The following expression is the law of motion for the measure of productivity:

$$\hat{\mu}(t + dt, \mathcal{Z}) = \int_{\mathcal{Z}} \hat{\mu}(t, z - \dot{z}dt) e^{-\delta dt} dz + \int_{\mathcal{Z}} \int_0^{dt} M(t) f(z - \dot{z}s) e^{-\delta s} ds dz$$

That is, the measure of firms with productivity $z \in \mathcal{Z}$ is the sum of the incumbent firms that had a productivity $z - \dot{z}dt$, dt periods ago, plus all the firms that were born and in

Figure 4: The Smooth Pasting Condition for $\sigma = 2$



period $t + dt$ had productivity $z \in \mathcal{Z}$. In the Appendix, we show that this expression can be reduced to

$$\mu(z) = Mf(z)dt + e^{-\delta dt}\mu(z - dz)$$

We can reduce this expression to

$$\delta\mu(z) = Mf(z) - \mu'(z)\dot{z}$$

For non exporters, that is, $z \in [1, z_x]$, this is

$$\delta\mu(z) = Mf(z) - \mu'(z)g_d z$$

This is a first order differential equation, with boundary condition $\mu(1) = Mf(1)$. The solution to this equation is

$$\mu(z) = z^{-\frac{\delta}{g_d}} [G_d(z) + Mf(1) - G_d(1)]$$

where $G_d(z) = \frac{M}{g_d} \int z^{\frac{\delta}{g_d}-1} f(z) dz$. Similarly, for exporters, that is, for $z > z_x$,

$$\delta\mu(z) = Mf(z) - \mu'(z)g_x z$$

The boundary condition is $\mu(z_x) = z_x^{-\frac{\delta}{\Delta_n}} [G_d(z) + Mf(1) - G_d(1)]$.

The solution is

$$\mu(z) = z^{-\frac{\delta}{g_x}} \left[G_x(z) + z_x^{\frac{\delta}{g_x}-\frac{\delta}{g_d}} [G_d(z_x) + Mf(1) - G_d(1)] - G_x(z_x) \right]$$

Gathering all together

$$\mu(z) = \begin{cases} z^{-\frac{\delta}{g_d}} [G_d(z) + Mf(1) - G_d(1)] & \text{if } z \leq z_x \\ z^{-\frac{\delta}{g_x}} \left[G_x(z) + z_x^{\frac{\delta}{g_x}-\frac{\delta}{g_d}} [G_d(z_x) + Mf(1) - G_d(1)] - G_x(z_x) \right] & \text{if } z > z_x \end{cases} \quad (14)$$

where $G_x(z) = \frac{M}{g_x} \int z^{\frac{\delta}{g_x}-1} f(z) dz$.

Proposition 2 *For a wide family of distributions $f(z)$, $\mu(z)$ features Zipf's law. That is, as z grows, $\mu(z)$ approaches a Pareto distribution.*

Proof: We next show that the upper tail of this distribution satisfies Zipf's law. For this, focus on the segment $z > z_x$, which is the upper tail. Notice that

$$\mu(z) = z^{-\frac{\delta}{g_x}} M \int z^{-\frac{\delta}{g_x}-1} f(z) dz + K M z^{-\frac{\delta}{g_x}} \text{ for } z > z_x$$

where K is a constant. Taking limits as z grows of the first term in the right hand side,

$$\lim_{z \rightarrow \infty} \frac{\frac{\delta}{g_x} M \int z^{-\frac{\delta}{g_x}-1} f(z) dz}{z^{-\frac{\delta}{g_x}}} = \lim_{z \rightarrow \infty} \frac{\frac{\delta}{g_x} M z^{-\frac{\delta}{g_x}-1} f(z) dz}{\frac{\delta}{g_x} z^{\frac{\delta}{g_x}-1}} = \lim_{z \rightarrow \infty} M f(z)$$

Thus,

$$\lim_{z \rightarrow \infty} \mu(z) = \lim_{z \rightarrow \infty} M \left(f(z) + K z^{-\frac{\delta}{g_x}} \right)$$

If $f(z)$ goes to zero sufficiently fast (faster than $z^{-\delta/g_x}$), then in the limit, $\mu(z)$ approaches a Pareto distribution, that is, Zipf's law holds. Examples of $f(z)$ that go fast enough to zero are the uniform distribution, the normal distribution, and the Pareto as long as the curvature parameter is larger than δ/g_x . $\square$

We can solve for the entire steady state equilibrium as a system of three equations and three unknowns. The unknowns are π , M , P . Given these variables, we can identify all the remaining variables in the model. The three equations that pin down these variables are the index price equation, the free entry condition, and labor market clearing. These equations are:

$$\begin{aligned}\kappa_E &= \int_1^{z_x} V_d(z) f(z) dz + \int_{z_x}^{\infty} [V_x(z) - \kappa_x] f(z) dz \\ 1 &= \int_1^{z_x} \left[\pi_d z^{\sigma-1} + \frac{\kappa_I z^{\sigma-1}}{2} g_d^2 \right] \mu(z) dz + \int_{z_x}^{\infty} \left[\pi_x z^{\sigma-1} + \frac{\kappa_I z^{\sigma-1}}{2} g_x^2 \right] \mu(z) dz \\ P^{1-\sigma} &= \int_1^{z_x} p(z)^{1-\sigma} \mu(z) dz + (1 + (1 + \tau)^{1-\sigma}) \int_{z_x}^{\infty} p(z)^{1-\sigma} \mu(z) dz\end{aligned}$$

To solve, notice that the free entry equation is a function only of π_d . Given π_d , labor market clearing is only a function of M . The last equation determines P .

2.3 Characterizing the Transitional Dynamics

We compute the equilibrium during the transition between steady states by solving a system of partial differential equations. These partial differential equations (PDEs) are given by the measure of exporters and non exporters at each point in time. We assume the economy is in the high trade cost steady state for all $t < 0$, and at $t = 0$ there is an unexpected (small) reduction in trade costs. Trade costs remain at this low value for all $t \geq 0$.

We make the simplifying assumption that new firms enter the economy with common productivity $z = 1$. This implies that the free entry condition is²

$$\begin{aligned}\kappa_E = V_d(1) &= \\ &= \int_0^{z_x} \left(\pi_d(t) - \frac{\kappa_1}{2} g_d(t) \right) e^{(g_d(t) - \rho - \delta)(\sigma-1)t} dt + \int_{z_x}^{\infty} \left(\pi_x(t) - \frac{\kappa_1}{2} g_x(t) \right) e^{(g_x(t) - \rho - \delta)(\sigma-1)t} dt\end{aligned}$$

This equation is key to our solution strategy. Suppose that $\pi_d(t) = \pi_{d1}$ for all $t \geq 0$. This would imply that $\pi_x(t) = \pi_{x1}$, $g_d(t) = g_{d1}$, $g_x(t) = g_{x1}$ for all $t \geq 0$. If this is the case, then the free entry condition is one equation and one unknown for all $t \geq 0$, where the unknown is π_{d1} . But it turns out that we know the solution to this equation. The equilibrium π_{d1} is the new steady state level of profits, since the same free entry

²Since the drop in trade costs is small enough, this condition holds at every point in time.

condition must be satisfied in the steady state with low trade costs.

If these levels of profits and growth rates satisfy the entire system of equations that characterize the equilibrium, then we have found an equilibrium transition path. We show in the Appendix that these equations are satisfied in equilibrium.

This simplifies the analysis a great deal. It implies that we can take profits and growth rates as given to solve for the PDEs that characterize the transition. We solve these PDEs in the Appendix. Here, we lay out our results, that is, the measure of firms along the transition as well as aggregate variables such as price indices and consumption levels.

Our solution strategy divides time into intervals of length t_1 , where t_1 is the time it takes a new born firm to become an exporter, that is, $e^{t_1 g_{d1}} = z_{x1}$. The reason for this is that we know that for all $t > t_1$, all firms born before the reduction in trade costs are exporting. This helps us solve the system of equations by grouping cohorts of firms according to their export status. For example, for $t \in (0, t_1)$, firms born before the reduction in trade costs are both exporters and non exporters, but for $t > t_1$, these are all exporters.

Let $\mu(t, z)$ denote the measure of firms with productivity z at time t . Essentially, the PDE to solve is

$$\mu_t(t, z) + \mu_z(t, z)g_i z = -\delta\mu(t, z) \quad (15)$$

where $i = x1$ if the firm is an exporter, and $i = d1$ if it is not. Notice that this equation is the same as the steady state equation where $\mu_t(t, z) = 0$. Equation 15 is the Kolmogorov forward equation along the transition. This can be solved given the value of g_i and an initial condition.

Thus, to solve, we need to take into account that different firms differ in their initial condition. This implies identifying different areas at different points in time and treat them separately. A full description of the procedure is in the Appendix.

We next describe the solution to these PDEs for two time intervals: $t \in (0, t_1)$ and $t \in (t_1, 2t_1)$:

For $t \in (0, t_1)$,

$$\begin{aligned}
\mu(t, z) &= M \left(t - \frac{1}{g_{1,d}} \log(z) \right) z^{-\frac{\delta}{g_{1,d}}} && \text{if } z \leq z^*(t) \\
&= M_0 e^{\frac{t\delta}{g_{0,d}} - \frac{g_{1,d} - g_{0,d}}{g_{0,d}} \frac{\delta}{g_{0,d}}} z^{-\frac{\delta}{g_{0,d}}} && \text{if } z^*(t) < z \leq z_{1,x} \\
&= M_0 z_{1,x}^{\frac{\delta}{g_{0,d}} \left(\frac{g_{1,d}}{g_{1,x}} - 1 \right)} e^{t\delta \left(\frac{g_{1,d}}{g_{0,d}} - 1 \right)} z^{-\delta \frac{g_{1,d}}{g_{1,x}g_{0,d}}} && \text{if } z_{1,x} < z \leq z^{1*}(t) \\
&= M_0 e^{\frac{t\delta}{g_{0,d}} \frac{g_{1,x} - g_{0,d}}{g_{0,d}} - \frac{\delta}{g_{0,d}}} z^{-\frac{\delta}{g_{0,d}}} && \text{if } z^{1*}(t) < z \leq z^{3*}(t) \\
&= M_0 e^{\frac{t\delta}{g_{0,x}} \frac{g_{1,x} - g_{0,x}}{g_{0,x}} - \frac{\delta}{g_{0,x}}} z^{-\frac{\delta}{g_{0,x}}} && \text{if } z > z^{3*}(t)
\end{aligned}$$

where $z^*(t) = e^{g_{1,d}t}$, $z^{1*}(t) = e^{g_{1,x}t} z_{1,x}$, and $z^{3*}(t) = e^{g_{1,x}t} z_{0,x}$. A subscript 0 indicates old steady state. A subscript 1 indicates the new steady state.

For $t_1 < t < 2t_1$

$$\begin{aligned}
\mu(t, z) &= M \left(t - \frac{\log(z)}{g_{1d}} \right) z^{-\frac{\delta}{g_{1d}}} && \text{if } z \leq z_{1x} \\
&= M \left(t - \frac{\log(z)}{g_{1x}} + \left(\frac{1}{g_{1x}} - \frac{1}{g_{1d}} \right) \log(z_{1x}) \right) z_{1x}^{\delta \left(\frac{1}{g_{1x}} - \frac{1}{g_{1d}} \right)} z^{-\frac{\delta}{g_{1x}}} && \text{if } z_{1x} \leq z \leq z^{2*}(t) \\
&= M_0 z_{1x}^{\frac{\delta}{g_{0d}} \left(\frac{g_{1d} - g_{1x}}{g_{1x}} \right)} e^{t\delta \left(\frac{g_{1d} - g_{0d}}{g_{0d}} \right)} z^{-\delta \frac{g_{1d}}{g_{1x}g_{0d}}} && \text{if } z^{2*}(t) \leq z \leq z^{1*}(t) \\
&= M_0 e^{\frac{t\delta}{g_{0d}} \left(\frac{g_{1x} - g_{0d}}{g_{0d}} \right) - \frac{\delta}{g_{0d}}} z^{-\frac{\delta}{g_{0d}}} && \text{if } z^{1*}(t) \leq z \leq z^{3*}(t) \\
&= M_0 e^{\frac{t\delta}{g_{0x}} \left(\frac{g_{1x} - g_{0x}}{g_{0x}} \right) - \frac{\delta}{g_{0x}}} z^{-\frac{\delta}{g_{0x}}} && \text{if } z > z^{3*}(t)
\end{aligned}$$

where $z^{2*}(t) = z_{1x} e^{g_{1x}t}$.

Notice that to know the measure of firms, we still need to determine entry at each point, that is, $M(t)$ for $t > 0$. We do this via the labor market clearing condition. The solution implies that entry takes the following shape

For $t \in (0, t_1)$,

$$M(t) = m_0 + m_1 e^{m_2 t} + m_3 e^{m_4}$$

And for $t \in (t_1, 2t_1)$,

$$M(t) = \tilde{m}_0 + \tilde{m}_1 e^{\tilde{m}_2 t} + \tilde{m}_3 e^{\tilde{m}_4}$$

where the m 's and $\tilde{m}$'s are constants. See the online appendix for details.

3 The Multi-Country Model

This section modifies the model to introduce many different countries, heterogeneous export costs across firms, corporate and labor taxes. We also assume that all firms are born with $z = 1$. The first modification allows us to calibrate the model to the data on the seven countries in the EFIGE database. The second modification implies that in equilibrium there will be some exporters smaller than some non exporters. We also add corporate and labor taxes. Corporate taxes come from the data. We calculate labor taxes as labor wedges, and we obtain these from OECD calculations for married couples with two kids. We assume innovation is not expensed, and hence is not deducted from taxes. We skip the characterisation of the steady state and the transitional dynamics, since it is similar as the two symmetric country case presented before.

Index countries by $i = 1, \dots, J$. Let L_i be labor in country i , κ_{Ii} be country specific innovation costs, and let τ_{ij} be the tariff in country j for a good produced in country i , with $\tau_{ii} = 0$. Corporate taxes are τ_{ci} and labor taxes τ_{li} . All other variables remain as before.

In addition to the equilibrium variables from before, we need to introduce wages in country i , w_i and $\pi_{ij}(z)$, the profits of a type z firm in country i exporting to j . We use $\pi_{ii}(z)$ to denote domestic profits. Profits and wages are before taxes.

It is straightforward to show

$$\pi_{ii}(z) = (w_i(1 + \tau_{li}))^{1-\sigma} z^{\sigma-1} P_i^{\sigma-1} I_i \sigma^{-\sigma} (\sigma - 1)^{\sigma-1}$$

Profits for an exporter in country i exporting to j are

$$\pi_{ij}(z) = \left(\frac{w_i(1 + \tau_{li})}{w_j(1 + \tau_{lj})} \right)^{1-\sigma} \pi_{jj}(z) (1 + \tau_{ij})^{1-\sigma}$$

Taxes affect growth rates as follows.

$$g_{di} = \frac{\rho + \delta}{\sigma - 1} \left(1 - \sqrt{1 - h_{di}} \right), \text{ where } h_{di} = \frac{2\pi_{di}(1 - \tau_{ci})(\sigma - 1)^2}{(\rho + \delta)^2 w_i(1 + \tau_{li})\kappa_I}$$

$$g_{xi} = \frac{\rho + \delta}{\sigma - 1} \left(1 - \sqrt{1 - h_{xi}} \right), \text{ where } h_{xi} = \frac{2\pi_{xi}(1 - \tau_{ci})(\sigma - 1)^2}{(\rho + \delta)^2 w_i(1 + \tau_{li})\kappa_I}$$

We model exporting as follows. Firms differ in their sunk export cost. Entrants draw a fixed export cost κ_x from a distribution $S(\kappa_x)$, with density $s(\kappa_x)$. Firms that pay this cost have access to all foreign markets, so in equilibrium an exporter will export to all

destinations at once.

The distribution $s(\kappa)$ is the same across countries. The export threshold depends on the draw of κ_x , and the value functions depend on this parameter, that is, $V_x(z, \kappa_x)$, $V_d(z, \kappa_x)$. For each z , define $\kappa_x(z)$ as the level of κ_x that makes that firm indifferent between exporting and not exporting

$$\kappa_x(z) = \Phi_i \hat{z}^{\sigma-1}$$

where $\Phi_i = \frac{\kappa_{Ii}(\rho+\delta-g_{di}(\sigma-1))(g_{xi}-g_{di})}{(\rho+\delta)(\sigma-1)}$.

To build the distribution of firms, we must incorporate the fact that a fraction of non exporters are always switching to the export sector, for all levels of z . In an interval dt , the measure of non exporters with productivity z that become exporters is

$$\begin{aligned} Pr(\hat{\kappa}_x(z - \dot{z}dt) < \kappa < \hat{\kappa}_x(z)) &= Pr(\hat{\kappa}_x(z) - \hat{\kappa}'_x(z)\dot{z}dt < \kappa < \hat{\kappa}_x(z)) = \\ &= s(\hat{\kappa}_x(z))\hat{\kappa}'_x(z)\dot{z}dt = s(z^{\sigma-1}\Phi)z^{\sigma-2}(\sigma-1)\Phi\dot{z}dt \end{aligned} \quad (16)$$

The distribution of non exporters is determined by entering firms, firms that were non exporters in the past and remain non exporters, minus firms that turned into exporters. A measure $(1 - S(\hat{\kappa}_z(z)))Mf(z)$ enter as non exporters. The fraction of firms that become exporters is given by (16). The distribution of non exporters is therefore

$$\mu_d(z) (\delta + s(z^{\sigma-1}\Phi)z^{\sigma-2}(\sigma-1)\Phi g_d z) = -\mu'_d(z)g_d z + (1 - S(\hat{\kappa}_x(z)))Mf(z)$$

Exporters include entrants, firms that were exporting in the past, and firms that become exporters.

$$\delta\mu_x(z) = -\mu'_x(z)g_x z + \mu_d(z)s(z^{\sigma-1}\Phi)z^{\sigma-2}(\sigma-1)\Phi g_d z + S(\hat{\kappa}_x(z))Mf(z)$$

The aggregate equations describing free entry, labor market clearing and the pricing

equation are

$$\begin{aligned}
(1 + \tau_{li})w_i\kappa_{Ei} &= \int_{\Phi_i}^{\infty} V_d(1, \kappa_x)s(\kappa_x)d\kappa_x + \int_0^{\Phi_i} (V_x(1, \kappa_x) - \kappa_x)s(\kappa_x)d\kappa_x \\
(1 + \tau_{li})w_iL_i &= \left(\frac{\pi_{ii}}{1 + \tau_{wi}} + \frac{w_i\kappa_{Ii}}{2}g_{id}^2 \right) \int z^{\sigma-1}\mu_{id}(z)dz + \\
&\quad \sum_j \left(\frac{\pi_{ij}}{1 + \tau_{wi}} + \frac{w_i\kappa_{Ii}}{2}g_{ix}^2 \right) \int z^{\sigma-1}\mu_{ix}(z)dz + w_i\kappa_{Ei} \\
P_i^{1-\sigma} &= \int_1^{\infty} p_i(z)^{1-\sigma}\mu_{di}(z)dz + \sum_{j=1}^J (1 + \tau_{ji})^{1-\sigma} \int_1^{\infty} p_j(z)\mu_{xj}(z)dz
\end{aligned}$$

The equation that is different from the simple model is free entry. Entrants become exporters immediately if their draw of $\kappa_x < \Phi_i$.

We impose trade balance. Exports from a firm z in country i to country j are $(1 + \tau_{ij})p_i(z)q_j((1 + \tau_{ij})p_i(z))$. Total exports from country i to country j are $\sigma \int \pi_{ij}(z)\mu_{xi}(z)dz$. Total country i exports are $\sum_{j \neq i} \sigma \int \pi_{ij}(z)\mu_{xi}(z)dz$. Similarly, imports into country i are $\sum_{j \neq i} \sigma \int \pi_{ji}(z)\mu_{xj}(z)dz$. Thus, trade balance is

$$\sum_{j \neq i} \int \pi_{ij}(z)\mu_{xi}(z)dz = \sum_{j \neq i} \int \pi_{ji}(z)\mu_{xj}(z)dz$$

4 Calibration

We set the total number of countries equal to 7 to replicate the available countries in the EFIGE database. We set $\rho = 0.04$ and $\delta = 0.10$. That is, a steady state risk free interest rate of 4% and assume that 10% of firms die each year.

We assume the following form for the distribution of sunk export costs:

$$s(\kappa) = \begin{cases} \frac{\kappa^{-1}}{\log(b) - \log(a)} & \text{if } a < \kappa < b \\ 0 & \text{otherwise} \end{cases}$$

This functional form allows us to derive the distribution of firms in closed form solution. We set $a = 0.5$ so that a fraction of entrants export immediately. We set $b = 10,000$, so that very few levels of z include only exporters. We set $\sigma = 2$ as in Ruhl (2008) and Rubini (2011), consistent with the international business cycle literature. The entry cost, κ_e does not affect our results.

Labor taxes come from the OECD's wedges for married couples with two children. Corporate taxes come from Wikipedia.³

We use the EFIGE database to calibrate the remaining parameters. We normalize labor in Germany equal to 1, and set labor in the remaining countries to match relative employment. We assume the variable export cost depends on the country of origin but not the country of destination, $\tau_{ij} = \tau_i$ for all $j \neq i$. The τ_i 's match the ratios of exports to total sales.⁴ Innovation costs κ_{Ii} match the slopes of the distributions of exporters with more than 19 employees in each country. We match this slopes for the largest firms in the model. The slope of the upper tail in the model converges to $1 - \delta/g_{xi}$. We calibrate κ_i so that this number matches the distribution in the data.

5 Results

Table 1 shows the values for the calibration of the key parameters.

Table 1: Calibrated Parameters

Country	L	κ_I	τ	τ_c	τ_l
Germany	1.00	326	0.65	0.29	0.33
U.K.	0.67	275	1.36	0.25	0.27
Italy	0.62	330	0.97	0.31	0.37
France	0.54	316	0.57	0.33	0.42
Spain	0.36	277	1.46	0.30	0.34
Austria	0.11	368	0.45	0.25	0.37
Hungary	0.10	370	0.69	0.10	0.36

Contrary to what a model without international trade would conclude, Italy and Spain do not have really high innovation costs. In fact, Spain has the lowest innovation cost in the sample. Their steep slope is more related to international trade issues. Their export costs are really high. This implies low export profits, and low growth rates, thus explaining the relatively large number of small firms. Additionally, the corporate taxes in these countries are relatively large, also reducing the growth rates. There are key differences between Italy and Spain. Italy has a higher innovation cost, while Spain has higher trade costs. In equilibrium, this implies that in spite of the lower trade cost in Italy, the slope is higher.

³http://en.wikipedia.org/wiki/Tax_rates_of_Europe

⁴The lack of data on bilateral exports does not allow for a calibration of individual τ_{ij} 's.

We can learn also from the behavior of the remaining countries. The smallest countries in the sample, Austria and Hungary, have the highest innovation costs. Austria compensates by being extremely export oriented, with the lowest export cost in the sample. It also has a very low relative corporate tax. Hungary compensates mainly with a really low corporate tax.

U.K. has unexpectedly large export costs. But can achieve healthy growth rates due to their low taxes, both labor and corporate. Germany and France are similar in many dimensions. They have similar innovation costs, trade costs, and taxes. France has somewhat lower trade costs, which compensates for the smaller domestic market.

The model performs very poorly in replicating the distribution of non exporters. While in the data the ratio of the slope of the distributions between exporters and non exporters is between 0.73 and 0.93, in the model these ratios are lower than 0.21. This is big a shortcoming of the model. We are currently working on this issue. One potential reason is fiscal distortions. Small firms often are subject to lower fiscal burden than larger firms. For example, in Italy, firms with less than 15 employees are not subject to firing costs, while larger firms are. policies applied to non-exporters. Another potential area to work with is the elasticity of substitution among goods. At this stage, and for tractability reasons, we use an elasticity of substitution equal to two. Increasing this elasticity might reduce the difference in exporter profits versus non exporter profits by increasing the elasticity of demand for each good. For tractability reasons, we only work with an elasticity parameter equal to two. We plan to experiment with larger numbers in the future.

6 Conclusion

The large availability of firm level data allows economists to analyze the distribution of firms in a country and derive conclusions based on its shape. Typically, models that focus on these distributions work under the assumption of closed economy. We have argued that this abstraction is very costly in countries that are open, such as European countries, both qualitatively and quantitatively.

In particular, we find that when analyzing the distribution of firms in Europe, a model of closed economy will wrongly conclude that innovation costs are too high in Italy and Spain (abstracting from other frictions such as taxes). We find that this conclusion is wrong. Spain has a really low innovation cost, similar to U.K.'s. And Italy's innovation cost is no larger than Germany's or France's.

Introducing trade allows for a more complete picture of the economies, and a better understanding of the data. Some countries that look very different by looking at their distributions, are in fact quite similar. Spain is actually very close to the U.K. Essentially, they only differ in their tax policy. Italy, Germany and France only differ in their trade costs.

Other countries look very similar, but are quite different. Austria and Germany have a similar distribution of firms. But Austria has really large innovation costs, that compensate for their low taxes and export costs. The U.K. is widely different to Germany and France, even when their distributions are similar. The similarity appears because of the combination of differences, that “cancel each other”. U.K. cancels out large trade costs with low innovation costs. Germany and France are more moderate in both areas.

Our analysis provides a useful tool for policymakers trying to make ends meet in Italy and Spain. An area of concern is trade. These countries need to focus their policies in areas that will lower their trade costs.

We conclude by pointing out that an open economy is extremely important to make inferences from the observed firm size distributions. Abstracting from trade will only result in misguided recommendations.

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Appendix A The Problem of Non Exporters

The steady state value function for a non exporter is

$$V_d(z) = \max_{\dot{z}} \int_0^{T_z} e^{-\rho t} \left[\pi_d z^{\sigma-1} - \frac{\kappa_1 z^{\sigma-1}}{2} \left(\frac{\dot{z}}{z} \right)^2 \right] dt + e^{-\rho T_z} [V(z_x) - \kappa_x]$$

We solve this in two stages. We first solve the first integral for an arbitrary length of time T , and then we show that T is determined by the time it takes firms to reach a level of productivity z_x .

Given T , the first problem is a problem of calculus of variations:

$$\max_{\dot{z}(t)} \int_0^{T_z} e^{-\rho t} \left[\pi_d z(t)^{\sigma-1} - \frac{\kappa_1 z(t)^{\sigma-1}}{2} \left(\frac{\dot{z}(t)}{z(t)} \right)^2 \right] dt$$

Subject to

$$z(0) = z, z(T_z) = z_x$$

To solve this, we make the following guess and verify it later:

$$\dot{z} = g_d z \Rightarrow \ddot{z} = g_d \dot{z}$$

Solving this functional problem as in Chiang (2000),

$$0 = -\kappa_1 z^{\sigma-2} g_d \frac{\dot{z}}{z} - \kappa_1 (\sigma-3) z^{\sigma-2} \left(\frac{\dot{z}}{z} \right)^2 + z^{\sigma-2} \rho \kappa_1 \frac{\dot{z}}{z} - z^{\sigma-2} (\sigma-1) \pi_d + z^{\sigma-2} (\sigma-3) \frac{\kappa_1}{2} \left(\frac{\dot{z}}{z} \right)^2$$

The solution to this equation is

$$\frac{\dot{z}}{z} = g_d = \frac{\rho}{\sigma-1} \left(1 - \sqrt{1 - h_d} \right) \quad (17)$$

where

$$h_d = \frac{2\pi_d (\sigma-1)^2}{\rho^2 \kappa_1}$$

Next we show that non exporters will wait until their productivity equals z_x to

become exporters. The value of a non exporter with productivity z for an arbitrary switching time T_z is

$$\begin{aligned}
V_d(z) &= \int_0^{T_z} e^{-\rho t} \left[\pi_d z(t)^{\sigma-1} - \frac{\kappa_1 z(t)^{\sigma-1}}{2} \left(\frac{\dot{z}(t)}{z(t)} \right)^2 \right] dt + e^{-\rho T_z} [V(z_x) - \kappa_x] \\
&= \int_0^{T_z} e^{-\rho t} \left[\pi_d z^{\sigma-1} e^{g_d(\sigma-1)t} - \frac{\kappa_1 z^{\sigma-1} e^{g_d(\sigma-1)t}}{2} g_d^2 \right] dt + e^{-\rho T_z} [B z^{\sigma-1} e^{\Delta(\sigma-1)T_z} - \kappa_x] \\
&= z^{\sigma-1} \frac{\pi_d - \kappa_1 g_d^2/2}{\rho - g_d(\sigma-1)} (1 - e^{-(\rho - g_d(\sigma-1))T_z}) + z^{\sigma-1} B e^{-(\rho - g_d(\sigma-1))T_z} - e^{-\rho T_z} \kappa_x
\end{aligned}$$

Maximize with respect to T_z :

$$z^{\sigma-1} e^{-(\rho - g_d(\sigma-1))T_z} (\pi_d - g_d^2 \kappa_1/2) - (\rho - g_d(\sigma-1)) z^{\sigma-1} B e^{-(\rho - g_d(\sigma-1))T_z} + \rho e^{-\rho T_z} \kappa_x = 0$$

The solution to this equation is $z e^{g_d T_z} = z_x$. Notice $\rho - g_d(\sigma-1) = \rho(\sqrt{1-h_d})$. Thus,

$$z_x^{\sigma-1} = \frac{\rho \kappa_x}{\rho(\sqrt{1-h_d})B + g_d^2 \kappa_1/2 - \pi_d}$$

Appendix B The Smooth Pasting Condition

Smooth pasting requires two conditions

$$V_x(z_x) - \kappa_x = V_d(z_x) \text{ and}$$

$$V'_x(z_x) = V'_d(z_x)$$

Additionally, to show that the smooth pasting is from above,

$$V'_x(z) \geq V'_d(z) \text{ for all } z \leq z_x$$

$$V_x(1) - \kappa_x \leq V_d(1)$$

The last inequality follows from the equilibrium condition that not all firms export.

To see the remaining conditions hold, rewrite the value functions as

$$V_x(z) = B z^{\sigma-1}$$

$$V_d(z) = C z^{\sigma-1} + (B - C) z_x^{\sigma-1 - \frac{\rho+\delta}{g_d}} z^{\frac{\rho+\delta}{g_d}} - \kappa_x z_x^{-\frac{\rho+\delta}{g_d}} z^{\frac{\rho+\delta}{g_d}}$$

where $B = \kappa_I \frac{\rho+\delta}{(\sigma-1)^2} (1 - \sqrt{1 - h_x})$ and $C = \frac{\kappa_I g_d}{\sigma-1}$.

First notice that $V_d(z_x) = V_x(z_x) - \kappa_x$:

$$V_d(z_x) = C z_x^{\sigma-1} + (B - C) z_x^{\sigma-1-\frac{\rho+\delta}{g_d}} z_x^{\frac{\rho+\delta}{g_d}} - \kappa_x z_x^{-\frac{\rho+\delta}{g_d}} z_x^{\frac{\rho+\delta}{g_d}} = B z_x^{\sigma-1} - \kappa_x = V_x(z_x)$$

Next consider the relation between $V'_x(z)$ and $V'_d(z)$.

$$V'_x(z) - V'_d(z) = (\sigma - 1)(B - C) z^{\sigma-2} - \frac{\rho + \delta}{g_d} z^{\frac{\rho+\delta}{g_d}-1} z_x^{-\frac{\rho+\delta}{g_d}} ((B - C) z_x^{\sigma-1} - \kappa_x)$$

From the derivation of z_x , we can obtain

$$z_x^{\sigma-1}(B - C) = \frac{(\rho + \delta)\kappa_x}{\rho + \delta - g_d(\sigma - 1)}$$

Inserting this above,

$$\begin{aligned} V'_x(z) - V'_d(z) &= (\sigma - 1)(B - C) z^{\sigma-2} - \frac{\rho + \delta}{g_d} z^{\frac{\rho+\delta}{g_d}-1} z_x^{-\frac{\rho+\delta}{g_d}} \kappa_x \left(\frac{g_d(\sigma - 1)}{\rho + \delta - g_d(\sigma - 1)} \right) \\ &= (\sigma - 1)(B - C) z^{\sigma-2} - \frac{(\rho + \delta) z^{\frac{\rho+\delta}{g_d}-1}}{(\rho + \delta) - g_d(\sigma - 1)} z_x^{-\frac{\rho+\delta}{g_d}} \kappa_x (\sigma - 1) \\ &= (\sigma - 1)(B - C) z^{\sigma-2} \left[1 - \left(\frac{z_x}{z} \right)^{\sigma-1-\frac{\rho+\delta}{g_d}} \right] \end{aligned} \quad (18)$$

Setting $z = z_x$, we show $V'_x(z_x) = V'_d(z_x)$. Smooth pasting from above implies $V'_x(z) \geq V'_d(z)$ for all $z \leq z_x$. This holds since $z \leq z_x$ and $\sigma - 1 - \frac{\rho+\delta}{g_d} < 0$.

□

Appendix C Deriving the Endogenous Distribution of Firms

Define $\mathcal{Z} = [z_1, z_2]$

$$\hat{\mu}(t + dt, \mathcal{Z}) = \int_{\mathcal{Z}} \hat{\mu}(t, z - \dot{z}dt) e^{-\delta dt} dz + \int_{\mathcal{Z}} \int_t^{t+dt} M(s) f(z - \dot{z}s) e^{-\delta s} ds dz$$

Taking limits as $z_1 \rightarrow z_2 \rightarrow z$

$$\hat{\mu}(t + dt, z) = \hat{\mu}(t, z - \dot{z}dt) e^{-\delta dt} + \int_t^{t+dt} M(s) f(z - \dot{z}s) e^{-\delta s} ds$$

For small dt , the following holds:

$$\begin{aligned}\hat{\mu}(t+dt, z) &\approx \hat{\mu}(t, z) + \hat{\mu}_1(t, z)dt \\ \hat{\mu}(t, z - \dot{z}dt) &\approx \hat{\mu}(t, z) - \hat{\mu}_2(t, z)\dot{z}dt \\ f(z - \dot{z}s) &\approx f(z) - f'(z)\dot{z}s \\ e^{-\delta dt} &\approx (1 - \delta dt)\end{aligned}$$

Thus,

$$\begin{aligned}\hat{\mu}(t, z) + \hat{\mu}_1(t, z)dt &= \hat{\mu}(t, z) - \hat{\mu}_2(t, z)\dot{z}dt - \delta dt (\hat{\mu}(t, z) + \hat{\mu}_2(t, z)\dot{z}dt) + \\ &\int_t^{t+dt} M(s)(f(z) - f'(z)\dot{z}s)(1 - \delta s)ds\end{aligned}$$

Note that in steady state $\hat{\mu}_1(t, z) = 0$ and $M(s) = M$. Putting all together,

$$\hat{\mu}(t, z) = \hat{\mu}(t, z) - \hat{\mu}_2(t, z)\dot{z}dt - \delta dt (\hat{\mu}(t, z) + \hat{\mu}_2(t, z)\dot{z}dt) + M \int_t^{t+dt} (f(z) - f'(z)\dot{z}s)(1 - \delta s)ds$$

Solving for the last integral

$$\begin{aligned}\int_t^{t+dt} f(z)ds &= f(z)dt & \int_t^{t+dt} -f'(z)\dot{z}sds &= -f'(z)\dot{z}(dt)^2/2 \\ \int_t^{t+dt} -\delta s f(z)ds &= -\delta f(z)(dt)^2/2 & \int_t^{t+dt} -\delta f'(z)\dot{z}s^2ds &= -\delta f'(z)\dot{z}(dt)^3/3\end{aligned}$$

Eliminating all the terms with dt elevated to a power larger than 1,

$$\hat{\mu}(t, z) = \hat{\mu}(t, z) - \hat{\mu}_2(t, z)\dot{z}dt - \delta \hat{\mu}(t, z)dt + M f(z)dt$$

Cancelling terms and dividing by dt ,

$$\delta \hat{\mu}(t, z) = M f(z) - \hat{\mu}_2(t, z)\dot{z}$$

Appendix D The Distribution in the Transition

Please see the online appendix at

https://sites.google.com/a/asu.edu/loris-rubini/Online_App_Piguillem_Rubini.pdf?attredirects=0&d=1.